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Finiteness principles for smooth convex functions

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ABSTRACT

Let $E \subset \mathbb{R}^n$ be a compact set, and $f : E \to \mathbb{R}$. How can we tell if there exists a convex extension $F \in C^{1,1}(\mathbb{R}^n)$ of f, i.e. satisfying $F|_E = f|_E$? Assuming such an extension exists, how small can one take the Lipschitz constant $\mathrm{Lip}(\nabla F) := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|\nabla F(x)-\nabla F(y)|}{|x-y|}$? We provide an answer to these questions for the class of strongly convex functions by proving that there exist constants $k^{\#} \in \mathbb{R}$ and $C > 0$ depending only on the dimension n, such that if for every subset $S \subset E$, $\#S \leq k^{\#}$, there exists an η-strongly convex function $F^S \in C^{1,1}(\mathbb{R}^n)$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$, then there exists an $\frac{\eta}{C}$-strongly convex function $F \in C^{1,1}_c(\mathbb{R}^n)$ satisfying $F|_E = f|_E$, and $\mathrm{Lip}(\nabla F) \leq CM^2/\eta$. Further, we prove a Finiteness Principle for the space of convex functions in $C^{1,1}(\mathbb{R})$ and that the sharp finiteness constant for this space is $k^{\#} = 5$.

1. Introduction

Let $C^{1,1}_c(\mathbb{R}^n)$ be the space of convex, differentiable functions with Lipschitz continuous gradient. We say that a function $F : \mathbb{R}^n \to \mathbb{R}$ is *η-strongly convex*, for $\eta \geq 0$, if the function $F(x) - \frac{\eta}{2}|x|^2$ is convex. Let $E \subset \mathbb{R}^n$ be compact and $f : E \to \mathbb{R}$. In this paper, we provide an answer to the following questions: Under what conditions

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on η and f does there exists an η-strongly convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ that is an extension of f, i.e., satisfying $F|_E = f|_E$? Assuming such an extension exists, how small can one take the Lipschitz constant $\mathrm{Lip}(\nabla F)$ for an η-strongly convex extension F of f? Recall the Lipschitz constant of a function $G : \mathbb{R}^n \to \mathbb{R}^d$ is defined as $\mathrm{Lip}(G) := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|G(x)-G(y)|}{|x-y|}$. We prove the following result:

Theorem 1. *Let $E \subset \mathbb{R}^n$ be compact, the constants η, M satisfy $M > \eta > 0$, and the function $f : E \to \mathbb{R}$. There exist $k^\# \in \mathbb{N}$ and $C > 0$ depending only on the dimension n such that the following holds: Suppose that for all $S \subset E$ satisfying $\#S \leq k^\#$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$. Then for any $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, there exists an η/p^2-strongly convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F|_E = f|_E$, and $\mathrm{Lip}(\nabla F) \leq C_1 q^2 M^2/\eta$.*

Fix a constant $C_0 > C_1 q^2$ and $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$; suppose the hypotheses of Theorem 1 are satisfied by E, f, M, and $\eta \in (\frac{C_1 q^2}{C_0} M, M)$. Then Theorem 1 produces an η/p^2-strongly convex extension of f, $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $\mathrm{Lip}(\nabla F) \leq C_0 M$. But if instead the hypotheses are satisfied by E, f, M, and η much smaller than M ($\eta \in [0, \frac{C_1 q^2}{C_0} M)$), we expect this theorem is not optimal. We conjecture that satisfaction of the hypotheses of Theorem 1 ensures the existence of a strongly convex extension of f, i.e. $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $\mathrm{Lip}(\nabla F) \leq CM$, where C depends on n, p, and q, but not on η or M. Indeed this is true in dimension $n = 1$, which is our second result:

Theorem 2. *Let $E \subset \mathbb{R}$ be compact, the constants η, M satisfy $M > \eta \geq 0$, and the function $f : E \to \mathbb{R}$. Suppose for every $S \subset E$ satisfying $\#S \leq k_1^\# = 5$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R})$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$. Then there exists an η-strongly convex function $F \in C_c^{1,1}(\mathbb{R})$ satisfying $F|_E = f|_E$ and $\mathrm{Lip}(\nabla F) \leq 5M$.*

Remark 1.1. In Theorem 2, no constant smaller than $k_1^\# = 5$ will suffice (the *sharp* finiteness constant for $C_c^{1,1}(\mathbb{R})$ is $k_1^\# = 5$). To see this, consider the following example: Let $E \subset \mathbb{R}$ be $E := \{-2, -1, 0, 1, 2\}$; for $x \in E$, let $f(x) := |x|$. For every set $S \subset E$ satisfying $\#S \leq 4$, one can construct a convex function $F^S \in C_c^{1,1}(\mathbb{R})$ satisfying $F|_S = f|_S$, but any convex extension of f, $F : \mathbb{R} \to \mathbb{R}$ must satisfy $F(x) = |x|$ for $x \in [-2, 2]$, which is not differentiable at $x = 0$. Thus, we must have $k_1^\# > 4$.

Our results are the first attempt to understand the constrained interpolation problem for *convex* functions in $C_c^{1,1}(\mathbb{R}^n)$.[2] We build on techniques used to understand whether a function has a smooth extension despite obstacles to their direct application.

Let $\mathbb{X}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ be a complete semi-normed space of continuous functions. Given a compact set $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$, how can we tell if there exists

[2] The constrained interpolation problem where the interpolating function is required to be non-negative has been studied by C. Fefferman, A. Israel, and K. Luli in [8], and K. Luli, and F. Jiang in [12] and [13].

$F \in \mathbb{X}(\mathbb{R}^n)$ extending f, that is satisfying $F|_E = f|_E$? In [14,15], Pavel Shvartsman answered this question for the linear space $\mathbb{X}(\mathbb{R}^n) = C^{1.1}(\mathbb{R}^n)$ through a *Finiteness Principle*.

We say that there is a *Finiteness Principle for* $\mathbb{X}(\mathbb{R}^n)$ if there exist $k \in \mathbb{N}$, $C > 0$ depending on $\mathbb{X}(\mathbb{R}^n)$ such that given $E \subset \mathbb{R}^n$ finite and $f : E \to \mathbb{R}$, if we assume for every $S \subset E$ satisfying $\#S \leq k$, there exists $F^S \in \mathbb{X}(\mathbb{R}^n)$ satisfying $F^S|_S = f|_S$ and $\|F^S\|_{X(\mathbb{R}^n)} \leq 1$, then there exists $F \in \mathbb{X}(\mathbb{R}^n)$ such that $F|_E = f|_E$ and $\|F\|_{\mathbb{X}(\mathbb{R}^n)} \leq C$.

Further, P. Shvartsman proved that the sharp finiteness constant for $C^{1,1}(\mathbb{R}^n)$ is $k = 3 \cdot 2^{n-1}$, and conjectured with Yuri Brudnyi that Finiteness Principles for the linear spaces $C^m(\mathbb{R}^n)$ and $C^{m-1,1}(\mathbb{R}^n)$ would hold in [3,4]. In [5,6] Charles Fefferman proved Finiteness Principles for these spaces ($C^{m-1,1}(\mathbb{R}^n)$ and $C^m(\mathbb{R}^n)$).

Theorems 1 and 2 are progress toward the proof of a Finiteness Principle for the non-linear space of smooth convex function $C^{1,1}_c(\mathbb{R}^n)$. Our hope is this work and the continued study of finiteness principles for smooth convex functions allow the development of algorithms for constructing smooth, convex extensions of a function (or its approximation) analogous to the work by C. Fefferman and Boaz Klartag in [9,10] for $C^m(\mathbb{R}^n)$.

Our proofs of Theorems 1 and 2 rely on an inequality relating the *jets* of a convex function in $C^{1,1}_c(\mathbb{R}^n)$. Let $\mathcal{P}$ be the space of real-valued affine (degree one) polynomials. For $F \in C^1(\mathbb{R}^n)$, we define the jet of F at x, $J_xF \in \mathcal{P}$ as $J_xF(y) := F(x) + \langle \nabla F(x), y - x\rangle$. Let the function $F \in C^{1,1}_c(\mathbb{R}^n)$ be convex; as a consequence of Taylor's inequality,

$$F(x) - J_yF(x) \geq \frac{1}{2\operatorname{Lip}(\nabla F)}|\nabla F(x) - \nabla F(y)|^2 \qquad (x, y \in \mathbb{R}^n).$$

In Section 2.1, we prove this inequality. In [2,1], Daniel Azagra, Erwan Le Gruyer, and Carlos Mudarra proved a partial converse to this inequality, criteria for convex $C^{1,1}$-extension of degree one polynomials defined on a closed set $E \subset \mathbb{R}^n$, which is a key component of our proofs:

Theorem 3 *(D. Azagra, E. Le Gruyer, and C. Mudarra [1], Theorem 2.4). Let $E \subset \mathbb{R}^n$ be closed and the polynomials $(P_x)_{x\in E} \subset \mathcal{P}$ satisfy for all $x, y \in E$,*

$$P_x(x) - P_y(x) \geq \frac{1}{2M}|\nabla P_x - \nabla P_y|^2. \tag{1.1}$$

Then there exists a convex function $F \in C^{1,1}_c(\mathbb{R}^n)$ satisfying $J_xF = P_x$ for all $x \in E$ and $\operatorname{Lip}(\nabla F) \leq M$*.*

We now give a sketch of the proof of Theorem 1. We write c, C, C', etc. to denote constants dependent only on the dimension n. By appealing to the Arzelà-Ascoli Theorem, we reduce to the case $E \subset \mathbb{R}^n$ finite. In Proposition 2.6, we prove that if $(P_x)_{x\in E}$ satisfy (1.1) and

$$P_x(y) + \frac{\eta}{2}|y-x|^2 \leq P_y(y) \quad \text{for all } x, y \in E, \tag{1.2}$$

then the conclusions of Theorem 3 hold for a *strongly* convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $\mathrm{Lip}(\nabla F) \leq CM$. Thus, given $f : E \to \mathbb{R}$, we aim to find $(P_x)_{x\in E} \subset \mathcal{P}$ satisfying $P_x(x) = f(x)$ and the inequalities (1.1) and (1.2) with uniform constants M and η for all $x, y \in E$. To do this, we introduce an approximation of the set of prospective jets of strongly-convex $C_c^{1,1}(\mathbb{R}^n)$ extensions of f. For $x \in E$, $\eta > 0$, let $\Gamma_\eta^E(x) \subset \mathcal{P}$ be

$$\Gamma_\eta^E(x) := \{P \in \mathcal{P} : P(x) = f(x) \text{ and } P(y) + \frac{\eta}{2}|y-x|^2 \leq f(y) \text{ for all } y \in E \setminus \{x\}\}.$$

Immediately, we see for an η-strongly convex extension $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F|_E = f|_E$ and $\mathrm{Lip}(\nabla F) \leq M$, we have $J_x F \in \Gamma_\eta^E(x)$ for all $x \in E$. We prove we can choose $(P_x)_{x\in E} \subset \mathcal{P}$ so that

$$P_x \in \Gamma_\eta^E(x) \qquad (x \in E), \text{ and} \tag{1.3}$$

$$\sup_{x,y\in E, x\neq y} \left\{ \frac{|\nabla P_x - \nabla P_y|}{|x-y|} \right\} \leq C'M. \tag{1.4}$$

Together (1.3) and (1.4) imply

$$f(y) - P_x(y) \geq \frac{\eta}{2}|y-x|^2 \geq \frac{\eta}{2(C'M)^2}|\nabla P_x - \nabla P_y|^2 \qquad (x, y \in E).$$

Hence, this choice of $(P_x)_{x\in E}$ satisfies (1.1) with a constant $(C'M)^2/\eta$, and we can apply Proposition 2.6.

To prove we can choose $(P_x)_{x\in E}$ satisfying (1.3) and (1.4), we use Helly's Theorem, (following P. Shvartsman in [14], [15], and C. Fefferman in [5], [6]) and a Finiteness Principle for Smooth Selection proved by C. Fefferman, Arie Israel, and Kevin Luli in [7]; see also C. Fefferman and P. Shvartsman's results in [11].

Theorem 4 *(Helly, see e.g. [17]). Let $\mathcal{J}$ be a finite family of convex subsets of $\mathbb{R}^d$, and suppose every $(d+1)$ elements of the family has non-empty intersection. Then the entire family has non-empty intersection. If $\mathcal{J}$ is infinite, the sets must also be compact for the result to follow.*

For $D \geq 1$, let $C^{0,1}(\mathbb{R}^n, \mathbb{R}^D)$ denote the Banach space of all $\mathbb{R}^D$-valued Lipschitz functions F on $\mathbb{R}^n$, for which the norm $\|F\|_{C^{0,1}(\mathbb{R}^n,\mathbb{R}^D)} = \sup_{x\in\mathbb{R}^n}\{|F(x)|\} + \mathrm{Lip}(F)$, is finite.

Theorem 5 *(C. Fefferman, A. Israel, and K. Luli (Theorem 3(B) of [7])). There exist $k_s^\# = k_s^\#(n, D) \in \mathbb{N}$ and $C^\# = C^\#(n, D) > 0$ such that the following holds: Let $E \subset \mathbb{R}^n$ be arbitrary. For each $x \in E$, let $K(x) \subset \mathbb{R}^D$ be a closed convex set. Suppose that for each $S \subset E$ with $\#S \leq k_s^\#$, there exists $F^S \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^D)$ with norm $\|F^S\|_{C^{0,1}(\mathbb{R}^n,\mathbb{R}^D)} \leq 1$,* *such that* $F^S(x) \in K(x)$ *for all* $x \in S$*. Then there exists* $F \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^D)$ *with norm* $\|F^S\|_{C^{0,1}(\mathbb{R}^n,\mathbb{R}^D)} \le C^{\#}$*, such that* $F(x) \in K(x)$ *for all* $x \in E$*.*

We assume the following Finiteness Hypothesis: for every $S \subset E$, $\#S \le k^{\#}$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \le M$, with $k^{\#} = k_s^{\#}(n+2)+1$ and $k_s^{\#} = k_s^{\#}(n,n)$ from Theorem 5. Using this hypothesis, we can apply Helly's Theorem to show the hypotheses of Theorem 5 are satisfied for the family of convex sets $(K(x) = \{\nabla P : P \in \Gamma_\eta^E(x)\})_{x\in E}$ in $\mathbb{R}^n$. Thus, we can apply Theorem 5 to produce a Lipschitz selection $G \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)$ from $(K(x))_{x\in E}$ satisfying $\mathrm{Lip}(G) \le C'M$. For $x \in E$, we let $P_x(x) := f(x)$ and $\nabla P_x := G(x)$, and as promised, $(P_x)_{x\in E}$ satisfies (1.3) and (1.4).

This concludes our sketch of the proof of Theorem 1. The rest of the paper is organized as follows: In Section 2, we adapt Theorems 3 and 5 to our setting and analyze sets approximating the set of jets of smooth convex extensions of the function f, including $\Gamma_\eta^E(x)$. In Section 3, we prove Theorem 1 for dimension $n \ge 1$. In Section 4, we detail technical estimates that hold only in dimension $n = 1$, and prove Theorem 2.

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1.2. Notation

Let $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$. We use the following notation:

$$\begin{aligned}
&|x| := |x|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2} && (x = (x_1, \ldots, x_n) \in \mathbb{R}^n);\\
&B(y,R) := \{x \in \mathbb{R}^n : |x-y| < R\} && (y \in \mathbb{R}^n,\ R \ge 0);\\
&D_{xy}^f := \frac{f(y)-f(x)}{y-x} && (x, y \in E, x \ne y,\ E \subset \mathbb{R}).
\end{aligned}$$

Let $\Omega \subset \mathbb{R}^n$ be a *domain* (i.e., a non-empty, connected open set), and let the vector-valued function $F : \Omega \to \mathbb{R}^D$. The Lipschitz constant of the function F is

$$\mathrm{Lip}(F;\Omega) := \sup_{x,y\in\Omega, x\ne y} \frac{|F(x)-F(y)|}{|x-y|}.$$

Where the domain Ω is evident, we write $\mathrm{Lip}(F)$ in place of $\mathrm{Lip}(F;\Omega)$.

For $m = 0$ or $m = 1$, let $C^m(\Omega)$ denote the Banach space of real-valued C^m functions F on Ω for which the norm

$$\|F\|_{C^m(\Omega)} = \sup_{x\in\Omega} \max_{|\alpha|\le m} |\partial^\alpha F(x)|$$

is finite, and $C^{m,1}(\Omega)$ denote the Banach space of real-valued C^m functions F on Ω with Lipshitz continuous gradient for which the norm

$$\|F\|_{C^{m,1}(\Omega)} = \|F\|_{C^m(\Omega)} + \mathrm{Lip}(\nabla^m F; \Omega)$$

is finite.

For $D \geq 1$, let $C^{m,1}(\mathbb{R}^n, \mathbb{R}^D)$ denote the Banach space of vector-valued C^m functions F on $\mathbb{R}^n$, for which the norm

$$\|F\|_{C^{m,1}(\mathbb{R}^n,\mathbb{R}^D)} = \sup_{x\in\mathbb{R}^n} \max_{|\alpha|\leq m} |\partial^\alpha F(x)| + \mathrm{Lip}(\nabla^m F; \mathbb{R}^n)$$

is finite.

Let $C^{m,1}_{loc}(\Omega)$ denote the space of functions F on $\mathbb{R}^n$ satisfying $\|F\|_{C^{m,1}(\Omega')} < \infty$ for all bounded open sets $\Omega' \subset\subset \Omega$.

Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let $C^{1,1}_c(\Omega) \subset C^{1,1}_{loc}(\Omega)$ denote the space of convex, differentiable functions with Lipschitz continuous gradient.

Let $\mathcal{P}$ be the space of real-valued affine (degree one) polynomials. For $F \in C^1(\mathbb{R}^n)$, we define the jet of F at x, $J_x F \in \mathcal{P}$ as

$$J_x F(y) := F(x) + \langle \nabla F(x), y - x\rangle.$$

For each $x \in \mathbb{R}^n$, the jet product $\odot_x$ on $\mathcal{P}$ is defined by

$$P \odot_x Q := J_x(P \cdot Q) \quad (P, Q \in \mathcal{P}).$$

Let $\mathcal{R}_x = (\mathcal{P}, \odot_x)$ be the ring of 1-jets of functions at $x \in \mathbb{R}^n$.

Let $E \subset \mathbb{R}^n$ and $P_x \in \mathcal{R}_x$ for all $x \in E$; then we say $(P_x)_{x\in E} \subset \mathcal{P}$ is a Whitney field on E. Let $Wh(E)$ be the set of all Whitney fields on E.

For $\gamma_x \in \mathcal{R}_x$, $\gamma_y \in \mathcal{R}_y$, we will say $\gamma_x \sim_M \gamma_y$ if the following inequalities are satisfied:

$$\gamma_x(x) - \gamma_y(x) \geq \frac{1}{2M}|\nabla\gamma_x - \nabla\gamma_y|^2 \tag{1.5}$$

$$\gamma_y(y) - \gamma_x(y) \geq \frac{1}{2M}|\nabla\gamma_x - \nabla\gamma_y|^2. \tag{1.6}$$

We write c, C, C', etc. to denote constants dependent only on the dimension n.

2. Technical tools

Let $E \subset \mathbb{R}^n$ be compact, and let $f : E \to \mathbb{R}$. We now introduce certain convex subsets of $\mathcal{R}_x$ that reflect constraints on the jet of a convex extension of f. For $S \subset E$ and $x \in S$, let $\Gamma^0(x; f)$, $\Gamma^S(x; f)$, and $\Gamma^S_\eta(x; f) \subset \mathcal{R}_x$ be

$$\Gamma^0(x; f) := \{P \in \mathcal{R}_x : P(x) = f(x)\}, \text{ and}$$
$$\Gamma^S(x; f) := \{P \in \Gamma^0(x; f) : P(y) \le f(y) \text{ for all } y \in S\}$$
$$\Gamma_\eta^S(x; f) = \{P \in \Gamma^0(x; f) : P(y) + \frac{\eta}{2}|y - x|^2 \le f(y) \text{ for all } y \in S \setminus \{x\}\} \tag{2.1}$$
$$= \bigcap_{y \in S\setminus\{x\}} \{P \in \Gamma^0(x; f) : P(y) + \frac{\eta}{2}|y - x|^2 \le f(y)\}.$$

Where the function f is evident, we will not write f; i.e., we write $\Gamma^0(x)$ in place of $\Gamma^0(x; f)$.

Any convex extension of the function f, $F : \mathbb{R}^n \to \mathbb{R}$, satisfies $\partial F(x) \subset \Gamma^E(x; f)$ for all $x \in E$, where $\partial F(x) := \{\xi \in \mathbb{R}^n : F(y) \ge F(x) + \langle \xi, y - x\rangle \text{ for all } y \in \Omega\}$ is the *subdifferential of F at x*.

The sets $\Gamma^0(x)$, $\Gamma^S(x)$, and $\Gamma_\eta^S(x)$ are convex subsets of $\mathcal{R}_x$; this property is immediate for $\Gamma^0(x)$. To see the set $\Gamma^S(x)$ is convex, notice if $P \in \{P \in \Gamma^0(x) : P(y) \le f(y)\}$ for $y \in S\setminus\{x\}$, then $P(x) = f(x)$ and ∇P satisfies the linear inequality $f(x)+\langle \nabla P, y-x\rangle \le f(y)$. Hence, $\{P \in \Gamma^0(x) : P(y) \le f(y)\}$ is convex, implying $\Gamma^S(x) = \bigcap_{y\in S\setminus\{x\}}\{P \in \Gamma^0(x) : P(y) \le f(y)\}$ is convex. Similarly, if $P \in \{P \in \Gamma^0(x) : P(y) + \frac{\eta}{2}|y - x|^2 \le f(y)\}$ for $y \in S \setminus \{x\}$, then $P(x) = f(x)$ and ∇P satisfies the linear inequality $f(x) + \langle \nabla P, x - y\rangle + \frac{\eta}{2}|y - x|^2 \le f(y)$, implying $\Gamma_\eta^S(x)$ is convex as an intersection of convex sets.

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ be compact and $f : E \to \mathbb{R}$. Suppose $\Gamma^E(x; f) \ne \emptyset$ for all $x \in E$. Then there exists a convex (and thus, locally Lipschitz) function $F : \mathbb{R}^n \to \mathbb{R}$ extending f.*

Proof. Let $F : \mathbb{R}^n \to \mathbb{R}$ be $F(x) := \sup_{y\in E}\{P_y(x) : P_y \in \Gamma^E(y)\}$; then $F|_E = f|_E$ and as the supremum of convex functions, F is convex. □

Lemma 2.2. *Let $E \subset \mathbb{R}^n$ be compact and $f : E \to \mathbb{R}$. Suppose $\Gamma_\eta^E(x; f) \ne \emptyset$ for all $x \in E$; then there exists an η-strongly convex function $F : \mathbb{R}^n \to \mathbb{R}$ extending f.*

Proof. Suppose $\Gamma_\eta^E(x; f) \ne \emptyset$ for all $x \in E$, then $\Gamma^E(x; g) \ne \emptyset$ for all $x \in E$, where $g(x) := f(x) - \frac{\eta}{2}|x|^2$. By Lemma 2.1 there exists a convex function $G : \mathbb{R}^n \to \mathbb{R}$ satisfying $G|_E = g|_E$. Thus, $F(x) := G(x) + \frac{\eta}{2}|x|^2$ is η-strongly convex and satisfies $F|_E = f|_E$. □

Lemma 2.3. *Let $E \subset \mathbb{R}^n$ be compact and $f : E \to \mathbb{R}$. Let $S \subset E$ and $F \in C_c^{1,1}(\mathbb{R}^n)$ be an η-strongly convex function satisfying $F|_S = f|_S$. Then $J_x F \in \Gamma_\eta^S(x; f)$ for all $x \in S$.*

Proof. Let $G : \mathbb{R}^n \to \mathbb{R}$ be $G(x) := F(x) - \frac{\eta}{2}|x|^2$. Because F is η-strongly convex, G is convex, implying for all $x, y \in S$, $J_x G(y) \le G(y)$; equivalently,

$$F(x) - \frac{\eta}{2}|x|^2 + \langle \nabla F(x) - \eta x, y - x\rangle \le F(y) - \frac{\eta}{2}|y|^2 \quad (x, y \in S).$$

This reduces to $J_xF(y) + \frac{\eta}{2}|y-x|^2 \leq F(y) = f(y)$, implying $J_xF \in \Gamma_\eta^S(x;f)$ for all $x \in S$. $\square$

2.1. An estimate on 1-jets of convex functions in $C_c^{1,1}$

Recall that $B(a,r)$ is the Euclidean ball of radius r centered at a: $B(a,r) := \{x \in \mathbb{R}^n : |x-a| < r\}$. For a function $F \in C^{1,1}(B(a,2R))$, we use Taylor's inequality to bound the difference between the function value and its jet evaluated at a point:

$$|F(z) - J_xF(z)| \leq \frac{1}{2}\mathrm{Lip}(\nabla F; B(a,2R))|x-z|^2 \qquad (x,z \in B(a,2R)), \tag{2.2}$$

where $\mathrm{Lip}(\nabla F; B(a,2R)) := \sup_{x\neq y, x,y\in B(a,2R)} \left\{\frac{|\nabla F(x)-\nabla F(y)|}{|x-y|}\right\}$. We use (2.2) in the following estimate on the jets of a convex function $F \in C_c^{1,1}(B(a,2R))$.

Lemma 2.4. *Let $F \in C_c^{1,1}(B(a,2R))$ be convex; then $J_xF \sim_M J_yF$ for all $x,y \in B(a,R)$, where $M = \mathrm{Lip}(\nabla F; B(a,2R))$.*

Proof. We adapt the proof of Proposition 3.2 in [2]. For F affine, the result is immediate. Suppose F is not affine. Let $M := \mathrm{Lip}(\nabla F; B(a,2R))$, and suppose there exist $x,y \in B(a,R)$ such that $J_xF \not\sim_M J_yF$. Then without loss of generality, we can assume

$$F(x) - F(y) - \langle \nabla F(y), x-y\rangle < \frac{1}{2M}|\nabla F(x) - \nabla F(y)|^2. \tag{2.3}$$

By translation (by y) and subtraction of an affine function ($z \mapsto F(y) + \nabla F(y)(z-y)$), we can assume $y = 0 \in B(a,R)$, $F(y) = 0$, and $\nabla F(y) = 0$. Because F is convex, this implies $F(z) \geq 0$ for $z \in B(a,2R)$, and (2.3) becomes

$$0 \leq F(x) < \frac{1}{2M}|\nabla F(x)|^2.$$

In particular, $\nabla F(x) \neq 0$. Because $\mathrm{Lip}(\nabla F; B(a,2R)) = M$, we have $|\nabla F(x)| = |\nabla F(x) - \nabla F(0)| \leq M|x|$. Hence, for $x \in B(a,R)$, $\big(x - \nabla F(x)/M\big) \in B(a,2R)$. From (2.2) evaluated at $z = x - \frac{\nabla F(x)}{M}$ and the previous inequality,

$$\begin{aligned} F\left(x - \nabla F(x)/M\right) &\leq F(x) + \langle \nabla F(x), x - \nabla F(x)/M - x\rangle + \frac{M}{2}\left|x - \nabla F(x)/M - x\right|^2 \\ &= F(x) - |\nabla F(x)|^2/M + |\nabla F(x)|^2/(2M) \\ &< \left(\frac{1}{2M} - \frac{1}{M} + \frac{1}{2M}\right)|\nabla F(x)|^2 < 0, \end{aligned}$$

but this contradicts our deduction that $F(z) \geq 0$ for $z \in B(a,2R)\setminus\{0\}$. Thus, the lemma holds. $\square$

For a convex function $F \in C_c^{1,1}(\mathbb{R}^n)$, we have $\mathrm{Lip}(\nabla F;\mathbb{R}^n) < \infty$, implying the following:

Corollary 2.5. *Let $F \in C_c^{1,1}(\mathbb{R}^n)$ be convex; then $J_xF \sim_M J_yF$ for all $x, y \in \mathbb{R}^n$, where $M = \mathrm{Lip}(\nabla F;\mathbb{R}^n)$.*

2.2. Estimates for $C_c^{1,1}$-convex extension of 1-jets

Let $E \subset \mathbb{R}^n$ be closed, and suppose the Whitney field $(\gamma_x)_{x\in E}$ satisfies $\gamma_x \sim_M \gamma_y$ for all $x, y \in E$. Then $(\gamma_x)_{x\in E}$ satisfies the hypotheses of Theorem 3, and we deduce there exists a convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $J_xF = \gamma_x$ for all $x \in E$ and $\mathrm{Lip}(\nabla F) \leq M$.

Under the same hypotheses (i.e., that the Whitney field $(\gamma_x)_{x\in E}$ satisfies $\gamma_x \sim_M \gamma_y$ for all $x, y \in E$), we can add (1.5) to (1.6) and apply the Cauchy-Schwartz inequality to deduce

$$|\nabla\gamma_y - \nabla\gamma_x||y - x| \geq \langle \nabla\gamma_y - \nabla\gamma_x, y - x\rangle \geq \frac{1}{M}|\nabla\gamma_y - \nabla\gamma_x|^2,$$

and thus, $|\nabla\gamma_y - \nabla\gamma_x| \leq M|y - x|$. The non-negativity of (1.6) implies

$$\begin{aligned}\gamma_x(x) - \gamma_y(x) \leq& \big(\gamma_x(x) - \gamma_y(x)\big) + \big(\gamma_y(y) - \gamma_x(y)\big)\\ =&\langle \nabla\gamma_y - \nabla\gamma_x, y - x\rangle \leq M|y - x|^2.\end{aligned}$$

Similarly, from the non-negativity of (1.5), we see if $\gamma_x \sim_M \gamma_y$,

$$\gamma_y(y) - \gamma_x(y) \leq M|y - x|^2. \tag{2.4}$$

This implies the following:

Remark 2.1. If the Whitney field $(\gamma_x)_{x\in E}$ satisfies $\gamma_x \sim_M \gamma_y$ for all $x, y \in E$, and $\sup_{x\in E}\{|\gamma_x(x)|\} + \sup_{x\in E}\{|\nabla\gamma_x|\} \leq M$, the polynomials $(\gamma_x)_{x\in E} \in Wh(E)$ satisfy the hypotheses of Whitney's Extension Theorem for $C^{1,1}(\mathbb{R}^n)$ with a constant M, implying there exists a function $G \in C^{1,1}(\mathbb{R}^n)$ satisfying $J_xG = \gamma_x$ for all $x \in E$, and $\|G\|_{C^{1,1}(\mathbb{R}^n)} \leq C(n)M$. But the function G need <u>not</u> be convex (see e.g. [16] for this version of Whitney's Extension Theorem).

Next we construct an example of constants $M \geq \eta > 0$, a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a choice of Whitney field $(\gamma_x)_{x\in E} \in Wh(E)$ satisfying $\gamma_x \in \Gamma_\eta^E(x)$ and $\gamma_x \sim_M \gamma_y$ for all $x, y \in E$, such that there does <u>not</u> exist an η-strongly convex function $F \in C_c^{1,1}(\mathbb{R})$ satisfying $\mathrm{Lip}(\nabla F) \leq M$ and $J_xF = \gamma_x$ for all $x \in E$.

Example: Let $E = \{0, 1\} \subset \mathbb{R}$, $\eta \in (0, 1/4)$, $M = 1$, $f(0) = 0$ and $f(1) = \eta/2$. Then the polynomials $\gamma_0 \in \mathcal{R}_0, \gamma_1 \in \mathcal{R}_1$ defined as $\gamma_0(x) = 0$, and $\gamma_1(x) = \frac{\eta}{2} + 2\eta(x-1)$ satisfy

$$\gamma_0(0) - \gamma_1(0) = \frac{3\eta}{2} \geq \frac{\eta}{2}|1-0|^2, \tag{2.5}$$

$$\gamma_0(0) - \gamma_1(0) = \frac{3\eta}{2} \geq \frac{1}{2}|2\eta-0|^2, \tag{2.6}$$

$$\gamma_1(1) - \gamma_0(1) = \frac{\eta}{2} = \frac{\eta}{2}|1-0|^2, \text{ and} \tag{2.7}$$

$$\gamma_1(1) - \gamma_0(1) = \frac{\eta}{2} \geq \frac{1}{2}|2\eta-0|^2. \tag{2.8}$$

Inequality (2.5) implies $\gamma_0 \in \Gamma_\eta^E(0)$; (2.7) implies $\gamma_1 \in \Gamma_\eta^E(1)$; and together (2.6) and (2.8) imply $\gamma_0 \sim_1 \gamma_1$.

Notice any η-strongly convex extension of γ_0 must lie above the function $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) := \frac{\eta}{2}|x|^2$. But $g(1) = \gamma_1(1) = \frac{\eta}{2}$ and $g'(1) = \eta < 2\eta = \nabla\gamma_1$. Thus, there is no η-strongly convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $J_0F = \gamma_0$, $J_1F = \gamma_1$. However, the following proposition implies that there is an η/p-strongly convex function satisfying these conditions, for any $p > 1$.

Proposition 2.6. *Let $E^* \subset \mathbb{R}^n$ be closed, $f^* : E^* \to \mathbb{R}$, and $M \geq \eta > 0$. Suppose $(\gamma_x)_{x\in E^*}$ satisfies $\gamma_x \in \Gamma_\eta^{E^*}(x; f^*)$ and $\gamma_x \sim_M \gamma_y$ for all $x, y \in E^*$. Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists an η/p-strongly convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $J_xF = \gamma_x$ for all $x \in E^*$ and $\mathrm{Lip}(\nabla F) \leq qM + \eta/p \leq (q+1)M$.*

We will turn to the proof of Proposition 2.6 momentarily. The following lemma explains the reduction in the strong convexity constant η of an extension, under these hypotheses:

Lemma 2.7. *Let $E \subset \mathbb{R}^n$ be closed, $f : E \to \mathbb{R}$, and $M \geq \eta > 0$. Suppose $(\gamma_x)_{x\in E}$ satisfies $\gamma_x \in \Gamma_\eta^E(x; f)$ and $\gamma_x \sim_M \gamma_y$ for all $x, y \in E$. Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in E$, let $P_x \in \mathcal{R}_x$ satisfy $P_x(x) = f(x) - \frac{\eta}{2p}|x|^2$ and $\nabla P_x = \nabla\gamma_x - \frac{\eta}{p}x$. Then $P_x \sim_{qM} P_y$ for all $x, y \in E$.*

Proof. Let $x, y \in E$. By translation (by y) and subtraction of the affine function γ_y, we can assume $y = 0$ and $\gamma_y = 0$. By hypothesis, $\gamma_x(x) = f(x)$; thus, $\gamma_x(z) = f(x) + \langle\nabla\gamma_x, z - x\rangle$. Then $\gamma_x \sim_M \gamma_y$ implies

$$f(x) \geq \frac{1}{2M}|\nabla\gamma_x|^2, \text{ and} \tag{2.9}$$

$$-f(x) + \langle\nabla\gamma_x, x\rangle \geq \frac{1}{2M}|\nabla\gamma_x|^2. \tag{2.10}$$

The conditions $\gamma_x \in \Gamma_\eta^E(x)$, $\gamma_y \in \Gamma_\eta^E(y)$ imply

$$f(x) \geq \frac{\eta}{2}|x|^2, \tag{2.11}$$

$$-f(x) + \langle\nabla\gamma_x, x\rangle \geq \frac{\eta}{2}|x|^2, \text{ and} \tag{2.12}$$

$$|\nabla \gamma_x||x| \geq \langle \nabla \gamma_x, x\rangle \geq \eta |x|^2, \tag{2.13}$$

where the final inequality comes from adding (2.11) to (2.12) and applying the Cauchy Schwartz inequality to the inner product. We want to prove that $P_x \sim_{qM} P_y$. By our normalizations, $P_y(y) = f(y) - \frac{\eta}{2p}|y|^2 = 0$, and $\nabla P_y = \nabla \gamma_y - \frac{\eta}{p} y = 0$, so $P_y = 0$. Thus, we want to show

$$f(x) - \frac{\eta}{2p}|x|^2 \geq \frac{1}{2qM}|\nabla \gamma_x - \frac{\eta}{p}x|^2, \text{ and} \tag{2.14}$$

$$-f(x) + \frac{\eta}{2p}|x|^2 - \langle \nabla \gamma_x - \frac{\eta}{p}x, -x\rangle \geq \frac{1}{2qM}|\nabla \gamma_x - \frac{\eta}{p}x|^2. \tag{2.15}$$

From (2.9) and (2.11), we have

$$f(x) - \frac{\eta}{2p}|x|^2 \geq \frac{1}{2qM}|\nabla \gamma_x|^2 + \left(1 - \frac{1}{q}\right)\frac{\eta}{2}|x|^2 - \frac{\eta}{2p}|x|^2, \tag{2.16}$$

and from (2.10) and (2.12),

$$\begin{aligned} -f(x) + \frac{\eta}{2p}|x|^2 - \langle \nabla \gamma_x - \frac{\eta}{p}x, -x\rangle &= -f(x) + \langle \nabla \gamma_x, x\rangle - \frac{\eta}{2p}|x|^2 \\ \geq \frac{1}{2qM}|\nabla \gamma_x|^2 + \left(1 - \frac{1}{q}\right)\frac{\eta}{2}|x|^2 - \frac{\eta}{2p}|x|^2. \end{aligned} \tag{2.17}$$

We next bound the terms on the right-hand side of (2.16) and (2.17) (which are the same). Because $\frac{1}{p} + \frac{1}{q} = 1$, we have:

$$\begin{aligned} \frac{1}{2qM}|\nabla \gamma_x|^2 + \left(1 - \frac{1}{q}\right)\frac{\eta}{2}|x|^2 - \frac{\eta}{2p}|x|^2 &= \frac{1}{2qM}|\nabla \gamma_x|^2 \\ &= \frac{1}{2qM}|\nabla \gamma_x - \frac{\eta}{p}x|^2 - \frac{\eta^2}{2qp^2M}|x|^2 + \frac{\eta}{qpM}\langle \nabla \gamma_x, x\rangle \\ &\overset{(2.13)}{\geq} \frac{1}{2qM}|\nabla \gamma_x - \frac{\eta}{p}x|^2 - \frac{\eta^2}{2qp^2M}|x|^2 + \frac{\eta^2}{qpM}|x|^2 \\ &\geq \frac{1}{2qM}|\nabla \gamma_x - \frac{\eta}{p}x|^2, \end{aligned}$$

where the last inequality uses the fact that $2p \geq 1$. Thus, we have proven (2.14) and (2.15). $\square$

Proof of Proposition 2.6. We apply Lemma 2.7: For $x \in E$, let $P_x \in \mathcal{R}_x$ satisfy $P_x(x) = f(x) - \frac{\eta}{2p}|x|^2$ and $\nabla P_x = \nabla \gamma_x - \frac{\eta}{p}x$. Then $P_x \sim_{qM} P_y$ for all $x, y \in E$. We apply Theorem 3 to the polynomials $(P_x)_{x \in E}$, to produce a convex function $G \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $J_x G = P_x$ for all $x \in E$ and $\mathrm{Lip}(\nabla G) \leq qM$. We define $F \in C^{1,1}(\mathbb{R}^n)$ as $F(x) = G(x) + \frac{\eta}{2p}|x|^2$. Because G is convex, F is $\frac{\eta}{p}$-strongly convex; $\mathrm{Lip}(\nabla F) \leq$

$qM + \frac{\eta}{p} \leq (q+1)M$, and $J_x F = \gamma_x$ for all $x \in E$. This completes the proof of the proposition. $\square$

2.3. Relevant convex subsets of $\mathcal{P}$

In this section we introduce convex sets of jets that are relevant to the $C_c^{1,1}(\mathbb{R}^n)$ extension problem. We establish the basic properties of these sets in the two lemmas below. We invoke Helly's theorem to prove Lemma 2.10 which states that the convex sets are non-empty when $f : E \to \mathbb{R}$ satisfies a finiteness hypothesis, as in the assumptions of Theorem 1.

For the rest of this section, we fix a finite set $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$. For $M > \eta > 0$, and $T \subset E$, we define the set $\widetilde{\Gamma}^1_\eta(T, M; f) \subset Wh(T)$ as

$$\widetilde{\Gamma}^1_\eta(T, M; f) := \bigcap_{z \in E} \left\{ (P_x)_{x \in T} \in Wh(T) : \begin{array}{l} \exists \text{ an } \eta\text{-strongly convex function } F \in C_c^{1,1}(\mathbb{R}^n) \text{ s.t.} \\ \mathrm{Lip}(\nabla F) \leq M,\ J_x F = P_x\ \forall x \in T,\ \text{and } F|_{\{z\} \cup T} = f|_{\{z\} \cup T} \end{array} \right\}.$$

Again, where the function f is evident, we will not write f; i.e., we write $\widetilde{\Gamma}^1_\eta(T, M)$ in place of $\widetilde{\Gamma}^1_\eta(T, M; f)$.

We establish the basic containments for these convex sets in the following.

Lemma 2.8. *Let $E \subset \mathbb{R}^n$ be finite and $f : E \to \mathbb{R}$. For all $T \subset E$, we have*

$$\widetilde{\Gamma}^1_\eta(T, M_0) \subset \widetilde{\Gamma}^1_\eta(T, M) \qquad (M > M_0 > \eta > 0), \text{ and} \tag{2.18}$$

$$\widetilde{\Gamma}^1_\eta(T, M) \subset \widetilde{\Gamma}^1_{\eta_0}(T, M) \qquad (M > \eta > \eta_0 > 0). \tag{2.19}$$

Let $(P_x)_{x \in T} \in \widetilde{\Gamma}^1_\eta(T, M)$; then

$$P_x \in \Gamma^E_\eta(x) \qquad \text{for all } x \in T. \tag{2.20}$$

Proof. Properties (2.18) and (2.19) are immediate from the definition of the set $\widetilde{\Gamma}^1_\eta(T, M)$. Let $(P_x)_{x \in T} \in \widetilde{\Gamma}^1_\eta(T, M)$ and $x \in T$. For $y \in E \setminus \{x\}$, there exists an η-strongly convex function $F^y \in C_c^{1,1}(\mathbb{R}^n)$ such that $J_x F = P_x$ and $F^y|_{\{x,y\}} = f|_{\{x,y\}}$. Thus, $P_x(y) + \frac{\eta}{2}|y - x|^2 \leq f(y)$. Because this is true for all $y \in E \setminus \{x\}$, we deduce $P_x \in \Gamma^E_\eta(x)$. $\square$

Lemma 2.9. *Let $E \subset \mathbb{R}^n$ be finite, $f : E \to \mathbb{R}$, and $M > \eta > 0$; for $T \subset E$, the set $\widetilde{\Gamma}^1_\eta(T, M) \subset Wh(T)$ is convex.*

Proof. For $T \subset E$, $z \in E$, define $K(T, z) \subset Wh(T)$ as

$$K(T, z) := \left\{ (P_x)_{x \in T} \in Wh(T) : \begin{array}{l} \exists \text{ an } \eta\text{-strongly convex function } F \in C_c^{1,1}(\mathbb{R}^n) \text{ s.t.} \\ \mathrm{Lip}(\nabla F) \leq M,\ J_x F = P_x\ \forall x \in T,\ \text{and } F|_{\{z\} \cup T} = f|_{\{z\} \cup T} \end{array} \right\}.$$

Let $F_1, F_2 \in C_c^{1,1}(\mathbb{R}^n)$ be η-strongly convex functions satisfying $\mathrm{Lip}(\nabla F_i) \leq M$ for $i = 1, 2$. Then $F_t := tF_1 + (1-t)F_2$ is η-strongly convex and satisfies $\mathrm{Lip}(\nabla F_t) \leq M$. Because $J_x F^t = tJ_x F_1 + (1-t)J_x F_2$, we see $(J_x F_1)_{x\in T}, (J_x F_2)_{x\in T} \in K(T, z)$ implies $(J_x F_t)_{x\in T} \in K(T, z)$. Thus, $K(T, z)$ is convex, and $\widetilde{\Gamma}^1_\eta(T, M) = \bigcap_{z\in E} K(T, z)$ is convex. $\square$

Lemma 2.10. *Let $E \subset \mathbb{R}^n$ be finite and $f : E \to \mathbb{R}$. Suppose $j, k \in \mathbb{N}$ satisfy $k \geq (n+2)j + 1$, and for every $S \subset E$ satisfying $\#S \leq k$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$. Then for every $T \subset E$ satisfying $\#T \leq j$, $\widetilde{\Gamma}^1_\eta(T, M) \neq \emptyset$.*

Proof. In the previous lemma, we proved that $\widetilde{\Gamma}^1_\eta(T, M) = \bigcap_{z\in E} K(T, z)$ is the intersection of a family of $\big((n+1)|T|\big)$-dimensional convex sets. Here we show that given $T \subset E$, every subfamily of $\big((n+1)|T| + 1\big)$ of the convex sets $K(T, z)$ has a non-empty intersection. Then we can apply Helly's Theorem (Theorem 4) to prove the intersection of the entire family is non-empty.

Let $T \subset E$ satisfy $\#T \leq j$. Let $z_1, \ldots, z_{(n+1)j+1}$ be points of E; then $S := \{z_1, \cdots, z_{(n+1)j+1}\}$ is contained in E and $\#(S \cup T) \leq (n+2)j + 1 \leq k$. By applying the hypothesis of the lemma to the set $S \cup T$, we deduce there exists an η-strongly convex function $F^{S\cup T} \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F^{S\cup T}|_{S\cup T} = f|_{S\cup T}$ and $\mathrm{Lip}(\nabla F^{S\cup T}) \leq M$. Thus,

$$
\begin{aligned}
&(J_x F^{S\cup T})_{x\in T} \\
&\in \bigcap_{i=1}^{(n+1)j+1} \left\{ (P_x)_{x\in T} \in Wh(T) : \begin{array}{l} \exists \text{ an } \eta\text{-strongly convex function } F \in C_c^{1,1}(\mathbb{R}^n) \text{ s.t.} \\ \mathrm{Lip}(\nabla F) \leq M,\ J_x F = P_x\ \forall x \in T, \text{ and } F|_{\{z_i\}\cup T} = f|_{\{z_i\}\cup T} \end{array} \right\} \\
&\quad = \bigcap_{i=1}^{(n+1)j+1} K(T, z_i).
\end{aligned}
$$

Thus, the intersection of any $\big((n+1)j + 1\big)$-element subfamily of $\{K(T, z) : z \in E\}$ is non-empty. We can then apply Helly's Theorem to conclude $\widetilde{\Gamma}^1_\eta(T, M) = \bigcap_{z\in E} K(T, z)$ is non-empty. $\square$

2.4. Lipschitz selection of gradient vectors

We want to make a Lipschitz selection of gradient vectors from the sets $(\{\nabla\gamma : \gamma \in \Gamma^E_\eta(x; f)\})_{x\in E}$. To do so, we adapt Theorem 5 to our setting:

Proposition 2.11. *There exist $k_s^\# \in \mathbb{N}$ and $C^\# > 0$ such that the following holds: Let $E \subset \mathbb{R}^n$ be finite. For each $x \in E$, let $K(x) \subset \mathbb{R}^n$ be a closed convex set. Suppose that for each $S \subset E$ with $\#S \leq k_s^\#$, there exists a Lipschitz function $F^S : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\mathrm{Lip}(F^S) \leq 1$ and $F^S(x) \in K(x)$ for all $x \in S$. Then there exists a $C^{0,1}$ (bounded and Lipschitz) function $F : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathrm{Lip}(F) \leq C^\#$, such that $F(x) \in K(x)$ for all $x \in E$.*

Proof. Let $k_s^\# := k_s^\#(n, D) = k_s^\#(n, n)$ from Theorem 5. Because $E \subset \mathbb{R}^n$ is finite, there exists $R_0 > 0$ such that $E \subset B(0, R_0)$. For each $S \subset E$ with $\#S \le k_s^\#$, let $F^S : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function satisfying $\mathrm{Lip}(F^S) \le 1$ and $F^S(x) \in K(x)$ for all $x \in S$. Each F^S is locally bounded, and there are finitely many S, so we can let $A_0 \in (0, \infty)$ be a uniform upper bound on $\sup_{x\in B(0,R_0)} |F^S(x)|$ for every $S \subset E$ with $\#(S) \le k_s^\#$. Thus,

$$\bigcup_{\substack{S\subset E,\\ \#S\le k_s^\#}} \left\{F^S(x) : x \in B(0, R_0)\right\} \subset B(0, A_0),$$

and because $\mathrm{Lip}(F^S) \le 1$, $\|F^S\|_{C^0(B(A_0+2R_0))} \le 2A_0 + R_0$ for any $S \subset E$ satisfying $\#S \le k_s^\#$.

Let the function $\theta \in C^\infty(\mathbb{R}^n)$ be a smooth bump function satisfying $0 \le \theta \le 1$, $\theta|_{B(0,R_0)} = 1|_{B(0,R_0)}$, $\mathrm{supp}(\theta) \subset B(0, A_0 + 2R_0)$, and $|\nabla\theta(x)| \le \frac{C_0}{(A_0+R_0)}$ for $x \in \mathbb{R}^n$. For $S \subset E$, $\#S \le k_s^\#$, let the function $\bar{F}^S : \mathbb{R}^n \to \mathbb{R}^n$ be $\bar{F}^S(x) := F^S(x)\theta(x)$. Then $\bar{F}^S \in C^{0,1}(\mathbb{R}^n; \mathbb{R}^n)$ satisfies

$$\begin{aligned}
&\|\bar{F}^S\|_{C^0(\mathbb{R}^n)} \le \|F^S\|_{C^0(B(A_0+2R_0))} \le 2A_0 + R_0,\\
&\bar{F}^S(x) \in K(x) \text{ for all } x \in S, \text{ and}\\
&\mathrm{Lip}(\bar{F}^S) \le \mathrm{Lip}(F^S) + \|F^S\|_{C^0(B(A_0+2R_0))} \cdot \mathrm{Lip}(\theta) \le 1 + \frac{(2A_0 + R_0)C_0}{A_0 + R_0} \le 1 + 2C_0.
\end{aligned}$$

Fix $N := 2(A_0 + R_0)$. Let $E' \subset \mathbb{R}^n$ and $K'(x) \subset \mathbb{R}^n$ for $x \in E$ be

$$\begin{aligned}
E' &= \frac{1}{N}E := \{y/N : y \in E\}, \text{ and}\\
K'(x) &:= \frac{1}{N(2 + 4C_0)}K(x) := \left\{\frac{y}{N(2 + 4C_0)} : y \in K(x)\right\}.
\end{aligned}$$

For $S' \subset E'$ satisfying $\#S' \le k_s^\#$, the set NS' is contained in E. Let the function $G^{S'} : \mathbb{R}^n \to \mathbb{R}^n$ be

$$G^{S'}(x) := \frac{1}{N(2 + 4C_0)}\bar{F}^{NS'}(Nx).$$

Then $|G^{S'}(x)| \le 1/2$ for all $x \in \mathbb{R}^n$, and $\mathrm{Lip}(G^{S'}) \le \frac{1}{2+4C_0}\mathrm{Lip}(\bar{F}^{NS'}) \le 1/2$, implying the function $G^{S'}$ satisfies:

$$\begin{aligned}
&\|G^{S'}\|_{C^{0,1}(\mathbb{R}^n,\mathbb{R}^n)} \le 1 \text{ and}\\
&G^{S'}(x) \in K'(Nx) \text{ for } x \in S'.
\end{aligned}$$

This result holds for all $S' \subset E'$ satisfying $\#S \le k_s^\#$. Thus, the convex sets $(K'(Nx))_{x\in E'}$ satisfy the hypotheses of Theorem 5 with $(n, D) = (n, n)$; applying Theorem 5, we deduce there exists $G \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $G(x) \in K'(Nx)$ for all $x \in E'$ and $\|G\|_{C^{0,1}(\mathbb{R}^n,\mathbb{R}^n)} \leq C_0^{\#}$, where $C_0^{\#} := C^{\#}(n,n)$ from Theorem 5. Let the function $F : \mathbb{R}^n \to \mathbb{R}^n$ be $F(x) := N(2+4C_0)G(\frac{x}{N})$; F satisfies $F(x) \in K(x)$ for all $x \in E$ and $\mathrm{Lip}(F) \leq C_0^{\#}(2+4C_0) =: C^{\#}$. This completes the proof of Proposition 2.11. $\square$

3. Main results in dimension $n \geq 1$ (Theorem 1)

We begin by proving a version of Theorem 1 for finite E. Using a compactness argument, we will then extend the result to arbitrary E.

Theorem 6. *Let $E \subset \mathbb{R}^n$ be finite, $M > \eta > 0$, $f : E \to \mathbb{R}$, and $k^{\#} = k_s^{\#}(n+2)+1$, where $k_s^{\#}$ is the constant from Proposition 2.11. Suppose for all $S \subset E$ satisfying $\#S \leq k^{\#}$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$. Then for all $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, there exists an $\frac{\eta}{p}$-strongly convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F|_E = f|_E$, and $\mathrm{Lip}(\nabla F) \leq C_3 M^2/\eta$, where $C_3 = (C^{\#})^2(q+1)$, and $C^{\#}$ is the constant in Proposition 2.11.*

Proof. By applying Lemma 2.10 with $j := k_s^{\#}$ and $k := k_s^{\#}(n+2)+1$, we see for every $T \subset E$ satisfying $\#T \leq k_s^{\#}$, $\widetilde{\Gamma}_\eta^1(T, M) \neq \emptyset$. From the definition of the set $\widetilde{\Gamma}_\eta^1(T, M)$, we deduce for $T \subset E$ satisfying $\#T \leq k_s^{\#}$, there exists an η-strongly convex function $F^T \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $\mathrm{Lip}(\nabla F^T) \leq M$, $F|_T = f|_T$, and $(J_x F^T)_{x \in T} \in \widetilde{\Gamma}_\eta^1(T, M)$. In light of (2.20), $J_x F^T \in \Gamma_\eta^E(x)$ for all $x \in T$. We summarize this result:

$$\begin{aligned}
&\text{For all } T \subset E \text{ satisfying } \#T \leq k_s^{\#} \\
&\text{there exists an } \eta\text{-strongly convex function } F^T \in C_c^{1,1}(\mathbb{R}^n) \\
&\text{satisfying } \mathrm{Lip}(\nabla F^T) \leq M \text{ and } J_x F^T \in \Gamma_\eta^E(x)\ \forall x \in T.
\end{aligned} \tag{3.1}$$

For $x \in E$, $\eta > 0$, let $\bar{\Gamma}_\eta^E(x) \subset \mathbb{R}^n$ be

$$\bar{\Gamma}_\eta^E(x) := \{\nabla P : P \in \Gamma_\eta^E(x)\}.$$

(See (2.1) for the definition of $\Gamma_\eta^E(x)$.) For $x \in E$, the set $\Gamma_\eta^E(x)$ is a convex subset of $\mathcal{R}_x$, so the set $\bar{\Gamma}_\eta^E(x)$ is a convex subset of $\mathbb{R}^n$.

Let $S \subset E$ satisfy $\#S \leq k_s^{\#}$. Define $G^S \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)$ as $G^S(x) := \nabla F^S(x)$, where $F^S \in C_c^{1,1}(\mathbb{R}^n)$ is the function produced by applying (3.1) to the set S. Then $G^S(x) \in \bar{\Gamma}_\eta^E(x)$ for all $x \in S$ and $\mathrm{Lip}(G^S) \leq M$. We apply Proposition 2.11 to produce $G \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $G(x) \in \bar{\Gamma}_\eta^E(x)$ for all $x \in E$ and

$$\mathrm{Lip}(G) \leq C^{\#} M. \tag{3.2}$$

For $x \in E$, define $\gamma_x \in \mathcal{R}_x$ as the polynomial satisfying $\gamma_x(x) = f(x)$ and $\nabla \gamma_x = G(x)$. Immediately, $\gamma_x \in \Gamma_\eta^E(x)$, so for all $x, y \in E$

$$\gamma_x(x) - \gamma_y(x) \geq \frac{\eta}{2}|x-y|^2, \text{ and} \tag{3.3}$$

$$\gamma_y(y) - \gamma_x(y) \geq \frac{\eta}{2}|x-y|^2. \tag{3.4}$$

In light of (3.2), $|\nabla\gamma_x - \nabla\gamma_y| \leq C^{\#}M|x-y|$. In combination with (3.3) and (3.4), we deduce for all $x, y \in E$,

$$\gamma_x(x) - \gamma_y(x) \geq \frac{\eta}{2(C^{\#}M)^2}|\nabla\gamma_x - \nabla\gamma_y|^2, \text{ and}$$

$$\gamma_y(y) - \gamma_x(y) \geq \frac{\eta}{2(C^{\#}M)^2}|\nabla\gamma_x - \nabla\gamma_y|^2,$$

implying $\gamma_x \sim_{\frac{M^2}{\eta}(C^{\#})^2} \gamma_y$ for all $x, y \in E$. Thus, we can apply Proposition 2.6 to produce an $\frac{\eta}{p}$-strongly convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F|_E = f|_E$ and $\mathrm{Lip}(\nabla F) \leq (C^{\#})^2(q+1)M^2/\eta$, where $C^{\#}$ is the constant from Proposition 2.11. This completes the proof of Theorem 6. □

Proof of Theorem 1. Fix $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Let $E \subset \mathbb{R}^n$ be compact. There exists $R > 1$ such that $E \subset B(0, R)$. For $A > 0$, let $\mathcal{B}(A) \subset C^1(B(0, 2R))$ be

$$\mathcal{B}(A) = \{F \in C_c^{1,1}(B(0, 2R)) : F \text{ is } \frac{\eta}{p}\text{-strongly convex, and } \|F\|_{C^{1,1}(B(0,2R))} \leq A\}.$$

The set $\mathcal{B}(A)$ is closed in the $C^1(B(0, 2R))$-topology. For any $A > 0$, $\mathcal{B}(A)$ is also bounded and equicontinuous in the $C^1(B(0, 2R))$-topology, implying by the Arzelà-Ascoli Theorem that $\mathcal{B}(A)$ is compact.

Let E' be a countable dense subset of E, and let $(E_i)_{i\in\mathbb{N}}$ be an increasing sequence of sets satisfying for $i \in \mathbb{N}$, $E_i \subset E'$, $\#E_i < \infty$, and $\bigcup_{i\in\mathbb{N}} E_i = E'$. By assumption, for all $S \subset E_i \subset E$ satisfying $\#S \leq k^{\#}$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$.

Therefore, for each $i \in \mathbb{N}$, we can apply Theorem 6 to produce $F_i \in C_c^{1,1}(\mathbb{R}^n)$, satisfying

$$\begin{aligned}
&F_i \text{ is an } \frac{\eta}{p}\text{-strongly convex function,}\\
&F_i|_{E_i} = f|_{E_i}, \text{ and}\\
&\mathrm{Lip}\left(\nabla F_i\right) \leq C_3\frac{M^2}{\eta},
\end{aligned}$$

where $C_3 = (C^{\#})^2(q+1)$. Restricting the domain of F_i to $B(0, 2R)$, we see for A large enough, $(F_i)_{i\in\mathbb{N}} \subset \mathcal{B}(A)$.

By the compactness of $\mathcal{B}(A)$, there exists a convergent subsequence $(F_{i_j})_{j\in\mathbb{N}} \to \bar{F} \in \mathcal{B}(A)$ in the C^1 topology. The function $\bar{F}$ satisfies $\bar{F} \in C_c^{1,1}(B(0, 2R))$,

$$\bar{F} \text{ is an } \frac{\eta}{p}\text{-strongly convex function,}$$
$$\bar{F}|_{E'} = f|_{E'}, \text{ and}$$
$$\operatorname{Lip}\left(\nabla \bar{F}; B(0,2R)\right) \leq C_3 \frac{M^2}{\eta},$$

where the last inequality follows because $\operatorname{Lip}\left(\nabla F_{i_j}\right) \leq C_3 \frac{M^2}{\eta}$ for all $j \in \mathbb{N}$. Further, because this convergence is uniform, $\bar{F}|_E = f|_E$.

We plan to apply Proposition 2.6 to $(J_x \bar{F})_{x \in B(0,R)}$, so we verify the hypotheses of the proposition with $E^* := \overline{B(0,R)}$ and $f^* := \bar{F}$: By Lemma 2.4, $J_x \bar{F} \sim_{\operatorname{Lip}(\nabla \bar{F}; B(0,2R))} J_y \bar{F}$ for all $x, y \in \overline{B(0,R)}$. Recall that in (2.1) we defined $\Gamma_\eta^{\overline{B(0,R)}}(x; \bar{F}) \subset Wh(\overline{B(0,R)})$ for $x \in \overline{B(0,R)}$ as

$$\Gamma_\eta^{\overline{B(0,R)}}(x; \bar{F}) := \bigcap_{y \in \overline{B(0,R)} \setminus \{x\}} \{P \in \mathcal{R}_x : P(x) = \bar{F}(x) \text{ and } P(y) + \frac{\eta}{2}|y-x|^2 \leq \bar{F}(y)\}.$$

Because $\bar{F}$ is $\frac{\eta}{p}$-strongly convex on $B(0,2R)$, $J_x \bar{F} \in \Gamma_{\frac{\eta}{p}}^{\overline{B(0,R)}}(x; \bar{F})$ for all $x \in \overline{B(0,R)}$.

We apply Proposition 2.6 to $(J_x \bar{F})_{x \in \overline{B(0,R)}}$ to produce an $\frac{\eta}{p^2}$-strongly convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $J_x F = J_x \bar{F}$ for all $x \in \overline{B(0,R)}$ and $\operatorname{Lip}(\nabla F) \leq (q+1) \operatorname{Lip}(\nabla \bar{F}; B(0,2R)) \leq C_1 q^2 \frac{M^2}{\eta}$. Because $E \subset B(0,R)$ and $J_x \bar{F} \in \Gamma_{\frac{\eta}{p}}^{\overline{B(0,R)}}(x; \bar{F})$ for $x \in B(0,R)$, this implies $F(x) = f(x)$ for all $x \in E$. This completes the proof of Theorem 1. $\square$

4. Main results in dimension $n = 1$ (Theorem 2)

We begin by proving a version of Theorem 2 for finite E and $\eta = 0$. In Section 4.3, we complete the proof of Theorem 2.

Theorem 7. *Let $E \subset \mathbb{R}$ be finite, and let the function $f : E \to \mathbb{R}$. Suppose that for every $S \subset E$ satisfying $\#S \leq 5$, there exists a function $F^S \in C_c^{1,1}(\mathbb{R})$ satisfying $F^S|_S = f|_S$ and $\operatorname{Lip}(\nabla F^S) \leq M$. Then there exists a function $F \in C_c^{1,1}(\mathbb{R})$ satisfying $F|_E = f|_E$ and $\operatorname{Lip}(\nabla F) \leq 2M$.*

4.1. Technical tools in dimension $n = 1$

Let $E \subset \mathbb{R}$ be finite and $f : E \to \mathbb{R}$. Assuming the validity of a finiteness hypothesis on f, as per Theorem 7, our aim is to find $(\gamma_x)_{x \in E} \in Wh(E)$ that satisfies $\gamma_x(x) = f(x)$ and $\gamma_x \sim_M \gamma_y$ for all $x, y \in E$, with a uniform constant M; then we can apply Theorem 3 to produce a convex extension of f in $C_c^{1,1}(\mathbb{R})$. In our first result, we deduce a transitivity property of the relation $\sim_M$ in one dimension. According to this, we only need to confirm the compatibility of γ_x at adjacent points of E.

Lemma 4.1. *Let $x, y, z \in \mathbb{R}$ satisfy $x < y < z$, and suppose $\gamma_x \in \mathcal{R}_x$, $\gamma_y \in \mathcal{R}_y$, $\gamma_z \in \mathcal{R}_z$ satisfy $\gamma_x \sim_M \gamma_y$, and $\gamma_y \sim_M \gamma_z$; then $\gamma_x \sim_M \gamma_z$.*

Proof. Suppose $x < y < z$, $\gamma_x \sim_M \gamma_y$, and $\gamma_y \sim_M \gamma_z$; we have

$$\begin{aligned}
&\frac{1}{2M}|\nabla\gamma_z - \nabla\gamma_x|^2 \\
&= \frac{1}{2M}\big(|\nabla\gamma_z - \nabla\gamma_y|^2 + |\nabla\gamma_y - \nabla\gamma_x|^2 + 2\langle\nabla\gamma_z - \nabla\gamma_y, \nabla\gamma_y - \nabla\gamma_x\rangle\big) \\
&\leq \big(\gamma_y(y) - \gamma_z(y)\big) + \big(\gamma_x(x) - \gamma_y(x)\big) + \frac{1}{M}\langle\nabla\gamma_z - \nabla\gamma_y, \nabla\gamma_y - \nabla\gamma_x\rangle \\
&= \gamma_x(x) - \gamma_z(x) + \left\langle \nabla\gamma_z - \nabla\gamma_y, x - y + \frac{1}{M}\big(\nabla\gamma_y - \nabla\gamma_x\big)\right\rangle. \qquad (4.1)
\end{aligned}$$

Because $x < y$ and $\gamma_x \sim_M \gamma_y$, we have $\frac{\nabla\gamma_y - \nabla\gamma_x}{M} \leq y - x$, which implies $x - y + \frac{1}{M}(\nabla\gamma_y - \nabla\gamma_x) \leq 0$. Because $\nabla\gamma_z - \nabla\gamma_y > 0$, the last term in (4.1) must be negative, implying $\frac{1}{2M}|\nabla\gamma_z - \nabla\gamma_x|^2 \leq \gamma_x(x) - \gamma_z(x)$. The proof of the inequality $\frac{1}{2M}|\nabla\gamma_z - \nabla\gamma_x|^2 \leq \gamma_z(z) - \gamma_x(z)$ follows analogously. $\square$

Remark 4.1. A transitivity result relying only on the configuration of points cannot be transferred to higher dimensions ($n > 1$) because we cannot ensure $\langle\nabla\gamma_z - \nabla\gamma_y, x - y + \frac{1}{M}\big(\nabla\gamma_y - \nabla\gamma_x\big)\rangle$ is non-positive without further hypotheses on γ_x, γ_y, and γ_z. Even if x, y, z are ordered points on a line in $\mathbb{R}^n$ (i.e. there exists $t \in (0,1)$ such that $y = tx + (1-t)z$), the quantity $\langle\nabla\gamma_z - \nabla\gamma_y, x - y + \frac{1}{M}\big(\nabla\gamma_y - \nabla\gamma_x\big)\rangle$ need not be non-positive. However, if we assume $x, y, z \in \mathbb{R}^n$ satisfy $\langle y - x, z - y\rangle > 0$ <u>and</u> $(\nabla\gamma_z - \nabla\gamma_y) = \lambda(\nabla\gamma_y - \nabla\gamma_x)$ for $\lambda > 0$, then we can show $\gamma_x \sim_M \gamma_y$, $\gamma_y \sim_M \gamma_z$ implies $\gamma_x \sim_M \gamma_z$.

Lemma 4.2. *Let $E \subset \mathbb{R}$ be finite, $f : E \to \mathbb{R}$, and $M > 0$. For distinct $x, y \in E$, let $\gamma_x \in \Gamma^0(x)$ satisfy*

$$0 \leq \langle D^f_{xy} - \nabla\gamma_x, y - x\rangle \leq \frac{M}{2}|y - x|^2. \qquad (4.2)$$

If $\gamma^x_y \in \Gamma^0(y)$ is defined by

$$\begin{aligned}
\nabla\gamma^x_y &= \nabla\gamma_x + \sqrt{2M\langle D^f_{xy} - \nabla\gamma_x, y - x\rangle} \qquad &&\text{if } x < y, \text{ and} \\
\nabla\gamma^x_y &= \nabla\gamma_x - \sqrt{2M\langle \nabla\gamma_x - D^f_{xy}, x - y\rangle} \qquad &&\text{if } x > y,
\end{aligned}$$

then $\gamma_x \sim_M \gamma^x_y$.

Proof. Suppose $x < y$. By definition of γ^x_y and the fact that $\gamma_x \in \Gamma^0(x)$, $\gamma^x_y \in \Gamma^0(y)$, we have

$$\frac{1}{2M}\left|\nabla\gamma_y^x - \nabla\gamma_x\right|^2 = \langle D_{xy}^f - \nabla\gamma_x, y-x\rangle = \gamma_y^x(y) - \gamma_x(y),$$

implying (1.6) holds with equality for $\gamma_y := \gamma_y^x$ and $\gamma_x := \gamma_x$.

We next show that (1.5) holds for $\gamma_y := \gamma_y^x$ and $\gamma_x := \gamma_x$, proving $\gamma_y^x \sim_M \gamma_x$. From (4.2) and the definition of γ_y^x, we have

$$\begin{aligned}\nabla\gamma_y^x &\le \nabla\gamma_x + M(y-x)\\ &\le \nabla\gamma_x + M(y-x) + \sqrt{-2M\langle D_{xy}^f - \nabla\gamma_x, y-x\rangle + M^2(y-x)^2},\end{aligned} \tag{4.3}$$

where the non-negativity of the term inside the square root also follows from (4.2). Further,

$$\begin{aligned}\nabla\gamma_y^x &= \nabla\gamma_x + \sqrt{2M\langle D_{xy}^f - \nabla\gamma_x, y-x\rangle}\\ &\ge \nabla\gamma_x + M(y-x) - \sqrt{-2M\langle D_{xy}^f - \nabla\gamma_x, y-x\rangle + M^2(y-x)^2}.\end{aligned} \tag{4.4}$$

The latter inequality follows from the observation that for $\omega > 0$, $\sqrt{4\omega t} \ge 2\omega - \sqrt{4\omega^2 - 4\omega t}$ for all $t \in [0,\omega]$. Let $\omega := \frac{M}{2}(y-x)$ and $t := D_{xy}^f - \nabla\gamma_x$ ((4.2) ensures $t \in [0,\omega]$), and the result follows.

For $\gamma_x \in \Gamma^0(x)$, $\gamma_y \in \Gamma^0(y)$, and $x < y$, (1.5) holds if

$$f(x) - f(y) - \langle\nabla\gamma_y, x-y\rangle \ge \frac{1}{2M}|\nabla\gamma_x - \nabla\gamma_y|^2.$$

This is expression is equivalent to a quadratic equation in the variable $\nabla\gamma_y$:

$$(\nabla\gamma_y)^2 + (\nabla\gamma_y)\left(-2\nabla\gamma_x + 2M(x-y)\right) + \left((\nabla\gamma_x)^2 + 2M(f(y)-f(x))\right) \le 0. \tag{4.5}$$

The discriminant $\Delta \in \mathbb{R}$ of this quadratic equation is non-negative thanks to (4.2); indeed,

$$\begin{aligned}\Delta &= \left(-2\nabla\gamma_x + 2M(x-y)\right)^2 - 4\left((\nabla\gamma_x)^2 + 2M(f(y)-f(x))\right)\\ &= 4M\left(-2\big(f(y) - f(x) - \langle\nabla\gamma_x, y-x\rangle\big) + M(y-x)^2\right)\\ &= -8M\langle D_{xy}^f - \nabla\gamma_x, y-x\rangle + 4M^2(y-x)^2 \ge 0.\end{aligned}$$

Thus, (4.5) is equivalent to

$$\begin{aligned}\nabla\gamma_y \in \Bigg[&\nabla\gamma_x + M(y-x) - \sqrt{-2M\langle D_{xy}^f - \nabla\gamma_x, y-x\rangle + M^2(y-x)^2},\\ &\nabla\gamma_x + M(y-x) + \sqrt{-2M\langle D_{xy}^f - \nabla\gamma_x, y-x\rangle + M^2(y-x)^2}\Bigg],\end{aligned}$$

which is valid for $\gamma_y := \gamma_y^x$ thanks to (4.3) and (4.4). This completes the proof of (1.5), and with it the proof that $\gamma_x \sim_M \gamma_y^x$.

The proof that if $x > y$ and $\gamma_y^x \in \Gamma^0(y)$ satisfies

$$\nabla \gamma_y^x = \nabla \gamma_x - \sqrt{2M\langle \nabla \gamma_x - D_{xy}^f, x - y\rangle},$$

then $\gamma_x \sim_M \gamma_y^x$ follows analogously. $\square$

4.2. Proof of Theorem 7

Proof of Theorem 7. Because E is finite, we enumerate $E = \{x_1, x_2, \dots, x_N\}$, with $x_1 < x_2 < \cdots < x_N$. We may assume $N > 5$, else the result is trivial. For distinct $i, j \in \{1, \dots, N\}$, let

$$D_{i,j} := D_{x_i x_j}^f = \frac{f(x_j) - f(x_i)}{x_j - x_i}.$$

By the finiteness hypothesis, the restriction of f to any 3 consecutive points of E admits a convex extension, hence

$$D_{1,2} \le D_{2,3} \le \cdots \le D_{N-1,N}.$$

For $i \in \{1, \dots, N-1\}$, let $P_i^\ell \in \Gamma^0(x_i)$, and for $i \in \{2, \dots, N\}$, let $P_i^r \in \Gamma^0(x_i)$ satisfy

$$\begin{aligned}
\nabla P_1^\ell &:= D_{1,2} - \frac{M}{2}(x_2 - x_1), \\
\nabla P_i^\ell &:= \max\left\{D_{i-1,i}, D_{i,i+1} - \frac{M}{2}(x_{i+1} - x_i)\right\} \qquad (i \in \{2, \dots, N-1\}), \qquad (4.6)\\
\nabla P_N^r &:= D_{N-1,N} + \frac{M}{2}(x_N - x_{N-1}), \text{ and} \\
\nabla P_i^r &:= \min\left\{D_{i,i+1}, D_{i-1,i} + \frac{M}{2}(x_i - x_{i-1})\right\} \qquad (i \in \{2, \dots, N-1\}). \qquad (4.7)
\end{aligned}$$

For $i \in \{2, \dots, N-1\}$, let $P_i^+, P_i^- \in \Gamma^0(x_i)$ satisfy

$$\nabla P_i^+ = \nabla P_{i-1}^\ell + \sqrt{2M\langle D_{i-1,i} - \nabla P_{i-1}^\ell, x_i - x_{i-1}\rangle}, \text{ and} \qquad (4.8)$$

$$\nabla P_i^- = \nabla P_{i+1}^r - \sqrt{2M\langle \nabla P_{i+1}^r - D_{i,i+1}, x_{i+1} - x_i\rangle}. \qquad (4.9)$$

We use the following monotonicity result in the estimates that follow.

Lemma 4.3. *For $\omega > 0$, the function $h(t) = -t + \sqrt{4\omega t}$ is an increasing function on $[0, \omega]$. Thus, for $0 \le t_1 \le t_2 \le \omega$, we have $h(t_1) \le h(t_2)$, and if $t_1 < t_2$ then $h(t_1) < h(t_2)$.*

Proof. Trivial. □

For $i \in \{2, \dots, N-1\}$, by definition (4.6), we have $0 \le D_{i-1,i} - \nabla P^{\ell}_{i-1} \le \frac{M}{2}(x_i - x_{i-1})$. So, we can apply Lemma 4.3 with $t_1 := D_{i-1,i} - \nabla P^{\ell}_{i-1}$ and $t_2 := \omega := \frac{M}{2}(x_i - x_{i-1})$; then $h(t_1) \le h(t_2)$, so

$$\begin{aligned} \nabla P_i^+ &= \nabla P^{\ell}_{i-1} + \sqrt{2M\langle D_{i-1,i} - \nabla P^{\ell}_{i-1}, x_i - x_{i-1}\rangle} \\ &\le D_{i-1,i} - \frac{M}{2}(x_i - x_{i-1}) + \sqrt{2M\left\langle \frac{M}{2}(x_i - x_{i-1}), x_i - x_{i-1}\right\rangle} \\ &= D_{i-1,i} + \frac{M}{2}(x_i - x_{i-1}) \qquad (i \in \{2, \dots, N-1\}). \end{aligned} \tag{4.10}$$

Similarly, by the definition (4.7) of P^r_{i+1}, we have $0 \le \nabla P^r_{i+1} - D_{i,i+1} \le \frac{M}{2}(x_{i+1} - x_i)$, so we can apply Lemma 4.3 with $t_1 := \nabla P^r_{i+1} - D_{i,i+1}$ and $t_2 := \omega := \frac{M}{2}(x_{i+1} - x_i)$; then $-h(t_1) \ge -h(t_2)$, so

$$\begin{aligned} \nabla P_i^- &= \nabla P^r_{i+1} - \sqrt{2M\langle \nabla P^r_{i+1} - D_{i,i+1}, x_{i+1} - x_i\rangle} \\ &\ge D_{i,i+1} + \frac{M}{2}(x_{i+1} - x_i) - \sqrt{2M\left\langle \frac{M}{2}(x_{i+1} - x_i), x_{i+1} - x_i\right\rangle} \\ &= D_{i,i+1} - \frac{M}{2}(x_{i+1} - x_i) \qquad (i \in \{2, \dots, N-1\}). \end{aligned} \tag{4.11}$$

Lemma 4.4. *For $i \in \{2, \dots, N-1\}$, $P^{\ell}_{i-1} \in \Gamma^0(x_{i-1})$ satisfies $\nabla P^{\ell}_{i-1} \ge \nabla P$ for all $P \in \Gamma^0(x_{i-1})$ satisfying $P \sim_M P_i^+$. Similarly, $P^r_{i+1} \in \Gamma^0(x_{i+1})$ satisfies $\nabla P^r_{i+1} \le \nabla P$ for all $P \in \Gamma^0(x_{i+1})$ satisfying $P \sim_M P_i^-$.*

Proof. Let $P \in \Gamma^0(x_{i-1})$ satisfy $P \sim_M P_i^+$, then from inequality (1.5) with $\gamma_x := P_i^+$ and $\gamma_y := P$ we have

$$\nabla P + \sqrt{2M\langle D_{i-1,i} - \nabla P, x_i - x_{i-1}\rangle} \ge \nabla P_i^+. \tag{4.12}$$

If, in addition, $\nabla P > \nabla P^{\ell}_{i-1}$, then $0 \le D_{i-1,i} - \nabla P < D_{i-1,i} - \nabla P^{\ell}_{i-1} \le \frac{M}{2}(x_i - x_{i-1})$, thanks to (4.6). So, we can apply Lemma 4.3 with $t_1 := D_{i-1,i} - \nabla P$, $t_2 := D_{i-1,i} - \nabla P^{\ell}_{i-1}$, and $\omega := \frac{M}{2}(x_i - x_{i-1})$ to see that $h(t_1) < h(t_2)$, i.e.,

$$\begin{aligned} &\nabla P + \sqrt{2M\langle D_{i-1,i} - \nabla P, x_i - x_{i-1}\rangle} \\ &< \nabla P^{\ell}_{i-1} + \sqrt{2M\langle D_{i-1,i} - \nabla P^{\ell}_{i-1}, x_i - x_{i-1}\rangle} = \nabla P_i^+. \end{aligned}$$

But this contradicts (4.12). Thus, if $P \sim_M P_i^+$, we must have $\nabla P \le \nabla P^{\ell}_{i-1}$. The proof that $\nabla P^r_{i+1} \le \nabla P$ for all $P \in \Gamma^0(x_{i+1})$ satisfying $P \sim_M P_i^-$ follows analogously. □

Lemma 4.5. *Suppose that for every $S \subset E$ satisfying $\#S \leq 5$, there exists a function $F^S \in C_c^{1,1}(\mathbb{R})$ satisfying $F^S|_S = f|_S$ and* $\mathrm{Lip}(\nabla F^S) \leq M$*. Then* $\max\{\nabla P_i^-, D_{i-1,i}\} \leq \min\{\nabla P_i^+, D_{i,i+1}\}$ *for $i \in \{2, \dots, N-1\}$.*

Proof. Let the sets $S_i \subset E$ ($i \in \{1, \dots, N\}$) be defined by $S_i := \{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ for $i \in \{3, \dots, N-2\}$, and $S_1, S_2 := S_3$, $S_N, S_{N-1} := S_{N-2}$. By the finiteness hypothesis, for $i \in \{1, \dots, N\}$, there exists a function $F^{S_i} \in C_c^{1,1}(\mathbb{R})$, satisfying $F^{S_i}|_{S_i} = f|_{S_i}$ and $\mathrm{Lip}(\nabla F^{S_i}) \leq M$. Because $F^{S_i}|_{S_i} = f|_{S_i}$,

$$D_{j-1,j} = \frac{F^{S_i}(x_j) - F^{S_i}(x_{j-1})}{x_j - x_{j-1}} \qquad (i \in \{2, \dots, N-1\}, j \in \{i, i+1\}).$$

Hence, the convexity of F^{S_i} implies

$$D_{i-1,i} \leq \nabla J_{x_i} F^{S_i} \leq D_{i,i+1} \qquad (i \in \{2, \dots, N-1\}). \tag{4.13}$$

We claim that $\nabla P_i^- \leq \nabla J_{x_i} F^{S_i} \leq \nabla P_i^+$ for $i \in \{2, \dots, N-1\}$. In combination with (4.13), this implies $\max\{\nabla P_i^-, D_{i-1,i}\} \leq \nabla J_{x_i} F^{S_i} \leq \min\{\nabla P_i^+, D_{i,i+1}\}$, proving the lemma.

For $i \in \{2, \dots, N\}$, Corollary 2.5 implies $J_{x_{i-1}} F^{S_i} \sim_M J_{x_i} F^{S_i}$. Letting $\gamma_x := J_{x_i} F^{S_i}$ and $\gamma_y := J_{x_{i-1}} F^{S_i}$ in (1.5), we see

$$\nabla J_{x_i} F^{S_i} \leq \nabla J_{x_{i-1}} F^{S_i} + \sqrt{2M \langle D_{i-1,i} - \nabla J_{x_{i-1}} F^{S_i}, x_i - x_{i-1} \rangle} \qquad (i \in \{2, \dots, N\}). \tag{4.14}$$

By Taylor's inequality (2.2) and the convexity of F^{S_i}, $0 \leq F^{S_i}(x_{i-2}) - J_{x_{i-1}} F^{S_i}(x_{i-2}) \leq \frac{M}{2}(x_{i-1} - x_{i-2})^2$ for $i \in \{3, \dots, N\}$. Dividing by the positive quantity $(x_{i-1} - x_{i-2})$, we see that

$$0 \leq \nabla J_{x_{i-1}} F^{S_i} - D_{i-2,i-1} \leq \frac{M}{2}(x_{i-1} - x_{i-2}) \qquad (i \in \{3, \dots, N\}). \tag{4.15}$$

Similarly, $0 \leq F^{S_i}(x_i) - J_{x_{i-1}} F^{S_i}(x_i) \leq \frac{M}{2}(x_i - x_{i-1})^2$ for $i \in \{2, \dots, N\}$, which implies

$$0 \leq D_{i-1,i} - \nabla J_{x_{i-1}} F^{S_i} \leq \frac{M}{2}(x_i - x_{i-1}) \qquad (i \in \{2, \dots, N\}). \tag{4.16}$$

By combining (4.15) and (4.16), and applying the definition (4.6) of ∇P_{i-1}^ℓ, we have

$$\nabla J_{x_{i-1}} F^{S_i} \geq \max\{D_{i-1,i} - \frac{M}{2}(x_i - x_{i-1}), D_{i-2,i-1}\} = \nabla P_{i-1}^\ell \qquad (i \in \{3, \dots, N\}). \tag{4.17}$$

When $i = 2$, (4.16) reads as

$$0 \leq D_{1,2} - \nabla J_{x_1} F^{S_2} \leq \frac{M}{2}(x_2 - x_1) = D_{1,2} - \nabla P_1^{\ell},$$

where the last equality is by the definition of ∇P_1^{ℓ}. Together with (4.17), we have

$$0 \leq D_{i-1,i} - \nabla J_{x_{i-1}} F^{S_i} \leq D_{i-1,i} - \nabla P_{i-1}^{\ell} \leq \frac{M}{2}(x_i - x_{i-1}) \qquad (i \in \{2, \ldots, N\}).$$

Thus, we can apply Lemma 4.3 with $t_1 := D_{i-1,i} - \nabla J_{x_{i-1}} F^{S_i}$, $t_2 := D_{i-1,i} - \nabla P_{i-1}^{\ell}$, and $\omega := \frac{M}{2}(x_i - x_{i-1})$ to see $h(t_1) \leq h(t_2)$, and in combination with (4.14),

$$\begin{aligned} \nabla J_{x_i} F^{S_i} &\leq \nabla J_{x_{i-1}} F^{S_i} + \sqrt{2M\langle D_{i-1,i} - \nabla J_{x_{i-1}} F^{S_i}, x_i - x_{i-1}\rangle} \\ &\leq \nabla P_{i-1}^{\ell} + \sqrt{2M\langle D_{i-1,i} - \nabla P_{i-1}^{\ell}, x_i - x_{i-1}\rangle} = \nabla P_i^{+} \qquad (i \in \{2, \ldots, N\}). \end{aligned}$$

By an analogous argument, we deduce $\nabla J_{x_i} F^{S_i} \geq \nabla P_i^{-}$ for $i \in \{1, \ldots, N-1\}$. This completes the proof of the claim, and as described, the lemma. □

We are prepared to choose $\gamma_i \in \Gamma^0(x_i)$ for $i \in \{1, \ldots, N\}$. We use the compatibility condition in Lemma 4.2 to inform our choice of derivative for $\gamma_1 \in \Gamma^0(x_1)$ (chosen so that $\gamma_1 \sim_M \gamma_2$) and $\gamma_N \in \Gamma^0(x_N)$ (chosen so that $\gamma_{N-1} \sim_M \gamma_N$). Let $(\gamma_i)_{i=1}^N \in Wh(E)$ be the unique Whitney field of polynomials satisfying

$$\gamma_i \in \Gamma^0(x_i), \text{ and}$$
$$\nabla \gamma_i = \begin{cases} \frac{1}{2}\big(\max\{\nabla P_i^{-}, D_{i-1,i}\} + \min\{\nabla P_i^{+}, D_{i,i+1}\}\big) & i \in \{2, \ldots, N-1\} \\ \nabla \gamma_2 - \sqrt{2M\langle \nabla \gamma_2 - D_{1,2}, x_2 - x_1\rangle} & i = 1 \\ \nabla \gamma_{N-1} + \sqrt{2M\langle D_{N-1,N} - \nabla \gamma_{N-1}, x_N - x_{N-1}\rangle} & i = N. \end{cases} \tag{4.18}$$

As a result of Lemma 4.5, for $i \in \{2, \ldots, N-1\}$, we have

$$\max\{\nabla P_i^{-}, D_{i-1,i}\} \leq \nabla \gamma_i \leq \min\{\nabla P_i^{+}, D_{i,i+1}\}. \tag{4.19}$$

Consequently, $\nabla \gamma_2 - D_{1,2} \geq 0$, and $D_{N-1,N} - \nabla \gamma_{N-1} \geq 0$, ensuring γ_1 and γ_2 are well-defined in (4.18).

In the next several results, we verify additional basic inequalities satisfied by $(\gamma_i)_{i=1}^N$.

From the previous lemma, we have $\nabla \gamma_i \in [D_{i-1,1}, D_{i,i+1}]$ for all $i \in \{2, \cdots, N-1\}$. Because the sequence of divided differences is non-decreasing, we have $\nabla \gamma_2 \leq \cdots \leq \nabla \gamma_{N-1}$. By inspection of the definitions of $\nabla \gamma_1$ and $\nabla \gamma_N$ in (4.18), we obtain the following result:

Corollary 4.6. *The polynomials $(\gamma_i)_{i=1}^N$ defined in (4.18) satisfy that their gradients are non-decreasing:*

$$\nabla \gamma_1 \leq \nabla \gamma_2 \leq \cdots \leq \nabla \gamma_N.$$

Lemma 4.7. *For $i \in \{3, \dots, N-1\}$, the polynomials $(\gamma_i)_{i=1}^N$ satisfy*

$$\nabla P_{i-1}^{\ell} \le \nabla \gamma_{i-1}, \ \textit{and} \tag{4.20}$$

$$\nabla \gamma_i \le \nabla P_i^r. \tag{4.21}$$

Proof. By inequality (4.19), $\nabla \gamma_{i-1} \ge \max\{\nabla P_{i-1}^-, D_{i-2,i-1}\} \ge D_{i-2,i-1}$. If $\nabla P_{i-1}^{\ell} = D_{i-2,i-1}$, then this implies $\nabla \gamma_{i-1} \ge \nabla P_{i-1}^{\ell}$. Else, by definition of P_{i-1}^{ℓ}, we have $\nabla P_{i-1}^{\ell} = D_{i-1,i} - \frac{M}{2}(x_i - x_{i-1})$. Then

$$\nabla \gamma_{i-1} \ge \max\{\nabla P_{i-1}^-, D_{i-2,i-1}\} \ge \nabla P_{i-1}^- \ge D_{i-1,i} - \frac{M}{2}(x_i - x_{i-1}) = \nabla P_{i-1}^{\ell},$$

where the last inequality is from (4.11). We have proved (4.20). The proof of (4.21) follows analogously. □

By applying Theorem 3, the next lemma will be used to complete the proof of Theorem 7.

Lemma 4.8. *The Whitney field $(\gamma_i)_{i=1}^N \in Wh(E)$ defined in (4.18) satisfies $\gamma_i \sim_{2M} \gamma_j$ for all $i, j \in \{1, \dots, N\}$.*

Proof. In light of Lemma 4.1, we only need to prove $\gamma_i \sim_{2M} \gamma_{i-1}$ for $i \in \{2, \dots, N\}$.

For $i \in \{2, \dots, N-1\}$, inequalities (4.19), (4.10), and (4.11) imply

$$0 \le \nabla \gamma_i - D_{i-1,i} \le \nabla P_i^+ - D_{i-1,i} \le \frac{M}{2}(x_i - x_{i-1}), \ \text{and} \tag{4.22}$$

$$0 \le D_{i,i+1} - \nabla \gamma_i \le D_{i,i+1} - \nabla P_i^- \le \frac{M}{2}(x_{i+1} - x_i). \tag{4.23}$$

Thus, we can apply Lemma 4.2 with $\gamma_x := \gamma_2$ and $y := x_1$ to see we have $\gamma_1 \sim_M \gamma_2$. Likewise, inequality (4.23) allows us to apply Lemma 4.2 with $\gamma_x := \gamma_{N-1}$ and $y := x_N$ to see $\gamma_{N-1} \sim_M \gamma_N$. Therefore, it suffices to show $\gamma_i \sim_{2M} \gamma_{i-1}$ for $i \in \{3, \dots, N-1\}$. We prove this by demonstrating that

$$\begin{aligned} &f(x_i) - f(x_{i-1}) - \langle \nabla \gamma_{i-1}, x_i - x_{i-1} \rangle \ge \frac{1}{4M} |\nabla \gamma_i - \nabla \gamma_{i-1}|^2 \\ &\quad (i \in \{3, \dots, N-1\}), \ \text{and} \end{aligned} \tag{4.24}$$

$$\begin{aligned} &f(x_i) - f(x_{i+1}) - \langle \nabla \gamma_{i+1}, x_i - x_{i+1} \rangle \ge \frac{1}{4M} |\nabla \gamma_i - \nabla \gamma_{i+1}|^2 \\ &\quad (i \in \{2, \dots, N-2\}). \end{aligned} \tag{4.25}$$

Indeed, the fact that (4.24) holds, and (4.25) holds with i replaced by $(i-1)$, implies that $\gamma_i \sim_{2M} \gamma_{i-1}$ for $i \in \{3, \cdots, N-1\}$.

We will next establish inequality (4.24) by splitting into the two cases below. Let $i \in \{3, \ldots, N-1\}$.

Case 1: Suppose that $\nabla P_{i-1}^- \leq \nabla P_{i-1}^\ell$. From the definition of P_{i-1}^ℓ in (4.6), we have $D_{i-2,i-1} \leq \nabla P_{i-1}^\ell$. Together, these inequalities imply

$$\begin{aligned}
\nabla \gamma_{i-1} &= \frac{1}{2}\big(\max\{\nabla P_{i-1}^-, D_{i-2,i-1}\} + \min\{\nabla P_{i-1}^+, D_{i-1,i}\}\big) \\
&\leq \frac{1}{2}\big(\nabla P_{i-1}^\ell + \min\{\nabla P_{i-1}^+, D_{i-1,i}\}\big) \\
&\leq \frac{1}{2}\big(\nabla P_{i-1}^\ell + D_{i-1,i}\big).
\end{aligned}$$

With inequality (4.20), we've shown $\nabla \gamma_{i-1} \in \left[\nabla P_{i-1}^\ell, \frac{\nabla P_{i-1}^\ell + D_{i-1,i}}{2}\right]$. In particular, $\nabla P_{i-1}^\ell \leq D_{i-1,i}$. In combination with (4.23),

$$0 \leq D_{i-1,i} - \frac{\nabla P_{i-1}^\ell + D_{i-1,i}}{2} \leq D_{i-1,i} - \nabla \gamma_{i-1} \leq \frac{M}{2}(x_i - x_{i-1}),$$

we can apply Lemma 4.3 with $t_1 := D_{i-1,i} - \frac{\nabla P_{i-1}^\ell + D_{i-1,i}}{2}$, $t_2 := D_{i-1,i} - \nabla \gamma_{i-1}$, and $\omega := M(x_i - x_{i-1})$; the map h is increasing, and in particular $h(t_2) \geq h(t_1)$, so

$$\begin{aligned}
\nabla \gamma_{i-1} + &\sqrt{4M(f(x_i) - f(x_{i-1}) - \langle \nabla \gamma_{i-1}, x_i - x_{i-1}\rangle)} \\
&= \nabla \gamma_{i-1} + \sqrt{4M\langle D_{i-1,i} - \nabla \gamma_{i-1}, x_i - x_{i-1}\rangle} \\
&\geq \frac{\nabla P_{i-1}^\ell + D_{i-1,i}}{2} + \sqrt{4M\left\langle D_{i-1,i} - \frac{\nabla P_{i-1}^\ell + D_{i-1,i}}{2}, x_i - x_{i-1}\right\rangle} \\
&\geq \nabla P_{i-1}^\ell + \sqrt{2M\langle D_{i-1,i} - \nabla P_{i-1}^\ell, x_i - x_{i-1}\rangle} = \nabla P_i^+, \qquad (4.26)
\end{aligned}$$

where the last inequality follows because $D_{i-1,i} \geq \nabla P_{i-1}^\ell$. From inequality (4.19) and Corollary 4.6, we have $\nabla P_i^+ \geq \nabla \gamma_i \geq \nabla \gamma_{i-1}$; we first use these inequalities and then (4.26) to bound $\frac{1}{4M}|\nabla \gamma_{i-1} - \nabla \gamma_i|^2 \leq \frac{1}{4M}|\nabla \gamma_{i-1} - \nabla P_i^+|^2 \leq f(x_i) - f(x_{i-1}) - \langle \nabla \gamma_{i-1}, x_i - x_{i-1}\rangle$, which is (4.24).

Case 2: Suppose $\nabla P_{i-1}^\ell < \nabla P_{i-1}^-$; thus, by the definition of P_{i-1}^ℓ in (4.6), $D_{i-2,i-1} \leq \nabla P_{i-1}^\ell < \nabla P_{i-1}^-$. Therefore,

$$\begin{aligned}
\nabla \gamma_{i-1} &= \frac{1}{2}\big(\max\{\nabla P_{i-1}^-, D_{i-2,i-1}\} + \min\{\nabla P_{i-1}^+, D_{i-1,i}\}\big) \\
&= \frac{1}{2}\big(\nabla P_{i-1}^- + \min\{\nabla P_{i-1}^+, D_{i-1,i}\}\big) \\
&\leq \frac{1}{2}\big(\nabla P_{i-1}^- + D_{i-1,i}\big).
\end{aligned}$$

From inequality (4.19), $\nabla\gamma_{i-1} \geq \max\{\nabla P^-_{i-1}, D_{i-2,i-1}\} \geq \nabla P^-_{i-1}$. We have shown $\nabla\gamma_{i-1} \in \left[\nabla P^-_{i-1}, \frac{\nabla P^-_{i-1}+D_{i-1,i}}{2}\right]$. In particular, $\nabla P^-_{i-1} \leq D_{i-1,i}$. In combination with (4.23),

$$0 \leq D_{i-1,i} - \frac{\nabla P^-_{i-1} + D_{i-1,i}}{2} \leq D_{i-1,i} - \nabla\gamma_{i-1} \leq D_{i-1,i} - \nabla P^-_{i-1} \leq \frac{M}{2}(x_i - x_{i-1}).$$

Thus, we can apply Lemma 4.3 with $t_1 := D_{i-1,i} - \frac{\nabla P^-_{i-1}+D_{i-1,i}}{2}$, $t_2 := D_{i-1,i} - \nabla\gamma_{i-1}$, and $\omega := M(x_i - x_{i-1})$ to see $h(t_2) \geq h(t_1)$, so that

$$\begin{aligned}
\nabla\gamma_{i-1}+&\sqrt{4M\langle D_{i-1,i} - \nabla\gamma_{i-1}, x_i - x_{i-1}\rangle} \\
&\geq \frac{\nabla P^-_{i-1} + D_{i-1,i}}{2} + \sqrt{4M\left\langle D_{i-1,i} - \frac{\nabla P^-_{i-1} + D_{i-1,i}}{2}, x_i - x_{i-1}\right\rangle} \\
&\geq \nabla P^-_{i-1} + \sqrt{2M\left\langle D_{i-1,i} - \nabla P^-_{i-1}, x_i - x_{i-1}\right\rangle}. \qquad (4.27)
\end{aligned}$$

According to (4.23), $0 \leq D_{i-1,i} - \nabla P^-_{i-1} \leq \frac{M}{2}(x_i - x_{i-1})$. This verifies the hypothesis of Lemma 4.2 for $\gamma_x := P^-_{i-1}$ and $y := x_i$. Thus, there exists $\tilde{\gamma}^{i-1}_i \in \Gamma^0(x_i)$ satisfying

$$\nabla\tilde{\gamma}^{i-1}_i = \nabla P^-_{i-1} + \sqrt{2M\left\langle D_{i-1,i} - \nabla P^-_{i-1}, x_i - x_{i-1}\right\rangle}$$

and $\tilde{\gamma}^{i-1}_i \sim_M P^-_{i-1}$. In light of Lemma 4.4, we must have $\nabla\tilde{\gamma}^{i-1}_i \geq \nabla P^r_i$. Hence, continuing from (4.27),

$$\begin{aligned}
&\nabla\gamma_{i-1} + \sqrt{4M\langle D_{i-1,i} - \nabla\gamma_{i-1}, x_i - x_{i-1}\rangle} \\
&\geq \nabla P^-_{i-1} + \sqrt{2M\left\langle D_{i-1,i} - \nabla P^-_{i-1}, x_i - x_{i-1}\right\rangle} \\
&= \nabla\tilde{\gamma}^{i-1}_i \geq \nabla P^r_i \geq \nabla\gamma_i, \qquad (4.28)
\end{aligned}$$

where the last inequality follows from (4.21). This is equivalent to (4.24). We have exhausted all cases proving (4.24).

The proof of inequality (4.25) follows analogously because of the symmetry of our choice of $\nabla\gamma_i := \frac{1}{2}\big(\max\{\nabla P^-_i, D_{i-1,i}\} + \min\{\nabla P^+_i, D_{i,i+1}\}\big)$ for $i \in \{2, \dots, N-1\}$ in light of Lemma 4.5. We summarize the proof briefly. Recall inequality (4.25) is

$$\begin{aligned}
f(x_i) - f(x_{i+1}) - \langle\nabla\gamma_{i+1}, x_i - x_{i+1}\rangle &= \langle\nabla\gamma_{i+1} - D_{i,i+1}, x_{i+1} - x_i\rangle \\
&\geq \frac{1}{4M}|\nabla\gamma_i - \nabla\gamma_{i+1}|^2 \quad (i \in \{2, \dots, N-2\}).
\end{aligned}$$

First, we suppose $\nabla P^+_{i+1} \geq \nabla P^r_{i+1}$; then $\nabla\gamma_{i+1} \in \left[\frac{\nabla P^r_{i+1}+D_{i,i+1}}{2}, \nabla P^r_{i+1}\right]$. We use Lemma 4.3 and inequality (4.19) to see (4.25) holds under the assumption $\nabla P^+_{i+1} \geq \nabla P^r_{i+1}$. Second, we suppose $\nabla P^+_{i+1} < \nabla P^r_{i+1}$ and, therefore, $\nabla\gamma_{i+1} \in \left[\frac{\nabla P^+_{i+1}+D_{i,i+1}}{2}, \nabla P^+_{i+1}\right]$. We use Lemmas 4.2-4.4 and inequality (4.19) to see (4.25) holds, exhausting all cases and completing the proof of (4.25).

This completes the proof of Lemma 4.8. □

By applying Theorem 3 to $(\gamma_i)_{i=1}^N \in Wh(E)$ and using that $\gamma_i(x_i) = f(x_i)$, we complete the proof of Theorem 7. □

4.3. Proof of Theorem 2

Theorem 2 is an immediate consequence of the following theorem:

Theorem 8. *Let $E \subset \mathbb{R}$ be compact, the function $f : E \to \mathbb{R}$, and $M > 0$. Suppose for every $S \subset E$ satisfying $\#S \leq k_1^\# = 5$, there exists a convex function $F^S \in C_c^{1,1}(\mathbb{R})$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$. Then there exists a convex function $F \in C_c^{1,1}(\mathbb{R})$ satisfying $F|_E = f|_E$ and $\mathrm{Lip}(\nabla F) \leq 2M$.*

To see Theorem 2 follows, we assume the hypotheses of Theorem 2: Let $E \subset \mathbb{R}$ be compact, and let the function $f : E \to \mathbb{R}$. Suppose for every $S \subset E$ satisfying $\#S \leq k_1^\# = 5$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R})$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$. Let $g : E \to \mathbb{R}$ be $g(x) := \frac{1}{1+\eta/M}(f(x) - \frac{\eta}{2}|x|^2)$, and for $S \subset E$ satisfying $\#S \leq k_1^\# = 5$, let $G^S : \mathbb{R} \to \mathbb{R}$ be $G^S(x) := \frac{1}{1+\eta/M}(F^S(x) - \frac{\eta}{2}|x|^2)$. The function G^S satisfies G^S is convex, $G^S|_S = g|_S$, and $\mathrm{Lip}(\nabla G^S) \leq \frac{1}{1+\eta/M}(\mathrm{Lip}(\nabla F^S) + \eta) \leq M$. Thus, $g : E \to \mathbb{R}$ satisfies the hypotheses of Theorem 8. Applying this theorem, we deduce there exists a convex function $G \in C_c^{1,1}(\mathbb{R})$ satisfying $G|_E = g|_E$ and $\mathrm{Lip}(\nabla G) \leq 2M$. Let the function $F : \mathbb{R} \to \mathbb{R}$ be $F(x) := (1 + \eta/M)G(x) + \frac{\eta}{2}|x|^2$. Then $F \in C_c^{1,1}(\mathbb{R})$ is η-strongly convex and satisfies $\mathrm{Lip}(\nabla F) \leq 2M + 3\eta$ and $F|_E = f|_E$. The conclusion of Theorem 2 follows.

Thus, our remaining task is to prove Theorem 8.

Proof of Theorem 8. Let $E \subset \mathbb{R}$ be compact. There exists $R > 1$ such that $E \subset B(0, R)$. For $A > 0$, let $\mathcal{B}(A) \subset C^1(B(0, 2R))$ be

$$\mathcal{B}(A) = \{F \in C_c^{1,1}(B(0, 2R)) : F \text{ is convex, and } \|F\|_{C^{1,1}(B(0,2R))} \leq A\}.$$

The set $\mathcal{B}(A)$ is closed in the $C^1(B(0, 2R))$-topology. For any $A > 0$, $\mathcal{B}(A)$ is also bounded and equicontinuous in the $C^1(B(0, 2R))$-topology, implying by the Arzelà-Ascoli Theorem that $\mathcal{B}(A)$ is compact.

Let E' be a countable dense subset of E, and let $(E_i)_{i\in\mathbb{N}}$ be an increasing sequence of sets satisfying for $i \in \mathbb{N}$, $E_i \subset E'$, $\#E_i < \infty$, and $\bigcup_{i\in\mathbb{N}} E_i = E'$. By assumption, for all

$S \subset E_i \subset E$ satisfying $\#S \leq 5$, there exists an η-strongly convex function $F^S \in C_c^{1,1}(\mathbb{R})$ satisfying $F^S|_S = f|_S$ and $\mathrm{Lip}(\nabla F^S) \leq M$. We apply Theorem 7 to produce a convex function $F_i \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $F_i|_{E_i} = f|_{E_i}$, and $\mathrm{Lip}(\nabla F_i) \leq 2M$. Restricting the domain of F_i to $B(0, 2R)$, we see for A large enough, $F_i \in \mathcal{B}(A)$ for all $i \in \mathbb{N}$. By the compactness of $\mathcal{B}(A)$, there exists a convergent subsequence $(F_{i_k})_{k\in\mathbb{N}} \to \bar{F} \in \mathcal{B}(A)$. The limiting function $\bar{F}$ satisfies $\mathrm{Lip}(\nabla \bar{F}; B(0, 2R)) \leq 2M$ and $\bar{F}|_{E'} = f|_{E'}$, and because this convergence is uniform $\bar{F}|_E = f|_E$. Because $\bar{F} \in \mathcal{B}(A)$, $\bar{F} \in C_c^{1,1}(B(0, 2R))$ is convex on $B(0, 2R)$.

By Corollary 2.5, $J_x\bar{F} \sim_{\mathrm{Lip}(\nabla \bar{F}; B(0,2R))} J_y\bar{F}$ for all $x, y \in \overline{B(0, R)}$. We apply Theorem 3 to $(J_x\bar{F})_{x\in\overline{B(0,R)}}$ to produce a convex function $F \in C_c^{1,1}(\mathbb{R}^n)$ satisfying $J_xF = J_x\bar{F}$ for all $x \in \overline{B(0, R)}$ (implying $F|_E = f|_E$) and $\mathrm{Lip}(\nabla F) \leq 2M$, completing the proof of Theorem 8. □

References

[1] D. Azagra, E. Le Gruyer, C. Mudarra, Explicit formulas for $C^{1,1}$ and $C^{1,\omega}_{\mathrm{conv}}$ extensions of 1-jets in Hilbert and superreflexive spaces, J. Funct. Anal. 274 (10) (2018) 3003–3032.

[2] D. Azagra, C. Mudarra, Whitney extension theorems for convex functions of the classes C^1 and $C^{1,\omega}$, Proc. Lond. Math. Soc. 114 (1) (2017) 133–158.

[3] Y.A. Brudnyi, P. Shvartsman, The traces of differentiable functions to subset of $\mathbb{R}^n$, in: Linear and Complex Analysis. Problem Book 3. Part II, in: Lecture Notes in Mathematics, vol. 1574, Springer-Verlag, 1984, pp. 279–281.

[4] Y.A. Brudnyi, P. Shvartsman, Generalizations of Whitney's extension theorem, Int. Math. Res. Not. (3) (1994) 129–139.

[5] C. Fefferman, A sharp form of Whitney's extension theorem, Ann. Math. 161 (1) (2005) 509–577.

[6] C. Fefferman, Whitney's extension problem for C^m, Ann. Math. 164 (1) (2006) 313–359.

[7] C. Fefferman, A. Israel, G.K. Luli, Finiteness principles for smooth selection, Geom. Funct. Anal. 26 (2016) 422–477.

[8] C. Fefferman, A. Israel, G.K. Luli, Interpolation of data by smooth nonnegative functions, Rev. Mat. Iberoam. 33 (1) (2017) 305–324.

[9] C. Fefferman, B. Klartag, Fitting a C^m-smooth function to data I, Ann. Math. 169 (1) (2009) 315–346.

[10] C. Fefferman, B. Klartag, Fitting a C^m-smooth function to data II, Rev. Mat. Iberoam. 25 (1) (2009) 49–273.

[11] C. Fefferman, P. Shvartsman, Sharp finiteness principles for Lipschitz selections, Geom. Funct. Anal. 28 (2018) 1641–1705.

[12] F. Jiang, G.K. Luli, Nonnegative $C^2(\mathbb{R}^2)$ interpolation, Adv. Math. 375 (2020).

[13] F. Jiang, G.K. Luli, Algorithms for nonnegative $C^2(\mathbb{R}^2)$ interpolation, Adv. Math. 385 (2021).

[14] P. Shvartsman, The traces of functions of two variables satisfying the Zygmund condition, in: Studies in the Theory of Functions of Several Real Variables, Yaroslav Gos. Univ., 1986, pp. 145–168 (in Russian).

[15] P. Shvartsman, On traces of functions of Zygmund classes, Sib. Math. J. 28 (1987) 853–863.

[16] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

[17] R. Webster, Convexity, Monographs in Mathematics, vol. 1, Oxford Science Publications, 1994.