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# Approximate extension in Sobolev space ☆

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ABSTRACT

Let $L^{m,p}(\mathbb{R}^n)$ be the homogeneous Sobolev space for $p \in (n, \infty)$, $\mu$ be a Borel regular measure on $\mathbb{R}^n$, and $L^{m,p}(\mathbb{R}^n) + L^p(d\mu)$ be the space of Borel measurable functions with finite seminorm $\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)} := \inf_{f_1+f_2=f}\{\|f_1\|^p_{L^{m,p}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |f_2|^p d\mu\}^{1/p}$. We construct a linear operator $T : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to L^{m,p}(\mathbb{R}^n)$, that nearly optimally decomposes every function in the sum space: $\|Tf\|^p_{L^{m,p}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |Tf - f|^p d\mu \leq C\|f\|^p_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)}$ with $C$ dependent on $m$, $n$, and $p$ only. For $E \subset \mathbb{R}^n$, let $L^{m,p}(E)$ denote the space of all restrictions to $E$ of functions $F \in L^{m,p}(\mathbb{R}^n)$, equipped with the standard trace seminorm. For $p \in (n, \infty)$, we construct a linear extension operator $T : L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)$ satisfying $Tf|_E = f|_E$ and $\|Tf\|_{L^{m,p}(\mathbb{R}^n)} \leq C\|f\|_{L^{m,p}(E)}$, where $C$ depends only on $n$, $m$, and $p$. We show these operators can be expressed through a collection of linear functionals whose supports have bounded overlap.



**Contents**



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## 1. Introduction

The homogeneous Sobolev space $L^{m,p}(\mathbb{R}^n)$, consists of all functions $F : \mathbb{R}^n \to \mathbb{R}$ whose distributional partial derivatives of order $m$ belong to $L^p(\mathbb{R}^n)$. We define the $L^{m,p}(\mathbb{R}^n)$ seminorm by

$$\|F\|_{L^{m,p}(\mathbb{R}^n)} = \max_{|\alpha|=m} \|\partial^{\alpha} F\|_{L^p(\mathbb{R}^n)} \qquad (F \in L^{m,p}(\mathbb{R}^n)).$$

Given a Borel regular measure $\mu$ on $\mathbb{R}^n$, let $L^p(d\mu)$ be the space of Borel measurable functions $g : \mathbb{R}^n \to \mathbb{R}$ with finite norm $\|g\|_{L^p(d\mu)} := \left(\int_{\mathbb{R}^n} g(x) d\mu\right)^{1/p} < \infty$. Let

$L^{m,p}(\mathbb{R}^n) + L^p(d\mu)$ be the space of Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ with finite seminorm

$$\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)} := \inf_{f_1+f_2=f} \left\{ \|f_1\|^p_{L^{m,p}(\mathbb{R}^n)} + \|f_2\|^p_{L^p(d\mu)} \right\}^{1/p} < \infty.$$

We consider the topic of nearly optimal decomposition of functions in this sum space. Precisely:

**Question 1.** *Can we construct a linear operator $T : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to L^{m,p}(\mathbb{R}^n)$ satisfying $\|Tf\|_{L^{m,p}(\mathbb{R}^n)} + \|f - Tf\|_{L^p(d\mu)} \leq C\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)}$, where $C$ is independent of $f$?*

**Question 2.** *Can we estimate $\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)}$?*

We answer these questions with our first theorem. Let $C^m_{loc}(\mathbb{R}^n)$ denote the space of all functions $F : \mathbb{R}^n \to \mathbb{R}$ with continuous derivatives up to order $m$. The function space $C^m(\mathbb{R}^n)$ consists of functions with continuous, bounded derivatives up to order $m$, with norm: $\|F\|_{C^m(\mathbb{R}^n)} = \max_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^n} \{|\partial^\alpha F(x)|\}$. Let $\mathcal{P}$ denote the space of real-valued $(m-1)^{\text{rst}}$ degree polynomials on $\mathbb{R}^n$. For $K \subset \mathbb{R}^n$, a *Whitney field* on $K$ is a tuple of polynomials $(P_x)_{x\in K}$ with $P_x \in \mathcal{P}$ for all $x \in K$. The space of Whitney fields on $K$ is denoted by

$$Wh(K) := \left\{ \vec{P} : \vec{P} = (P_x)_{x\in K},\ P_x \in \mathcal{P} \text{ for all } x \in K \right\}.$$

If $F$ is a $C^{m-1}$-function defined on a neighborhood of a point $y \in \mathbb{R}^n$, then we write $J_y(F)$ (the "jet" of $F$ at $y$) for the $(m-1)^{\text{rst}}$ degree Taylor polynomial of $F$ at $y$.

If $n < p < \infty$, then the Sobolev embedding theorem implies that $L^{m,p}(\mathbb{R}^n) \subset C^{m-1}_{loc}(\mathbb{R}^n)$. We define a semi-norm on Whitney fields, $\vec{S} \in Wh(K)$,

$$\|\vec{S}\|_{L^{m,p}(K)} = \inf\{\|F\|_{L^{m,p}(\mathbb{R}^n)} :\ F \in L^{m,p}(\mathbb{R}^n),\ J_x(F) = S_x \text{ for all } x \in K\}. \quad (1.1)$$

By definition, if there does not exist $F \in L^{m,p}(\mathbb{R}^n)$ with $J_x(F) = S_x$ for all $x \in K$, then $\|\vec{S}\|_{L^{m,p}(K)} = +\infty$.

Below, we write $\mathrm{Cl}(X)$ to denote the closure of a subset $X \subset \mathbb{R}^n$, and we write $\mathrm{supp}(\mu) \subset \mathbb{R}^n$ to denote the support of a measure $\mu$.

**Theorem 1.** *Let $\mu$ be a compactly supported Borel regular measure on $\mathbb{R}^n$, $m \in \mathbb{N}$, and $n < p < \infty$. Then there exists a linear operator $T : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to L^{m,p}(\mathbb{R}^n)$ and a map $M : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to \mathbb{R}$ satisfying for all $f \in L^{m,p}(\mathbb{R}^n) + L^p(d\mu)$:*

$$\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)} \leq \|Tf\|_{L^{m,p}(\mathbb{R}^n)} + \|Tf - f\|_{L^p(d\mu)} \leq C \cdot \|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)};$$

$$c \cdot Mf \leq \|Tf\|_{L^{m,p}(\mathbb{R}^n)} + \|Tf - f\|_{L^p(d\mu)} \leq C \cdot Mf;\ \text{and}$$

$$Mf = \left( \|\vec{S}(f)\|^p_{L^{m,p}(K)} + \sum_{\ell \in \mathbb{N}} \Big( \int_{A_\ell} |\zeta_\ell(f) - f|^p d\mu + |\psi_\ell(f)|^p \Big) \right)^{1/p}, \tag{1.2}$$

*where $K \subset \mathrm{Cl}(\mathrm{supp}(\mu))$, $\vec{S} : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to Wh(K)$ is a linear map, and for each $\ell \in \mathbb{N}$, $A_\ell \subset \mathrm{supp}(\mu)$ is a Borel set, $\zeta_\ell : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to L^p(d\mu)$ and $\psi_\ell : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to \mathbb{R}$ are linear maps. The constants $c$ and $C$ depend on $m$, $n$, and $p$ but are independent of $f$ and $\mu$.*

We introduce the notion of *constructibility* in Theorems 3 and 4 to further describe the structure of the operator $T$ and map $M$.

As an application of Theorem 1, we construct a linear extension operator from the trace space $L^{m,p}(E)$ ($E \subset \mathbb{R}^n$ arbitrary) to $L^{m,p}(\mathbb{R}^n)$ for $p \in (n, \infty)$. Let $(\mathbb{X}(\mathbb{R}^n), \|\cdot\|_{\mathbb{X}(\mathbb{R}^n)})$ be a complete semi-normed linear space of continuous functions. For $E \subset \mathbb{R}^n$, let $\mathbb{X}(E)$ be the space of restrictions to $E$ of functions in $\mathbb{X}(\mathbb{R}^n)$, equipped with the trace semi-norm:

$$\begin{aligned} &\mathbb{X}(E) := \{f : E \to \mathbb{R} : \exists F \in \mathbb{X}(\mathbb{R}^n),\ F|_E = f\}, \text{ with} \\ &\|f\|_{\mathbb{X}(E)} := \inf\{\|F\|_{\mathbb{X}(\mathbb{R}^n)} : F \in \mathbb{X}(\mathbb{R}^n) \text{ and } F|_E = f\}. \end{aligned}$$

A function $F : \mathbb{R}^n \to \mathbb{R}$ satisfying $F|_E = f$ is an *extension* of $f$. A linear map $T : \mathbb{X}(E) \to \mathbb{X}(\mathbb{R}^n)$ satisfying $Tf|_E = f$ and $\|Tf\|_{\mathbb{X}(\mathbb{R}^n)} \le C\|f\|_{\mathbb{X}(E)}$, where $C$ is independent of $f$, is a *bounded linear extension operator*. We pose the following questions about the trace and extension problems in $\mathbb{X}(\mathbb{R}^n)$:

**Question 3.** *Given $E \subset \mathbb{R}^n$, does there exist a bounded linear extension operator $T : \mathbb{X}(E) \to \mathbb{X}(\mathbb{R}^n)$ satisfying $Tf|_E = f|_E$ and $\|Tf\|_{\mathbb{X}(\mathbb{R}^n)} \le C\|f\|_{\mathbb{X}(E)}$ for all $f : E \to \mathbb{R}$, where $C$ is independent of $E$ and $f$?*

**Question 4.** *Can we estimate $\|f\|_{\mathbb{X}(E)}$?*

For $\mathbb{X}(\mathbb{R}^n) = L^{m,p}(\mathbb{R}^n)$, $n < p < \infty$, and $E$ arbitrary, C. Fefferman, A. Israel, and G.K. Luli prove the existence of a bounded linear extension operator, as in Question 3, in [13] and [12]. When $E$ is finite, they introduce the concept of *assisted bounded depth* to describe the structure of the operator $T$ and they give an approximate formula for the trace norm $\|f\|_{\mathbb{X}(E)}$ in this case. When $E$ is arbitrary, they prove the existence of a bounded linear extension operator $T$ by taking the Banach limit of operators extending $f$ from a sequence of finite subsets of $E$. Consequently, their extension operator $T$ loses all of its structural properties when $E$ is arbitrary. To prove Theorem 2 below, we provide a direct construction of a bounded linear extension operator $T$, valid when $E$ is arbitrary. In Theorem 5, we describe the structure of the extension operator and an approximate formula for the trace norm through the notion of constructibility, which is a generalization of the notion of assisted bounded depth to the setting when $E$ is arbitrary.

The question of optimal decomposition in the sum space $L^{m,p}(\mathbb{R}^n) + L^p(d\mu)$ can be phrased as the problem of *approximate extension and interpolation* in Sobolev spaces, a topic of independent interest. Suppose an experimenter collects data defining a function $f : E \to \mathbb{R}$ ($E$ finite). Rather than assume $f$ is the restriction to the set $E$ of a function in the space $L^{m,p}(\mathbb{R}^n)$, we assume $f$ lies *near* a function in the space $L^{m,p}(\mathbb{R}^n)$. Let the function $\mu : E \to [0, \infty]$ represent the confidence in data collected; where the experimenter has high confidence in the data, we expect $\mu$ to be very large, and where the experimenter lacks confidence, we expect $\mu$ to approach 0. Then we wish to estimate

$$\inf_{F \in L^{m,p}(\mathbb{R}^n)} \left\{ \|F\|^p_{L^{m,p}(\mathbb{R}^n)} + \sum_{x \in E} |F(x) - f(x)|^p \mu(x) \right\} \tag{1.3}$$

(subject to the convention that if $|F(x) - f(x)| = 0$ and $\mu(x) = \infty$, then $|F(x) - f(x)|^p \mu(x) = 0 \cdot \infty = 0$), and to construct a function $Tf \in L^{m,p}(\mathbb{R}^n)$ satisfying,

$$\begin{aligned} &\|Tf\|^p_{L^{m,p}(\mathbb{R}^n)} + \sum_{x \in E} |Tf(x) - f(x)|^p \mu(x) \\ &\quad \le C \cdot \inf_{F \in L^{m,p}(\mathbb{R}^n)} \left\{ \|F\|^p_{L^{m,p}(\mathbb{R}^n)} + \sum_{x \in E} |F(x) - f(x)|^p \mu(x) \right\}. \end{aligned}$$

In this setting, we can interpret Theorem 1 as giving a construction for a near-optimal approximate extension $Tf$ of the function $f$ subject to the confidence, represented by $\mu$, and an approximate formula for the optimal value of (1.3). When $\mu : E \to \mathbb{R}$ is defined as $\mu(x) \equiv \infty$ for all $x \in E$, the expression (1.3) is finite only if $F|_E = f$ (i.e., $F$ is an exact interpolant of $f$ on $E$). This problem then encompasses the problem of interpolation in Sobolev spaces, which was studied by Israel, Luli, and Fefferman in [14], [15], and [16].

We use Theorem 1 to construct a bounded linear extension operator for $L^{m,p}(\mathbb{R}^n)$ ($n < p < \infty$): Let $E \subset \mathbb{R}^n$ be a bounded Borel set, and $f : E \to \mathbb{R}$ be Borel measurable. Define a Borel measure $\mu_E$ on $\mathbb{R}^n$ so that, for all Borel sets $A \subset \mathbb{R}^n$,

$$\mu_E(A) = \begin{cases} \infty & \text{if } A \cap E \neq \emptyset, \\ 0 & \text{if } A \cap E = \emptyset. \end{cases} \tag{1.4}$$

We apply Theorem 1 to the measure $\mu_E$ to produce a linear operator $T$ and map $M$. We make a few observations. We shall identify $f : E \to \mathbb{R}$ with a function on $\mathbb{R}^n$, using its extension by zero. First, note that $f \in L^{m,p}(E)$ if and only if $f \in L^{m,p}(\mathbb{R}^n) + L^p(d\mu_E)$; furthermore, $\|f\|_{L^{m,p}(E)} = \|f\|_{L^{m,p}(\mathbb{R}^n) + L^p(d\mu_E)}$. We also have that $\|F\|_{L^{m,p}(\mathbb{R}^n)} + \|F - f\|_{L^p(d\mu_E)}$ is finite if and only if $F \in L^{m,p}(\mathbb{R}^n)$ and $F = f$ on $E$, and in this case we have $\|F - f\|_{L^p(d\mu_E)} = 0$. Therefore, the map $T : L^{m,p}(\mathbb{R}^n) + L^p(d\mu_E) \to L^{m,p}(\mathbb{R}^n)$, given in Theorem 1, is a bounded linear extension operator on the trace space $L^{m,p}(E)$. Further, because $\mu_E(\{x\}) = \infty$ for $x \in E$, $Mf$ is finite if and only if $\zeta_\ell(f) - f \equiv 0 \in L^p(d\mu_E)$ for all $\ell \in \mathbb{N}$. Consequently, by applying Theorem 1, we obtain the following result:

**Theorem 2.** *Let $E \subset \mathbb{R}^n$ be a compact set, $m \in \mathbb{N}$, and $n < p < \infty$. There exists a bounded linear extension operator $T : L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)$ and a map $M : L^{m,p}(E) \to \mathbb{R}$ satisfying for all $f \in L^{m,p}(E)$,*

$$\begin{aligned} &Tf = f \text{ on } E\\ &\|f\|_{L^{m,p}(E)} \leq \|Tf\|_{L^{m,p}(\mathbb{R}^n)} \leq C \cdot \|f\|_{L^{m,p}(E)};\\ &c \cdot Mf \leq \|Tf\|_{L^{m,p}(\mathbb{R}^n)} \leq C \cdot Mf; \text{ and}\\ &Mf = \big( \sum_{\ell \in \mathbb{N}} |\psi_\ell(f)|^p + \|\vec{S}(f)\|^p_{L^{m,p}(K)} \big)^{1/p}, \end{aligned}$$

*where $K \subset E$, $\vec{S} : L^{m,p}(E) \to Wh(K)$ is a linear map, and for each $\ell \in \mathbb{N}$, $\psi_\ell : L^{m,p}(E) \to \mathbb{R}$ is a bounded linear functional. The constants $c$ and $C$ depend on $m$, $n$, and $p$ but are independent of $f$ and $E$.*

Given a real normed linear space $\mathbb{X}$, we denote the dual space of $\mathbb{X}$ by $\mathbb{X}^* := \{f : \mathbb{X} \to \mathbb{R} : \ f \text{ is a bounded linear functional } \}$.

When $E$ is finite, Fefferman, Israel, and Luli [13,12] introduce the notion of assisted bounded depth to describe how the values of their extension and its derivatives up to order $m-1$ rely on a collection of linear functionals contained in $L^{m,p}(E)^*$. Below, in Theorems 3 and 4, we express the operator $T$ and the map $M$ through a collection of functionals in $(L^{m,p}(\mathbb{R}^n) + L^p(d\mu))^*$ whose supports have bounded overlap, generalizing the notion of assisted bounded depth to the setting of the optimal decomposition problem (Questions 1 and 2). In Theorem 5, we translate this to the setting of the extension problem (Questions 3 and 4), where the operator $T$ and the map $M$ can be expressed through a collection of functionals in $L^{m,p}(E)^*$ whose supports have bounded overlap.

Recall the support of a functional $\omega \in (L^{m,p}(\mathbb{R}^n) + L^p(d\mu))^*$ is defined as

$$\operatorname{supp}(\omega) := \left( \bigcup \left\{ E \subset \mathbb{R}^n : \begin{array}{l} E \text{ is open, and for all } f \in L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \\ \text{satisfying } \operatorname{supp}(f) \subset E, \ \omega(f) = 0. \end{array} \right\} \right)^c.$$

We write $|S|$ for the cardinality of a set $S$. A collection $\Pi$ of sets has *$A$-bounded overlap* ($A \geq 1$) provided that $|\{\pi \in \Pi : x \in \pi\}| \leq A$ for all $x$. With this notation, we now state the refined version of Theorem 1:

**Theorem 3.** *The linear operator $T$ and map $M$ in Theorem 1 can be chosen to be $\Omega$-constructible, in the following sense:*

*There exists a collection of linear functionals $\Omega = \{\omega_r\}_{r \in \Upsilon} \subset (L^{m,p}(\mathbb{R}^n) + L^p(d\mu))^*$, such that the collection of sets $\{\operatorname{supp}(\omega_r)\}_{r \in \Upsilon}$ has $C'$-bounded overlap, and for each $x \in \mathbb{R}^n$, there exists a finite subset $\Upsilon_x \subset \Upsilon$ and a collection of polynomials $\{v_{r,x}\}_{r \in \Upsilon_x} \subset \mathcal{P}$ such that $|\Upsilon_x| \leq C$ and*

$$J_x T f = \sum_{r \in \Upsilon_x} \omega_r(f) \cdot v_{r,x}. \tag{1.5}$$

*Recall that the map $M$ is defined in (1.2) in terms of linear maps $(\zeta_\ell)_{\ell\in\mathbb{N}}$, $(\psi_\ell)_{\ell\in\mathbb{N}}$, and $\vec{S} = (S_x)_{x\in K}$. Then the following holds:*

***(1)*** *For each $\ell \in \mathbb{N}$ and $y \in \operatorname{supp}(\mu)$, there exists a finite subset $\bar{\Upsilon}_{\ell,y} \subset \Upsilon$ and constants $\{\eta^\ell_{s,y}\}_{s\in\bar{\Upsilon}_{\ell,y}} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_{\ell,y}| \le C$, and the map $f \mapsto \zeta_\ell(f)(y)$ has the form*

$$\zeta_\ell(f)(y) = \sum_{s\in\bar{\Upsilon}_{\ell,y}} \eta^\ell_{s,y} \cdot \omega_s(f).$$

***(2)*** *For each $\ell \in \mathbb{N}$, there exists a finite subset $\bar{\Upsilon}_\ell \subset \Upsilon$ and constants $\{\eta^\ell_s\}_{s\in\bar{\Upsilon}_\ell} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_\ell| \le C$, and the map $\psi_\ell$ has the form*

$$\psi_\ell(f) = \sum_{s\in\bar{\Upsilon}_\ell} \eta^\ell_s \cdot \omega_s(f).$$

***(3)*** *For $y \in K$, there exist $\{\omega^\alpha_y\}_{\alpha\in\mathcal{M}} \subset \Omega$ satisfying for all $\alpha \in \mathcal{M}$, $\operatorname{supp}(\omega^\alpha_y) \subset \{y\}$, and the map $f \mapsto S_y(f)$ has the form*

$$S_y(f) = \sum_{\alpha\in\mathcal{M}} \omega^\alpha_y(f) \cdot v_\alpha,$$

*where $\{v_\alpha\}_{\alpha\in\mathcal{M}}$ is a basis for $\mathcal{P}$.*

*The constants $C$ and $C'$ depend on $m$, $n$, and $p$ but are independent of $f$ and $\mu$.*

When $\mu$ is a finite measure (i.e., $\mu(\mathbb{R}^n) < \infty$) the formulas for $M$ and $T$ are simpler. Precisely,

**Theorem 4.** *Let $\mu$ be a finite Borel regular measure on $\mathbb{R}^n$ with compact support. Then the map $M : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to \mathbb{R}$ in Theorem 1 satisfies:*

$$Mf = \Big(\sum_{\ell=1}^{K} \left( \int_{A_\ell} |\zeta_\ell(f) - f|^p d\mu + |\psi_\ell(f)|^p + |\psi_\ell(f)|^p \right) \Big)^{1/p},$$

*where for each $\ell$, $A_\ell \subset \operatorname{supp}(\mu)$ is a Borel set, $\zeta_\ell : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to L^p(d\mu)$, and $\psi_\ell : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to \mathbb{R}$ are linear maps. Further the collection $\Omega \subset (L^{m,p}(\mathbb{R}^n) + L^p(d\mu))^*$ used to express $M$ and $T$ in Theorem 3 consists of a finite collection of linear functionals.*

As promised, we next adapt the notion of constructibility to describe the structure of the linear extension operator in Theorem 2.

**Theorem 5.** *The linear operator $T$ in Theorem 2 is $\Omega$-constructible. Precisely, there exists a collection of linear functionals $\Omega = \{\omega_r\}_{r\in\Upsilon} \subset L^{m,p}(E)^*$, such that the collection of sets $\{\mathrm{supp}(\omega_r)\}_{r\in\Upsilon}$ has $C'$-bounded overlap, and for each $x \in \mathbb{R}^n$, there exists a finite subset $\Upsilon_x \subset \Upsilon$ and a collection of polynomials $\{v_{r,x}\}_{r\in\Upsilon_x} \subset \mathcal{P}$ such that $|\Upsilon_x| \le C$ and*

$$J_x T f = \sum_{r\in\Upsilon_x} \omega_r(f) \cdot v_{r,x}. \tag{1.6}$$

*And the map $M$ in Theorem 2 is $\Omega$-constructible:*

***(1)*** *For each $\ell \in \mathbb{N}$, there exists a finite subset $\bar{\Upsilon}_\ell \subset \Upsilon$ and constants $\{\eta_s^\ell\}_{s\in\bar{\Upsilon}_\ell} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_\ell| \le C$, and the map $\psi_\ell$ has the form*

$$\psi_\ell(f) = \sum_{s\in\bar{\Upsilon}_\ell} \eta_s^\ell \cdot \omega_s(f).$$

***(2)*** *For $y \in K$, there exist $\{\omega_y^\alpha\}_{\alpha\in\mathcal{M}} \subset \Omega$ satisfying for all $\alpha \in \mathcal{M}$, $\mathrm{supp}(\omega_y^\alpha) \subset \{y\}$, and the map $f \mapsto S_y(f)$ has the form*

$$S_y(f) = \sum_{\alpha\in\mathcal{M}} \omega_y^\alpha(f) \cdot v_\alpha,$$

*where $\{v_\alpha\}_{\alpha\in\mathcal{M}}$ is a basis for $\mathcal{P}$.*

*The constants $C$ and $C'$ depend on $m$, $n$, and $p$ but are independent of $f$ and $E$.*

Let $E \subset \mathbb{R}^n$ be finite. A linear map which is $\Omega$-constructible as defined in (1.6) has $\Omega$-assisted bounded depth (see [13] for the definition of assisted bounded depth operators). Consequently, the extension operator we construct in Theorem 5 has the same good structural property as the extension operator in Theorem 3 of [13] when $E$ is finite.

A pair of complete semi-normed spaces $(A, B)$ is said to be compatible if there exists a Hausdorff topological vector space $\mathscr{H}$ into which both $A$ and $B$ can be continuously embedded (see [3], p. 153). For a compatible couple $(A, B)$, the sum $A + B$ is the set of elements $f \in \mathscr{H}$ that can be represented as the sum, $f = f_1 + f_2$, of elements $f_1 \in A$ and $f_2 \in B$. Under the norm $\|f\|_{A+B} = \inf_{f_1+f_2=f}\{\|f_1\|_A + \|f_2\|_B\}$, $A+B$ is a complete semi-normed linear space. We can generate interpolation spaces for the couple $(A, B)$ via the real method, by calculating the $K$-functional:

$$K\big(t; f : (A, B)\big) := \inf\left\{\|f_1\|_A + t \cdot \|f_2\|_B : f = f_1 + f_2\right\}.$$

The Banach couple, $(A, B)$, is *$C, K$-linearized* if there exists a constant $C$ independent of $t$ such that for all $t > 0$, for all $f \in A+B$, there exists an almost optimal decomposition $f = F_1^t + F_2^t$, where $F_1^t, F_2^t$ depend linearly on $f$:

$$\|F_1^t\|_A + t\|F_2^t\|_B \le C \cdot \|f\|_{K(t;f:(A,B))}.$$

The pair $(L^{m,p}(\mathbb{R}^n), L^p(d\mu))$ is a compatible couple, and

$$\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)} \leq K\Big(1; f : (L^{m,p}(\mathbb{R}^n), L^p(d\mu))\Big) \leq C\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(d\mu)},$$

where $C = C(n, p)$.

Furthermore,

$$\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(t^p d\mu)} \leq K\Big(t; f : (L^{m,p}(\mathbb{R}^n), L^p(d\mu))\Big) \leq C\|f\|_{L^{m,p}(\mathbb{R}^n)+L^p(t^p d\mu)},$$

where $C = C(n, p)$.

Theorem 1 immediately implies:

**Theorem 6.** *Let $\mu$ be a compactly supported Borel regular measure on $\mathbb{R}^n$. Then the Banach couple $(L^{m,p}(\mathbb{R}^n), L^p(d\mu))$ is $C, K$-linearized, with $C$ independent of $\mu$.*

### *1.1. Background*

P. Shvartsman considered Questions 1 and 2, providing a solution when $\mu$ is a $\sigma$-finite Borel measure, $m = 1$, and $p \in (n, \infty)$ in [22].

H. Whitney gave an answer to Questions 3 and 4 for the function space $\mathbb{X}(\mathbb{R}^n) = C^m(\mathbb{R}^n)$ in the case $n = 1$ ([25]). For $\omega \in (0, 1]$, the space $C^{m,\omega}(\mathbb{R}^n)$ consists of $C^m(\mathbb{R}^n)$ functions with $\omega$-Hölder continuous $m^{th}$ order derivatives and finite norm:

$$\|F\|_{C^{m,\omega}(\mathbb{R}^n)} = \|F\|_{C^m(\mathbb{R}^n)} + \max_{|\alpha|=m} \sup_{x,y\in\mathbb{R}^n, x\neq y} \left\{ \frac{|\partial^\alpha F(x) - \partial^\alpha F(y)|}{|x-y|^\omega} \right\}.$$

Fefferman solved Questions 3 and 4 for $\mathbb{X}(\mathbb{R}^n) = C^m(\mathbb{R}^n)$ and $\mathbb{X}(\mathbb{R}^n) = C^{m,\omega}(\mathbb{R}^n)$ for $m, n \in \mathbb{N}$, and $\omega \in (0, 1]$ in [7], [8], [9], and [10]. His work built on theory developed by G. Glaeser, Y. Brudnyi, Shvartsman, E. Bierstone, P. Milman, and W. Pawłucki in [19], [4], [5], and [1]. Fefferman considered questions of approximate extension and utilized this relaxation of the extension problem in the spaces $C^m(\mathbb{R}^n)$ and $C^{m,\omega}(\mathbb{R}^n)$ in [6–8] and [11], and in his work with B. Klartag in [17] and [18].

For $\mathbb{X}(\mathbb{R}^n) = L^{m,p}(\mathbb{R}^n)$, Shvartsman used the classical Whitney extension operator to answer Questions 3 and 4 for $p \in (n, \infty)$ and $m = 1$ in [21]. Israel [20] and Shvartsman [23] independently answered Questions 3 and 4 when $m = 2$, $n < p < \infty$, and $E$ is finite. Fefferman, Israel, and Luli extended the method of [20] to $m \in \mathbb{N}$, $n < p < \infty$ in [13] and [12]. However, they describe structural properties of the extension operator $T$ only for finite $E \subset \mathbb{R}^n$.

### *1.2. Overview of the proof of Theorem 1*

Let $\mu$ be a Borel regular measure satisfying that $\text{supp}(\mu) \subset Q$ for a cube $Q \subset \mathbb{R}^n$. In order to construct a linear operator approximately extending a function from the support of $\mu$, we study the freedom we have to define a Sobolev function at each point of $Q$. In light of the Sobolev embedding theorem on $L^{m,p}(\mathbb{R}^n)$ for $p > n$, we can study the set of prospective $(m-1)$-jets of approximate extensions and utilize the inductive framework introduced by Fefferman in [7]. We define $\mathscr{J}$-functionals to encode how well a Sobolev function approximates a function $f : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to \mathbb{R}$ in terms of the measure $\mu$. When the measure $\mu$ is finite, each step of the induction utilizes a *finite* Calderón Zygmund (CZ) decomposition of $Q$ that identifies cubes where we can use the inductive hypothesis to solve the approximate extension problem locally. Then we can use the techniques of Israel in [20] and Israel, Luli, and Fefferman in [13] to ensure global compatibility of these local solutions and patch them together.

When the measure $\mu$ is not finite, the CZ decomposition need not be finite, implying there is a subset of $Q$ we call *keystone points* where we cannot use an inductive hypothesis to produce a local solution. How can we define an approximate extension on this set so that we produce a Sobolev function on $Q$? This is a new problem for extension in Sobolev space, though Whitney navigates the boundary between an infinite decomposition and its complement in his $C^m$ extension theorem in [25]. In Section 4.2, we identify the set of keystone points $K_p$, and consider its properties. In Section 6.3, we show how the measure $\mu$ restricts a prospective approximate extension on $K_p$, relying on a new estimate (Lemma 5.2). Then we use a Sobolev function to define an approximate extension on $K_p$ that is compatible with the extension defined on the CZ cubes. In Section 5.3.2, we show that this function defined piecewise on $K_p$ and the CZ cubes is, in fact, a function in $L^{m,p}(Q)$ through a characterization of Sobolev space due to Brudnyi in [2]. This follows the work of Shvartsman in [21], who used [2] to give an intrinsic characterization of the trace space $L^{1,p}(E)$ for $E \subset \mathbb{R}^n$. In Sections 8-10, we establish that the constructed approximate extension is optimal using properties of the map $M : L^{m,p}(\mathbb{R}^n) + L^p(d\mu) \to \mathbb{R}$ in Theorem 1.

## 2. Notation and further theory

### 2.1. Notation

Fix integers $m, n \geq 1$ and a real number $p > n$. Unless we say otherwise, constants written $c, c', C, C'$, etc. depend only on $m, n$, and $p$. They are called "universal" constants. The lower case letters denote small (universal) constants while the upper case letters denote large (universal) constants. Given a parameter $\xi$, we write $c(\xi)$, $C(\xi)$, etc., to denote constants depending only on $m, n, p$, and $\xi$.

For non-negative quantities $A, B$, we write $A \simeq B$, $A \lesssim B$, or $A \gtrsim B$ to indicate that $cB \leq A \leq CB$, $A \leq CB$, or $A \geq cB$, respectively, for universal constants $0 < c < C$. Given a parameter $\xi$, we write $A \simeq_\xi B$, $A \lesssim_\xi B$, or $A \gtrsim_\xi B$ to indicate that $c(\xi)B \leq A \leq C(\xi)B$, $A \leq C(\xi)B$, or $A \geq c(\xi)B$, respectively, for constants $0 < c(\xi) < C(\xi)$.

When $\varphi_y$ is an indexed family of functions, an expression of the form $\partial^\alpha \varphi_y(y)$ will always mean $\partial_z^\alpha \varphi_y(z)|_{z=y}$, and never $\partial_z^\alpha \varphi_z(z)|_{z=y}$

A cube $Q \subset \mathbb{R}^n$ is a set of the form:

$$Q = a + (-\delta, \delta]^n \quad (a \in \mathbb{R}^n, \delta > 0) .$$

The sidelength of $Q$ is denoted $\delta_Q := 2\delta$, while the center of $Q$ is denoted $\mathrm{ctr}(Q) := a$. For $\gamma > 0$ let $\gamma Q$ be the cube having the same center as $Q$ but with sidelength $\gamma \delta_Q$. A *dyadic cube* $Q \subset \mathbb{R}^n$ has the form:

$$\begin{aligned} Q = &\left(j_1 \cdot 2^k, (j_1+1) \cdot 2^k\right] \times \left(j_2 \cdot 2^k, (j_2+1) \cdot 2^k\right] \times \cdots \\ &\times \left(j_n \cdot 2^k, (j_n+1) \cdot 2^k\right], \ (j_1, j_2, \dots, j_n \in \mathbb{Z}, \ k \in \mathbb{Z}) . \end{aligned}$$

To *bisect* a cube $Q \subset \mathbb{R}^n$ is to partition it into $2^n$ disjoint subcubes of sidelength $\frac{1}{2}\delta_Q$. These subcubes are called the children of $Q$. If $Q \subsetneq Q'$ are dyadic cubes we say that $Q'$ is an ancestor of Q. Every dyadic cube $Q$ has a smallest ancestor called its parent, which we denote by $Q^+$.

A rectangular box in $\mathbb{R}^n$ is a set of the form $R = \prod_{j=1}^n I_j$, where each $I_j \subset \mathbb{R}$ is an interval of length $\delta_j > 0$. We refer to $\delta_1, \dots, \delta_n$ as the sidelengths of $R$. If the sidelengths of $R$ differ by at most a constant factor $\eta \geq 1$ (i.e., $\delta_j \leq \eta \delta_i$ for all $i, j$), then we say $R$ is *$\eta$-non-degenerate.*

We use the following notation:

$$\begin{aligned}
&|x| := |x|_\infty = \max\{|x_1|, \dots, |x_n|\} && (x = (x_1, \dots, x_n) \in \mathbb{R}^n); \\
&\mathrm{dist}(x, \Omega) := \inf\{|x - y| : y \in \Omega\} && (\Omega \subset \mathbb{R}^n, \ x \in \mathbb{R}^n); \\
&\mathrm{dist}(\Omega', \Omega) := \inf\{|x - y| : x \in \Omega', \ y \in \Omega\} && (\Omega, \Omega' \subset \mathbb{R}^n); \\
&B(\Omega, R) := \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega) \leq R\} && (\Omega \subset \mathbb{R}^n, R > 0); \\
&\mathrm{diam}(S) := \sup\{|x - y| : x, y \in S\} && (S \subset \mathbb{R}^n); \text{ and} \\
&|S| := \text{cardinality of } S && (S \subset \mathbb{R}^n).
\end{aligned}$$

In particular, since we use the $\ell^\infty$ norm on $\mathbb{R}^n$, we have $\mathrm{diam}(Q) = \delta_Q$ for any cube $Q$.

The analogous metric quantities defined with respect to the Euclidean norm $|x|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$ are denoted by $\mathrm{dist}_2(x, \Omega)$, $\mathrm{dist}_2(\Omega', \Omega)$, $B_2(\Omega, R)$, and $\mathrm{diam}_2(S)$.

Given a subset $K \subset \mathbb{R}^n$, we write $\mathrm{int}(K)$ to denote the interior of $K$, and we write $\mathrm{Cl}(K)$ to denote the closure of $K$.

We write $\mathcal{M}$ for the collection of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, with $\alpha_i \geq 0$ for all $i$, of order $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq m - 1$. If $\alpha$ and $\beta$ are multi-indices, then $\delta_{\alpha\beta}$ denotes the Kronecker delta: $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$; $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, then $\alpha! := \prod_{j=1}^n \alpha_j!$.

Let $\mathcal{P}$ denote the space of real-valued $(m-1)^{\mathrm{rst}}$ degree polynomials on $\mathbb{R}^n$. Then $\mathcal{P}$ is a vector space of dimension $D := \dim(\mathcal{P})$.

If $F$ is a $C^{m-1}$ function on a neighborhood of a point $y \in \mathbb{R}^n$, then we write $J_y(F) \in \mathcal{P}$ (the "jet" of $F$ at $y$) for the $(m-1)^{\mathrm{rst}}$ degree Taylor polynomial of $F$ at $y$, given by

$$J_y(F)(z) = \sum_{\alpha \in \mathcal{M}} \frac{\partial^\alpha F(y)}{\alpha!}(z-y)^\alpha.$$

For each $x \in \mathbb{R}^n$, the jet product $\odot_x$ on $\mathcal{P}$ is defined by

$$P \odot_x Q := J_x(P \cdot Q) \quad (P, Q \in \mathcal{P}).$$

For $x \in \mathbb{R}^n$, $\delta > 0$, define a norm on $\mathcal{P}$:

$$|P|_{x,\delta} = \Big( \sum_{\alpha \in \mathcal{M}} |\partial^\alpha P(x)|^p \cdot \delta^{n+(|\alpha|-m)p} \Big)^{1/p} \quad (P \in \mathcal{P}). \tag{2.1}$$

For $x' \in \mathbb{R}^n$, we have the Taylor expansion

$$\partial^\alpha P(x) = \sum_{|\gamma| \le m-1-|\alpha|} \frac{1}{\gamma!} \partial^{\alpha+\gamma} P(x') \cdot (x-x')^\gamma \quad (|\alpha| \le m-1).$$

Thus, the norms defined in (2.1) satisfy the inequality

$$|P|_{x,\delta} \le C' |P|_{x',\delta} \quad (x, x' \in \mathbb{R}^n, |x-x'| \le C\delta). \tag{2.2}$$

And by computation, for $\delta' > \delta$,

$$|P|_{x,\delta'} \le |P|_{x,\delta} \le (\delta'/\delta)^{m-n/p} |P|_{x,\delta'}. \tag{2.3}$$

The homogeneous Hölder space $C^{m-1,1-n/p}(\mathbb{R}^n)$ is the space of $(m-1)$-times differentiable functions $F : \mathbb{R}^n \to \mathbb{R}$, with finite semi-norm,

$$\|F\|_{C^{m-1,1-n/p}(\mathbb{R}^n)} := \max_{|\alpha|=m-1} \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|\partial^\alpha F(x) - \partial^\alpha F(y)|}{|x-y|^{1-n/p}}.$$

Given a set $K \subset \mathbb{R}^n$, we let $Wh(K)$ be the space of Whitney fields on $K$, namely, the set of all collections of polynomials $\vec{P} = (P_x)_{x \in K}$, where $P_x \in \mathcal{P}$ for all $x \in K$.

Let $\mu$ be a Borel regular measure on $\mathbb{R}^n$. For a Borel set $S \subset \mathbb{R}^n$, we define the restricted measure $\mu|_S$ by $\mu|_S(A) := \mu(A \cap S)$ for all Borel sets $A \subset \mathbb{R}^n$.

Write $\mathrm{meas}_\mu(\mathbb{R}^n)$ for the vector space of all equivalence classes of $\mu$-measurable functions $f : \mathbb{R}^n \to \mathbb{R}$, with functions identified in an equivalence class if they agree on the complement of a set of $\mu$-measure 0.

For a $\mu$-measurable function $f : \mathbb{R}^n \to \mathbb{R}$, we define:

$$\|F\|_{\mathscr{J}(f,\mu)} := \Big( \|F\|^p_{L^{m,p}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |F-f|^p d\mu \Big)^{1/p} \qquad F \in L^{m,p}(\mathbb{R}^n). \tag{2.4}$$

We remark that $\|\cdot\|_{\mathscr{J}(f,\mu)}$ is not a norm on $L^{m,p}(\mathbb{R}^n)$. Rather, $\|\cdot\|_{\mathscr{J}(f,\mu)}$ is a $[0,\infty]$-valued functional on $L^{m,p}(\mathbb{R}^n)$, satisfying the convexity condition,

$$\|\lambda_1 F_1 + \lambda F_2\|_{\mathscr{J}(\lambda_1 f_1 + \lambda_2 f_2, \mu)} \leq \lambda_1 \|F_1\|_{\mathscr{J}(f_1,\mu)} + \lambda_2 \|F_2\|_{\mathscr{J}(f_2,\mu)}.$$

We write $\mathscr{J}(\mu) := L^{m,p}(\mathbb{R}^n) + L^p(d\mu)$ for the sum space defined in the introduction. Then $\mathscr{J}(\mu)$ is a complete seminormed vector space. We can characterize the seminorm[1] on this space using the functionals in (2.4):

$$\mathscr{J}(\mu) := \left\{ f : f \text{ is } \mu\text{-measurable, and } \|f\|_{\mathscr{J}(\mu)} < \infty \right\}, \text{ with}$$
$$\|f\|_{\mathscr{J}(\mu)} := \inf \left\{ \|F\|_{\mathscr{J}(f,\mu)} : F \in L^{m,p}(\mathbb{R}^n) \right\}.$$

Next, we introduce localized variants of the above functionals.

Let $R \subset \mathbb{R}^n$ be a rectangular box, and let $\delta > 0$.

Given $F \in L^{m,p}(R)$, a $\mu$-measurable function $f$, and $P \in \mathcal{P}$, we define

$$\|F, P\|_{\mathscr{J}_*(f,\mu;R)} := \left( \|F\|^p_{L^{m,p}(R)} + \int_R |F - f|^p d\mu + \|F - P\|^p_{L^p(R)} / \operatorname{diam}(R)^{mp} \right)^{1/p}; \text{ and}$$
$$\|f, P\|_{\mathscr{J}_*(\mu;R)} := \inf \left\{ \|F, P\|_{\mathscr{J}_*(f,\mu;R)} : F \in L^{m,p}(R) \right\}. \tag{2.5}$$

Given $F \in L^{m,p}(\mathbb{R}^n)$, a $\mu$-measurable function $f$, and $P \in \mathcal{P}$, we define:

$$\|F, P\|_{\mathscr{J}(f,\mu;\delta)} := \left( \|F\|^p_{L^{m,p}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |F - f|^p d\mu + \|F - P\|^p_{L^p(\mathbb{R}^n)} / \delta^{mp} \right)^{1/p}; \text{ and}$$
$$\|f, P\|_{\mathscr{J}(\mu;\delta)} := \inf \left\{ \|F, P\|_{\mathscr{J}(f,\mu;\delta)} : F \in L^{m,p}(\mathbb{R}^n) \right\}. \tag{2.6}$$

These are $[0,\infty]$-valued functionals on the spaces $L^{m,p}(R) \times \mathcal{P}$, $\text{meas}_\mu(\mathbb{R}^n) \times \mathcal{P}$, $L^{m,p}(\mathbb{R}^n) \times \mathcal{P}$, and $\text{meas}_\mu(\mathbb{R}^n) \times \mathcal{P}$, respectively. We make use of the fourth functional to define the seminormed vector space:

$$\mathscr{J}(\mu;\delta) = \left\{ (f, P) : f \in \mathscr{J}(\mu),\ P \in \mathcal{P},\ \|f, P\|_{\mathscr{J}(\mu;\delta)} < \infty \right\}. \tag{2.7}$$

Generally, $\mathscr{J}(\mu;\delta)$ is a subspace of $\mathscr{J}(\mu) \times \mathcal{P}$. Later, we will show that $\mathscr{J}(\mu;\delta) = \mathscr{J}(\mu) \times \mathcal{P}$ if $\text{supp}(\mu)$ is compact (see Lemma 2.10).

We can make comparisons between the different localized functionals. First, immediately from the definitions, if $\operatorname{diam}(R) \simeq \delta$ and $F \in L^{m,p}(\mathbb{R}^n)$, then

$$\|F, P\|_{\mathscr{J}_*(f,\mu;R)} \lesssim \|F, P\|_{\mathscr{J}(f,\mu;\delta)}. \tag{2.8}$$

Furthermore, the $\mathscr{J}(\mu;\delta)$ and $\mathscr{J}(f,\mu;\delta)$ functionals are monotone in $\delta$ in the sense that:

[1] A seminorm on a vector space $X$ is a $[0,\infty)$-valued functional $\|\cdot\|_X$ on $X$ satisfying the conditions: $\|f_1 + f_2\|_X \leq \|f_1\|_X + \|f_2\|_X$, and $\|\lambda f\|_X = |\lambda| \|f\|_X$ for $\lambda \in \mathbb{R}$. There is no requirement that $\|f\|_X = 0 \implies f = 0$ for a seminorm.

$$\|F, P\|_{\mathscr{J}(f,\mu;\delta)} \leq \|F, P\|_{\mathscr{J}(f,\mu;\delta')},$$
$$\|f, P\|_{\mathscr{J}(\mu;\delta)} \leq \|f, P\|_{\mathscr{J}(\mu;\delta')} \qquad (\delta \geq \delta'). \tag{2.9}$$

*2.2. Further elementary inequalities*

The next result is immediate from the definitions of the $\mathscr{J}$-functionals, due to the sublinearity of the $L^{m,p}$-norm, $L^p$-norm, and $L^p(d\mu)$-norm

**Lemma 2.1** *(Sublinearity of $\mathscr{J}$-functionals). Let $\mu$ be a Borel regular measure on $\mathbb{R}^n$, and let $\delta > 0$. Then $\|\cdot,\cdot\|_{\mathscr{J}(\cdot,\mu;\delta)}$ and $\|\cdot,\cdot\|_{\mathscr{J}(\mu;\delta)}$ are sublinear: Let $f_1, f_2$ be $\mu$-measurable functions. Given $F_1, F_2 \in L^{m,p}(\mathbb{R}^n)$, $\lambda > 0$, $P_1, P_2 \in \mathcal{P}$,*

$$\|F_1 + \lambda F_2, P_1 + \lambda P_2\|_{\mathscr{J}(f_1+\lambda f_2,\mu;\delta)} \leq \|F_1, P_1\|_{\mathscr{J}(f_1,\mu;\delta)} + \lambda\|F_2, P_2\|_{\mathscr{J}(f_2,\mu;\delta)}; \text{ and} \tag{2.10}$$

$$\|f_1 + \lambda f_2, P_1 + \lambda P_2\|_{\mathscr{J}(\mu;\delta)} \leq \|f_1, P_1\|_{\mathscr{J}(\mu;\delta)} + \lambda\|f_2, P_2\|_{\mathscr{J}(\mu;\delta)}. \tag{2.11}$$

Our assumption moving forward is that $n < p < \infty$, so we can apply the Sobolev inequality on $L^{m,p}(\mathbb{R}^n)$. We let $U \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We shall consider the setting where $U$ is either all of $\mathbb{R}^n$, or the union of two $\eta$-non-degenerate rectangular boxes with a common interior point. This includes, for instance, the case when $U$ is a cube, when $\eta = 1$.

**Sobolev Inequality.** (see [13]) For $F \in L^{m,p}(U)$, there exists a constant $C$ such that

$$|\partial^\alpha (J_y(F) - F)(x)| \leq C \cdot |x - y|^{m-|\alpha|-n/p} \cdot \|F\|_{L^{m,p}(U)} \quad (x, y \in U, |\alpha| \leq m - 1).$$

If $U = \mathbb{R}^n$, the constant $C$ is determined by $m$, $n$, and $p$ alone. If $U$ is the union of two $\eta$-non-degenerate rectangular boxes with a common interior point, then $C$ is dependent on $\eta$ as well. As an immediate consequence, we have for $F \in L^{m,p}(U)$ and $y \in U$,

$$\|F - J_y(F)\|_{L^p(U)} / \operatorname{diam}(U)^m \leq C\|F\|_{L^{m,p}(U)}. \tag{2.12}$$

Next we establish a relationship between the $L^p$ norm and the $|\cdot|_{x,\delta}$ norm of polynomials.

**Lemma 2.2.** *Let $R_1$ and $R_2$ be two $\eta$-non-degenerate rectangular boxes with a common interior point, such that* $\operatorname{diam}(R_1) \simeq \operatorname{diam}(R_2) \simeq \delta$. *Let $U = R_1 \cup R_2$. Then for all $y \in U$ and $P \in \mathcal{P}$,*

$$\|P\|_{L^p(U)}/\delta^m \simeq_\eta |P|_{y,\delta}.$$

**Proof.** Observe $\operatorname{diam}(U) \leq \operatorname{diam}(R_1) + \operatorname{diam}(R_2) \leq C\delta$, so the Lebesgue measure of $U$ is at most $C\delta^n$. We begin by writing the polynomial $P$ in a Taylor expansion at the basepoint $y$; then because for all $x \in U$, we have $|x - y| \leq \operatorname{diam}(U) \leq C\delta$, we deduce

$$\begin{aligned}
\|P\|^p_{L^p(U)}/\delta^{mp} &= \int_U |P(x)|^p dx/\delta^{mp} \\
&= \int_U \left| \sum_{\alpha\in\mathcal{M}} \partial^\alpha P(y)(x-y)^\alpha/\alpha! \right|^p dx/\delta^{mp} \\
&\le C \int_U \sum_{\alpha\in\mathcal{M}} |\partial^\alpha P(y)|^p \delta^{|\alpha|p} dx/\delta^{mp} \\
&\le C' \sum_{\alpha\in\mathcal{M}} |\partial^\alpha P(y)|^p \delta^{n-(m-|\alpha|)p} \\
&\le C''|P|^p_{y,\delta},
\end{aligned}$$

where $C, C', C''$ are universal constants.

To show the reverse inequality, we may assume by rescaling and translation ($x \mapsto C\cdot\eta x/\delta + y'$) that $y \in Q_0 = (0,1]^n$, $Q_0 \subset R_1$ or $Q_0 \subset R_2$, and $\delta = \eta$ in the statement of the lemma. Without loss of generality, suppose $y \in R_1$. In light of (2.3), we have $|P|_{y,\eta} \simeq_\eta |P|_{y,1}$. Because $\mathcal{P}$ is a finite dimensional vector space, all norms on $\mathcal{P}$ are equivalent; in particular, we have $|P|_{y,1} \simeq \|P\|_{L^p(Q_0)}$. Consequently

$$|P|_{y,\eta} \simeq_\eta |P|_{y,1} \simeq \|P\|_{L^p(Q_0)} \lesssim_\eta \|P\|_{L^p(U)}/\eta^m,$$

proving the lemma. □

**Lemma 2.3.** *Let $U$ be the union of two $\eta$-non-degenerate rectangular boxes with a common interior point, $U = R_1\cup R_2$, such that $\operatorname{diam}(R_1) \simeq \operatorname{diam}(R_2) \simeq \delta$. There exists a constant $C$ depending on $m, n, p, \eta$ such that the following holds.*

*Let $x \in U$, $F \in L^{m,p}(U)$, and $P \in \mathcal{P}$. For $|\alpha| \le m-1$,*

$$|\partial^\alpha(F-P)(x)| \le C\left(\|F\|_{L^{m,p}(U)} + \|F-P\|_{L^p(U)}/\delta^m\right)\delta^{m-|\alpha|-n/p}. \tag{2.13}$$

*If $\mu$ is a Borel regular measure on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mu$-measurable function, then for $F \in L^{m,p}(U)$,*

$$|\partial^\alpha(F-P)(x)| \le C(\|F,P\|_{\mathscr{J}_*(f,\mu;R_1)} + \|F,P\|_{\mathscr{J}_*(f,\mu;R_2)})\delta^{m-|\alpha|-n/p}, \tag{2.14}$$

*while for $F \in L^{m,p}(\mathbb{R}^n)$,*

$$|\partial^\alpha(F-P)(x)| \le C\|F,P\|_{\mathscr{J}(f,\mu;\delta)}\delta^{m-|\alpha|-n/p}. \tag{2.15}$$

**Proof.** Let $F \in L^{m,p}(U)$. To prove (2.13) we will show that

$$\sup_{x\in U}\max_{\alpha\in\mathcal{M}} |\partial^\alpha F(x)|\delta^{|\alpha|+n/p-m} \le C\left(\|F\|_{L^{m,p}(U)} + \|F\|_{L^p(U)}/\delta^m\right). \tag{2.16}$$

By replacing $F$ by $F - P$ in (2.16), we deduce (2.13).

By the Sobolev Inequality, and since $\mathrm{diam}(U) \lesssim \delta$, for any $x, y \in U$, we have

$$|(F - J_y(F))(x)| \leq C_s' \|F\|_{L^{m,p}(U)} \delta^{m-n/p},$$

and hence by integrating,

$$\|F - J_y(F)\|_{L^p(U)} \leq C_s \|F\|_{L^{m,p}(U)} \delta^m,$$

where $C_s'$ and $C_s$ depend on $m$, $n$, $p$, and $\eta$. If $F|_U = 0$, then both sides of (2.16) are zero, and the inequality holds true. Thus we may assume $F|_U \neq 0$.

For sake of contradiction, suppose (2.16) does not hold with a constant $C = \Lambda$, for $\Lambda > 0$ to be determined momentarily. Let $y \in U$, $\beta \in \mathcal{M}$ be the argument of

$$\sup_{x \in U} \max_{\alpha \in \mathcal{M}} |\partial^\alpha F(x)| \delta^{|\alpha| + n/p - m}.$$

Then

$$|\partial^\beta F(y)| \delta^{|\beta| + n/p - m} \geq \Lambda \left( \|F\|_{L^{m,p}(U)} + \|F\|_{L^p(U)} / \delta^m \right). \tag{2.17}$$

Applying Lemma 2.2 to $J_y(F)$ and from the definition (2.1), we have

$$\|J_y(F)\|_{L^p(U)} / \delta^m \simeq |J_y(F)|_{y,\delta} \simeq |\partial^\beta F(y)| \delta^{|\beta| + n/p - m}.$$

Consequently, there exists a universal constant $c'$ such that

$$\|J_y(F)\|_{L^p(U)} / \delta^m \geq c' |\partial^\beta (F)(y)| \delta^{|\beta| + n/p - m}.$$

Choose $\Lambda > \frac{1 + C_s}{c'}$. Then from (2.17) and the Sobolev Inequality, we have

$$\begin{aligned}
&|\partial^\beta (F)(y)| \delta^{|\beta| + n/p - m} \\
&\quad > \frac{1 + C_s}{c'} \left( \|F\|_{L^{m,p}(U)} + \|F\|_{L^p(U)} / \delta^m \right) \\
&\quad > \frac{1 + C_s}{c'} \|F\|_{L^{m,p}(U)} + \frac{1}{c'} \|F\|_{L^p(U)} / \delta^m \\
&\quad \geq \frac{1 + C_s}{c'} \|F\|_{L^{m,p}(U)} + \frac{1}{c'} \Big( \|J_y(F)\|_{L^p(U)} / \delta^m - \|F - J_y(F)\|_{L^p(U)} / \delta^m \Big) \\
&\quad \geq \frac{1 + C_s}{c'} \|F\|_{L^{m,p}(U)} + \frac{1}{c'} \Big( \|J_y(F)\|_{L^p(U)} / \delta^m - C_s \|F\|_{L^{m,p}(U)} \Big) \\
&\quad \geq \frac{1}{c'} \|F\|_{L^{m,p}(U)} + |\partial^\beta F(y)| \delta^{|\beta| + n/p - m}.
\end{aligned}$$

This implies $\|F\|_{L^{m,p}(U)} < 0$, a contradiction. So (2.16) must hold, which implies (2.13).

Note that (2.14) follows from (2.13) and the definition of the $\mathcal{J}_*(\cdots)$ functional. Finally, (2.15) follows from (2.8) and (2.14). $\square$

**Lemma 2.4.** *Let $Q$ be a cube, let $\mu$ be a Borel regular measure on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) \subset Q$, let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mu$-measurable, let $P \in \mathcal{P}$, and let $\eta \in [.001, 100]$. Let $\theta$ be a $C^\infty$ function satisfying $\operatorname{supp}(\theta) \subset (1+\eta)Q$, $\theta|_Q = 1$, and $|\partial^\alpha \theta(x)| \leq C\delta_Q^{-|\alpha|}$ for all $|\alpha| \leq m$.*

*Let $F \in L^{m,p}((1+\eta)Q)$. Define $\bar{F} := \theta \cdot F + (1-\theta)P$. Then $\bar{F} \in L^{m,p}(\mathbb{R}^n)$, $\bar{F}|_Q = F|_Q$, and*

$$\|\bar{F}\|_{L^{m,p}(\mathbb{R}^n)} \lesssim \|F\|_{L^{m,p}((1+\eta)Q)} + \|F - P\|_{L^p((1+\eta)Q)}/\delta_Q^m, \tag{2.18}$$

$$\|\bar{F}, P\|_{\mathscr{J}(f,\mu;\delta_Q)} \lesssim \|F, P\|_{\mathscr{J}_*(f,\mu;(1+\eta)Q)}. \tag{2.19}$$

**Proof.** Proof of (2.18): Note that $\bar{F}$ agrees with the $(m-1)^{\text{rst}}$ degree polynomial $P$ on $\mathbb{R}^n \setminus (1+\eta)Q$. Thus,

$$\begin{aligned}
\|\bar{F}\|^p_{L^{m,p}(\mathbb{R}^n)} &= \|\bar{F}\|^p_{L^{m,p}((1+\eta)Q)} \\
&= \|\bar{F} - P\|^p_{L^{m,p}((1+\eta)Q)} \\
&= \|(F-P)\theta\|^p_{L^{m,p}((1+\eta)Q)} \\
&\lesssim \int\limits_{(1+\eta)Q} \sum_{|\alpha|+|\beta|=m} |\partial^\alpha (F-P)(x)|^p \cdot |\partial^\beta \theta(x)|^p dx \\
&\overset{(2.13)}{\lesssim} \|F\|^p_{L^{m,p}((1+\eta)Q)} + \|F-P\|^p_{L^p((1+\eta)Q)}/\delta_Q^{mp}.
\end{aligned}$$

Proof of (2.19): Note that $\bar{F} - P = 0$ on $\mathbb{R}^n \setminus (1+\eta)Q$, $\bar{F} - P = \theta(P - F)$ on $(1+\eta)Q$, and $\bar{F} = F$ on $Q \supset \operatorname{supp}(\mu)$. Thus, we can use (2.18) to bound

$$\begin{aligned}
\|\bar{F}, P\|^p_{\mathscr{J}(f,\mu;\delta_Q)} &= \|\bar{F}\|^p_{L^{m,p}((1+\eta)Q)} + \int\limits_{\mathbb{R}^n} |\bar{F} - f|^p d\mu + \|\bar{F} - P\|^p_{L^p((1+\eta)Q)}/\delta_Q^{mp} \\
&\overset{(2.18)}{\lesssim} \|F\|^p_{L^{m,p}((1+\eta)Q)} + \|F - P\|^p_{L^p((1+\eta)Q)}/\delta_Q^{mp} + \int\limits_{\mathbb{R}^n} |F - f|^p d\mu \\
&\qquad + \|\theta(F-P)\|^p_{L^p((1+\eta)Q)}/\delta_Q^{mp} \\
&\lesssim \|F\|^p_{L^{m,p}((1+\eta)Q)} + \int\limits_{\mathbb{R}^n} |F - f|^p d\mu + \|F - P\|^p_{L^p((1+\eta)Q)}/\delta_Q^{mp} \\
&= \|F, P\|^p_{\mathscr{J}_*(f,\mu;(1+\eta)Q)}. \quad \square
\end{aligned}$$

**Lemma 2.5.** *Suppose $R_1$ and $R_2$ are $\eta$-non-degenerate rectangular boxes, satisfying $R_1 \subset R_2$ and $\operatorname{diam}(R_2) \leq \eta \operatorname{diam}(R_1)$, for $\eta \geq 1$. For any $P \in \mathcal{P}$,*

$$\|P\|_{L^p(R_2)} \simeq_\eta \|P\|_{L^p(R_1)}. \tag{2.20}$$

*Here, the constants in $\simeq_\eta$ depend on $m$, $n$, $p$, and $\eta$.*

**Proof.** Let $x \in R_1$. By applying Lemma 2.2 for both $U = R_2$ and $U = R_1$, and inequality (2.3), for $P \in \mathcal{P}$,

$$\|P\|_{L^p(R_2)} \lesssim_\eta \operatorname{diam}(R_2)^m |P|_{x,\operatorname{diam}(R_2)} \lesssim_\eta \operatorname{diam}(R_1)^m |P|_{x,\operatorname{diam}(R_1)} \lesssim_\eta \|P\|_{L^p(R_1)},$$

and since $R_1 \subset R_2$, also $\|P\|_{L^p(R_1)} \le \|P\|_{L^p(R_2)}$, proving $\|P\|_{L^p(R_2)} \simeq_\eta \|P\|_{L^p(R_1)}$. □

**Lemma 2.6.** *Suppose $R_1$ and $R_2$ are $\eta$-non-degenerate rectangular boxes, satisfying $R_1 \subset R_2$, for $\eta \ge 1$. For any $H \in L^{m,p}(R_2)$ and $P \in \mathcal{P}$,*

$$\|H - P\|_{L^p(R_2)}/\operatorname{diam}(R_2)^m \lesssim_\eta \|H\|_{L^{m,p}(R_2)} + \|H - P\|_{L^p(R_1)}/\operatorname{diam}(R_1)^m. \tag{2.21}$$

*Here, the constants in $\lesssim_\eta$ depend on $m$, $n$, $p$, and $\eta$.*

**Proof.** Let $x \in R_1$. Repeatedly applying the triangle inequality, (2.3), (2.12) and Lemma 2.2, we have

$$\begin{aligned}
&\|H - P\|_{L^p(R_2)}/\operatorname{diam}(R_2)^m \\
&\quad \le \|H - J_xH\|_{L^p(R_2)}/\operatorname{diam}(R_2)^m + \|J_xH - P\|_{L^p(R_2)}/\operatorname{diam}(R_2)^m \\
&\quad \lesssim_\eta \|H\|_{L^{m,p}(R_2)} + \|J_xH - P\|_{L^p(R_2)}/\operatorname{diam}(R_2)^m \\
&\quad \simeq_\eta \|H\|_{L^{m,p}(R_2)} + |J_xH - P|_{x,\operatorname{diam}(R_2)} \\
&\quad \le \|H\|_{L^{m,p}(R_2)} + |J_xH - P|_{x,\operatorname{diam}(R_1)} \\
&\quad \simeq_\eta \|H\|_{L^{m,p}(R_2)} + \|J_xH - P\|_{L^p(R_1)}/\operatorname{diam}(R_1)^m \\
&\quad \le \|H\|_{L^{m,p}(R_2)} + \|J_xH - H\|_{L^p(R_1)}/\operatorname{diam}(R_1)^m + \|H - P\|_{L^p(R_1)}/\operatorname{diam}(R_1)^m \\
&\quad \lesssim_\eta \|H\|_{L^{m,p}(R_2)} + \|H\|_{L^{m,p}(R_1)} + \|H - P\|_{L^p(R_1)}/\operatorname{diam}(R_1)^m.
\end{aligned}$$

completing the proof of (2.21). □

The next result is an immediate consequence of Stein's Sobolev Extension Theorem for minimally smooth domains (see [24]).

**Sobolev Extension Theorem for Cubes [24].** Let $Q \subset \mathbb{R}^n$ be a cube. Let $F \in L^{m,p}(Q)$. There exists a linear operator $E : L^{m,p}(Q) \to L^{m,p}(\mathbb{R}^n)$ satisfying $EF|_Q = F|_Q$, and $\|EF\|_{L^{m,p}(\mathbb{R}^n)} \le C\|F\|_{L^{m,p}(Q)}$, where $C$ is independent of $m$, $n$, $p$, and $Q$.

**Lemma 2.7.** *Let $Q \subset \mathbb{R}^n$ be a cube and $K \subset Q$. Let $\vec{S} \in Wh(K)$ satisfy $\|\vec{S}\|_{L^{m,p}(K)} < \infty$. Then $\|\vec{S}\|_{L^{m,p}(K)} \simeq \inf\{\|F\|_{L^{m,p}(Q)} : F \in L^{m,p}(Q),\ J_x(F) = S_x \text{ for all } x \in K\}$. Hence, for all $F \in L^{m,p}(Q)$ satisfying $J_xF = S_x$ for all $x \in K$, we have*

$$\|\vec{S}\|_{L^{m,p}(K)} \lesssim \|F\|_{L^{m,p}(Q)}.$$

**Proof.** Let $\epsilon_1 > 0$; then there exists $F_1 \in L^{m,p}(\mathbb{R}^n)$ satisfying $J_x F_1 = S_x$ for all $x \in K$ and $\|F_1\|_{L^{m,p}(\mathbb{R}^n)} \leq \|\vec{S}\|_{L^{m,p}(K)} + \epsilon_1$. By restricting $F_1$ to $Q$ and letting $\epsilon_1 \to 0$, we deduce $\inf\{\|F\|_{L^{m,p}(Q)} : F \in L^{m,p}(Q),\ J_x(F) = S_x \text{ for all } x \in K\} \leq \|\vec{S}\|_{L^{m,p}(K)}$. Let $\epsilon_2 > 0$; then there exists $G \in L^{m,p}(Q)$ satisfying $J_x G = S_x$ for all $x \in K$, and $\|G\|_{L^{m,p}(Q)} \leq \inf\{\|F\|_{L^{m,p}(Q)} : F \in L^{m,p}(Q),\ J_x(F) = S_x \text{ for all } x \in K\} + \epsilon_2$. Applying Stein's extension operator to $G$, we deduce

$$\begin{aligned}
\|\vec{S}\|_{L^{m,p}(K)} &\leq \|EG\|_{L^{m,p}(\mathbb{R}^n)} \\
&\leq C\|G\|_{L^{m,p}(Q)} \\
&\leq \inf\{\|F\|_{L^{m,p}(Q)} : F \in L^{m,p}(Q),\ J_x(F) = S_x \text{ for all } x \in K\} + \epsilon_2.
\end{aligned}$$

Letting $\epsilon_2 \to 0$, we have $\|\vec{S}\|_{L^{m,p}(K)} \lesssim \inf\{\|F\|_{L^{m,p}(Q)} : F \in L^{m,p}(Q),\ J_x(F) = S_x \text{ for all } x \in K\}$, proving the lemma. □

**Lemma 2.8.** *Let $Q \subset \mathbb{R}^n$ be a cube, let $\mu$ be a Borel regular measure on $\mathbb{R}^n$ with* $\operatorname{supp}(\mu) \subset Q$*, let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mu$-measurable, and let $P \in \mathcal{P}$. Then*

$$\|f, P\|_{\mathscr{J}_*(\mu;Q)} \simeq \|f, P\|_{\mathscr{J}(\mu;\delta_Q)}; \text{ and} \tag{2.22}$$

$$\|f\|_{\mathscr{J}(\mu)} \simeq \inf_{P \in \mathcal{P}} \|f, P\|_{\mathscr{J}_*(\mu;Q)}. \tag{2.23}$$

**Proof.** Proof of (2.22): Given $F \in L^{m,p}(Q)$, we will show there exists $\bar{F} \in L^{m,p}(\mathbb{R}^n)$ such that $\bar{F}|_Q = F|_Q$, and

$$\|\bar{F}, P\|_{\mathscr{J}(f,\mu;\delta_Q)} \lesssim \|F, P\|_{\mathscr{J}_*(f,\mu;Q)}. \tag{2.24}$$

Define $\widetilde{F} := E(F)$, where $E : L^{m,p}(Q) \to L^{m,p}(\mathbb{R}^n)$ is the Sobolev extension operator for cubes, so $\widetilde{F}|_Q = F|_Q$, and

$$\|\widetilde{F}\|_{L^{m,p}(1.1Q)} \leq \|\widetilde{F}\|_{L^{m,p}(\mathbb{R}^n)} \lesssim \|F\|_{L^{m,p}(Q)}. \tag{2.25}$$

Let $\bar{F} := \theta \cdot \widetilde{F} + (1-\theta)P$, where $\theta$ is a $C^\infty$ function satisfying $\operatorname{supp}(\theta) \subset 1.1Q$, $\theta|_Q = 1$, and $|\partial^\alpha \theta(x)| \leq C\delta_Q^{-|\alpha|}$ for all $|\alpha| \leq m$. Then from Lemma 2.4, we have $\bar{F} \in L^{m,p}(\mathbb{R}^n)$ and $\bar{F} = \widetilde{F} = F$ on $Q \supsetneq \operatorname{supp}(\mu)$. Applying this, (2.19), (2.21) (with $R_1 = Q$, $R_2 = 1.1Q$), and (2.25), we have

$$\begin{aligned}
\|\bar{F}, P\|^p_{\mathscr{J}(f,\mu;\delta_Q)} &\overset{(2.19)}{\lesssim} \|\widetilde{F}, P\|^p_{\mathscr{J}_*(f,\mu;1.1Q)} \\
&= \|\widetilde{F}\|^p_{L^{m,p}(1.1Q)} + \int_{\mathbb{R}^n} |\widetilde{F} - f|^p d\mu + \|\widetilde{F} - P\|^p_{L^p(1.1Q)}/\delta_Q^{mp} \\
&\overset{(2.21)}{\lesssim} \|\widetilde{F}\|^p_{L^{m,p}(1.1Q)} + \int_{\mathbb{R}^n} |\widetilde{F} - f|^p d\mu + \|\widetilde{F} - P\|^p_{L^p(Q)}/\delta_Q^{m}
\end{aligned}$$

$$= \|\widetilde{F}\|^p_{L^{m,p}(1.1Q)} + \int\limits_{\mathbb{R}^n} |F-f|^p d\mu + \|F-P\|^p_{L^p(Q)}/\delta_Q^m$$

$$\overset{(2.25)}{\lesssim} \|F\|^p_{L^{m,p}(Q)} + \int\limits_{\mathbb{R}^n} |F-f|^p d\mu + \|F-P\|^p_{L^p(Q)}/\delta_Q^m$$

$$= \|F,P\|^p_{\mathscr{J}_*(f,\mu;Q)}.$$

This proves (2.24).

By definition of the $\mathscr{J}(\mu;\delta_Q)$ functional, as an infimum, we have that $\|\bar{F},P\|_{\mathscr{J}(f,\mu;\delta_Q)} \geq \|f,P\|_{\mathscr{J}(\mu;\delta_Q)}$ for any $\bar{F} \in L^{m,p}(\mathbb{R}^n)$. Thus, from (2.24), we have for any $F \in L^{m,p}(Q)$,

$$\|f,P\|_{\mathscr{J}(\mu;\delta_Q)} \lesssim \|F,P\|_{\mathscr{J}_*(f,\mu;Q)}.$$

Taking the infimum over functions $F \in L^{m,p}(Q)$, we deduce that

$$\|f,P\|_{\mathscr{J}(\mu;\delta_Q)} \lesssim \|f,P\|_{\mathscr{J}_*(\mu;Q)}.$$

On the other hand, using (2.8), we take the infimum over functions $F \in L^{m,p}(\mathbb{R}^n)$, implying that

$$\inf_{F\in L^{m,p}(\mathbb{R}^n)} \|F,P\|_{\mathscr{J}(f,\mu;\delta_Q)} \gtrsim \inf_{F\in L^{m,p}(\mathbb{R}^n)} \|F,P\|_{\mathscr{J}_*(f,\mu;Q)},$$

deducing $\|f,P\|_{\mathscr{J}(\mu;\delta_Q)} \gtrsim \|f,P\|_{\mathscr{J}_*(\mu;Q)}$.

Proof of (2.23): Let $x \in Q$; by (2.12), we have

$$\begin{aligned}
&\big\{ \inf_{P\in\mathcal{P}} \|f,P\|_{\mathscr{J}_*(\mu;Q)} \big\}^p \\
&\quad = \inf_{P\in\mathcal{P}} \big\{ \inf_{F\in L^{m,p}(Q)} \{\|F,P\|_{\mathscr{J}_*(f,\mu;Q)}\}^p \big\} \\
&\quad = \inf_{F\in L^{m,p}(Q)} \big\{ \inf_{P\in\mathcal{P}} \{\|F,P\|^p_{\mathscr{J}_*(f,\mu;Q)}\} \big\} \\
&\quad \leq \inf_{F\in L^{m,p}(Q)} \{\|F,J_x(F)\|^p_{\mathscr{J}_*(f,\mu;Q)}\} \\
&\quad = \inf_{F\in L^{m,p}(Q)} \big\{ \|F\|^p_{L^{m,p}(Q)} + \int\limits_Q |F-f|^p d\mu + \|F-J_x(F)\|^p_{L^p(Q)}/\delta_Q^{mp} \big\} \\
&\quad \lesssim \inf_{F\in L^{m,p}(Q)} \big\{ \|F\|^p_{L^{m,p}(Q)} + \int\limits_Q |F-f|^p d\mu \big\} \\
&\quad \lesssim \inf_{F\in L^{m,p}(\mathbb{R}^n)} \big\{ \|F\|^p_{L^{m,p}(Q)} + \int\limits_Q |F-f|^p d\mu \big\} \\
&\quad \lesssim \|f\|^p_{\mathscr{J}(\mu)}.
\end{aligned}$$

Let $F \in L^{m,p}(Q)$ and $P \in \mathcal{P}$. Define $\bar{F} = EF$, where $E : L^{m,p}(Q) \to L^{m,p}(\mathbb{R}^n)$ is the Sobolev extension for cubes, so $\bar{F}|_Q = F|_Q$ and $\|\bar{F}\|_{L^{m,p}(\mathbb{R}^n)} \lesssim \|F\|_{L^{m,p}(Q)}$. Because $\operatorname{supp}(\mu) \subsetneq Q$, we have

$$\begin{aligned}\|\bar{F}\|^p_{\mathscr{J}(f,\mu)} &= \|\bar{F}\|^p_{L^{m,p}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |\bar{F} - f|^p d\mu \\ &\lesssim \|F\|^p_{L^{m,p}(Q)} + \int_Q |F - f|^p d\mu \\ &\lesssim \|F, P\|^p_{\mathscr{J}(f,\mu;Q)}.\end{aligned}$$

Taking the infimum over $F \in L^{m,p}(Q)$ and $P \in \mathcal{P}$, we have $\|f\|_{\mathscr{J}(\mu)} \lesssim \inf_{P\in\mathcal{P}} \|f, P\|_{\mathscr{J}_*(\mu;Q)}$.

This completes the proof of (2.23). $\square$

**Corollary 2.9.** *Let $R$ be a $\eta$-non-degenerate rectangular box, let $\mu$ be a Borel measure with* $\operatorname{supp}(\mu) \subset R$*, let $f : \mathbb{R}^n \to \mathbb{R}$, and let $P \in \mathcal{P}$. Then*

$$\|f, P\|_{\mathscr{J}_*(\mu;R)} \simeq_\eta \|f, P\|_{\mathscr{J}(\mu;\operatorname{diam}(R))}. \tag{2.26}$$

**Lemma 2.10.** *If* $\operatorname{supp}(\mu)$ *is compact,*

$$\mathscr{J}(\mu; \delta) = \mathscr{J}(\mu) \times \mathcal{P}.$$

**Proof.** Suppose $\operatorname{supp}(\mu)$ is compact. Then there exists $Q \subset \mathbb{R}^n$ such that $\operatorname{supp}(\mu) \subset Q$. By definition, if $(f, P) \in \mathscr{J}(\mu; \delta)$ then $(f, P) \in \mathscr{J}(\mu) \times \mathcal{P}$. Suppose $(f, P) \in \mathscr{J}(\mu) \times \mathcal{P}$; we will show there exists $\bar{F} \in L^{m,p}(\mathbb{R}^n)$ such that $\|\bar{F}, P\|_{\mathscr{J}(f,\mu;\delta)} < \infty$, implying $(f, P) \in \mathscr{J}(\mu; \delta)$. Because $f \in \mathscr{J}(\mu)$, there exists $F \in L^{m,p}(\mathbb{R}^n)$ such that $\|F\|_{\mathscr{J}(f,\mu)} < \infty$. Let $\theta$ be a $C^\infty$ function satisfying $\operatorname{supp}(\theta) \subset (1.1)Q$, $\theta|_Q = 1$, and $|\partial^\alpha\theta(x)| \le C\delta_Q^{-|\alpha|}$ for all $|\alpha| \le m$. Define $\bar{F} := \theta \cdot F + (1 - \theta)P$. Then from (2.18), $\bar{F} \in L^{m,p}(\mathbb{R}^n)$, $\bar{F}|_Q = F|_Q$,

$$\|\bar{F}, P\|_{\mathscr{J}(f,\mu;\delta_Q)} \lesssim \|F, P\|_{\mathscr{J}_*(f,\mu;1.1Q)}.$$

Consequently, for $x \in Q$, we apply (2.12) to deduce

$$\begin{aligned}&\|\bar{F}, P\|^p_{\mathscr{J}(f,\mu;\delta)} \\ &\quad\le (1 + \delta_Q^{mp}/\delta^{mp})\|\bar{F}, P\|^p_{\mathscr{J}(f,\mu;\delta_Q)} \\ &\quad\lesssim (1 + \delta_Q^{mp}/\delta^{mp})\|F, P\|^p_{\mathscr{J}_*(f,\mu;1.1Q)}. \\ &\quad= (1 + \delta_Q^{mp}/\delta^{mp})\Big(\|F\|^p_{L^{m,p}(1.1Q)} + \int_{1.1Q} |F - f|^p d\mu + \|F - P\|^p_{L^p(1.1Q)}/\delta_Q^{mp}\Big) \\ &\quad\lesssim (1 + \delta_Q^{mp}/\delta^{mp})\Big(\|F\|^p_{L^{m,p}(1.1Q)} + \int_{1.1Q} |F - f|^p d\mu + \|F - J_x(F)\|^p_{L^p(1.1Q)}/\delta_Q^{mp}\end{aligned}$$

$$
\begin{aligned}
&\quad + \|J_x(F) - P\|^p_{L^p(1.1Q)}/\delta_Q^{mp}\Big) \\
&\lesssim (1 + \delta_Q^{mp}/\delta^{mp})\Big(\|F\|^p_{L^{m,p}(1.1Q)} + \int\limits_{1.1Q} |F - f|^p d\mu + \|J_x(F) - P\|^p_{L^p(1.1Q)}/\delta_Q^{mp}\Big) \\
&\lesssim (1 + \delta_Q^{mp}/\delta^{mp})\Big(\|F\|^p_{\mathscr{J}(f,\mu)} + \|J_x(F) - P\|^p_{L^p(1.1Q)}/\delta_Q^{mp}\Big) < \infty. \quad \square
\end{aligned}
$$

*2.2.1. Characterization of Sobolev space by local polynomial approximation*

**Definition 2.1** *(Packing).* Let $Q_0 \subset \mathbb{R}^n$ be a cube. A collection of cubes $\pi$ is a packing of $Q_0$ provided the following conditions hold:

1. $Q \subset Q_0$ for all $Q \in \pi$.
2. $\mathrm{int}(Q) \cap \mathrm{int}(Q') = \emptyset$ for all distinct $Q, Q' \in \pi$.

We write $\Pi(Q_0)$ to denote the collection of all packings of $Q_0$.

**Definition 2.2** *(Congruent $\delta$-packing).* Given a cube $Q \subset \mathbb{R}^n$, we say $\pi$ is a congruent $\delta$-packing of $Q$ if it is a finite set of disjoint cubes of equal sidelength, $\delta$, contained in $Q$. Let

$$
\Pi_{\simeq}(Q) := \{\pi : \pi \text{ is a congruent } \delta\text{-packing of } Q,\ \delta \le \delta_Q\},
$$

**Definition 2.3** *(Polynomial Approximation Error).* Given $F \in L^p_{loc}(\mathbb{R}^n)$, and a measurable set, $S \subset \mathbb{R}^n$, we define the local approximation error of $F$ for $S$ as

$$
E(F, S) := \inf_{P \in \mathcal{P}} \|F - P\|_{L^p(S)}.
$$

In [2], A. Brudnyi characterizes Sobolev Space with the following result (Theorem 4 of Section 4):

**Proposition 2.11** *(Brudnyi, A. Yu.). Let $F \in L^p(Q)$. Suppose there exists $\lambda > 0$ such that*

$$
\sup_{\pi \in \Pi_{\simeq}(Q)} \delta_Q^{-m} \Big\{ \sum_{\widehat{Q} \in \pi} \big(E(F, \bar{Q})\big)^p \Big\}^{1/p} \le \lambda.
$$

*Then $F \in L^{m,p}(Q)$, and $\|F\|_{L^{m,p}(Q)} \lesssim \lambda$.*

If $\pi$ is a congruent $\delta$-packing of $Q$ then $\delta_{\bar{Q}} \le \delta_Q$ for all $\bar{Q} \in \pi$. Therefore, we obtain the following corollary of Brudnyi's result:

**Corollary 2.12.** *Let $F \in L^p(Q)$. Suppose there exists $\lambda > 0$ such that*

$$\sup_{\pi \in \Pi_{\simeq}(Q)} \left\{ \sum_{\bar{Q} \in \pi} \left( E(F, \bar{Q}) / \delta_{\bar{Q}}^m \right)^p \right\}^{1/p} \leq \lambda.$$

*Then $F \in L^{m,p}(Q)$, and $\|F\|_{L^{m,p}(Q)} \lesssim \lambda$.*

*2.2.2. Consequence of the classical Whitney extension theorem*

**Lemma 2.13.** *Let $F : Q \to \mathbb{R}$, and suppose there exists a Whitney field $\vec{P} \in Wh(Q)$ and a constant $A \geq 0$ satisfying $P_x(x) = F(x)$ for all $x \in Q$ and $|P_x - P_y|_{x,|x-y|} \leq A$ for all $x, y \in Q$.*

*Then $F \in C^{m-1,1-n/p}(Q)$, $J_x F = P_x$ for all $x \in Q$, and*

$$\|F\|_{C^{m-1,1-n/p}(Q)} \leq CA,$$

*for a constant $C$ determined by $m, n, p$.*

**Proof.** Observe that the condition $|P_x - P_y|_{x,|x-y|} \leq A$ $(x, y \in Q)$ implies that the Whitney field $\vec{P}$ satisfies the hypothesis of the classical Whitney extension theorem (see [19], [24]) for $C^{m-1,\alpha}$, $\alpha = 1 - n/p$. Thus, there exists a function $G : \mathbb{R}^n \to \mathbb{R}$ such that $\|G\|_{C^{m-1,\alpha}(\mathbb{R}^n)} \leq CA$, $J_x G = P_x$ for all $x \in Q$. In particular, $G(x) = J_x G(x) = P_x(x) = F(x)$ for all $x \in Q$. Hence, $F = G|_Q \in C^{m-1,\alpha}(Q)$ and $\|F\|_{C^{m-1,\alpha}(Q)} \leq \|G\|_{C^{m-1,\alpha}(\mathbb{R}^n)} \leq CA$. Meanwhile, because $F = G$ on $Q$, we have $J_x F = J_x G = P_x$ for all $x \in Q$. $\square$

## 3. Structure of the proof

Here we will take first steps toward the proof of Theorems 1 and 3. We will state the Extension Theorem for $(\mu, \delta)$, whose proof will occupy much of the remainder of this paper. In Section 11. we will show that Theorems 1 and 3 follow from this result.

*3.1. Plan for the proof*

Let $\mu$ be a Borel measure on $\mathbb{R}^n$ with compact support and let $\delta > 0$. We will prove the following theorem:

**Proposition 3.1** *(Extension Theorem for $(\mu, \delta)$). Suppose* $\mathrm{diam}(\mathrm{supp}(\mu)) < \delta$. *There exist a linear map $T : \mathscr{J}(\mu; \delta) \to L^{m,p}(\mathbb{R}^n)$, a map $M : \mathscr{J}(\mu; \delta) \to \mathbb{R}_+$, $K \subset \mathrm{Cl}(\mathrm{supp}(\mu))$, a linear map $\vec{S} : \mathscr{J}(\mu) \to Wh(K)$, and countable collections of Borel sets $\{A_\ell\}_{\ell \in \mathbb{N}} \subset$* $\mathrm{supp}(\mu)$*, and of linear maps $\{\phi_\ell\}_{\ell \in \mathbb{N}}$, $\phi_\ell : \mathscr{J}(\mu; \delta) \to \mathbb{R}$, and $\{\lambda_\ell\}_{\ell \in \mathbb{N}}$, $\lambda_\ell : \mathscr{J}(\mu; \delta) \to L^p(d\mu)$, that satisfy for each $(f, P_0) \in \mathscr{J}(\mu; \delta)$,*

***(i)*** $\|f, P_0\|_{\mathscr{J}(\mu;\delta)} \leq \|T(f, P_0), P_0\|_{\mathscr{J}(f,\mu;\delta)} \leq C \cdot \|f, P_0\|_{\mathscr{J}(\mu;\delta)};$ (3.1)

***(ii)*** $c \cdot M(f, P_0) \leq \|T(f, P_0), P_0\|_{\mathscr{J}(f,\mu;\delta)} \leq C \cdot M(f, P_0);$ *and* (3.2)

***(iii)*** $M(f, P_0) = \Big( \sum_{\ell \in \mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, P_0) - f|^p d\mu + \sum_{\ell \in \mathbb{N}} |\phi_\ell(f, P_0)|^p + \|\vec{S}(f)\|^p_{L^{m,p}(K)} \Big)^{1/p}.$ (3.3)

*The map $T$ is $\Omega'$-constructible in the following sense:*

*There exists a collection of linear functionals $\Omega' = \{\omega_s\}_{s \in \Upsilon} \subset \mathscr{J}(\mu)^*$ such that collection of sets $\{\mathrm{supp}(\omega_s)\}_{s \in \Upsilon}$ has $C$-bounded overlap, with $\mathrm{supp}(\omega_s) \subset \mathrm{supp}(\mu)$ for each $s \in \Upsilon$. Further, for each $y \in \mathbb{R}^n$, there exists a finite subset $\Upsilon_y \subset \Upsilon$ and a collection of polynomials $\{v_{s,y}\}_{s \in \Upsilon_y} \subset \mathcal{P}$ such that $|\Upsilon_y| \leq C$ and*

$$J_y T(f, P_0) = \sum_{s \in \Upsilon_y} \omega_s(f) \cdot v_{s,y} + \widetilde{\omega}_y(P_0), \tag{3.4}$$

*where $\widetilde{\omega}_y : \mathcal{P} \to \mathcal{P}$ is a linear map.*

*Further, the map $M$ is $\Omega'$-constructible:*

***(1)*** *For each $\ell \in \mathbb{N}$ and $y \in \mathrm{supp}(\mu)$, there exists a finite subset $\bar{\Upsilon}_{\ell,y} \subset \Upsilon$ and constants $\{\eta^\ell_{s,y}\}_{s \in \bar{\Upsilon}_{\ell,y}} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_{\ell,y}| \leq C$, and the map $(f, P_0) \mapsto \lambda_\ell(f, P_0)(y)$ has the form*

$$\lambda_\ell(f, P_0)(y) = \sum_{s \in \bar{\Upsilon}_{\ell,y}} \eta^\ell_{s,y} \cdot \omega_s(f) + \widetilde{\lambda}_{y,\ell}(P_0),$$

*where $\widetilde{\lambda}_{y,\ell} : \mathcal{P} \to \mathbb{R}$ is a linear functional.*

***(2)*** *For each $\ell \in \mathbb{N}$, there exists a finite subset $\bar{\Upsilon}_\ell \subset \Upsilon$ and constants $\{\eta^\ell_s\}_{s \in \bar{\Upsilon}_\ell} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_\ell| \leq C$, and the map $\phi_\ell$ has the form*

$$\phi_\ell(f, P_0) = \sum_{s \in \bar{\Upsilon}_\ell} \eta^\ell_s \cdot \omega_s(f) + \widetilde{\lambda}_\ell(P_0),$$

*where $\widetilde{\lambda}_\ell : \mathcal{P} \to \mathbb{R}$ is a linear functional.*

***(3)*** *For $y \in K$, there exist $\{\omega^\alpha_y\}_{\alpha \in \mathcal{M}} \subset \Omega'$ satisfying for all $\alpha \in \mathcal{M}$, $\mathrm{supp}(\omega^\alpha_y) \subset \{y\}$, and the map $f \mapsto S_y(f)$ has the form*

$$S_y(f) = \sum_{\alpha \in \mathcal{M}} \omega^\alpha_y(f) \cdot v_\alpha,$$

*where $\{v_\alpha\}_{\alpha \in \mathcal{M}}$ is a basis for $\mathcal{P}$.*

### *3.1.1. Order relation on labels*

To prove Proposition 3.1, we study the shape of symmetric, convex subsets of $\mathcal{P}$ that vary as we restrict the domain of the measure $\mu$. The shape of $\sigma \subset \mathcal{P}$ will be defined by a multi-index set $\mathcal{A} \subset \mathcal{M}$. We will sometimes refer to multi-index sets as *labels*.

Given distinct elements $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathcal{M}$, let $k \in \{0, \dots, n\}$ be maximal subject to the condition $\sum_{i=1}^k \alpha_i \neq \sum_{i=1}^k \beta_i$. We write $\alpha < \beta$ if

$$\sum_{i=1}^{k} \alpha_i < \sum_{i=1}^{k} \beta_i.$$

Given distinct multi-index sets $\mathcal{A}, \overline{\mathcal{A}} \subset \mathcal{M}$, we write $\mathcal{A} < \overline{\mathcal{A}}$ if the minimal element of the symmetric difference $\mathcal{A} \Delta \overline{\mathcal{A}}$ (under the order $<$ on elements) lies in $\mathcal{A}$. The following properties hold:

- If $\alpha, \beta \in \mathcal{M}$ and $|\alpha| < |\beta|$ then $\alpha < \beta$
- If $\mathcal{A} \subsetneq \overline{\mathcal{A}} \subset \mathcal{M}$ then $\overline{\mathcal{A}} < \mathcal{A}$. In particular, the empty set is maximal and the whole set $\mathcal{M}$ is minimal under the order on multi-index sets.

*3.1.2. Polynomial bases*

A subset $\sigma$ of a vector space $V$ is symmetric provided that $v \in \sigma \implies -v \in \sigma$.

**Definition 3.1.** Given a symmetric, convex set $\sigma \subset \mathcal{P}$, $\mathcal{A} \subset \mathcal{M}$, $\epsilon > 0$, $x \in \mathbb{R}^n$, and $\delta > 0$, we say $(P^\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{P}$ *forms an* $(\mathcal{A}, x, \epsilon, \delta)$*-basis for* $\sigma$ if the following are satisfied:

**(i)** $P^\alpha \in \epsilon \delta^{n/p+|\alpha|-m} \sigma$ for $\alpha \in \mathcal{A}$;

**(ii)** $\partial^\beta P^\alpha(x) = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{A}$; and

**(iii)** $\left|\partial^\beta P^\alpha(x)\right| \leq \epsilon \delta^{|\alpha|-|\beta|}$ for $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$ s.t. $\beta > \alpha$.

Evidently, the basis property for symmetric convex sets is monotone in $\sigma, \epsilon, \delta$ in the following sense:

$$\begin{aligned}&\text{Suppose that } (P^\alpha)_{\alpha \in \mathcal{A}} \text{ forms an } (\mathcal{A}, x, \epsilon, \delta)\text{-basis for } \sigma. \text{ Then } (P^\alpha)_{\alpha \in \mathcal{A}} \\ &\text{forms an } (\mathcal{A}, x, \epsilon', \delta')\text{-basis for } \sigma' \text{ for } \epsilon' \geq \epsilon,\ \sigma' \supset \sigma, \text{ and } 0 < \delta' \leq \delta.\end{aligned} \tag{3.5}$$

**Lemma 3.2.** *Suppose $(P^\alpha)_{\alpha \in \mathcal{A}}$ forms an $(\mathcal{A}, x, \epsilon, \delta)$-basis for a convex set $\sigma \subset \mathcal{P}$. Then for $k > 1$, $(P^\alpha)_{\alpha \in \mathcal{A}}$ forms an $(\mathcal{A}, x, k^m \cdot \epsilon, k \cdot \delta)$-basis for $\sigma$.*

**Proof.** We verify Property **(i)** from Definition 3.1 first: For $\alpha \in \mathcal{M}$,

$$\epsilon \delta^{n/p+|\alpha|-m} = (k^m \cdot \epsilon) \delta^{n/p+|\alpha|} (k \cdot \delta)^{-m} \leq (k^m \cdot \epsilon)(k \cdot \delta)^{n/p+|\alpha|-m},$$

implying if $P^\alpha \in \epsilon \delta^{n/p+|\alpha|-m} \sigma$ then $P^\alpha \in (k^m \cdot \epsilon)(\delta \cdot k)^{n/p+|\alpha|-m} \sigma$. Property **(ii)** is independent of $\epsilon$ and $\delta$. Fix $\alpha, \beta \in \mathcal{M}$ such that $\beta > \alpha$. Then we see Property **(iii)** holds:

$$|\partial^\beta P^\alpha(x)| \leq \epsilon \delta^{|\alpha|-|\beta|} = (k^m \cdot \epsilon)(k \cdot \delta)^{|\alpha|-|\beta|} k^{|\beta|-|\alpha|-m} \leq (k^m \cdot \epsilon)(k \cdot \delta)^{|\alpha|-|\beta|},$$

so $(P^\alpha)_{\alpha \in \mathcal{A}}$ forms an $(\mathcal{A}, x, k^m \cdot \epsilon, k \cdot \delta)$-basis for $\sigma$. $\square$

Given $x \in \mathbb{R}^n$, $\mu$, and $\delta > 0$, define symmetric convex subsets of $\mathcal{P}$:

$$\sigma_J(x,\mu) = \left\{P \in \mathcal{P} : \exists F \in L^{m,p}(\mathbb{R}^n) \text{ s.t. } J_x(F) = P,\ \|F\|_{\mathscr{J}(0,\mu)} \leq 1\right\}.$$
$$\sigma(\mu,\delta) = \left\{P \in \mathcal{P} : \exists F \in L^{m,p}(\mathbb{R}^n) \text{ s.t. } \|F,P\|_{\mathscr{J}(0,\mu;\delta)} \leq 1\right\}.$$

**Lemma 3.3.** *Suppose* $\mathrm{supp}(\mu) \subset Q$. *Then there exists* $C_0 > 0$ *such that for all* $x \in Q$, $\sigma_J(x,\mu) \subset C_0 \cdot \sigma(\mu,\delta_Q)$. *In particular, if* $(P^\alpha)_{\alpha\in\mathcal{A}}$ *forms an* $(\mathcal{A}, x, \epsilon/C_0, \delta_Q)$*-basis for* $\sigma_J(x,\mu)$*, then* $(P^\alpha)_{\alpha\in\mathcal{A}}$ *forms an* $(\mathcal{A}, x, \epsilon, \delta_Q)$*-basis for* $\sigma(\mu,\delta_Q)$.

**Proof.** Let $P \in \sigma_J(x,\mu)$. Then there exists $F \in L^{m,p}(\mathbb{R}^n)$ such that $J_x(F) = P$ and

$$\|F\|^p_{\mathscr{J}(0,\mu)} = \|F\|^p_{L^{m,p}(\mathbb{R}^n)} + \int_Q |F|^p d\mu \leq 1.$$

By (2.19) there exists $F' \in L^{m,p}(\mathbb{R}^n)$ satisfying $F'|_Q = F|_Q$ and

$$\|F',P\|_{\mathscr{J}(0,\mu;\delta_Q)} \leq C\|F,P\|_{\mathscr{J}_*(0,\mu;1.1Q)}.$$

By (2.12), we have $\|F-P\|_{L^p(1.1Q)}/\delta_Q^m \leq C\|F\|_{L^{m,p}(1.1Q)}$, and thus

$$\|F,P\|_{\mathscr{J}_*(0,\mu;1.1Q)} \leq C'\|F\|_{\mathscr{J}(0,\mu)} \leq C'.$$

Combining the above inequalities, we have $\|F',P\|_{\mathscr{J}(0,\mu;\delta_Q)} \leq C_0$. Therefore, $P \in C_0 \cdot \sigma(\mu,\delta_Q)$. This proves $\sigma_J(x,\mu) \subset C_0 \cdot \sigma(\mu,\delta_Q)$.

The basis result follows, for if $P^\alpha \in \frac{\epsilon}{C_0}\delta_Q^{m-n/p}\sigma_J(x,\mu)$ ($\alpha \in \mathcal{A}$) then $P^\alpha \in \epsilon\delta_Q^{m-n/p}\sigma(\mu,\delta_Q)$. $\square$

### *3.2. The induction*

#### *3.2.1. The Main Lemma*

Fix a collection of multi-indices $\mathcal{A} \subset \mathcal{M}$. Let $C_0 > 0$ be the universal constant defined in Lemma 3.3. We prove the following by induction with respect to the multi-index set $\mathcal{A}$.

**Lemma 3.4** *(Main Lemma for* $\mathcal{A}$*). Fix* $\mathcal{A} \subset \mathcal{M}$. *There exists a constant* $\epsilon = \epsilon(\mathcal{A}) > 0$, *depending on* $\mathcal{A}, m, n$, *and* $p$, *such that the following holds: Suppose* $\mu$ *is a Borel regular measure on* $\mathbb{R}^n$ *with compact support, satisfying* $\mathrm{diam}(\mathrm{supp}(\mu)) < \delta$, *and suppose:*

$$\textit{For all } x \in \mathrm{supp}(\mu),\ \sigma_J(x,\mu) \textit{ has an } (\mathcal{A}, x, \epsilon/C_0, 10\delta)\textit{-basis.} \tag{3.6}$$

*Then the Extension Theorem for* $(\mu,\delta)$ *(Proposition 3.1) is true.*

*Furthermore, for* $\mathcal{A} \neq \emptyset$, *in the Extension Theorem for* $(\mu,\delta)$, *one can take* $K = \emptyset$, *and so the functional* $M : \mathscr{J}(\mu;\delta) \to \mathbb{R}_+$ *has the form*

$$M(f,P_0)=\Big(\sum_{\ell\in\mathbb{N}}\int_{A_\ell}|\lambda_\ell(f,P_0)-f|^p d\mu+\sum_{\ell\in\mathbb{N}}|\phi_\ell(f,P_0)|^p\Big)^{1/p}. \tag{3.7}$$

Note that condition (3.6) in the Main Lemma for $\mathcal{A}=\emptyset$ holds vacuously. Thus the Main Lemma for $\mathcal{A}=\emptyset$ implies the Extension Theorem for $(\mu,\delta)$. Thus we have reduced the proof of the Extension Theorem for $(\mu,\delta)$ to the task of proving the Main Lemma for $\mathcal{A}$, for each $\mathcal{A}\subset\mathcal{M}$. We proceed by induction and establish the following:
Base Case: The Main Lemma for $\mathcal{M}$ holds.
Induction Step: Let $\mathcal{A}\subset\mathcal{M}$ with $\mathcal{A}\neq\mathcal{M}$. Suppose that the Main Lemma for $\mathcal{A}'$ holds for each $\mathcal{A}'<\mathcal{A}$. Then the Main Lemma for $\mathcal{A}$ holds.

*3.2.2. Proof of the base case*

**Proof.** Fix $(\mu,\delta)$ as in the Main Lemma for $\mathcal{M}$, satisfying (3.6) for $\mathcal{A}=\mathcal{M}$. Given that $\operatorname{diam}(\operatorname{supp}(\mu))<\delta$, we can fix a cube $Q\subset\mathbb{R}^n$ with $\delta_Q=\delta$ and $\operatorname{supp}(\mu)\subset Q$. By (3.6), $\sigma_J(x,\mu)$ has an $(\mathcal{M},x,\epsilon/C_0,10\delta_Q)$-basis for all $x\in\operatorname{supp}(\mu)$. We will fix the choice of $\epsilon(\mathcal{M})$ momentarily. By Lemma 3.3, $\sigma(\mu,10\delta_Q)$ has an $(\mathcal{M},x,\epsilon,10\delta_Q)$-basis for all $x\in\operatorname{supp}(\mu)$. Specifically, $1\in\epsilon(10\delta_Q)^{n/p-m}\cdot\sigma(\mu,10\delta_Q)$, implying there exists $G\in L^{m,p}(\mathbb{R}^n)$ satisfying

$$\Big(\int_Q|G|^p d\mu\Big)^{1/p}\le\|G,1\|_{\mathscr{J}(0,\mu;10\delta_Q)}\le\epsilon(10\delta_Q)^{n/p-m}. \tag{3.8}$$

By (2.15), for $x\in Q$,

$$\begin{aligned}|G(x)-1|&\le C\|G,1\|_{\mathscr{J}(0,\mu;\delta_Q)}\delta_Q^{m-n/p}\\&\lesssim\|G,1\|_{\mathscr{J}(0,\mu;10\delta_Q)}\delta_Q^{m-n/p}\\&\lesssim\epsilon.\end{aligned}$$

We now fix $\epsilon$ small enough so that the previous inequality implies $|G(x)-1|<1/2$ for $x\in Q$. Then $|G(x)|>1/2$ for $x\in Q$, and $\Big(\int_Q|G|^p d\mu\Big)^{1/p}\ge C'\mu(Q)^{1/p}$, and by (3.8)

$$\mu(Q)^{1/p}\le C\epsilon\delta_Q^{n/p-m}. \tag{3.9}$$

Let $(f,P_0)\in\mathscr{J}(\mu;\delta_Q)$. Define $T(f,P_0):=P_0$. Then for any $F\in L^{m,p}(\mathbb{R}^n)$, we use the assumption $\operatorname{supp}(\mu)\subset Q$ to deduce that

$$\|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;\delta_Q)}=\left(\int_Q|P_0-f|^p d\mu\right)^{1/p}$$

$$\leq \left(\int_Q |P_0 - F|^p d\mu\right)^{1/p} + \left(\int_Q |F - f|^p d\mu\right)^{1/p}$$

$$\overset{(2.15)}{\lesssim} \left(\int_Q \|F, P_0\|^p_{\mathscr{J}(f,\mu;\delta_Q)} \delta_Q^{mp-n} d\mu\right)^{1/p} + \left(\int_Q |F - f|^p d\mu\right)^{1/p}$$

$$\leq \|F, P_0\|_{\mathscr{J}(f,\mu;\delta_Q)} \delta_Q^{m-n/p} \mu(Q)^{1/p} + \left(\int_Q |F - f|^p d\mu\right)^{1/p}$$

$$\overset{(3.9)}{\leq} (1 + C\epsilon)\|F, P_0\|_{\mathscr{J}(f,\mu;\delta_Q)}.$$

Because $F$ is arbitrary, we can conclude $\|T(f, P_0), P_0\|_{\mathscr{J}(f,\mu;\delta_Q)} \leq C\|f, P_0\|_{\mathscr{J}(\mu;\delta_Q)}$. When $\mathcal{A} = \mathcal{M}$, we take $K = \emptyset$, as promised in Lemma 3.4, so the functional $M$ has the form (3.7). Define the family $\{\phi_\ell\}_{\ell\in\mathbb{N}}$ by $\phi_\ell(f, P_0) = 0$ for $\ell \geq 1$.

Define the family $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$, by $\lambda_\ell(f, P_0) = 0$ for $\ell \geq 2$, and $\lambda_1(f, P_0) = P_0$, and define the sets $A_\ell = \emptyset$ for $\ell \geq 2$, and $A_1 = \mathrm{supp}(\mu)$. For these families of linear maps, the functional $M$ in (3.7) is given by $M(f, P_0) = \|P_0 - f\|_{L^p(d\mu)}$. Note the above chain of inequalities implies that $P_0 - f \in L^p(d\mu)$, and

$$M(f, P_0) = \left(\int_{\mathbb{R}^n} |P_0 - f|^p d\mu\right)^{1/p} = \|T(f, P_0), P_0\|_{\mathscr{J}(f,\mu;\delta_Q)}.$$

Let $\Omega' = \emptyset$; immediately, the collection $\{\mathrm{supp}(\omega)\}_{\omega\in\Omega'}$ has bounded (empty) overlap. For $y \in \mathrm{supp}(\mu)$, define $\widetilde{\omega}_y(P) := P(y)$; then $\lambda_1(f, P_0)(y) = \widetilde{\omega}_y(P_0)$, indicating $M$ is $\Omega'$-constructible. And $J_y T(f, P_0) = \widetilde{\omega}_y(P_0) = P_0$, so $T$ is $\Omega'$-constructible. $\square$

*3.2.3. Technical lemmas*

The following linear algebra lemmas are adapted from Sections 3 and 4 of [13], relying on (2.10) and the inequality $\|F\|_{L^{m,p}(\mathbb{R}^n)} \leq \|F, P\|_{\mathscr{J}(0,\mu)}$.

**Lemma 3.5.** *There exist universal constants $c_1 \in (0, 1)$ and $C_1 \geq 1$ so that the following holds.*
*Suppose we are given the following:*

- **(D1)** *Real numbers $\epsilon_1 \in (0, c_1]$ and $\epsilon_2 \in (0, \epsilon_1^{2D+2}]$.*
- **(D2)** *A lengthscale $\delta > 0$.*
- **(D3)** *A collection of multi-indices $\mathcal{A} \subset \mathcal{M}$.*
- **(D4)** *A Borel regular measure $\mu$ and a bounded, non-empty set $E \subset \mathbb{R}^n$, satisfying $\mathrm{supp}(\mu) \subset E$ and $\mathrm{diam}(E) \leq 10\delta$.*

**(D5)** *A family of polynomials* $(\widetilde{P}_x^\alpha)_{\alpha\in\mathcal{A}}$ *that forms an* $(\mathcal{A}, x, \epsilon_2, \delta)$*-basis for* $\sigma_J(x,\mu)$ *for each* $x \in E$.

**(D6)** *A point* $x_0 \in E$, *a multi-index* $\bar\beta \in \mathcal{M} \setminus \mathcal{A}$, *and* $\widetilde{P}_{x_0}^{\bar\beta} \in \mathcal{P}$ *satisfying*

$$\widetilde{P}_{x_0}^{\bar\beta} \in \epsilon_2 \delta^{|\bar\beta|+n/p-m} \sigma_J(x_0,\mu); \tag{3.10}$$

$$\partial^{\bar\beta} \widetilde{P}_{x_0}^{\bar\beta}(x_0) = 1; \tag{3.11}$$

$$|\partial^\beta \widetilde{P}_{x_0}^{\bar\beta}(x_0)| \le \epsilon_1 \delta^{|\bar\beta|-|\beta|} \qquad (\beta \in \mathcal{M},\ \beta > \bar\beta);\ \textit{and} \tag{3.12}$$

$$|\partial^\beta \widetilde{P}_{x_0}^{\bar\beta}(x_0)| \le \epsilon_1^{-D} \delta^{|\bar\beta|-|\beta|} \qquad (\beta \in \mathcal{M}). \tag{3.13}$$

**(D7)** *For all* $x \in E$, *the basis* $(\widetilde{P}_x^\alpha)_{\alpha\in\mathcal{A}}$ *in* ***(D5)*** *satisfies*

$$|\partial^\beta \widetilde{P}_x^\alpha(x)| \le \epsilon_1^{-D-1} \delta^{|\alpha|-|\beta|} \qquad (\alpha \in \mathcal{A},\ \alpha < \bar\beta,\ \beta \in \mathcal{M}). \tag{3.14}$$

*Then there exists* $\overline{\mathcal{A}} < \mathcal{A}$ *so that for every* $x \in E$, $\sigma_J(x,\mu)$ *contains an* $(\overline{\mathcal{A}}, x, C_1\epsilon_1, \delta)$*-basis.*

**Proof.** Let $c_1$ be a sufficiently small constant, to be determined later. By rescaling it suffices to assume that $\delta = 1$. Our hypothesis tells us that $\epsilon_1 \le c_1$, $\epsilon_2 \le \epsilon_1^{2D+2}$, and that $(\widetilde{P}_x^\alpha)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, x, \epsilon_2, 1)$-basis for $\sigma_J(x,\mu)$, for each $x \in E$. That is,

$$\widetilde{P}_x^\alpha \in \epsilon_2 \cdot \sigma_J(x,\mu); \tag{3.15}$$

$$\partial^\beta \widetilde{P}_x^\alpha(x) = \delta_{\alpha\beta} \qquad (\alpha,\beta \in \mathcal{A});\ \text{and} \tag{3.16}$$

$$|\partial^\beta \widetilde{P}_x^\alpha(x)| \le \epsilon_2 \qquad (\alpha \in \mathcal{A}, \beta \in \mathcal{M},\ \beta > \alpha). \tag{3.17}$$

By (3.10) (for $\delta = 1$), there exists $\varphi^{\bar\beta} \in L^{m,p}(\mathbb{R}^n)$ satisfying

$$J_{x_0}\varphi^{\bar\beta} = \widetilde{P}_{x_0}^{\bar\beta};\ \text{and} \tag{3.18}$$

$$\|\varphi^{\bar\beta}\|_{L^{m,p}(\mathbb{R}^n)} \le \|\varphi^{\bar\beta}\|_{\mathscr{J}(0,\mu)} \le \epsilon_2. \tag{3.19}$$

Fix $y \in E$ and define $\widetilde{P}_y^{\bar\beta} := J_y\varphi^{\bar\beta}$. Then the definition of $\sigma_J(\cdot,\cdot)$ and (3.19) imply

$$\widetilde{P}_y^{\bar\beta} \in \epsilon_2 \cdot \sigma_J(y,\mu). \tag{3.20}$$

We have $|x_0 - y| \le \operatorname{diam}(E) \le 10$, so by the Sobolev Inequality, (3.18), (3.19), (3.11), and (3.12),

$$\begin{aligned} |\partial^\beta \widetilde{P}_y^{\bar\beta}(y) - \delta_{\beta\bar\beta}| &\le |\partial^\beta \varphi^{\bar\beta}(y) - \partial^\beta J_{x_0}\varphi^{\bar\beta}(y)| + |\partial^\beta \widetilde{P}_{x_0}^{\bar\beta}(y) - \delta_{\beta\bar\beta}| \\ &\le C\epsilon_2 + C\epsilon_1 < C\epsilon_1 \qquad (\beta \in \mathcal{M},\ \beta \ge \bar\beta). \end{aligned} \tag{3.21}$$

Similarly, from (3.13), we estimate:

$$\begin{aligned} |\partial^\beta \widetilde{P}^{\bar\beta}_y(y)| &\le |\partial^\beta \widetilde{P}^{\bar\beta}_y(y) - \delta_{\beta\bar\beta}| + 1 \\ &\le C\epsilon_1^{-D} \qquad (\beta \in \mathcal{M}). \end{aligned} \tag{3.22}$$

Define $\overline{\mathcal{A}} = \{\alpha \in \mathcal{A} : \alpha < \bar\beta\} \cup \{\bar\beta\}$. Then because $\bar\beta \notin \mathcal{A}$, the minimal element of $\mathcal{A}\Delta\overline{\mathcal{A}}$ is $\bar\beta$. Thus, we have $\overline{\mathcal{A}} < \mathcal{A}$, by definition of the order relation on multiindex set. Define

$$P^{\bar\beta}_y := \widetilde{P}^{\bar\beta}_y - \sum_{\gamma \in \overline{\mathcal{A}} \setminus \{\bar\beta\}} \partial^\gamma \widetilde{P}^{\bar\beta}_y(y) \widetilde{P}^\gamma_y.$$

Notice that $\overline{\mathcal{A}} \setminus \{\bar\beta\} \subset \mathcal{A}$; thus (3.16) (for $x = y$) implies that

$$\partial^\alpha P^{\bar\beta}_y(y) = 0 \qquad (\alpha \in \overline{\mathcal{A}} \setminus \{\bar\beta\}).$$

And (3.20), (3.22), and (3.15) imply that

$$P^{\bar\beta}_y \in (\epsilon_2 + C\epsilon_1^{-D}\epsilon_2) \cdot \sigma_J(y,\mu) \subset C\epsilon_1^{-D}\epsilon_2 \cdot \sigma_J(y,\mu).$$

Since $\bar\beta$ is the maximal element of $\overline{\mathcal{A}}$, it follows that for any $\beta \ge \bar\beta$ and any $\gamma \in \overline{\mathcal{A}} \setminus \{\bar\beta\}$, we have $\beta > \gamma$. Thus (3.21), (3.22), and (3.17) imply that

$$|\partial^\beta P^{\bar\beta}_y(y) - \delta_{\beta\bar\beta}| \le C(\epsilon_1 + \epsilon_1^{-D}\epsilon_2) \qquad (\beta \in \mathcal{M},\ \beta \ge \bar\beta). \tag{3.23}$$

And (3.22) and (3.14) imply that

$$|\partial^\beta P^{\bar\beta}_y(y)| \le C\epsilon_1^{-2D-1} \qquad (\beta \in \mathcal{M}).$$

Recall that $\epsilon_2 \le \epsilon_1^{D+1}$ and $\epsilon_1 \le c_1$. We now fix $c_1$ to be a small universal constant, so that (3.23) yields $\partial^{\bar\beta} P^{\bar\beta}_y(y) \in [1/2, 2]$. We then define $\hat{P}^{\bar\beta}_y = P^{\bar\beta}_y / \partial^{\bar\beta} P^{\bar\beta}_y(y)$. The above properties of $P^{\bar\beta}_y$ give that

$$\hat{P}^{\bar\beta}_y \in C\epsilon_1^{-D}\epsilon_2 \cdot \sigma_J(y,\mu); \tag{3.24}$$

$$\partial^\beta \hat{P}^{\bar\beta}_y(y) = \delta_{\beta\bar\beta} \qquad (\beta \in \overline{\mathcal{A}}); \tag{3.25}$$

$$|\partial^\beta \hat{P}^{\bar\beta}_y(y)| \le C(\epsilon_1 + \epsilon_1^{-D}\epsilon_2) \qquad (\beta \in \mathcal{M},\ \beta > \bar\beta); \text{ and} \tag{3.26}$$

$$|\partial^\beta \hat{P}^{\bar\beta}_y(y)| \le C\epsilon_1^{-2D-1} \qquad (\beta \in \mathcal{M}). \tag{3.27}$$

For each $\alpha \in \overline{\mathcal{A}} \setminus \{\bar\beta\}$, we define $\hat{P}^\alpha_y = \widetilde{P}^\alpha_y - \left(\partial^{\bar\beta} \widetilde{P}^\alpha_y(y)\right) \hat{P}^{\bar\beta}_y$. Note that $\alpha < \bar\beta$, and hence $|\partial^{\bar\beta} \widetilde{P}^\alpha_y(y)| \le \epsilon_2 \le 1$, thanks to (3.17). From (3.15) and (3.24), we now obtain

$$\hat{P}^\alpha_y \in C'\epsilon_1^{-D}\epsilon_2 \cdot \sigma_J(y,\mu). \tag{3.28}$$

From (3.17) and (3.27),

$$|\partial^\beta \hat{P}_y^\alpha(y)| \leq |\partial^\beta \widetilde{P}_y^\alpha(y)| + |\partial^{\bar{\beta}} \widetilde{P}_y^\alpha(y)| \cdot |\partial^\beta \hat{P}_y^{\bar{\beta}}(y)| \leq \epsilon_2 + \epsilon_2 \cdot C\epsilon_1^{-2D-1} \leq C'\epsilon_1^{-2D-1}\epsilon_2,\ (\beta \in \mathcal{M},\ \beta > \alpha). \tag{3.29}$$

Recall that $\overline{\mathcal{A}} = \{\alpha \in \mathcal{A} : \alpha < \bar{\beta}\} \cup \{\bar{\beta}\}$. From (3.16) and (3.25), we have

$$\partial^\beta \hat{P}_y^\alpha(y) = \delta_{\alpha\beta} \qquad (\beta \in \overline{\mathcal{A}}). \tag{3.30}$$

By now varying the point $y \in E$, we deduce from (3.24)-(3.26) and (3.28)-(3.30) that $\sigma_J(y,\mu)$ contains an $\left(\overline{\mathcal{A}}, y, C \cdot (\epsilon_1 + \epsilon_1^{-2D-1}\epsilon_2), 1\right)$-basis for each $y \in E$. Since $\epsilon_2 \leq \epsilon_1^{2D+2}$, the conclusion of the Lemma is immediate. □

**Lemma 3.6.** *Let $c_1$ and $C_1$ be the constants from Lemma 3.5. Suppose we are given data*

$$\left(\epsilon_1, \epsilon_2, \delta, \mathcal{A}, \mu, E, \left(\widetilde{P}_x^\alpha\right)_{\alpha\in\mathcal{A}, x\in E}\right),$$

*satisfying **(D1)-(D5)** of Lemma 3.5, and the family of polynomials $(\widetilde{P}_x^\alpha)_{\alpha\in\mathcal{A},x\in E}$ also satisfies*

$$\max\left\{|\partial^\beta \widetilde{P}_x^\alpha(x)|\delta^{|\beta|-|\alpha|} : x \in E,\ \alpha \in \mathcal{A},\ \beta \in \mathcal{M}\right\} \geq \epsilon_1^{-D-1}. \tag{3.31}$$

*Then there exists $\bar{\beta} \in \mathcal{M} \setminus \mathcal{A}$ so that **(D7)** is satisfied, and additionally there exist $x_0 \in E$ and $\widetilde{P}_{x_0}^{\bar{\beta}} \in \mathcal{P}$ so that **(D6)** is satisfied. Hence, there exists $\overline{\mathcal{A}} < \mathcal{A}$ so that for every $x \in E$, $\sigma_J(x,\mu)$ contains an $(\overline{\mathcal{A}}, x, C_1\epsilon_1, \delta)$-basis.*

**Proof.** By rescaling it suffices to assume that $\delta = 1$. Our hypotheses tell us that $\epsilon_1 \leq c_1$, $\epsilon_2 \leq \epsilon_1^{2D+2}$, and that $(\widetilde{P}_x^\alpha)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, x, \epsilon_2, 1)$-basis for $\sigma_J(x,\mu)$, for each $x \in E$. That is,

$$\widetilde{P}_x^\alpha \in \epsilon_2 \cdot \sigma_J(x,\mu); \tag{3.32}$$

$$\partial^\beta \widetilde{P}_x^\alpha(x) = \delta_{\alpha\beta} \qquad (\alpha, \beta \in \mathcal{A});\ \text{and} \tag{3.33}$$

$$|\partial^\beta \widetilde{P}_x^\alpha(x)| \leq \epsilon_2 \qquad (\alpha \in \mathcal{A}, \beta \in \mathcal{M},\ \beta > \alpha). \tag{3.34}$$

For each $\alpha \in \mathcal{A}$, we define $Z_\alpha = \max\left\{|\partial^\beta \widetilde{P}_x^\alpha(x)| : x \in E, \beta \in \mathcal{M}\right\}$. Then hypothesis (3.31) is equivalent to $\max\{Z_\alpha : \alpha \in \mathcal{A}\} \geq \epsilon_1^{-D-1}$. Let $\bar{\alpha} \in \mathcal{A}$ be the minimal index with $Z_{\bar{\alpha}} \geq \epsilon_1^{-D-1}$. Thus,

$$Z_\alpha < \epsilon_1^{-D-1},\ \text{for all } \alpha \in \mathcal{A} \text{ with } \alpha < \bar{\alpha}, \tag{3.35}$$

and there exist $x_0 \in E$ and $\beta_0 \in \mathcal{M}$ with

$$\epsilon_1^{-D-1} \le Z_{\bar{\alpha}} = |\partial^{\beta_0} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)|. \tag{3.36}$$

Thus, (3.33) and (3.34) imply that $\beta_0 \neq \bar{\alpha}$ and $\beta_0 \le \bar{\alpha}$, respectively. Therefore, $\beta_0 < \bar{\alpha}$. By definition of $Z_{\bar{\alpha}}$ we also have

$$|\partial^{\beta} \widetilde{P}^{\bar{\alpha}}_{y}(y)| \le |\partial^{\beta_0} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)|, \text{ for all } y \in E_2 \text{ and } \beta \in \mathcal{M}. \tag{3.37}$$

Let the elements of $\mathcal{M}$ between $\beta_0$ and $\bar{\alpha}$ be ordered as follows:

$$\beta_0 < \beta_1 < \cdots < \beta_k = \bar{\alpha}.$$

Note that $k+1 \le |\mathcal{M}| = D$. Define

$$a_i = |\partial^{\beta_i} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)|, \text{ for } i = 0, \ldots, k.$$

Then (3.33) and (3.36) imply that $a_k = 1$ and $a_0 \ge \epsilon_1^{-D-1}$. Choose $r \in \{0, \ldots, l\}$ with $a_r \epsilon_1^{-r} = \max\{a_l \epsilon_1^{-l} : 0 \le l \le k\}$. Note that $a_0 \ge \epsilon_1^{-D-1} > a_k \epsilon_1^{-k}$ which implies $r \neq k$. Moreover, we have

$$a_r \ge \epsilon_1^{D} a_0 \text{ and } a_r \ge \epsilon_1^{-1} a_i \text{ for } i = r+1, \ldots, k. \tag{3.38}$$

Define $\bar{\beta} = \beta_r \in \mathcal{M}$. Then (3.38) states that

$$|\partial^{\bar{\beta}} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \ge \epsilon_1^{D} |\partial^{\beta_0} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \ge \epsilon_1^{-1}. \tag{3.39}$$

Also, (3.34) and (3.39) imply that

$$|\partial^{\beta} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \le \epsilon_2 \le 1 \le \epsilon_1 |\partial^{\bar{\beta}} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \qquad (\beta \in \mathcal{M},\ \beta > \bar{\beta}).$$

For $\bar{\beta} < \beta \le \bar{\alpha}$, (3.38) implies that $|\partial^{\beta} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \le \epsilon_1 |\partial^{\bar{\beta}} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)|$. Thus we have,

$$|\partial^{\beta} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \le \epsilon_1 |\partial^{\bar{\beta}} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \qquad (\beta \in \mathcal{M},\ \beta > \bar{\beta}). \tag{3.40}$$

By (3.39) we have $|\partial^{\bar{\beta}} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| > 1$. Hence, (3.33) and (3.34) imply that

$$\bar{\beta} < \bar{\alpha} \text{ and } \bar{\beta} \notin \mathcal{A}. \tag{3.41}$$

Define $\widetilde{P}^{\bar{\beta}}_{x_0} = \widetilde{P}^{\bar{\alpha}}_{x_0} / \partial^{\bar{\beta}} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)$, which satisfies

$$\widetilde{P}^{\bar{\beta}}_{x_0} \in \epsilon_2 \sigma_J(x_0, \mu) \qquad \text{from (3.32) and } |\partial^{\bar{\beta}} \widetilde{P}^{\bar{\alpha}}_{x_0}(x_0)| \ge 1; \tag{3.42}$$

$$|\partial^{\beta} \widetilde{P}^{\bar{\beta}}_{x_0}(x_0)| \le \epsilon_1 \qquad (\beta \in \mathcal{M},\ \beta > \bar{\beta}), \text{ from (3.40)}; \tag{3.43}$$

$$|\partial^{\beta} \widetilde{P}^{\bar{\beta}}_{x_0}(x_0)| \le \epsilon_1^{-D} \qquad (\beta \in \mathcal{M}), \text{ from (3.37) and (3.39); and} \tag{3.44}$$

$$|\partial^{\bar{\beta}} \widetilde{P}^{\bar{\beta}}_{x_0}(x_0)| = 1. \tag{3.45}$$

For $x \in E$, from (3.35) and the definition of $Z_\alpha$,

$$|\partial^\beta \widetilde{P}_x^\alpha(x)| \le \epsilon_1^{-D-1} \qquad (\alpha \in \mathcal{A},\ \alpha < \bar{\beta},\ \beta \in \mathcal{M}) \tag{3.46}$$

Now the hypotheses of Lemma 3.5 hold with

$$\left(\epsilon_1, \epsilon_2, \delta, \mathcal{A}, \mu, E, \left(\widetilde{P}_x^\alpha\right)_{\alpha\in\mathcal{A}, x\in E}, x_0, \bar{\beta}, \widetilde{P}_{x_0}^{\bar{\beta}}\right)$$

satisfying **(D1)-(D7)** due to (3.42)-(3.46). Hence, there exists $\overline{\mathcal{A}} < \mathcal{A}$ so that for every $x \in E$, $\sigma_J(x, \mu)$ contains an $(\overline{\mathcal{A}}, x, C_1\epsilon_1, \delta)$-basis. □

**Definition 3.2.** Let $S \ge 1$, $\epsilon \in (0, 1)$, and $\mathcal{A} \subset \mathcal{M}$ be given. A matrix $(B_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ is called $(S, \epsilon)$ near-triangular if

$$|B_{\alpha\beta} - \delta_{\alpha\beta}| \le \epsilon \qquad \alpha, \beta \in \mathcal{M},\ \alpha \le \beta; \text{ and}$$
$$|B_{\alpha\beta}| \le S \qquad \alpha \in \mathcal{A},\ \beta \in \mathcal{M}.$$

**Lemma 3.7** *(Lemma 3.4 of [13]). Given $R \ge 1$, there exist constants $c_2 \ge 0$, $C_2 \ge 1$ depending only on $R, m, n$, so that the following holds. Suppose we are given $\epsilon_2 \in (0, c_2]$, $x \in \mathbb{R}^n$, a symmetric convex subset $\sigma \subset \mathcal{P}$ and a family of polynomials $(P^\alpha)_{\alpha\in\mathcal{A}} \subset \mathcal{P}$, such that*

$$P^\alpha \in \epsilon_2\sigma \qquad \alpha \in \mathcal{A}; \tag{3.47}$$
$$|\partial^\beta P^\alpha(x) - \delta_{\alpha\beta}| \le \epsilon_2 \qquad \alpha \in \mathcal{A}, \beta \in \mathcal{M}, \beta \ge \alpha;\ \textit{and} \tag{3.48}$$
$$|\partial^\beta P^\alpha(x)| \le R \qquad \alpha \in \mathcal{A}, \beta \in \mathcal{M}. \tag{3.49}$$

*Then there exists a $(C_2, C_2\epsilon_2)$ near-triangular matrix $B = (B_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$, so that if we define*

$$\widetilde{P}^\alpha := \sum_{\beta\in\mathcal{A}} B_{\alpha\beta} P^\beta,\ \alpha \in \mathcal{A},$$

*then $(\widetilde{P}^\alpha)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, x, C_2\epsilon_2, 1)$-basis for $\sigma$. Furthermore $|\partial^\beta \widetilde{P}^\alpha(x)| \le C_2$ for every $\alpha \in \mathcal{A}$ and every $\beta \in \mathcal{M}$.*

### *3.3. The inductive hypothesis*

Fix $\mathcal{A} \subset \mathcal{M}$, $\mathcal{A} \ne \mathcal{M}$. We will impose the inductive hypothesis that the Main Lemma holds for all $\mathcal{A}' < \mathcal{A}$. Our task is to prove the Main Lemma for $\mathcal{A}$. The inductive hypothesis will be a standing assumption until we complete the proof of the Main Lemma for $\mathcal{A}$ in Section 10.

We will assume the value of $\epsilon = \epsilon(\mathcal{A})$ in the Main Lemma for $\mathcal{A}$ is less than a small enough constant determined by $m$ and $n$ and eventually determine such a constant. Let $(\mu, \delta)$ be as in the statement of the Main Lemma for $\mathcal{A}$ (Lemma 3.4). By rescaling and translating, we may assume without loss of generality that

$$\begin{aligned} &\operatorname{diam}(\operatorname{supp}(\mu)) < \delta = 1/10, \\ &\operatorname{supp}(\mu) \subset \frac{1}{10} Q^\circ, \quad Q^\circ = (0,1]^n. \end{aligned} \tag{3.50}$$

From hypothesis (3.6) in the Main Lemma for $\mathcal{A}$, with $\delta = 1/10$, we have:

$$\text{For every } x \in \operatorname{supp}(\mu),\ \sigma_J(x,\mu) \text{ has an } (\mathcal{A}, x, \epsilon/C_0, 1)\text{-basis.} \tag{3.51}$$

The inductive hypothesis states that the Main Lemma for $\mathcal{A}'$ is true for every $\mathcal{A}' < \mathcal{A}$. Let $\epsilon(\mathcal{A}')$ be the constants arising in the Main Lemma for $\mathcal{A}'$ ($\mathcal{A}' < \mathcal{A}$). Define

$$\epsilon_0 := \min\{\epsilon(\mathcal{A}') : \mathcal{A}' < \mathcal{A}\}. \tag{3.52}$$

By the inductive hypothesis, for $\mathcal{A}' < \mathcal{A}$ and any Borel regular measure $\widehat{\mu}$ on $\mathbb{R}^n$,

$$\begin{aligned} &\text{if } \sigma_J(x, \widehat{\mu}) \text{ admits an } (\mathcal{A}', x, \epsilon_0/C_0, 10\widehat{\delta})\text{-basis, for each } x \in \operatorname{supp}(\widehat{\mu}), \\ &\text{for some } \widehat{\delta} \geq \operatorname{diam}(\operatorname{supp}(\widehat{\mu})), \text{then the Extension Theorem for } (\widehat{\mu}, \widehat{\delta}) \text{ holds.} \end{aligned} \tag{3.53}$$

We will assume that $\epsilon < \epsilon_0$.

Suppose that there exists $\overline{\mathcal{A}} < \mathcal{A}$ such that $\sigma_J(x,\mu)$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 1)$-basis for every $x \in \operatorname{supp}(\mu)$. Then by the validity of the Main Lemma for $\overline{\mathcal{A}}$, the Extension Theorem for $(\mu, \delta)$ holds (see Proposition 3.1). Note that $\overline{\mathcal{A}} \neq \emptyset$ because $\emptyset$ is maximal under the order on multiindex sets. Thus, by the Main Lemma for $\overline{\mathcal{A}}$, in the conclusion of the Extension Theorem for $(\mu, \delta)$ one can take $K = \emptyset$, and so the functional $M : \mathscr{J}(\mu; \delta) \to \mathbb{R}_+$ has the form (3.7). Therefore, we have proven the Main Lemma for $\mathcal{A}$ in the case that there exists $\overline{\mathcal{A}}$ as above. Therefore, we may now assume:

$$\begin{aligned} &\text{For every } \overline{\mathcal{A}} < \mathcal{A}, \text{ there exists } x \in \operatorname{supp}(\mu) \text{ such that} \\ &\qquad \sigma_J(x,\mu) \text{ does not contain an } (\overline{\mathcal{A}}, x, \epsilon_0/C_0, 1)\text{-basis.} \end{aligned} \tag{3.54}$$

*3.3.1. Auxiliary polynomials*

**Lemma 3.8.** *Let $\mu$ be a Borel regular measure on $\mathbb{R}^n$ satisfying (3.50), (3.51), (3.54). Then for all $x \in 100Q^\circ$, there exists a family of polynomials $(P_x^\alpha)_{\alpha \in \mathcal{A}}$ such that*

$$(P_x^\alpha)_{\alpha \in \mathcal{A}} \text{ forms an } (\mathcal{A}, x, C\epsilon/C_0, 1)\text{-basis for } \sigma_J(x,\mu), \text{ and} \tag{3.55}$$

$$|\partial^\beta P_x^\alpha(x)| \leq C \quad (\alpha \in \mathcal{A},\ \beta \in \mathcal{M}). \tag{3.56}$$

**Proof.** From (3.51), for $y \in \operatorname{supp}(\mu)$, there exists $(\widetilde{P}^\alpha_y)_{\alpha\in\mathcal{A}}$ that form an $(\mathcal{A}, y, \epsilon/C_0, 1)$-basis for $\sigma_J(y, \mu)$. As a consequence of (3.54), we have

$$|\partial^\beta \widetilde{P}^\alpha_y(y)| \le C \quad \alpha \in \mathcal{A},\ \beta \in \mathcal{M},\ y \in \operatorname{supp}(\mu). \tag{3.57}$$

To see this, suppose (3.57) fails for the constant $C = \epsilon_1^{-D-1}$, where $\epsilon_1 \le \min\{c_1, \epsilon_0/(C_0C_1)\}$ (and $c_1, C_1$ are the constants from Lemma 3.6). That is, suppose

$$\max\left\{|\partial^\beta \widetilde{P}^\alpha_y(y)| : y \in \operatorname{supp}(\mu),\ \alpha \in \mathcal{A},\ \beta \in \mathcal{M}\right\} > \epsilon_1^{-D-1}.$$

We may assume that $\epsilon \le \epsilon_1^{2D+2}$. Then the hypotheses of Lemma 3.6 hold with parameters

$$\left(\epsilon_1, \epsilon_2, \delta, \mathcal{A}, \mu, E, (\widetilde{P}^\alpha_x)_{\alpha\in\mathcal{A}, x\in E}\right) := \left(\epsilon_1, \epsilon, 1, \mathcal{A}, \mu, \operatorname{supp}(\mu), (\widetilde{P}^\alpha_y)_{\alpha\in\mathcal{A}, y\in\operatorname{supp}(\mu)}\right).$$

Thus we find $\overline{\mathcal{A}} < \mathcal{A}$ so that $\sigma_J(y, \mu)$ contains an $(\overline{\mathcal{A}}, y, C_1\epsilon_1, 1)$-basis for each $y \in \operatorname{supp}(\mu)$. Since $C_1\epsilon_1 \le \epsilon_0/C_0$, this contradicts (3.54), which concludes our proof of (3.57).

Now fix $x_0 \in \operatorname{supp}(\mu)$. Then we have shown $\sigma_J(x_0, \mu)$ has an $(\mathcal{A}, x_0, \epsilon/C_0, 1)$-basis, $\{\widetilde{P}^\alpha_{x_0}\}_{\alpha\in\mathcal{A}}$ satisfying the inequalities (3.57) for $y = x_0$. That is, for $\alpha \in \mathcal{A}$,

$$\partial^\beta \widetilde{P}^\alpha_{x_0}(x_0) = \delta_{\alpha\beta} \qquad (\beta \in \mathcal{A}); \tag{3.58}$$

$$|\partial^\beta \widetilde{P}^\alpha_{x_0}(x_0)| \le \epsilon/C_0 \qquad (\beta \in \mathcal{M},\ \beta > \alpha);\ \text{and} \tag{3.59}$$

$$|\partial^\beta \widetilde{P}^\alpha_{x_0}(x_0)| \le C \qquad (\beta \in \mathcal{M}). \tag{3.60}$$

Furthermore, since $\widetilde{P}^\alpha_{x_0} \in (\epsilon/C_0)\sigma_J(x_0, \mu)$ for each $\alpha \in \mathcal{A}$ there exists

$$\varphi^\alpha \in L^{m,p}(\mathbb{R}^n) \text{ with } J_{x_0}\varphi^\alpha = \widetilde{P}^\alpha_{x_0} \text{ and } \|\varphi^\alpha\|_{\mathscr{J}(0,\mu)} \le \epsilon/C_0. \tag{3.61}$$

For $x \in 100Q^\circ$, define

$$\hat{P}^\alpha_x := J_x\varphi^\alpha. \tag{3.62}$$

Because $\|\varphi^\alpha\|_{\mathscr{J}(0,\mu)} \le \epsilon/C_0$, we have

$$\hat{P}^\alpha_x \in \epsilon/C_0 \cdot \sigma_J(x, \mu) \subset C\epsilon\sigma_J(x, \mu). \tag{3.63}$$

From the Sobolev Inequality, since $\hat{P}^\alpha_x = J_x\varphi^\alpha$ and $\widetilde{P}^\alpha_{x_0} = J_{x_0}\varphi^\alpha$, and since $|x - x_0| \le 100\delta_{Q^\circ} = 100$,

$$\begin{aligned} |\partial^\beta \hat{P}^\alpha_x(x) - \delta_{\alpha\beta}| &\le |\partial^\beta \widetilde{P}^\alpha_{x_0}(x) - \delta_{\alpha\beta}| + |\partial^\beta J_{x_0}\varphi^\alpha(x) - \partial^\beta \hat{P}^\alpha_x(x)| \\ &\le \left| \sum_{0\le\gamma\le m-1-|\beta|} \partial^{\beta+\gamma} \widetilde{P}^\alpha_{x_0}(x_0)(x - x_0)^\gamma/\gamma! - \delta_{\alpha\beta} \right| \\ &\quad + C\|\varphi^\alpha\|_{L^{m,p}(Q^\circ)}\delta_{Q^\circ}^{m-|\beta|-n/p}. \end{aligned} \tag{3.64}$$

From (3.58), (3.59), (3.61), (3.64), and since $|x - x_0| \leq 100$, for $\beta \in \mathcal{M}$ satisfying $\beta \geq \alpha$, we have

$$|\partial^\beta \hat{P}^\alpha_x(x) - \delta_{\alpha\beta}| \leq C\epsilon. \tag{3.65}$$

From (3.60), (3.61), (3.64), and since $|x - x_0| \leq 100$, for $\beta \in \mathcal{M}$,

$$|\partial^\beta \hat{P}^\alpha_x(x) - \delta_{\alpha\beta}| \leq C, \text{ hence } |\partial^\beta \hat{P}^\alpha_x(x)| \leq C + 1. \tag{3.66}$$

Fix a constant $R$, determined by $m$, $n$, and $p$, so that $R = C + 1$, with $C$ from (3.66). We fix $c_2$ and $C_2$ for the universal constants in Lemma 3.7 determined by this choice of $R$. By taking $\epsilon$ small enough, we may assume $\epsilon_2 := C\epsilon$ in (3.63), (3.65) satisfies $\epsilon_2 < c_2$. Then (3.63), (3.65), and (3.66) allow us to apply Lemma 3.7 to the family of polynomials $(\hat{P}^\alpha_x)_{\alpha\in\mathcal{A}}$, with $\sigma = \sigma_J(x, \mu)$, $\epsilon_2 = C\epsilon$ and $R = C + 1$. Thus, there exists a $(C_2, C_2 \cdot C\epsilon)$-near triangular matrix $A^x = (A^x_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ such that if we define $P^\alpha_x \in \mathcal{P}$ as

$$P^\alpha_x := \sum_{\beta\in\mathcal{A}} A^x_{\alpha\beta} \cdot \hat{P}^\beta_x \quad (\alpha \in \mathcal{A}), \tag{3.67}$$

then $(P^\alpha_x)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, x, C_2C\epsilon, 1)$-basis for $\sigma_J(x, \mu)$ and $|\partial^\beta P^\alpha_x(x)| \leq C_2$ for all $\beta \in \mathcal{A}$. Thus, the family of polynomials $(P^\alpha_x)_{\alpha\in\mathcal{A}}$ satisfies (3.55)-(3.56). $\square$

For $\alpha \in \mathcal{A}$, $\beta \in \mathcal{M}$ satisfying $\beta > \alpha$, $x, y \in 100Q^\circ$, and $P^\alpha_y \in \mathcal{P}$ satisfying (3.55) and (3.56), we use the Taylor expansion to bound

$$\begin{aligned} \left|\partial^\beta P^\alpha_y(x)\right| &= \left|\partial^\beta \sum_{\gamma\in\mathcal{M}} \partial^\gamma P^\alpha_y(y)(x-y)^\gamma/\gamma!\right| \\ &= \left|\sum_{\eta+\beta\in\mathcal{M}} \partial^{\eta+\beta} P^\alpha_y(y)(x-y)^\eta/\eta!\right| \\ &\leq C\epsilon/C_0. \end{aligned} \tag{3.68}$$

**Proposition 3.9.** *For each $x \in 100Q^\circ$, there exists a $(C, C\epsilon)$-near triangular matrix $A^x = (A^x_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$, and there exists a corresponding family of functions $(\varphi^\alpha_x)_{\alpha\in\mathcal{A}} \subset L^{m,p}(\mathbb{R}^n)$ given by*

$$\varphi^\alpha_x := \sum_{\beta\in\mathcal{A}} A^x_{\alpha\beta} \cdot \varphi^\beta; \tag{3.69}$$

*where $(\varphi^\alpha)_{\alpha\in\mathcal{A}}$ is a family of functions satisfying*

$$\|\varphi^\alpha\|_{\mathscr{J}(0,\mu)} \leq \epsilon/C_0, \tag{3.70}$$

*and the family $(\varphi^\alpha_x)_{\alpha\in\mathcal{A}}$ also satisfies*

$$J_x \varphi_x^\alpha = P_x^\alpha; \ \textit{and} \tag{3.71}$$

$$\|\varphi_x^\alpha\|_{\mathscr{J}(0,\mu)} \le C\epsilon/C_0. \tag{3.72}$$

**Proof.** We follow the proof of Lemma 3.8, and let $(\varphi^\alpha)_{\alpha\in\mathcal{A}}$ be the family of functions satisfying (3.61). Accordingly, we have $\|\varphi^\alpha\|_{\mathscr{J}(0,\mu)} \le \epsilon/C_0$, giving (3.70).

For $x \in 100Q^\circ$, let $A^x = (A^x_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ be the $(C, C\epsilon)$-near triangular matrix defined just before (3.67). Then define $\varphi_x^\alpha$ ($\alpha \in \mathcal{A}$) as in (3.69). Then (3.71) follows by applying $J_x$ to both sides of (3.69) and applying (3.62) and (3.67).

Note $|A^x_{\alpha\beta}| \le C$ for $\alpha, \beta \in \mathcal{A}$, since $A^x$ is $(C, C\epsilon)$-near triangular. Thus, (3.72) follows by sublinearity of the $\mathscr{J}(0,\mu)$ functional; indeed, $\|F\|_{\mathscr{J}(0,\mu)} = (\|F\|^p_{L^{m,p}(\mathbb{R}^n)} + \|F\|^p_{L^p(d\mu)})^{1/p}$ is a norm on $L^{m,p}(\mathbb{R}^n) \cap L^p(d\mu)$. □

*3.3.2. Reduction to monotonic $\mathcal{A}$*

**Definition 3.3** *(Monotonic labels).* A collection of multi-indices $\mathcal{A} \subset \mathcal{M}$ is monotonic if

$$\alpha \in \mathcal{A} \text{ and } |\gamma| \le m - 1 - |\alpha| \text{ implies } \alpha + \gamma \in \mathcal{A}.$$

If the above property fails, we say that $\mathcal{A}$ is non-monotonic.

In this section we follow Section 4.2 of [13] to deduce the monotonicity of $\mathcal{A}$ using assumption (3.54) and condition (3.51) for $\mathcal{A}$.

For the sake of contradiction, we assume that $\mathcal{A}$ is non-monotonic. We will show there exists $\overline{\mathcal{A}} < \mathcal{A}$ such that for all $x \in \operatorname{supp}(\mu)$, $\sigma_J(x,\mu)$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 1)$-basis, contradicting (3.54). Thus our proof of the Induction Step is reduced to the case of monotonic $\mathcal{A}$.

Let $\alpha_0 \in \mathcal{A}$, $\gamma \in \mathcal{M}$ satisfy

$$0 < |\gamma| \le m - 1 - |\alpha_0| \text{ and } \bar{\alpha} := \alpha_0 + \gamma \in \mathcal{M} \setminus \mathcal{A}. \tag{3.73}$$

Define $\overline{\mathcal{A}} = \mathcal{A} \cup \{\bar{\alpha}\}$. Note that $\alpha_0 < \bar{\alpha}$, and the only element of $\overline{\mathcal{A}} \Delta \mathcal{A}$ is $\bar{\alpha}$, which is in $\overline{\mathcal{A}}$, so $\overline{\mathcal{A}} < \mathcal{A}$ by definition of the order on multiindex sets.

For fixed $y \in \operatorname{supp}(\mu) \subset \frac{1}{10}Q^\circ$, we let $(P_y^\alpha)_{\alpha\in A}$ satisfy (3.55) and (3.56). We now define

$$P_y^{\bar{\alpha}} := P_y^{\alpha_0} \odot_y q^y, \qquad \text{where } q^y(x) := \frac{\alpha_0!}{\bar{\alpha}!}(x-y)^\gamma. \tag{3.74}$$

Expanding out this product, we have

$$P_y^{\bar{\alpha}}(x) = \frac{\alpha_0!}{\bar{\alpha}!} \sum_{|\omega| \le m-1-|\gamma|} \frac{1}{\omega!} \partial^\omega P_y^{\alpha_0}(y)(x-y)^{\omega+\gamma}.$$

Note that $\omega = \alpha_0$ arises in the sum above, thanks to (3.73). Also, the terms with $\omega + \gamma > \bar{\alpha} = \alpha_0 + \gamma$ correspond precisely to $\omega > \alpha_0$, by definition of the order. The following properties are now immediate from (3.55) and (3.56):

$$\partial^{\bar{\alpha}} P_y^{\bar{\alpha}}(y) = 1; \tag{3.75}$$

$$|\partial^{\beta} P_y^{\bar{\alpha}}(y)| \leq C'\epsilon \qquad (\beta \in \mathcal{M},\ \beta > \bar{\alpha});\ \text{and} \tag{3.76}$$

$$|\partial^{\beta} P_y^{\bar{\alpha}}(y)| \leq C' \qquad (\beta \in \mathcal{M}). \tag{3.77}$$

From (3.55), we have $P_y^{\alpha_0} \in C\epsilon \cdot \sigma_J(y,\mu)$, so there exists $\varphi \in L^{m,p}(\mathbb{R}^n)$ satisfying

$$\|\varphi\|_{\mathscr{J}(0,\mu)} = \left( \|\varphi\|^p_{L^{m,p}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |\varphi|^p d\mu \right)^{1/p} \leq C\epsilon;\ \text{and} \tag{3.78}$$

$$J_y\varphi = P_y^{\alpha_0}. \tag{3.79}$$

Let $\theta \in C_0^\infty(Q^\circ)$ satisfy $\theta|_{0.99Q^\circ} = 1$ and $|\partial^\alpha \theta(x)| \leq C$ for $x \in Q^\circ$ and $|\alpha| \leq m$. Define $\bar{\varphi} : \mathbb{R}^n \to \mathbb{R}$ as $\bar{\varphi} := \theta \cdot (\varphi q^y) + (1-\theta) \cdot P_y^{\bar{\alpha}}$. Then $J_y\bar{\varphi} = J_y(\varphi q^y) = P_y^{\alpha_0} \odot_y q^y = P_y^{\bar{\alpha}}$, since $y \in \frac{1}{10}Q^\circ$. Note that $\|q^y\|_{L^\infty(Q^\circ)} \leq C$. Thus, from (3.50) and (3.78) we deduce

$$\int_{\mathbb{R}^n} |\bar{\varphi}|^p d\mu \leq \int_{\frac{1}{10}Q^\circ} |\varphi q^y|^p d\mu \leq C \int_{\frac{1}{10}Q^\circ} |\varphi|^p d\mu \leq C(C\epsilon)^p.$$

We will show $\|\bar{\varphi}\|_{L^{m,p}(\mathbb{R}^n)} \leq C\epsilon$, implying that $\|\bar{\varphi}\|_{\mathscr{J}(0,\mu)} \leq C'\epsilon$ and $P_y^{\bar{\alpha}} \in C'\epsilon \cdot \sigma_J(y,\mu)$.

Since $\bar{\varphi}$ agrees with an $(m-1)^{\text{rst}}$ degree polynomial on $\mathbb{R}^n \setminus Q^\circ$, and since the partial derivatives of $\theta$ are uniformly bounded, we have

$$\begin{aligned}
\|\bar{\varphi}\|_{L^{m,p}(\mathbb{R}^n)} &= \|\bar{\varphi}\|_{L^{m,p}(Q^\circ)} \\
&= \|\bar{\varphi} - P_y^{\bar{\alpha}}\|_{L^{m,p}(Q^\circ)} \\
&= \|\theta \cdot (\varphi q^y - P_y^{\bar{\alpha}})\|_{L^{m,p}(Q^\circ)} \\
&\leq C \sum_{|\beta| \leq m} \|\partial^\beta(\varphi q^y - P_y^{\bar{\alpha}})\|_{L^p(Q^\circ)}.
\end{aligned} \tag{3.80}$$

As a consequence of (3.74) and (3.79), $P_y^{\bar{\alpha}} = J_y(\varphi q^y)$. Hence, by the Sobolev Inequality, for $x \in Q^\circ$, $|\beta| \leq m$,

$$|\partial^\beta(\varphi q^y - P_y^{\bar{\alpha}})(x)| \leq C\|\varphi q^y\|_{L^{m,p}(Q^\circ)}. \tag{3.81}$$

For $x \in Q^\circ$, $|\beta| = m$, because $q^y$ is a degree $|\gamma|$ polynomial, we have

$$
\begin{aligned}
|\partial^\beta(\varphi q^y)(x)| &\le \sum_{\omega+\omega'=\beta} |\partial^{\omega'}\varphi(x)|\cdot|\partial^\omega q^y(x)| \\
&= |\partial^\beta\varphi(x)||q^y(x)| + \sum_{\substack{\omega+\omega'=\beta,\\ 0<|\omega|\le|\gamma|}} \Big(|\partial^{\omega'}(\varphi - P_y^{\alpha_0})(x)| + |\partial^{\omega'}P_y^{\alpha_0}(x)|\Big)\cdot|\partial^\omega q^y(x)|.
\end{aligned} \tag{3.82}
$$

Furthermore, for $(\omega,\omega')$ as in the sum above, $0<|\omega|\le|\gamma|$ implies $m-1\ge|\omega'|\ge m-|\gamma|>|\alpha_0|$ (see (3.73)). Recall estimate (3.68): $|\partial^{\omega'}P_y^{\alpha_0}(x)|\le C\epsilon$, and note that by construction, we have $\|\partial^\omega q^y\|_{L^\infty(Q^\circ)}\le C$. This, in combination with the Sobolev Inequality, (3.78), and (3.79) allows us to further reduce the sum on the right hand side of inequality (3.82). When $x\in Q^\circ$ and $|\beta|=m$:

$$
\sum_{\substack{\omega+\omega'=\beta,\\ 0<|\omega|\le|\gamma|}} \Big(|\partial^{\omega'}(\varphi - P_y^{\alpha_0})(x)| + |\partial^{\omega'}P_y^{\alpha_0}(x)|\Big)\cdot|\partial^\omega q^y(x)| \le C\cdot\big(\|\varphi\|_{L^{m,p}(Q^\circ)} + C\epsilon\big) \le C'\epsilon.
$$

Substituting this into (3.82), we have for $x\in Q^\circ$ and $|\beta|=m$,

$$
|\partial^\beta(\varphi q^y)(x)| \le |\partial^\beta\varphi(x)||q^y(x)| + C'\epsilon.
$$

By integrating over $x\in Q^\circ$ and again applying (3.78) and $\|q^y\|_{L^\infty(Q^\circ)}\le C$, we deduce that $\|\varphi q^y\|_{L^{m,p}(Q^\circ)}\le C\epsilon$. Thus, keeping in mind (3.80) and (3.81), we have

$$
\begin{aligned}
\|\bar\varphi\|_{L^{m,p}(\mathbb{R}^n)} &\le C\|\varphi q^y\|_{L^{m,p}(Q^\circ)} \\
&\le C'\epsilon.
\end{aligned}
$$

Because $J_y\bar\varphi = P_y^{\bar\alpha}$, this completes our proof that

$$
P_y^{\bar\alpha} \in C'\epsilon\cdot\sigma_J(y,\mu). \tag{3.83}
$$

Due to (3.55), (3.56), (3.75)-(3.77), and (3.83), the family $(P_y^\alpha)_{\alpha\in\overline{\mathcal{A}}}$ satisfies (3.47)-(3.49) with $R$ equal to a universal constant, $\epsilon_2 = C'\epsilon$, and $\sigma=\sigma_J(y,\mu)$. We may assume $C'\epsilon<c_2$, where $c_2$ comes from Lemma 3.7. Then the hypotheses of Lemma 3.7 hold; hence, $\sigma_J(y,\mu)$ contains an $(\overline{\mathcal{A}},y,C_2C'\epsilon,1)$-basis for $C_2\ge1$, a universal constant. Recall our assumption that $\epsilon$ is less than a small enough constant determined by $m$ and $n$. We may assume $\epsilon<\epsilon_0/(C'C_0C_2)$, so $\sigma_J(y,\mu)$ contains an $(\overline{\mathcal{A}},y,\epsilon_0/C_0,1)$-basis. Since $y\in\operatorname{supp}(\mu)$ is arbitrary, this contradicts (3.54).

This completes the proof by contradiction, and establishes that $\mathcal{A}\subset\mathcal{M}$ is monotonic.

## 4. The Calderón-Zygmund decomposition

Recall we have fixed a multi-index set $\mathcal{A}\subset\mathcal{M}$, $\mathcal{A}\ne\mathcal{M}$ and a Borel regular measure $\mu$ on $\mathbb{R}^n$ satisfying the conditions (3.50), (3.51), and (3.54). By the results of Section 3.3.2, we deduce that $\mathcal{A}$ is monotonic. Our goal is to prove the Extension Theorem for $(\mu, \delta = 1/10)$. Recall we have defined constants $\epsilon_0$ in (3.52) and $C_0$ in Lemma 3.3.

### *4.1. Defining the decomposition*

**Definition 4.1** *(OK Cubes).* A dyadic cube $Q \subset Q^\circ = (0,1]^n$ is OK if there exists $\overline{\mathcal{A}} < \mathcal{A}$ such that for every $x \in \operatorname{supp}(\mu) \cap 3Q$, $\sigma_J(x, \mu|_{3Q})$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 30\delta_Q)$-basis.

**Definition 4.2** *(Calderón-Zygmund Cubes).* A dyadic cube $Q \subset Q^\circ$ is CZ if $Q$ is OK and every dyadic cube $Q' \subset Q^\circ$ that properly contains $Q$ is not OK.

If $Q, Q' \subset Q^\circ$ are CZ and $Q \neq Q'$ then $Q \cap Q' = \emptyset$. Indeed, since CZ cubes are dyadic cubes, either $Q \subsetneq Q'$, $Q' \subsetneq Q$, or $Q \cap Q' \neq \emptyset$. The first case is impossible, since, according to the definition of CZ cubes if $Q$ is CZ and $Q \subsetneq Q'$ then $Q'$ is not OK, hence $Q'$ is not CZ. Similarly, the second case is impossible. Therefore, $Q \cap Q' \neq \emptyset$, as claimed.

We write $CZ^\circ = \{Q_i\}_{i \in I}$ to denote the collection of all CZ cubes. The index set $I$ may be countable or finite. In contrast to [13], the CZ cubes do not necessarily form a partition of $Q^\circ$; however, if $\mu$ is a finite measure the CZ cubes do form a partition of $Q^\circ$ (see Lemma 4.3 below).

**Lemma 4.1.** *Suppose $Q, Q' \subset Q^\circ$ are dyadic cubes. Suppose $Q'$ is OK and $3Q \subset 3Q'$. Then $Q$ is OK.*

**Proof.** We have $\sigma_J(x, \mu|_{3Q'}) \subset \sigma_J(x, \mu|_{3Q})$; indeed, this follows by the definition of $\sigma_J(\cdot, \cdot)$, and because $\|F\|_{\mathscr{J}(0, \mu|_{3Q})} \leq \|F\|_{\mathscr{J}(0, \mu|_{3Q'})}$ for $3Q \subset 3Q'$ and $F \in L^{m,p}(\mathbb{R}^n)$. Because $Q'$ is OK, there exists $\overline{\mathcal{A}} < \mathcal{A}$ such that for all $x \in \operatorname{supp}(\mu) \cap 3Q'$, $\sigma_J(x, \mu|_{3Q'})$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 30\delta_{Q'})$-basis. Since $\delta_Q \leq \delta_{Q'}$, and thanks to (3.5), we have that $\sigma_J(x, \mu|_{3Q})$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 30\delta_Q)$-basis for any $x \in 3Q$. This indicates $Q$ is OK. $\square$

Because of Lemma 3.3, for every $Q \in CZ^\circ$, there exists $\overline{\mathcal{A}} < \mathcal{A}$ such that for every $x \in \operatorname{supp}(\mu) \cap 3Q$, $\sigma(\mu|_{3Q}, \delta_Q)$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0, \delta_Q)$-basis.

**Lemma 4.2.** *Let $Q \subset Q^\circ$ be a dyadic cube. If $\mu(3Q)^{1/p} \leq \frac{\epsilon_0}{C_0}(30\delta_Q)^{n/p-m}$ then $Q$ is OK.*

**Proof.** Suppose $Q \subset Q^\circ$ satisfies $\mu(3Q)^{1/p} \leq \epsilon_0/C_0(30\delta_Q)^{n/p-m}$. Let $F^0 = P^0 = 1$. Then for $y \in \operatorname{supp}(\mu) \cap 3Q$, $J_y F^0 = P^0$, and

$$\|F^0\|_{\mathscr{J}(0,\mu|_{3Q})} = \left(\|F^0\|^p_{L^{m,p}(\mathbb{R}^n)} + \|F^0\|^p_{L^p(d\mu|_{3Q})}\right)^{1/p} = \mu(3Q)^{1/p} \leq \frac{\epsilon_0}{C_0}(30\delta_Q)^{n/p-m}.$$

So for all $y \in \operatorname{supp}(\mu) \cap 3Q$, $P^0 \in \frac{\epsilon_0}{C_0}(30\delta_Q)^{n/p-m} \cdot \sigma_J(y, \mu|_{3Q})$, and $(P^0)$ forms a $(\{0\}, y, \epsilon_0/C_0, 30\delta_Q)$-basis for $\sigma_J(y, \mu|_{3Q})$. Because 0 is the minimal element of $\mathcal{M}$ and

$0 \notin \mathcal{A}$ ($\mathcal{A}$ is monotonic and $\mathcal{A} \neq \mathcal{M}$ because $\mathcal{M}$ is minimal in the order), $\{0\} < \mathcal{A}$. Hence, $Q$ is OK. $\square$

**Lemma 4.3.** *The decomposition $CZ^\circ$ is not equal to $\{Q^\circ\}$. In particular, each $Q \in CZ^\circ$ has a (unique) dyadic parent $Q^+ \subset Q^\circ$.*

*Furthermore, if $\mu$ is finite, the CZ cubes form a non-trivial, finite partition of $Q^\circ$ into dyadic cubes.*

**Proof.** If $Q^\circ$ is OK, then there exists $\overline{\mathcal{A}} < \mathcal{A}$ such that for every $x \in \operatorname{supp}(\mu)$, $\sigma_J(x, \mu)$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 1)$-basis, contradicting (3.54). So $Q^\circ \notin CZ^\circ$.

Suppose $\mu$ is finite. Define $H : \mathbb{R}_+ \to \mathbb{R}$ as

$$H(\delta) := \max_{x \in Q^\circ} \{\mu\left(B(x, 3\delta)\right)^{1/p} \cdot \delta^{m-n/p}\}.$$

Because $\mu$ is finite, $\lim_{\delta \to 0^+} H(\delta) = H(0) = 0$. Further, H is non-increasing, so there exists $\delta_1 > 0$ such that if $\delta < \delta_1$ then $H(\delta) \leq H(\delta_1) < \frac{\epsilon_0}{30^{m-n/p} C_0}$. Let $Q \subset Q^\circ$ satisfy $\delta_Q < \delta_1$, and fix $x \in Q$. Then

$$\begin{aligned} \mu(3Q)^{1/p} &\leq \mu(B(x, 3\delta_Q))^{1/p} \\ &\leq H(\delta_Q) \cdot \delta_Q^{n/p-m} \\ &\leq \frac{\epsilon_0}{C_0} (30\delta_Q)^{n/p-m}. \end{aligned}$$

We apply Lemma 4.2 to deduce that $Q$ is OK, so every dyadic subcube of $Q^\circ$ of sidelength smaller than $\delta_1$ is OK. Therefore, $CZ^\circ$ is a finite partition of $Q^\circ$. $\square$

As a corollary of Lemma 4.2, we can bound below the measure of the dyadic parent of any cube in $CZ^\circ$:

**Corollary 4.4.** *For $Q \in CZ^\circ$, we have*

$$\mu(3Q^+)^{1/p} > (\epsilon_0/C_0) \cdot (30\delta_Q^+)^{n/p-m}.$$

Two cubes, $Q_i, Q_j \in CZ^\circ$ are called "neighbors" if their closures satisfy $\mathrm{Cl}(Q_i) \cap \mathrm{Cl}(Q_j) \neq \emptyset$. (In particular, any CZ cube is neighbors with itself.) We denote this relation by $Q_i \leftrightarrow Q_j$, or $i \leftrightarrow j$ $(i, j \in I)$.

**Lemma 4.5** *(Good Geometry). If $Q, Q' \in CZ^\circ$ satisfy $Q \leftrightarrow Q'$, then*

$$\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q. \tag{4.1}$$

**Proof.** For the sake of contradiction, suppose that $Q, Q' \in CZ^\circ$ satisfy

$$\mathrm{Cl}(Q) \cap \mathrm{Cl}(Q') \neq \emptyset \text{ and } 4\delta_Q \leq \delta_{Q'}.$$

Since $Q^+, Q'$ are dyadic cubes and $\delta_{Q^+} = 2\delta_Q \leq \frac{1}{2}\delta_{Q'}$, we have $3Q^+ \subset 3Q'$. Because $Q'$ is OK, by Lemma 4.1, we have that $Q^+$ is OK, contradicting $Q \in CZ^\circ$. $\square$

**Lemma 4.6** *(More Good Geometry). For each $Q \in CZ^\circ$, the following properties hold.*

*If $Q' \in CZ^\circ$ is such that $(1.3)Q' \cap (1.3)Q \neq \emptyset$, then $Q \leftrightarrow Q'$. Consequently:*

$$\text{Each point } x \in \mathbb{R}^n \text{ belongs to at most } C(n) \text{ of the cubes } 1.3Q \text{ with } Q \in CZ^\circ. \tag{4.2}$$

$$\text{If } \mathrm{Cl}(Q) \cap \partial Q^\circ \neq \emptyset, \text{ then } \delta_Q \geq \frac{1}{20}\delta_{Q^\circ}. \tag{4.3}$$

**Proof.** Proof of (4.2): Fix $Q \in CZ^\circ$. Suppose that $Q' \in CZ^\circ$ does not neighbor $Q$. Let $\delta = \max\{\delta_Q/2, \delta_{Q'}/2\}$. Then (4.1) implies that $\mathrm{dist}(Q, Q') \geq \delta$. Thus,

$$(1.3)Q \cap (1.3)Q' \subset B(Q, (0.3)\delta) \cap B(Q', (0.3)\delta) = \emptyset.$$

Proof of (4.3): For the sake of contradiction, suppose $\mathrm{Cl}(Q) \cap \partial Q^\circ \neq \emptyset$ and $\delta_Q < \frac{1}{20}\delta_{Q^\circ}$. Therefore, $9Q \subset \mathbb{R}^n \setminus \frac{1}{10}Q^\circ$. Note that $3Q^+ \subset 9Q$. Because $\mathrm{supp}(\mu) \subset \frac{1}{10}Q^\circ$, we have $\mu(3Q^+) = 0$, so by Lemma 4.2, $Q^+$ is OK. This is a contradiction. $\square$

**Remark.** For $\eta \in [1, 100]$ and $Q_i \in CZ^\circ$, $\eta Q_i \cap Q^\circ$ is a $C$-non-degenerate rectangular box, for $C = C(n)$, satisfying

$$\mathrm{diam}(\eta Q_i \cap Q^\circ) \simeq \delta_{Q_i}.$$

**Definition 4.3.** A cube $Q \in CZ^\circ$ is called a keystone cube if for any $Q' \in CZ^\circ$ with $Q' \cap 100Q \neq \emptyset$, we have $\delta_Q \leq \delta_{Q'}$.

We will use the following notation:

$$CZ^\circ = \{Q_i\}_{i\in I} := \{\text{the collection of CZ cubes}\} \in \Pi(Q^\circ);$$

$$K_{CZ} := \bigcup_{Q \in CZ^\circ} Q; \text{ and}$$

$$CZ_{key} = \{Q_s\}_{s \in \bar{I}} := \{Q_s \in CZ^\circ : Q_s \text{ is a keystone cube}\}.$$

For $i \in I$ ($I$ is the indexing set of $CZ^\circ$ as above), let $x_i$ denote the center of $Q_i$, i.e., $x_i := \mathrm{ctr}(Q_i)$. We denote the set of CZ basepoints by

$$\mathfrak{B}_{CZ} = \{x_i\}_{i \in I}.$$

**Lemma 4.7.** *Let $Q_s$ be a keystone cube. Then*

$$\left|\{s' \in \bar{I} : 10Q_s \cap 10Q_{s'} \neq \emptyset\}\right| \leq C.$$

**Proof.** Let $s, s' \in \bar{I}$ with $10Q_s \cap 10Q_{s'} \neq \emptyset$. Suppose, without loss of generality, $\delta_{Q_s} \leq \delta_{Q_{s'}}$. Then $10Q_s \cap 10Q_{s'} \neq \emptyset$ implies $100Q_{s'} \cap Q_s \neq \emptyset$. But because $Q_{s'}$ is keystone, this implies $\delta_{Q_{s'}} \leq \delta_{Q_s}$. So $\delta_{Q_s} = \delta_{Q_{s'}}$. There are at most $C$ dyadic cubes $Q_{s'}$ that satisfy this condition and $10Q_s \cap 10Q_{s'} \neq \emptyset$ for fixed $s$. This completes the proof. □

*4.2. Keystone points*

**Definition 4.4.** We define the set of keystone points as $K_p := Q^\circ \setminus K_{CZ}$.

**Lemma 4.8.** *The set $K_p$ is closed.*

**Proof.** Let $x \in K_p$. Then every dyadic cube containing $x$ is <u>not OK</u>. As a consequence of Lemma 4.2,

$$\mu(B(x,\eta)) = \infty \qquad (\text{for all } \eta > 0,\ x \in K_p). \tag{4.4}$$

To see this, suppose for sake of contradiction that $\mu(B(x,\eta)) \leq A < \infty$ for some $\eta > 0$. Then, consider a dyadic cube $Q \subset Q^\circ$ with $x \in Q$ and $3Q \subset B(x,\eta)$. Then $\mu(3Q) \leq A$. If $\delta_Q$ is sufficiently small, depending on $A, m, n, p$, it follows that $\mu(3Q) \leq (\epsilon_0/C_0)^p (30\delta_Q)^{n-mp}$, since $n - mp < 0$, and hence $Q$ is OK by Lemma 4.2. Thus, $x$ belongs to an OK cube, so $x$ belongs to a CZ cube, hence $x \in K_{CZ}$, contradicting that $x \in K_p$.

In combination with (3.50), (4.4) implies $K_p \subset \mathrm{Cl}(\frac{1}{10}Q^\circ) \subset \frac{1}{9}Q^\circ$.

Let $Q \in CZ^\circ$. We claim that for any $x \in 2Q \cap Q^\circ$, there exists $Q' \in CZ^\circ$ satisfying $x \in Q'$ – in particular, $2Q \cap Q^\circ \subset K_{CZ}$. To see this, let $Q'' \subset Q^\circ$ be a dyadic cube satisfying $x \in Q''$ and $\delta_{Q''} < \delta_Q/100$. Then $3Q'' \subset 3Q$. Since $Q$ is OK, we have that $Q''$ is OK, thanks to Lemma 4.1. Hence, by definition of the CZ decomposition, $Q''$ must have a dyadic ancestor $Q'$ with $Q' \in CZ^\circ$. Then $x \in Q'' \subset Q'$, completing the proof of the claim.

Let $x \in K_{CZ}$ be arbitrary. Then $x \in Q$ for some $Q \in CZ^\circ$. Note that

$$B(x, \delta_Q/3) \cap Q^\circ \subset 2Q \cap Q^\circ \subset K_{CZ}.$$

Hence, $K_{CZ}$ is relatively open in $Q^\circ$. Thus, $K_p = Q^\circ \setminus K_{CZ}$ is relatively closed in $Q^\circ$. Since $K_p \subset \frac{1}{9}Q^\circ$, we have that $K_p$ is closed. □

Below, we write $|S|$ for the Lebesgue measure of a measurable set $S \subset \mathbb{R}^n$.

**Lemma 4.9.** *If $\mathcal{A} \neq \emptyset$ then $|K_p| = 0$.*

**Proof.** Suppose for sake of contradiction that $|K_p| > 0$ and $\mathcal{A} \neq \emptyset$. Recall from (3.55) and (3.56) in Section 3.3.1 that for all $x \in Q^\circ$, there exists a family of polynomials, $(P_x^\alpha)_{\alpha\in\mathcal{A}}$ such that $(P_x^\alpha)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, x, C\epsilon/C_0, 1)$-basis for $\sigma_J(x,\mu)$ satisfying

$$|\partial^\beta P_x^\alpha(x)| \leq C \qquad (\alpha \in \mathcal{A},\ \beta \in \mathcal{M}). \tag{4.5}$$

Let $x_0 \in K_p$ be a Lebesgue point of $\chi_{K_p}$; then $\lim_{r\to 0} \frac{|K_p \cap B(x_0,r)|}{|B(x_0,r)|} = 1$. We will fix universal constants $c_1, c_2 \in (0,1)$ (determined only by $m$, $n$, and $p$) momentarily. Choose any $r \in (0,1)$ satisfying $r^{1-n/p} < c_2/(2\epsilon)$ and $\frac{|K_p\cap B(x_0,r)|}{|B(x_0,r)|} > 1 - c_1^n$. Since $\mathcal{A} \neq \emptyset$, we may fix $\alpha \in \mathcal{A}$. Let

$$\begin{aligned} M &:= \max_{\beta\in\mathcal{M}} \max_{z\in B(x_0,r)} |\partial^\beta P_{x_0}^\alpha(z)| r^{|\beta|}, \\ y &:= \arg \max_{z\in B(x_0,r/2)} |P_{x_0}^\alpha(z)|. \end{aligned}$$

By Bernstein's Inequality for polynomials, there exists a universal constant $C > 0$, such that $\|\partial^\beta P(z)\|_{L^\infty(B(x_0,r))} \leq Cr^{-|\beta|}\|P\|_{L^\infty(B(x_0,r/2))}$ for all multi-indices $\beta \in \mathcal{M}$ and $(m-1)^{\text{rst}}$ degree polynomials $P \in \mathcal{P}$. Thus, by the definition of $M$,

$$|P_{x_0}^\alpha(y)| = \|P_{x_0}^\alpha\|_{L^\infty(B(x_0,r/2))} \geq c'M \tag{4.6}$$

for a universal constant $c' \in (0,1)$. For any $\eta \in (0,1/2)$ we have $B(y,\eta r) \subset B(x_0,r)$. Thus, since $|\nabla P_{x_0}(z)| \leq \sqrt{n}M/r$ for $z \in B(x_0,r)$ (see the definition of $M$), by the mean value theorem,

$$|P_{x_0}^\alpha(y) - P_{x_0}^\alpha(x)| \leq (\sqrt{n}M/r)|y-x| \leq (\sqrt{n}M/r)\eta r = \sqrt{n}M\eta \ \text{ for } x \in B(y,\eta r).$$

We fix the universal constant $c_1 \in (0,1/2)$ given by $c_1 = \frac{c'}{4\sqrt{n}}$. Taking $\eta = c_1$ in the above, and using (4.6), we learn that $B(y,c_1 r) \subset B(x_0,r)$, and

$$|P_{x_0}^\alpha(x)| \geq |P_{x_0}^\alpha(y)| - |P_{x_0}^\alpha(x) - P_{x_0}^\alpha(y)| > c'M/2 \ \text{ for } x \in B(y,c_1 r).$$

Furthermore, by the basis property of $(P_{x_0}^\alpha)$, we have $\partial^\alpha P_{x_0}^\alpha(x_0) = 1$, and so $M \geq r^{|\alpha|} \geq r^{m-1}$ (see the definition of $M$). Hence,

$$|P_{x_0}^\alpha(x)| > c'r^{m-1}/2 \ \text{ for } x \in B(y,c_1 r). \tag{4.7}$$

Because $P_{x_0}^\alpha \in C\epsilon\cdot\sigma_J(x_0,\mu)$, there exists $\varphi_{x_0}^\alpha \in L^{m,p}(\mathbb{R}^n)$ satisfying $J_{x_0}\varphi_{x_0}^\alpha = P_{x_0}^\alpha$ and $\|\varphi_{x_0}^\alpha\|_{L^{m,p}(\mathbb{R}^n)} \leq \|\varphi_{x_0}^\alpha\|_{\mathscr{J}(0,\mu)} \leq C\epsilon$. If $x \in B(y,c_1 r)$ then $x \in B(x_0,r)$, so $|x - x_0| \leq r$. By the Sobolev Inequality and (4.7),

$$\begin{aligned} |\varphi_{x_0}^\alpha(x)| &\geq |P_{x_0}^\alpha(x)| - C\|\varphi_{x_0}^\alpha\|_{L^{m,p}(\mathbb{R}^n)} r^{m-n/p} \\ &> c'r^{m-1}/2 - C'\epsilon r^{m-n/p} \end{aligned}$$

$$= C' r^{m-1}(c_2 - \epsilon r^{1-n/p}) \geq C' c_2 r^{m-1}/2, \quad \text{for } x \in B(y, c_1 r), \tag{4.8}$$

where we now fix the universal constant $c_2 := c'/(2C')$ so that the equality in the last line is valid, and we recall our choice of $r$ satisfying $r^{1-n/p} < c_2/(2\epsilon)$ to justify the inequality in the last line.

Because $\frac{|K_p \cap B(x_0,r)|}{|B(x_0,r)|} > 1 - c_1^n$ and $B(y, c_1 r) \subset B(x_0, r)$, we have that $K_p \cap \operatorname{int}(B(y, c_1 r)) \neq \emptyset$. Thus, by (4.4) we learn that $\mu(B(y, c_1 r)) = \infty$. In combination with (4.8), this implies

$$\int_{B(y,c_1 r)} |\varphi_{x_0}^{\alpha}|^p d\mu = \infty,$$

contradicting $\|\varphi_{x_0}^{\alpha}\|_{\mathscr{J}(0,\mu)} \leq C\epsilon$. This completes the proof by contradiction, and the lemma follows. $\square$

**Lemma 4.10.** *For $Q_i \in CZ^\circ$, we have $\operatorname{int}(3Q_i) \cap K_p = \emptyset$, and consequently $\delta_{Q_i} \leq \operatorname{dist}(Q_i, K_p)$.*

**Proof.** For sake of contradiction suppose there exists $Q_i \in CZ^\circ$ with $\operatorname{int}(3Q_i) \cap K_p \neq \emptyset$. Then there exists a dyadic cube $Q$ satisfying $3Q \subset 3Q_i$ and $Q \cap K_p \neq \emptyset$. The cube $Q_i$ is OK, thus $Q$ is OK, thanks to Lemma 4.1. Thus, $Q \subset K_{CZ}$, so $Q \cap K_p$ must be empty, a contradiction. $\square$

**Lemma 4.11.** *There exist universal constants $C \geq 1$ and $c \in (0,1)$ such that the following holds. Let $Q \in CZ^\circ$. Then there exists a sequence of cubes, $\mathscr{S}(Q) \subset CZ^\circ$, that either is (i) finite, satisfying, $\mathscr{S}(Q) = \{Q^k\}_{k=1}^L$, where $Q^L$ is a keystone cube, or (ii) infinite, satisfying $\mathscr{S}(Q) = \{Q^k\}_{k \in \mathbb{N}}$ and for $\operatorname{ctr}(Q^k) = x^k$, $\lim_{k\to\infty} x^k = x' \in K_p$. Regardless, the sequence $\mathscr{S}(Q) = \{Q^k\}$ satisfies*

$$Q^1 = Q, \tag{4.9}$$

$$Q^k \leftrightarrow Q^{k+1} \qquad (\textit{for } Q^k, Q^{k+1} \in \mathscr{S}(Q)); \ \textit{and} \tag{4.10}$$

$$\delta_{Q^k} \leq C \cdot c^{k-\ell} \delta_{Q^\ell} \qquad (\textit{for } Q^k, Q^\ell \in \mathscr{S}(Q),\ k \geq \ell). \tag{4.11}$$

**Proof.** Let $Q \in CZ^\circ$. If $Q$ is a keystone cube, then $\mathscr{S}(Q) = \{Q^1 = Q\}$ satisfies the conclusion of the lemma.

Suppose $Q$ is not a keystone cube; then there exists $Q' \in CZ^\circ$ satisfying $100Q \cap Q' \neq \emptyset$ and $\delta_{Q'} \leq \frac{1}{2}\delta_Q$. Let $Q^{1,0} := Q$ and

$$Q^{2,0} \in \arg\min_{Q'} \left\{ \operatorname{dist}(Q^{1,0}, Q') : \begin{array}{l} Q' \in CZ^\circ,\ 100Q^{1,0} \cap Q' \neq \emptyset, \\ \text{and } \delta_{Q'} \leq (1/2)\delta_{Q^{1,0}} \end{array} \right\}.$$

We call $Q^{2,0}$ constructed by this procedure a “junior partner” of $Q^{1,0}$.

Let $s : [0,1] \to \mathbb{R}^n$ be an affine map with $s(0) \in \mathrm{Cl}(Q^{1,0})$, $s(1) \in \mathrm{Cl}(Q^{2,0})$, and $|s(1) - s(0)|_2 = \mathrm{dist}_2(Q^{1,0}, Q^{2,0})$. Then $s((0,1))$ meets finitely many CZ cubes, $Q^{1,1}, \dots, Q^{1,K_1}$, where $K_1 < C$, $Q^{1,k} \leftrightarrow Q^{1,k+1}$, $Q^{1,K_1} \leftrightarrow Q^{2,0}$, and $\delta_Q \le \delta_{Q^{1,i}} \le C\delta_Q$ for $i \in 1, \dots, K_1$, and $\delta_{Q^{2,0}} \le \frac{1}{2}\delta_{Q^{1,0}}$. Hence, after removing repeated cubes, $\{Q = Q^{1,0}, Q^{1,1}, \dots, Q^{1,K_1}, Q^{2,0}\}$ satisfies $Q^{1,k} \leftrightarrow Q^{1,k+1}$, and $\delta_{Q^{1,k}} \le C \cdot c^{k-l}\delta_{Q^{1,l}}$ for $0 \le l \le k$. If $Q^{2,0}$ is a keystone cube, we stop, producing a finite sequence $\mathscr{S}(Q)$ (which we relabel $\{Q^k\}_{k=1}^L$). Otherwise, we repeat this process.

We construct $\mathscr{S}(Q)$ by concatenating the sequences of cubes connecting successive junior partners, $\mathscr{S}(Q) := \{Q^{1,0}, \dots, Q^{1,K_1}, Q^{2,0}, \dots, Q^{2,K_2}, Q^{3,0}, \dots\}$. Relabel $\mathscr{S}(Q) = \{Q^k\}_{k=1}^L$ if a junior partner is keystone, stopping the process and producing a finite sequence that terminates at a keystone cube. Otherwise, if no junior partner is keystone, relabel $\mathscr{S}(Q) = \{Q^k\}_{k\in\mathbb{N}}$. By construction, $\mathscr{S}(Q)$ satisfies (4.9) and (4.10), Also, $\mathscr{S}(Q)$ satisfies (4.11) because the subsequence of junior partners in $\mathscr{S}(Q)$ satisfies $\delta_{Q^{k,0}} \le 2^{j-k}\delta_{Q^{j,0}}$ for $k > j$, and because there are at most $C$ many cubes in $\mathscr{S}(Q)$ connecting consecutive junior partners.

It remains to verify conclusion (ii) of the lemma in the event that $\mathscr{S}(Q)$ is infinite. Suppose $\mathscr{S}(Q)$ is infinite. Let $\mathscr{S}(Q) = \{Q^k\}_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$, define $x^k := \mathrm{ctr}(Q^k)$. Because of (4.11),

$$\lim_{k\to\infty} \delta_{Q^k} = 0. \tag{4.12}$$

Because of (4.10) and (4.11), we have for $j > k$,

$$\begin{aligned} |x^k - x^j| &\le 3\sum_{i=k}^{j} \delta_{Q^i} \\ &\le 3\sum_{i=k}^{j} Cc^{i-k}\delta_{Q^k} \\ &\lesssim \delta_{Q^k}. \end{aligned}$$

In light of (4.12), this proves the sequence $\{x^k\}_{k\in\mathbb{N}}$ is Cauchy, and there exists $x' \in \mathrm{Cl}(Q^\circ)$ such that $x^k \xrightarrow{k\to\infty} x'$. Because of (4.3), $x' \notin \partial Q^\circ$.

It remains to show that $x' \in K_p$. Suppose for sake of contradiction that $x' \notin K_p$; then $x' \in Q'$ for some $Q' \in CZ^\circ$. Since $x^k = \mathrm{ctr}(Q^k) \to x' \in Q'$ as $k \to \infty$, there exists $K_0 \in \mathbb{N}$ such that for $k > K_0$, $Q^k$ belongs to the neighbor set of $Q'$, $Q^k \in \mathcal{N}(Q') := \{Q'' \in CZ^\circ : Q'' \leftrightarrow Q'\}$. But by Good Geometry of $CZ^\circ$, any cube $Q'' \in \mathcal{N}(Q')$ has sidelength $\delta_{Q''} \ge \frac{1}{2}\delta_{Q'}$, contradicting (4.12). Hence, $x' \in K_p$, as desired. $\square$

### 4.3. Partition of unity

We have defined the collection of CZ cubes, $CZ^\circ = \{Q_i\}_{i\in I}$, such that $K_{CZ} = \bigcup_{i\in I} Q_i$. Due to the Good Geometry of $CZ^\circ$, (4.1), (4.2), and (4.3), there exists a partition of unity, $\{\theta_i\}_{i\in I} \subset C^\infty(K_{CZ})$, satisfying

**(POU1)** $0 \leq \theta_i \leq 1$;
**(POU2)** $\theta_i$ vanishes on $K_{CZ} \setminus (1.1)Q_i$;
**(POU3)** $|\partial^\alpha \theta_i| \leq C\delta_{Q_i}^{-|\alpha|}$ whenever $|\alpha| \leq m$; and
**(POU4)** $\sum_{i\in I} \theta_i \equiv 1$ on $K_{CZ}$.

For the construction of such a partition of unity, see [24], page 170.

### 4.4. Local extension operators

In this section, we apply the inductive hypothesis to construct local extension operators for functions defined on Borel subsets $E_i$ of $3Q_i$.

Let $Q_i \in CZ^\circ$ and $E_i \subset 3Q_i$ be Borel. Because $Q_i$ is OK, there exists $\mathcal{A}_i < \mathcal{A}$ such that for all $x \in \mathrm{supp}(\mu)\cap 3Q_i$, $\sigma_J(x, \mu|_{3Q_i})$ contains an $(\mathcal{A}_i, x, \epsilon/C_0, 30\delta_{Q_i})$-basis. Because $E_i \subset 3Q_i$ we have $\sigma_J(x, \mu|_{3Q_i}) \subset \sigma_J(x, \mu|_{E_i})$. Since $\epsilon < \epsilon_0$, for all $x \in \mathrm{supp}(\mu|_{E_i})$,

$$\sigma_J(x, \mu|_{E_i}) \text{ contains an } (\mathcal{A}_i, x, \epsilon_0/C_0, 10 \cdot (3\delta_{Q_i}))\text{-basis.}$$

The restricted measure $\mu|_{E_i}$ is Borel regular, and $\mathrm{diam}(\mathrm{supp}(\mu|_{E_i})) \leq 3\delta_{Q_i}$ (since $E_i \subset 3Q_i$), so by the consequence of the inductive hypothesis, (3.53), the Extension Theorem for $(\mu|_{E_i}, 3\delta_{Q_i})$ holds. Because the seminorms for the spaces $\mathscr{J}(\mu|_{E_i}; 3\delta_{Q_i})$ and $\mathscr{J}(\mu|_{E_i}; \delta_{Q_i})$ are equivalent up to universal constant factors, we have that the Extension Theorem for $(\mu|_{E_i}, \delta_{Q_i})$ holds.

Thus, there exist a linear map $T_i : \mathscr{J}(\mu|_{E_i}; \delta_{Q_i}) \to L^{m,p}(\mathbb{R}^n)$, a map $M_i : \mathscr{J}(\mu|_{E_i}; \delta_{Q_i}) \to \mathbb{R}_+$, and countable collections of Borel sets $\{A^i_\ell\}_{\ell\in\mathbb{N}}, A^i_\ell \subset \mathrm{supp}(\mu|_{E_i})$, and of linear maps $\{\phi^i_\ell : \mathscr{J}(\mu|_{E_i}; \delta_{Q_i}) \to \mathbb{R}\}_{\ell\in\mathbb{N}}$, and $\{\lambda^i_\ell : \mathscr{J}(\mu|_{E_i}; \delta_{Q_i}) \to L^p(d\mu)\}_{\ell\in\mathbb{N}}$, that satisfy for each $(f, P) \in \mathscr{J}(\mu|_{E_i}; \delta_{Q_i})$:

**(AL1)** $\|f, P\|_{\mathscr{J}(\mu|_{E_i}; \delta_{Q_i})} \leq \|T_i(f, P), P\|_{\mathscr{J}(f, \mu|_{E_i}; \delta_{Q_i})} \leq C \cdot \|f, P\|_{\mathscr{J}(\mu|_{E_i}; \delta_{Q_i})}$;

**(AL2)** $c \cdot M_i(f, P) \leq \|T_i(f, P), P\|_{\mathscr{J}(f, \mu|_{E_i}; \delta_{Q_i})} \leq C \cdot M_i(f, P)$; and

**(AL3)** $M_i(f, P) = \Big(\sum_{\ell\in\mathbb{N}} \int_{A^i_\ell} |\lambda^i_\ell(f, P) - f|^p d\mu + \sum_{\ell\in\mathbb{N}} |\phi^i_\ell(f, P)|^p\Big)^{1/p}$.

We obtain the particular form **(AL3)** for $M_i$ because $\mathcal{A}_i \neq \emptyset$, a consequence of the fact that $\mathcal{A}_i < \mathcal{A}$ and $\emptyset$ is maximal under the order on multi-index sets. Thus, the map $M_i$ has the form (3.7).

Further, from the Extension Theorem for $(\mu|_{E_i}, \delta_{Q_i})$ we know that the maps $T_i$ and $M_i$ are $\Omega_i'$-constructible. Thus there exists a collection of linear functionals $\Omega_i' = \{\omega_s^i\}_{s\in\Upsilon^i} \subset \mathscr{J}(\mu|_{E_i})^*$, satisfying that the collection of sets $\{\mathrm{supp}(\omega_s^i)\}_{s\in\Upsilon^i}$ has $C$-bounded overlap, and for each $x \in \mathbb{R}^n$, there exists a finite subset $\Upsilon_x^i \subset \Upsilon^i$ and a collection of polynomials $\{v_{s,x}^i\}_{s\in\Upsilon_x^i} \subset \mathcal{P}$ such that $|\Upsilon_x^i| \le C$ and

$$\textbf{(AL4)}\ J_x T_i(f,P) = \sum_{s\in\Upsilon_x^i} \omega_s^i(f)\cdot v_{s,x}^i + \widetilde{\omega}_x^i(P),$$

where $\widetilde{\omega}_x^i : \mathcal{P} \to \mathcal{P}$ is a linear map.

Similarly, for each $\ell \in \mathbb{N}$ and $y \in \mathrm{supp}(\mu)$, there exists a finite subset $\bar{\Upsilon}_{\ell,y}^i \subset \Upsilon^i$ and constants $\{\eta_{s,y}^{\ell,i}\}_{s\in\bar{\Upsilon}_{\ell,y}^i} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_{\ell,y}^i| \le C$, and the map $(f,P) \mapsto \lambda_\ell(f,P)(y)$ has the form

$$\textbf{(AL5)}\ \lambda_\ell^i(f,P)(y) = \sum_{s\in\bar{\Upsilon}_{\ell,y}^i} \eta_{s,y}^{\ell,i}\cdot\omega_s^i(f) + \widetilde{\lambda}_{y,\ell}^i(P),$$

where $\widetilde{\lambda}_{y,\ell}^i : \mathcal{P} \to \mathbb{R}$ is a linear functional.

And for each $\ell \in \mathbb{N}$, there exists a finite subset $\bar{\Upsilon}_\ell^i \subset \Upsilon^i$ and constants $\{\eta_s^{\ell,i}\}_{s\in\bar{\Upsilon}_\ell^i} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_\ell^i| \le C$, and the map $\phi_\ell^i$ has the form

$$\textbf{(AL6)}\ \phi_\ell^i(f,P) = \sum_{s\in\bar{\Upsilon}_\ell^i} \eta_s^{\ell,i}\cdot\omega_s^i(f) + \widetilde{\lambda}_\ell^i(P),$$

where $\widetilde{\lambda}_\ell^i : \mathcal{P} \to \mathbb{R}$ is a linear functional.

**Remark 4.1.** Notice that $\mathscr{J}(\mu) \subset \mathscr{J}(\mu|_{E_i})$ with an inequality of norms, i.e., $\|f\|_{\mathscr{J}(\mu|_{E_i})} \le \|f\|_{\mathscr{J}(\mu)}$ for any Borel measurable $f : \mathbb{R}^n \to \mathbb{R}$. Thus, $\mathscr{J}(\mu|_{E_i})^*$ naturally embeds in $\mathscr{J}(\mu)^*$. In particular, the $\Omega_i'$ may be regarded as a family of functionals in $\mathscr{J}(\mu)^*$.

If $\omega \in \mathscr{J}(\mu)^*$ then $\mathrm{supp}(\omega) \subset \mathrm{supp}(\mu)$. Therefore, for any choice of $E_i \subset 3Q_i$, the collection of functionals $\Omega_i' = \{\omega_s^i\}_{s\in\Upsilon^i}$ satisfies

$$\mathrm{supp}(\omega_s^i) \subset \mathrm{supp}(\mu|_{E_i}) \subset \mathrm{supp}(\mu)\cap \mathrm{Cl}(E_i) \quad \text{for all } i \in I, s \in \Upsilon^i. \tag{4.13}$$

## 5. Preliminary estimates and technical tools

### 5.1. Estimates for auxiliary polynomials

Recall from (3.55) and (3.56), we have constructed $(P_x^\alpha)_{\alpha\in\mathcal{A}}$, for all $x \in 100Q^\circ$, such that

$$\begin{aligned}&(P_x^\alpha)_{\alpha\in\mathcal{A}} \text{ forms an } (\mathcal{A}, x, \bar{C}\epsilon, 1)\text{-basis for } \sigma_J(x,\mu), \text{ with}\\ &|\partial^\beta P_x^\alpha(x)| \le \bar{C} \text{ for } \alpha\in\mathcal{A}, \beta\in\mathcal{M}.\end{aligned} \tag{5.1}$$

Here, $\bar{C}$ is a universal constant, determined only by $m$, $n$, $p$. As in Section 12 of [13], we have:

**Lemma 5.1.** *There exists a universal constant $C_3$ such that the following holds. Let $Q \in CZ^\circ$, and let $\widehat{\delta} \in [\delta_Q/2, 1]$. Then*

$$|\partial^\beta P_y^\alpha(y)| \le C_3 \widehat{\delta}^{|\alpha|-|\beta|} \ \ (\alpha\in\mathcal{A},\ \beta\in\mathcal{M},\ y\in 3Q^+). \tag{5.2}$$

*In particular,*

$$|\partial^\beta P_y^\alpha(y)| \le C_3 \delta_Q^{|\alpha|-|\beta|} \ \ (\alpha\in\mathcal{A},\ \beta\in\mathcal{M},\ y\in 3Q^+). \tag{5.3}$$

**Proof.** Inequality (5.3) follows from (5.2) by setting $\widehat{\delta} = \delta_Q$.

We now prove inequality (5.2).

If $\widehat{\delta} \in [1/4, 1]$ then (5.2) follows from (5.1) because $3Q^+ \subset 10Q \subset 100Q^\circ$ for $Q \in CZ^\circ$. So we may assume $\widehat{\delta} \in [\delta_Q/2, 1/4]$.

Let $C' := 30^m \bar{C}$. By (5.1) and Lemma 3.2, $(P_x^\alpha)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, x, C'\epsilon, 30)$-basis for $\sigma_J(x,\mu)$ for $x \in 100Q^\circ$. By (3.5), since $120\widehat{\delta} \le 30$, $\sigma_J(x,\mu) \subset \sigma_J(x, \mu|_{3Q^+})$, and $10Q \subset 100Q^\circ$, we have that

$$(P_x^\alpha)_{\alpha\in\mathcal{A}} \text{ forms an } (\mathcal{A}, x, C'\epsilon, 120\widehat{\delta})\text{-basis for } \sigma_J(x,\mu|_{3Q^+}) \text{ for all } x \in 10Q. \tag{5.4}$$

Define $\epsilon_1 = \min\{c_1, \epsilon_0/(C_0C_1)\}$, a universal constant, where $c_1, C_1$ are the constants from Lemma 3.6, and $C_0$ is the constant from Lemma 3.3. For the sake of contradiction, suppose (5.2) fails to hold for a sufficiently large constant $C_3$. Thus, we may assume

$$\max\{|\partial^\beta P_z^\alpha(z)| \cdot (120\widehat{\delta})^{|\beta|-|\alpha|} : z \in 3Q^+,\ \alpha\in\mathcal{A},\ \beta\in\mathcal{M}\} > \epsilon_1^{-D-1}. \tag{5.5}$$

By taking $\epsilon$ small enough, we may assume that $C'\epsilon \le \epsilon_1^{2D+2}$. We claim that the hypotheses of Lemma 3.6 hold with the parameters

$$\left(\epsilon_1, \epsilon_2, \delta, \mathcal{A}, \mu, E, \left(\widetilde{P}_x^\alpha\right)_{\alpha\in\mathcal{A}, x\in E}\right) := \left(\epsilon_1, C'\epsilon, 120\widehat{\delta}, \mathcal{A}, \mu|_{3Q^+}, \mathrm{Cl}(3Q^+), \left(P_x^\alpha\right)_{\alpha\in\mathcal{A}, x\in 3Q^+}\right).$$

Specifically, we have to check the conditions **(D1)-(D5)** in Lemma 3.5 and condition (3.31) for the above choice of parameters. Note that **(D1)** is satisfied because $\epsilon_1 \le c_1$ and $C'\epsilon \le \epsilon_1^{2D+2}$, while **(D2)** and **(D3)** do not mention any conditions on the parameters, hence are trivially satisfied. Further, **(D4)** is satisfied because $\mathrm{supp}(\mu|_{3Q^+}) \subset \mathrm{Cl}(3Q^+)$ and $\mathrm{diam}(3Q^+) = 6\delta_Q \le 120\widehat{\delta}$. Note **(D5)** is satisfied due to (5.4), since $\mathrm{Cl}(3Q^+) \subset 10Q$. Finally, (3.31) is satisfied thanks to (5.5).

Thus, we may apply Lemma 3.6 to deduce that there exists $\overline{\mathcal{A}} < \mathcal{A}$ such that $\sigma_J(x, \mu|_{3Q^+})$ contains an $(\overline{\mathcal{A}}, x, C_1\epsilon_1, 120\widehat{\delta})$-basis for all $x \in 3Q^+$. We apply (3.5), using that $\delta_{Q^+} = 2\delta_Q \leq 4\widehat{\delta}$ and $C_1\epsilon_1 \leq \epsilon_0/C_0$, to deduce that $\sigma_J(x, \mu|_{3Q^+})$ contains an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 30\delta_{Q^+})$-basis for all $x \in 3Q^+$. Therefore, the cube $Q^+$ is OK (see Definition 4.1), contradicting that $Q \in CZ^\circ$. This completes the proof of (5.2). $\square$

**Lemma 5.2.** *There exists a universal constant $C_4$ such that the following holds. Let $x \in K_p$. For $\delta \in (0,1)$,*

$$|\partial^\beta P_x^\alpha(x)| \leq C_4 \delta^{|\alpha|-|\beta|} \qquad (\alpha \in \mathcal{A}, \beta \in \mathcal{M}). \tag{5.6}$$

*In particular $\partial^\beta P_x^\alpha(x) = 0$ for $|\alpha| > |\beta|$.*

**Proof.** Let $x \in K_p$ and $\delta \in (0,1)$. Recall from (5.1), we have

$$|\partial^\beta P_x^\alpha(x)| \leq C \qquad (\alpha \in \mathcal{A},\ \beta \in \mathcal{M}). \tag{5.7}$$

For $|\beta| \geq |\alpha|$, (5.7) implies (5.6). So it suffices to prove (5.6) for $|\alpha| > |\beta|$. We will prove that $\partial^\beta P_x^\alpha(x) = 0$ for $|\alpha| > |\beta|$. This will complete the proof of (5.6), and with it, the proof of the lemma.

Suppose for sake of contradiction that there exist $\beta \in \mathcal{M}$, $\alpha \in \mathcal{A}$, $|\alpha| > |\beta|$ such that $|\partial^\beta P_x^\alpha(x)| > \eta > 0$. Fix $\epsilon_1 < \min\{c_1, \epsilon_0/(30^m C_0 C_1)\}$, for $c_1$ and $C_1$ as in Lemma 3.5, and $C_0$ is the constant from Lemma 3.3. Fix $\delta_1 < \eta\epsilon_1^{D+1}$ with $\delta_1 < 1$. Fix a dyadic cube $Q \subset Q^\circ$ with $x \in Q$ and $\delta_Q < \delta_1$. Observe that $\delta_Q^{|\beta|-|\alpha|} > \delta_1^{|\beta|-|\alpha|} \geq \delta_1^{-1}$. So,

$$\begin{aligned}\max\left\{|\partial^{\hat\beta} P_y^{\hat\alpha}(y)|\delta_Q^{|\hat\beta|-|\hat\alpha|} : y \in 3Q,\ \hat\alpha \in \mathcal{A},\ \hat\beta \in \mathcal{M}\right\} &\geq |\partial^\beta P_x^\alpha(x)|\delta_Q^{|\beta|-|\alpha|} \\ &> \eta\delta_1^{-1} > \epsilon_1^{-D-1}.\end{aligned} \tag{5.8}$$

We fix $\epsilon_2 < \epsilon_1^{2D+2}$. From (5.1), and since $5Q \subset 100Q^\circ$, $\left(P_y^\alpha\right)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, y, C\epsilon, 1)$-basis for $\sigma_J(y, \mu)$ for all $y \in 5Q$. We can assume $C\epsilon < \epsilon_2$. Note that $\delta_Q < \delta_1 < 1$. From (3.5), $\left(P_y^\alpha\right)_{\alpha\in\mathcal{A}}$ forms an $(\mathcal{A}, y, \epsilon_2, \delta_Q)$-basis for $\sigma_J(y, \mu|_{3Q})$ for all $y \in 5Q$. In combination with (5.8), we see the hypotheses of Lemma 3.6 hold with parameters

$$\left(\epsilon_1, \epsilon_2, \delta, \mathcal{A}, \mu, E, \left(\widetilde{P}_x^\alpha\right)_{\alpha\in\mathcal{A}, x\in E}\right) = \left(\epsilon_1, \epsilon_2, \delta_Q, \mathcal{A}, \mu|_{3Q}, \mathrm{Cl}(3Q), (P_x^\alpha)_{\alpha\in\mathcal{A}, x\in 3Q}\right).$$

Hence there exists $\overline{\mathcal{A}} < \mathcal{A}$ so that for every $y \in 3Q$, $\sigma_J(y, \mu|_{3Q})$ contains an $(\overline{\mathcal{A}}, y, C_1\epsilon_1, \delta_Q)$-basis. Because $C_1\epsilon_1 < \epsilon_0/(30^m C_0)$, we apply Lemma 3.2 to deduce that for every $y \in 3Q$, $\sigma_J(y, \mu|_{3Q})$ contains an $(\overline{\mathcal{A}}, y, \epsilon_0/C_0, 30\delta_Q)$-basis, indicating that $Q$ is OK, thus $Q$ is contained in a CZ cube. But $x \in Q$, so $x \in K_{CZ}$. This contradicts that $x \in K_p$, completing the proof of (5.6) by contradiction. $\square$

In Proposition 3.9 of Section 3.3.1, we defined a family of functions $(\varphi_x^\alpha)_{\alpha\in\mathcal{A},x\in Q^\circ} \subset L^{m,p}(\mathbb{R}^n)$, related to the $(P_x^\alpha)_{\alpha\in\mathcal{A},x\in Q^\circ}$, satisfying (3.69)-(3.72). In particular, $J_x\varphi_x^\alpha = P_x^\alpha$ for $x \in Q^\circ$, $\alpha \in \mathcal{A}$.

**Lemma 5.3.** *We have*

$$P_x^\alpha(x) = \varphi_x^\alpha(x) = 0 \quad \text{for } x \in K_p,\ \alpha \in \mathcal{A}.$$

**Proof.** Recall that $\mathcal{A} \subset \mathcal{M}$, $\mathcal{A} \neq \mathcal{M}$, and $\mathcal{A}$ is monotonic. Therefore, $\mathcal{A}$ does not contain the zero multi-index. Hence, $|\alpha| > 0$ for $\alpha \in \mathcal{A}$. Due to (5.6), we have $|P_x^\alpha(x)| \le C\delta^{|\alpha|}$ for all $\delta \in (0,1)$. Hence, $P_x^\alpha(x) = 0$. The result follows. □

*5.2. Estimates for local solutions*

Recall the norm defined on $\mathcal{P}$:

$$|P|_{x,\delta} = \Big( \sum_{|\alpha|\le m-1} |\partial^\alpha P(x)|^p \cdot \delta^{n+(|\alpha|-m)p} \Big)^{1/p}.$$

Recall $I$ is the indexing set for the CZ decomposition $CZ^\circ = \{Q_i\}_{i\in I}$. For $P \in \mathcal{P}$ and $i \in I$, define

$$|P|_i := |P|_{x_i,\delta_{Q_i}}. \tag{5.9}$$

By applying (2.14) to the measure $\mu|_{Q_i}$ and domain $U = Q_i$, for $F \in L^{m,p}(\mathbb{R}^n)$ and $x \in Q_i$,

$$|J_xF - P|_{x,\delta_{Q_i}} \le C\|F,P\|_{\mathscr{J}_*(f,\mu|_{Q_i};Q_i)}. \tag{5.10}$$

**Lemma 5.4.** *Let $Q_i \in CZ^\circ$. Then for $P \in \mathcal{P}$,*

$$\|0,P\|_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} \lesssim |P|_i = |P|_{x_i,\delta_{Q_i}}. \tag{5.11}$$

*Moreover, if $P \in \mathcal{P}$ satisfies $\partial^\alpha P(x_i) = 0$ for all $\alpha \in \mathcal{A}$, then*

$$\|0,P\|_{\mathscr{J}(\mu|_{9Q_i};\delta_{Q_i})} \simeq |P|_i = |P|_{x_i,\delta_{Q_i}}. \tag{5.12}$$

**Proof.** For the proof of (5.11), let $F_0 = (1-\theta)P$, where $\theta \in C^\infty(\mathbb{R}^n)$, $\theta|_{9Q_i} = 1$, $\mathrm{supp}(\theta) \subset 10Q_i$, and $|\partial^\alpha\theta(x)| \le C\delta_{Q_i}^{-|\alpha|}$. Then

$$\begin{aligned}\|0,P\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} &\le \|0,P\|^p_{\mathscr{J}(\mu|_{9Q_i};\delta_{Q_i})} \\ &\le \|F_0,P\|^p_{\mathscr{J}(0,\mu|_{9Q_i};\delta_{Q_i})}\end{aligned}$$

$$= \|F_0\|^p_{L^{m,p}(\mathbb{R}^n)} + \int\limits_{9Q_i} |F_0|^p d\mu + \|F_0 - P\|^p_{L^p(\mathbb{R}^n)}/\delta_{Q_i}^{mp}$$

$$= \|F_0 - P\|^p_{L^{m,p}(\mathbb{R}^n)} + \int\limits_{9Q_i} |F_0|^p d\mu + \|F_0 - P\|^p_{L^p(\mathbb{R}^n)}/\delta_{Q_i}^{mp}.$$

Note $F_0 = 0$ on $9Q_i$, and $F_0 = P$ on $\mathbb{R}^n \setminus 10Q_i$. Thus, continuing from the above bounds, and using the Taylor expansion $P(x) = \sum_{|\alpha| \le m-1} P(x_i) \cdot (x - x_i)^\alpha/\alpha!$,

$$\begin{aligned}\|0, P\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} &\le \|\theta P\|^p_{L^{m,p}(10Q_i)} + \|\theta P\|^p_{L^p(10Q_i)}/\delta_{Q_i}^{mp} \\ &\le C|P|_i^p.\end{aligned}$$

This completes the proof of (5.11).

We now prove (5.12). Note, the proof of (5.11) above shows that $\|0, P\|_{\mathscr{J}(\mu|_{9Q_i};\delta_{Q_i})} \lesssim |P|_i$. Thus, it suffices to establish the reverse inequality. We assume for contradiction that there exists $P' \in \mathcal{P}$, $P' \neq 0$ satisfying $\partial^\alpha P'(x_i) = 0$ for all $\alpha \in \mathcal{A}$ and

$$\|0, P'\|_{\mathscr{J}(\mu|_{9Q_i};\delta_{Q_i})} \le \epsilon_1^{D+1}|P'|_i, \tag{5.13}$$

for a universal constant $\epsilon_1$. We will later choose $\epsilon_1 > 0$ small enough so that we reach a contradiction. Define

$$P := P' \cdot \left(\max_{\beta \in \mathcal{M}} \left\{|\partial^\beta P'(x_i)|\delta_{Q_i}^{|\beta|+n/p-m}\right\}\right)^{-1}.$$

Note that

$$\max_{\beta \in \mathcal{M}} \left\{|\partial^\beta P(x_i)|\delta_{Q_i}^{|\beta|+n/p-m}\right\} = 1, \tag{5.14}$$

and thus $|P|_i = |P|_{x_i,\delta_{Q_i}} \le C$. Further, since $P = \gamma P'$ for some $\gamma \in \mathbb{R}$, from (5.13), $\|0, P\|_{\mathscr{J}(\mu|_{9Q_i};\delta_{Q_i})} \le \epsilon_1^{D+1}|P|_i \le C\epsilon_1^{D+1}$, and so

$$P \in C\epsilon_1^{D+1}\sigma(\mu|_{9Q_i}, \delta_{Q_i}). \tag{5.15}$$

Also, $P$ satisfies $\partial^\alpha P(x_i) = 0$ for all $\alpha \in \mathcal{A}$. For each integer $\ell \ge 0$, we define $\Delta_\ell \subset \mathcal{M}$ by

$$\Delta_\ell = \left\{\alpha \in \mathcal{M} : |\partial^\alpha P(x_i)|\, \delta_{Q_i}^{|\alpha|+n/p-m} \in (\epsilon_1^\ell, 1]\right\}.$$

Note that $\Delta_\ell \subset \Delta_{\ell+1}$ for each $\ell \ge 0$ and that $\Delta_\ell \neq \emptyset$ for $\ell \ge 1$. Since $\mathcal{M}$ contains $D$ elements, there exists $0 \le \ell_* \le D$ with $\Delta_{\ell_*} = \Delta_{\ell_*+1} \neq \emptyset$. Let $\bar{\alpha} \in \mathcal{M}$ be the maximal element of $\Delta_{\ell_*}$. Because $P$ satisfies $\partial^\alpha P(x_i) = 0$, we have $\bar{\alpha} \notin \mathcal{A}$. Further, because $\bar{\alpha} \in \Delta_{\ell_*}$, we have

$$\left|\partial^{\bar{\alpha}} P(x_i)\right| \delta_{Q_i}^{|\bar{\alpha}|+n/p-m} > \epsilon_1^{\ell_*}; \text{ and} \tag{5.16}$$

$$\left|\partial^{\beta} P(x_i)\right| \delta_{Q_i}^{|\beta|+n/p-m} \leq \epsilon_1^{\ell_*+1} \quad (\beta \in \mathcal{M}, \beta > \bar{\alpha}), \tag{5.17}$$

where (5.17) follows because for every $\beta \in \mathcal{M}$ with $\beta > \bar{\alpha}$, we have $\beta \notin \Delta_{\ell_*} = \Delta_{\ell_*+1}$. Define $P^{\bar{\alpha}} := P \cdot \left(\partial^{\bar{\alpha}} P(x_i)\right)^{-1}$. Then because $\ell_* \leq D$, from (5.14), (5.15), (5.16), and (5.17), we have

$$P^{\bar{\alpha}} \in C\epsilon_1 \delta_{Q_i}^{|\bar{\alpha}|+n/p-m} \cdot \sigma(\mu|_{9Q_i}, \delta_{Q_i}); \tag{5.18}$$

$$\partial^{\bar{\alpha}} P^{\bar{\alpha}}(x_i) = 1; \tag{5.19}$$

$$|\partial^{\beta} P^{\bar{\alpha}}(x_i)| \leq \epsilon_1 \delta_{Q_i}^{|\bar{\alpha}|-|\beta|} \qquad (\beta \in \mathcal{M}, \beta > \bar{\alpha}); \text{ and} \tag{5.20}$$

$$|\partial^{\beta} P^{\bar{\alpha}}(x_i)| \leq \epsilon_1^{-D} \delta_{Q_i}^{|\bar{\alpha}|-|\beta|} \qquad (\beta \in \mathcal{M}). \tag{5.21}$$

From (5.18), there exists $\varphi^{\bar{\alpha}} \in L^{m,p}(\mathbb{R}^n)$ satisfying

$$\begin{aligned} \|\varphi^{\bar{\alpha}}, P^{\bar{\alpha}}\|_{\mathscr{J}(0,\mu|_{9Q_i};\delta_{Q_i})} &= \left(\|\varphi^{\bar{\alpha}}\|^p_{L^{m,p}(\mathbb{R}^n)} + \int\limits_{9Q_i} |\varphi^{\bar{\alpha}}|^p d\mu + \|\varphi^{\bar{\alpha}} - P^{\bar{\alpha}}\|^p_{L^p(\mathbb{R}^n)} / \delta_{Q_i}^{mp}\right)^{1/p} \\ &\leq C\epsilon_1 \delta_{Q_i}^{|\bar{\alpha}|+n/p-m}. \end{aligned} \tag{5.22}$$

For $x \in \operatorname{supp}(\mu) \cap 9Q_i$, set $P_x^{\bar{\alpha}} := J_x \varphi^{\bar{\alpha}}$.

Because $\|\varphi^{\bar{\alpha}}\|_{\mathscr{J}(0,\mu|_{9Q_i})} \leq \|\varphi^{\bar{\alpha}}, P^{\bar{\alpha}}\|_{\mathscr{J}(0,\mu|_{9Q_i};\delta_{Q_i})}$, we have

$$P_x^{\bar{\alpha}} \in C\epsilon_1 \delta_{Q_i}^{|\bar{\alpha}|+n/p-m} \cdot \sigma_J(x, \mu|_{9Q_i}). \tag{5.23}$$

Also, $|x - x_i| \leq C\delta_{Q_i}$, so for $\beta \in \mathcal{M}$ we have

$$\begin{aligned} |\partial^{\beta} P_x^{\bar{\alpha}}(x) - \partial^{\beta} P^{\bar{\alpha}}(x_i)| &= |\partial^{\beta} \varphi^{\bar{\alpha}}(x) - \partial^{\beta} P^{\bar{\alpha}}(x_i)| \\ &\leq |\partial^{\beta} \varphi^{\bar{\alpha}}(x) - \partial^{\beta} P^{\bar{\alpha}}(x)| + |\partial^{\beta} P^{\bar{\alpha}}(x) - \partial^{\beta} P^{\bar{\alpha}}(x_i)| \\ &\overset{(2.15)}{\lesssim} \|\varphi^{\bar{\alpha}}, P^{\bar{\alpha}}\|_{\mathscr{J}(0,\mu|_{9Q_i};\delta_{Q_i})} \delta_{Q_i}^{m-|\beta|-n/p} \\ &\quad + \left| \sum_{0<|\omega|\leq m-1-|\beta|} \partial^{\beta+\omega} P^{\bar{\alpha}}(x_i) \frac{(x-x_i)^{\omega}}{\omega!} \right| \\ &\overset{(5.22)}{\lesssim} \epsilon_1 \delta_{Q_i}^{|\bar{\alpha}|-|\beta|} + \left| \sum_{0<|\omega|\leq m-1-|\beta|} \partial^{\beta+\omega} P^{\bar{\alpha}}(x_i) \frac{(x-x_i)^{\omega}}{\omega!} \right| \\ &\overset{(5.20),(5.21)}{\lesssim} \begin{cases} \epsilon_1 \delta_{Q_i}^{|\bar{\alpha}|-|\beta|} & \text{if } \beta \geq \bar{\alpha} \\ \epsilon_1^{-D} \delta_{Q_i}^{|\bar{\alpha}|-|\beta|} & \text{for any } \beta, \end{cases} \end{aligned} \tag{5.24}$$

where in the last line we have used that $\beta+\omega>\beta$ provided that $0<|\omega|\le m-1-|\beta|$, and hence, $\beta+\omega>\bar{\alpha}$ provided $\beta\ge\bar{\alpha}$. We insert (5.24) into (5.19)-(5.21) to deduce,

$$|\partial^\beta P_x^{\bar{\alpha}}(x)-\delta_{\bar{\alpha}\beta}|\le C\epsilon_1\delta_{Q_i}^{|\bar{\alpha}|-|\beta|} \qquad (\beta\in\mathcal{M},\beta\ge\bar{\alpha});\ \text{and} \tag{5.25}$$

$$|\partial^\beta P_x^{\bar{\alpha}}(x)|\le C\epsilon_1^{-D}\delta_{Q_i}^{|\bar{\alpha}|-|\beta|} \qquad (\beta\in\mathcal{M}) \tag{5.26}$$

We will use $\bar{\alpha}\in\mathcal{M}$ and $P^{\bar{\alpha}}$ to construct $\overline{\mathcal{A}}<\mathcal{A}$ and an $(\overline{\mathcal{A}},x,\epsilon_0/C_0,\delta_{Q_i^+})$-basis for $\sigma_J(x,\mu|_{3Q_i^+})$ for all $x\in\operatorname{supp}(\mu)\cap 3Q_i^+$, indicating that $Q_i^+$ is OK, a contradiction. So (5.13) cannot hold, and we have $\|0,P\|_{\mathscr{J}(\mu|_{9Q_i};\delta_{Q_i})}\simeq|P|_i$ for all $P\in\mathcal{P}$. This next portion of the proof follows Section 13 of [13].

Fix $x\in\operatorname{supp}(\mu)\cap 3Q_i^+\subset Q^\circ$. Recall from (5.1) and (5.3), that the auxiliary polynomials $(P_x^\alpha)_{\alpha\in\mathcal{A}}$ form an $(\mathcal{A},x,C\epsilon,1)$-basis for $\sigma_J(x,\mu)$, satisfying:

$$P_x^\alpha\in C\epsilon\cdot\sigma_J(x,\mu); \tag{5.27}$$

$$\partial^\beta P_x^\alpha(x)=\delta_{\alpha\beta} \qquad (\alpha,\beta\in\mathcal{A}); \tag{5.28}$$

$$|\partial^\beta P_x^\alpha(x)|\le C\epsilon \qquad (\alpha\in\mathcal{A},\beta\in\mathcal{M},\beta>\alpha); \tag{5.29}$$

$$|\partial^\beta P_x^\alpha(x)|\le C \qquad (\alpha\in\mathcal{A},\beta\in\mathcal{M});\ \text{and} \tag{5.30}$$

$$|\partial^\beta P_x^\alpha(x)|\lesssim\delta_{Q_i}^{|\alpha|-|\beta|} \qquad (\alpha\in\mathcal{A},\beta\in\mathcal{M}). \tag{5.31}$$

Define

$$\widetilde{P}_x^{\bar{\alpha}}:=P_x^{\bar{\alpha}}-\sum_{\alpha\in\mathcal{A},\alpha<\bar{\alpha}}\partial^\alpha P_x^{\bar{\alpha}}(x)\cdot P_x^\alpha.$$

We have $\sigma_J(x,\mu)\subset\sigma_J(x,\mu|_{9Q_i})$, $\delta_{Q_i}<1$, and $m-n/p>m-1\ge|\alpha|$. So from (5.23), (5.26), and (5.27), we have

$$\begin{aligned}\widetilde{P}_x^{\bar{\alpha}}&\in\Big(C\epsilon_1\delta_{Q_i}^{|\bar{\alpha}|+n/p-m}+\sum_{\alpha\in\mathcal{A},\alpha<\bar{\alpha}}(C\epsilon_1^{-D}\delta_{Q_i}^{|\bar{\alpha}|-|\alpha|})(C\epsilon)\Big)\cdot\sigma_J(x,\mu|_{9Q_i})\\ \implies\ \widetilde{P}_x^{\bar{\alpha}}&\in C\left(\epsilon_1+\epsilon_1^{-D}\epsilon\right)\delta_{Q_i}^{|\bar{\alpha}|+n/p-m}\cdot\sigma_J(x,\mu|_{9Q_i}).\end{aligned} \tag{5.32}$$

For $\beta\in\mathcal{A}$, $\beta<\bar{\alpha}$, from (5.28),

$$\partial^\beta\widetilde{P}_x^{\bar{\alpha}}(x)=\partial^\beta P_x^{\bar{\alpha}}(x)-\partial^\beta P_x^{\bar{\alpha}}(x)=0. \tag{5.33}$$

For $\beta\in\mathcal{M}$, $\beta\ge\bar{\alpha}$, from (5.25), (5.26), and (5.29),

$$\begin{aligned}|\partial^\beta\widetilde{P}_x^{\bar{\alpha}}(x)-\delta_{\beta\bar{\alpha}}|&\le|\partial^\beta P_x^{\bar{\alpha}}(x)-\delta_{\beta\bar{\alpha}}|+|\partial^\beta(P_x^{\bar{\alpha}}-\widetilde{P}_x^{\bar{\alpha}})(x)|\\&\le|\partial^\beta P_x^{\bar{\alpha}}(x)-\delta_{\beta\bar{\alpha}}|+\sum_{\alpha\in\mathcal{A},\alpha<\bar{\alpha}}|\partial^\alpha P_x^{\bar{\alpha}}(x)||\partial^\beta P_x^\alpha(x)|\end{aligned}$$

$$\le C\epsilon_1 \delta_{Q_i}^{|\bar\alpha|-|\beta|} + \sum_{\alpha\in\mathcal{A},\alpha<\bar\alpha} (C\epsilon_1^{-D}\delta_{Q_i}^{|\bar\alpha|-|\alpha|})(C\epsilon)$$

$$\lesssim (\epsilon_1 + \epsilon_1^{-D}\epsilon)\delta_{Q_i}^{|\bar\alpha|-|\beta|}, \tag{5.34}$$

where the last inequality uses that $\delta_{Q_i} < 1$ and $|\beta| \ge |\alpha|$ for $\beta \ge \bar\alpha > \alpha$.

For $\beta \in \mathcal{M}$ arbitrary, from (5.26), (5.31),

$$|\partial^\beta \widetilde{P}_x^{\bar\alpha}(x)| \le |\partial^\beta P_x^{\bar\alpha}(x)| + \sum_{\alpha\in\mathcal{A},\alpha<\bar\alpha} |\partial^\alpha P_x^{\bar\alpha}(x)||\partial^\beta P_x^{\alpha}(x)|$$

$$\le C\epsilon_1^{-D}\delta_{Q_i}^{|\bar\alpha|-|\beta|} + \sum_{\alpha\in\mathcal{A},\alpha<\bar\alpha} (C\epsilon_1^{-D}\delta_{Q_i}^{|\bar\alpha|-|\alpha|})(C\delta_{Q_i}^{|\alpha|-|\beta|})$$

$$\lesssim \epsilon_1^{-D}\delta_{Q_i}^{|\bar\alpha|-|\beta|}. \tag{5.35}$$

Define

$$\overline{\mathcal{A}} = \{\bar\alpha\} \cup \{\alpha \in \mathcal{A} : \alpha < \bar\alpha\}.$$

Then the minimal element of the symmetric difference $\overline{\mathcal{A}}\Delta\mathcal{A}$ is $\bar\alpha$, which is in $\overline{\mathcal{A}}$. So $\overline{\mathcal{A}} < \mathcal{A}$, by definition of the order relation on multiindex sets. We may assume $\epsilon < \epsilon_1^{D+1}$. Then, for small enough $\epsilon_1$, (5.34) implies $\partial^{\bar\alpha}\widetilde{P}_x^{\bar\alpha}(x) \ge 1/2$. So $\hat{P}_x^{\bar\alpha} := \widetilde{P}_x^{\bar\alpha}/(\partial^{\bar\alpha}\widetilde{P}_x^{\bar\alpha}(x))$ is well-defined, and due to (5.32)-(5.35), this polynomial satisfies:

$$\hat{P}_x^{\bar\alpha} \in C(\epsilon_1 + \epsilon_1^{-D}\epsilon)\delta_{Q_i}^{|\bar\alpha|+n/p-m} \cdot \sigma_J(x, \mu|_{9Q_i}); \tag{5.36}$$

$$\partial^\beta \hat{P}_x^{\bar\alpha}(x) = \delta_{\beta\bar\alpha} \qquad (\beta \in \overline{\mathcal{A}}); \tag{5.37}$$

$$|\partial^\beta \hat{P}_x^{\bar\alpha}(x)| \le C(\epsilon_1 + \epsilon_1^{-D}\epsilon)\delta_{Q_i}^{|\bar\alpha|-|\beta|} \qquad (\beta \in \mathcal{M}, \beta > \bar\alpha);\ \text{and} \tag{5.38}$$

$$|\partial^\beta \hat{P}_x^{\bar\alpha}(x)| \le C\epsilon_1^{-D}\delta_{Q_i}^{|\bar\alpha|-|\beta|} \qquad (\beta \in \mathcal{M}). \tag{5.39}$$

For $\alpha \in \overline{\mathcal{A}} \setminus \{\bar\alpha\}$, define

$$\hat{P}_x^\alpha = P_x^\alpha - \partial^{\bar\alpha}P_x^\alpha(x) \cdot \hat{P}_x^{\bar\alpha}.$$

Notice, if $\alpha \in \overline{\mathcal{A}} \setminus \{\bar\alpha\}$ then $\alpha < \bar\alpha$. Also, $\delta_{Q_i} < 1$. Thus, thanks to (5.29), $|\partial^{\bar\alpha}P_x^\alpha(x)| \le C\epsilon \le C\epsilon\delta_{Q_i}^{|\alpha|-|\bar\alpha|}$. Therefore, from (5.27)-(5.30), and (5.36)-(5.39), for any $\alpha \in \overline{\mathcal{A}} \setminus \{\bar\alpha\}$,

$$\hat{P}_x^\alpha \in \left[C\epsilon + C\epsilon\delta_{Q_i}^{|\alpha|-|\bar\alpha|}(\epsilon_1 + \epsilon_1^{-D}\epsilon)\delta_{Q_i}^{|\bar\alpha|+n/p-m}\right] \cdot \sigma_J(x, \mu|_{9Q_i}); \tag{5.40}$$

$$\partial^\beta \hat{P}_x^\alpha(x) = \delta_{\alpha\beta} \qquad (\beta \in \overline{\mathcal{A}});\ \text{and} \tag{5.41}$$

$$|\partial^\beta \hat{P}_x^\alpha(x)| \le C\epsilon + C\epsilon\delta_{Q_i}^{|\alpha|-|\bar\alpha|}C\epsilon_1^{-D}\delta_{Q_i}^{|\bar\alpha|-|\beta|} \qquad (\beta \in \mathcal{M}, \beta > \alpha). \tag{5.42}$$

We suppose $\epsilon < \epsilon_1^{D+1}$. Since $|\alpha| + n/p - m < 0$, and $\delta_{Q_i} < 1$, (5.40) implies,

$$\hat{P}^{\alpha}_x \in C(\epsilon_1 + \epsilon_1^{-D}\epsilon)\delta_{Q_i}^{|\alpha|+n/p-m} \cdot \sigma_J(x, \mu|_{9Q_i}). \tag{5.43}$$

Similarly, because $\delta_{Q_i} < 1$ and $|\alpha| - |\beta| \leq 0$ for $\beta > \alpha$, (5.42) implies

$$|\partial^\beta \hat{P}^{\alpha}_x(x)| \leq C\epsilon_1 \delta_{Q_i}^{|\alpha|-|\beta|} \quad (\beta \in \mathcal{M}, \beta > \alpha). \tag{5.44}$$

Recall $x \in \mathrm{supp}(\mu) \cap 3{Q_i}^+$ is arbitrary. Together, (5.36)-(5.39), (5.41), (5.43), and (5.44) imply that, for each $x \in \mathrm{supp}(\mu) \cap 3{Q_i}^+$, $\{\hat{P}^\alpha_x\}_{\alpha \in \overline{\mathcal{A}}}$ forms an $(\overline{\mathcal{A}}, x, C'(\epsilon_1 + \epsilon_1^{-D}\epsilon), 30\delta_{{Q_i}^+})$-basis for $\sigma_J(x, \mu|_{9Q_i})$, and hence for $\sigma_J(x, \mu|_{3{Q_i}^+})$. Fix a universal constant $\epsilon_1 > 0$, small enough so that the preceding arguments hold, and so that $\epsilon_1 < \frac{\epsilon_0}{2C'C_0}$. Recall that we have assumed $\epsilon < \epsilon_1^{D+1}$. Thus, $\{\hat{P}^\alpha_x\}_{\alpha \in \overline{\mathcal{A}}}$ forms an $(\overline{\mathcal{A}}, x, \epsilon_0/C_0, 30\delta_{{Q_i}^+})$-basis for $\sigma_J(x, \mu|_{3{Q_i}^+})$ for each $x \in \mathrm{supp}(\mu) \cap 3{Q_i}^+$, indicating that ${Q_i}^+$ is OK. But this contradicts the assumption that $Q_i$ is a CZ cube. We have reached the desired contradiction. This completes the proof of (5.12), and with it, the proof of the lemma. □

### *5.3. Patching estimates*

#### *5.3.1. Patching estimate on $K_{CZ}$*

Recall that we have defined a collection of disjoint dyadic cubes $CZ^\circ = \{Q_i\}_{i \in I} \in \Pi(Q^\circ)$ contained in $Q^\circ = (0,1]^n$. We set $K_{CZ} = \bigcup_{i \in I} Q_i$. Then $K_{CZ}$ is a relatively open subset of $Q^\circ$. Indeed, we showed that $K_p$ is a closed set in $\mathbb{R}^n$, and $K_p \subset \mathrm{supp}(\mu) \subset \frac{1}{10}Q^\circ$.

We associate to each cube $Q_i \in CZ^\circ$ a basepoint $x_i = \mathrm{ctr}(Q_i)$. We define the polynomial norms $|P|_i := |P|_{x_i, \delta_{Q_i}}$.

Recall that a Whitney field $\vec{P} \in Wh(\mathfrak{B}_{CZ})$ is an indexed collection of polynomials, $\vec{P} = \{P_x\}_{x \in \mathfrak{B}_{CZ}}$, associated to the CZ basepoints $\mathfrak{B}_{CZ} = \{x_i\}_{i \in I}$.

For a relatively open set $\Omega \subset Q^\circ$ with $K_{CZ} \subset \Omega$, and for $\vec{P} \in Wh(\mathfrak{B}_{CZ})$, $f \in \mathscr{J}(\mu)$, and $F \in L^{m,p}(\Omega)$, we define:

$$\|F, \vec{P}\|_{\mathscr{J}_*(f,\mu;\Omega,CZ^\circ)} = \left( \|F\|^p_{L^{m,p}(\Omega)} + \int_\Omega |F - f|^p d\mu + \sum_{i \in I} \|F - P_{x_i}\|^p_{L^p(Q_i)} / \delta_{Q_i}^{mp} \right)^{1/p}; \text{ and} \tag{5.45}$$

$$\|f, \vec{P}\|_{\mathscr{J}_*(\mu;\Omega,CZ^\circ)} = \inf\left\{ \|F, \vec{P}\|_{\mathscr{J}_*(f,\mu;\Omega,CZ^\circ)} : F \in L^{m,p}(\Omega) \right\}. \tag{5.46}$$

We define the seminormed vector space:

$$\mathscr{J}_*(\mu; \Omega, CZ^\circ) = \left\{ (f, \vec{P}) : f \in \mathscr{J}(\mu),\ \vec{P} \in Wh(\mathfrak{B}_{CZ}),\ \|f, \vec{P}\|_{\mathscr{J}_*(\mu;\Omega,CZ^\circ)} < \infty \right\}. \tag{5.47}$$

In the previous definitions, we have in mind to take $\Omega = K_{CZ}$ or $\Omega = Q^\circ$.

**Lemma 5.5.** *Suppose we are given a collection of functions $\{G_i\}_{i \in I} \subset L^{m,p}(\mathbb{R}^n)$ and a Whitney field $\vec{P} = (P_{x_i})_{i \in I} \in Wh(\mathfrak{B}_{CZ})$.*

*Define $G : K_{CZ} \to \mathbb{R}$ by $G(x) = \sum_{i \in I} G_i(x) \cdot \theta_i(x)$, where $\{\theta_i\}_{i \in I}$ is a partition of unity satisfying* ***(POU1)-(POU4)*** *(see Section 4.3). Then*

$$\|G, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} \lesssim \sum_{i \in I} \|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i \cap Q^\circ};1.1Q_i \cap Q^\circ)} + \sum_{i,i' \in I,\, i \leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i. \tag{5.48}$$

*Here, we write $i \leftrightarrow i'$ to denote that $\mathrm{Cl}(Q_i) \cap \mathrm{Cl}(Q_{i'}) \neq \emptyset$, and the $\mathscr{J}_*$-functionals on the right-hand side of (5.48) are defined in (2.5).*

**Proof.** For $x \in Q_{i'}$, if $x \in \mathrm{supp}(\theta_i)$ then $x \in 1.1Q_i$, hence $1.1Q_i \cap Q_{i'} \neq \emptyset$ and thus $i \leftrightarrow i'$ by the good geometry of the CZ cubes. Thus, by the condition $\sum_i \theta_i = 1$ on $K_{CZ}$, we deduce that $G(x) = G_{i'}(x) + \sum_{i \in I,\, i \leftrightarrow i'} (G_i - G_{i'})(x) \cdot \theta_i(x)$. By the Leibniz rule, for any multiindex $\gamma$ such that $|\gamma| = m$, we have

$$\partial^\gamma G(x) = \partial^\gamma G_{i'}(x) + \sum_{(\alpha,\beta):\alpha+\beta=\gamma} \sum_{i \in I:\, i \leftrightarrow i'} \partial^\beta (G_i - G_{i'})(x) \partial^\alpha \theta_i(x).$$

Then, taking $p$'th powers, summing on $\gamma$ with $|\gamma| = m$, and integrating over $x \in Q_{i'}$, we have

$$\|G\|^p_{L^{m,p}(Q_{i'})} \lesssim \|G_{i'}\|^p_{L^{m,p}(Q_{i'})} + \sum_{(\alpha,\beta):|\alpha|+|\beta|=m} \sum_{i \in I: i \leftrightarrow i'} \int_{Q_{i'}} |\partial^\beta (G_i - G_{i'})(x)|^p |\partial^\alpha \theta_i(x)|^p dx.$$

Now note, by **(POU1)-(POU4)**, $|\partial^\alpha \theta_i(x)| \lesssim \delta_{Q_i}^{-|\alpha|}$, and $\theta_i$ is supported on $1.1Q_i$. So,

$$\begin{aligned} \|G\|^p_{L^{m,p}(Q_{i'})} &\lesssim \|G_{i'}\|^p_{L^{m,p}(Q_{i'})} \\ &\quad + \sum_{(\alpha,\beta):|\alpha|+|\beta|=m} \sum_{i \in I: i \leftrightarrow i'} \delta_{Q_i}^{-|\alpha| p} \int_{Q_{i'} \cap 1.1Q_i} |\partial^\beta (G_i - G_{i'})(x)|^p dx. \end{aligned} \tag{5.49}$$

If $\alpha = 0$ in the previous sum, then $|\beta| = m$, so

$$\int_{Q_{i'} \cap 1.1Q_i} |\partial^\beta (G_i - G_{i'})(x)|^p dx \leq C\big(\|G_{i'}\|^p_{L^{m,p}(Q_{i'})} + \|G_i\|^p_{L^{m,p}(1.1Q_i \cap Q^\circ)}\big).$$

Now suppose $|\alpha| > 0$ in the previous sum. Then $|\beta| \leq m - 1$, and

$$|\partial^\beta (G_i - G_{i'})(x)| \leq |\partial^\beta (G_i - P_{x_i})(x)| + |\partial^\beta (P_{x_i} - P_{x_{i'}})(x)| + |\partial^\beta (G_{i'} - P_{x_{i'}})(x)|.$$

Thus, by integrating over $x \in Q_{i'} \cap 1.1Q_i$, we can apply (2.14) twice, on the rectangle $1.1Q_i \cap Q^\circ$ and on the square $Q_{i'}$, to obtain

$$\int_{Q_{i'}\cap 1.1Q_i} |\partial^\beta(G_i - G_{i'})(x)|^p dx$$
$$\leq C\Big(\|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ};1.1Q_i\cap Q^\circ)}\delta_{Q_i}^{mp-|\beta|p}$$
$$+\|G_{i'}, P_{x_{i'}}\|^p_{\mathscr{J}_*(f,\mu|_{Q_{i'}};Q_{i'})}\delta_{Q_i}^{mp-|\beta|p} + \max_{x\in 1.1Q_i\cap Q_{i'}} |\partial^\beta(P_{x_i} - P_{x_{i'}})(x)|^p\delta^n_{Q_i}\Big).$$

Now because $|\alpha| + |\beta| = m$,

$$\delta_{Q_i}^{-|\alpha|p}\int_{Q_{i'}\cap 1.1Q_i} |\partial^\beta(G_i - G_{i'})(x)|^p dx \leq C\cdot\Big(\|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ};1.1Q_i\cap Q^\circ)}$$
$$+\|G_{i'}, P_{x_{i'}}\|^p_{\mathscr{J}_*(f,\mu|_{Q_{i'}};Q_{i'})} + \max_{x\in 1.1Q_i\cap Q_{i'}} |\partial^\beta(P_{x_i} - P_{x_{i'}})(x)|^p\delta_{Q_i}^{n+|\beta|p-mp}\Big).$$

We use (2.2) to bound the third term in the parentheses by $|P_{x_i} - P_{x_{i'}}|^p_{x_i,\delta_{Q_i}} = |P_{x_i} - P_{x_{i'}}|^p_i$. Returning to (5.49),

$$\|G\|^p_{L^{m,p}(Q_{i'})} \lesssim \|G_{i'}\|^p_{L^{m,p}(Q_{i'})} + \sum_{i\in I, i\leftrightarrow i'}\Big(\|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ};1.1Q_i\cap Q^\circ)}$$
$$+\|G_{i'}, P_{x_{i'}}\|^p_{\mathscr{J}_*(f,\mu|_{Q_{i'}};Q_{i'})} + |P_{x_i} - P_{x_{i'}}|^p_i\Big).$$

We can bound $\|G_{i'}\|^p_{L^{m,p}(Q_{i'})} \leq \|G_{i'}, P_{x_{i'}}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_{i'}\cap Q^\circ};1.1Q_{i'}\cap Q^\circ)}$. Thus, by summing on $i' \in I$, and using that for each $i \in I$ there are at most $C$ many $i' \in I$ with $i \leftrightarrow i'$, we have

$$\begin{aligned}\|G\|^p_{L^{m,p}(K_{CZ})} &= \sum_{i'\in I}\|G\|^p_{L^{m,p}(Q_{i'})}\\ &\lesssim \sum_{i\in I}\|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ};1.1Q_i\cap Q^\circ)}\\ &\quad+\sum_{(i,i'):i\leftrightarrow i'}|P_{x_i} - P_{x_{i'}}|^p_i.\end{aligned} \tag{5.50}$$

Because $\theta_i \leq 1$, $\mathrm{supp}(\theta_i) \subset 1.1Q_i$, and $\sum\theta_i = 1$ on $K_{CZ}$, we have

$$\begin{aligned}\int_{K_{CZ}} |G - f|^p d\mu &= \int_{K_{CZ}}\Big|\sum_{i\in I}(G_i - f)\cdot\theta_i\Big|^p d\mu\\ &\lesssim \sum_{i\in I}\int_{1.1Q_i\cap Q^\circ}|G_i - f|^p d\mu\\ &\lesssim \sum_{i\in I}\|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ};1.1Q_i\cap Q^\circ)}.\end{aligned} \tag{5.51}$$

First applying that $\sum \theta_{i'} = 1$ on $K_{CZ}$, and then $x \in \mathrm{supp}(\theta_i') \implies x \in 1.1Q_{i'}$, with (4.1), (4.2), and Lemma 2.2, we obtain,

$$\begin{aligned}
&\sum_{i\in I} \|G - P_{x_i}\|^p_{L^p(Q_i)}/\delta_{Q_i}^{mp} \\
&\quad = \sum_{i\in I} \Big\| \sum_{i'\in I: i'\leftrightarrow i} (G_{i'} - P_{x_i})\theta_{i'} \Big\|^p_{L^p(Q_i)}/\delta_{Q_i}^{mp} \\
&\quad \lesssim \sum_{i\in I} \sum_{i'\in I: i'\leftrightarrow i} \Big[ \|G_{i'} - P_{x_{i'}}\|^p_{L^p(1.1Q_{i'}\cap Q^\circ)}/\delta_{Q_{i'}}^{mp} + \|P_{x_i} - P_{x_{i'}}\|^p_{L^p(Q_i)}/\delta_{Q_i}^{mp} \Big] \\
&\quad \lesssim \sum_{i\in I} \|G_i - P_{x_i}\|^p_{L^p(1.1Q_i\cap Q^\circ)}/\delta_{Q_i}^{mp} + \sum_{(i,i'): i'\leftrightarrow i} |P_{x_i} - P_{x_{i'}}|_i^p \\
&\quad \lesssim \sum_{i\in I} \|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ}; 1.1Q_i\cap Q^\circ)} + \sum_{(i,i'): i'\leftrightarrow i} |P_{x_i} - P_{x_{i'}}|_i^p. \qquad (5.52)
\end{aligned}$$

From (5.50), (5.51), and (5.52), we conclude,

$$\begin{aligned}
\|G, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} &= \|G\|^p_{L^{m,p}(K_{CZ})} + \int_{K_{CZ}} |G - f|^p d\mu + \sum_{i\in I} \|G - P_{x_i}\|^p_{L^p(Q_i)}/\delta_{Q_i}^{mp} \\
&\lesssim \sum_{i\in I} \|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ}; 1.1Q_i\cap Q^\circ)} + \sum_{i'\leftrightarrow i} |P_{x_i} - P_{x_{i'}}|_i^p. \qquad \square
\end{aligned}$$

*5.3.2. Patching estimate on $Q^\circ$*

Recall that $K_{CZ} \subset Q^\circ$ and $K_p = Q^\circ \setminus K_{CZ}$. We showed that $K_p$ is a closed set in $\mathbb{R}^n$, and $K_p \subset \mathrm{supp}(\mu) \subset \frac{1}{10}Q^\circ$.

We have defined $\mathfrak{B}_{CZ} = \{x_i\}_{i\in I}$, the set of all CZ basepoints, with $x_i$ the center of $Q_i$ for each $Q_i \in CZ^\circ$.

Given $\vec{S} \in Wh(K_p)$, $\vec{P} \in Wh(\mathfrak{B}_{CZ})$, we regard $(\vec{P}, \vec{S}) \in Wh(\mathfrak{B}_{CZ} \cup K_p)$ as a Whitney field on $K_p \cup \mathfrak{B}_{CZ}$.

**Lemma 5.6.** *Fix a collection of functions $\{G_i\}_{i\in I} \subset L^{m,p}(\mathbb{R}^n)$, and two Whitney fields $\vec{R} = (R_{x_i})_{i\in I} \in Wh(\mathfrak{B}_{CZ})$ and $\vec{S} = (S_x)_{x\in E} \in Wh(K_p)$. Let $G : Q^\circ \to \mathbb{R}^n$ be defined as*

$$G(x) = \begin{cases} \sum_{i\in I} G_i(x) \cdot \theta_i(x) & x \in K_{CZ} \\ S_x(x) & x \in K_p, \end{cases}$$

*where $\{\theta_i\}_{i\in I}$ is a partition of unity satisfying* ***(POU1)-(POU4)*** *(see Section 4.3).*

*If $(G, \vec{R}, \vec{S})$ satisfy the conditions*

$$\|G, \vec{R}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|\vec{S}\|_{L^{m,p}(K_p)} < \infty \text{ and } (\vec{S}, \vec{R}) \in C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ}),$$

*then $G \in L^{m,p}(Q^\circ)$, $J_xG = S_x$ for all $x \in K_p$, and*

$$\|G\|_{L^{m,p}(Q^\circ)} \lesssim \|G, \vec{R}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|\vec{S}\|_{L^{m,p}(K_p)}. \tag{5.53}$$

**Remark 5.1.** In later applications of Lemma 5.6, the hypothesis $\|G, \vec{R}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} < \infty$ will be verified using Lemma 5.5.

**Proof.** By assumption, $\|\vec{S}\|_{L^{m,p}(K_p)} < \infty$, hence for any $\eta > 0$, there exists $H \in L^{m,p}(\mathbb{R}^n)$ satisfying $J_x H = S_x$ for all $x \in K_p$ and $\|H\|_{L^{m,p}(\mathbb{R}^n)} \leq \|\vec{S}\|_{L^{m,p}(K_p)} + \eta$.

By assumption, $\|G, \vec{R}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} < \infty$. In particular, $G \in L^{m,p}(K_{CZ})$. Thus, $J_x G \in \mathcal{P}$ is well-defined for $x \in K_{CZ}$.

Define $\vec{P} \in Wh(Q^\circ)$ as

$$P_x := \begin{cases} J_x G & x \in K_{CZ} \\ S_x = J_x H & x \in K_p. \end{cases}$$

By definition of $G$, observe that

$$P_x(x) = G(x) \text{ for all } x \in Q^\circ. \tag{5.54}$$

Fix $\delta > 0$, and fix a cube $\widehat{Q} \subset \mathbb{R}^n$ with $\delta_{\widehat{Q}} \leq \delta$. We will show that for all $x, y \in \widehat{Q} \cap Q^\circ$ we have

$$\begin{aligned} &|P_x - P_y|_{y,\delta} \\ &\quad \lesssim \|H\|_{L^{m,p}(21\widehat{Q})} + \sup_{Q_i \subseteq 35\widehat{Q}} \{\|G, R_{x_i}\|_{\mathscr{J}_*(f,\mu;Q_i)}\} \\ &\qquad + \sup\left\{\|G\|_{L^{m,p}(\operatorname{int}(B(z,r))\cap Q^\circ)} : z \in 7\widehat{Q}, r \leq 7\delta, \operatorname{int}(B(z,r)) \cap Q^\circ \subset K_{CZ}\right\}. \end{aligned} \tag{5.55}$$

Here, as usual, $B(z,r) = \{w \in \mathbb{R}^n : |w - z| \leq r\}$, and $|\cdot|$ is the $\ell^\infty$ (sup) metric on $\mathbb{R}^n$. Thus, $B(z,r)$ is a closed cube centered at $z$ of sidelength $2r$.

Fix $x, y, \widehat{Q}, \delta$ as above. We shall split the proof of (5.55) into cases depending on the relative positions of $x$, $y$, and $K_p$. Observe that

$$|x - y| \leq \delta_{\widehat{Q}} \leq \delta. \tag{5.56}$$

Case 1: Suppose $x, y \in K_p$. We apply (2.3) and the Sobolev Inequality on $\widehat{Q}$,

$$|P_x - P_y|_{y,\delta} = |J_x H - J_y H|_{y,\delta} \leq |J_x H - J_y H|_{y,|x-y|} \lesssim \|H\|_{L^{m,p}(\widehat{Q})}. \tag{5.57}$$

This completes the proof of (5.55) in Case 1.

Case 2: Suppose $x, y \in K_{CZ}$ satisfy $\operatorname{dist}(y, K_p) > 3|y - x|$ or $\operatorname{dist}(x, K_p) > 3|y - x|$. Because $|P|_{y,\delta} \simeq |P|_{x,\delta}$ for $\delta \geq |x - y|$ by (2.2), we may assume without loss of generality the first case occurs. By the triangle inequality, $B(y, |y-x|) \subset \mathbb{R}^n \setminus K_p$. Thus, we can apply (2.3) and the Sobolev Inequality on $B(y, |y-x|) \cap Q^\circ \subset K_{CZ}$, and obtain

$$|P_x - P_y|_{y,\delta} = |J_x G - J_y G|_{y,\delta} \leq |J_x G - J_y G|_{y,|x-y|} \lesssim \|G\|_{L^{m,p}(B(y,|y-x|)\cap Q^\circ)}.$$

Therefore, we can upper bound $|P_x - P_y|_{y,\delta}$ by the second supremum in (5.55). This completes the proof of (5.55) in Case 2.

Case 3: Suppose $y \in K_{CZ}$ and $x \in K_p$, or $y \in K_p$ and $x \in K_{CZ}$. As in Case 2, without loss of generality, $y \in K_{CZ}$ and $x \in K_p$. Because $K_p$ is closed (see Lemma 4.8), there exists $z_y \in K_p$ satisfying $\operatorname{dist}(y, K_p) = |z_y - y|$. Because $x \in K_p$, and from (5.56), we have

$$|z_y - y| \leq |x - y| \leq \delta_{\widehat{Q}} \leq \delta; \tag{5.58}$$

$$B(y, |z_y - y|) \subset 3\widehat{Q}; \tag{5.59}$$

$$|x - z_y| \leq |x - y| + |y - z_y| \leq 2\delta_{\widehat{Q}} \leq 2\delta. \tag{5.60}$$

Because $z_y$ is a closest point of $K_p$ to $y$, $\operatorname{int} B(y, |z_y - y|) \subset \mathbb{R}^n \setminus K_p$. We write $[y, z)$ for the segment $\{y + t(z-y) : 0 \leq t < 1\}$. Then there exists a sequence $\{z_y^k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ satisfying

$$z_y^k \in [y, z_y) \subset \operatorname{int} B(y, |z_y - y|) \subset \mathbb{R}^n \setminus K_p \text{ for all } k \in \mathbb{N}; \tag{5.61}$$

$$\lim_{k\to\infty} z_y^k = z_y. \tag{5.62}$$

Observe that

$$\operatorname{dist}(z_y^k, K_p) \leq |z_y^k - z_y| \to 0 \text{ as } k \to \infty. \tag{5.63}$$

By (5.61) and (5.58), we have

$$|z_y^k - y| \leq |z_y - y| \leq |x - y| \leq \delta_{\widehat{Q}} \leq \delta \qquad (k \in \mathbb{N}). \tag{5.64}$$

Observe that $z_y^k \in [y, z_y) \subset Q^\circ$ by convexity of $Q^\circ$, so $z_y^k \in (\mathbb{R}^n \setminus K_p) \cap Q^\circ = K_{CZ}$. Thus, we can define a map $\tau_y : \mathbb{N} \to I$, such that

$$\tau_y(k) = i \qquad \text{if } z_y^k \in Q_i \in CZ^\circ.$$

Consequently,

$$\delta_{Q_{\tau_y(k)}} \leq \operatorname{dist}(z_y^k, K_p) \leq |z_y^k - z_y| \leq \operatorname{dist}(y, K_p) = |y - z_y| \leq |y - x| \leq \delta_{\widehat{Q}} \leq \delta \quad (k \in \mathbb{N}); \tag{5.65}$$

$$Q_{\tau_y(k)} \subset B(y, 2|y - x|) \subset 5\widehat{Q} \quad (k \in \mathbb{N}). \tag{5.66}$$

For the proof of (5.65), we use Lemma 4.10, which implies $\delta_{Q_{\tau_y(k)}} \leq \operatorname{dist}(Q_{\tau_y(k)}, K_p) \leq \operatorname{dist}(z_y^k, K_p)$, where the second inequality uses that $z_y^k \in Q_{\tau_y(k)}$; further, $\operatorname{dist}(z_y^k, K_p) \leq |z_y^k - z_y| \leq \operatorname{dist}(y, K_p) = |y - z_y|$ because $z_y^k$ is on the segment connecting $y$ to a nearest point $z_y$ of $K_p$, and the remaining inequalities in (5.65) are immediate from (5.58). Lastly, the first inclusion of (5.66) uses that $z_y^k \in Q_{\tau_y(k)}$, $|z_y^k - y| \leq |x - y|$ and $\delta_{Q_{\tau_y(k)}} \leq |x - y|$ (see (5.64) and (5.65)); the second inclusion of (5.66) uses that $y \in \widehat{Q}$ and $|y - x| \leq \delta_{\widehat{Q}}$ (see (5.58)).

Let $k \in \mathbb{N}$. (We will later send $k \to \infty$.) Because $x \in K_p$ and $y \in K_{CZ}$, we have $P_x = S_x$ and $P_y = J_y G$. By (5.64), we have $|y - z_y^k| \leq |y - z_y| \leq |y - x| \leq \delta$. By applying the triangle inequality, and then (2.2),

$$\begin{aligned} &|P_x - P_y|_{y,\delta} \\ &\quad = |S_x - J_y G|_{y,\delta} \\ &\quad \leq |S_x - S_{z_y}|_{y,\delta} + |S_{z_y} - R_{x_{\tau_y(k)}}|_{y,\delta} + |R_{x_{\tau_y(k)}} - J_{z_y^k} G|_{y,\delta} + |J_{z_y^k} G - J_y G|_{y,\delta} \\ &\quad \overset{(2.2)}{\lesssim} |S_x - S_{z_y}|_{x,\delta} + |S_{z_y} - R_{x_{\tau_y(k)}}|_{z_y,\delta} + |R_{x_{\tau_y(k)}} - J_{z_y^k} G|_{z_y^k,\delta} + |J_{z_y^k} G - J_y G|_{y,\delta}. \end{aligned} \tag{5.67}$$

We analyze the four terms on the right-hand side of (5.67), one by one.

From (5.58), since $y \in \widehat{Q}$, we deduce $z_y \in 3\widehat{Q}$. Also, note that $|x - z_y| \leq 2\delta$, according to (5.60). We apply (2.3), and then the Sobolev Inequality on $3\widehat{Q}$ to estimate

$$|S_x - S_{z_y}|_{x,\delta} \lesssim |S_x - S_{z_y}|_{x,|x-z_y|} = |J_x H - J_{z_y} H|_{x,|x-z_y|} \lesssim \|H\|_{L^{m,p}(3\widehat{Q})}. \tag{5.68}$$

Because $(\vec{S}, \vec{R}) \in C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})$, we have $R_{x_i} \to S_z$ whenever $x_i \in \mathfrak{B}_{CZ}$, $x_i \to z$, $z \in K_p$. Observe, $|x_{\tau_y(k)} - z_y^k| \leq \delta_{Q_{\tau_y(k)}}$ (as both $x_{\tau_y(k)}$ and $z_y^k$ belong to $Q_{\tau_y(k)}$). Further, $\delta_{Q_{\tau_y(k)}} \leq \operatorname{dist}(z_y^k, K_p) \to 0$ as $k \to \infty$ (see (5.63) and (5.65)). When combined with (5.62), this implies $x_{\tau_y(k)} \to z_y$ as $k \to \infty$. Hence, $R_{x_{\tau_y(k)}} \to S_{z_y}$ as $k \to \infty$. Thus,

$$\lim_{k\to\infty} |S_{z_y} - R_{x_{\tau_y(k)}}|_{z_y,\delta} = 0. \tag{5.69}$$

From (5.65), $\delta_{Q_{\tau_y(k)}} \leq \delta$, So, applying (2.3) and then (5.10), using that $z_y^k$ belongs to $Q_{\tau_y(k)}$, we have

$$|R_{x_{\tau_y(k)}} - J_{z_y^k} G|_{z_y^k,\delta} \leq |R_{x_{\tau_y(k)}} - J_{z_y^k} G|_{z_y^k,\delta_{Q_{\tau_y(k)}}} \lesssim \|G, R_{x_{\tau_y(k)}}\|_{\mathscr{J}_*(f,\mu;Q_{\tau_y(k)})}. \tag{5.70}$$

We recall that $|z_y^k - y| \leq \delta$ (see (5.64)). Also note that by (5.61), $\operatorname{int}(B(y, |z_y - y|)) \cap Q^\circ \subset (\mathbb{R}^n \setminus K_p) \cap Q^\circ = K_{CZ}$. So we can apply (2.3) and then the Sobolev inequality on $\operatorname{int}(B(y, |z_y - y|)) \cap Q^\circ$, and deduce

$$|J_{z_y^k} G - J_y G|_{y,\delta} \leq |J_{z_y^k} G - J_y G|_{y,|z_y^k - y|} \lesssim \|G\|_{L^{m,p}(\operatorname{int}(B(y,|z_y-y|))\cap Q^\circ)}. \tag{5.71}$$

Recalling (5.66), (5.58), (5.61), we let $k \to \infty$ in (5.67), and use (5.68)-(5.71) to conclude

$$|P_x - P_y|_{y,\delta} \lesssim \|H\|_{L^{m,p}(3\widehat{Q})} + \sup_{Q_i \subseteq 5\widehat{Q}} \{\|G, R_{x_i}\|_{\mathscr{J}_*(f,\mu;Q_i)}\} + \sup\left\{\|G\|_{L^{m,p}(\mathrm{int}(B(z,r))\cap Q^\circ)} : z \in \widehat{Q}, r \le \delta, \mathrm{int}(B(z,r)) \cap Q^\circ \subset K_{CZ}\right\}. \tag{5.72}$$

This completes the proof of inequality (5.55) in Case 3.

Case 4: Suppose $x, y \in K_{CZ}$ satisfy $\mathrm{dist}(y, K_p) \le 3|y-x|$ and $\mathrm{dist}(x, K_p) \le 3|y-x|$. Because $K_p$ is closed, there exist $z_x, z_y \in K_p$ such that $\mathrm{dist}(x, K_p) = |z_x - x|$ and $\mathrm{dist}(y, K_p) = |z_y - y|$. Hence, we have

$$|z_x - x|, |z_y - y| \le 3|y-x| \le 3\delta_{\widehat{Q}} \le 3\delta. \tag{5.73}$$

Consequently,

$$|z_x - z_y| \le |z_x - x| + |x-y| + |y - z_y| \le 7\delta_{\widehat{Q}} \le 7\delta, \tag{5.74}$$

and $z_x, z_y \in 7\widehat{Q}$ because $x, y \in \widehat{Q}$. Considering (5.73) and (5.74), we apply (2.2) and (2.3) to estimate

$$|P_x - P_y|_{y,\delta} \lesssim |P_x - P_{z_x}|_{x,7\delta} + |P_{z_x} - P_{z_y}|_{z_x,7\delta} + |P_{z_y} - P_y|_{y,7\delta}.$$

Because $z_x, z_y \in K_p$, while $z, y \in K_{CZ}$, we apply inequalities (5.57) of Case 1 and (5.72) of Case 3 with the cube $7\widehat{Q}$ playing the role of $\widehat{Q}$, and $7\delta$ playing the role of $\delta$, to further reduce this:

$$\begin{aligned} &|P_x - P_y|_{y,\delta} \\ &\quad\lesssim \|H\|_{L^{m,p}(21\widehat{Q})} + \sup_{Q_i \subseteq 35\widehat{Q}} \{\|G, R_{x_i}\|_{\mathscr{J}_*(f,\mu;Q_i)}\} \\ &\qquad + \sup\left\{\|G\|_{L^{m,p}(\mathrm{int}(B(z,r))\cap Q^\circ)} : z \in 7\widehat{Q}, r \le 7\delta, \mathrm{int}(B(z,r)) \cap Q^\circ \subset K_{CZ}\right\}. \end{aligned}$$

This completes the proof of the inequality (5.55) in Case 4.

Since Cases 1–4 are exhaustive, we have proven (5.55).

We prepare to apply Corollary 2.12 to show that $G \in L^{m,p}(Q^\circ)$. Fix $\delta > 0$ and fix a congruent $\delta$-packing $\pi \in \Pi_{\simeq}(Q^\circ)$. Thus, $\pi$ is a family of cubes in $Q^\circ$ of equal sidelength $\delta$, with pairwise disjoint interiors.

For the next calculation we use the terminology of local approximation error, $E(G, Q)$, in Section 2.2.1. For $\widehat{Q} \in \pi$, $z \in \widehat{Q}$, $\delta = \delta_{\widehat{Q}}$, we have

$$
\begin{aligned}
E(G,\widehat{Q})/\delta^m \leq \|G-P_z\|_{L^p(\widehat{Q})}/\delta^m \leq \|G-P_z\|_{L^\infty(\widehat{Q})}\delta^{n/p-m} \\
&\leq \sup_{x,y\in\widehat{Q}}\{|G(y)-P_x(y)|\}\delta^{n/p-m} \\
&\leq \sup_{x,y\in\widehat{Q}}|P_y-P_x|_{y,\delta},
\end{aligned}
$$

where the last line follows from (5.54). Combining this with (5.55),

$$
\begin{aligned}
\sum_{\widehat{Q}\in\pi}\big(E(G,\widehat{Q})/\delta^m\big)^p &\lesssim \sum_{\widehat{Q}\in\pi}\sup_{x,y\in\widehat{Q}}|P_y-P_x|^p_{y,\delta} \\
&\lesssim \sum_{\widehat{Q}\in\pi}\Big(\|H\|_{L^{m,p}(21\widehat{Q})} + \sup_{Q_i\subseteq 35\widehat{Q}}\{\|G,R_{x_i}\|_{\mathscr{J}_*(f,\mu;Q_i)}\} \\
&\quad + \sup_{\substack{z\in 7\widehat{Q},r\leq 7\delta \\ \mathrm{int}(B(z,r))\cap Q^\circ\subset K_{CZ}}}\{\|G\|_{L^{m,p}(\mathrm{int}(B(z,r))\cap Q^\circ)}\}\Big)^p \\
&\lesssim \|G,\vec{R}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|H\|^p_{L^{m,p}(\mathbb{R}^n)} \\
&\lesssim \|G,\vec{R}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + (\|\vec{S}\|_{L^{m,p}(K_p)}+\eta)^p,
\end{aligned}
$$

where the second to last inequality follows because $y\in 7\widehat{Q},\ r\leq 7\delta \implies B(y,r)\subset 35\widehat{Q}$, and because $\{35\widehat{Q}\}_{\widehat{Q}\in\pi}$ has bounded overlap (recall $\pi$ consists of cubes with pairwise disjoint interiors and equal sidelength), and the last inequality follows because $\|H\|_{L^{m,p}(\mathbb{R}^n)} \leq \|\vec{S}\|_{L^{m,p}(K_p)}+\eta$. We let $\eta\to 0$, then take the supremum over $\pi\in\Pi_{\simeq}(Q^\circ)$ and apply Corollary 2.12 to conclude $G\in L^{m,p}(Q^\circ)$ and

$$
\|G\|_{L^{m,p}(Q^\circ)} \lesssim \|G,\vec{R}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|\vec{S}\|_{L^{m,p}(K_p)}.
$$

Next, we will show $J_xG=S_x$ for all $x\in K_p$. We claim that $\vec{P}\in Wh(Q^\circ)$ satisfies $|P_x-P_y|_{y,|x-y|}\leq A<\infty$ for all $x,y\in Q^\circ$. Let $x,y\in Q^\circ$, and fix $Q\subset Q^\circ$ with $x,y\in Q$ and $\delta_Q=|x-y|$. By (5.55), we have

$$
\begin{aligned}
|P_x-P_y|_{y,|x-y|} &\lesssim \|H\|_{L^{m,p}(21Q)} + \sup_{Q_i\subseteq 35Q}\{\|G,R_{x_i}\|_{\mathscr{J}_*(f,\mu;Q_i)}\} \\
&\quad + \sup_{\substack{z\in 7Q,r\leq 7\delta \\ \mathrm{int}(B(z,r))\cap Q^\circ\subset K_{CZ}}}\{\|G\|_{L^{m,p}(\mathrm{int}(B(z,r))\cap Q^\circ)}\} \\
&\lesssim \|H\|_{L^{m,p}(\mathbb{R}^n)} + \|G,\vec{R}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|G\|_{L^{m,p}(K_{CZ})} \\
&\lesssim \|H\|_{L^{m,p}(\mathbb{R}^n)} + \|G,\vec{R}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} < \infty.
\end{aligned}
$$

Since $G(x)=P_x(x)$ for $x\in Q^\circ$ (see (5.54), we can apply Lemma 2.13 to deduce $J_xG=P_x$ for all $x\in Q^\circ$. Since $P_x=S_x$ for $x\in K_p$, we have $J_xG=S_x$ for all $x\in K_p$. This completes the proof of Lemma 5.6. □

*5.3.3. Patching estimates for restriction of $\mu$*

We state variants of the last two lemmas for the restriction of the measure $\mu$ to a Borel set $E \subset \mathbb{R}^n$. Their proofs follow from the proofs of Lemma 5.5 and Lemma 5.6 with the measure $\mu$ replaced by $\mu|_E$.

**Lemma 5.7.** *Suppose we are given a collection of functions $\{G_i\}_{i\in I} \subset L^{m,p}(\mathbb{R}^n)$, a Whitney field $\vec{P} = (P_{x_i})_{i\in I} \in Wh(\mathfrak{B}_{CZ})$, and a Borel set $E \subset Q^\circ$.*

*Define $G : K_{CZ} \to \mathbb{R}$ by $G(x) = \sum_{i\in I} G_i(x) \cdot \theta_i(x)$, where $\{\theta_i\}_{i\in I}$ is a partition of unity satisfying* ***(POU1)-(POU4)*** *(see Section 4.3). Then*

$$\|G, \vec{P}\|^p_{\mathscr{J}_*(f,\mu|_E;K_{CZ},CZ^\circ)} \lesssim \sum_{i\in I} \|G_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap E};1.1Q_i\cap Q^\circ)} + \sum_{i,i'\in I,\ i\leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i.$$

**Lemma 5.8.** *Fix a collection of functions $\{G_i\}_{i\in I} \subset L^{m,p}(\mathbb{R}^n)$, and Whitney fields $\vec{R} = (R_{x_i})_{i\in I} \in Wh(\mathfrak{B}_{CZ})$ and $\vec{S} = (S_x)_{x\in E} \in Wh(K_p)$. Let $E \subset Q^\circ$ be a Borel set. Let $G : Q^\circ \to \mathbb{R}^n$ be defined as*

$$G(x) = \begin{cases} \sum_{i\in I} G_i(x) \cdot \theta_i(x) & x \in K_{CZ} \\ S_x(x) & x \in K_p, \end{cases}$$

*where $\{\theta_i\}_{i\in I}$ is a partition of unity satisfying* ***(POU1)-(POU4)****.*

*If $(G, \vec{R}, \vec{S})$ satisfies the conditions $\|G, \vec{R}\|_{\mathscr{J}_*(f,\mu|_E;K_{CZ},CZ^\circ)} + \|\vec{S}\|_{L^{m,p}(K_p)} < \infty$ and $(\vec{S}, \vec{R}) \in C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})$ then $G \in L^{m,p}(Q^\circ)$, $J_x G = S_x$ for all $x \in K_p$, and*

$$\|G\|_{L^{m,p}(Q^\circ)} \lesssim \|G, \vec{R}\|_{\mathscr{J}_*(f,\mu|_E;K_{CZ},CZ^\circ)} + \|\vec{S}\|_{L^{m,p}(K_p)}.$$

## 6. Further constraints on extension

Let $Q^\circ$, $CZ^\circ = \{Q_i\}_{i\in I}$, $K_{CZ}$, and $K_p$ be as defined in Section 4. Let $\mathfrak{B}_{CZ} = \{x_i\}_{i\in I}$ be the set of all CZ basepoints.

*6.1. Definition and properties of the space $\mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$*

For $(f, \vec{P}, \vec{S}) \in \mathscr{J}(\mu) \times Wh(\mathfrak{B}_{CZ}) \times Wh(K_p)$, we define:

$$\|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} = \inf\left\{ \|F, \vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} : \begin{array}{l} F \in L^{m,p}(Q^\circ) \text{ and} \\ J_x(F) = S_x \text{ for all } x \in K_p \end{array} \right\}. \tag{6.1}$$

Here, we have used the $\mathscr{J}_*(f, \mu; \Omega, CZ^\circ)$ functional defined in (5.45), with $\Omega = Q^\circ$.

We define the seminormed vector space:

$$\mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p) = \left\{ (f, \vec{P}, \vec{S}) : \begin{matrix} f \in \mathscr{J}(\mu),\ \vec{P} \in Wh(\mathfrak{B}_{CZ}),\ \vec{S} \in Wh(K_p), \\ \|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)} < \infty \end{matrix} \right\}. \tag{6.2}$$

The next result gives a compatibility condition between the Whitney fields $\vec{P}$ and $\vec{S}$ whenever $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$.

**Proposition 6.1.** *Let $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$. Then $(\vec{P}, \vec{S}) \in Wh(\mathfrak{B}_{CZ} \cup K_p)$, and*

$$\|(\vec{P}, \vec{S})\|_{C^{m-1,1-n/p}(\mathfrak{B}_{CZ} \cup K_p)} \leq C_H \|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)}. \tag{6.3}$$

*Here, $C_H > 0$ is a universal constant, determined only by $m$, $n$, and $p$.*

**Proof.** Let $\eta > 0$ be arbitrary. Let $H \in L^{m,p}(Q^\circ)$ satisfy $J_x H = S_x$ for all $x \in K_p$, and

$$\|H, \vec{P}\|_{\mathscr{J}_*(f,\mu; Q^\circ, CZ^\circ)} \leq \|f, \vec{P}, \vec{S}\|_{\mathscr{J}(\mu; Q^\circ, K_{CZ}; K_p)} + \eta. \tag{6.4}$$

<u>Case 1:</u> As a consequence of the Sobolev Inequality, if $x, y \in K_p$, then

$$|S_x - S_y|_{y,|y-x|} = |J_x H - J_y H|_{y,|y-x|} \lesssim \|H\|_{L^{m,p}(Q^\circ)} \lesssim \|H, \vec{P}\|_{\mathscr{J}_*(f,\mu; Q^\circ, CZ^\circ)}. \tag{6.5}$$

<u>Case 2:</u> If $x \in K_p$ and $y \in \mathfrak{B}_{CZ}$, then $y = x_i$ for some $i \in I$, and thanks to Lemma 4.10, $\delta_{Q_i} \leq \operatorname{dist}(x_i, K_p) \leq |x - x_i|$. Hence from the Sobolev Inequality, (2.3), and (5.10),

$$\begin{aligned} |S_x - P_{x_i}|_{x_i,|x-x_i|} &\leq |J_x H - J_{x_i} H|_{x_i,|x-x_i|} + |J_{x_i} H - P_{x_i}|_{x_i,\delta_{Q_i}} \\ &\lesssim \|H\|_{L^{m,p}(Q^\circ)} + \|H, P_{x_i}\|_{\mathscr{J}_*(f,\mu;Q_i)} \\ &\lesssim \|H, \vec{P}\|_{\mathscr{J}_*(f,\mu; Q^\circ, CZ^\circ)}. \end{aligned} \tag{6.6}$$

Consequently, using (2.2), we get that $|S_x - P_{x_i}|_{x,|x-x_i|} \lesssim \|H, \vec{P}\|_{\mathscr{J}_*(f,\mu; Q^\circ, CZ^\circ)}$.
<u>Case 3:</u> Similar to Case 2, if $x, y \in \mathfrak{B}_{CZ}$ are distinct, then $y = x_i$, $x = x_j$ for some $i, j \in I$, and $\delta_{Q_i}, \delta_{Q_j} \leq C|x_i - x_j|$. From the Sobolev Inequality, (2.2), (2.3), and (5.10),

$$\begin{aligned} |P_{x_j} - P_{x_i}|_{x_i,|x_j-x_i|} &\leq |J_{x_j} H - J_{x_i} H|_{x_i,|x_j-x_i|} + |J_{x_i} H - P_{x_i}|_{x_i,\delta_{Q_i}} + |J_{x_j} H - P_{x_j}|_{x_j,\delta_{Q_j}} \\ &\lesssim \|H\|_{L^{m,p}(Q^\circ)} + \|H, P_{x_i}\|_{\mathscr{J}_*(f,\mu;Q_i)} + \|H, P_{x_j}\|_{\mathscr{J}_*(f,\mu;Q_j)} \\ &\lesssim \|H, \vec{P}\|_{\mathscr{J}_*(f,\mu; Q^\circ, CZ^\circ)}. \end{aligned} \tag{6.7}$$

From (6.5), (6.6), and (6.7), we have

$$\|(\vec{P}, \vec{S})\|_{C^{m-1,1-n/p}(\mathfrak{B}_{CZ} \cup K_p)} \lesssim \|H, \vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)}.$$

In light of (6.4), and since $\eta > 0$ is arbitrary, we have proven inequality (6.3). □

### 6.2. Coherency

We start by introducing two pieces of terminology. Recall we have fixed a multiindex set $\mathcal{A} \subset \mathcal{M}$.

**Definition 6.1** *(Coherency).* Let $K \subset \mathbb{R}^n$ and $P_0 \in \mathcal{P}$. We say that a Whitney field $(P_x)_{x \in K} \in Wh(K)$ is coherent with $P_0$ if $\partial^\alpha P_x(x) = \partial^\alpha P_0(x)$ for all $x \in K$, and for all $\alpha \in \mathcal{A}$.

**Definition 6.2** *($K$-Coherency).* Let $K \subset Q^\circ \subset \mathbb{R}^n$ and $P_0 \in \mathcal{P}$. We say that a function $F \in C^{m-1,1-n/p}(Q^\circ)$ is $K$-coherent with $P_0$ if $\partial^\alpha J_x F(x) = \partial^\alpha F(x) = \partial^\alpha P_0(x)$ for all $x \in K$, and for all $\alpha \in \mathcal{A}$.

The next result will be used in Section 6.3, to give the proofs of the lemmas therein.

**Proposition 6.2.** *For* $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$,

$$\inf\left\{\|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} : \begin{array}{l} \vec{P} \in Wh(\mathfrak{B}_{CZ}) \text{ and } \vec{S} \in Wh(K_p) \text{ satisfy} \\ (\vec{P}, \vec{S}) \in Wh(\mathfrak{B}_{CZ} \cup K_p) \text{ is coherent with } P_0 \end{array}\right\}$$
$$\leq C\|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})}.$$

The rest of this section is devoted to the proof of Proposition 6.2.

#### 6.2.1. Proof of Proposition 6.2

Let $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$ be given.

Given $G \in L^{m,p}(\mathbb{R}^n)$, we will define a function $F \in L^{m,p}(Q^\circ)$, and the Whitney fields $\vec{P} \in Wh(\mathfrak{B}_{CZ})$, and $\vec{S} \in Wh(K_p)$, which satisfy the following properties:

$$\|F, \vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \leq C\|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}, \tag{6.8}$$

$$(\vec{P}, \vec{S}) \text{ is coherent with } P_0, \tag{6.9}$$

$$J_x F = S_x \text{ for all } x \in K_p. \tag{6.10}$$

Notice that (6.8) and (6.10) imply that $\|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \leq C\|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}$. By taking the infimum in this inequality over $G \in L^{m,p}(\mathbb{R}^n)$, the proposition follows. For $G \in L^{m,p}(\mathbb{R}^n)$,

$$\|G - P_0, 0\|_{\mathscr{J}(f-P_0,\mu;\delta_{Q^\circ})} = \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.$$

Therefore, in the proof of (6.8)–(6.10) it suffices to assume that

$$P_0 = 0.$$

We now explain how to construct $F, \vec{P}, \vec{S}$ that satisfy (6.8)–(6.10) for $P_0 = 0$. For reference, see (6.16) for the definition of $F$, see (6.14) for the definition of $\vec{P} = (P_{x_i})_{i \in I}$, and see (6.17) for the definition of $\vec{S} = (S_x)_{x \in K_p}$.

We shall make use of the auxiliary polynomials $P_x^\alpha \in \mathcal{P}$ and functions $\varphi_x^\alpha \in L^{m,p}(\mathbb{R}^n)$ ($x \in Q^\circ$, $\alpha \in \mathcal{A}$) defined in Section 3.3.1. Recall $(P_x^\alpha)$ satisfies (3.55)-(3.56) and $(\varphi_x^\alpha)$ satisfies (3.69)-(3.72), while $J_x \varphi_x^\alpha = P_x^\alpha$. Each $\varphi_x^\alpha$ is defined in terms of another family of functions $(\varphi^\beta)_{\beta \in \mathcal{A}}$, via (3.69) which states that

$$\varphi_x^\alpha = \sum_{\beta \in \mathcal{A}} A_{\alpha\beta}^x \cdot \varphi^\beta,$$

where $(A_{\alpha\beta}^x)_{\alpha,\beta \in \mathcal{M}}$ is a $(C, C\epsilon)$ near-triangular matrix. The inverse of a near-triangular matrix is near-triangular with comparable parameters. So, for any $x, y \in Q^\circ$ and $\alpha \in \mathcal{A}$, we can write $\varphi_y^\alpha$ as a bounded linear combination of $(\varphi_x^\beta)_{\beta \in \mathcal{A}}$. Precisely, there exist coefficients $(\omega_{\alpha\beta}^{xy})_{\alpha,\beta \in \mathcal{A}, x,y \in Q^\circ} \subset \mathbb{R}$ such that

$$\begin{aligned} &\varphi_y^\alpha = \sum_{\beta \in \mathcal{A}} \omega_{\alpha\beta}^{xy} \cdot \varphi_x^\beta \qquad && (\alpha \in \mathcal{A}), \text{ and} \\ &|\omega_{\alpha\beta}^{xy}| \leq C && (\alpha, \beta \in \mathcal{A}). \end{aligned} \tag{6.11}$$

From (3.71), we have $J_{x_i} \varphi_{x_i}^\alpha = P_{x_i}^\alpha$. It follows from equation (5.3) in Lemma 5.1 that the functions $\varphi_{x_i}^\alpha$ are locally bounded on $3Q_i^+$. Indeed, by taking $Q = Q_i$ and $y = x_i$ in (5.3), we have

$$|\partial^\beta \varphi_{x_i}^\alpha(x_i)| = |\partial^\beta P_{x_i}^\alpha(x_i)| \leq C \delta_{Q_i}^{|\alpha|-|\beta|} \qquad (\alpha \in \mathcal{A},\ \beta \in \mathcal{M},\ i \in I). \tag{6.12}$$

We shall define functions $F_i$ on the CZ cubes, and patch them together on $K_{CZ}$ using a partition of unity. Define $F_i : \mathbb{R}^n \to \mathbb{R}$ and $P_{x_i} \in \mathcal{P}$ for each $i \in I$ as

$$F_i(x) = G(x) - \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(x_i) \cdot \varphi_{x_i}^\alpha(x); \text{ and} \tag{6.13}$$

$$P_{x_i} = J_{x_i} F_i = J_{x_i} G - \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(x_i) \cdot P_{x_i}^\alpha. \tag{6.14}$$

From (3.55), the polynomials $(P_{x_i}^\alpha)_{\alpha \in \mathcal{A}}$ satisfy $\partial^\beta P_{x_i}^\alpha(x_i) = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{A}$. Thus,

$$\partial^\beta P_{x_i}(x_i) = \partial^\beta F_i(x_i) = 0 \text{ for } i \in I, \beta \in \mathcal{A}. \tag{6.15}$$

Now we define $F : Q^\circ \to \mathbb{R}$.

$$F(x) = \begin{cases} \sum_{i \in I} F_i(x) \cdot \theta_i(x) & x \in K_{CZ}, \\ G(x) & x \in K_p. \end{cases} \tag{6.16}$$

where $\{\theta_i\}_{i \in I}$ is a partition of unity satisfying **(POU1)-(POU4)** (see Section 4.3). By construction, $F \in C^{m-1}_{loc}(K_{CZ})$, and hence, $J_y F$ is well-defined for $y \in K_{CZ}$.

For any $y \in Q^\circ$, we define the polynomial

$$S_y := \begin{cases} J_y F & y \in K_{CZ}, \\ J_y G - \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(y) P^\alpha_y & y \in K_p. \end{cases} \tag{6.17}$$

By restricting the family of polynomials $(S_y)_{y \in Q^\circ}$ to $K_p$, we obtain the Whitney field $\vec{S} \in Wh(K_p)$.

The basis $(P^\alpha_y)_{\alpha \in \mathcal{A}}$ satisfies $\partial^\beta P^\alpha_y(y) = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{A}$. Hence, $\partial^\beta S_y(y) = 0$ for $y \in K_p$, $\beta \in \mathcal{A}$. Thus,

$$(S_y)_{y \in K_p} \text{ is coherent with } P_0 = 0. \tag{6.18}$$

The cutoff functions $\theta_i$ satisfy $x_i \in \operatorname{supp}(\theta_j)$ only if $j = i$, and $J_{x_i} \theta_i = 1$, so from (6.17), (6.16), (6.14), $S_{x_i} = J_{x_i} F = J_{x_i} F_i = P_{x_i}$. Thus,

$$S_{x_i} = P_{x_i} \qquad (i \in I). \tag{6.19}$$

By Lemma 5.3, we have $P^\alpha_y(y) = 0$ for $y \in K_p$, so by definition of $F$ (see (6.16)), we have $S_y(y) = J_y G(y) = G(y) = F(y)$ for $y \in K_p$. Evidently, also $S_y(y) = F(y)$ for $y \in K_{CZ}$. Thus,

$$S_y(y) = F(y) \qquad (y \in Q^\circ). \tag{6.20}$$

**Lemma 6.3.** *Let $x, y \in K_p \cup \mathfrak{B}_{CZ}$. Let $U \subset \mathbb{R}^n$ be a domain such that $x, y \in U$, and such that $U$ is the union of two $\eta$-non-degenerate boxes with an interior point in common (in particular, $U$ can be a cube, with $\eta = 1$). Then*

$$|S_x - S_y|_{y,|y-x|} \lesssim_\eta \|G\|_{L^{m,p}(U)} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta \in \mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(U)}, \tag{6.21}$$

*where $(\varphi^\beta)_{\beta \in \mathcal{A}}$ is as defined in Proposition 3.9 of Section 3.3.1, and the constants in $\lesssim_\eta$ depend only on $m, n, p$ and $\eta$.*

**Proof.** We define $F_y \in L^{m,p}(\mathbb{R}^n)$ $(y \in Q^\circ)$ by

$$F_y(x) = G(x) - \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(y) \cdot \varphi^\alpha_y(x). \tag{6.22}$$

Observe, by definition of the polynomial $S_y$ in (6.17) and by property (3.71) of $\varphi_y^\alpha$, that $S_y = J_y F_y$ for $y \in K_p$. On the other hand, note $F_{x_i} = F_i$ defined in (6.13). Hence, by (6.19), $S_{x_i} = P_{x_i} = J_{x_i} F_i = J_{x_i} F_{x_i}$ for $i \in I$. Thus,

$$S_x = J_x F_x \qquad (x \in K_p \cup \mathfrak{B}_{CZ}).$$

Because $P_0 = 0$, by (2.14), we have

$$|\partial^\beta G(z)| = |\partial^\beta (G - P_0)(z)| \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \quad (\beta \in \mathcal{A},\ z \in Q^\circ). \tag{6.23}$$

For distinct $x, y \in K_p \cup \mathfrak{B}_{CZ}$, we want to bound

$$|S_y - S_x|_{y,|x-y|} = \Big( \sum_{\gamma \in \mathcal{M}} |\partial^\gamma (S_y - S_x)(y)|^p |x-y|^{|\gamma|p + n - mp} \Big)^{1/p}.$$

For $\gamma \in \mathcal{M}$, we have

$$\begin{aligned} |\partial^\gamma (S_y - S_x)(y)| &= |\partial^\gamma (F_y - J_x F_x)(y)| \\ &\le |\partial^\gamma (F_x - F_y)(y)| + |\partial^\gamma (J_x F_x - F_x)(y)| \end{aligned} \tag{6.24}$$

Now apply (6.22) and (6.11) to estimate

$$\begin{aligned} |\partial^\gamma (F_x - F_y)(y)| &= \Big| \partial^\gamma \Big( \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(x) \varphi_x^\alpha - \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(y) \varphi_y^\alpha \Big)(y) \Big| \\ &= \Big| \partial^\gamma \Big( \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(x) \sum_{\beta \in \mathcal{A}} (\omega_{\alpha\beta}^{yx} \cdot \varphi_y^\beta) - \sum_{\beta \in \mathcal{A}} \partial^\beta G(y) \cdot \varphi_y^\beta \Big)(y) \Big| \\ &\le \sum_{\beta \in \mathcal{A}} \Big| \sum_{\alpha \in \mathcal{A}} \partial^\alpha G(x) \cdot \omega_{\alpha\beta}^{yx} - \partial^\beta G(y) \Big| |\partial^\gamma \varphi_y^\beta (y)| \\ &= \sum_{\beta \in \mathcal{A}} |\partial^\beta F_x(y)| |\partial^\gamma \varphi_y^\beta (y)| \end{aligned} \tag{6.25}$$

Because $\mathcal{A}$ is monotonic, if $\beta \in \mathcal{A}$ and $\beta + \alpha \in \mathcal{M}$ then $\beta + \alpha \in \mathcal{A}$, so

$$\partial^\beta J_x F_x(y) = \sum_{\beta + \alpha \in \mathcal{M}} \partial^{\beta+\alpha} F_x(x) \frac{(y-x)^\alpha}{\alpha!} = 0 \qquad (\beta \in \mathcal{A}). \tag{6.26}$$

We claim that $|\partial^\gamma \varphi_y^\beta(y)| \lesssim |y - x|^{|\beta| - |\gamma|}$ for $\gamma \in \mathcal{M}$, $\beta \in \mathcal{A}$. Indeed, this estimate follows by (5.6) if $y \in K_p$, and by (5.2) if $y = x_i$ for some $i \in I$ (note: if $y = x_i$ then $x \notin Q_i$, as $y, x$ are distinct, hence $\delta_{Q_i}/2 \le |y - x| \le 1$). Thus, using (6.26) in (6.25), and then using the Sobolev Inequality, we bound

$$\begin{aligned}|\partial^\gamma(F_x - F_y)(y)| &\leq \sum_{\beta\in\mathcal{A}} |\partial^\beta F_x(y)||\partial^\gamma \varphi_y^\beta(y)| \\ &= \sum_{\beta\in\mathcal{A}} |\partial^\beta (F_x - J_x F_x)(y)||\partial^\gamma \varphi_y^\beta(y)| \\ &\lesssim \sum_{\beta\in\mathcal{A}} \|F_x\|_{L^{m,p}(U)}|y-x|^{m-|\beta|-n/p}|y-x|^{|\beta|-|\gamma|} \\ &\lesssim \|F_x\|_{L^{m,p}(U)}|y-x|^{m-|\gamma|-n/p}. \end{aligned} \tag{6.27}$$

On the other hand, by the Sobolev Inequality,

$$|\partial^\gamma(J_x(F_x) - F_x)(y)| \lesssim \|F_x\|_{L^{m,p}(U)}|x-y|^{m-|\gamma|-n/p}.$$

Combining the previous inequalities in (6.24), we have

$$|\partial^\gamma(S_y - S_x)(y)| \lesssim \|F_x\|_{L^{m,p}(U)}|x-y|^{m-|\gamma|-n/p} \quad (\gamma\in\mathcal{M}). \tag{6.28}$$

From (3.69), we have $\varphi_x^\alpha := \sum_{\beta\in\mathcal{A}} A^x_{\alpha\beta}\cdot\varphi^\beta$, where $(A^x_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ is a $(C, C\epsilon)$-near triangular matrix, and in particular $|A^x_{\alpha\beta}| \leq C$. Hence $\|\varphi^\alpha_x\|_{L^{m,p}(U)} \lesssim \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(U)}$ for all $\alpha\in\mathcal{A}$; applying this and (6.23), in the definition of $F_x$, we have

$$\begin{aligned}\|F_x\|_{L^{m,p}(U)} &\leq \|G\|_{L^{m,p}(U)} + \Big\| \sum_{\alpha\in\mathcal{A}} \partial^\alpha G(x)\varphi_x^\alpha \Big\|_{L^{m,p}(U)} \\ &\lesssim \|G\|_{L^{m,p}(U)} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(U)}. \end{aligned} \tag{6.29}$$

Substituting (6.29) into (6.28),

$$|\partial^\gamma(S_y - S_x)(y)|\cdot|x-y|^{|\gamma|+n/p-m} \lesssim \|G\|_{L^{m,p}(U)} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(U)}. \tag{6.30}$$

Thus, by definition of the norm $|\cdot|_{y,|x-y|}$, we conclude that

$$\begin{aligned}|S_y - S_x|^p_{y,|x-y|} &= \sum_{\gamma\in\mathcal{M}} |\partial^\gamma(S_x - S_y)(y)|^p|y-x|^{|\gamma|p+n-mp} \\ &\lesssim \|G\|^p_{L^{m,p}(U)} + \|G, P_0\|^p_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}\cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|^p_{L^{m,p}(U)}. \end{aligned}$$

This completes the proof of (6.21). □

**Lemma 6.4.** *The function $F|_{K_{CZ}} : K_{CZ} \to \mathbb{R}$ and Whitney field $\vec{P} \in Wh(\mathfrak{B}_{CZ})$ satisfy*

$$\|F, \vec{P}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}. \tag{6.31}$$

**Proof.** By (5.48),

$$\|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} \lesssim \sum_{i \in I} \|F_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i \cap Q^\circ};1.1Q_i \cap Q^\circ)} + \sum_{i \leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i$$

Because $J_{x_i} F_i = P_{x_i}$, by (2.12), $\|F_i - P_{x_i}\|_{L^p(Q_i)}/\delta^m_{Q_i} \lesssim \|F_i\|_{L^{m,p}(Q_i)}$, hence we have

$$\|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} \lesssim \sum_{i \in I} \left[ \|F_i\|^p_{L^{m,p}(1.1Q_i \cap Q^\circ)} + \int_{1.1Q_i} |F_i - f|^p d\mu \right] + \sum_{i \leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i. \tag{6.32}$$

We now bound the term $\sum_{i \leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i$ in (6.32). Recall from (6.19) that $P_{x_i} = S_{x_i}$. If $i \leftrightarrow i'$ then $Q_i$ and $Q_{i'}$ are neighboring CZ cubes, hence, $U_{ii'} := (1.1Q_i \cup 1.1Q_{i'}) \cap Q^\circ$ is the union of two $C$-non-degenerate boxes with a common interior point, and $x_i, x_{i'} \in U_{ii'}$. Note also $|x_i - x_{i'}| \simeq \delta_{Q_i}$, by the good geometry of the CZ decomposition. Hence, from Lemma 6.3, applied with $U = U_{ii'}$, for any $i \leftrightarrow i'$,

$$\begin{aligned} |P_{x_i} - P_{x_{i'}}|^p_i = |S_{x_i} - S_{x_{i'}}|^p_{x_i,\delta_{Q_i}} &\simeq |S_{x_i} - S_{x_{i'}}|^p_{x_i,|x_i - x_{i'}|} \\ &\lesssim \|G\|^p_{L^{m,p}((1.1Q_i \cup 1.1Q_{i'}) \cap Q^\circ)} \\ &\quad + \|G, P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)} \sum_{\beta \in \mathcal{A}} \|\varphi^\beta\|^p_{L^{m,p}((1.1Q_i \cup 1.1Q_{i'}) \cap Q^\circ)}. \end{aligned} \tag{6.33}$$

By summing (6.33) over all $i, i' \in I$ with $i \leftrightarrow i'$, and by using the bounded overlap of the regions $(1.1Q_i \cup 1.1Q_{i'}) \cap Q^\circ \subset Q^\circ$, we find that

$$\sum_{i \leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i \lesssim \|G\|^p_{L^{m,p}(Q^\circ)} + \|G, P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)} \sum_{\beta \in \mathcal{A}} \|\varphi^\beta\|^p_{L^{m,p}(Q^\circ)}.$$

Further, from (3.70) we have $\|\varphi^\beta\|_{L^{m,p}(Q^\circ)} \leq \|\varphi^\beta\|_{\mathscr{J}(0,\mu)} \lesssim 1$ for $\beta \in \mathcal{A}$. Also, $\|G\|_{L^{m,p}(Q^\circ)} \leq \|G, P_0\|_{\mathscr{J}_*(f,\mu;Q^\circ)}$. Hence,

$$\sum_{i \leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i \lesssim \|G, P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}. \tag{6.34}$$

Recall $P_0 = 0$. For $\alpha \in \mathcal{A}$, by (2.14),

$$|\partial^\alpha G(x_i)| = |\partial^\alpha (G - P_0)(x_i)| \leq C \|G, P_0\|_{\mathscr{J}_*(f,\mu;Q^\circ)}. \tag{6.35}$$

From (3.69), we have $\varphi^\alpha_{x_i} := \sum_{\beta \in \mathcal{A}} A^{x_i}_{\alpha\beta} \cdot \varphi^\beta$ ($\alpha \in \mathcal{A}$), where $(A^{x_i}_{\alpha\beta})_{\alpha,\beta \in \mathcal{A}}$ is a $(C, C\epsilon)$-near triangular matrix. In particular, $|A^{x_i}_{\alpha\beta}| \leq C$. We also have, from (3.70), $\|\varphi^\beta\|_{L^p(d\mu)} \leq$

$\|\varphi^\beta\|_{\mathscr{J}(0,\mu)} \lesssim 1$ for $\beta \in \mathcal{A}$. Thus, by definition of $F_i$ (see (6.13)), and by the triangle inequality, we deduce that

$$\begin{aligned}
&\sum_{i\in I}\int_{1.1Q_i}|F_i - f|^p d\mu \\
&\lesssim \sum_{i\in I}\left[\int_{1.1Q_i}|G-f|^p d\mu + \sum_{\alpha\in\mathcal{A}}|\partial^\alpha G(x_i)|^p \sum_{\beta\in\mathcal{A}}|A^{x_i}_{\alpha\beta}|^p \cdot \int_{1.1Q_i}|\varphi^\beta|^p d\mu\right] \\
&\lesssim \sum_{i\in I}\left[\int_{1.1Q_i}|G-f|^p d\mu + \|G,P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}\sum_{\beta\in\mathcal{A}}\int_{1.1Q_i}|\varphi^\beta|^p d\mu\right] \\
&\lesssim \int_{Q^\circ}|G-f|^p d\mu + \|G,P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}\sum_{\beta\in\mathcal{A}}\int_{Q^\circ}|\varphi^\beta|^p d\mu \\
&\lesssim \|G,P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}. \qquad (6.36)
\end{aligned}$$

Similarly, by instead using that $\|\varphi^\beta\|_{L^{m,p}(Q^\circ)} \leq \|\varphi^\beta\|_{\mathscr{J}(0,\mu)} \lesssim 1$ for $\beta \in \mathcal{A}$, and by definition of $F_i$, we have

$$\begin{aligned}
&\sum_{i\in I}\|F_i\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)} \\
&\lesssim \sum_{i\in I}\left[\|G\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)} + \sum_{\alpha\in\mathcal{A}}|\partial^\alpha G(x_i)|^p\sum_{\beta\in\mathcal{A}}|A^{x_i}_{\alpha\beta}|^p\|\varphi^\beta\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)}\right] \\
&\lesssim \sum_{i\in I}\left[\|G\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)} + \|G,P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}\sum_{\beta\in\mathcal{A}}\|\varphi^\beta\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)}\right] \\
&\lesssim \|G\|^p_{L^{m,p}(Q^\circ)} + \|G,P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}\sum_{\beta\in\mathcal{A}}\|\varphi^\beta\|^p_{L^{m,p}(Q^\circ)} \\
&\lesssim \|G,P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}. \qquad (6.37)
\end{aligned}$$

We substitute inequalities (6.34), (6.36), and (6.37) into (6.32) to deduce:

$$\|F,\vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} \lesssim \|G,P_0\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)}.$$

We apply (2.8) to conclude:

$$\|F,\vec{P}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} \lesssim \|G,P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.$$

This completes the proof of (6.31). The proof of the lemma is complete. □

**Lemma 6.5.** *The polynomials $S_x$ ($x \in Q^\circ$) defined in (6.17) satisfy*

$$\sup_{x,y\in Q^\circ} |S_x - S_y|_{x,|y-x|} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.$$

**Proof.** Let $x, y \in Q^\circ$. We will prove the desired inequality by considering the geometry of $x$ and $y$ in relation to $K_p$ and $K_{CZ}$:

Case 1: Suppose $x, y \in K_{CZ}$. Let $y \in Q_i \in CZ^\circ$, $x \in Q_j \in CZ^\circ$. Suppose $\delta_{Q_i} > 21|y-x|$; then $x \in 1.1Q_i$. By applying the Sobolev inequality, we have

$$|S_x - S_y|_{x,|y-x|} = |J_xF - J_yF|_{x,|y-x|} \lesssim \|F\|_{L^{m,p}((1.1Q_i)\cap Q^\circ)} \leq \|F\|_{L^{m,p}(K_{CZ})}. \tag{6.38}$$

Similarly, if $\delta_{Q_j} > 21|y-x|$; then $y \in 1.1Q_j$. By applying the Sobolev inequality, we have

$$|S_x - S_y|_{x,|y-x|} = |J_xF - J_yF|_{x,|y-x|} \lesssim \|F\|_{L^{m,p}((1.1Q_j)\cap Q^\circ)} \leq \|F\|_{L^{m,p}(K_{CZ})}. \tag{6.39}$$

Now suppose $\delta_{Q_i}, \delta_{Q_j} \leq 21|y-x|$; then $|x_i - x_j| \leq \delta_{Q_i} + |x-y| + \delta_{Q_j} \leq 43|x-y|$. We apply (2.2), the Sobolev Inequality, and Lemma 6.3 with $U = Q^\circ$ to deduce

$$\begin{aligned} |S_x - S_y|_{x,|y-x|} &\lesssim |S_y - S_{x_i}|_{y,\delta_{Q_i}} + |S_{x_i} - S_{x_j}|_{x_i,|x_i-x_j|} + |S_{x_j} - S_x|_{x,\delta_{Q_j}} \\ &= |J_yF - J_{x_i}F|_{y,\delta_{Q_i}} + |S_{x_i} - S_{x_j}|_{x_i,|x_i-x_j|} + |J_{x_j}F - J_xF|_{x,\delta_{Q_j}} \\ &\lesssim \|F\|_{L^{m,p}(Q_i)} + \|F\|_{L^{m,p}(Q_j)} + \|G\|_{L^{m,p}(Q^\circ)} \\ &\quad + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(Q^\circ)} \\ &\lesssim \|F\|_{L^{m,p}(K_{CZ})} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}, \end{aligned} \tag{6.40}$$

where in the last line we use (3.70). By (6.31), we have $\|F\|_{L^{m,p}(K_{CZ})} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}$. Considering (6.38), (6.39), and (6.40), we conclude for $x, y \in K_{CZ}$,

$$|S_x - S_y|_{x,|y-x|} \lesssim \|F\|_{L^{m,p}(K_{CZ})} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}. \tag{6.41}$$

Case 2: Suppose $x \in K_p$, $y \in K_{CZ}$. Let $y \in Q_i \in CZ^\circ$. Because $|x - x_i| \lesssim |y-x|$ and $\delta_{Q_i} \leq |y-x|$ (see Lemma 4.10), we can apply (2.2) and then the Sobolev Inequality to deduce

$$\begin{aligned} |S_x - S_y|_{x,|y-x|} &\lesssim |S_x - S_{x_i}|_{x,|x_i-x|} + |S_{x_i} - S_y|_{x_i,\delta_{Q_i}} \\ &= |S_x - S_{x_i}|_{x,|x_i-x|} + |J_{x_i}F - J_yF|_{x_i,\delta_{Q_i}} \\ &\lesssim |S_x - S_{x_i}|_{x,|x_i-x|} + \|F\|_{L^{m,p}(Q_i)}. \end{aligned} \tag{6.42}$$

We apply Lemma 6.3 with $U = Q^\circ$, recalling for $\alpha \in \mathcal{A}$, we have $\|\varphi^\alpha\|_{L^{m,p}(Q^\circ)} \leq \epsilon/C_0$ (see (3.70)), to deduce

$$|S_x - S_{x_i}|_{x,|x_i-x|} \lesssim \|G\|_{L^{m,p}(Q^\circ)} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(Q^\circ)}$$
$$\lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}. \tag{6.43}$$

Substituting this into (6.42), and using again that $\|F\|_{L^{m,p}(K_{CZ})} \leq \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}$, we have

$$|S_x - S_y|_{x,|y-x|} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.$$

Case 3: Suppose $x, y \in K_p$; then as in (6.43) in Case 2, we can apply Lemma 6.3 to deduce

$$|S_x - S_y|_{x,|y-x|} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.$$

This completes the proof of Lemma 6.5. □

**Lemma 6.6.** *The function $F$ defined in (6.16) belongs to $L^{m,p}(Q^\circ)$, and*

$$\|F\|_{L^{m,p}(Q^\circ)} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}. \tag{6.44}$$

**Proof.** We will use Corollary 2.12. Fix $\pi \in \Pi_{\simeq}(Q^\circ)$, a congruent $\delta$-packing of $Q^\circ$, with $\delta \leq \delta_{Q^\circ}$.

For $\widehat{Q} \in \pi$, $\delta_{\widehat{Q}} = \delta$, by Hölder's inequality,

$$E(F, \widehat{Q})\delta^{-m} \leq \sup_{x\in\widehat{Q}} \|F - S_x\|_{L^p(\widehat{Q})}\delta^{-m} \leq \sup_{x\in\widehat{Q}} \|F - S_x\|_{L^\infty(\widehat{Q})}\delta^{n/p-m}$$
$$\leq \sup_{x,y\in\widehat{Q}} |S_y - S_x|_{y,\delta}, \tag{6.45}$$

where the last line follows because $F(y) = S_y(y)$ for $y \in Q^\circ$ – see (6.20).

Fix $\widehat{Q} \in \pi$ and $x, y \in \widehat{Q}$. Then, $|x - y| \leq \delta_{\widehat{Q}} = \delta$.

If $\widehat{Q} \subset K_{CZ}$, then by the Sobolev Inequality,

$$|S_x - S_y|_{y,\delta} = |J_x F - J_y F|_{y,\delta} \lesssim \|F\|_{L^{m,p}(\widehat{Q})}. \tag{6.46}$$

Now suppose $\widehat{Q} \cap K_p \neq \emptyset$. We will show

$$|S_x - S_y|_{y,\delta} \lesssim \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(100\widehat{Q}\cap Q^\circ)}$$
$$+ \|G\|_{L^{m,p}(100\widehat{Q}\cap Q^\circ)} + \sup_{1.1Q_i\subset 100\widehat{Q}} \|F\|_{L^{m,p}(1.1Q_i\cap Q^\circ)}. \tag{6.47}$$

Case 1: Let $x, y \in K_p$. Then apply (6.21) with $U = \widehat{Q}$,

$$|S_x - S_y|_{y,\delta} \lesssim \|G\|_{L^{m,p}(\widehat{Q})} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta \in \mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(\widehat{Q})},$$

which implies (6.47).

Case 2: Let $y \in Q_i \in CZ^\circ$ and $\delta_{Q_i} > 21|x-y|$ or $x \in Q_j \in CZ^\circ$ and $\delta_{Q_j} > 21|x-y|$. Because $|P|_{x,\delta} \simeq |P|_{y,\delta}$ for $|x-y| \leq \delta$, we may assume without loss of generality that $y \in Q_i$, $\delta_{Q_i} > 21|x-y|$. Then $x \in 1.1Q_i \subset K_{CZ}$. As $x, y \in K_{CZ}$, we have $S_x = J_x F$ and $S_y = J_y F$. Because $\widehat{Q} \cap K_p \neq \emptyset$ and $x \in \widehat{Q}$, it follows that $\operatorname{dist}(x, K_p) \leq \delta_{\widehat{Q}}$. Also, by Lemma 4.10), we have $\operatorname{dist}(Q_i, K_p) \geq \delta_{Q_i}$. Thus,

$$\delta_{\widehat{Q}} \geq \operatorname{dist}(x, K_p) \geq \operatorname{dist}(Q_i, K_p) \geq \delta_{Q_i}.$$

Since $Q_i \cap \widehat{Q} \neq \emptyset$, we have $1.1Q_i \subset 100\widehat{Q}$, and by the Sobolev Inequality,

$$|S_x - S_y|_{y,\delta} = |J_x F - J_y F|_{y,\delta} \lesssim \|F\|_{L^{m,p}(1.1Q_i \cap Q^\circ)},$$

completing the proof of (6.47).

Case 3: Let $y \in Q_i \in CZ^\circ$ and $\delta_{Q_i} \leq 21|x-y| \leq 21\delta_{\widehat{Q}} = 21\delta$. Then $1.1Q_i \subset 100\widehat{Q}$. Recall $J_{x_i}F = S_{x_i}$. Then from (2.2), By the Sobolev inequality,

$$|S_{x_i} - S_y|_{y,\delta} = |J_{x_i}F - J_y F|_{y,\delta} \lesssim \|F\|_{L^{m,p}(Q_i)}.$$

Thus, by the triangle inequality and (2.2),

$$|S_x - S_y|_{y,\delta} \lesssim |S_x - S_{x_i}|_{x,\delta} + |S_{x_i} - S_y|_{y,\delta} \lesssim |S_x - S_{x_i}|_{x,\delta} + \|F\|_{L^{m,p}(Q_i)}. \quad (6.48)$$

We continue this bound by splitting into subcases.

Subcase 3a: Suppose $x \in K_p$. Note that $|x - x_i| \leq |x-y| + |y - x_i| \leq |x-y| + \delta_{Q_i} \leq 22\delta$. Further, $x, x_i \in 100\widehat{Q} \cap Q^\circ$. Thus, from (2.3), and (6.21) with $U = 100\widehat{Q} \cap Q^\circ$,

$$\begin{aligned} |S_{x_i} - S_x|_{x,\delta} &\lesssim |S_{x_i} - S_x|_{x,|x-x_i|} \\ &\lesssim \|G\|_{L^{m,p}(100\widehat{Q} \cap Q^\circ)} + \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta \in \mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(100\widehat{Q} \cap Q^\circ)}. \end{aligned}$$

Substituting this into (6.48), we have

$$\begin{aligned} |S_x - S_y|_{y,\delta} \lesssim \|F\|_{L^{m,p}(Q_i)} &+ \|G\|_{L^{m,p}(100\widehat{Q} \cap Q^\circ)} \\ &+ \|G, P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta \in \mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(100\widehat{Q} \cap Q^\circ)}, \end{aligned}$$

completing the proof of (6.47).

Subcase 3b: Suppose $x \notin K_p$. Then $x \in Q_j$ for some $Q_j \in CZ^\circ$ and because of Case 2, we can assume $\delta_{Q_j} \leq 21|x-y| \leq 21\delta$. Hence, $1.1Q_j \subset 100\widehat{Q}$. Also, $|x - x_j| \leq \delta_{Q_j} \leq 21\delta$, and

$$|x_i - x_j| \leq |x_j - x| + |x-y| + |y - x_i| \leq \delta_{Q_j} + |x-y| + \delta_{Q_i} \leq 43\delta.$$

Thus, from (2.2), (2.3), and (6.21) with $U = 100\widehat{Q} \cap Q^\circ$, and the Sobolev Inequality,

$$\begin{aligned}
|S_x - S_{x_i}|_{x,\delta} &\lesssim |S_{x_i} - S_{x_j}|_{x_j,\delta} + |S_{x_j} - S_x|_{x,\delta} \\
&\lesssim |S_{x_i} - S_{x_j}|_{x_j,|x_i - x_j|} + |J_{x_j}F - J_x F|_{x,\delta} \\
&\lesssim \|G\|_{L^{m,p}(100\widehat{Q}\cap Q^\circ)} + \|G,P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(100\widehat{Q}\cap Q^\circ)} \\
&\quad + \|F\|_{L^{m,p}(Q_j)}.
\end{aligned} \tag{6.49}$$

By substituting (6.49) into (6.48), we have

$$\begin{aligned}
|S_x - S_y|_{y,\delta} \lesssim \|F\|_{L^{m,p}(Q_i)} + \|F\|_{L^{m,p}(Q_j)} + \|G\|_{L^{m,p}(100\widehat{Q}\cap Q^\circ)} \\
+ \|G,P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|_{L^{m,p}(100\widehat{Q}\cap Q^\circ)}.
\end{aligned}$$

This completes the proof of (6.47).

Consequently, using (6.45), (6.46), and (6.47), we have

$$\begin{aligned}
&\sum_{\widehat{Q}\in\pi} \left(E(F,\widehat{Q})/\delta_{\widehat{Q}}^m\right)^p \\
&\lesssim \sum_{\widehat{Q}\in\pi} \sup_{x,y\in\widehat{Q}} |S_x - S_y|_{y,\delta}^p \\
&\lesssim \sum_{\widehat{Q}\in\pi,\widehat{Q}\cap K_p\neq\emptyset} \Bigg( \sup_{1.1Q_i\subset 100\widehat{Q}} \|F\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)} + \|G\|^p_{L^{m,p}(100\widehat{Q}\cap Q^\circ)} \\
&\quad + \|G,P_0\|^p_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|^p_{L^{m,p}(100\widehat{Q}\cap Q^\circ)} \Bigg) + \sum_{\widehat{Q}\in\pi,\widehat{Q}\subset K_{CZ}} \|F\|^p_{L^{m,p}(\widehat{Q})} \\
&\lesssim \|F\|^p_{L^{m,p}(K_{CZ})} + \|G\|^p_{L^{m,p}(Q^\circ)} + \|G,P_0\|^p_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \cdot \sum_{\beta\in\mathcal{A}} \|\varphi^\beta\|^p_{L^{m,p}(Q^\circ)},
\end{aligned} \tag{6.50}$$

where the last inequality follows because for $Q_i \in CZ^\circ$, $|\{\widehat{Q} \in \pi : 1.1Q_i \subset 100\widehat{Q}\}| \leq C$, and since $\{100\widehat{Q} : \widehat{Q} \in \pi\}$ has $C$-bounded overlap (because $\pi$ is a congruent packing). From (3.70), we have $\|\varphi^\beta\|_{L^{m,p}(Q^\circ)} \leq \epsilon/C_0$ for all $\beta \in \mathcal{A}$. From (6.31), we have $\|F\|_{L^{m,p}(K_{CZ})} \lesssim \|G,P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}$. Therefore, from (6.50),

$$\sum_{\widehat{Q}\in\pi} \left(E(F,\widehat{Q})/\delta_{\widehat{Q}}^{m}\right)^p \lesssim \|G,P_0\|^p_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.$$

Now by taking the supremum over $\pi \in \Pi_{\simeq}(Q^\circ)$, and applying Corollary 2.12, we conclude $F \in L^{m,p}(Q^\circ)$ and

$$\|F\|_{L^{m,p}(Q^\circ)} \lesssim \|G,P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.$$

This completes the proof of the lemma. □

We complete the section by giving the proof of conditions (6.8) – (6.10).

We apply inequalities (6.31) and (6.44) and the identity $F|_{K_p} = G|_{K_p}$ (see (6.16)) to bound

$$\begin{aligned}\|F,\vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} &\lesssim \|F,\vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|F\|^p_{L^{m,p}(Q^\circ)} + \int_{K_p} |F-f|^p d\mu \\ &\lesssim \|G,P_0\|^p_{\mathscr{J}(f,\mu;\delta_{Q^\circ})}.\end{aligned}$$

This establishes the inequality (6.8) for $P_0 = 0$, as desired.

As a consequence of Lemma 6.5, the Whitney field $(S_y)_{y\in Q^\circ} \in Wh(Q^\circ)$ defined in (6.17) is in $C^{m-1,1-n/p}(Q^\circ)$, and thanks to (6.20) it satisfies $S_x(x) = F(x)$ for all $x \in Q^\circ$. Due to Lemma 2.13, the function $F$ satisfies

$$J_x F = S_x, \quad x \in Q^\circ. \tag{6.51}$$

Then (6.10) holds, thanks to (6.51). From (6.18), we have $\vec{S} = (S_y)_{y\in K_p}$ is coherent with $P_0 = 0$. From (6.15), we have $\vec{P}$ is coherent with $P_0 = 0$. Thus, $(\vec{P},\vec{S}) \in Wh(\mathfrak{B}_{CZ} \cup K_p)$ is coherent with $P_0 = 0$, proving (6.9). This completes the proof of (6.8) – (6.10), thus completing the proof of Proposition 6.2.

## *6.3. Keystone point jets*

Let $(f,P_0) \in \mathscr{J}(\mu;\delta_{Q^\circ})$. The goal of this section is to associate to the data $(f,P_0)$ a Whitney field $\vec{R}^* \in Wh(K_p)$, determined linearly by $(f,P_0)$, and satisfying the properties outlined below in Lemma 6.7, Lemma 6.8, and Corollary 6.9. These results will be used later, in Section 9 and Section 10.

Recall that we have fixed a multi-index set $\mathcal{A} \subset \mathcal{M}$. In the previous subsection we introduced the following notation: Given $K \subset \mathbb{R}^n$, a Whitney field $\vec{R} = (R_x)_{x\in K} \in Wh(K)$ is coherent with $P_0$ provided that $\partial^\alpha R_x(x) = \partial^\alpha P_0(x)$ for all $x \in K$, $\alpha \in \mathcal{A}$. A function $H \in C^{m-1,1-n/p}(\mathbb{R}^n)$ is $K$-coherent with $P_0$ provided that $\partial^\alpha H(x) = \partial^\alpha P_0(x)$ for all $x \in K$ and for all $\alpha \in \mathcal{A}$.

**Lemma 6.7.** *For each* $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$, *there exists a Whitney field* $\vec{R}^* = \vec{R}^*(f, P_0) \in Wh(K_p)$ *with the following properties.*

1. $\vec{R}^*$ *is coherent with* $P_0$.
2. *If* $H \in L^{m,p}(\mathbb{R}^n)$ *satisfies* $\|H\|_{\mathscr{J}(f,\mu)} < \infty$ *and* $H$ *is* $K_p$*-coherent with* $P_0$, *then*

$$J_x H = R_x^* \quad \forall x \in K_p. \tag{6.52}$$

3. $\|\vec{R}^*\|^p_{L^{m,p}(K_p)} + \int\limits_{K_p} |R_x^*(x) - f(x)|^p d\mu \lesssim \|f, P_0\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})}$.
4. $\vec{R}^*$ *depends linearly on* $(f, P_0)$.

**Proof.** Due to Proposition 6.2, for any $\eta > 0$ there exists $H_1 \in \mathscr{J}(f, \mu)$ that is $K_p$-coherent with $P_0$ and satisfies $\|H_1\|_{\mathscr{J}(f,\mu)} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta < \infty$. Define $\vec{R}^* \in Wh(K_p)$ as

$$\vec{R}^* = (R_x^*)_{x \in K_p} := (J_x H_1)_{x \in K_p}. \tag{6.53}$$

Next, we verify properties 1–4 of the lemma. In particular, we show that $\vec{R}^*$ is independent of $\eta$ and the choice of $H_1$ (property 2), and furthermore that $\vec{R}^*$ is a linear function of $(f, P_0)$ (property 4).

**Proof of property 1:** By definition, $R_x^* = J_x H_1$ for $x \in K_p$, where $H_1$ is $K_p$-coherent with $P_0$. Thus, $\vec{R}^*$ is coherent with $P_0$.

**Proof of property 2:**

We shall make use of the auxiliary polynomials $\left(P_y^\alpha\right)_{\alpha \in \mathcal{A}}$ $(y \in Q^\circ)$, defined in Lemma 3.8. In particular, from (3.55) we know that $\left(P_y^\alpha\right)_{\alpha \in \mathcal{A}}$ forms an $(\mathcal{A}, y, C\epsilon/C_0, 1)$-basis for $\sigma_J(y, \mu)$ for all $y \in 100Q^\circ$.

Fix $\epsilon_1 > 0$ satisfying $\epsilon_1 < \min\{c_1, \epsilon_0/(30^m C_0 C_1)\}$ and $\epsilon_1 < \min\{(50^m C_3)^{-1/(D+1)}, C_4^{-1/(D+1)}\}$ where $C_3$ is from (5.2), $C_4$ is from (5.6), $c_1$ and $C_1$ are from Lemma 3.5, and $D = |\mathcal{M}|$. Fix $\epsilon_2 < \epsilon_1^{2D+2}$.

For the sake of contradiction, suppose that (6.52) does not hold. Thus, there exist $x \in K_p$ and $H_2 \in L^{m,p}(\mathbb{R}^n)$ satisfying $\|H_2\|_{\mathscr{J}(f,\mu)} < \infty$ such that

$$\partial^\alpha (H_1 - H_2)(x) = 0 \text{ for all } \alpha \in \mathcal{A}, \text{ but } J_x(H_1 - H_2) \neq 0. \tag{6.54}$$

Let

$$\mathcal{B} := \{\beta \in \mathcal{M} \setminus \mathcal{A} : \ \partial^\beta (H_1 - H_2)(x) \neq 0\} = \{\beta_i : i \in \{1, \dots, k\}\},$$

where $\beta_i$ are ordered:

$$\beta_1 < \beta_2 < \dots < \beta_k.$$

Note that (6.54) implies $\mathcal{B}$ is nonempty. Also, $k = |\mathcal{B}| \leq |\mathcal{M}| = D$. Choose $\delta \in (0,1)$ such that for all $\beta \in \mathcal{B}$,

$$\delta^{m-|\beta|-n/p} \left( \|H_1\|_{\mathscr{J}(f,\mu)} + \|H_2\|_{\mathscr{J}(f,\mu)} \right) < \epsilon_2 |\partial^\beta (H_1 - H_2)(x)|. \tag{6.55}$$

Choose a dyadic cube $Q \subset Q^\circ$ with $x \in Q$ and $\delta_Q < \delta < 1$. Let

$$j := \arg\max\{ |\partial^{\beta_i}(H_1 - H_2)(x)| \cdot \delta_Q^{|\beta_i|} \epsilon_1^{-i} : i \in \{1, \ldots, k\}\}.$$

Then for all $i \in \{1, \cdots, k\}$,

$$\frac{|\partial^{\beta_i}(H_1 - H_2)(x)|}{|\partial^{\beta_j}(H_1 - H_2)(x)|} \leq \epsilon_1^{i-j} \delta_Q^{|\beta_j|-|\beta_i|} \leq \begin{cases} \epsilon_1 \delta_Q^{|\beta_j|-|\beta_i|} & \text{if } \beta_i > \beta_j,\ i > j, \\ \epsilon_1^{-D} \delta_Q^{|\beta_j|-|\beta_i|} & \text{if } \beta_i < \beta_j,\ i < j. \end{cases} \tag{6.56}$$

Define $P_x^{\beta_j} := J_x(H_1 - H_2)/(\partial^{\beta_j}(H_1 - H_2)(x))$. By definition of $\mathcal{B}$ and (6.54), $\partial^\beta P_x^{\beta_j}(x) = 0$ for $\beta \in \mathcal{M} \setminus \mathcal{B} = \mathcal{A}$. Combining this with (6.56), we have

$$\partial^{\beta_j} P_x^{\beta_j}(x) = 1; \tag{6.57}$$

$$|\partial^\beta P_x^{\beta_j}(x)| \leq \epsilon_1 \delta_Q^{|\beta_j|-|\beta|} \qquad (\beta \in \mathcal{M},\ \beta > \beta_j); \text{ and} \tag{6.58}$$

$$|\partial^{\beta_i} P_x^{\beta_j}(x)| \leq \epsilon_1^{-D} \delta_Q^{|\beta_j|-|\beta_i|} \qquad (\beta \in \mathcal{M}). \tag{6.59}$$

Then from (6.55) and since $\delta_Q < \delta$, if $\varphi := (H_1 - H_2)/|\partial^{\beta_j}(H_1 - H_2)(x)|$ then $J_x\varphi = P_x^{\beta_j}$ and

$$\begin{aligned} \|\varphi\|_{\mathscr{J}(0,\mu|_{3Q})} = \frac{\|H_1 - H_2\|_{\mathscr{J}(0,\mu|_{3Q})}}{|\partial^{\beta_j}(H_1 - H_2)(x)|} &\leq \frac{\|H_1\|_{\mathscr{J}(f,\mu|_{3Q})} + \|H_2\|_{\mathscr{J}(f,\mu|_{3Q})}}{\delta_Q^{m-|\beta_j|-n/p}(\|H_1\|_{\mathscr{J}(f,\mu)} + \|H_2\|_{\mathscr{J}(f,\mu)})} \cdot \epsilon_2 \\ &\leq \epsilon_2 \delta_Q^{|\beta_j|+n/p-m}. \end{aligned}$$

Hence,

$$P_x^{\beta_j} \in \epsilon_2 \delta_Q^{|\beta_j|+n/p-m} \cdot \sigma_J(x, \mu|_{3Q}). \tag{6.60}$$

From (6.57)-(6.60), we see that $(\delta, x_0, \bar{\beta}, \widetilde{P}_{x_0}^{\bar{\beta}}, \mu) := (\delta_Q, x, \beta_j, P_x^{\beta_j}, \mu|_{3Q})$ satisfies **(D6)** of Lemma 3.5.

In (3.55), we saw that $\left(P_y^\alpha\right)_{\alpha \in \mathcal{A}}$ forms an $(\mathcal{A}, y, C\epsilon, 1)$-basis for $\sigma_J(y, \mu)$ for all $y \in 100Q^\circ$. We can assume $C\epsilon < \epsilon_2$. If $Q \subset Q^\circ$ then $4Q \subset 100Q^\circ$. So, by (3.5),

$$\left(P_y^\alpha\right)_{\alpha \in \mathcal{A}} \text{ forms an } (\mathcal{A}, y, \epsilon_2, \delta_Q)\text{-basis for } \sigma_J(y, \mu|_{3Q}) \text{ for all } y \in 10Q. \tag{6.61}$$

Fix $y \in 10Q$. Suppose first $y \in K_{CZ}$, so that $y \in Q_i$ for some $Q_i \in CZ^\circ$. Because $Q$ is not OK, we must have $\delta_{Q_i}/100 \leq \delta_Q < 1$, and so, by applying (5.2) for the cube $Q_i$ and for some $\widehat{\delta} \in [\delta_{Q_i}/2, 1]$ satisfying $\delta_Q \leq \widehat{\delta} \leq 50\delta_Q$, we deduce that

$$|\partial^\beta P^\alpha_y(y)| \le C_3 \widehat{\delta}^{|\alpha|-|\beta|} \le 50^m C_3 \delta_Q^{|\alpha|-|\beta|} \qquad (\alpha \in \mathcal{A},\ \beta \in \mathcal{M}). \tag{6.62}$$

Now suppose $y \in K_p$. Then from (5.6) with $\delta = \delta_Q$,

$$|\partial^\beta P^\alpha_y(y)| \le C_4 \delta_Q^{|\alpha|-|\beta|} \qquad (\alpha \in \mathcal{A},\ \beta \in \mathcal{M}). \tag{6.63}$$

Because $\epsilon_1 < \min\{(50^m C_3)^{-1/(D+1)}, C_4^{-1/(D+1)}\}$, in light of (6.62) and (6.63), we have

$$|\partial^\beta P^\alpha_y(y)| \le \epsilon_1^{-D-1} \cdot \delta_Q^{|\alpha|-|\beta|} \qquad (y \in 10Q, \alpha \in \mathcal{A}, \beta \in \mathcal{M}) \tag{6.64}$$

It is now evident from (6.57)-(6.60), (6.61), and (6.64) that properties **(D1)-(D7)** of Lemma 3.5 hold with parameters:

$$\begin{aligned}&\left(\epsilon_1, \epsilon_2, \delta,\ \mathcal{A}, \mu, E, \left(\widetilde{P}^\alpha_x\right)_{\alpha\in\mathcal{A}, x\in E}, x_0, \bar{\beta}, \widetilde{P}^{\bar{\beta}}_{x_0}\right)\\ &\quad := \left(\epsilon_1, \epsilon_2, \delta_Q, \mathcal{A}, \mu|_{3Q}, 10Q, (P^\alpha_x)_{\alpha\in\mathcal{A}, x\in 10Q}, x, \beta_j, P^{\beta_j}_x\right).\end{aligned}$$

Thus there exists $\overline{\mathcal{A}} < \mathcal{A}$ so that for every $y \in 10Q$, $\sigma_J(y, \mu|_{3Q})$ contains an $(\overline{\mathcal{A}}, y, C_1\epsilon_1, \delta_Q)$-basis. Because $C_1\epsilon_1 < \epsilon_0/(30^m C_0)$, we apply Lemma 3.2 to deduce that for every $y \in 3Q$, $\sigma_J(y, \mu|_{3Q})$ contains an $(\overline{\mathcal{A}}, y, \epsilon_0/C_0, 30\delta_Q)$-basis, indicating that $Q$ is OK. This contradicts that $x \in K_p$ and $x \in Q$.

This completes the proof by contradiction of (6.52). So we have proven property 2.

**Proof of property 3:** By definition, $R^*_x(x) = H_1(x)$ for all $x \in K_p$, where $\|H_1\|_{\mathscr{J}(f,\mu)} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ}} + \eta$, and $\eta > 0$ is arbitrary. By property 2, $\vec{R}^*$ is independent of $\eta$. Then:

$$\begin{aligned}\int_{K_p} |R^*_x(x) - f(x)|^p d\mu(x) &= \int_{K_p} |H_1 - f|^p d\mu \le \int_{\mathbb{R}^n} |H_1 - f|^p d\mu \le \|H_1\|^p_{\mathscr{J}(f,\mu)}\\ &\lesssim (\|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta)^p.\end{aligned}$$

Furthermore, since $R^*_x = J_x H_1$ for all $x \in K_p$, by definition of the $L^{m,p}(K_p)$ trace norm on $Wh(K_p)$, we have

$$\|\vec{R}^*\|_{L^{m,p}(K_p)} \le \|H_1\|_{L^{m,p}(\mathbb{R}^n)} \le \|H_1\|_{\mathscr{J}(f,\mu)} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta.$$

Now let $\eta \to 0$ in the previous inequalities. This completes the proof of property 3.

**Proof of property 4:** To complete the proof of the lemma, we will show that $\vec{R}^* = \vec{R}^*(f, P_0)$ depends linearly on $(f, P_0)$.

Let $(f_1, P_1)$, $(f_2, P_2) \in \mathscr{J}(\mu; \delta_{Q^\circ})$. Fix $H_j \in \mathscr{J}(f_j, \mu)$ such that $H_j$ is $K_p$-coherent with $P_j$ for $j = 1, 2$. By property 3, we have that $\vec{R}^*(f_j, P_j) = (J_x H_j)_{x\in K_p}$ for $j = 1, 2$.

Then, for $\lambda \in \mathbb{R}$, $H_1 + \lambda H_2 \in \mathscr{J}(f_1 + \lambda f_2, \mu)$, and for $\alpha \in \mathcal{A}$, $x \in K_p$,

$$\begin{aligned}\partial^\alpha (H_1 + \lambda H_2)(x) &= \partial^\alpha H_1(x) + \lambda \partial^\alpha H_2(x) \\ &= \partial^\alpha P_1(x) + \lambda \partial^\alpha P_2(x),\end{aligned}$$

indicating that $H_1 + \lambda H_2$ is $K_p$-coherent with $P_1 + \lambda P_2$. Because of (6.52),

$$\vec{R}^*(f_1 + \lambda f_2, P_1 + \lambda P_2) = \{J_x(H_1 + \lambda H_2)\}_{x \in K_p} = \vec{R}^*(f_1, P_1) + \lambda \vec{R}^*(f_2, P_2).$$

Therefore, the map $(f, P_0) \mapsto \vec{R}^*(f, P_0)$ is linear, completing the proof of property 4. $\square$

**Lemma 6.8.** *Let $x \in K_p$ and $r > 0$. Set $\mu_{x,r} := \mu|_{B(x,r)}$. Fix $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$. Suppose $H \in L^{m,p}(\mathbb{R}^n)$, $\|H\|_{\mathscr{J}(f,\mu_{x,r})} < \infty$, and $H$ is $K_p$-coherent with $P_0$. Then $R^*_x(f, P_0) = J_x H$.*

**Proof.** We employ a proof by contradiction, following the proof of property 2 of Lemma 6.7, with the measure $\mu$ replaced by $\mu_{x,r}$ in this proof. We make one change in our previous proof: When we choose the dyadic cube $Q \subset Q^\circ$ with $x \in Q$, we impose the additional condition $\delta_Q < r/3$. This condition implies $3Q \subset B(x, r)$, so that $\mu_{x,r}|_{3Q} = \mu|_{3Q}$. Following our previous proof, we reach the conclusion that $\sigma_J(y, \mu_{x,r}|_{3Q})$ contains an $(\overline{\mathcal{A}}, y, \epsilon_0/C_0, 30\delta_Q)$-basis for all $y \in 3Q$, for some $\overline{\mathcal{A}} < \mathcal{A}$. So, $\sigma_J(y, \mu|_{3Q})$ contains an $(\overline{\mathcal{A}}, y, \epsilon_0/C_0, 30\delta_Q)$-basis for all $y \in 3Q$, indicating that $Q$ is OK, a contradiction. $\square$

As a consequence of Proposition 6.2, the Whitney field $\vec{R}^* \in Wh(K_p)$ satisfies the following condition:

**Corollary 6.9.** *For $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$, and for $\vec{R}^* = \vec{R}^*(f, P_0)$ as in Lemma 6.7,*

$$\inf \left\{ \|f, \vec{P}, \vec{R}^*\|_{\mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)} : \begin{array}{l} \vec{P} \in Wh(\mathfrak{B}_{CZ}) \textit{ satisfies} \\ \vec{P} \textit{ is coherent with } P_0 \end{array} \right\} \lesssim \|f, P_0\|_{\mathscr{J}(\mu; \delta_{Q^\circ})}.$$

## 7. Optimal local extension

In this section, we prove a general result, Lemma 7.2, on the optimization of certain $L^p$-type norms by linear maps. In Section 7.2, we apply this result to construct a Whitney field $\vec{R}'$ on the keystone basepoint set $\mathfrak{B}_{key} := \{x_s\}_{s \in \bar{I}}$. We shall apply Lemma 7.2 once more, later, in Section 11, when we give the proofs of the main theorems.

*7.1. Optimization by linear maps*

**Lemma 7.1** *(Linear Map Lemma). Let $(\nu, X, \Sigma)$ be a measure space, and let $V$ be a vector space. Let $k \geq 1$, let $p \geq 1$, and let $\Lambda : V \times \mathbb{R}^k \to L^p(d\nu)$ be a linear map. Then there exists a linear map $\xi : V \to \mathbb{R}^k$, satisfying:*

$$\int_X |\Lambda(v, \xi(v))|^p d\nu \leq C \inf_{w \in \mathbb{R}^k} \int_X |\Lambda(v, w)|^p d\nu, \qquad \textit{for all } v \in V, \tag{7.1}$$

*where $C$ depends only on $k$ and $p$.*

**Proof.** We will show this is true when $k = 1$; then this can be iterated for the full result. We factor the linear map $\Lambda : V \times \mathbb{R} \to L^p(d\nu)$, as follows: $\Lambda(v, w) = \hat{\Lambda}(v) - a \cdot w$ for a linear map $\hat{\Lambda} : V \to L^p(d\nu)$ and $a \in L^p(d\nu)$. Note, if $\|a\|_{L^p(d\nu)} = 0$ then we can take $\xi(v) = 0$, and the conclusion of the lemma will be satisfied. Thus, we may assume $\|a\|_{L^p(d\nu)} \neq 0$. Define $\xi : V \to \mathbb{R}$:

$$\xi(v) := \frac{\int_{\{a \neq 0\}} \frac{\hat{\Lambda}(v)}{a} |a|^p d\nu}{\int_{\{a \neq 0\}} |a|^p d\nu}.$$

Note the convergence of the integral in the numerator, by Hölder's inequality. Then $v \mapsto \xi(v)$ is linear.

For any $w \in \mathbb{R}$, we claim that $\int_X |\Lambda(v, \xi(v))|^p d\nu \leq C \int_X |\Lambda(v, w)|^p d\nu$. To see this, first note

$$\begin{aligned}
\int_X |\Lambda(v, w)|^p d\nu &= \int_X |\hat{\Lambda}(v) - a \cdot w|^p d\nu \\
&= \int_{\{a=0\}} |\hat{\Lambda}(v)|^p d\nu + \int_{\{a \neq 0\}} |\hat{\Lambda}(v) - a \cdot w|^p d\nu.
\end{aligned}$$

We bound

$$\begin{aligned}
&\int_{\{a \neq 0\}} |\hat{\Lambda}(v) - a \cdot \xi(v)|^p d\nu \\
&= \int_{\{a \neq 0\}} \left| \frac{\hat{\Lambda}(v)}{a} - \xi(v) \right|^p |a|^p d\nu \\
&\leq 2^p \cdot \Big( \int_{\{a \neq 0\}} \left| \frac{\hat{\Lambda}(v)}{a} - w \right|^p |a|^p d\nu + \int_{\{a \neq 0\}} \left| \frac{\int_{\{a \neq 0\}} \frac{\hat{\Lambda}(v)}{a} |a|^p d\nu}{\int_{\{a \neq 0\}} |a|^p d\nu} - w \right|^p |a|^p d\nu \Big)
\end{aligned}$$

$$\leq (1+2^p)\cdot \int\limits_{\{a\neq 0\}} \left|\frac{\hat{\Lambda}(v)}{a} - w\right|^p |a|^p d\nu$$
$$= (1+2^p)\cdot \int\limits_{\{a\neq 0\}} \left|\hat{\Lambda}(v) - a\cdot w\right|^p d\nu,$$

where the last inequality follows by applying Jensen's Inequality to the second term in the previous line. Thus, adding $\int_{\{a=0\}} |\hat{\Lambda}(v)|^p d\nu$ to both sides of the above estimate, we obtain (7.1) in the case $k=1$, with constant $C = 1+2^p$. By iterating $k$ times, we reach the conclusion of the lemma with a constant $C=(1+2^p)^k$. □

**Lemma 7.2** *(Linear Map Lemma II). Let $\mu_0$ be a Borel regular measure on $\mathbb{R}^n$, and let $V$ be a vector space. Let $k \geq 1$, and for each $\ell \in \mathbb{N}$, let $\lambda_\ell : V \times \mathbb{R}^k \to L^p(d\mu_0)$ be a linear map, and let $\phi_\ell : V \times \mathbb{R}^k \to \mathbb{R}$ be a linear functional. Let $\Psi : V \times \mathbb{R}^k \to \mathbb{R}^N$ be a linear map, such that $w \mapsto \Psi(0,w)$ is surjective ($N \leq k$).*

*Suppose that the functional*

$$M(v,w) := \sum_{\ell=1}^{\infty} \|\lambda_\ell(v,w)\|^p_{L^p(d\mu_0)} + \sum_{\ell=1}^{\infty} |\phi_\ell(v,w)|^p$$

*is finite for every $(v,w) \in V \times \mathbb{R}^k$.*

*Then there exists a linear map $\xi : V \to \mathbb{R}^k$, satisfying:*

$$\Psi(v,\xi(v)) = 0 \in \mathbb{R}^N,$$
$$M(v,\xi(v)) \leq C \inf\{M(v,w) : w \in \mathbb{R}^k,\ \Psi(v,w)=0\}, \qquad \text{for all } v \in V,$$

*where $C$ depends only on $k$ and $p$.*

**Proof.** We first suppose $N=0$, i.e., the constraint map $\Psi$ is trivial. To prove Lemma 7.2 in this case, we apply Lemma 7.1 to the product measure $\nu = (\mu_0 + \delta_z) \times \mu_1$ on the space $X = (\mathbb{R}^n \sqcup \{z\}) \times \mathbb{N}$, where $\mu_0$ is a given Borel regular measure on $\mathbb{R}^n$, $\delta_z$ is a Dirac delta measure supported at the point $z$ ($z \notin \mathbb{R}^n$), and $\mu_1$ is the counting measure on $\mathbb{N}$.

Now suppose $N > 0$. Write $\Psi(v,w) = \Psi_1(v) - \Psi_2(w)$, for linear maps $\Psi_1 : V \to \mathbb{R}^N$ and $\Psi_2 : \mathbb{R}^k \to \mathbb{R}^N$. The condition that $w \mapsto \Psi(0,w)$ is surjective ensures that $\Psi_2$ is surjective. By Gaussian elimination, after possibly permuting the coordinates of $w = (w_1,\ldots,w_k)$, the constraint set

$$\{(v,w) : \Psi(v,w) = 0\} = \{(v,w) : \Psi_2(w) = \Psi_1(v)\} \subset V \times \mathbb{R}^k$$

can be written in the form

$$\{(v,w^1,w^2) : v \in V,\ \ w^1 = (w_1,\ldots,w_\ell) \in \mathbb{R}^\ell,$$
$$w^2 = (w_{\ell+1},\ldots,w_k) = \psi(v,w_1,\ldots,w_\ell) \in \mathbb{R}^{k-\ell}\}$$

for some $\ell \leq k$, and for a linear map $\psi : V \times \mathbb{R}^\ell \to \mathbb{R}^{\ell-k}$. (The condition that $\Psi_2$ is surjective ensures that the linear system $\Psi_2(w) = \Psi_1(v)$ admits a solution $w$ for every $v \in V$.)

We obtain the conclusion of the lemma then, by applying the version of Lemma 7.2 without constraints to the functional $\widetilde{M} : V \times \mathbb{R}^\ell \to \mathbb{R}_+$ given by

$$\widetilde{M}(v, w_1, \ldots, w_\ell) := M(v, w_1, \ldots, w_\ell, \psi(v, w_1, \ldots, w_\ell)). \quad \square$$

*7.2. Local extension*

Fix a keystone cube $Q_s$. (See Definition 4.3.) We will construct a polynomial $R'_{x_s} \in \mathcal{P}$ that is coherent with $P_0$ and approximately minimizes the expression $\|f, R'_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s}, \delta_{Q_s})}$.

**Lemma 7.3.** *Let $Q_s \in CZ_{key}$, and let $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$. There exists $R'_{x_s} \in \mathcal{P}$ that depends linearly on $(f|_{9Q_s}, P_0)$ and satisfies $\partial^\alpha R'_{x_s}(x_s) = \partial^\alpha P_0(x_s)$ for all $\alpha \in \mathcal{A}$, and*

$$\|f, R'_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s}, \delta_{Q_s})} \lesssim \inf_{R \in \mathcal{P}} \left\{ \|f, R\|_{\mathscr{J}(\mu|_{9Q_s}, \delta_{Q_s})} : \partial^\alpha R(x_s) = \partial^\alpha P_0(x_s) \;\; \forall \alpha \in \mathcal{A} \right\}. \tag{7.2}$$

**Proof.** Due to the good geometry of the CZ decomposition, $\delta_Q \to 0$ as $Q \to x$, $Q \in CZ^\circ$, $x \in K_p$. If $100Q_s \cap K_p \neq \emptyset$, then $100Q_s$ intersects CZ cubes of arbitrarily small sidelength, due to the previous remark, which contradicts that $Q_s$ is a keystone cube. Therefore, $100Q_s \cap K_p = \emptyset$.

Let $CZ_s := \{\widehat{Q}^j\}_{1 \leq j \leq k}$ be the collection of all $Q \in CZ^\circ$ satisfying $1.1Q \cap 100Q_s \neq \emptyset$, and with $\widehat{Q}^1 = Q_s$. Let $x^j := \mathrm{ctr}(\widehat{Q}^j)$ and $D_j := 1.1\widehat{Q}^j \cap 9Q_s$ for $j = 1, \ldots, k$. Then $\{D_j\}_{1 \leq j \leq k}$ is a cover of $9Q_s$ by axis-parallel rectangles.

Let $P^1 \in \mathcal{P}$, and let $\vec{P}_* := (P^j)_{2 \leq j \leq k} \in \mathcal{P}^{k-1}$ be a $(k-1)$-tuple of elements of $\mathcal{P}$.

We apply **(AL1)-(AL3)** (see Section 4.4) to the CZ cube $\widehat{Q}^i$ and the Borel set $E_i = D_i \subset 3\widehat{Q}^i$. So, for each $i = 1, 2, \ldots, k$, there exist a linear map $T_i : \mathscr{J}(\mu|_{D_i}) \times \mathcal{P} \to L^{m,p}(\mathbb{R}^n)$, a functional $M_i : \mathscr{J}(\mu|_{D_i}) \times \mathcal{P} \to \mathbb{R}_+$, and countable collections of Borel sets $\{A^i_\ell\}_{\ell \in \mathbb{N}}$ with

$$A^i_\ell \subset \mathrm{supp}(\mu|_{D_i}) \subset \mathrm{supp}(\mu|_{9Q_s}), \tag{7.3}$$

and of linear maps $\phi^i_\ell : \mathscr{J}(\mu|_{D_i}) \times \mathcal{P} \to \mathbb{R}$, and $\lambda^i_\ell : \mathscr{J}(\mu|_{D_i}) \times \mathcal{P} \to L^p(d\mu)$ $(\ell \in \mathbb{N})$, that satisfy: For all $f \in \mathscr{J}(\mu|_{D_i})$ and $P \in \mathcal{P}$,

**(i)** $M_i(f, P) \simeq \|T_i(f, P), P\|_{\mathscr{J}(f, \mu|_{D_i}; \delta_{\widehat{Q}^i})} \simeq \|f, P\|_{\mathscr{J}(\mu|_{D_i}; \delta_{\widehat{Q}^i})}$; and

**(ii)** $M_i(f, P) = \Big( \sum_{\ell \in \mathbb{N}} \int_{A^i_\ell} |\lambda^i_\ell(f, P) - f|^p d\mu + \sum_{\ell \in \mathbb{N}} |\phi^i_\ell(f, P)|^p \Big)^{1/p} < \infty.$

We now show that: For any $f \in \mathscr{J}(\mu|_{9Q_s})$ and $P^1 \in \mathcal{P}$,

$$\|f, P^1\|^p_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} \simeq \inf_{\vec{P}_* \in \mathcal{P}^{k-1}} \left\{ \sum_{i=1}^k \|f, P^i\|^p_{\mathscr{J}(\mu|_{D_i};\delta_{\widehat{Q}^i})} + \sum_{i,j=1}^k |P^i - P^j|^p_{x^i,\delta_{\widehat{Q}^i}} \right\}. \tag{7.4}$$

We prepare to apply Lemma 5.8 for the proof of the upper bound in (7.4).

We fix $P^1 \in \mathcal{P}$, and let $\vec{P}_* = (P^j)_{2\leq j\leq k}$ be arbitrary.

We define a Whitney field $\vec{P} = (P_{x_i})_{i\in I} \in Wh(\mathfrak{B}_{CZ})$ by letting

$$P_{x_i} = \begin{cases} P^j \text{ if } Q_i = \widehat{Q}^j, \text{ some } j = 1,\dots,k; \text{ else} \\ P^1 \text{ if } Q_i \in CZ^\circ \setminus \{\widehat{Q}^j\}_{1\leq j\leq k}. \end{cases}$$

Define $\vec{S} \in Wh(K_p)$ by letting $S_x = P^1$ for all $x \in K_p$. Define

$$F_i(x) = \begin{cases} T_j(f, P^j) & \text{if } Q_i = \widehat{Q}^j, \text{ some } j = 1,\dots,k \\ P^1(x) & \text{if } Q_i \in CZ^\circ \setminus \{\widehat{Q}^j\}_{1\leq j\leq k}. \end{cases}$$

Let $\{\theta_i\}_{i\in I}$ be a partition of unity satisfying **(POU1)-(POU4)** (see Section 4.3). Define $F : Q^\circ \to \mathbb{R}$ as

$$F(x) = \begin{cases} \sum_{i\in I} \theta_i(x)\cdot F_i(x) & x \in K_{CZ} \\ S_x(x) = P^1(x) & x \in K_p. \end{cases}$$

Because $\vec{S}$ is constant on $K_p$, $\|\vec{S}\|_{L^{m,p}(K_p)} = 0$. Further, $(\vec{P}, \vec{S}) \in C^{m-1,1-n/p}(\mathfrak{B}_{CZ} \cup K_p)$, because $(\vec{P}, \vec{S})$ is constant except at finitely many points. Hence by Lemma 5.8, if $\|F, \vec{P}\|_{\mathscr{J}_*(f,\mu|_{9Q_s};K_{CZ},CZ^\circ)} < \infty$ then $F \in L^{m,p}(Q^\circ)$ and

$$\|F\|_{L^{m,p}(Q^\circ)} \lesssim \|F, \vec{P}\|_{\mathscr{J}_*(f,\mu|_{9Q_s};K_{CZ},CZ^\circ)}. \tag{7.5}$$

We prepare to estimate $\|F, \vec{P}\|_{\mathscr{J}_*(f,\mu|_{9Q_s};K_{CZ},CZ^\circ)}$ and establish it to be finite. By Lemma 5.7,

$$\begin{aligned} \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};K_{CZ},CZ^\circ)} \lesssim &\sum_{i\in I} \|F_i, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s\cap 1.1Q_i};1.1Q_i\cap Q^\circ)} \\ &+ \sum_{i\leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_{x_i,\delta_{Q_i}}. \end{aligned} \tag{7.6}$$

From the definitions of $F_i$ and $P_{x_i}$, we see that all but finitely many of the terms in either of the preceding sums are equal to zero. Indeed, for $Q_i \notin CZ_s = \{\widehat{Q}^j\}_{1\leq j\leq k}$, $F_i = P_{x_i} = P^1$ and the support of the restricted measure $\mu|_{9Q_s}$, is disjoint from $1.1Q_i$.

Therefore, $\|F_i, P_{x_i}\|_{\mathscr{J}_*(f,\mu|_{9Q_s\cap 1.1Q_i};1.1Q_i\cap Q^\circ)} = 0$ for $Q_i \notin CZ_s$. Furthermore, by definition of $P_{x_i}$, note that $P_{x_i} \neq P_{x_{i'}}$ implies either $Q_i \in CZ_s$ or $Q_{i'} \in CZ_s$. Hence, the sum on the right-hand side of (7.6) is finite. By reindexing the sums in (7.6) to be over $j \in \{1, 2, \dots, k\}$, and using (7.5), we conclude that

$$\begin{aligned}\|F\|^p_{L^{m,p}(Q^\circ)} &\lesssim \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};K_{CZ},CZ^\circ)} \\ &\lesssim \sum_{j=1}^{k} \|T_j(f, P^j), P^j\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s\cap 1.1\widehat{Q}^j};1.1\widehat{Q}^j\cap Q^\circ)} + \sum_{j,j'=1}^{k} |P^j - P^{j'}|^p_{x^j,\delta_{\widehat{Q}^j}}.\end{aligned} \tag{7.7}$$

Note that $9Q_s \cap \mathrm{supp}(\mu) \subset 9Q_s \cap Q^\circ \subset K_{CZ}$, since $Q_s$ is keystone. Hence,

$$\int_{9Q_s} |F - f|^p d\mu \leq \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};K_{CZ},CZ^\circ)}. \tag{7.8}$$

Also, by Lemma 2.6,

$$\begin{aligned}\|F - P^1\|^p_{L^p(9Q_s\cap Q^\circ)}/\delta^{mp}_{Q_s} &\lesssim \|F\|^p_{L^{m,p}(9Q_s\cap Q^\circ)} + \|F - P^1\|^p_{L^p(Q_s)}/\delta^{mp}_{Q_s} \\ &\lesssim \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};K_{CZ},CZ^\circ)},\end{aligned} \tag{7.9}$$

where the last line follows because $\widehat{Q}^1 = Q_s$, and so $P_{x_s} = P^1$ by definition of $\vec{P}$.

Combining (7.7), (7.8), and (7.9), we have

$$\begin{aligned}&\|F, P^1\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};9Q_s\cap Q^\circ)} \\ &\simeq \|F\|^p_{L^{m,p}(9Q_s\cap Q^\circ)} + \int_{9Q_s} |F - f|^p d\mu + \|F - P^1\|^p_{L^p(9Q_s\cap Q^\circ)}/\delta^{mp}_{Q_s} \\ &\lesssim \sum_{j=1}^{k} \|T_j(f, P^j), P^j\|^p_{\mathscr{J}_*(f,\mu|_{D_j};1.1\widehat{Q}^j\cap Q^\circ)} + \sum_{j,j'=1}^{k} |P^j - P^{j'}|^p_{x^j,\delta_{\widehat{Q}^j}}.\end{aligned}$$

By property **(i)** of $T_j$, and estimate (2.8), relating the $\mathscr{J}_*$ and $\mathscr{J}$-functionals,

$$\|T_j(f, P^j), P^j\|^p_{\mathscr{J}_*(f,\mu|_{D_j};1.1\widehat{Q}^j\cap Q^\circ)} \lesssim \|T_j(f, P^j), P^j\|_{\mathscr{J}(f,\mu|_{D_j};\delta_{\widehat{Q}^j})} \simeq \|f, P^j\|_{\mathscr{J}(\mu|_{D_j};\delta_{\widehat{Q}^j})},$$

and thus,

$$\|F, P^1\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};9Q_s\cap Q^\circ)} \lesssim \sum_{j=1}^{k} \|f, P^j\|^p_{\mathscr{J}(\mu|_{D_j};\delta_{\widehat{Q}^j})} + \sum_{j,j'=1}^{k} |P^j - P^{j'}|^p_{x^j,\delta_{\widehat{Q}^j}}.$$

Thanks to (2.26) and the definition of the $\mathscr{J}_*$-functional as an infimum,

$$\|f,P^1\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} \simeq \|f,P^1\|_{\mathscr{J}_*(\mu|_{9Q_s};9Q_s\cap Q^\circ)} \leq \|F,P^1\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};9Q_s\cap Q^\circ)}.$$

Combining the above inequalities, we take the infimum over $P^2,\dots,P^k \in \mathcal{P}$, and obtain the upper bound in (7.4):

$$\|f,P^1\|^p_{\mathscr{J}(\mu|_{9Q_s},\delta_{Q_s})} \lesssim \inf\left\{\sum_{j=1}^k \|f,P^j\|^p_{\mathscr{J}(\mu|_{D_j};\delta_{\widehat{Q}^j})} + \sum_{j,j'=1}^k |P^j - P^{j'}|^p_{x^j,\delta_{\widehat{Q}^j}} : P^2,\dots,P^k \in \mathcal{P}\right\}.$$

Now, to complete the proof of (7.4), we'll show the reverse inequality:

$$\|f,P^1\|^p_{\mathscr{J}(\mu|_{9Q_s},\delta_{Q_s})} \gtrsim \inf\left\{\sum_{j=1}^k \|f,P^j\|^p_{\mathscr{J}(\mu|_{D_j};\delta_{\widehat{Q}^j})} + \sum_{j,j'=1}^k |P^j - P^{j'}|^p_{x^j,\delta_{\widehat{Q}^j}} : P^2,\dots,P^k \in \mathcal{P}\right\}.$$

By taking $P^j = P^1$ for $j = 2,\dots,k$, we learn that

$$\begin{aligned}&\inf\Big\{\sum_{j=1}^k \|f,P^j\|^p_{\mathscr{J}(\mu|_{D_j};\delta_{\widehat{Q}^j})} + \sum_{j,j'=1}^k |P^j - P^{j'}|^p_{x^j,\delta_{\widehat{Q}^j}} : P^2,\dots,P^k \in \mathcal{P}\Big\}\\ &\quad\leq \sum_{j=1}^k \|f,P^1\|^p_{\mathscr{J}(\mu|_{D_j};\delta_{\widehat{Q}^j})} \simeq \sum_{j=1}^k \|f,P^1\|^p_{\mathscr{J}(\mu|_{D_j};\delta_{Q_s})},\end{aligned}$$

where the last line uses that $\delta_{Q_s} \simeq \delta_{\widehat{Q}^j}$ for $j = 1,2,\dots,k$. Finally, note that $D_j \subset 9Q_s$, so

$$\sum_{j=1}^k \|f,P^1\|^p_{\mathscr{J}(\mu|_{D_j};\delta_{Q_s})} \leq k\|f,P^1\|^p_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})}.$$

Since $k$ is bounded by a universal constant, this completes the proof of (7.4).

We prepare to apply (7.4) to construct the polynomial $R'_{x_s}$. By definition of the $|\cdot|_{x,\delta}$ norm (2.1), and conditions **(i)** and **(ii)** relating to properties of $T_i$ and $M_i$, we approximate the expression inside the infimum in (7.4) as follows:

$$\begin{aligned}&\sum_{i=1}^k \|f,P^i\|^p_{\mathscr{J}(\mu|_{D_i};\delta_{\widehat{Q}^i})} + \sum_{i,j=1}^k |P^i - P^j|^p_{x^i,\delta_{\widehat{Q}^i}}\\ &\quad\simeq \sum_{i=1}^k \sum_{\ell\in\mathbb{N}} \Big(\int_{A^i_\ell} |\lambda^i_\ell(f,P^i) - f|^p d\mu + |\phi^i_\ell(f,P^i)|^p\Big) + \sum_{i,j=1}^k \sum_{\alpha\in\mathcal{M}} c_{\alpha,i,j}|\partial^\alpha(P^i - P^j)(x^i)|^p,\end{aligned} \tag{7.10}$$

for some constants $c_{\alpha,i,j} \geq 0$. For $i, j \in \{1, 2, \dots, k\}$, we define $\phi^{i,j,\alpha} : \mathscr{J}(\mu|_{9Q_s}) \times \mathcal{P} \times \mathcal{P}^{k-1} \to \mathbb{R}$ as $\phi^{i,j,\alpha}(f, P^1, \vec{P}_*) := c_{\alpha,i,j}^{1/p} \partial^\alpha (P^i - P^j)(x^i)$. By substituting these functionals into the right-hand side of (7.10) and reindexing the sum, we have

$$\sum_{i=1}^{k} \|f, P^i\|^p_{\mathscr{J}(\mu|_{D_i};\delta_{\widehat{Q}^i})} + \sum_{i,j=1}^{k} |P^i - P^j|^p_{x^i,\delta_{\widehat{Q}^i}} \simeq M(f, P^1, \vec{P}_*)^p, \text{ where} \tag{7.11}$$

$$M(f, P^1, \vec{P}_*)^p := \sum_{\ell \in \mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, P^1, \vec{P}_*) - f|^p d\mu + |\phi_\ell(f, P^1, \vec{P}_*)|^p,$$

for countable collections of Borel sets $\{A_\ell\}_{\ell \in \mathbb{N}}$ with $A_\ell \subset \operatorname{supp}(\mu|_{9Q_s})$ (see (7.3)), and of linear maps $\phi_\ell : \mathscr{J}(\mu|_{9Q_s}) \times \mathcal{P} \times \mathcal{P}^{k-1} \to \mathbb{R}$, and $\lambda_\ell : \mathscr{J}(\mu|_{9Q_s}) \times \mathcal{P} \times \mathcal{P}^{k-1} \to L^p(d\mu)$ ($\ell \in \mathbb{N}$). In combination with (7.4),

$$\|f, P^1\|^p_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} \simeq \inf \left\{ M(f, P^1, \vec{P}_*) : \vec{P}_* \in \mathcal{P}^{k-1} \right\}. \tag{7.12}$$

According to (7.11), the functional $M(f, P^1, \vec{P}_*)$ is finite-valued for every $(f, P^1, \vec{P}_*) \in \mathscr{J}(\mu|_{9Q_s}) \times \mathcal{P} \times \mathcal{P}^{k-1}$. So we are justified to apply Lemma 7.2 (with a trivial constraint map $\Psi$) to determine $\vec{P}_* = (P^2, \dots, P^k) := \xi(f, P^1) \in \mathcal{P}^{k-1}$ depending linearly on $(f|_{9Q_s}, P^1)$ and satisfying

$$M(f, P^1, \xi(f, P^1)) \leq C \cdot \inf \left\{ M(f, P^1, \vec{P}_*) : \vec{P}_* \in \mathcal{P}^{k-1} \right\}, \tag{7.13}$$

where $C$ is a constant determined by $p$, $k$, and $D$. Observe that both $k$ and $D = \dim \mathcal{P}$ are bounded by constants depending on $m$ and $n$. Hence, $C$ is bounded by a constant determined by $p$, $m$, and $n$.

From (7.12) and (7.13),

$$\begin{aligned} &\inf_{P^1 \in \mathcal{P}} \left\{ \|f, P^1\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} : \partial^\alpha P^1(x_s) = \partial^\alpha P_0(x_s) \ \forall \alpha \in \mathcal{A} \right\} \\ &\qquad \simeq \inf_{P^1 \in \mathcal{P}} \left\{ M(f, P^1, \xi(f, P^1)) : \partial^\alpha P^1(x_s) = \partial^\alpha P_0(x_s) \ \forall \alpha \in \mathcal{A} \right\}. \end{aligned} \tag{7.14}$$

We apply Lemma 7.2 again, to the functional $M(f, P_0, P^1) := M(f, P^1, \xi(f, P^1))$, which is independent of $P_0$, where the constraint map $\Psi : \mathscr{J}(\mu) \times \mathcal{P} \times \mathcal{P} \to \mathbb{R}^{|\mathcal{A}|}$ is chosen so that $\Psi(f, P_0, P^1) = 0$ encodes the constraints $\partial^\alpha P^1(x_s) = \partial^\alpha P_0(x_s)$ (all $\alpha \in \mathcal{A}$). Thus we determine $R_{x_s} = \xi_s(f, P_0) \in \mathcal{P}$ depending linearly on $(f|_{9Q_s}, P_0)$ and satisfying $\partial^\alpha R_{x_s}(x_s) = \partial^\alpha P_0(x_s)$ for all $\alpha \in \mathcal{A}$, and

$$M(f, R_{x_s}, \xi(f, R_{x_s})) \leq C \cdot \inf_{P^1 \in \mathcal{P}} \left\{ M(f, P^1, \xi(f, P^1)) : \partial^\alpha P^1(x_s) = \partial^\alpha P_0(x_s) \ \forall \alpha \in \mathcal{A} \right\},$$

where $C = C(p, D)$. Therefore, in light of (7.14), $R'_{x_s} := R_{x_s}$ satisfies (7.2). This completes the construction and verification of the properties of the polynomial $R'_{x_s}$. $\square$

## 8. Decomposition of the functional

**Lemma 8.1.** *Let $Q^\circ$, $CZ^\circ$, $K_{CZ}$, and $K_p$ be defined as in Section 4. Let $\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)$ be the space defined in (6.2).*

***Existence of a Bounded Linear Map.*** *There exists a linear map $T : \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p) \to L^{m,p}(Q^\circ)$ satisfying the following conditions. For any $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$,*

$$J_x T(f, \vec{P}, \vec{S}) = S_x \text{ for all } x \in K_p, \text{ and} \tag{8.1}$$

$$\|T(f, \vec{P}, \vec{S}), \vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \leq C \cdot \|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}. \tag{8.2}$$

***Characterization of the Function Space.*** *For $f \in \mathscr{J}(\mu)$, $\vec{P} \in Wh(\mathfrak{B}_{CZ})$ and $\vec{S} \in Wh(K_p)$, consider the functionals $\mathcal{S}_0(f, \vec{P}, \vec{S}) \leq \mathcal{S}_1(f, \vec{P}, \vec{S}) \leq \mathcal{S}_2(f, \vec{P}, \vec{S})$ valued in $[0, \infty]$, given by*

$$\begin{aligned} \mathcal{S}_0(f, \vec{P}, \vec{S}) &:= \sum_{i \in I} \|f, P_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i \leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|_i^p + \int_{K_p} |S_x(x) - f(x)|^p d\mu, \\ \mathcal{S}_1(f, \vec{P}, \vec{S}) &:= \mathcal{S}_0(f, \vec{P}, \vec{S}) + 1_{\mathcal{A}=\emptyset} \cdot \|\vec{S}\|^p_{L^{m,p}(K_p)}, \\ \mathcal{S}_2(f, \vec{P}, \vec{S}) &:= \mathcal{S}_0(f, \vec{P}, \vec{S}) + \|\vec{S}\|^p_{L^{m,p}(K_p)}. \end{aligned} \tag{8.3}$$

*Then $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$ if and only if*

$$(\vec{P}, \vec{S}) \in C^{m-1,1-n/p}(\mathfrak{B}_{CZ} \cup K_p) \quad \text{and} \quad \mathcal{S}_2(f, \vec{P}, \vec{S}) < \infty. \tag{8.4}$$

*Further, for all $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$,*

$$\|f, \vec{P}, \vec{S}\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \simeq \mathcal{S}_1(f, \vec{P}, \vec{S}). \tag{8.5}$$

*In (8.3), the indicator term $1_{\mathcal{A}=\emptyset}\|\vec{R^*}\|^p_{L^{m,p}(K_p)}$ is included in the expression $\mathcal{S}_1(f, \vec{R}, \vec{R}^*)$ if and only if $\mathcal{A} = \emptyset$.*

**Proof.** We establish half of (8.4). Specifically, we show that $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$ implies

$$(\vec{P}, \vec{S}) \in C^{m-1,1-n/p}(\mathfrak{B}_{CZ} \cup K_p) \quad \text{and} \quad \mathcal{S}_2(f, \vec{P}, \vec{S}) \lesssim \|f, \vec{P}, \vec{S}\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} < \infty. \tag{8.6}$$

Let $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$; then from (6.3), we have $(\vec{P}, \vec{S}) \in C^{m-1,1-n/p}(\mathfrak{B}_{CZ} \cup K_p)$. Let $\eta > 0$; then there exists $H \in L^{m,p}(Q^\circ)$ satisfying

$$J_xH = S_x \text{ for all } x \in K_p, \text{ and} \tag{8.7}$$

$$\|H, \vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \leq \|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \eta < \infty. \tag{8.8}$$

As a consequence of Lemma 2.7, we have

$$\begin{aligned}\|\vec{S}\|_{L^{m,p}(K_p)} \lesssim \|H\|_{L^{m,p}(Q^\circ)} \leq \|H, \vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\ \lesssim \|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \eta < \infty.\end{aligned} \tag{8.9}$$

Also,

$$\begin{aligned}\int\limits_{K_p} |S_x(x) - f(x)|^p d\mu \leq \int |H - f|^p d\mu \leq \|H, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\ \lesssim (\|f, \vec{P}, \vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \eta)^p < \infty.\end{aligned} \tag{8.10}$$

By Lemma 2.6,

$$\|H - P_{x_i}\|^p_{L^p(1.1Q_i \cap Q^\circ)}/\delta^{mp}_{Q_i} \lesssim \|H - P_{x_i}\|^p_{L^p(Q_i)}/\delta^{mp}_{Q_i} + \|H\|^p_{L^{m,p}(1.1Q_i \cap Q^\circ)}. \tag{8.11}$$

Because of (4.1), (2.2), and (2.3), it holds that $|P|_i \simeq |P|_{i'}$ for all $P \in \mathcal{P}$ if $i \leftrightarrow i'$. Thus, by the Sobolev Inequality, (5.10), and the bounded overlap of the cubes $\{1.1Q_i\}_{i\in I}$, we have

$$\begin{aligned}\sum_{i\leftrightarrow i'} |P_{x_i} - P_{x_{i'}}|^p_i &\overset{(2.2)}{\lesssim} \sum_{i\leftrightarrow i'} \left\{|P_{x_i} - J_{x_i}H|^p_i + |J_{x_i}H - J_{x_{i'}}H|^p_i + |J_{x_{i'}}H - P_{x_{i'}}|^p_{i'}\right\} \\ &\overset{\mathrm{SI}/(5.10)}{\lesssim} \sum_{i\in I} \|H, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{Q_i};Q_i)} + \sum_{i\leftrightarrow i'} \|H\|^p_{L^{m,p}((1.1Q_i \cup 1.1Q_{i'})\cap Q^\circ)} \\ &\lesssim \sum_{i\in I} \|H, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{Q_i};Q_i)} + \|H\|^p_{L^{m,p}(K_{CZ})}.\end{aligned} \tag{8.12}$$

We apply (2.26) (for the $C$-non-degenerate rectangular box $1.1Q_i \cap Q^\circ$), (8.12), (8.11), and (8.8) to estimate

$$\begin{aligned}\sum_{i\in I} \|f, P_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} &\overset{(2.26)}{\lesssim} \sum_{i\in I} \|f, P_{x_i}\|^p_{\mathscr{J}_*(\mu|_{1.1Q_i};1.1Q_i\cap Q^\circ)} \\ &\leq \sum_{i\in I} \|H, P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i};1.1Q_i\cap Q^\circ)} \\ &= \sum_{i\in I} \left[\|H\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)} + \int\limits_{1.1Q_i} |H - f|^p d\mu + \|H - P_{x_i}\|^p_{L^p(1.1Q_i\cap Q^\circ)}/\delta^{mp}_{Q_i}\right]\end{aligned}$$

$$\overset{(8.11)}{\lesssim} \sum_{i\in I}\Big[\|H\|^p_{L^{m,p}(1.1Q_i\cap Q^\circ)} + \int_{1.1Q_i} |H-f|^p d\mu + \|H-P_{x_i}\|^p_{L^p(Q_i)}/\delta_{Q_i}^{mp}\Big]$$

$$\lesssim \|H\|^p_{L^{m,p}(K_{CZ})} + \int_{K_{CZ}} |H-f|^p d\mu + \sum_{i\in I}\|H-P_{x_i}\|^p_{L^p(Q_i)}/\delta_{Q_i}^{mp}$$

$$= \|H,\vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)}$$

$$\overset{(8.8)}{\leq} (\|f,\vec{P},\vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}+\eta)^p. \tag{8.13}$$

Combining (8.9), (8.10), (8.12), and (8.13), and letting $\eta \to 0$, we have established (8.6) and consequently the sufficiency of (8.4).

Next we describe the construction of the map $T$. By applying **(AL1)-(AL3)** (see Section 4.4) to the cube $Q_i \in CZ^\circ$ and subset $E_i = 1.1Q_i \subset 3Q_i$, we obtain the existence of a linear map $T_i : \mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i}) \to L^{m,p}(\mathbb{R}^n)$, a functional $M_i : \mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i}) \to \mathbb{R}_+$, and countable collections of Borel sets $\{A^i_\ell\}_{\ell\in\mathbb{N}}$, $A^i_\ell \subset \mathrm{supp}(\mu|_{1.1Q_i})$, and of linear maps $\{\phi^i_\ell : \mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i}) \to \mathbb{R}\}_{\ell\in\mathbb{N}}$, and $\{\lambda^i_\ell : \mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i}) \to L^p(d\mu)\}_{\ell\in\mathbb{N}}$, with the following properties.

Given $(f,P_{x_i}) \in \mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})$,

$$\|f,P_{x_i}\|_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} \simeq \|T_i(f,P_{x_i}),P_{x_i}\|_{\mathscr{J}(f,\mu|_{1.1Q_i};\delta_{Q_i})} \simeq M_i(f,P_{x_i}); \text{ and} \tag{8.14}$$

$$M_i(f,P_{x_i}) = \Big(\sum_{\ell\in\mathbb{N}}\int_{A^i_\ell} |\lambda^i_\ell(f,P_{x_i})-f|^p d\mu + \sum_{\ell\in\mathbb{N}}|\phi^i_\ell(f,P_{x_i})|^p\Big)^{1/p}. \tag{8.15}$$

Further, the maps $T_i$ and $M_i$ are $\Omega'_i$-constructible for a family of linear functionals $\Omega'_i \subset \mathscr{J}(\mu|_{1.1Q_i})^*$, i.e., the maps satisfy **(AL4)-(AL6)** for $E_i = 1.1Q_i$

Given $(f,\vec{P},\vec{S}) \in \mathscr{J}(\mu)\times Wh(\mathfrak{B}_{CZ})\times Wh(K_p)$, define a function $T(f,\vec{P},\vec{S}) : Q^\circ \to \mathbb{R}$ as

$$T(f,\vec{P},\vec{S})(x) = \begin{cases} \sum_{i\in I} T_i(f,P_{x_i})(x)\cdot\theta_i(x) & x\in K_{CZ} \\ S_x(x) & x\in K_p \end{cases} \tag{8.16}$$

where $\{\theta_i\}_{i\in I}$ satisfy **(POU1)-(POU4)** (see Section 4.3). We apply (5.48), (2.8), and (8.14) to deduce

$$\|T(f,\vec{P},\vec{S}),\vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)}$$

$$\overset{(5.48)}{\lesssim} \sum_{i\in I}\|T_i(f,P_{x_i}),P_{x_i}\|^p_{\mathscr{J}_*(f,\mu|_{1.1Q_i\cap Q^\circ};1.1Q_i\cap Q^\circ)} + \sum_{i'\leftrightarrow i}|P_{x_i}-P_{x_{i'}}|^p_i$$

$$\overset{(2.8)}{\lesssim} \sum_{i\in I}\|T_i(f,P_{x_i}),P_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i'\leftrightarrow i}|P_{x_i}-P_{x_{i'}}|^p_i$$

$$\overset{(8.14)}{\lesssim} \sum_{i \in I} \|f, P_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i' \leftrightarrow i} |P_{x_i} - P_{x_{i'}}|_i^p. \tag{8.17}$$

We have not proved the map $T$ is bounded yet, but we return to the proof of the necessity of (8.4): Suppose $(\vec{P}, \vec{S}) \in C^{m-1,1-n/p}(\mathfrak{B}_{CZ} \cup K_p)$ and $\mathcal{S}_2(f, \vec{P}, \vec{S}) < \infty$. In light of (8.17) and the definition of the $\mathcal{S}_2$ functional,

$$\|T(f, \vec{P}, \vec{S}), \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|\vec{S}\|^p_{L^{m,p}(K_p)} \lesssim \mathcal{S}_2(f, \vec{P}, \vec{S}) < \infty.$$

Consequently, we may apply Lemma 5.6 to deduce that the function $T(f, \vec{P}, \vec{S})$ defined in (8.16) satisfies

$$T(f, \vec{P}, \vec{S}) \in L^{m,p}(Q^\circ), \tag{8.18}$$

$$J_x T(f, \vec{P}, \vec{S}) = S_x \text{ for all } x \in K_p, \text{ and} \tag{8.19}$$

$$\begin{aligned} \|T(f, \vec{P}, \vec{S})\|_{L^{m,p}(Q^\circ)} &\lesssim \|T(f, \vec{P}, \vec{S}), \vec{P}\|_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|\vec{S}\|_{L^{m,p}(K_p)} \\ &< \infty. \end{aligned} \tag{8.20}$$

Because $T(f, \vec{P}, \vec{S})(x) = S_x(x)$ for $x \in K_p$, we have

$$\int_{K_p} |T(f, \vec{P}, \vec{S}) - f|^p d\mu = \int_{K_p} |S_x(x) - f(x)|^p d\mu(x) \leq \mathcal{S}_2(f, \vec{P}, \vec{S}) < \infty. \tag{8.21}$$

In combination with (8.18)-(8.20) and (8.17), we bound

$$\begin{aligned} &\|f, \vec{P}, \vec{S}\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \\ &\quad \leq \|T(f, \vec{P}, \vec{S}), \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\ &\quad \leq \|T(f, \vec{P}, \vec{S}), \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \|T(f, \vec{P}, \vec{S})\|^p_{L^{m,p}(Q^\circ)} + \int_{K_p} |T(f, \vec{P}, \vec{S}) - f|^p d\mu \\ &\quad \lesssim \sum_{i \in I} \|f, P_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i' \leftrightarrow i} |P_{x_i} - P_{x_{i'}}|_i^p + \|\vec{S}\|^p_{L^{m,p}(K_p)} + \int_{K_p} |S_x(x) - f(x)|^p d\mu \\ &\quad \lesssim \mathcal{S}_2(f, \vec{P}, \vec{S}) < \infty, \end{aligned} \tag{8.22}$$

completing the proof of (8.4).

Suppose $(f, \vec{P}, \vec{S}) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$; we proved the extension operator $T(f, \vec{P}, \vec{S})$ defined in (8.16) satisfies $J_x T(f, \vec{P}, \vec{S}) = S_x$ for all $x \in K_p$, and

$$\mathcal{S}_2(f, \vec{P}, \vec{S}) \lesssim \|f, \vec{P}, \vec{S}\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \tag{8.23}$$

(see (8.6)). In combination with (8.22), this proves (8.1) and (8.2):

$$\|T(f,\vec{P},\vec{S}),\vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \lesssim \|f,\vec{P},\vec{S}\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)},$$

as well as (8.5) in the case $\mathcal{A}=\emptyset$: In this case, $\mathcal{S}_2(f,\vec{P},\vec{S})=\mathcal{S}_1(f,\vec{P},\vec{S})$. Together with (8.22) and (8.23), we have

$$\|f,\vec{P},\vec{S}\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \simeq \mathcal{S}_1(f,\vec{P},\vec{S}).$$

Suppose $\mathcal{A}\neq\emptyset$. Thanks to Lemma 4.9, the set $K_p$ has Lebesgue measure 0, and so

$$\|T(f,\vec{P},\vec{S})\|_{L^{m,p}(Q^\circ)} = \|T(f,\vec{P},\vec{S})\|_{L^{m,p}(K_{CZ})} \qquad (\mathcal{A}\neq\emptyset),$$

and therefore,

$$\|T(f,\vec{P},\vec{S}),\vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} = \|T(f,\vec{P},\vec{S})\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \int\limits_{K_p} |T(f,\vec{P},\vec{S})-f|^p d\mu. \tag{8.24}$$

Combining (8.17), (8.21), (8.24),

$$\begin{aligned}
&\|T(f,\vec{P},\vec{S}),\vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\
&\overset{(8.24)}{=} \|T(f,\vec{P},\vec{S}),\vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \int\limits_{K_p} |T(f,\vec{P},\vec{S})-f|^p d\mu \\
&\overset{(8.21)}{=} \|T(f,\vec{P},\vec{S}),\vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)} + \int\limits_{K_p} |S_x(x)-f(x)|^p d\mu(x) \\
&\overset{(8.17)}{\lesssim} \sum_{i\in I} \|f,P_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i\leftrightarrow i'} |P_{x_i}-P_{x_{i'}}|^p_i + \int\limits_{K_p} |S_x(x)-f(x)|^p d\mu(x) \\
&= \mathcal{S}_1(f,\vec{P},\vec{S}).
\end{aligned} \tag{8.25}$$

Therefore,

$$\|f,\vec{P},\vec{S}\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \le \|T(f,\vec{P},\vec{S}),\vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \lesssim \mathcal{S}_1(f,\vec{P},\vec{S}).$$

Combining this estimate with (8.23) and $\mathcal{S}_1(f,\vec{P},\vec{S}) \le \mathcal{S}_2(f,\vec{P},\vec{S})$, we have proven (8.5) under the assumption that $\mathcal{A}\neq\emptyset$, completing the proof of Lemma 8.1. $\square$

## 9. Optimal Whitney field

We consider the set of all Calderón-Zygmund cubes $CZ^\circ=\{Q_i\}_{i\in I}$ in $Q^\circ$. We denote $K_{CZ}=\bigcup_{i\in I} Q_i$, and $K_p = Q^\circ \setminus K_{CZ}$. Then $CZ^\circ$ is a partition of the set $K_{CZ}$ into disjoint dyadic cubes. We denote $CZ_{key} = \{Q_s\}_{s\in\bar{I}} \subset \{Q_i\}_{i\in I}$ for the set of keystone cubes (see page 42 for the definition and basic properties of keystone cubes). Recall that $\mathfrak{B}_{CZ} = \{x_i\}_{i\in I}$ and $\mathfrak{B}_{key} = \{x_s\}_{s\in\bar{I}}$ are the sets of centers of CZ cubes, and keystone cubes, respectively.

For each $Q_i \in CZ^\circ$, we apply Lemma 4.11 to produce a sequence of CZ cubes $\mathscr{S}(Q_i)$. Then either $\mathscr{S}(Q_i) = \{Q^{i,k}\}_{k=1}^L$ and $Q^{i,L} \in CZ_{key}$ or $\mathscr{S}(Q_i) = \{Q^{i,k}\}_{k\in\mathbb{N}}$ and $x^{i,k} \to x \in K_p$ as $k \to \infty$, where $x^{i,k}$ is the center of $Q^{i,k}$. In either case, $Q^{i,1} = Q_i$, $Q^{i,k} \leftrightarrow Q^{i,k+1}$ for each $k$, and for $1 \leq \ell \leq k$,

$$\delta_{Q^{i,k}} \leq C \cdot c^{k-\ell} \delta_{Q^{i,\ell}}, \tag{9.1}$$

for universal constants $C \geq 1$ and $c \in (0,1)$.

Now, define a mapping $\kappa : \mathfrak{B}_{CZ} \to \mathfrak{B}_{key} \cup K_p$ by

$$\kappa(x_i) = \begin{cases} x_s & \mathscr{S}(Q_i) = \{Q^{i,k}\}_{k=1}^L, \text{ and } Q^{i,L} = Q_s \in CZ_{key} \\ x & \mathscr{S}(Q_i) = \{Q^{i,k}\}_{k\in\mathbb{N}}, \text{ and } \lim_{k\to\infty} x^{i,k} = x \in K_p. \end{cases} \tag{9.2}$$

The next lemma contains further elementary properties of the sequences $\mathscr{S}(Q_i)$, and of the mapping $\kappa$, that will be used throughout the section.

**Lemma 9.1.** *For each $i \in I$, let $x_i$ be the center of $Q_i$, and let $x^{i,j}$ be the center of the cube $Q^{i,j}$ in the sequence $\mathscr{S}(Q_i)$.*

1. *If $\kappa(x_i) \neq x_i$ then $|x_i - \kappa(x_i)| \simeq \delta_{Q_i}$.*
2. *$|x_i - \kappa(x_i)| \lesssim |x_i - x|$ for any $x \in K_p$.*
3. *If $Q^{i,j} \in \mathscr{S}(Q_i)$ then $Q^{i,j} \subset CQ_i$ and $|x^{i,j} - \kappa(x_i)| \lesssim \delta_{Q^{i,j}}$.*
4. *For any keystone cube $Q_s$ and $x_i \in \kappa^{-1}(x_s)$, $Q_s \subset CQ_i$, and in particular $\delta_{Q_i} \geq c\delta_{Q_s}$. Furthermore, for any fixed $\delta > 0$ and $s \in \bar{I}$,*

$$|\{i \in I : x_i \in \kappa^{-1}(x_s), \delta_{Q_i} = \delta\}| \leq C.$$

**Proof.** (1): Because $\kappa(x_i) \neq x_i$, the sequence $\mathscr{S}(Q_i)$ has length at least 2, and $\kappa(x_i)$ does not belong to $\mathrm{int}(Q_i)$ (either $\kappa(x_i) \in Q_s$ for $s \neq i$, or $\kappa(x_i) \in K_p$). Recall the sequence $\mathscr{S}(Q_i) = \{Q^{i,k}\}_{k\geq 1}$ (finite or infinite) of CZ cubes satisfies $Q^{i,k} \leftrightarrow Q^{i,k+1}$ and $Q^{i,1} = Q_i$. Hence, $|x^{i,k} - x^{i,k+1}| \lesssim \delta_{Q^{i,k}}$, $x^{i,1} = x_i$, and either $x^{i,L} = \kappa(x_i)$ for some $L < \infty$ or $x^{i,k} \to \kappa(x_i)$ as $k \to \infty$. We use inequality (9.1) for $\ell = 1$ to bound:

$$|x_i - \kappa(x_i)| \leq \sum_{k\geq 1} |x^{i,k} - x^{i,k+1}| \lesssim \sum_{k\geq 1} \delta_{Q^{i,k}} \lesssim \sum_{k\geq 1} \delta_{Q_i} c^k \lesssim \delta_{Q_i}.$$

Because $x_i = \mathrm{ctr}(Q_i)$ and $\kappa(x_i)$ does not belong to $\mathrm{int}(Q_i)$, we also have $\delta_{Q_i} \lesssim |x_i - \kappa(x_i)|$.

(2): Let $x \in K_p$. We may assume $\kappa(x_i) \neq x_i$. Because $Q_i \subset K_{CZ}$ and $x_i = \mathrm{ctr}(Q_i)$, we have $\mathrm{dist}(x_i, K_p) \gtrsim \delta_{Q_i}$. In combination with (1),

$$|x - x_i| \geq \mathrm{dist}(x_i, K_p) \gtrsim \delta_{Q_i} \gtrsim |x_i - \kappa(x_i)|.$$

(3): Let $Q^{i,j} \in \mathscr{S}(Q_i)$. From (9.1), $\delta_{Q^{i,j}} \lesssim \delta_{Q_i}$ and $\mathrm{dist}(Q^{i,j}, Q_i) \lesssim \sum_{k=1}^{j} \delta_{Q^{i,k}} \lesssim \delta_{Q_i}$, implying there exists $C > 0$ such that $Q^{i,j} \subset CQ_i$. We use (1) to bound

$$|x^{i,j} - \kappa(x_i)| \leq |x^{i,j} - x_i| + |x_i - \kappa(x_i)| \lesssim \delta_{Q_i}.$$

(4): If $Q_s$ is a keystone cube and $x_i \in \kappa^{-1}(x_s)$ then $\mathscr{S}(Q_i)$ is finite and $Q_s = Q^{i,L}$ for some $L < \infty$. In light of (9.1), $\delta_{Q_s} = \delta_{Q^{i,L}} \lesssim \delta_{Q_i}$. From (3) applied with $j = L$, we have $Q_s \subset CQ_i$. By a volume counting argument,

$$|\{x_i \in \kappa^{-1}(x_s) : \delta_{Q_i} = \delta\}| \leq |\{Q_i : \delta_{Q_i} = \delta \text{ and } |x_i - x_s| \simeq \delta\}| \leq C. \quad \square$$

The next result concerns a certain operator mapping Whitney fields on $K_p \cup \mathfrak{B}_{CZ}$ to Whitney fields on $\mathfrak{B}_{CZ}$.

**Lemma 9.2.** *Given $\vec{P}^* = (P_x^*)_{x \in K_p} \in Wh(K_p)$ and $\vec{P} = (P_{x_i})_{i \in I} \in Wh(\mathfrak{B}_{CZ})$, define $\vec{P}^{**} \in Wh(\mathfrak{B}_{CZ})$ by*

$$P_{x_i}^{**} := \begin{cases} P_{\kappa(x_i)} & \kappa(x_i) \in \mathfrak{B}_{key} \\ P^*_{\kappa(x_i)} & \kappa(x_i) \in K_p. \end{cases}$$

*If $(\vec{P}^*, \vec{P}) \in C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})$, then for $F \in L^{m,p}(Q^\circ)$,*

$$\sum_{i \in I} |P_{x_i} - P_{x_i}^{**}|_i^p \lesssim \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)}. \tag{9.3}$$

**Proof.** From Lemma 9.1, $|x_i - \kappa(x_i)| \lesssim \delta_{Q_i}$. Therefore, from (2.2), we have

$$\begin{aligned}\sum_{i \in I} |P_{x_i} - P_{x_i}^{**}|_i^p = \sum_{i \in I} |P_{x_i} - P_{x_i}^{**}|^p_{x_i,\delta_{Q_i}} &\lesssim \sum_{i \in I} |P_{x_i} - P_{x_i}^{**}|^p_{\kappa(x_i),\delta_{Q_i}} \\ &= \sum_{i \in I} \sum_{\alpha \in \mathcal{M}} |\partial^\alpha (P_{x_i} - P_{x_i}^{**})(\kappa(x_i))|^p \delta_{Q_i}^{n-mp+|\alpha|p}.\end{aligned} \tag{9.4}$$

We will compare polynomials $P^{i,k}$ ($k \geq 1$) associated to the cubes $Q^{i,k} \in \mathscr{S}(Q_i)$ to bound (9.4). We define these polynomials in terms of the Whitney field $\vec{P} \in Wh(\mathfrak{B}_{CZ})$, Note that each $Q^{i,k} \in \mathscr{S}(Q_i)$ is in $CZ^\circ$, hence, $Q^{i,k} = Q_j$ for $j \in I$ – then we define $P^{i,k} = P_{x_j}$.

Evidently, $Q^{i,1} = Q_i$, so

$$P^{i,1} = P_{x_i}.$$

If $\mathscr{S}(Q_i) = \{Q^{i,k}\}_{1\leq k\leq L}$ is finite, then its terminal cube is a keystone cube, $Q^{i,L} = Q_s$, for $\kappa(x_i) = x_s \in \mathfrak{B}_{key}$. In this case, $P^{i,L} = P_{\kappa(x_i)} = P^{**}_{x_i}$. So,

$$P^{i,L} = P^{**}_{x_i} \text{ if } \mathscr{S}(Q_i) = \{Q^{i,k}\}_{1\leq k\leq L} \text{ is finite.}$$

If $\mathscr{S}(Q_i) = \{Q^{i,k}\}_{k\geq 1}$ is infinite, then $x^{i,k} = \mathrm{ctr}(Q^{i,k})$ converges to $x = \kappa(x_i) \in K_p$ as $k \to \infty$. Because $(\vec{P}^*, \vec{P}) \in C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})$, also $P^{i,k}$ converges to $P^*_x$ as $k \to \infty$. But $P^*_x = P^{**}_{x_i}$, by definition of $\vec{P}^{**}$. Thus,

$$\lim_{k\to\infty} P^{i,k} = P^{**}_{x_i} \text{ if } \mathscr{S}(Q_i) \text{ is infinite.}$$

Evidently, by use of the telescoping series formula, and by the above properties, for $\alpha \in \mathcal{M}$ and $i \in I$,

$$|\partial^\alpha(P_{x_i} - P^{**}_{x_i})(\kappa(x_i))| \leq \sum_{k\geq 1} |\partial^\alpha(P^{i,k} - P^{i,k+1})(\kappa(x_i))|,$$

where we write $\sum_{k\geq 1}$ to indicate the summation $\sum_{k=1}^{L-1}$ in the case $\mathscr{S}(Q_i) = \{Q^{i,k}\}_{1\leq k\leq L}$ is finite, and the summation $\sum_{k=1}^{\infty}$ in the case $\mathscr{S}(Q_i)$ is infinite

Fix a universal constant $\epsilon' \in (0, 1-n/p)$, and let $p' \in (1,\infty)$ be the conjugate exponent of $p \in (1,\infty)$, so that $1/p + 1/p' = 1$. We apply Hölder's inequality to deduce

$$\begin{aligned}
&|\partial^\alpha(P_{x_i} - P^{**}_{x_i})(\kappa(x_i))| \\
&\quad\leq \Big(\sum_{k\geq 1} |\partial^\alpha(P^{i,k} - P^{i,k+1})(\kappa(x_i))|^p \delta_{Q^{i,k}}^{-mp+n+|\alpha|p+\epsilon' p}\Big)^{\frac{1}{p}} \cdot \Big(\sum_{k\geq 1} \delta_{Q^{i,k}}^{(m-n/p-|\alpha|-\epsilon')p'}\Big)^{\frac{1}{p'}} \\
&\quad\lesssim \delta_{Q_i}^{m-n/p-|\alpha|-\epsilon'} \big(\sum_{k\geq 1} |\partial^\alpha(P^{i,k} - P^{i,k+1})(\kappa(x_i))|^p \delta_{Q^{i,k}}^{-mp+n+|\alpha|p+\epsilon' p}\big)^{\frac{1}{p}},
\end{aligned} \tag{9.5}$$

where the last inequality uses that $m - n/p - |\alpha| - \epsilon' > 0$ for $|\alpha| \leq m-1$, so (9.1) implies

$$\big(\sum_{k\geq 1} \delta_{Q^{i,k}}^{(m-n/p-|\alpha|-\epsilon')p'}\big)^{1/p'} \lesssim \delta_{Q_i}^{m-n/p-|\alpha|-\epsilon'}.$$

Because $Q^{i,k} \in \mathscr{S}(Q_i)$, we have $|x^{i,k} - \kappa(x_i)| \lesssim \delta_{Q^{i,k}}$ (see Lemma 9.1). By substituting (9.5) into (9.4), using the definition of the $|\cdot|_{\kappa(x_i),\delta_{Q^{i,k}}}$ norm, and then applying (2.2),

$$\begin{aligned}
\sum_{i\in I} |P_{x_i} - P^{**}_{x_i}|_i^p &\lesssim \sum_{i\in I} \delta_{Q_i}^{-\epsilon' p} \sum_{k\geq 1} \sum_{\alpha\in\mathcal{M}} |\partial^\alpha(P^{i,k} - P^{i,k+1})(\kappa(x_i))|^p \delta_{Q^{i,k}}^{-mp+n+|\alpha|p+\epsilon' p} \\
&= \sum_{i\in I} \delta_{Q_i}^{-\epsilon' p} \sum_{k\geq 1} |P^{i,k} - P^{i,k+1}|_{\kappa(x_i),\delta_{Q^{i,k}}} \delta_{Q^{i,k}}^{\epsilon' p} \\
&\lesssim \sum_{i\in I} \delta_{Q_i}^{-\epsilon' p} \sum_{k\geq 1} |P^{i,k} - P^{i,k+1}|^p_{x^{i,k},\delta_{Q^{i,k}}} \cdot \delta_{Q^{i,k}}^{\epsilon' p}
\end{aligned}$$

$$\lesssim \sum_{\substack{j,j'\in I\\ j\leftrightarrow j'}} \sum_{\substack{i\in I\\ Q_j\in\mathscr{S}(Q_i)}} \left(\frac{\delta_{Q_j}}{\delta_{Q_i}}\right)^{\epsilon' p} |P_{x_j} - P_{x_{j'}}|^p_{x_j,\delta_{Q_j}}. \tag{9.6}$$

For fixed $Q_j \in CZ^\circ$ and any dyadic length scale $\delta > 0$,

$$|\{i \in I : Q_j \in \mathscr{S}(Q_i), \delta_{Q_i} = \delta\}| \le |\{i \in I : Q_j \subset CQ_i,\ \delta_{Q_i} = \delta\}| \le C,$$

where the first inequality uses property 3 in Lemma 9.1, and the second inequality uses that $Q_i$ and $Q_j$ are dyadic cubes. Furthermore, by the first inequality above,

$$|\{i \in I : Q_j \in \mathscr{S}(Q_i),\ \delta_{Q_i} = \delta\}| = 0 \text{ if } \delta < c\delta_{Q_j}.$$

Thus we may continue from (9.6):

$$\begin{aligned}
\sum_{i\in I} |P_{x_i} - P^{**}_{x_i}|^p_i &\lesssim \sum_{j\leftrightarrow j'} |P_{x_j} - P_{x_{j'}}|^p_j \sum \left\{ \left(\frac{\delta_{Q_j}}{\delta_{Q_i}}\right)^{\epsilon' p} : i \in I,\ Q_j \in \mathscr{S}(Q_i) \right\} \\
&\lesssim \sum_{j\leftrightarrow j'} |P_{x_j} - P_{x_{j'}}|^p_j \sum \left\{ \left(\frac{\delta_{Q_j}}{\delta}\right)^{\epsilon' p} : \delta \text{ a dyadic length scale, } \delta \ge c\delta_{Q_j} \right\} \\
&\lesssim \sum_{j\leftrightarrow j'} |P_{x_j} - P_{x_{j'}}|^p_j, \qquad (9.7)
\end{aligned}$$

where the final inequality follows because we are computing the sum of a geometric series, and $\epsilon' > 0$ is a universal constant, dependent on $m$, $n$, and $p$. From the triangle inequality, the Sobolev Inequality, (2.14), and (2.2), for $F \in L^{m,p}(Q^\circ)$,

$$\begin{aligned}
&\sum_{j\leftrightarrow j'} |P_{x_j} - P_{x_{j'}}|^p_j \\
&\lesssim \sum_{j\leftrightarrow j'} \left[ |P_{x_j} - J_{x_j}(F)|^p_j + |J_{x_{j'}}(F) - P_{x_{j'}}|^p_j + \|F\|^p_{L^{m,p}((1.1Q_j \cup 1.1Q_{j'})\cap Q^\circ)} \right] \\
&\lesssim \sum_{j\leftrightarrow j'} \left[ \|F, P_{x_j}\|^p_{\mathscr{J}_*(f,\mu;Q_j)} + \|F\|^p_{L^{m,p}((1.1Q_j \cup 1.1Q_{j'})\cap Q^\circ)} \right] \\
&\lesssim \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)}.
\end{aligned}$$

Substituting this into (9.7), we complete the proof of the lemma:

$$\sum_{i\in I} |P_{x_i} - P^{**}_{x_i}|^p_i \lesssim \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;K_{CZ},CZ^\circ)}. \qquad \square$$

### *9.1. Whitney fields on $K_p$ and $\mathfrak{B}_{CZ}$*

Fix the data $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$.

From this data, we defined a Whitney field $\vec{R}^* \in Wh(K_p)$ in Section 6.3. The defining properties of $\vec{R}^*$ are stated in Lemma 6.7. We also defined a family of polynomials $R'_{x_s} \in \mathcal{P}$ $(s \in \bar{I})$, satisfying the conditions in Lemma 7.3. This Whitney field and these polynomials depend linearly on $(f, P_0)$, and they are coherent with $P_0$ in the sense that $\partial^\alpha R^*_x(x) = \partial^\alpha P_0(x)$ for all $x \in K_p$, $\alpha \in \mathcal{A}$, while $\partial^\alpha R'_{x_s}(x_s) = \partial^\alpha P_0(x_s)$ for all $s \in \bar{I}$, $\alpha \in \mathcal{A}$. Because $\mathcal{A}$ is monotonic it holds that if $\partial^\alpha P(x) = 0$ for all $\alpha \in \mathcal{A}$, then $\partial^\alpha P \equiv 0$ on $\mathbb{R}^n$ for all $\alpha \in \mathcal{A}$, for any $P \in \mathcal{P}$. Therefore,

$$\partial^\alpha (R'_{x_s} - P_0) \equiv 0, \;\; \partial^\alpha (R^*_x - P_0) \equiv 0 \text{ for all } x \in K_p, s \in \bar{I}, \alpha \in \mathcal{A}. \tag{9.8}$$

We define the Whitney field $\vec{R} = (R_x)_{x \in \mathfrak{B}_{CZ}} \in Wh(\mathfrak{B}_{CZ})$ $(\mathfrak{B}_{CZ} = \{x_i\}_{i \in I})$ by

$$R_{x_i} := \begin{cases} R'_{x_s} & \text{if } \kappa(x_i) = x_s \in \mathfrak{B}_{key} \\ R^*_x & \text{if } \kappa(x_i) = x \in K_p, \end{cases} \tag{9.9}$$

where $\kappa : \mathfrak{B}_{CZ} \to \mathfrak{B}_{key} \cup K_p$ is the mapping defined in (9.2).

Evidently, the Whitney field $(\vec{R}^*, \vec{R}) \in Wh(K_p \cup \mathfrak{B}_{CZ})$ depends linearly on $(f, P_0)$. From (9.8), it holds that $\partial^\alpha (R_{x_i} - P_0) \equiv 0$ for $\alpha \in \mathcal{A}$ and $x_i \in \mathfrak{B}_{CZ}$ – thus, $\vec{R}$ is coherent with $P_0$. As mentioned above, $\vec{R}^*$ is coherent with $P_0$. So, $(\vec{R}^*, \vec{R})$ is coherent with $P_0$.

**Lemma 9.3.** *The Whitney field $(\vec{R}^*, \vec{R}) \in Wh(K_p \cup \mathfrak{B}_{CZ})$ satisfies*

$$\|(\vec{R}^*, \vec{R})\|_{C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})} \leq C \|f, P_0\|_{\mathscr{J}(\mu; \delta_{Q^\circ})}.$$

**Proof.** Let $\eta > 0$. From Corollary 6.9, there exists $H \in L^{m,p}(Q^\circ)$ and $\vec{P} \in Wh(\mathfrak{B}_{CZ})$ satisfying $J_x H = R^*_x$ for all $x \in K_p$, $\vec{P}$ is coherent with $P_0$, and

$$\|H, \vec{P}\|_{\mathscr{J}_*(f, \mu; Q^\circ, CZ^\circ)} \leq \|f, \vec{P}, \vec{R}^*\|_{\mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)} + \eta/2 \lesssim \|f, P_0\|_{\mathscr{J}(\mu; \delta_{Q^\circ})} + \eta. \tag{9.10}$$

Consequently, since $\vec{R}^* \in Wh(K_p)$ is coherent with $P_0$, also $H$ is $K_p$-coherent with $P_0$. From Proposition 6.1 and (9.10), we have

$$\|(\vec{R}^*, \vec{P})\|_{C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})} \lesssim \|f, P_0\|_{\mathscr{J}(\mu; \delta_{Q^\circ})} + \eta. \tag{9.11}$$

To prove Lemma 9.3, we must prove the following inequalities:

$$\begin{cases} |R^*_x - R^*_y|_{x, |x-y|} \leq C \|f, P_0\|_{\mathscr{J}(\mu; \delta_{Q^\circ})} & x, y \in K_p \text{ distinct;} \\ |R^*_x - R_{x_i}|_{x_i, |x - x_i|} \leq C \|f, P_0\|_{\mathscr{J}(\mu; \delta_{Q^\circ})} & x \in K_p, \; i \in I; \\ |R_{x_i} - R_{x_j}|_{x_j, |x_j - x_i|} \leq C \|f, P_0\|_{\mathscr{J}(\mu; \delta_{Q^\circ})} & i, j \in I \text{ distinct.} \end{cases} \tag{9.12}$$

We proceed with the proof of (9.12) in cases, below.

Case 1: For distinct $x, y \in K_p$, we apply (9.11):

$$|R_x^* - R_y^*|_{x,|y-x|} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta.$$

By letting $\eta \to 0$, we deduce the first line of (9.12).

Case 2: Suppose $x \in K_p$ and $i \in I$; we will prove the second line of (9.12): First, suppose that $\kappa(x_i) = y \in K_p$. Then by definition of $\vec{R}$ (see (9.9)), $R_{x_i} = R_y^*$. By part 2 of Lemma 9.1, $|x-y| \leq |x-x_i| + |x_i - y| \lesssim |x - x_i|$. So, by (2.2), and the analysis of Case 1,

$$|R_x^* - R_{x_i}|_{x,|x_i-x|} = |R_x^* - R_y^*|_{x,|x_i-x|} \lesssim |R_x^* - R_y^*|_{x,|x-y|} \lesssim \|f, P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})}. \tag{9.13}$$

This estimate proves the second line of (9.12) for $\kappa(x_i) = y \in K_p$.

Next suppose that $\kappa(x_i) = x_s \in \mathfrak{B}_{key}$. By definition of $\vec{R}$, $R_{x_i} = R'_{x_s}$, with $x_s \in \mathfrak{B}_{CZ}$ the center of the keystone cube $Q_s$. We apply (9.11) to deduce

$$\begin{aligned} |R_x^* - R_{x_i}|_{x_i,|x_i-x|} &\leq |R_x^* - P_{x_i}|_{x_i,|x_i-x|} + |R_{x_i} - P_{x_i}|_{x_i,|x_i-x|} \\ &\lesssim \|f, P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})} + |R_{x_i} - P_{x_i}|_{x_i,|x_i-x|} + \eta. \end{aligned} \tag{9.14}$$

By part 2 of Lemma 9.1, since $x \in K_p$, $|x_i - x_s| = |x_i - \kappa(x_i)| \lesssim |x - x_i|$, and by part 4 of Lemma 9.1, $\delta_{Q_s} \lesssim \delta_{Q_i} \leq |x - x_i|$, so we can apply (2.2) and (9.11) to deduce

$$\begin{aligned} |R_{x_i} - P_{x_i}|_{x_i,|x_i-x|} &= |R'_{x_s} - P_{x_i}|_{x_i,|x_i-x|} \\ &\lesssim |R'_{x_s} - P_{x_s}|_{x_s,|x_i-x|} + |P_{x_s} - P_{x_i}|_{x_s,|x_i-x_s|} \\ &\lesssim |R'_{x_s} - P_{x_s}|_{x_s,\delta_{Q_s}} + \|f, P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})} + \eta. \end{aligned} \tag{9.15}$$

Because $R'_{x_s}, P_{x_s} \in \mathcal{P}$ are each coherent with $P_0$, and $\mathcal{A}$ is monotonic, we have $\partial^\alpha(R'_{x_s} - P_{x_s}) \equiv 0$ for all $\alpha \in \mathcal{A}$. We apply (2.2), (5.12), and the triangle inequality to bound

$$\begin{aligned} |R'_{x_s} - P_{x_s}|_{x_s,\delta_{Q_s}} &\lesssim \|0, R'_{x_s} - P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} \\ &\lesssim \|f, R'_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} + \|f, P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s});\delta_{Q_s})} \\ &\lesssim \|f, P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})}, \end{aligned} \tag{9.16}$$

where the last inequality follows by the defining properties of $R'_{x_s}$ (see Lemma 7.3), since $P_{x_s}$ is coherent with $P_0$. We apply (2.26) (for the $C$-non-degenerate rectangular box $9Q_s \cap Q^\circ$) and (2.21) (for $R_1 = Q_s$, $R_2 = 9Q_s \cap Q^\circ$), and finally (9.10), to estimate

$$\begin{aligned} \|f, P_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_s},\delta_{Q_s})} &\overset{(2.26)}{\lesssim} \|f, P_{x_s}\|^p_{\mathscr{J}_*(\mu|_{9Q_s};9Q_s \cap Q^\circ)} \\ &\leq \|H, P_{x_s}\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};9Q_s\cap Q^\circ)} \end{aligned}$$

$$
\begin{aligned}
&= \|H\|^p_{L^{m,p}(9Q_s\cap Q^\circ)} + \int\limits_{9Q_s} |H-f|^p d\mu + \|H-P_{x_s}\|^p_{L^p(9Q_s\cap Q^\circ)}/\delta_{Q_s}^{mp} \\
&\overset{(2.21)}{\lesssim} \|H\|^p_{L^{m,p}(Q^\circ)} + \int\limits_{9Q_s} |H-f|^p d\mu + \|H-P_{x_s}\|^p_{L^p(Q_s)}/\delta_{Q_s}^{mp} \\
&\leq \|H,\vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\
&\overset{(9.10)}{\lesssim} (\|f,P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})}+\eta)^p. \qquad (9.17)
\end{aligned}
$$

Combining (9.14)–(9.17), we have for $\kappa(x_i)=x_s\in\mathfrak{B}_{key}$,

$$
|R^*_x - R_{x_i}|_{x_i,|x_i-x|} \lesssim \|f,P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})} + 3\eta.
$$

By letting $\eta\to 0$, we complete the proof of the second line of (9.12).

Case 3: Let $i,j\in I$ be distinct. Then $x_i\in Q_i\in CZ^\circ$ and $x_j\in Q_j\in CZ^\circ$. We aim to prove the third line of (9.12): Suppose $\kappa(x_i)=x\in K_p$, so that $R_{x_i}=R^*_x$. By Lemma 9.1, $|x-x_i|\simeq\delta_{Q_i}\lesssim|x_i-x_j|$ and $|x-x_j|\leq|x-x_i|+|x_i-x_j|\lesssim|x_i-x_j|$, so from (2.2),

$$
|R_{x_i}-R_{x_j}|_{x_j,|x_j-x_i|} \lesssim |R^*_x - R_{x_j}|_{x_j,|x-x_j|} \lesssim \|f,P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})},
$$

where the last inequality follows by the analysis of Case 2. This proves the third line of (9.12) if $\kappa(x_i)\in K_p$.

If $\kappa(x_j)\in K_p$, then by (2.2), $|R_{x_i}-R_{x_j}|_{x_j,|x_j-x_i|}\lesssim|R_{x_i}-R_{x_j}|_{x_i,|x_j-x_i|}$, and we repeat the preceding analysis to prove the third line of (9.12) in this case.

Now suppose $\kappa(x_i)=x_s,\kappa(x_j)=x_t\in\mathfrak{B}_{key}$. Then $R_{x_i}=R'_{x_s}$ and $R_{x_j}=R'_{x_t}$. Because $\delta_{Q_s}\lesssim\delta_{Q_i}\simeq|x_i-x_s|\lesssim|x_i-x_j|$ (see Lemma 9.1), we can apply (2.2) and (9.11) to deduce

$$
\begin{aligned}
|R_{x_i}-P_{x_i}|_{x_i,|x_i-x_j|} &= |R'_{x_s}-P_{x_i}|_{x_i,|x_i-x_j|} \\
&\lesssim |R'_{x_s}-P_{x_s}|_{x_s,|x_i-x_j|} + |P_{x_s}-P_{x_i}|_{x_s,|x_i-x_s|} \\
&\lesssim |R'_{x_s}-P_{x_s}|_{x_s,\delta_{Q_s}} + \|f,P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})}+\eta. \qquad (9.18)
\end{aligned}
$$

From (9.16) and (9.17), we have

$$
|R'_{x_s}-P_{x_s}|_{x_s,\delta_{Q_s}} \lesssim \|f,P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})}+\eta. \qquad (9.19)
$$

Combining (9.18), (9.19), we learn that

$$
|R_{x_i}-P_{x_i}|_{x_i,|x_i-x_j|} \lesssim \|f,P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})}+2\eta.
$$

Similarly,

$$|R_{x_j} - P_{x_j}|_{x_j,|x_i-x_j|} \lesssim \|f, P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})} + 2\eta.$$

We apply the triangle inequality and (2.2), and then the preceding estimates and (9.11) to deduce

$$\begin{aligned}|R_{x_i} - R_{x_j}|_{x_i,|x_i-x_j|} &\lesssim |R_{x_i} - P_{x_i}|_{x_i,|x_i-x_j|} + |P_{x_i} - P_{x_j}|_{x_i,|x_i-x_j|} + |P_{x_j} - R_{x_j}|_{x_j,|x_i-x_j|} \\ &\lesssim \|f, P_0\|_{\mathscr{J}(\mu,\delta_{Q^\circ})} + 5\eta.\end{aligned}$$

By letting $\eta \to 0$, we complete the proof of the third line of (9.12) in Case 3, concluding the proof of the lemma. □

**Proposition 9.4.** *For $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ})$, we have $(f, \vec{R}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$, and*

$$\|f, \vec{R}, \vec{R}^*\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \leq C\|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})}.$$

*Furthermore, $(\vec{R}, \vec{R}^*)$ depends linearly on $(f, P_0)$.*

**Proof.** From Lemma 6.7, Lemma 7.3, and the definition of $\vec{R}$ in (9.9), $(\vec{R}, \vec{R}^*)$ depends linearly on $(f, P_0)$.

Because of Corollary 6.9, it suffices to show that for any $\vec{P} \in Wh(\mathfrak{B}_{CZ})$, satisfying $\vec{P}$ is coherent with $P_0$ and $(f, \vec{P}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$, we have

$$\|f, \vec{R}, \vec{R}^*\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \leq C\|f, \vec{P}, \vec{R}^*\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}. \tag{9.20}$$

Let $\vec{P} = \{P_{x_i}\}_{i\in I} \in Wh(\mathfrak{B}_{CZ})$ satisfy $\vec{P}$ is coherent with $P_0$ and $(f, \vec{P}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$. Observe that the Whitney field $(\vec{R}^*, \vec{P}) \in Wh(K_p \cup \mathfrak{B}_{CZ})$ is in the class $C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})$, thanks to Proposition 6.1. Define $\vec{P}^{**} \in Wh(\mathfrak{B}_{CZ})$ by

$$P^{**}_{x_i} := \begin{cases} P_{\kappa(x_i)} & \text{if } \kappa(x_i) \in \mathfrak{B}_{key} \\ R^*_{\kappa(x_i)} & \text{if } \kappa(x_i) \in K_p. \end{cases} \tag{9.21}$$

We will demonstrate that $(f, \vec{P}^{**}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$. To see this, suppose that $F \in L^{m,p}(Q^\circ)$ is arbitrary, satisfying $J_x(F) = R^*_x$ for all $x \in K_p$. Recalling (5.45), and applying (9.3) to the pair of Whitney fields $(\vec{R}^*, \vec{P})$, we have

$$\begin{aligned}&\|F, \vec{P}^{**}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\ &\quad= \|F\|^p_{L^{m,p}(Q^\circ)} + \int_{Q^\circ} |F - f|^p d\mu + \sum_{i\in I} \|F - P^{**}_{x_i}\|^p_{L^p(Q_i)}/\delta^{mp}_{Q_i} \\ &\quad\lesssim \|F\|^p_{L^{m,p}(Q^\circ)} + \int_{Q^\circ} |F - f|^p d\mu + \sum_{i\in I} \left\{\|F - P_{x_i}\|^p_{L^p(Q_i)}/\delta^{mp}_{Q_i} + |P_{x_i} - P^{**}_{x_i}|^p_i\right\}\end{aligned}$$

$$= \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} + \sum_{i\in I} |P_{x_i} - P^{**}_{x_i}|^p_i$$
$$\overset{(9.3)}{\lesssim} \|F, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)}.$$

Here, the first $\lesssim$ uses the triangle inequality as well as the fact that $\|P\|_{L^p(Q_i)}\delta_{Q_i}^{-m} \simeq |P|_i$ for $P \in \mathcal{P}$. By taking the infimum in the above inequality over $F \in L^{m,p}(Q^\circ)$ satisfying $J_x(F) = R^*_x$ for all $x \in K_p$,

$$\|f, \vec{P}^{**}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \lesssim \|f, \vec{P}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} < \infty. \tag{9.22}$$

Thus, $(f, \vec{P}^{**}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$, as claimed.

From Lemma 9.3, we have $(\vec{R}^*, \vec{R}) \in C^{m-1,1-n/p}(K_p \cup \mathfrak{B}_{CZ})$. Therefore, by Lemma 8.1, to demonstrate that $(f, \vec{R}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$ it suffices to prove that

$$\begin{aligned}\mathcal{S}_2(f, \vec{R}, \vec{R}^*) := \sum_{i\in I} \|f, R_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i\leftrightarrow i'} |R_{x_i} - R_{x_{i'}}|^p_i \\ + \|\vec{R}^*\|^p_{L^{m,p}(K_p)} + \int_{K_p} |R^*_x(x) - f(x)|^p d\mu < \infty.\end{aligned} \tag{9.23}$$

By the triangle inequality, (5.11), (2.2), (8.5), (8.3), and (9.22),

$$\begin{aligned}
&\sum_{i\in I} \|f, R_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i\leftrightarrow i'} |R_{x_i} - R_{x_{i'}}|^p_i \\
&\quad\lesssim \sum_{i\in I} \left\{ \|f, P^{**}_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \|0, P^{**}_{x_i} - R_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} \right\} \\
&\qquad\qquad + \sum_{i\leftrightarrow i'} \left\{ |R_{x_i} - P^{**}_{x_i}|^p_i + |P^{**}_{x_i} - P^{**}_{x_{i'}}|^p_i + |R_{x_{i'}} - P^{**}_{x_{i'}}|^p_i \right\} \\
&\quad\overset{(5.11),(2.2)}{\lesssim} \sum_{i\in I} \left\{ \|f, P^{**}_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + |P^{**}_{x_i} - R_{x_i}|^p_i \right\} \\
&\qquad\qquad + \sum_{i\leftrightarrow i'} \left\{ |R_{x_i} - P^{**}_{x_i}|^p_i + |P^{**}_{x_i} - P^{**}_{x_{i'}}|^p_i + |R_{x_{i'}} - P^{**}_{x_{i'}}|^p_{i'} \right\} \\
&\quad\lesssim \sum_{i\in I} \|f, P^{**}_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i\leftrightarrow i'} |P^{**}_{x_i} - P^{**}_{x_{i'}}|^p_i + \sum_{i\in I} |P^{**}_{x_i} - R_{x_i}|^p_i \\
&\quad\overset{(8.5),(8.3)}{\lesssim} \|f, \vec{P}^{**}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i \\
&\quad\overset{(9.22)}{\lesssim} \|f, \vec{P}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i.
\end{aligned} \tag{9.24}$$

We wish to bound the sum $\sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i$. If $\kappa(x_i) \in K_p$, then $R_{x_i} = R^*_{\kappa(x_i)} = P^{**}_{x_i}$, by (9.9) and (9.21), and so the summand vanishes. Else, suppose $\kappa(x_i) \in \mathfrak{B}_{key}$. Then $R_{x_i} = R'_{\kappa(x_i)}$ and $P^{**}_{x_i} = P_{\kappa(x_i)}$, by (9.9) and (9.21). Hence,

$$\sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i = \sum_{i\in I:\kappa(x_i)\in\mathfrak{B}_{key}} |R'_{\kappa(x_i)} - P_{\kappa(x_i)}|^p_i. \tag{9.25}$$

For $\kappa(x_i) \in \mathfrak{B}_{key}$, we have $|x_i - \kappa(x_i)| \le C\delta_{Q_i}$ (see Lemma 9.1), so, by recalling the definition of the $|\cdot|_i$ polynomial norm, and by (2.2),

$$\begin{aligned}
&\sum_{i\in I:\kappa(x_i)\in\mathfrak{B}_{key}} |R'_{\kappa(x_i)} - P_{\kappa(x_i)}|^p_{x_i,\delta_{Q_i}} \\
&\le \sum_{i\in I:\kappa(x_i)\in\mathfrak{B}_{key}} |R'_{\kappa(x_i)} - P_{\kappa(x_i)}|^p_{\kappa(x_i),\delta_{Q_i}} \\
&= \sum_{s\in\bar{I}} \sum_{x_i\in\kappa^{-1}(x_s)} |P_{x_s} - R'_{x_s}|^p_{x_s,\delta_{Q_i}} \\
&= \sum_{s\in\bar{I}} \sum_{x_i\in\kappa^{-1}(x_s)} \sum_{|\beta|\le m-1} |\partial^\beta(P_{x_s} - R'_{x_s})(x_s)|^p \cdot \delta_{Q_i}^{|\beta|p+n-mp}.
\end{aligned}$$

By Lemma 9.1, for any $s \in \bar{I}$ and $x_i \in \kappa^{-1}(x_s)$, we have $\delta_{Q_i} \ge c\delta_{Q_s}$; furthermore, for any dyadic lengthscale $\delta > 0$, and fixed $s$,

$$|\{i \in I : x_i \in \kappa^{-1}(x_s), \delta_{Q_i} = \delta\}| \le C.$$

Thus, in combination with the inequality, $|\beta|p + n - mp < 0$ for all $\beta \in \mathcal{M}$, we have

$$\sum_{x_i\in\kappa^{-1}(x_s)} \delta_{Q_i}^{|\beta|p+n-mp} \lesssim \delta_{Q_s}^{|\beta|p+n-mp}.$$

So, using the previous three equation lines, we reduce (9.25),

$$\sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i \lesssim \sum_{s\in\bar{I}} \sum_{|\beta|\le m-1} |\partial^\beta(P_{x_s} - R'_{x_s})(x_s)|^p \delta_{Q_s}^{|\beta|p+n-mp} = \sum_{s\in\bar{I}} |P_{x_s} - R'_{x_s}|^p_s. \tag{9.26}$$

Because $\vec{P}$ and $\vec{R}$ in $Wh(\mathfrak{B}_{CZ})$ are both coherent with $P_0$, we can manipulate the right-hand side of the inequality (9.26) using (5.12), (2.10), (7.2), and (2.26), to obtain

$$\begin{aligned}
\sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i &\overset{(5.12)}{\lesssim} \sum_{s\in\bar{I}} \|0, P_{x_s} - R'_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_s},\delta_{Q_s})} \\
&\overset{(2.10)}{\lesssim} \sum_{s\in\bar{I}} \|f, P_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_s},\delta_{Q_s})} + \|f, R'_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_s},\delta_{Q_s})}
\end{aligned}$$

$$\overset{(7.2)}{\lesssim} \sum_{s\in\bar{I}} \|f, P_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_s},\delta_{Q_s})}$$

$$\overset{(2.26)}{\lesssim} \sum_{s\in\bar{I}} \|f, P_{x_s}\|^p_{\mathscr{J}_*(\mu|_{9Q_s};10Q_s\cap Q^\circ)}. \tag{9.27}$$

Consider an arbitrary $H \in L^{m,p}(Q^\circ)$ satisfying $J_x H = R^*_x$ for all $x \in K_p$. Recall from Lemma 4.7, for any keystone cube $Q_s$, $\left|\{s' \in \bar{I} : 10Q_s \cap 10Q_{s'} \neq \emptyset\}\right| \leq C$. So from (9.27), we have

$$\begin{aligned}\sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i &\lesssim \sum_{s\in\bar{I}} \|H, P_{x_s}\|^p_{\mathscr{J}_*(f,\mu|_{10Q_s};10Q_s\cap Q^\circ)} \\ &\lesssim \|H\|^p_{L^{m,p}(Q^\circ)} + \int |H-f|^p d\mu + \sum_{s\in\bar{I}} \|H - P_{x_s}\|^p_{L^p(10Q_s\cap Q^\circ)}/\delta^{mp}_{Q_s}.\end{aligned} \tag{9.28}$$

Now apply (2.21) (with $R_1 = Q_s$ and $R_2 = 10Q_s \cap Q^\circ$), to obtain

$$\|H - P_{x_s}\|^p_{L^p(10Q_s\cap Q^\circ)}/\delta^{mp}_{Q_s} \lesssim \|H - P_{x_s}\|^p_{L^p(Q_s)}/\delta^{mp}_{Q_s} + \|H\|^p_{L^{m,p}(10Q_s\cap Q^\circ)}.$$

Then summing on $s$, and using the bounded overlap condition on $\{10Q_s : s \in \bar{I}\}$ again, we obtain

$$\sum_{s\in\bar{I}} \|H - P_{x_s}\|^p_{L^p(10Q_s\cap Q^\circ)}/\delta^{mp}_{Q_s} \lesssim \sum_{s\in\bar{I}} \|H - P_{x_s}\|^p_{L^p(Q_s)}/\delta^{mp}_{Q_s} + \|H\|^p_{L^{m,p}(Q^\circ)}$$

Using this in (9.28),

$$\begin{aligned}\sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i &\lesssim \|H\|^p_{L^{m,p}(Q^\circ)} + \int |H-f|^p d\mu + \sum_{i\in I} \|H - P_{x_i}\|^p_{L^p(Q_i)}/\delta^{mp}_{Q_i} \\ &= \|H, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)}.\end{aligned}$$

Taking the infimum with respect to $H \in L^{m,p}(Q^\circ)$ satisfying $J_x H = R^*_x$ for all $x \in K_p$, we have

$$\sum_{i\in I} |R_{x_i} - P^{**}_{x_i}|^p_i \lesssim \|f, \vec{P}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}. \tag{9.29}$$

Furthermore, for any $H \in L^{m,p}(Q^\circ)$ satisfying $J_x H = R^*_x$ for all $x \in K_p$, we have

$$\begin{aligned}\|\vec{R}^*\|^p_{L^{m,p}(K_p)} + \int_{K_p} |R^*_x(x) - f(x)|^p d\mu &\lesssim \|H\|^p_{L^{m,p}(Q^\circ)} + \int_{K_p} |H-f|^p d\mu \\ &\leq \|H, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)}.\end{aligned}$$

For the first inequality, we have used Lemma 2.7. The second inequality follows by definition of the $\mathscr{J}_*(f, \mu; Q^\circ, CZ^\circ)$ functional. Now, taking the infimum over all $H$ in the previous inequality,

$$\|\vec{R}^*\|^p_{L^{m,p}(K_p)} + \int_{K_p} |R^*_x(x) - f(x)|^p d\mu \lesssim \|f, \vec{P}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}. \tag{9.30}$$

From (9.24), (9.29), (9.30), we conclude that $\mathcal{S}_2(f, \vec{R}, \vec{R}^*)$ defined in (9.23) satisfies

$$\mathcal{S}_2(f, \vec{R}, \vec{R}^*) \lesssim \|f, \vec{P}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} < \infty.$$

Thus, we have proven (9.23), as desired.

We conclude by Lemma 8.1 that $(f, \vec{R}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$ and

$$\|f, \vec{R}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \simeq \mathcal{S}_1(f, \vec{R}, \vec{R}^*) \leq \mathcal{S}_2(f, \vec{R}, \vec{R}^*) \lesssim \|f, \vec{P}, \vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}.$$

This completes the proof of (9.20). This completes the proof of the proposition. □

## 10. Proof of the Main Lemma for $\mathcal{A}$

The next lemmas tell us that the Main Lemma for $\mathcal{A}$ is true. That is, we shall establish the Extension Theorem for $(\mu, \delta)$. Recall, we have rescaled and translated $\mu$ and $\delta$, so that $\mathrm{diam}(\mathrm{supp}(\mu)) < \delta = 1/10$ (see (3.50)).

**Lemma 10.1.** *There exist a linear map $T : \mathscr{J}(\mu; 1/10) \to L^{m,p}(\mathbb{R}^n)$, a map $M : \mathscr{J}(\mu; 1/10) \to \mathbb{R}_+$, $K \subset \mathrm{Cl}(\mathrm{supp}(\mu))$, a linear map $\vec{S} : \mathscr{J}(\mu) \to Wh(K)$, and a countable collection of linear maps $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$, $\lambda_\ell : \mathscr{J}(\mu; 1/10) \to L^p(d\mu)$, that satisfy for each $(f, P_0) \in \mathscr{J}(\mu; 1/10)$, (3.1), (3.2), and (3.3) hold with $\delta = 1/10$. Furthermore, if $\mathcal{A} \neq \emptyset$ then $K = \emptyset$, and so the map $M$ satisfies (3.7).*

**Proof.** We recall from Proposition 9.4, there exists $(\vec{R}, \vec{R}^*)$ depending linearly on $(f, P_0)$, so that $(f, \vec{R}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$, and

$$\|f, \vec{R}, \vec{R}^*\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \leq C\|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})}. \tag{10.1}$$

Because $(f, \vec{R}, \vec{R}^*) \in \mathscr{J}_*(\mu; Q^\circ, CZ^\circ; K_p)$, we can apply Lemma 8.1 to produce $T(f, \vec{R}, \vec{R}^*) \in L^{m,p}(Q^\circ)$ satisfying $J_x T(f, \vec{R}, \vec{R}^*) = R_x$ for $x \in K_p$, and

$$\|T(f, \vec{R}, \vec{R}^*), \vec{R}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \leq C\|f, \vec{R}, \vec{R}^*\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}. \tag{10.2}$$

Define $T : \mathscr{J}(\mu; \delta) \to L^{m,p}(\mathbb{R}^n)$ as $T(f, P_0) := \theta \cdot T(f, \vec{R}, \vec{R}^*) + (1 - \theta)P_0$, where $\theta$ is a smooth cutoff function satisfying $\mathrm{supp}(\theta) \subset Q^\circ$, $\theta|_{0.99Q^\circ} = 1$, $|\theta(x)| \leq 1$, and

$|\partial^\alpha \theta(x)| \le C$ for $\alpha \in \mathcal{M}$. Then we apply (2.19) and the definition of the $\mathscr{J}_*(f,\mu;Q^\circ)$ and $\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)$ functionals to estimate

$$\begin{aligned}\|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} &\lesssim \|T(f,\vec{R},\vec{R}^*),P_0\|_{\mathscr{J}_*(f,\mu;Q^\circ)}\\ &\lesssim \|T(f,\vec{R},\vec{R}^*),\vec{R}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} + \|T(f,\vec{R},\vec{R}^*)-P_0\|_{L^p(Q^\circ)}.\end{aligned} \tag{10.3}$$

Let $Q_1 \in CZ^\circ$ satisfy $\delta_{Q_1} \ge c$ (such a cube exists by (4.3) – recall $\delta_{Q^\circ}=1$). Then, using (2.20) and (2.21) with $R_1 = Q_1$, $R_2 = Q^\circ$,

$$\begin{aligned}&\|T(f,\vec{R},\vec{R}^*)-P_0\|_{L^p(Q^\circ)}\\ &\quad\le \|T(f,\vec{R},\vec{R}^*)-R_{x_1}\|_{L^p(Q^\circ)} + \|R_{x_1}-P_0\|_{L^p(Q^\circ)}\\ &\quad\lesssim \|T(f,\vec{R},\vec{R}^*)\|_{L^{m,p}(Q^\circ)} + \|T(f,\vec{R},\vec{R}^*)-R_{x_1}\|_{L^p(Q_1)} + \|R_{x_1}-P_0\|_{L^p(Q_1)}\\ &\quad\lesssim \|T(f,\vec{R},\vec{R}^*)\|_{L^{m,p}(Q^\circ)} + \|T(f,\vec{R},\vec{R}^*)-R_{x_1}\|_{L^p(Q_1)}/\delta_{Q_1}^m + \|R_{x_1}-P_0\|_{L^p(Q_1)}\\ &\quad\lesssim \|T(f,\vec{R},\vec{R}^*),\vec{R}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} + \|R_{x_1}-P_0\|_{L^p(Q_1)}.\end{aligned} \tag{10.4}$$

Combining (10.1), (10.2), (10.3), and (10.4), we have

$$\begin{aligned}\|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} &\lesssim \|f,\vec{R},\vec{R}^*\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \|R_{x_1}-P_0\|_{L^p(Q_1)}\\ &\lesssim \|f,P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \|R_{x_1}-P_0\|_{L^p(Q_1)}.\end{aligned} \tag{10.5}$$

From Corollary 6.9, for $\eta > 0$ there exist $H \in L^{m,p}(Q^\circ)$ and $\vec{P} \in Wh(\mathfrak{B}_{CZ})$ satisfying $J_x H = R_x^*$ for all $x \in K_p$, $\vec{P}$ is coherent with $P_0$, and

$$\|H,\vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \le \|f,\vec{P},\vec{R}^*\|_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \eta/2 \lesssim \|f,P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta. \tag{10.6}$$

Consequently, since $\vec{R}^*$ is coherent with $P_0$, also $H$ is $K_p$-coherent with $P_0$.

We now work to estimate the term $\|R_{x_1}-P_0\|_{L^p(Q_1)}$ in (10.5).

*Case I:* First suppose that $\kappa(x_1) = x \in K_p$. Recalling the definition of $\vec{R} = (R_{x_i})_{i\in I}$ in (9.9), we have $R_{x_1} = R_x^* = J_x H$. Due to the fact that $\delta_{Q_1} \simeq 1$, we have $\|P\|_{L^p(Q_1)} \simeq |P|_{x_1,\delta_{Q_1}}$ for any polynomial $P \in \mathcal{P}$. Also, because $\vec{R}^*$ is coherent with $P_0$, and $\mathcal{A}$ is monotonic, we have $\partial^\alpha(R_x^* - P_0) \equiv 0$ for all $\alpha \in \mathcal{A}$. By these remarks, (5.12), and the triangle inequality, we have

$$\begin{aligned}\|R_{x_1}-P_0\|^p_{L^p(Q_1)} = \|R_x^*-P_0\|^p_{L^p(Q_1)} \simeq |R_x^*-P_0|^p_{x_1,\delta_{Q_1}} &\overset{(5.12)}{\simeq} \|0,R_x^*-P_0\|^p_{\mathscr{J}(\mu|_{9Q_1};\delta_{Q_1})}\\ &\le \|f,R_x^*\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \|f,P_0\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})}.\end{aligned}$$

Continuing by using (2.22), (2.12), and (10.6), we obtain

$$
\begin{aligned}
\|f, R_x^*\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})} &\overset{(2.22)}{\lesssim} \|f, R_x^*\|^p_{\mathscr{J}_*(\mu;Q^\circ)} \\
&\le \|H, J_x H\|^p_{\mathscr{J}_*(f,\mu;Q^\circ)} \\
&= \|H\|^p_{L^{m,p}(Q^\circ)} + \|H - f\|^p_{L^p(d\mu)} + \|H - J_x H\|^p_{L^p(Q^\circ)} \\
&\overset{(2.12)}{\lesssim} \|H\|^p_{L^{m,p}(Q^\circ)} + \|H - f\|^p_{L^p(d\mu)} \\
&\le \|H, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\
&\overset{(10.6)}{\lesssim} (\|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta)^p.
\end{aligned}
$$

Combining the previous two lines, and letting $\eta \to 0$, we have

$$\|R_{x_1} - P_0\|_{L^p(Q_1)} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} \qquad (\text{if } \kappa(x_1) \in K_p). \tag{10.7}$$

*Case II:* Next suppose that $\kappa(x_1) = x_s \in \mathfrak{B}_{key}$. Recalling the definition of $\vec{R} = (R_{x_i})_{i\in I}$ in (9.9), we have $R_{x_1} = R'_{x_s}$, where $Q_s$ is a keystone cube, and $\delta_{Q_s} \lesssim \delta_{Q_1} \simeq 1$. We have

$$\|R_{x_1} - P_0\|_{L^p(Q_1)} = \|R'_{x_s} - P_0\|_{L^p(Q_1)} \le \|R'_{x_s} - P_{x_s}\|_{L^p(Q_1)} + \|P_{x_s} - P_0\|_{L^p(Q_1)}. \tag{10.8}$$

In what follows, we bound the two terms on the right-hand side of (10.8).

We first look to bound the term $\|P_{x_s} - P_0\|_{L^p(Q_1)}$ on the right-hand side of (10.8). Because $\vec{P}$ is coherent with $P_0$, in particular, $P_{x_s}$ is coherent with $P_0$. Hence, because $\mathcal{A}$ is monotonic, $\partial^\alpha(P_{x_s} - P_0) \equiv 0$ for all $\alpha \in \mathcal{A}$. We apply (5.12), the triangle inequality, and $\delta_{Q_1} \simeq \delta_{Q^\circ} = 1$ to deduce

$$
\begin{aligned}
\|P_{x_s} - P_0\|_{L^p(Q_1)} &\lesssim |P_{x_s} - P_0|_{x_1,\delta_{Q_1}} \lesssim \|0, P_{x_s} - P_0\|_{\mathscr{J}(\mu|_{9Q_1};\delta_{Q_1})} \\
&\le \|f, P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_1};\delta_{Q_1})} + \|f, P_0\|_{\mathscr{J}(\mu|_{9Q_1};\delta_{Q_1})} \\
&\lesssim \|f, P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_1};\delta_{Q_1})} + \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})}.
\end{aligned} \tag{10.9}
$$

Applying (2.26) (note that $9Q_1 \cap Q^\circ$ is $C$-non-degenerate),

$$
\begin{aligned}
\|f, P_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_1};\delta_{Q_1})} &\lesssim \|f, P_{x_s}\|^p_{\mathscr{J}_*(\mu;9Q_1\cap Q^\circ)} \\
&\le \|H, P_{x_s}\|^p_{\mathscr{J}_*(f,\mu;9Q_1\cap Q^\circ)} \\
&= \|H\|^p_{L^{m,p}(9Q_1\cap Q^\circ)} + \int_{9Q_1} |H - f|^p d\mu + \|H - P_{x_s}\|^p_{L^p(9Q_1\cap Q^\circ)}/\delta_{Q_1}^{mp}.
\end{aligned} \tag{10.10}
$$

By applying (2.21), with $R_1 = Q_s$ and $R_2 = 9Q_1 \cap Q^\circ$, we have

$$\|H - P_{x_s}\|_{L^p(9Q_1 \cap Q^\circ)}/\delta_{Q_1}^m \lesssim \|H\|_{L^{m,p}(9Q_1 \cap Q^\circ)} + \|H - P_{x_s}\|_{L^p(Q_s)}/\delta_{Q_s}^m. \tag{10.11}$$

Substituting (10.11) into (10.10), and using (10.6), we see

$$\begin{aligned}\|f, P_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_1};\delta_{Q_1})} &\lesssim \|H\|^p_{L^{m,p}(Q^\circ)} + \|H - P_{x_s}\|^p_{L^p(Q_s)}/\delta_{Q_s}^{mp} + \int_{9Q_1} |H - f|^p d\mu \\ &\lesssim \|H, \vec{P}\|^p_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \overset{(10.6)}{\lesssim} (\|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta)^p.\end{aligned}$$

Substituting the previous equation into (10.9), we have

$$\|P_{x_s} - P_0\|_{L^p(Q_1)} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta. \tag{10.12}$$

We next look to bound the term $\|R'_{x_s} - P_{x_s}\|_{L^p(Q_1)}$ on the right-hand side of (10.8). Because $R'_{x_s}$ and $P_{x_s}$ are each coherent with $P_0$, and because $\mathcal{A}$ is monotonic, $\partial^\alpha(R'_{x_s} - P_{x_s}) \equiv 0$ for all $\alpha \in \mathcal{A}$. Using $\delta_{Q_s} \lesssim \delta_{Q_1} \simeq 1$, $|x_s - x_1| \lesssim \delta_{Q_1}$, (2.2) and (2.3), (5.12), and the triangle inequality, we deduce

$$\begin{aligned}\|R'_{x_s} - P_{x_s}\|_{L^p(Q_1)} \simeq |R'_{x_s} - P_{x_s}|_{x_1,\delta_{Q_1}} &\lesssim |R'_{x_s} - P_{x_s}|_{x_s,\delta_{Q_s}} \\ &\lesssim \|0, R'_{x_s} - P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} \\ &\leq \|f, R'_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} + \|f, P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})}\end{aligned} \tag{10.13}$$

Recalling our choice of $R'_{x_s}$ in Lemma 7.3 (because $P_{x_s}$ is assumed to be coherent with $P_0$), we must have

$$\|f, R'_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} \lesssim \|f, P_{x_s}\|_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})}. \tag{10.14}$$

Applying (2.26) (note: $10Q_s \cap Q^\circ$ is $C$-non-degenerate), (2.21) with $R_1 = Q_s$ and $R_2 = 10Q_s \cap Q^\circ$, and (10.6), we have

$$\begin{aligned}\|f, P_{x_s}\|^p_{\mathscr{J}(\mu|_{9Q_s};\delta_{Q_s})} &\overset{(2.26)}{\lesssim} \|f, P_{x_s}\|^p_{\mathscr{J}_*(\mu|_{9Q_s};10Q_s\cap Q^\circ)} \leq \|H, P_{x_s}\|^p_{\mathscr{J}_*(f,\mu|_{9Q_s};10Q_s\cap Q^\circ)} \\ &\lesssim \|H\|^p_{L^{m,p}(10Q_s\cap Q^\circ)} + \int_{10Q_s} |H - f|^p d\mu + \|H - P_{x_s}\|^p_{L^p(10Q_s\cap Q^\circ)}/\delta_{Q_s}^{m,p} \\ &\overset{(2.21)}{\lesssim} \|H\|^p_{L^{m,p}(10Q_s\cap Q^\circ)} + \int_{10Q_s} |H - f|^p d\mu + \|H - P_{x_s}\|^p_{L^p(Q_s)}/\delta_{Q_s}^{m,p} \\ &\leq \|H, \vec{P}\|_{\mathscr{J}_*(f,\mu;Q^\circ,CZ^\circ)} \\ &\overset{(10.6)}{\lesssim} \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} + \eta.\end{aligned} \tag{10.15}$$

Substituting (10.12), (10.13), (10.14), and (10.15) into (10.8), and letting $\eta \to 0$ in (10.6), we have

$$\|R_{x_1} - P_0\|_{L^p(Q_1)} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} \qquad (\text{if } \kappa(x_1) \in \mathfrak{B}_{key}). \tag{10.16}$$

That completes the analysis of Cases I and II.

Combining (10.7) and (10.16), since either $\kappa(x_1) \in \mathfrak{B}_{key}$ or $\kappa(x_1) \in K_p$, we have shown

$$\|R_{x_1} - P_0\|_{L^p(Q_1)} \lesssim \|f, P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})}. \tag{10.17}$$

Using this in (10.5), we conclude that

$$\begin{aligned}\|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;1/10)} &\simeq \|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \lesssim \|f,P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} \\ &\simeq \|f,P_0\|_{\mathscr{J}(\mu;1/10)},\end{aligned} \tag{10.18}$$

where we have used that $\delta_{Q^\circ} = 1 \simeq 1/10$. Thus, we have proven the upper bound $\|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;\delta)} \leq C \cdot \|f,P_0\|_{\mathscr{J}(\mu;\delta)}$ in (3.1), for $\delta = 1/10$. The matching lower bound is an immediate consequence of the definition of the $\mathscr{J}(\cdots)$ functional.

We now prepare to approximate the quantity $\|f,P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})}$. From (10.1) and (10.17) we have

$$\|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \|R_{x_1} - P_0\|^p_{L^p(Q_1)} \lesssim \|f,P_0\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})}.$$

Furthermore, from (10.5) we have

$$\begin{aligned}\|f,P_0\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})} &\leq \|T(f,P_0),P_0\|^p_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \\ &\lesssim \|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + \|R_{x_1} - P_0\|^p_{L^p(Q_1)}.\end{aligned}$$

Combining the above two inequalities, and using $\|R_{x_1} - P_0\|_{L^p(Q_1)} \simeq |R_{x_1} - P_0|_{x_1,\delta_{Q_1}}$, we have

$$\|f,P_0\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})} \simeq \|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + |R_{x_1} - P_0|^p_{x_1,\delta_{Q_1}}. \tag{10.19}$$

In Lemma 8.1 we constructed an equivalent expression $\mathcal{S}_1(f,\vec{R},\vec{R}^*)$ for $\|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}$. Namely, we showed that $\|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} \simeq \mathcal{S}_1(f,\vec{R},\vec{R}^*)$, with

$$\begin{aligned}\mathcal{S}_1(f,\vec{R},\vec{R}^*) = \sum_{i\in I} \|f,R_{x_i}\|^p_{\mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})} + \sum_{i\leftrightarrow i'} |R_{x_i} - R_{x_{i'}}|^p_i \\ + \int_{K_p} |R^*_x(x) - f(x)|^p d\mu + 1_{\mathcal{A}=\emptyset}\|\vec{R}^*\|^p_{L^{m,p}(K_p)}.\end{aligned}$$

The indicator term $1_{\mathcal{A}=\emptyset}\|\vec{R}^*\|^p_{L^{m,p}(K_p)}$ is included in the expression $\mathcal{S}_1(f,\vec{R},\vec{R}^*)$ if and only if $\mathcal{A}=\emptyset$.

Next we apply (8.14) and (8.15) to replace the $\Sigma_{i\in I}$ sum in $\mathcal{S}_1$ by an equivalent expression, involving the countable collections of Borel sets $\{A^i_\ell\}_{\ell\in\mathbb{N}}, A^i_\ell\subset \operatorname{supp}(\mu|_{1.1Q_i})$, and of linear maps $\{\phi^i_\ell : \mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})\to\mathbb{R}\}_{\ell\in\mathbb{N}}$, and $\{\lambda^i_\ell : \mathscr{J}(\mu|_{1.1Q_i};\delta_{Q_i})\to L^p(d\mu)\}_{\ell\in\mathbb{N}}$ (see (8.14) and (8.15)). We also use the definition of the $|\cdot|_i$ polynomial norm to show that

$$\begin{aligned}
&\|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}\simeq \widehat{\mathcal{S}}_1(f,\vec{R},\vec{R}^*),\\
&\widehat{\mathcal{S}}_1(f,\vec{R},\vec{R}^*):=\sum_{i\in I}\sum_{\ell\in\mathbb{N}}\Big(\int_{A^i_\ell}|\lambda^i_\ell(f,R_{x_i})-f|^p d\mu+|\phi^i_\ell(f,R_{x_i})|^p\Big)\\
&\qquad+\sum_{i\leftrightarrow i'}\sum_{\alpha\in\mathcal{M}} c_{\alpha,i,i'}|\partial^\alpha(R_{x_i}-R_{x_{i'}})(x_i)|^p\\
&\qquad+\int_{K_p}|R^*_x(x)-f(x)|^p d\mu+1_{\mathcal{A}=\emptyset}\cdot\|\vec{R}^*\|^p_{L^{m,p}(K_p)},
\end{aligned}\tag{10.20}$$

for some constants $c_{\alpha,i,i'}\ge 0$.

Recall that the Whitney fields $\vec{R}=(R_{x_i})$ and $\vec{R}^*=(R^*_x)$ depend linearly on $(f,P_0)$. Thus, for $i\leftrightarrow i'$, we can define the linear functional $\phi^{i,i',\alpha}:\mathscr{J}(\mu;\delta_{Q^\circ})\to\mathbb{R}$ by

$$\phi^{i,i',\alpha}(f,P_0):=c^{1/p}_{\alpha,i,i'}\cdot\partial^\alpha(R_{x_i}-R_{x_{i'}})(x_i).\tag{10.21}$$

Also, define the map $\lambda':\mathscr{J}(\mu;\delta_{Q^\circ})\to L^p(d\mu)$ by

$$\lambda'(f,P_0)(x)=\begin{cases}R^*_x(x) & x\in K_p\\ 0 & x\in\mathbb{R}^n\setminus K_p.\end{cases}\tag{10.22}$$

We have

$$\int_{K_p}|R^*_x(x)-f(x)|^p d\mu=\int_{K_p}|\lambda'(f,P_0)-f|^p d\mu.$$

Making substitutions of these maps in equation (10.20), we have

$$\begin{aligned}
&\|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)}\\
&\quad\simeq 1_{\mathcal{A}=\emptyset}\cdot\|\vec{R}^*\|^p_{L^{m,p}(K_p)}+\sum_{i\in I}\sum_{\ell\in\mathbb{N}}\Big(\int_{A^i_\ell}|\lambda^i_\ell(f,R_{x_i})-f|^p d\mu+|\phi^i_\ell(f,R_{x_i})|^p\Big)\\
&\qquad+\sum_{i\leftrightarrow i'}\sum_{\alpha\in\mathcal{M}}|\phi^{i,i',\alpha}(f,P_0)|^p+\int_{K_p}|\lambda'(f,P_0)-f|^p d\mu.
\end{aligned}\tag{10.23}$$

Returning to (10.19), we write

$$|R_{x_1} - P_0|^p_{x_1,\delta_{Q_1}} = \sum_{\alpha\in\mathcal{M}} |\zeta_\alpha(f,P_0)|^p, \tag{10.24}$$

for linear functionals $\zeta_\alpha : \mathscr{J}(\mu;\delta_{Q^\circ}) \to \mathbb{R}$ of the form

$$\zeta_\alpha(f,P_0) := c_\alpha(\partial^\alpha(R_{x_1} - P_0)(x_1)) \qquad (\alpha \in \mathcal{M}). \tag{10.25}$$

Combining (10.19), (10.23), and (10.24), and reindexing the sums,

$$\begin{aligned}
\|f,P_0\|^p_{\mathscr{J}(\mu;\delta_{Q^\circ})} &\simeq \|f,\vec{R},\vec{R}^*\|^p_{\mathscr{J}_*(\mu;Q^\circ,CZ^\circ;K_p)} + |R_{x_1} - P_0|^p_{x_1,\delta_{Q_1}} \\
&\simeq 1_{\mathcal{A}=\emptyset}\cdot\|\vec{R}^*\|^p_{L^{m,p}(K_p)} + \sum_{i\in I}\sum_{\ell\in\mathbb{N}}\Big(\int_{A^i_\ell}|\lambda^i_\ell(f,R_{x_i}) - f|^p d\mu + |\phi^i_\ell(f,R_{x_i})|^p\Big) \\
&\quad + \sum_{i\leftrightarrow i'}\sum_{\alpha\in\mathcal{M}}|\phi^{i,i',\alpha}(f,P_0)|^p + \int_{K_p}|\lambda'(f,P_0) - f|^p d\mu + \sum_{\alpha\in\mathcal{M}}|\zeta_\alpha(f,P_0)|^p \\
&= 1_{\mathcal{A}=\emptyset}\cdot\|\vec{R}^*\|^p_{L^{m,p}(K_p)} + \sum_{\ell\in\mathbb{N}}\int_{A_\ell}|\lambda_\ell(f,P_0) - f|^p d\mu + \sum_{\ell\in\mathbb{N}}|\phi_\ell(f,P_0)|^p,
\end{aligned} \tag{10.26}$$

for certain Borel sets $A_\ell \subset \mathrm{supp}(\mu)$ and linear maps $\phi_\ell : \mathscr{J}(\mu;\delta_{Q^\circ}) \to \mathbb{R}$, $\lambda_\ell : \mathscr{J}(\mu;\delta_{Q^\circ}) \to L^p(d\mu)$ $(\ell \in \mathbb{N})$.

We define the map $M : \mathscr{J}(\mu;\delta_{Q^\circ}) \to \mathbb{R}_+$ as the $(1/p)^{th}$ power of the right hand side of (10.26). By the above, and the fact that $\|f,P_0\|_{\mathscr{J}(\mu;\delta_{Q^\circ})} \simeq \|f,P_0\|_{\mathscr{J}(\mu;1/10)}$ and $\|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;\delta_{Q^\circ})} \simeq \|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;1/10)}$, we have

$$M(f,P_0) \simeq \|f,P_0\|_{\mathscr{J}(\mu;1/10)} \simeq \|T(f,P_0),P_0\|_{\mathscr{J}(f,\mu;1/10)},$$

establishing (3.2).

Next we show that $M$ has the form (3.3).

First suppose $\mathcal{A} = \emptyset$. Then, by Lemma 6.7, if $H \in L^{m,p}(\mathbb{R}^n)$ and $\|H\|_{\mathscr{J}(f,\mu)} < \infty$, then $\vec{R}^* = (R^*_x)_{x\in K_p} = (J_x H)_{x\in K_p}$ – indeed, any $H$ is $K_p$-coherent with $P_0$, vacuously, if $\mathcal{A} = \emptyset$. This remark shows that $\vec{R}^*(f,P_0) = \vec{R}^*(f)$ depends only on $f$ when $\mathcal{A} = \emptyset$. By definition of $M(f,P_0)$ via (10.26),

$$M(f,P_0) = \left(\|\vec{R}^*(f)\|^p_{L^{m,p}(K_p)} + \sum_{\ell\in\mathbb{N}}\int_{A_\ell}|\lambda_\ell(f,P_0) - f|^p d\mu + \sum_{\ell\in\mathbb{N}}|\phi_\ell(f,P_0)|^p\right)^{1/p}.$$

Therefore, $M(f,P_0)$ has the form (3.3), with $K := K_p$, and with $\vec{S}(f) := \vec{R}^*(f)$ depending linearly on $f$.

If $\mathcal{A} \neq \emptyset$ then $M(f, P_0) = \left(\sum_{\ell \in \mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, P_0) - f|^p d\mu + \sum_{\ell \in \mathbb{N}} |\phi_\ell(f, P_0)|^p\right)^{1/p}$ via (10.26), so $M$ has the form (3.3) with $K = \emptyset$. In this case, the map $M$ has the form in (3.7).

Thus, the maps $T$ and $M$ satisfy (3.1)–(3.3), while $M$ has the form (3.7) provided $\mathcal{A} \neq \emptyset$, as desired. $\square$

In the previous lemma, we established the main properties of $M$ and $T$ in the Extension Theorem for $(\mu, \delta = 1/10)$. Next, we establish properties of the Whitney field $\vec{R}^*(f, P_0) \in Wh(K_p)$. In particular, the next lemma is a tool used in the proof that the map $T$ is $\Omega'$-constructible.

**Lemma 10.2.** *For $x \in K_p$, the linear map $(f, P_0) \in \mathscr{J}(\mu; \delta_{Q^\circ}) \mapsto R^*_x(f, P_0) \in \mathcal{P}$ defined in Lemma 6.7 has the form*

$$R^*_x(f, P_0) = \sum_{\alpha \in \mathcal{M}} \omega^\alpha_x(f) \cdot v_\alpha + \widetilde{\omega}_x(P_0),$$

*where $\widetilde{\omega}_x : \mathcal{P} \to \mathcal{P}$ is a linear map, $\{v_\alpha\}_{\alpha \in \mathcal{M}}$ is a basis for $\mathcal{P}$, and $\omega^\alpha_x : \mathscr{J}(\mu) \to \mathbb{R}$ ($\alpha \in \mathcal{M}$) are linear functionals satisfying* $\mathrm{supp}(\omega^\alpha_x) \subset \{x\}$.

**Proof.** Fix $x \in K_p$. We write $R^*_x(f, P_0) = \lambda_x(f) + \widetilde{\omega}_x(P_0)$ for linear maps $\lambda_x : \mathscr{J}(\mu) \to \mathcal{P}$ and $\widetilde{\omega}_x : \mathcal{P} \to \mathcal{P}$. Fix a basis $\{v_\alpha\}_{\alpha \in \mathcal{M}}$ for $\mathcal{P}$, and write

$$\lambda_x(f) = \sum_{\alpha \in \mathcal{M}} \omega^\alpha_x(f) \cdot v_\alpha$$

for linear functionals $\omega^\alpha_x \in \mathscr{J}(\mu)^*$. To demonstrate that $\mathrm{supp}(\omega^\alpha_x) \subset \{x\}$, we will show for any open neighborhood $U \ni x$ and any $f_1, f_2 \in \mathscr{J}(\mu; \delta_{Q^\circ})$ satisfying

$$f_1|_U = f_2|_U, \tag{10.27}$$

we have $\lambda_{x,\alpha}(f) = \lambda_{x,\alpha}(f)$ ($\alpha \in \mathcal{M}$). Equivalently, it suffices to show that for any $f_1, f_2$ as in (10.27), we have

$$R^*_x(f_1, P_0) = R^*_x(f_2, P_0).$$

Because of (10.27), there exists $\eta > 0$ such that $f_1|_{B(x,\eta)} = f_2|_{B(x,\eta)}$. Set $\mu_{x,\eta} = \mu|_{B(x,\eta)}$. Thus, for any $H \in L^{m,p}(\mathbb{R}^n)$, $\|H\|_{\mathscr{J}(f_1, \mu_{x,\eta})} = \|H\|_{\mathscr{J}(f_2, \mu_{x,\eta})}$. Due to Proposition 6.2, there exists $H \in L^{m,p}(\mathbb{R}^n)$ such that $\|H\|_{\mathscr{J}(f_1, \mu)} < \infty$ and $H$ is $K_p$-coherent with $P_0$. According to the above, $\|H\|_{\mathscr{J}(f_j, \mu_{x,\eta})} < \infty$ for $j = 1, 2$. We apply Lemma 6.8 to deduce that $R^*_x(f_1, P_0) = J_x H = R^*_x(f_2, P_0)$, concluding the proof. $\square$

At the end of the proof of Lemma 10.1, we showed that if $\mathcal{A} = \emptyset$ then $\vec{R}^*(f, P_0) = \vec{R}^*(f)$ is independent of $P_0$. Therefore, we have:

**Corollary 10.3.** *For $\mathcal{A} = \emptyset$, $\vec{R}^*(f, P_0) = \vec{R}^*(f)$, and consequently, for $x \in K_p$,*

$$R^*_x(f) = \sum_{\alpha \in \mathcal{M}} \omega^\alpha_x(f) \cdot v_\alpha,$$

*where $\{v_\alpha\}_{\alpha \in \mathcal{M}}$ is any basis for $\mathcal{P}$, and $\omega^\alpha_x : \mathscr{J}(\mu) \to \mathbb{R}$ ($\alpha \in \mathcal{M}$) are linear functionals satisfying $\operatorname{supp}(\omega^\alpha_x) \subset \{x\}$.*

In the next two lemmas, we verify that the maps $T$ and $M$ constructed in Lemma 10.1 are $\Omega'$-constructible, as claimed in the Extension Theorem for $(\mu, \delta)$, where $\Omega'$ is a set of linear functionals on $\mathscr{J}(\mu)$ whose supports have bounded overlap.

**Lemma 10.4.** *The map $T$ in Lemma 10.1 is $\Omega'$-constructible: There exists a collection of linear functionals $\Omega' = \{\omega_t\}_{t \in \Upsilon} \subset \mathscr{J}(\mu)^*$ satisfying the collection of sets $\{\operatorname{supp}(\omega_t)\}_{t \in \Upsilon}$ has $C$-bounded overlap, and for each $y \in \mathbb{R}^n$, there exists a finite subset $\Upsilon_y \subset \Upsilon$ and a collection of polynomials $\{v_{t,y}\}_{t \in \Upsilon_y} \subset \mathcal{P}$ such that $|\Upsilon_y| \leq C$ and*

$$J_y T(f, P_0) = \sum_{t \in \Upsilon_y} \omega_t(f) \cdot v_{t,y} + \widetilde{\omega}_y(P_0), \tag{10.28}$$

*where $\widetilde{\omega}_y : \mathcal{P} \to \mathcal{P}$ is a linear map.*

**Proof.** Any functional $\hat{\omega} \in \mathscr{J}(\mu; \delta)^*$ admits a unique decomposition $\hat{\omega} = \omega + \widetilde{\omega}$, where $\omega \in \mathscr{J}(\mu)^*$ and $\widetilde{\omega} \in \mathcal{P}^*$. We have defined $T : \mathscr{J}(\mu; \delta) \to L^{m,p}(\mathbb{R}^n)$ as $T(f, P_0) := \theta \cdot T(f, \vec{R}, \vec{R}^*) + (1 - \theta) P_0$, where $\theta$ is a smooth function satisfying $\theta \equiv 1$ on $0.9Q^\circ$ and $\operatorname{supp}(\theta) \subset Q^\circ$. Consequently,

$$J_y T(f, P_0) = \begin{cases} P_0 & y \in (Q^\circ)^c \\ J_y T(f, \vec{R}, \vec{R}^*) \odot_y J_y \theta + (1 - J_y \theta) \odot_y P_0 & y \in Q^\circ. \end{cases} \tag{10.29}$$

Here, $\odot_y$ is the jet product on $\mathcal{P}$ defined by $P \odot_y P' = J_y(PP')$. Recall, from (8.16) in the proof of Lemma 8.1, we defined $T(f, \vec{R}, \vec{R}^*) \in L^{m,p}(Q^\circ)$ as

$$T(f, \vec{R}, \vec{R}^*)(x) = \begin{cases} \sum_{i \in I} T_i(f, R_{x_i})(x) \cdot \theta_i(x) & x \in K_{CZ} \\ R^*_x(x) & x \in K_p \end{cases} \tag{10.30}$$

where $\{\theta_i\}_{i \in I}$ is a partition of unity satisfying **(POU1)-(POU4)** (see Section 4.3), the maps $T_i(f, R_{x_i})$ are defined just above (8.14), and the Whitney fields $\vec{R} = (R_{x_i})_{i \in I}$, $\vec{R}^* = (R^*_y)_{y \in K_p}$ are defined in (9.9) and Lemma 6.7, depending linearly on $(f, P_0)$. By Lemma 8.1,

$$J_y T(f, \vec{R}, \vec{R}^*) = R^*_y \text{ for all } y \in K_p. \tag{10.31}$$

In Lemma 10.2 we describe the form of the linear maps $(f, P_0) \to R^*_y(f, P_0)$ $(y \in K_p)$. There exists a basis $\{v^\gamma\}_{\gamma\in\mathcal{M}}$ for $\mathcal{P}$, a linear map $\widetilde{\omega}_y : \mathcal{P} \to \mathcal{P}$, and linear functionals $\omega^\gamma_y : \mathscr{J}(\mu) \to \mathbb{R}$ satisfying $\mathrm{supp}(\omega^\gamma_y) \subset \{y\}$ $(\gamma \in \mathcal{M})$, such that

$$R^*_y(f, P_0) = \sum_{\gamma\in\mathcal{M}} \omega^\gamma_y(f) \cdot v^\gamma + \widetilde{\omega}_y(P_0) \qquad (y \in K_p). \tag{10.32}$$

In (9.9), we defined $\vec{R} = (R_{x_i})_{i\in I} \in Wh(\mathfrak{B}_{CZ})$ by

$$R_{x_i} := \begin{cases} R'_{x_s} & \kappa(x_i) = x_s \in \mathfrak{B}_{key} \\ R^*_z & \kappa(x_i) = z \in K_p, \end{cases} \tag{10.33}$$

We now discuss the form of the linear maps $(f, P_0) \mapsto R'_{x_s}(f, P_0)$ $(s \in \bar{I})$, defined in Lemma 7.3. Recall that $R'_{x_s} \in \mathcal{P}$ depends linearly on $(f|_{9Q_s}, P_0)$. Thus, given a basis $\{v^\gamma\}_{\gamma\in\mathcal{M}}$ for $\mathcal{P}$, there exist linear functionals $\omega^\gamma_{x_s} : \mathscr{J}(\mu|_{9Q_s}) \to \mathbb{R}$ satisfying

$$\mathrm{supp}(\omega^\gamma_{x_s}) \subset \mathrm{supp}(\mu|_{9Q_s}) \subset 10Q_s \quad (\gamma \in \mathcal{M}), \tag{10.34}$$

and there exists a linear map $\widetilde{\omega}_{x_s} : \mathcal{P} \to \mathcal{P}$, such that

$$R'_{x_s}(f, P_0) = \sum_{\gamma\in\mathcal{M}} \omega^\gamma_{x_s}(f) \cdot v^\gamma + \widetilde{\omega}_{x_s}(P_0). \tag{10.35}$$

We now discuss the form of the linear maps $T_i$, defined just above (8.14). For each $i \in I$, the map $T_i$ is $\Omega'_i$-constructible, satisfying **(AL4)** for $E_i = 1.1Q_i$ (see Section 4.4). Thus, there exists a collection of linear functionals

$$\Omega'_i = \{\omega^i_t\}_{t\in\Upsilon^i} \subset \mathscr{J}(\mu|_{1.1Q_i})^* \subset \mathscr{J}(\mu)^* \quad \text{(see Remark 4.1)}, \tag{10.36}$$

such that the collection of sets $\{\mathrm{supp}(\omega^i_t)\}_{t\in\Upsilon^i}$ has $C$-bounded overlap, and by (4.13),

$$\mathrm{supp}(\omega^i_t) \subset \mathrm{supp}(\mu|_{1.1Q_i}) \subset 1.3Q_i \text{ for all } i \in I, t \in \Upsilon^i. \tag{10.37}$$

Further, for each $y \in \mathbb{R}^n$, there exists a finite subset $\Upsilon^i_y \subset \Upsilon^i$ and a collection of polynomials $\{v^i_{t,y}\}_{t\in\Upsilon^i} \subset \mathcal{P}$ such that $|\Upsilon^i_y| \le C$ and

$$J_y T_i(f, R_{x_i}) = \sum_{t\in\Upsilon^i_y} \omega^i_t(f) \cdot v^i_{t,y} + \widetilde{\omega}^i_y(R_{x_i}), \tag{10.38}$$

where $\widetilde{\omega}^i_y : \mathcal{P} \to \mathcal{P}$ is a linear map.

We define collections of functionals in $\mathscr{J}(\mu)^*$:

$\Omega_0 := \{\omega^\gamma_z : z \in K_p,\ \gamma \in \mathcal{M}\}$ indexed by $\Upsilon_0 := \{(z, \gamma) : z \in K_p,\ \gamma \in \mathcal{M}\}$;

$\Omega_1 := \{\omega^\gamma_{x_s} : x_s \in \mathfrak{B}_{key},\ \gamma \in \mathcal{M}\}$ indexed by $\Upsilon_1 := \{(x_s, \gamma) : x_s \in \mathfrak{B}_{key},\ \gamma \in \mathcal{M}\}$; and

$\Omega_2 := \bigcup_{i \in I} \Omega'_i = \{\omega^i_t : i \in I,\ t \in \Upsilon^i\}$ indexed by $\Upsilon_2 := \{(i,t) : i \in I,\ t \in \Upsilon^i\}$.

We define the complete collection of functionals as

$$\Omega' := \Omega_0 \cup \Omega_1 \cup \Omega_2 \subset \mathscr{J}(\mu)^*, \tag{10.39}$$

indexed by

$$\Upsilon = \Upsilon_0 \cup \Upsilon_1 \cup \Upsilon_2.$$

Because $\mathrm{supp}(\omega^\gamma_z) \subset \{z\}$, $\{\mathrm{supp}(\omega) : \omega \in \Omega_0\}$ has $C$-bounded overlap for $C = |\mathcal{M}|$, a universal constant.

For $x_s \in \mathfrak{B}_{key}$, $\mathrm{supp}(\omega^\gamma_{x_s}) \subset 10Q_s$ (see (10.34)). Because of Lemma 4.7, $|\{s' \in \bar{I} : 10Q_s \cap 10Q_{s'} \neq \emptyset\}| \leq C$ for fixed $s$. Thus, the supports of the functionals in $\Omega_1$ have $C$-bounded overlap.

From (4.2), $x \in 1.3Q$ for at most $C$ cubes $Q \in CZ^\circ$. The supports of the functionals in each collection $\Omega'_i$ are contained in $1.3Q_i$ (see (10.37)), and have $C$-bounded overlap. Thus, the supports of the functionals in $\Omega_2$ have $C$-bounded overlap.

We now show that the map $T : \mathscr{J}(\mu; \delta_{Q^\circ}) \to L^{m,p}(\mathbb{R}^n)$ is $\Omega'$-constructible. We do so by verifying the identity (10.28) for each $y \in \mathbb{R}^n$, for the set of functionals $\Omega' \subset \mathscr{J}(\mu)^*$ defined above.

For $y \in \mathbb{R}^n \setminus Q^\circ$, we have by (10.29),

$$J_y T(f, P_0) = P_0.$$

Thus, the identity (10.28) holds with $\Upsilon_y = \emptyset$ and $\widetilde{\omega}_y(P_0) = P_0$.

For $y \in K_p$, we have $J_y\theta = 1$ (recall $K_p \subset \frac{1}{9}Q^\circ$ and $\theta \equiv 1$ on $0.9Q^\circ$). So, by (10.29), (10.31), (10.32),

$$J_y T(f, P_0) = R^*_y(f, P_0) = \sum_{\gamma \in \mathcal{M}} \omega^\gamma_y(f) \cdot v^\gamma + \widetilde{\omega}_y(P_0).$$

Thus, the identity (10.28) holds with $\Upsilon_y = \{(y, \gamma) : \gamma \in \mathcal{M}\} \subset \Upsilon$.

Finally, suppose $y \in Q^\circ \setminus K_p$. Then $y \in Q_j \in CZ^\circ$ for a unique $j \in I$. Since $\mathrm{supp}(\theta_i) \subset 1.1Q_i$, by the good geometry of the CZ cubes, we have that $y \in \mathrm{supp}(\theta_i)$ if and only if $i \leftrightarrow j$. Thus,

$$J_y T(f, \vec{R}, \vec{R}^*) = J_y\Big(\sum_{i : i \leftrightarrow j} T_i(f, R_{x_i}) \cdot \theta_i\Big) = \sum_{i : i \leftrightarrow j} J_y T_i(f, R_{x_i}) \odot_y J_y\theta_i. \tag{10.40}$$

If $i \in I$ is such that $\kappa(x_i) = z \in K_p$, then $R_{x_i} = R^*_z$ by definition of $R_{x_i}$ in (10.33). So, by (10.32),

$$R_{x_i}(f, P_0) = \sum_{\gamma \in \mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot v^\gamma + \widetilde{\omega}_{\kappa(x_i)}(P_0) \quad (\text{if } \kappa(x_i) \in K_p). \tag{10.41}$$

On the other hand, if $i \in I$ is such that $\kappa(x_i) = x_s \in \mathfrak{B}_{key}$, then $R_{x_i} = R'_{x_s}$ by definition of $R_{x_i}$ in (10.33). So, by (10.35),

$$R_{x_i}(f, P_0) = \sum_{\gamma \in \mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot v^\gamma + \widetilde{\omega}_{\kappa(x_i)}(P_0) \quad (\text{if } \kappa(x_i) \in \mathfrak{B}_{key}). \tag{10.42}$$

Using the above identities in (10.38), we have, for all $i \in I$,

$$\begin{aligned} J_y T_i(f, R_{x_i}) &= \sum_{t \in \Upsilon^i_y} \omega^i_t(f) v^i_{t,y} + \widetilde{\omega}^i_y \Big( \sum_{\gamma \in \mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot v^\gamma + \widetilde{\omega}_{\kappa(x_i)}(P_0) \Big) \\ &= \sum_{t \in \Upsilon^i_y} \omega^i_t(f) v^i_{t,y} + \sum_{\gamma \in \mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot \widetilde{\omega}^i_y(v^\gamma) + \widetilde{\omega}^i_y\big(\widetilde{\omega}_{\kappa(x_i)}(P_0)\big). \end{aligned} \tag{10.43}$$

We return to the formula (10.40). We substitute (10.43) into (10.40) to write, for $y \in Q_j$,

$$\begin{aligned} J_y T(f, \vec{R}, \vec{R}^*) = &\sum_{i: i \leftrightarrow j} \sum_{t \in \Upsilon^i_y} \omega^i_t(f) \cdot \big\{ v^i_{t,y} \odot_y J_y \theta_i \big\} \\ &+ \sum_{i: i \leftrightarrow j} \sum_{\gamma \in \mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot \big\{ \widetilde{\omega}^i_y(v^\gamma) \odot_y J_y \theta_i \big\} + \widetilde{\omega}^i_y\big(\widetilde{\omega}_{\kappa(x_i)}(P_0)\big) \odot_y J_y \theta_i. \end{aligned}$$

Using (10.29), we write, for $y \in Q_j$,

$$\begin{aligned} J_y T(f, P_0) = &\sum_{i: i \leftrightarrow j} \sum_{t \in \Upsilon^i_y} \omega^i_t(f) \cdot \big\{ v^i_{t,y} \odot_y J_y \theta_i \odot_y J_y \theta \big\} \\ &+ \sum_{i: i \leftrightarrow j} \sum_{\gamma \in \mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot \big\{ \widetilde{\omega}^i_y(v^\gamma) \odot_y J_y \theta_i \odot_y J_y \theta \big\} \\ &+ \widetilde{\omega}^i_y\big(\widetilde{\omega}_{\kappa(x_i)}(P_0)\big) \odot_y J_y \theta_i \odot_y J_y \theta + (1 - J_y \theta) \odot_y P_0. \end{aligned} \tag{10.44}$$

For $y \in Q^\circ \setminus K_p$, there is a unique $j = j(y) \in I$ so that $y \in Q_j$; we define,

$$\begin{aligned} \Upsilon_{1,y} &:= \big\{ (i,t) : i \in I,\ i \leftrightarrow j(y),\ t \in \Upsilon^i_y \big\}; \\ \Upsilon_{2,y} &:= \{ (z, \gamma) : \kappa(x_i) = z \in K_p \text{ for some } i \in I,\ i \leftrightarrow j(y),\ \gamma \in \mathcal{M} \}; \text{ and} \\ \Upsilon_{3,y} &:= \{ (x_s, \gamma) : \kappa(x_i) = x_s \in \mathfrak{B}_{key} \text{ for some } i \in I,\ i \leftrightarrow j(y),\ \gamma \in \mathcal{M} \}. \end{aligned}$$

Note that the last term on the right-hand side of (10.44) is just a linear map applied to $P_0$. By splitting the second sum on $i \in I$ on the right-hand side of (10.44) into cases depending on whether $\kappa(x_i) = z \in K_p$ or $\kappa(x_i) = x_s \in \mathfrak{B}_{key}$, and using the definitions of the index sets $\Upsilon_{r,y}$ $(r = 1, 2, 3)$, we have

$$\begin{aligned} J_y T(f, P_0) = &\sum_{(i,t)\in\Upsilon_{1,y}} \omega_t^i(f) \cdot \left(v_{t,y}^i \odot_y J_y(\theta_i) \odot_y J_y(\theta)\right) \\ &+ \sum_{(z,\gamma)\in\Upsilon_{2,y}} \omega_z^\gamma(f) \cdot \sum_{i\in I:\ \kappa(x_i)=z, i\leftrightarrow j} \widetilde{\omega}_y^i(v_z^\gamma) \odot_y J_y(\theta_i) \odot_y J_y(\theta) \\ &+ \sum_{(x_s,\gamma)\in\Upsilon_{3,y}} \omega_{x_s}^\gamma(f) \cdot \sum_{i\in I:\ \kappa(x_i)=x_s, i\leftrightarrow j} \widetilde{\omega}_y^i(v^\gamma) \odot_y J_y(\theta_i) \odot_y J_y(\theta) + \widetilde{\omega}_y(P_0), \end{aligned} \tag{10.45}$$

for some linear map $\widetilde{\omega}_y : \mathcal{P} \to \mathcal{P}$.

Using (10.45), we see that the identity (10.28) holds with $\Upsilon_y := \Upsilon_{1,y} \cup \Upsilon_{2,y} \cup \Upsilon_{3,y}$. Note that

$$|\Upsilon_y| \leq C,$$

because $|\mathcal{M}| = D$, $|\{i \in I : i \leftrightarrow j\}| \leq C$ for fixed $j$, from (4.2), and $|\Upsilon_y^i| \leq C$ for fixed $(i, y)$. $\square$

**Lemma 10.5.** *The map $M$ defined in Lemma 10.1 is $\Omega'$-constructible, where $\Omega' = \{\omega_s\}_{s\in\Upsilon}$ is the collection defined in Lemma 10.4. Precisely, the objects $\lambda_\ell$, $\phi_\ell$, and $\vec{S}$ arising in the description of $M$ in (3.3), satisfy the following:*

***(1)*** *For each $\ell \in \mathbb{N}$ and $y \in \operatorname{supp}(\mu)$, there exists a finite subset $\bar{\Upsilon}_{\ell,y} \subset \Upsilon$ and constants $\{\eta_{s,y}\}_{s\in\bar{\Upsilon}_{\ell,y}}$ such that $|\bar{\Upsilon}_{\ell,y}| \leq C$, and the map $(f, P_0) \mapsto \lambda_\ell(f, P_0)(y)$ has the form*

$$\lambda_\ell(f, P_0)(y) = \sum_{s\in\bar{\Upsilon}_{\ell,y}} \eta_{s,y}\omega_s(f) + \widetilde{\lambda}_{y,\ell}(P_0), \tag{10.46}$$

*where $\widetilde{\lambda}_{y,\ell} : \mathcal{P} \to \mathbb{R}$ is a linear functional.*

***(2)*** *For each $\ell \in \mathbb{N}$, there exists a finite subset $\bar{\Upsilon}_\ell \subset \Upsilon$ and constants $\{\eta_s\}_{s\in\bar{\Upsilon}_\ell}$ such that $|\bar{\Upsilon}_\ell| \leq C$, and the map $\phi_\ell$ has the form*

$$\phi_\ell(f, P_0) = \sum_{s\in\bar{\Upsilon}_\ell} \eta_s\omega_s(f) + \widetilde{\lambda}_\ell(P_0), \tag{10.47}$$

*where $\widetilde{\lambda}_\ell : \mathcal{P} \to \mathbb{R}$ is a linear functional.*

***(3)*** *For $y \in K$, there exist $\{\omega_y^\alpha\}_{\alpha\in\mathcal{M}} \subset \Omega'$ satisfying for all $\alpha \in \mathcal{M}$, $\operatorname{supp}(\omega_y^\alpha) \subset \{y\}$, and the map $f \mapsto S_y(f)$ has the form*

$$S_y(f) = \sum_{\alpha \in \mathcal{M}} \omega_y^\alpha(f) \cdot v_\alpha, \tag{10.48}$$

*where* $\{v_\alpha\}_{\alpha \in \mathcal{M}}$ *is a basis for* $\mathcal{P}$*.*

**Proof.** In the proof of Lemma 10.1, we defined $M : \mathscr{J}(\mu; \delta) \to \mathbb{R}$ as the $(1/p)^{th}$ power of the right hand side of (10.26). Also, we defined $K = \emptyset$ for $\mathcal{A} \neq \emptyset$, and $K = K_p$, $\vec{S}(f) = \vec{R}^*(f)$ for $\mathcal{A} = \emptyset$. Then formula (10.48) follows immediately from Corollary 10.3, in the case $\mathcal{A} = \emptyset$; while formula (10.48) is vacuously true if $\mathcal{A} \neq \emptyset$ (for then $K = \emptyset$).

We now investigate the form of the various terms arising on the right-hand side of (10.26). We shall demonstrate that each of these terms either involves a map of the form (10.46) or (10.47).

We first recall the form of the maps $\lambda_\ell^i(f, P)$ and $\phi_\ell^i(f, P)$ arising in **(AL5)** and **(AL6)** for $E_i = 1.1Q_i$ (see Section 4.4). These maps arise in the reindexed sums on the right-hand side of (10.26).

By hypothesis **(AL5)**, for each $\ell \in \mathbb{N}$, $i \in I$, and $y \in \operatorname{supp}(\mu)$, there exists a finite subset $\bar{\Upsilon}_{\ell,y}^i \subset \Upsilon^i$ and constants $\{\eta_{s,y}^{\ell,i}\}_{s \in \bar{\Upsilon}_{\ell,y}^i} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_{\ell,y}^i| \leq C$, and the map $(f, P) \mapsto \lambda_\ell^i(f, P)(y)$ has the form

$$\lambda_\ell^i(f, P)(y) = \sum_{s \in \bar{\Upsilon}_{\ell,y}^i} \eta_{s,y}^{\ell,i} \cdot \omega_s^i(f) + \widetilde{\lambda}_{y,\ell}^i(P), \tag{10.49}$$

where $\widetilde{\lambda}_{y,\ell}^i : \mathcal{P} \to \mathbb{R}$ is a linear functional. Also, from **(AL6)**, for each $\ell \in \mathbb{N}$, and $i \in I$, there exists a finite subset $\bar{\Upsilon}_\ell^i \subset \Upsilon^i$ and constants $\{\eta_s^{\ell,i}\}_{s \in \bar{\Upsilon}_\ell^i} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_\ell^i| \leq C$, and the map $(f, P) \to \phi_\ell^i(f, P)$ has the form

$$\phi_\ell^i(f, P) = \sum_{s \in \bar{\Upsilon}_\ell^i} \eta_s^{\ell,i} \cdot \omega_s^i(f) + \widetilde{\lambda}_\ell^i(P), \tag{10.50}$$

where $\widetilde{\lambda}_\ell^i : \mathcal{P} \to \mathbb{R}$ is a linear functional.

For each $i \in I$, either $\kappa(x_i) \in K_p$ or $\kappa(x_i) \in \mathfrak{B}_{key}$. In (10.41) and (10.42), we have shown

$$R_{x_i}(f, P_0) = \sum_{\gamma \in \mathcal{M}} \omega_{\kappa(x_i)}^\gamma(f) \cdot v^\gamma + \widetilde{\omega}_{\kappa(x_i)}(P_0), \tag{10.51}$$

where $\{v^\gamma\}_{\gamma \in \mathcal{M}}$ is a basis for $\mathcal{P}$, $\omega_{\kappa(x_i)}^\gamma$ $(\gamma \in \mathcal{M})$ are linear functionals in $\Omega'$ (for the definition of $\Omega'$, see line (10.39) and the containing paragraph), and where $\widetilde{\omega}_{\kappa(x_i)} : \mathcal{P} \to \mathcal{P}$ is a linear map. Consequently,

$$\partial^\alpha[R_{x_i}(f, P_0)] = \sum_{\gamma \in \mathcal{M}} \omega_{\kappa(x_i)}^\gamma(f) \cdot \partial^\alpha[v^\gamma] + \partial^\alpha[\widetilde{\omega}_{\kappa(x_i)}(P_0)]. \tag{10.52}$$

For $j \in \mathbb{N}$, after the reindexing in (10.26), a map $\lambda_j : \mathscr{J}(\mu;\delta) \to L^p(d\mu)$ arising in (10.26) is either of the form $\lambda_j(f,P_0) = \lambda^i_\ell(f,R_{x_i})$ for $i \in I$ and $\ell \in \mathbb{N}$, or $\lambda_j(f,P_0) = \lambda'(f,P_0)$, where $\lambda'$ is defined in (10.22).

In the former case, we substitute (10.51) into (10.49) and write

$$\begin{aligned}
\lambda^i_\ell(f,R_{x_i})(y) &= \sum_{s\in\bar{\Upsilon}^i_{\ell,y}} \eta^{\ell,i}_{s,y}\cdot\omega^i_s(f) + \widetilde{\lambda}^i_{y,\ell}(R_{x_i}) \\
&= \sum_{s\in\bar{\Upsilon}^i_{\ell,y}} \eta^{\ell,i}_{s,y}\cdot\omega^i_s(f) + \widetilde{\lambda}^i_{y,\ell}\big(\sum_{\gamma\in\mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f)\cdot v^\gamma + \widetilde{\omega}_{\kappa(x_i)}(P_0)\big) \\
&= \sum_{s\in\bar{\Upsilon}^i_{\ell,y}} \eta^{\ell,i}_{s,y}\cdot\omega^i_s(f) + \sum_{\gamma\in\mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f)\widetilde{\lambda}^i_{y,\ell}(v^\gamma) + \widetilde{\lambda}^i_{y,\ell}\big(\widetilde{\omega}_{\kappa(x_i)}(P_0)\big).
\end{aligned}$$

Above, the number of terms in the two sums on the right-hand side is at most $|\bar{\Upsilon}^i_{\ell,y}| + |\mathcal{M}| \le C$, and each of the functionals $\omega^i_s(f)$ and $\omega^\gamma_{\kappa(x_i)}(f)$ belongs to the family $\Omega' = \{\omega_s\}_{s\in\Upsilon}$ defined in (10.39). Further, the third term on the right-hand side is a linear function of $P_0$. This proves the identity (10.46) for $\lambda_j$, provided that $\lambda_j(f,P_0) = \lambda^i_\ell(f,R_{x_i})$.

Now suppose $\lambda_j(f,P_0) = \lambda'(f,P_0)$ with $\lambda'$ defined in (10.22). If $x \in \mathbb{R}^n \setminus K_p$ then $\lambda'(f,P_0)(x) = 0$. If $x \in K_p$, then from Lemma 10.2,

$$\begin{aligned}
\lambda'(f,P_0)(x) &= R^*_x(f,P_0)(x) \\
&= \sum_{\alpha\in\mathcal{M}} \omega^\alpha_x(f)\cdot v_\alpha(x) + \widetilde{\omega}_x(P_0)(x),
\end{aligned}$$

where $\widetilde{\omega}_x : \mathcal{P} \to \mathcal{P}$ is a linear map. Above, the number of terms in the sum on the right-hand side is at most $|\mathcal{M}| \le C$, and each of the functionals $\omega^\alpha_x(f)$ belongs to the family $\Omega' = \{\omega_s\}_{s\in\Upsilon}$ defined in (10.39). Further, the second term on the right-hand side is a linear function of $P_0$. Therefore, we have succeeded in verifying the identity (10.46) for $\lambda_j$, provided that $\lambda_j(f,P_0) = \lambda'(f,P_0)$.

From (10.26) for $j \in \mathbb{N}$, a map $\phi_j : \mathscr{J}(\mu;\delta) \to \mathbb{R}$ is either of the form $\phi_j(f,P_0) = \phi^i_\ell(f,R_{x_i})$ for $i \in I$ and $\ell \in \mathbb{N}$, $\phi_j(f,P_0) = \phi^{i,i',\alpha}(f,P_0)$ for $i \leftrightarrow i'$ and $\alpha \in \mathcal{M}$ (see (10.21)), or $\phi_j(f,P_0) = \zeta_\alpha(f,P_0)$ for $\alpha \in \mathcal{M}$ (see (10.25)).

In the first case, if $\phi_j(f,P_0) = \phi^i_\ell(f,R_{x_i})$ for $i \in I$ and $\ell \in \mathbb{N}$, we substitute (10.51) into (10.50):

$$\begin{aligned}
\phi^i_\ell(f,R_{x_i}) &= \sum_{s\in\bar{\Upsilon}^i_\ell} \eta^{\ell,i}_s\cdot\omega^i_s(f) + \widetilde{\lambda}^i_\ell(R_{x_i}) \\
&= \sum_{s\in\bar{\Upsilon}^i_\ell} \eta^{\ell,i}_s\cdot\omega^i_s(f) + \widetilde{\lambda}^i_\ell\big(\sum_{\gamma\in\mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f)\cdot v^\gamma + \widetilde{\omega}_{\kappa(x_i)}(P_0)\big) \\
&= \sum_{s\in\bar{\Upsilon}^i_\ell} \eta^{\ell,i}_s\cdot\omega^i_s(f) + \sum_{\gamma\in\mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f)\widetilde{\lambda}^i_\ell(v^\gamma) + \widetilde{\lambda}^i_\ell\big(\widetilde{\omega}_{\kappa(x_i)}(P_0)\big).
\end{aligned}$$

In the expression above, the last term is a linear function of $P_0$, and the number of terms in both sums is at most $|\bar{\Upsilon}^i_\ell| + |\mathcal{M}| \leq C$. Further, each of the functionals $\omega^i_s(f)$ and $\omega^\gamma_{\kappa(x_i)}$ belongs to the family $\Omega' = \{\omega_s\}_{s\in\Upsilon}$ defined in (10.39). This proves the identity (10.47) for $\phi_j$, provided that $\phi_j(f, P_0) = \phi^i_\ell(f, R_{x_i})$.

Suppose $\phi_j(f, P_0) = \phi^{i,i',\alpha}(f, P_0) = c^{1/p}_{\alpha,i,i'}\partial^\alpha_x(R_{x_i} - R_{x_{i'}})(x_i)$ for $i \leftrightarrow i'$ and $\alpha \in \mathcal{M}$ (see (10.21)). Using (10.52), we have

$$\phi^{i,i',\alpha}(f, P_0) = c^{1/p}_{\alpha,i,i'} \sum_{\gamma\in\mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot \partial^\alpha_x[v^\gamma](x_i) - c^{1/p}_{\alpha,i,i'} \sum_{\gamma\in\mathcal{M}} \omega^\gamma_{\kappa(x_{i'})}(f) \cdot \partial^\alpha_x[v^\gamma](x_i) + c^{1/p}_{\alpha,i,i'} \cdot \partial^\alpha_x[\widetilde{\omega}_{\kappa(x_i)}(P_0) - \widetilde{\omega}_{\kappa(x_{i'})}(P_0)](x_i).$$

In the expression above, the last term is a linear function of $P_0$, and the number of terms in both sums is at most $2|\mathcal{M}| \leq C$. Further, each of the functionals $\omega^\gamma_{\kappa(x_i)}$ and $\omega^\gamma_{\kappa(x_{i'})}$ belongs to the family $\Omega' = \{\omega_s\}_{s\in\Upsilon}$ defined in (10.39). This proves the identity (10.47) for $\phi_j$, provided that $\phi_j(f, P_0) = \phi^{i,i',\alpha}(f, P_0)$.

Finally, suppose $\phi_j(f, P_0) = \zeta_\alpha(f, P_0) := c_\alpha\partial^\alpha(R_{x_1} - P_0)(x_1)$ for $\alpha \in \mathcal{M}$. Using (10.52) at $x = x_1$, we have

$$\zeta_\alpha(f, P_0) = c_\alpha \sum_{\gamma\in\mathcal{M}} \omega^\gamma_{\kappa(x_i)}(f) \cdot \partial^\alpha v^\gamma(x_1) + c_\alpha \left(\partial^\alpha\widetilde{\omega}_{\kappa(x_i)}(P_0)(x_1) - \partial^\alpha P_0(x_1)\right).$$

In the expression above, the last term is a linear function of $P_0$, and the number of terms in the sum is at most $|\mathcal{M}| \leq C$. Further, each of the functionals $\omega^\gamma_{\kappa(x_i)}$ belongs to the family $\Omega' = \{\omega_s\}_{s\in\Upsilon}$ defined in (10.39). Therefore, we have verified the identity (10.47) for $\phi_j$, provided that $\phi_j(f, P_0) = \zeta_\alpha(f, P_0)$.

This completes the proof that the map $M$ is $\Omega'$-constructible. □

The previous lemmas complete the inductive argument, begun in Section 3.3, by showing that under the assumption that the Main Lemma holds for all $\mathcal{A}' < \mathcal{A}$, the Main Lemma also holds for $\mathcal{A}$.

## 11. Proofs of the main theorems

**Proofs of Theorems 1 and 3.** Let $\mu$ be a compactly supported Borel regular measure on $\mathbb{R}^n$.
Fix $\delta > \mathrm{diam}(\mathrm{supp}(\mu))$, so that $\mathrm{supp}(\mu)$ is contained in a cube of sidelength $\delta$. From (2.22) and (2.23),

$$\|f\|_{\mathscr{J}(\mu)} \simeq \inf_{P\in\mathcal{P}} \|f, P\|_{\mathscr{J}(\mu;\delta)}. \tag{11.1}$$

Per the Extension Theorem for $(\mu, \delta)$ (Proposition 3.1), there exist a linear map $T : \mathscr{J}(\mu; \delta) \to L^{m,p}(\mathbb{R}^n)$, a map $M : \mathscr{J}(\mu; \delta) \to \mathbb{R}_+$, a set $K \subset \mathrm{Cl}(\mathrm{supp}(\mu))$, a linear map

$\vec{S} : \mathscr{J}(\mu) \to Wh(K)$, and countable families of Borel sets $\{A_\ell\}_{\ell\in\mathbb{N}}, A_\ell \subset \operatorname{supp}(\mu)$, and linear maps $\{\phi_\ell : \mathscr{J}(\mu;\delta) \to \mathbb{R}\}_{\ell\in\mathbb{N}}$, and $\{\lambda_\ell : \mathscr{J}(\mu;\delta) \to L^p(d\mu)\}_{\ell\in\mathbb{N}}$, that satisfy the conclusion of the extension theorem, as described below.

For each $(f, P_0) \in \mathscr{J}(\mu;\delta)$,

**(i)** $\|f, P_0\|_{\mathscr{J}(\mu;\delta)} \le \|T(f, P_0), P_0\|_{\mathscr{J}(f,\mu;\delta)} \le C \cdot \|f, P_0\|_{\mathscr{J}(\mu;\delta)}$; (11.2)

**(ii)** $c \cdot M(f, P_0) \le \|T(f, P_0), P_0\|_{\mathscr{J}(f,\mu;\delta)} \le C \cdot M(f, P_0)$; and (11.3)

**(iii)** $M(f, P_0) = \Big(\sum_{\ell\in\mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, P_0) - f|^p d\mu + \sum_{\ell\in\mathbb{N}} |\phi_\ell(f, P_0)|^p + \|\vec{S}(f)\|^p_{L^{m,p}(K)}\Big)^{1/p}.$ (11.4)

Hence,

$$\begin{aligned}\inf_{P\in\mathcal{P}} \|f, P\|_{\mathscr{J}(\mu;\delta)} &\simeq \inf_{P\in\mathcal{P}} M(f, P)\\ &= \inf_{P\in\mathcal{P}} \Big(\sum_{\ell\in\mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, P) - f|^p d\mu + \sum_{\ell\in\mathbb{N}} |\phi_\ell(f, P)|^p + \|\vec{S}(f)\|^p_{L^{m,p}(K)}\Big)^{1/p}.\end{aligned} \tag{11.5}$$

We apply Lemma 7.2 (with trivial constraint map $\Psi$, i.e., with $N = 0$) to compute $\xi(f) \in \mathcal{P}$, depending linearly on $f$ and satisfying:

$$\begin{aligned}&\Big(\sum_{\ell\in\mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, \xi(f)) - f|^p d\mu + \sum_{\ell\in\mathbb{N}} |\phi_\ell(f, \xi(f))|^p\Big)^{1/p}\\ &\qquad\lesssim \inf_{P\in\mathcal{P}} \Big(\sum_{\ell\in\mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, P) - f|^p d\mu + \sum_{\ell\in\mathbb{N}} |\phi_\ell(f, P)|^p\Big)^{1/p}.\end{aligned} \tag{11.6}$$

Define $Mf := M(f, \xi(f))$. Then

$$\begin{aligned}Mf &= \Big(\sum_{\ell\in\mathbb{N}} \int_{A_\ell} |\lambda_\ell(f, \xi(f)) - f|^p d\mu + |\phi_\ell(f, \xi(f))|^p\Big)^{1/p}\\ &= \Big(\sum_{\ell\in\mathbb{N}} \Big(\int_{A_\ell} |\zeta_\ell(f) - f|^p d\mu + |\psi_\ell(f)|^p\Big)\Big)^{1/p}.\end{aligned} \tag{11.7}$$

for linear maps $\zeta_\ell : \mathscr{J}(\mu) \to L^p(d\mu)$, and linear functionals $\psi_\ell : \mathscr{J}(\mu) \to \mathbb{R}$ ($\ell \in \mathbb{N}$), defined by $\zeta_\ell(f) = \lambda_\ell(f, \xi(f))$ and $\psi_\ell(f) = \phi_\ell(f, \xi(f))$.

Because of (11.2)-(11.4), $Tf := T(f, \xi(f))$ satisfies

$$\|Tf, \xi(f)\|_{\mathscr{J}(f,\mu;\delta)} \simeq \|f, \xi(f)\|_{\mathscr{J}(\mu;\delta)} \simeq M(f, \xi(f)) = Mf. \tag{11.8}$$

Together, (11.1) and (11.5)-(11.8) imply that

$$\|f\|_{\mathscr{J}(\mu)} \simeq \|Tf\|_{\mathscr{J}(f,\mu)} \simeq Mf,$$

completing the proof of Theorem 1.

Per the Extension Theorem for $(\mu, \delta)$, there exists a collection of linear functionals $\Omega' = \{\omega_s\}_{s\in\Upsilon} \subset \mathscr{J}(\mu)^*$ such that the collection of sets $\{\mathrm{supp}(\omega_s)\}_{s\in\Upsilon}$ has $C$-bounded overlap, and for each $y \in \mathbb{R}^n$, there exists a finite subset $\Upsilon_y \subset \Upsilon$ and a collection of polynomials $\{v_{s,y}\}_{s\in\Upsilon_y} \subset \mathcal{P}$ such that $|\Upsilon_y| \leq C$ and

$$J_yT(f, P_0) = \sum_{s\in\Upsilon_y} \omega_s(f) \cdot v_{s,y} + \widetilde{\omega}_y(P_0) \qquad \text{for } (f, P_0) \in \mathscr{J}(\mu; \delta), \tag{11.9}$$

where $\widetilde{\omega}_y : \mathcal{P} \to \mathcal{P}$ is a linear map.

We have defined $T(f) := T(f, \xi(f))$. Because $\xi : \mathscr{J}(\mu) \to \mathcal{P}$ is a linear map, we can write

$$\xi(f) = \sum_{\gamma\in\mathcal{M}} \omega_\gamma(f) \cdot v^\gamma, \tag{11.10}$$

for linear functionals $\omega_\gamma : \mathscr{J}(\mu) \to \mathbb{R}$ $(\gamma \in \mathcal{M})$, and where $\{v^\gamma\}_{\gamma\in\mathcal{M}}$ is a basis for $\mathcal{P}$. Substituting this into (11.9), we have

$$\begin{aligned} J_yT(f) = J_yT(f, \xi(f)) &= \sum_{s\in\Upsilon_y} \omega_s(f) \cdot v_{s,y} + \widetilde{\omega}_y\Big(\sum_{\gamma\in\mathcal{M}} \omega_\gamma(f) \cdot v^\gamma\Big) \\ &= \sum_{s\in\Upsilon_y} \omega_s(f) \cdot v_{s,y} + \sum_{\gamma\in\mathcal{M}} \omega_\gamma(f) \cdot \widetilde{\omega}_y(v^\gamma). \end{aligned} \tag{11.11}$$

Define the collection of functionals $\Omega := \Omega' \cup \{\omega_\gamma\}_{\gamma\in\mathcal{M}}$. Because $|\mathcal{M}| = D$ and the collection of supports of the functionals in $\Omega'$ has $C$-bounded overlap, the collection of supports of the functionals in $\Omega$ has $C'$-bounded overlap. Because $|\Upsilon_y| \leq C$, the number of terms in both sums in (11.11) is at most $C$. Therefore, $J_yT(f)$ is of the desired form in Theorem 3 (see (1.5)). That is, we have shown that $T$ is $\Omega$-constructible.

We have defined $Mf := M(f, \xi(f))$. Next we verify that the map $M$ is $\Omega$-constructible for the set $\Omega = \mathscr{J}(\mu)^*$ defined above. We shall verify conditions **(1)**–**(3)** of Theorem 3.

From the Extension Theorem for $(\mu, \delta)$, for $y \in K$, there exists $\{\omega_y^\alpha\}_{\alpha\in\mathcal{M}} \subset \Omega'$ satisfying for all $\alpha \in \mathcal{M}$, $\mathrm{supp}(\omega_y^\alpha) \subset \{y\}$, and the map $f \mapsto S_y(f)$ has the form

$$S_y(f) = \sum_{\alpha\in\mathcal{M}} \omega_y^\alpha(f) \cdot v_\alpha,$$

where $\{v_\alpha\}_{\alpha\in\mathcal{M}}$ is a basis for $\mathcal{P}$. This implies condition **(3)** of Theorem 3, because $\Omega' \subset \Omega$.

To prove that $M$ is $\Omega$-constructible, we must show that the maps $\zeta_\ell : \mathscr{J}(\mu) \to L^p(d\mu)$ and $\psi_\ell : \mathscr{J}(\mu) \to \mathbb{R}$ in (11.7) satisfy conditions **(1)** and **(2)** of Theorem 3.

From the Extension Theorem for $(\mu, \delta)$, we have for each $\ell \in \mathbb{N}$ and $y \in \operatorname{supp}(\mu)$, there exists a finite subset $\bar{\Upsilon}_{\ell,y} \subset \Upsilon$ and constants $\{\eta^\ell_{s,y}\}_{s\in\bar{\Upsilon}_{\ell,y}} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_{\ell,y}| \leq C$, and the map $\lambda_\ell$ has the form

$$\lambda_\ell(f, P)(y) = \sum_{s\in\bar{\Upsilon}_{\ell,y}} \eta^\ell_{s,y} \cdot \omega_s(f) + \widetilde{\lambda}_{y,\ell}(P),$$

where $\widetilde{\lambda}_{y,\ell} : \mathcal{P} \to \mathbb{R}$ is a linear functional. Using (11.10) in the previous equation, we deduce that the map $f \mapsto \zeta_\ell(f)(y)$ has the form

$$\begin{aligned}
\zeta_\ell(f)(y) = \lambda_\ell(f, \xi(f))(y) &= \sum_{s\in\bar{\Upsilon}_{\ell,y}} \eta^\ell_{s,y} \cdot \omega_s(f) + \widetilde{\lambda}_{y,\ell}\Bigg(\sum_{\gamma\in\mathcal{M}} \omega_\gamma(f) \cdot v^\gamma\Bigg) \\
&= \sum_{s\in\bar{\Upsilon}_{\ell,y}} \eta^\ell_{s,y} \cdot \omega_s(f) + \sum_{\gamma\in\mathcal{M}} \omega_\gamma(f) \cdot \widetilde{\lambda}_{y,\ell}(v^\gamma).
\end{aligned}$$

Therefore, $\zeta_\ell(f)$ has the form stated in condition **(1)** of Theorem 3, for the set $\Omega = \Omega' \cup \{\omega_\gamma\}_{\gamma\in\mathcal{M}}$.

Similarly, from the Extension Theorem for $(\mu, \delta)$, for $\ell \in \mathbb{N}$, there exists a finite subset $\bar{\Upsilon}_\ell \subset \Upsilon$ and constants $\{\eta^\ell_s\}_{s\in\bar{\Upsilon}_\ell} \subset \mathbb{R}$ such that $|\bar{\Upsilon}_\ell| \leq C$, and the functional $\phi_\ell$ has the form

$$\phi_\ell(f, P) = \sum_{s\in\bar{\Upsilon}_\ell} \eta^\ell_s \cdot \omega_s(f) + \widetilde{\lambda}_\ell(P),$$

where $\widetilde{\lambda}_\ell : \mathcal{P} \to \mathbb{R}$ is a linear functional. Again, using (11.10), we deduce that the map $\psi_\ell$ has the form

$$\begin{aligned}
\psi_\ell(f) = \phi_\ell(f, \xi(f)) &= \sum_{s\in\bar{\Upsilon}_\ell} \eta^\ell_s \cdot \omega_s(f) + \widetilde{\lambda}_\ell\Bigg(\sum_{\gamma\in\mathcal{M}} \omega_\gamma(f) \cdot v^\gamma\Bigg) \\
&= \sum_{s\in\bar{\Upsilon}_\ell} \eta^\ell_s \cdot \omega_s(f) + \sum_{\gamma\in\mathcal{M}} \omega_\gamma(f) \cdot \widetilde{\lambda}_\ell(v^\gamma).
\end{aligned}$$

Therefore, $\psi_\ell(f)$ has the form stated in condition **(2)** of Theorem 3, for the set $\Omega = \Omega' \cup \{\omega_\gamma\}_{\gamma\in\mathcal{M}}$.

This completes the proof of Theorem 3. $\square$

**Proof of Theorem 4.** Let $\mu$ be a finite Borel regular measure on $\mathbb{R}^n$ with compact support. By rescaling, we may assume $\operatorname{supp}(\mu) \subset \frac{1}{10}Q^\circ$ for $Q^\circ = (0, 1]^n$. In the proof of the Extension Theorem for $(\mu, \delta = 1/10)$, the Calderón-Zygmund decomposition $CZ^\circ$

defined in Section 4 is finite and $\bigcup_{Q \in CZ^\circ} Q = Q^\circ$ (see Lemma 4.3). Therefore, $K_p = \emptyset$ (see Definition 4.4). Consequently, $Mf : \mathscr{J}(\mu) \to \mathbb{R}$ in Theorem 1 has the form

$$Mf = \Big( \sum_{\ell=1}^{K} \int_{A_\ell} |\zeta_\ell(f) - f|^p d\mu + |\psi_\ell(f)|^p \Big)^{1/p},$$

where for each $\ell$, $A_\ell \subset \mathrm{supp}(\mu)$ is a Borel set, $\zeta_\ell : \mathscr{J}(\mu) \to L^p(d\mu)$, and $\psi_\ell : \mathscr{J}(\mu) \to \mathbb{R}$ are linear maps, and $K$ is finite. Again because the CZ decomposition is finite, in Theorem 3, we have that the collection of linear functionals $\Omega = \{\omega_r\}_{r \in \Upsilon} \subset \mathscr{J}(\mu)^*$ is finite ($|\Upsilon| < \infty$). $\square$

**Proofs of Theorems 2 and 5.** Theorems 2 and 5 are immediate consequences of Theorems 1 and 3 applied to the Borel measure $\mu = \mu_E$ in (1.4). For details, see the discussion after (1.4). $\square$

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