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HOCHSCHILD HOMOLOGY OF MOD-p MOTIVIC COHOMOLOGY OVER ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT. We perform Hochschild homology calculations in the algebro-geometric setting of motives over algebraically closed fields. The homotopy ring of motivic Hochschild homology contains torsion classes that arise from the mod-p motivic Steenrod algebra and generating functions defined on the natural numbers with finite non-empty support. Under Betti realization, we recover Bökstedt's calculation of the topological Hochschild homology of finite prime fields.

1. INTRODUCTION

Hochschild (aka derived Hochschild or Shukla) homology is in a precise sense *the* homology theory of associative algebras [38, Section 3], and so plays an important role from a purely ring-theoretic perspective, classifying extensions and so on. However, Hochschild homology rose in prominence in the 1980s via its cyclic structure as explored by Connes and Tsygan and its subsequent connection to (rational) algebraic K-theory. To include torsion phenomena (and wider applications), Goodwillie and Waldhausen conjectured that the differential of algebraic K-theory should correspond to some form of Hochschild homology of algebras over the sphere spectrum (ring spectra). When Bökstedt succeeded in extending the definition of Hochschild homology to cover algebras over the sphere spectrum ("topological Hochschild homology"), he also managed to calculate its values at the prime fields, revealing the striking periodicity which has been fundamental to much of the subsequent development. Later, the first author and McCarthy confirmed Goodwillie and Waldhausen's conjecture, leading to further advances and, ultimately, many calculations of algebraic K-theory.

In this paper, we try to emulate Bökstedt: we define Hochschild homology for algebras over the *motivic* sphere spectrum and calculate its values at prime fields. This is interesting for many reasons. Firstly, the motivic version over $\mathrm{Spec}(\mathbb{C})$ sheds light on the topological one, giving "reasons" for some of the relations from the classical case. Secondly, if we are to investigate the "number theory" of rings over the motivic sphere spectrum, we should access invariants of a K-theoretic nature with an ambitious goal of repeating the success in (equivariant) stable homotopy theory. We hope that this paper is a tiny step in the right direction. Our definition follows the interpretation of

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Hochschild homology as the homology of associative algebras in the spirit of Quillen. There are different options, but we do not pursue them in this paper.

Hochschild homology is a fundamental derived invariant for algebras and rings [11]. For a smooth algebra A over a field k of characteristic zero, such as the coordinate ring of a smooth affine variety, the Hochschild-Kostant-Rosenberg theorem identifies the Hochschild homology groups $\mathbf{HH}_n(A/k)$ with the Kähler differentials $\Omega^n_{A/k}$ of derivations. Using Hochschild homology, **HH**, one constructs many other derived invariants such as Connes' cyclic homology **HC**. We refer to [31] for background. Bökstedt-Hsiang-Madsen [9] pioneered the refined theory of topological Hochschild homology **THH** and topological cyclic homology **TC**; their trace methods remain of significant interest in algebraic K-theory, see [13], [20], and the modern viewpoint adapted to ∞-categories in [36].

Motivic homotopy theory is an $\mathbb{A}^1$-invariant homotopy theory for algebraic varieties, originally developed by Morel and Voevodsky in the 1990s [35], and motivated by the spectacular work of Voevodsky and Rost resolving the Milnor and Bloch-Kato conjectures relating Milnor K-theory with Galois cohomology [52], [54], and quadratic forms [37], [45]. Since then, this framework has shown itself to be a valuable setting for studying algebro-geometric cohomology theories, with applications to algebraic geometry, number theory, and algebraic topology. See [27] and [30] for recent surveys.

In this paper, we study Hochschild homology in the motivic setting. Let $\mathscr{R}$ be a motivic ring spectrum such as algebraic cobordism, homotopy algebraic K-theory, or motivic cohomology [49]. Working in the stable motivic homotopy category $\mathbf{SH}(F)$ of a field F, we define the motivic Hochschild homology $\mathbf{MHH}(\mathscr{R})$ of $\mathscr{R}$ as the derived tensor product

$$\mathscr{R} \wedge_{\mathscr{R}\wedge\mathscr{R}^{\mathrm{op}}} \mathscr{R}. \tag{1}$$

The primary purpose of this paper is to calculate the homotopy ring $\mathbf{MHH}_\star(\mathbb{F}_p)$ of motivic Hochschild homology of $\mathbf{M}\mathbb{F}_p$ over algebraically closed fields—the Suslin-Voevodsky motivic cohomology ring spectrum for p any prime number. When the base field admits an embedding into the complex numbers $\mathbb{C}$, the Betti realization functor allows us to compare our **MHH** calculations with Bökstedt's pioneering work in [8] on topological Hochschild homology of the corresponding topological Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{F}_p$. Additively, $\mathbf{THH}(\mathbb{F}_p)$ splits as sum of $\mathbf{H}\mathbb{F}_p$'s in the stable homotopy category. However, this is not the case for $\mathbf{MHH}(\mathbb{F}_p)$ and $\mathbf{M}\mathbb{F}_p$. The source of this extra layer of complexity is the abundance of τ-torsion elements in the homotopy ring. Here τ is a canonical class in the mod-p motivic cohomology of F, which maps to the unit element in singular cohomology under Betti realization.

We express the homotopy ring $\mathbf{MHH}_\star(\mathbb{F}_p)$ in terms of algebra generators $\tau, \mu_i, \mathrm{x}_{S,f}$ arising from the mod-p motivic Steenrod algebra [23], [53], and generating endofunctions $f : \mathbb{N} \circlearrowleft$ with finite non-empty support containing some subset $S \subset \mathbb{N}$. It is the τ-power in the equation $\mu_i^p = \tau^{p-1}\mu_{i+1}$ that gives rise to the infinity of τ-torsion classes $\mathrm{x}_{S,f}$ not witnessed topologically in $\mathbf{THH}_\star(\mathbb{F}_p)$.

Theorem 1.1. *Over an algebraically closed field of exponential characteristic $\neq p$, there is an algebra isomorphism*

$$\mathbf{MHH}_\star(\mathbb{F}_p) \cong \mathbb{F}_p[\tau, \mu_i, \mathrm{x}_{S,f}]_{i\in\mathbb{N},\,(S\subset \operatorname{supp} f,\, f:\,\mathbb{N}\circlearrowleft)}/\mathcal{I} \tag{2}$$

with the ideal of relations

$$\mathcal{J} = \left(\begin{array}{c} \mu_i^p - \tau^{p-1}\mu_{i+1}, \\ \tau^{p-1}\mathsf{x}_{S,f}, \\ \mathsf{x}_{S,f}\cdot \mathsf{x}_{T,g} - \sum_{u\in \operatorname{supp}(f+g)-S\cup T} \epsilon_u \cdot \mathsf{x}_{S\cup T\cup\{u\},f+g} \end{array} \right).$$

Here the support of f is a finite non-empty subset of the natural numbers and $S \subset \operatorname{supp} f \subset \mathbb{N}$ does not contain the minimal element of $\operatorname{supp} f$. The coefficient $\epsilon_u \in \mathbb{F}_p$ is given explicitly in Definition 2.12. The algebra generators have bidegrees given by $|\tau| = (0, -1)$, $|\mu_i| = (2p^i, p^i - 1)$, and

$$|\mathsf{x}_{S,f}| = (|S| + 1)(-1, p - 1) + p \sum_{j\in \operatorname{supp} f} f(j)(2p^j, p^j - 1).$$

Since the homotopy of $\mathbf{MHH}(\mathbb{F}_p)$ is not a free module over the homotopy of $\mathbf{M}\mathbb{F}_p$, we deduce a non-splitting of the motivic Hochschild homology in $\mathbf{M}\mathbb{F}_p$-modules.

Corollary 1.2. *The motivic Hochschild homology of $\mathbb{F}_p$ does not split as a wedge of suspensions of $\mathbf{M}\mathbb{F}_p$.*

This gives a surprising obstruction to classical results about topological Hochschild homology and Thom spectra. Mahowald showed that the Eilenberg–MacLane spectrum $\mathbf{H}\mathbb{F}_2$ is a Thom spectrum of a double loop map with source $\Omega^2 S^3$ [32]. Behrens–Wilson showed that an analogous result is true C_2-equivariantly, with the base now $\Omega^{2,1}S^{3,1}$ [6]. Blumberg–Cohen–Schlichtkrull showed that the topological Hochschild homology of Thom spectra are Thom spectra, and when the topological η vanishes, these split as smash products of the original Thom spectrum and a space related to the classifying space of the base [7]. Equivariantly, classically and C_2-equivariantly, this splits as a wedge of smash powers of spheres. Putting this all together, we cannot have that all of these results hold in the motivic setting.

As a guide to this paper, we outline the proof of Theorem 1.1 and explain how the algebra generators arise in our context. The key idea in proving our results is to study the τ-inversion and mod-τ^{p-1} reduction of $\mathbf{MHH}(\mathbb{F}_p)$, and then analyze how their homotopy classes conspire to describe the integral homotopy ring. We review some background and set our notation in Section 2. Remark 2.5 gives a Lefschetz Principle for the homotopy ring of $\mathbf{MHH}(\mathbb{F}_p)$, which reduces our computation to the case of complex numbers. In Section 3, we divide the proof of Theorem 1.1 into the following steps.

Step 1 The dual motivic Steenrod algebra of our ground field F at p, see (9), contains classes τ_i for $i \geq 0$. Theorem 3.3 calculates the τ-inverted or étale motivic Hochschild homology

(3)
$$\mathbf{MHH}_\star(\mathbb{F}_p)[\tau^{-1}] \cong \mathbb{F}_p[\tau^{\pm 1}, \mu_i]_{i\geq 0}/(\mu_i^p - \tau^{p-1}\mu_{i+1}) \cong \mathbb{F}_p[\mu, \tau^{\pm 1}] \cong \mathbf{THH}_*(\mathbb{F}_p)[\tau^{\pm 1}].$$

Here the generator μ has bidegree $(2, 0)$. The "homology suspension" classes $\mu_i := \sigma\tau_i$, see Section 2.2, generate the non-τ-torsion part of $\mathbf{MHH}_\star(\mathbb{F}_p)$ subject to the relation $\mu_i^p = \tau^{p-1}\mu_{i+1}$.

Step 2 Theorem 3.6 calculates the mod-τ^{p-1} motivic Hochschild homology

(4)
$$\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1} := \pi_\star(\mathbf{MHH}(\mathbb{F}_p)/\tau^{p-1}) \cong \Big(\bigotimes_{i\geq 0} \Gamma_{\mathbb{F}_p}(\bar{\mu}_i) \otimes \Lambda_{\mathbb{F}_p}(\bar{\lambda}_{i+1})\Big) \otimes \mathbb{F}_p[\tau]/\tau^{p-1}.$$

The bidegrees of the generators are $|\bar{\lambda}_i| = (2p^i - 1, p^i - 1)$, $|\bar{\mu}_i| = (2p^i, p^i - 1)$. The divided powers algebra generator $\bar{\mu}_i$ is the image of $\mu_i \in \mathbf{MHH}_\star(\mathbb{F}_p)$. It turns out that (4) coincides with the E^2 page of the Tor spectral sequence for $\mathbf{MHH}(\mathbb{F}_p)/\tau^{p-1}$. In fact, the said Tor spectral sequence collapses at E^2 with no multiplicative extensions.

Step 3 Lemma 3.9 shows that the τ^{p-1}-Bockstein of $\gamma_j\bar{\mu}_i$ equals $\bar{\lambda}_{i+1}\gamma_{j-p}\bar{\mu}_i$. First, we establish the case $j = p$, and the rest follows by shuffle products in the bar construction of $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$. Here, the τ^{p-1}-Bockstein on $\mathbf{MHH}_\star(\mathbb{F}_p)$ is the composite of the canonical boundary and quotient maps in

$$(5) \qquad \bar{\partial} : \mathbf{MHH}_{*+1,*}(\mathbb{F}_p)/\tau^{p-1} \xrightarrow{\partial} \mathbf{MHH}_{*,*+p-1}(\mathbb{F}_p) \xrightarrow{q} \mathbf{MHH}_{*,*+p-1}(\mathbb{F}_p)/\tau^{p-1}.$$

In Corollary 3.10, we conclude the Bockstein homology of $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$ is isomorphic to the graded commutative $\mathbb{F}_{p,\tau} := \mathbb{F}_p[\tau]/\tau^{p-1}$-algebra $\bigoplus_{i\geq 0}\Lambda_{\mathbb{F}_{p,\tau}}(\bar{\mu}_i)$.

Step 4 Lemma 3.13 shows that the τ-torsion classes in $\mathbf{MHH}_\star(\mathbb{F}_p)$ inject into $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$ with image that of the τ^{p-1}-Bockstein $\bar{\partial}$ (degrees are made explicit through generating functions). Moreover, the reduction map q sends the image of the boundary ∂ isomorphically to the image of the Bockstein $\bar{\partial}$.

Step 5 If $f : \mathbb{N} \circlearrowleft$ has finite support and $S \subseteq \operatorname{supp} f$, we set

$$\chi_{S,f} = \left(\prod_{m\in S} \bar{\lambda}_{m+1}\gamma_{pf(m)-p}\bar{\mu}_m\right)\left(\prod_{n\notin S}\gamma_{pf(n)}\bar{\mu}_n\right) \in \mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}.$$

We define the τ-torsion algebra generators in Theorem 1.1 by

$$\mathsf{x}_{S,f} = \partial\chi_{S,f} \in \mathbf{MHH}_\star(\mathbb{F}_p).$$

In particular, $\chi_{\emptyset,0} = 1$, $\chi_{\emptyset,p^j\delta_n} = \gamma_{p^{j+1}}\bar{\mu}_n$ and $\chi_{\{m\},\delta_m} = \lambda_{m+1}$. Here $\delta_n : \mathbb{N} \circlearrowleft$ is zero except for $\delta_n(n) = 1$. Applying the Bockstein operation ∂ to $\chi_{S,f}$ yields $\mathsf{x}_{S,f} = \sum_{n\in \operatorname{supp}(f)-S}\chi_{S\cup\{n\},f}$ since $\bar{\partial}\gamma_n\bar{\mu}_i = \bar{\lambda}_{i+1}\gamma_{n-p}\bar{\mu}_i$, $\bar{\partial}\bar{\lambda}_i = 0$, and $\bar{\partial}$ is a derivation. Since the classes $\bar{\mu}_i$, $\chi_{S,f}$, and the $\bar{\partial}$ cycles $\lambda_{i+1} = \bar{\partial}\gamma_p\bar{\mu}_i$ generate $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$, the classes $\bar{\mu}_i$ and $\mathsf{x}_{S,f}$ generate the boundary.

Step 6 By combining the τ-inverted and mod-τ^{p-1} calculations we finally deduce Theorem 1.1. The power operations in the dual motivic Steenrod algebra give rise to the relation $\mu_i^p = \tau^{p-1}\mu_{i+1}$. The Bockstein calculation $\mathsf{x}_{S,f} = \partial\chi_{S,f}$ implies the vanishing $\tau^{p-1}\mathsf{x}_{S,f} = 0$. Corollary 2.15 shows the multiplicative relation between the $\mathsf{x}_{S,f}$ classes follows from a similar formula for the $\chi_{S,f}$ classes. We refer to Definition 2.12 for the entity ϵ_u.

For example, at the prime $p = 2$, we obtain the relations

$$\mathsf{x}_{\delta_0+\delta_1}\mathsf{x}_{\delta_2} + \mathsf{x}_{\delta_1+\delta_2}\mathsf{x}_{\delta_0} + \mathsf{x}_{\delta_2+\delta_0}\mathsf{x}_{\delta_1} = 0,$$

$$\mathsf{x}_{\delta_0+\delta_1}\mathsf{x}_{\delta_1+\delta_2} = \mathsf{x}_{2\delta_1}\mathsf{x}_{\delta_0+\delta_2}.$$

Theorem 1.1 admits a succinct reformulation in terms of naturally induced pullback squares of commutative $\mathbb{F}_p[\tau]$-algebras given in Section 3.3.1 and Section 3.4. For example, when $p = 2$, we note the pullback square of commutative $\mathbb{F}_2[\tau]$-algebras

$$\begin{array}{ccc} \mathbf{MHH}_\star(\mathbb{F}_2) & \longrightarrow & \mathbb{F}_2[\tau, \mu_i]/(\mu_i^2 - \tau\mu_{i+1}) \\ \downarrow & & \downarrow \\ \mathbb{F}_2[\bar{\mu}_i, \mathsf{x}_{S,f}]/\mathcal{I} & \longrightarrow & \mathbb{F}_2[\bar{\mu}_i]/(\bar{\mu}_i^2) \end{array}$$

where the ideal of relations is given by (see Section 2.5 for the definition of t_{f+g})

$$\mathcal{I} = \left(\bar{\mu}_i^2, \mathsf{x}_{S,f} \cdot \mathsf{x}_{T,g} - \sum_{t_{f+g} \neq u \in \mathrm{supp}(f+g) - S \cup T} \epsilon_u \cdot \mathsf{x}_{S \cup T \cup \{u\}, f+g} \right).$$

Our calculation shows the left vertical map in the pullback is an isomorphism on τ-torsion classes. Furthermore, the upper horizontal map is an injection on non-τ-torsion classes. An analogous result holds for all odd primes.

1.1. **Notation.** This paper uses the following notation.

p, F	prime number, base field of exponential characteristic $\mathsf{e}(F) \neq p$
$\mathbf{SH}(F)$	stable motivic homotopy category of F
$\mathbf{CAlg}(F)$	commutative motivic ring spectra of F
$\mathscr{R}$	motivic ring spectrum
$H^\star, h^\star$	(bigraded) integral, mod-p motivic cohomology groups of F
K_*^M, k_*^M	(graded) integral, mod-p Milnor K-groups of F
$\mathbb{M}_\star$	mod-p motivic homology ring of F
$\mathcal{A}_\star$	dual motivic Steenrod algebra of F at p
$\mathbb{F}_{p,\tau}$	shorthand for $\mathbb{F}_p[\tau]/\tau^{p-1}$
Γ, Λ	divided power and exterior algebras

2. Motivic Hochschild homology

Definition 2.1. Let $\mathscr{R}$ be a motivic ring spectrum. The *motivic Hochschild homology* of an $\mathscr{R}$-bimodule $\mathscr{M}$ is the derived smash product

$$\mathbf{MHH}(\mathscr{R}; \mathscr{M}) \coloneqq \mathscr{M} \wedge_{\mathscr{R} \wedge \mathscr{R}^{\mathrm{op}}} \mathscr{R} \tag{6}$$

in $\mathbf{SH}(F)$.

When $\mathscr{R} = \mathscr{M}$, the derived tensor product (6) specializes to $\mathbf{MHH}(\mathscr{R})$ in (1). If $\mathscr{R} \to \mathscr{Q}$ is a map of motivic ring spectra and $\mathscr{M}$ is a $\mathscr{Q}$-$\mathscr{R}$ bimodule, then reassociating the smash factors implies the equivalence

$$\mathbf{MHH}(\mathscr{R}; \mathscr{M}) \simeq \mathscr{M} \wedge_{\mathscr{Q} \wedge \mathscr{R}^{\mathrm{op}}} \mathscr{Q}. \tag{7}$$

In the following, we assume that $\mathscr{R}$ is a cofibrant commutative motivic ring spectrum in any of the model categorical approaches to $\mathbf{SH}(F)$ as in [14], [21], [24], [28] (this assumption is superfluous in the ∞-category of motivic spectra [41]). Commutative motivic ring spectra are cotensored over motivic spaces via the free-forgetful adjunction $\mathcal{F} \dashv \mathcal{U}$ between $\mathbf{SH}(F)$ and commutative motivic ring spectra $\mathbf{CAlg}(F)$: if $\mathscr{X}$ is a motivic space, then $\mathscr{X} \otimes \mathscr{R}$ is the coequalizer of

$$\mathcal{F}(\mathscr{X}_+ \wedge \mathcal{U}\mathscr{R}) \leftleftarrows \mathcal{F}(\mathscr{X}_+ \wedge \mathcal{U}\mathcal{F}\mathcal{U}\mathscr{R}).$$

Here we use the canonical maps $\mathcal{UFUR} \to \mathcal{UR}$, $\mathcal{X}_+ \wedge \mathcal{UF}(\mathcal{UR}) \to \mathcal{UF}(\mathcal{X}_+ \wedge \mathcal{UR})$, and $\mathcal{FUF} \to \mathcal{F}$. We will only need the special case of simplicial sets or topological spaces. The case of finite simplicial sets is particularly transparent since it derives from the relation $\{1,\dots,n\}\otimes\mathcal{R} = \mathcal{R}^{\wedge n}$. The assignment $\mathcal{X} \mapsto \mathcal{X}\otimes\mathcal{R}$ from motivic spaces to motivic ring spectra has several useful properties which generalize from the topological setting and which we will use freely.

- $\mathcal{X} \mapsto \mathcal{X} \otimes \mathcal{R}$ is $\mathbb{A}^1$-homotopy invariant and preserves coproducts (and so, in particular, sends pushouts to smashes).
- $* \otimes \mathcal{R} \cong \mathcal{R}$, $S^0 \otimes \mathcal{R} \cong \mathcal{R} \wedge \mathcal{R}$ and (since S^1 is the derived pushout of $* \leftarrow S^0 \to *$)
$$\mathbf{MHH}(\mathcal{R};\mathcal{M}) \simeq \mathcal{M} \wedge_{\mathcal{R}} (S^1 \otimes \mathcal{R}).$$
- The product on $\mathcal{X} \otimes \mathcal{R}$ is induced by the fold $\mathcal{X} \coprod \mathcal{X} \to \mathcal{X}$.
- Choosing a point $* \to \mathcal{X}$ makes $\mathcal{X} \otimes \mathcal{R}$ an augmented commutative $\mathcal{R}$-algebra.
- The inclusion $\{-1,1\} \subseteq \{-1,0,1\} \cong \{0,-1\}\vee\{0,1\}$ induces the comultiplication $\mathcal{R}\wedge\mathcal{R} \to \mathcal{R}\wedge\mathcal{R}\wedge\mathcal{R} \cong (\mathcal{R}\wedge\mathcal{R})\wedge_{\mathcal{R}}(\mathcal{R}\wedge\mathcal{R})$ and the nontrivial automorphism $\{-1,1\} \to \{-1,1\}$ gives the anti-involution of the "dual Steenrod $\mathcal{R}$-Hopf algebroid" $S^0 \otimes \mathcal{R} = \mathcal{R} \wedge \mathcal{R}$ (alge*broid* since the maps involved are not pointed, and so there is no guarantee that the units corresponding to the two choices of base points will coincide). The suspensions of these maps give the pinch map
$$S^1 \cong [-1,1] \coprod_{\{-1,1\}} * \to [-1,1] \coprod_{\{-1,0,1\}} * \cong S^1 \vee S^1$$
and the flip map $S^1 \to S^1$, both of which are pointed maps, inducing the $\mathcal{R}$-Hopf algebroid structure
$$\psi : S^1 \otimes \mathcal{R} \longrightarrow (S^1 \vee S^1) \otimes \mathcal{R} \cong (S^1 \otimes \mathcal{R}) \wedge_{\mathcal{R}} (S^1 \otimes \mathcal{R}), \quad \chi : S^1 \otimes \mathcal{R} \cong S^1 \otimes \mathcal{R}$$
on the "motivic Hochschild homology"—to implement this using finite simplicial models of the circle, one subdivides as in [1].
Hence, if $\mathbf{MHH}_\star(\mathcal{R})$ is flat over $\mathcal{R}_\star$, which will turn out not to be true for $\mathcal{R} = \mathbf{M}\mathbb{F}_p$, we get an $\mathcal{R}_\star$-Hopf algebra structure on $\mathbf{MHH}_\star(\mathcal{R})$.
- The tensor with spaces in the category of motivic spectra is $\mathcal{X} \mapsto \mathcal{X}_+ \wedge \mathcal{R}$ and the universal property defines a unique map of motivic spectra
$$\sigma^+ : \mathcal{X}_+ \wedge \mathcal{R} \to \mathcal{X} \otimes \mathcal{R}.$$
If X is a set considered as a motivic space, the inclusion of the points $\{x\} \subseteq X$ induces the desired map $X_+ \wedge \mathcal{R} \cong \bigvee_{\{x\}\in X}\{x\} \otimes \mathcal{R} \to X \otimes \mathcal{R}$. If X is already pointed, the basepoint in X makes $X \otimes \mathcal{R}$ an $\mathcal{R}$-algebra, giving rise to the free extension to an $\mathcal{R}$-linear map
$$\sigma : \mathcal{R} \wedge X_+ \wedge \mathcal{R} \xrightarrow{1\wedge\sigma^+} \mathcal{R} \wedge X \otimes \mathcal{R} \xrightarrow{\text{mult}} X \otimes \mathcal{R}. \tag{8}$$
- If $\mathcal{A}$ is a commutative $\mathcal{R}$-algebra, then the internal hom $\mathcal{X} \mapsto \mathcal{A}^{\mathcal{X}} = \hom_{\mathcal{R}}(\mathcal{X}_+ \wedge \mathcal{R}, \mathcal{A})$ is a cotensor (does not depend on $\mathcal{R}$). The unit of the adjunction
$$\alpha^{\mathcal{R}} : \mathcal{A} \to (\mathcal{X} \otimes^{\mathcal{R}} \mathcal{A})^{\mathcal{X}}$$

is a map of commutative $\mathscr{R}$-algebras. (Here $\mathscr{X} \otimes^{\mathscr{R}} \mathcal{A}$ is the tensor in the category of commutative $\mathscr{R}$-algebras of the motivic space $\mathscr{X}$ with $\mathcal{A}$.) In the category of $\mathscr{R}$-modules, the adjoint of $\alpha^{\mathscr{R}}$ takes the form

$$\sigma^{\mathscr{R}} : (\mathscr{X}_+ \wedge \mathscr{R}) \wedge_{\mathscr{R}} \mathcal{A} \xrightarrow{1\wedge\alpha^{\mathscr{R}}} (\mathscr{X}_+ \wedge \mathscr{R}) \wedge_{\mathscr{R}} (\mathscr{X} \otimes^{\mathscr{R}} \mathcal{A})^{\mathscr{X}} \xrightarrow{\mathrm{ev}} (\mathscr{X} \otimes^{\mathscr{R}} \mathcal{A}),$$

where ev is the evaluation. Composition gives an $\mathscr{R}$-algebra map $\nu : \mathscr{R}^{\mathscr{X}} \wedge_{\mathscr{R}} (\mathscr{X} \otimes^{\mathscr{R}} \mathcal{A}) \to (\mathscr{X} \otimes^{\mathscr{R}} \mathcal{A})^{\mathscr{X}}$.

Assume that $\mathscr{X} = X$ is a finite cell complex and that $\pi_\star(X_+ \wedge \mathscr{R})$ is a finitely generated free $\pi_\star\mathscr{R}$-module with basis $\mathcal{B}$. Then ν is an equivalence, and the composite (we identify $\pi_\star(\mathscr{R}^X \wedge_{\mathscr{R}} (X \otimes^{\mathscr{R}} \mathcal{A}))$ with the given target)

$$\nu_\star^{-1}\alpha_\star^{\mathscr{R}} : \pi_\star\mathcal{A} \xrightarrow{\alpha^{\mathscr{R}}} \pi_\star(X \otimes^{\mathscr{R}} \mathcal{A})^X \xrightarrow{\nu_\star^{-1}} \hom_{\pi_\star\mathscr{R}}(\pi_\star(X_+ \wedge \mathscr{R}), \pi_\star\mathscr{R}) \otimes_{\pi_\star\mathscr{R}} \pi_\star(X \otimes^{\mathscr{R}} \mathcal{A})$$

satisfies

$$\nu_\star^{-1}\alpha_\star(a) = \sum_{x\in\mathcal{B}} x^\vee \otimes \sigma_\star^{\mathscr{R}}(x \otimes a).$$

Here $x^\vee$ is the basis element dual to x and $x \otimes a \in \pi_\star(X_+ \wedge \mathscr{R}) \otimes_{\pi_\star\mathscr{R}} \pi_\star\mathcal{A} = \pi_\star((X_+ \wedge \mathscr{R}) \wedge_{\mathscr{R}} \mathcal{A})$. We will use this formula in Lemma 2.3 to get a relation in $\mathbf{MHH}(\mathbb{F}_p)$ (in the topological case, see [1, §5] for $X = S^1$ using the circle action).

In the category of commutative $\mathscr{R}$-algebras, note that $\mathscr{X} \otimes^{\mathscr{R}} (\mathscr{R} \wedge \mathscr{R}) \cong \mathscr{R} \wedge (\mathscr{X} \otimes \mathscr{R})$ is the tensor of $\mathscr{X}$ with $S^0 \otimes \mathscr{R} = \mathscr{R} \wedge \mathscr{R}$ with its left $\mathscr{R}$-algebra structure, and there is a commutative diagram

$$\begin{array}{ccc} (\mathscr{X}_+ \wedge \mathscr{R}) \wedge_{\mathscr{R}} (\mathscr{R} \wedge \mathscr{R}) & \xrightarrow{\sigma^{\mathscr{R}}} & \mathscr{X} \otimes^{\mathscr{R}} (\mathscr{R} \wedge \mathscr{R}) \\ \Vert\cong & & \Vert\cong \\ \mathscr{R} \wedge \mathscr{X}_+ \wedge \mathscr{R} & \xrightarrow{1\wedge\sigma^+} & \mathscr{R} \wedge \mathscr{X} \otimes \mathscr{R}, \end{array}$$

where the vertical isomorphisms are the associators.

2.1. **Comparison of simplicial models.** It will be convenient to make explicit some of the simplicial models and how they interact (see [1] for a homological version in the topological setting). In this subsection, let $I = \Delta[1]$ be the simplicial interval with boundary $S^0 = \partial\Delta[1]$ and let $S^1 = I \coprod_{S^0} *$ be the simplicial circle. The subdivision of the circle relevant to the comultiplication is $dS^1 = (I \coprod I) \coprod_{S^0 \coprod S^0} S^0$ with weak equivalence $dS^1 \to S^1$ given by sending the first interval to the base point. The pinch map $\nabla : dS^1 \to S^1 \vee S^1$ identifies the endpoints. It is sometimes convenient to write dS^1 as $* \coprod_{S^0}(I \times S^0) \coprod_{S^0} *$. Under the canonical isomorphism $\mathscr{R} = * \otimes \mathscr{R}$ we get an identification $S^1 \otimes \mathscr{R} = (I \otimes^{\mathscr{R}} \mathscr{R}) \wedge_{S^0 \otimes \mathscr{R}} \mathscr{R}$ which is a concrete model for the derived smash $\mathscr{R} \wedge^L_{\mathscr{R}\wedge\mathscr{R}} \mathscr{R}$ and

$$dS^1 \otimes \mathscr{R} = ((I \textstyle\coprod I) \otimes \mathscr{R}) \wedge_{(S^0 \coprod S^0)\otimes\mathscr{R}} S^0 \otimes \mathscr{R} \cong \mathscr{R} \wedge_{\mathscr{R}\wedge\mathscr{R}} I \otimes (\mathscr{R} \wedge \mathscr{R}) \wedge_{\mathscr{R}\wedge\mathscr{R}} \mathscr{R}.$$

Let $\mathscr{R} \to \mathcal{A}$ be a cofibration of cofibrant commutative motivic ring spectra. Let $X \otimes^{\mathscr{R}} \mathcal{A}$ be the tensor in the category of commutative $\mathscr{R}$-algebras of the space X (all smashes involved are over $\mathscr{R}$). If $\mathscr{M}$ and $\mathscr{N}$ are $\mathcal{A}$-modules, then the derived smash $\mathscr{M} \wedge^L_{\mathcal{A}} \mathscr{N}$ is conveniently modeled as $\mathscr{M} \wedge_{\mathcal{A}} (I \otimes^{\mathscr{R}} \mathcal{A}) \wedge_{\mathcal{A}} \mathscr{N}$, often referred to as the "two-sided bar construction over $\mathscr{R}$". Note that this does not depend on $\mathscr{R}$, in the sense that the map $\mathscr{M} \wedge_{\mathcal{A}} (I \otimes \mathcal{A}) \wedge_{\mathcal{A}} \mathscr{N} \to \mathscr{M} \wedge_{\mathcal{A}} (I \otimes^{\mathscr{R}} \mathcal{A}) \wedge_{\mathcal{A}} \mathscr{N}$ is an equivalence. In the special case $\mathcal{A} = S^0 \otimes \mathscr{R} = \mathscr{R} \wedge \mathscr{R}$ we get an identification between the tensor with the subdivided circle and the bar construction $S^1 \otimes \mathscr{R} \cong \mathscr{R} \wedge_{\mathcal{A}} (I \otimes^{\mathscr{R}} \mathcal{A}) \wedge_{\mathcal{A}} \mathscr{R}$ and $dS^1 \otimes \mathscr{R} \cong \mathscr{R} \wedge_{\mathcal{A}} (I \otimes \mathcal{A}) \wedge_{\mathcal{A}} \mathscr{R}$. If one wishes to write the comultiplication

$$\psi\colon S^1 \otimes \mathscr{R} \xleftarrow{\sim} dS^1 \otimes \mathscr{R} \xrightarrow{\nabla \otimes 1} (S^1 \vee S^1) \otimes \mathscr{R} \xrightarrow{\cong} (S^1 \otimes \mathscr{R}) \wedge_{\mathscr{R}} (S^1 \otimes \mathscr{R})$$

in terms of the bar construction, a concrete way is to use the equivalence $I \otimes \mathcal{A} \to \mathcal{A}$ and the augmentation $I \otimes \mathcal{A} \to \mathscr{R}$ as in the diagram

$$\begin{array}{ccc} \mathscr{R}\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}\mathscr{R} & \xleftarrow{\sim} & \mathscr{R}\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}\mathscr{R} \\ & & \downarrow \\ (\mathscr{R}\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}\mathscr{R}) \wedge_{\mathscr{R}} (\mathscr{R}\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}\mathscr{R}) & \xleftarrow{\cong} & \mathscr{R}\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}\mathscr{R}\wedge_{\mathcal{A}}(I \otimes \mathcal{A})\wedge_{\mathcal{A}}\mathscr{R}. \end{array}$$

This formula only uses the augmentation $\mathcal{A} \to \mathscr{R}$ and not specifically that $\mathcal{A} = \mathscr{R} \wedge \mathscr{R}$. One may replace the $\otimes$ by $\otimes^{\mathscr{R}}$ if convenient.

2.2. **Some classes coming from the dual motivic Steenrod algebra.** Let $\mathcal{A}_\star = \pi_\star(\mathbf{M}\mathbb{F}_p \wedge \mathbf{M}\mathbb{F}_p)$ be the dual motivic Steenrod algebra of our ground field F at p,

$$\mathcal{A}_\star = \begin{cases} \mathbb{M}_\star[\xi_i, \tau_i]_{i \geq 0}/(\tau_i^2 - \rho(\tau_{i+1} - \tau_0\xi_{i+1}) - \tau\xi_{i+1}) & p = 2 \\ \mathbb{M}_\star[\xi_i]_{i \geq 0} \otimes_{\mathbb{M}_\star} \Lambda_{\mathbb{M}_\star}(\tau_i)_{i \geq 0} & p \neq 2 \end{cases} \tag{9}$$

(where $\mathbb{M}_\star$ is the mod-p motivic homology ring of F; τ and ρ are discussed below), whose Hopf algebroid structure is given in [23, §5.1], [40, §5], [53, §12]. Our notation indicates that τ_i is an exterior class when $p \neq 2$. By convention we set $\xi_0 = 1$. The bidegrees of the generators in (9) are given by

$$|\xi_i| = (2p^i - 2, p^i - 1), \qquad |\tau_i| = (2p^i - 1, p^i - 1).$$

The coproducts of the generators are defined by

$$\psi(\xi_i) = \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \xi_j, \qquad \psi(\tau_i) = \tau_i \otimes 1 + \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \tau_j. \tag{10}$$

The left unit is the canonical inclusion. When $p = 2$, the right unit is determined by

$$\eta_R(\rho) = \rho, \qquad \eta_R(\tau) = \tau + \rho\tau_0$$

for the canonical classes $\tau \in \mathbb{M}_{0,-1} \cong \mu_2(F)$ and $\rho \in \mathbb{M}_{-1,-1} \cong F^\times/(F^\times)^2$. The mod 2 Bockstein on τ equals ρ. While τ is always nontrivial—being the class of $-1 \in \mu_2(F)$—we have $\rho = 0$ if $\sqrt{-1} \in F$. The graded mod-2 Milnor K-theory ring $k_*^M \subseteq \mathbb{M}_\star$ of the base field F is comprised of primitive elements. The element τ is not primitive in general. If F contains a primitive pth root of unity so that $\mathbb{M}_{0,-1} \cong \mathbb{Z}/p\{\tau\}$, then $\mathbb{M}_\star \cong k_*^M[\tau]$ by the norm residue isomorphism [52], [54]. We shall also use the antipodal generators

$$c(\tau_i) = -\tau_i - \sum_{j=0}^{i-1} \xi_{i-j}^{p^j} c(\tau_j), \qquad c(\xi_i) = -\xi_i - \sum_{j=1}^{i-1} \xi_{i-j}^{p^j} c(\xi_j) \tag{11}$$

detailed in [23, §5]. *For legibility, we will abuse notation by implicitly using the antipodal classes* (11) *in our computations.* Voevodsky defines in [51, §3.1] the mod-p rigid motivic Steenrod algebra

$$\mathcal{A}_\star^{\mathrm{rig}} := \bigotimes_{i\geq 0} \mathrm{S}_{\mathbb{F}_p}(\xi_{i+1}) \otimes \Lambda_{\mathbb{F}_p}(\tau_i). \tag{12}$$

The equation (10) gives the coproducts of the generators. For $p \neq 2$, this is the dual topological Steenrod algebra at p.

Remark 2.2. Suppose $\overline{F}$ is an algebraically closed field of positive characteristic $\neq p$. Its ring of Witt vectors $W(\overline{F})$ is a Henselian local ring with residue field $\overline{F}$. Let $\overline{K}$ denote an algebraic closure of the quotient field K of $W(\overline{F})$. We note that $\overline{K}$ has characteristic zero. The natural maps

$$\overline{K} \leftarrow W(\overline{F}) \rightarrow \overline{F}$$

induce isomorphisms on $\mathbb{M}_\star$ and $\mathcal{A}_\star$ according to [55, §4,5,6]. These algebra isomorphisms preserve the classes τ_i and ξ_i. Moreover, $\mathbb{M}_\star$ and $\mathcal{A}_\star$ are invariant under extensions of algebraically closed fields of characteristic zero.

The structure of the dual Steenrod algebra has some direct consequences for motivic Hochschild homology. Recall from (8) the suspension operation

$$\sigma\colon\ \mathbf{M}\mathbb{F}_p \wedge S^1_+ \wedge \mathbf{M}\mathbb{F}_p \longrightarrow \mathbf{M}\mathbb{F}_p \wedge \mathbf{MHH}(\mathbb{F}_p) \xrightarrow{\text{mult}} \mathbf{MHH}(\mathbb{F}_p). \tag{13}$$

We note the isomorphisms

$$\begin{aligned}\pi_\star(\mathbf{M}\mathbb{F}_p \wedge S^1_+ \wedge \mathbf{M}\mathbb{F}_p) &\cong \pi_\star((\mathbf{M}\mathbb{F}_p \wedge S^1_+) \wedge_{\mathbf{M}\mathbb{F}_p} (\mathbf{M}\mathbb{F}_p \wedge \mathbf{M}\mathbb{F}_p))\\ &\cong H_*(S^1_+;\mathbb{M}_\star) \otimes_{\mathbb{M}_\star} \mathcal{A}_\star\\ &\cong H_*(S^1_+;\mathbb{F}_p) \otimes \mathcal{A}_\star.\end{aligned}$$

If $s_1 \in H_1(S^1_+;\mathbb{F}_p)$ is the standard generator and $\zeta \in \mathcal{A}_{s,w}$, we let "$\sigma\zeta$" denote the "homology suspension" of ζ, namely the image of $s_1 \otimes \zeta$ in $\mathbf{MHH}_{s+1,w}(\mathbb{F}_p)$ under the composite in (13) and also in $\pi_{s+1,w}(\mathbf{M}\mathbb{F}_p \wedge \mathbf{MHH}(\mathbb{F}_p))$ under the first map in (13).

We now show two relations useful in the forthcoming spectral sequence calculations. Remark 2.7 generalizes our second formula to all base fields when $p = 2$.

Lemma 2.3. *In the motivic Hochschild homology* $\mathbf{MHH}_\star(\mathbb{F}_p)$ *of an algebraically closed field of exponential characteristic* $\neq p$*, we have the relations*

$$\tau^{p-1}\sigma\tau_{i+1} = (\sigma\tau_i)^p, \qquad \tau^{p-1}\sigma\xi_{i+1} = 0$$

for all $i \geq 0$.

Proof. Suppose that $\mathscr{R}$ is a commutative $\mathbf{M}\mathbb{F}_p$-algebra and let $\alpha\colon\ S^{s,w} \to \mathscr{R}$ represent a class in $\pi_{s,w}\mathscr{R}$. We write $E\Sigma_p$ for the nerve of the translation category of the symmetric group Σ_p on p letters.

The "power operation"

$$P(\alpha)\colon\ \pi_\star(\mathbf{M}\mathbb{F}_p \wedge ((E\Sigma_p)_+ \wedge_{\Sigma_p} (S^{s,w})^{\wedge p})) \to \pi_\star\mathscr{R}$$

is the homotopy of the $\mathbf{M}\mathbb{F}_p$-module map $\mathbf{M}\mathbb{F}_p \wedge ((E\Sigma_p)_+ \wedge_{\Sigma_p} (S^{s,w})^{\wedge p}) \to \mathscr{R}$ adjoint to the composite

$$E\Sigma_p \wedge_{\Sigma_p} (S^{s,w})^{\wedge p} \xrightarrow{\mathrm{id}\wedge\alpha^p} E\Sigma_p \wedge_{\Sigma_p} \mathscr{R}^{\wedge p} \xrightarrow{E\Sigma_p\to *} \mathscr{R}^{\wedge p}/\Sigma_p \xrightarrow{\text{mult}} \mathscr{R}.$$

Precomposing with the map

$$H_*(B\Sigma_p;\mathbb{F}_p)\otimes \pi_\star(\mathbf{M}\mathbb{F}_p\wedge S^{s,w})\to \pi_\star(\mathbf{M}\mathbb{F}_p\wedge((E\Sigma_p)_+\wedge_{\Sigma_p}(S^{s,w})^{\wedge p}))$$

defined on the chain level (via the monoidal Quillen equivalence between $\mathbf{M}\mathbb{F}_p$-modules and motives with mod-p coefficients, see [16], [23], [42], [43]) as the diagonal for the Suslin-Voevodsky motivic complex [33], [50]

$$C_*(E\Sigma_p,\mathbb{F}_p)/\Sigma_p\otimes \mathbb{Z}/p(w)[s]\to C_*(E\Sigma_p,\mathbb{F}_p)\otimes_{\Sigma_p}\mathbb{Z}/p(w)[s]^{\otimes p}$$

and evaluation at the classical choice of generator of $H_i(BC_p;\mathbb{F}_p)\subseteq H_i(\Sigma_p,\mathbb{F}_p)$ gives us the (topological) Dyer-Lashof operation $Q_i(\alpha)\in\pi_{i+ps,pw}\mathscr{R}$ on α. We do the usual shift to upper indexing with $Q^r(\alpha)\doteq Q_{(2r-s)(p-1)}(\alpha)$ (for p odd; $Q^r(\alpha)=Q_{r-s}(\alpha)$ for $p=2$) so that $Q^0(\alpha)=\alpha$ and $Q^s(\alpha)=\alpha^p$ when $2r=s$ (for p odd; $r=s$ for $p=2$). We refer to [29, §1.5] for a survey of Dyer-Lashof operations.

In Section 2.1 we set $\mathscr{R}=\mathbf{M}\mathbb{F}_p$ and $\mathcal{A}=\mathscr{R}\wedge\mathscr{R}$ and let X be any space with finite basis $\{x\}$ for the homology $H_*(X;\mathbb{F}_p)$. Recall that $\alpha^{\mathscr{R}}:\mathcal{A}\to(X\otimes^{\mathscr{R}}\mathcal{A})^X$ and $\nu:\mathscr{R}^X\wedge_{\mathscr{R}}(X\otimes^{\mathscr{R}}\mathcal{A})\to X\otimes^{\mathscr{R}}\mathcal{A}$ are maps of commutative $\mathscr{R}$-algebras. For appropriate r and s, we obtain

$$Q^r\nu_\star^{-1}\alpha_\star^{\mathscr{R}}=\nu_\star^{-1}\alpha_\star^{\mathscr{R}}Q^s$$

and so

$$\sum_x\sum_{a+b=s}Q^ax^\vee\otimes Q^b\sigma^{\mathscr{R}}(x\otimes a)=\sum_x x^\vee\otimes\sigma^{\mathscr{R}}(x\otimes Q^sa).$$

When $X=S^1$ the Dyer-Lashof operations $Q^a(x^\vee)$ are trivial for $a\neq 0$ and so we get

$$Q^s\sigma^{\mathscr{R}}(x\otimes a)=\sigma^{\mathscr{R}}(x\otimes Q^sa).$$

Restricting to the generator $x=s_1\in H_1(S^1;\mathbb{F}_p)$ and multiplying down to homotopy, we get the crucial formula

$$Q^s\sigma=\sigma Q^s:\ \pi_{s,w}(\mathbf{M}\mathbb{F}_p\wedge\mathbf{M}\mathbb{F}_p)\to\mathbf{MHH}_{s+1+2s(p-1),pw}(\mathbb{F}_p)$$

for p odd, and

$$Q^s\sigma=\sigma Q^s:\ \pi_{s,w}(\mathbf{M}\mathbb{F}_2\wedge\mathbf{M}\mathbb{F}_2)\to\mathbf{MHH}_{1+2s,2w}(\mathbb{F}_2).$$

By construction, the power operations are preserved under base change. Over any algebraically closed field, we claim there is a relation (due to Steinberger [10, III.2] in the topological setting)

$$\tau^{p-1}\tau_{i+1}=Q^{p^i}\tau_i\in\pi_{2p^{i+1}-1,p^{i+1}-p}(\mathbf{M}\mathbb{F}_p\wedge\mathbf{M}\mathbb{F}_p). \tag{14}$$

By Remark 2.2 and rigidity, it suffices to know that the relation holds over the complex numbers, which follows by Betti realization to the topological situation (the motivic correction factor τ^{p-1} ensures the weights agree). Thus, for the antipodal classes (11), we obtain the first formula

$$\tau^{p-1}\sigma\tau_{i+1}=\sigma Q^{p^i}\tau_i=Q^{p^i}\sigma\tau_i=(\sigma\tau_i)^p. \tag{15}$$

We use this result to prove the vanishing of $(1\wedge\sigma^+)_\star(s_1\otimes\tau^{p-1}\xi_{i+1})$ for the map $1\wedge\sigma^+:\ \mathbf{M}\mathbb{F}_p\wedge S^1_+\wedge\mathbf{M}\mathbb{F}_p\to\mathbf{M}\mathbb{F}_p\wedge\mathbf{MHH}(\mathbb{F}_p)$. This relation will be shown in homology from which the homotopy versions follow by the $\mathbf{M}\mathbb{F}_p$-algebra structure mult: $\mathbf{M}\mathbb{F}_p\wedge\mathbf{MHH}(\mathbb{F}_p)\to\mathbf{MHH}(\mathbb{F}_p)$ (splitting the inclusion of homotopy in homology). Let $\beta:\ \mathbf{M}\mathbb{F}_p\to S^1\wedge\mathbf{M}\mathbb{F}_p$ be the p-Bockstein, i.e., the $\mathbf{M}\mathbb{F}_p$-linear boundary map in the fiber sequence of Eilenberg-MacLane spectra associated with the short exact sequence $p\mathbb{Z}/p^2 \subseteq \mathbb{Z}/p^2 \to \mathbb{Z}/p$. For any commutative ring spectrum $\mathscr{R}$, the map $(\beta \wedge 1)_\star : \pi_\star(\mathbf{M}\mathbb{F}_p \wedge \mathscr{R}) \to \pi_\star(S^1 \wedge \mathbf{M}\mathbb{F}_p \wedge \mathscr{R})$ is a derivation (since $p\mathbb{Z}/p^2 \subseteq \mathbb{Z}/p^2$ is a square zero ideal) and as usual we allow ourselves the shorthand β for $(\beta \wedge 1)_*$. By construction of the motivic Steenrod algebra, see [23, §5], [53, §9], the generators in the dual motivic Steenrod algebra $\mathcal{A}_\star = \pi_\star(\mathbf{M}\mathbb{F}_p \wedge \mathbf{M}\mathbb{F}_p)$ are connected via

$$\xi_{i+1} = \beta\tau_{i+1}.$$

The diagram

$$\begin{array}{ccc} \mathbf{M}\mathbb{F}_p \wedge S^1_+ \wedge \mathbf{M}\mathbb{F}_p & \xrightarrow{1\wedge\sigma^+} & \mathbf{M}\mathbb{F}_p \wedge \mathbf{MHH}(\mathbb{F}_p) \\ \downarrow{\scriptstyle \beta\wedge 1} & & \downarrow{\scriptstyle \beta\wedge 1} \\ S^1 \wedge \mathbf{M}\mathbb{F}_p \wedge S^1_+ \wedge \mathbf{M}\mathbb{F}_p & \xrightarrow{1\wedge\sigma^+} & S^1 \wedge \mathbf{M}\mathbb{F}_p \wedge \mathbf{MHH}(\mathbb{F}_p) \end{array}$$

commutes, and since the power operations commute with $\sigma^{\mathbf{M}\mathbb{F}_p} = 1 \wedge \sigma^+$ we get for p odd (where $\beta\tau = 0$) that

$$\begin{aligned} 0 &= \mathrm{mult}\beta[\sigma_\star^{\mathbf{M}\mathbb{F}_p}(s_1 \otimes \tau_i)]^p = \mathrm{mult}\beta\sigma_\star^{\mathbf{M}\mathbb{F}_p}(s_1 \otimes \tau^{p-1}\tau_{i+1}) \\ &= \sigma\beta\tau^{p-1}\tau_{i+1} = \sigma\tau^{p-1}\xi_{i+1} = \tau^{p-1}\sigma\xi_{i+1}. \end{aligned}$$

For $p = 2$, we will see that the last formula follows directly from the d^1-differentials in the Tor-spectral sequence, but we may also use the Bockstein and compute ($\rho = 0$ since the base field is algebraically closed)

$$\begin{aligned} 0 &= \mathrm{mult}\beta[\sigma_\star^{\mathbf{M}\mathbb{F}_2}(s_1 \otimes \tau_i)]^2 = \mathrm{mult}\beta\sigma_\star^{\mathbf{M}\mathbb{F}_2}(s_1 \otimes \tau\tau_{i+1}) \\ &= \sigma(\beta(\tau\tau_{i+1})) = \sigma(\beta(\tau)\tau_{i+1} + \tau\beta(\tau_{i+1})) = \sigma(\rho\tau_{i+1} + \tau\xi_{i+1}) = \tau\sigma\xi_{i+1}. \end{aligned}$$

This finishes the proof. □

2.3. **Tor spectral sequence for motivic Hochschild homology.** A motivic spectrum is cellular if it belongs to the smallest full subcategory of the stable motivic homotopy category which is closed under homotopy colimits and contains the motivic spheres $S^{p,q}$ for all $p, q \in \mathbb{Z}$, see [12, §2.8]. The cellularity assumption is central in motivic homotopy theory, see, e.g., [46, §2.3]. It is, moreover, needed for running the motivic Tor spectral sequence (we refer to [18] for the topological setting).

We begin by relating the integral Tor spectral sequence to the bar construction. Our setup is a map of motivic ring spectra $\mathscr{R} \to \mathscr{Q}$ and an $\mathscr{Q}$-$\mathscr{R}$ bimodule $\mathscr{M}$. We assume that $\mathscr{R}$ is a commutative motivic ring spectrum and $\mathcal{A} = \mathscr{Q} \wedge \mathscr{R}$ is a cofibrant $\mathscr{R}$-algebra. If $\mathcal{A}$ is commutative and the modules $\mathscr{M}$ and $\mathscr{Q}$ are commutative $\mathcal{A}$-algebras, then the homotopy colimit of the simplicial object $\{[s] \mapsto \mathscr{M} \wedge_{\mathscr{R}} \mathcal{A}^{\wedge^s_{\mathscr{R}}} \wedge_{\mathscr{R}} \mathscr{Q}\}$ is isomorphic to $(\mathscr{M} \wedge_{\mathscr{R}} \mathscr{Q}) \otimes_{\mathcal{A}} (S^1 \otimes \mathcal{A})$ *in the category of commutative $\mathscr{R}$-algebras*. Moreover, the derived smash product $\mathscr{M} \wedge_{\mathcal{A}} \mathscr{Q}$ in (7) is the homotopy colimit of the diagram over the opposite simplex category Δ^{op} given by

$$[s] \mapsto \mathscr{M} \wedge_{\mathscr{R}} \mathcal{A}^{\wedge^s_{\mathscr{R}}} \wedge_{\mathscr{R}} \mathscr{Q}.$$

The skeletal filtration yields the E^1 page of the Tor spectral sequence, which—if $\pi_\star\mathcal{A}$ is flat over $\pi_\star\mathscr{R}$—takes the form

$$E^1_{s,\star} = B_s(\pi_\star\mathscr{M}, \pi_\star\mathcal{A}, \pi_\star\mathscr{Q}) = \pi_\star\mathscr{M} \otimes_{\pi_\star\mathscr{R}} \pi_\star\mathcal{A}^{\otimes^s_{\pi_\star\mathscr{R}}} \otimes_{\pi_\star\mathscr{R}} \pi_\star\mathscr{Q}.$$

It is conventional to denote the generators of the bar complex by $[m_0|a_1|\dots|a_s|n_{s+1}]$. When $\mathscr{M} = \mathscr{Q} = \mathscr{R}$ we abbreviate $[1|a_1|\dots|a_s|1]$ to $[a_1|\dots|a_s]$. The homology of $E^1_{s,\star}$ computes the E^2 page of the Tor spectral sequence (16). We recall the d^1 differential is given by the alternating sum of the face maps

$$d_i[m_0|a_1|\dots|a_s|n_{s+1}] = \begin{cases} [m_0 \cdot a_1|a_2|\dots|a_s|n_{s+1}] & i = 0 \\ [m_0|a_1|\dots|a_i \cdot a_{i+1}|\dots|a_s|n_{s+1}] & 0 < i < s \\ [m_0|a_1|\dots|a_{s-1}|a_s \cdot n_{s+1}] & i = s. \end{cases}$$

The isomorphim $\mathscr{M} \wedge_{\mathscr{R}} (\mathscr{Q} \wedge \mathscr{R})^{\wedge^s_{\mathscr{R}}} \wedge_{\mathscr{R}} \mathscr{Q} \cong (\mathscr{M} \wedge_{\mathscr{R}} \mathscr{Q}) \wedge \mathscr{Q}^{\wedge s}$ given by multiplication relates our description of the derived smash product to the Hochschild homology style description

$$[s] \mapsto (\mathscr{M} \wedge_{\mathscr{R}} \mathscr{Q}) \wedge \mathscr{Q}^{\wedge s}.$$

Proposition 2.4. *Let $\mathscr{R} \to \mathscr{Q}$ be a map of cellular motivic ring spectra, where $\mathscr{R}$ is commutative and $\mathcal{A} = \mathscr{Q} \wedge \mathscr{R}$ is a cofibrant $\mathscr{R}$-algebra, and let $\mathscr{M}$ be an $\mathscr{Q}$-$\mathscr{R}$ bimodule. Then the skeletal filtration on the simplicial circle gives rise to a strongly convergent trigraded Tor spectral sequence*

$$E^2_{h,t,w} = \mathbf{Tor}^{\pi_\star(\mathcal{A})}_{h,t,w}(\pi_\star\mathscr{M}, \pi_\star\mathscr{S}) \Rightarrow \mathbf{MHH}_{h+t,w}(\mathscr{R};\mathscr{M}). \tag{16}$$

Here, h is the homological grading on the torsion product and (t, w) is the internal grading for the bigraded motivic homotopy groups in topological degree t and weight w. The differentials are of the form

$$d^r : E^r_{*,*,*} \to E^r_{*-r,*+r-1,*}.$$

If $\mathscr{R}$ is commutative and $\mathscr{Q}$ and $\mathscr{M}$ are commutative $\mathscr{R}$-algebras, then the Tor spectral sequence is a spectral sequence of $\mathscr{R}_\star$-algebras with the multiplicative structure on the E^1 page given by the shuffle product introduced by Eilenberg-MacLane [15]. *The pinch map on the circle induces the Hopf-algebra structure on the torsion groups on the E^2 page. If $E^2, \dots, E^r$ are all flat over $\mathscr{R}_\star$ for $2 \leq r$, then the E^r page inherits an $\mathscr{R}_\star$-Hopf algebra structure; in particular, the r-th differential $d^r : E^r \to E^r$ satisfies the "co-Leibniz" rule in the sense that it commutes with the coproduct $\psi : E^r \to E^r \otimes E^r$.*

Proof. This follows from (7), [1, §4], [12, Proposition 7.7], [39, §2]. □

The suspension map $\sigma : \mathscr{R} \wedge S^1 \wedge \mathscr{R} \to S^1 \otimes \mathscr{R}$ has a simple interpretation under the isomorphism

$$S^1 \otimes \mathscr{R} \cong \{[s] \mapsto (\mathscr{R} \wedge_{\mathscr{R}} \mathscr{R}) \wedge (\mathscr{R} \wedge \mathscr{R})^{\wedge_{\mathscr{R}} s}\}.$$

It is the map from $S^1 \wedge (\mathscr{R} \wedge \mathscr{R}) = \{[s] \mapsto \bigvee_{\{1,\dots,s\}} (\mathscr{R} \wedge \mathscr{R})\}$ sending the ith summand to the inclusion on the ith factor (and units elsewhere). In particular, if $x \in \pi_{d,w}(\mathscr{R} \wedge \mathscr{R})$, then $\sigma_\star x \in \pi_{d+1,w}(S^1 \otimes \mathscr{R})$ is the class represented by $[x] \in E^1_{1,d,w}$.

The Hopkins-Morel equivalence shown by Hoyois [22, Proposition 8.1] implies the cellularity assumption in Proposition 2.4 holds for $\mathscr{M} = \mathscr{R} = \mathscr{S} = \mathbf{M}\mathbb{F}_p$ since the base scheme F is a field of exponential characteristic $\mathrm{e}(F) \neq p$. In this case, we have the Tor spectral sequence

$$E^2_{h,t,w} = \mathbf{Tor}^{\mathcal{A}_\star}_{h,t,w}(\mathbb{M}_\star, \mathbb{M}_\star) \Rightarrow \mathbf{MHH}_{h+t,w}(\mathbf{M}\mathbb{F}_p). \tag{17}$$

Remark 2.5. By Remark 2.2 and (17) it follows that, for algebraically closed fields, $\mathbf{MHH}_\star(\mathbf{M}\mathbb{F}_p)$ is independent of the exponential characteristic $\neq p$.

2.4. **Torsion products.** We shall repeatedly make use of some torsion product computations, see [39, §6], and Section 1.1 for our notation.

Lemma 2.6.

(i) *For the polynomial algebra* $\mathbb{F}_p[\mathsf{x}]$ *on a generator* x *in even degree* d*, there is an* $\mathbb{F}_p$*-bialgebra isomorphism*

$$\mathbf{Tor}_*^{\mathbb{F}_p[\mathsf{x}]}(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\sigma\mathsf{x}).$$

Here, $\sigma\mathsf{x}$ *in degree* $(1, d)$ *is a coalgebra primitive represented in the bar complex by* $[\mathsf{x}]$.

(ii) *For the exterior algebra* $\Lambda_{\mathbb{F}_p}(\mathsf{x})$ *on a generator* x *in odd degree* d*, there is an* $\mathbb{F}_p$*-bialgebra isomorphism*

$$\mathbf{Tor}_*^{\Lambda_{\mathbb{F}_p}(\mathsf{x})}(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\sigma\mathsf{x}).$$

Here, $\gamma_j\sigma\mathsf{x}$ *in degree* (j, dj) *is represented in the bar complex by* $[\mathsf{x}|\mathsf{x}|\dots|\mathsf{x}]$ *and has coproduct*

$$\psi(\gamma_k\sigma\mathsf{x}) = \sum_{i+j=k} \gamma_i\sigma\mathsf{x} \otimes \gamma_j\sigma\mathsf{x}.$$

Remark 2.7. As an example, let us give a direct proof of the relation $\tau\sigma\xi_{i+1} = \rho\sigma\tau_{i+1} \in \mathbf{MHH}_\star(\mathbb{F}_2)$ shown for algebraically closed fields in Lemma 2.3. Consider the E^1 page

$$E^1_{s,\star} = \mathcal{A}_\star^{\otimes_{\mathbb{M}_\star} s} \cong \mathbb{M}_\star \otimes_{\mathbb{M}_\star} \mathcal{A}_\star^{\otimes_{\mathbb{M}_\star} s} \otimes_{\mathbb{M}_\star} \mathbb{M}_\star$$

of the spectral sequence for $\mathbf{MHH}(\mathbb{F}_2)$, where $\mathcal{A}_\star = \pi_\star(\mathbf{M}\mathbb{F}_2 \wedge \mathbf{M}\mathbb{F}_2)$ is the dual Steenrod algebra. Then

$$d^1[\tau_i|\tau_i] = [\tau_i^2] = \tau[\xi_{i+1}] + \rho[\tau_{i+1}] + \rho[\tau_0\xi_{i+1}]$$

and $d^1[\tau_0|\xi_{i+1}] = [\tau_0\xi_{i+1}]$, so that $\tau[\xi_{i+1}] + \rho[\tau_{i+1}]$ is a boundary. Hence $\tau\sigma\xi_{i+1} = \rho\sigma\tau_{i+1}$.

With the notation $\gamma_j\sigma\tau_i = [\tau_i|\dots|\tau_i] \in E^1_{j,\star}$ and $\sigma x = [x]$ the shuffle product yields an explicit formula for the d^1 differentials

$$\begin{aligned} d^1\gamma_{j+2}\sigma\tau_i &= \sum_{a=1}^{j-1} [\tau_i|\dots|\tau_i^2|\dots|\tau_i] \\ &= [\tau_i^2] \cdot [\tau_i|\dots|\tau_i] \\ &= (\tau[\xi_{i+1}] + \rho[\tau_{i+1}] + \rho[\tau_0\xi_{i+1}]) \cdot [\tau_i|\dots|\tau_i] \\ &= (\tau\sigma\xi_{i+1} + \rho\sigma\tau_{i+1} + \rho\sigma(\tau_0\xi_{i+1})) \cdot \gamma_j\sigma\tau_i. \end{aligned}$$

When the ground field contains a square root of -1, so that $\rho = 0$, we get the formula

$$d^1\gamma_{j+2}\sigma\tau_i = \tau\sigma\xi_{i+1} \cdot \gamma_j\sigma\tau_i.$$

Conversely, for odd primes p, we can use Lemma 2.3 to *deduce* differentials by a simple weight count—simplifying the corresponding topological argument. Lemma 2.6 tells us that

$$E^2_{*,\star} = \mathbb{M}_\star \otimes_{\mathbb{F}_p} \bigotimes_{i\geq 0} \Lambda_{\mathbb{F}_p}(\sigma\xi_{i+1}) \otimes \Gamma_{\mathbb{F}_p}(\sigma\tau_i).$$

We know that $\tau^{p-1}\sigma\xi_{i+1}$ has to be hit by a differential. When the ground field is algebraically closed, $\mathbb{M}_\star = \mathbb{F}_p[\tau]$ with $\tau \in \mathbb{M}_{0,-1}$. In this case the source of the differential hitting $\tau^{p-1}\sigma\xi_{i+1}$ must come from linear combinations of monomials in $\sigma\xi_i$s and $\gamma_j\sigma\tau_i$s of total degree $2p^{i+1}$ and weight at least $p^{i+1}-p$. A quick count shows that the only monomial with sufficient weight is $\gamma_p\sigma\tau_i$, and so we have the relation (described up to a unit in $\mathbb{F}_p$)

$$d^{p-1}\gamma_p\sigma\tau_i \doteq \tau^{p-1}\sigma\xi_{i+1}.$$

2.5. **A Bockstein type complex.** We end the section by doing an entirely algebraic exercise that will be needed later. Let p be any prime and consider the commutative differential graded $\mathbb{F}_p$-algebra (C,∂), where

$$C = \bigotimes_{i\geq 0} \Gamma_{\mathbb{F}_p}(\bar{\mu}_i)\otimes \Lambda_{\mathbb{F}_p}(\bar{\lambda}_{i+1})$$

and $\partial\colon C\to C$ is the derivation generated by $\partial(\gamma_{j+p}\bar{\mu}_i) = \bar{\lambda}_{i+1}\gamma_j\bar{\mu}_i$ and $\partial(\bar{\lambda}_{i+1}) = 0$. Set $B^\partial = \operatorname{im}\partial$, $Z^\partial = \ker\partial$ and $H^\partial = Z^\partial/B^\partial$—this subsection aims to calculate these. In our application C will be the mod-τ^{p-1} motivic Hochschild homology of $\mathbb{F}_p$ (the reader may recognize it as $\operatorname{Tor}_*^{\mathcal{A}_\star^{\mathrm{rig}}}(\mathbb{F}_p,\mathbb{F}_p)$) and ∂ will be derived from a Bockstein.

We first fix some notation. For each non-empty finite set of natural numbers $S\subseteq\mathbb{N}$, we choose an element $t_S\in S$ with the property that $t_{S\cup T}\in\{t_S,t_T\}$. The minimum, $t_S=\min S$, is a good choice, but many others exist. Down the road, such a choice amounts to a particular choice of basis, and there is no reason to prefer one over the other, except that in concrete examples, some can be more convenient. If the function $f\colon\mathbb{N}\circlearrowleft$ has finite non-empty support, $\operatorname{supp}f=\{n\in\mathbb{N}\,|\,f(n)\neq 0\}$, we write $t_f = t_{\operatorname{supp}f}$. For every $j\in\mathbb{N}$, let $\delta_j\colon\mathbb{N}\circlearrowleft$ be the function with $\operatorname{supp}\delta_j=\{j\}$ and $\delta_j(j)=1$.

Definition 2.8. Let J denote the set of pairs (S,f), where the function $f\colon\mathbb{N}\circlearrowleft$ has finite support and $S\subseteq\operatorname{supp}f$. The subset $K\subseteq J$ consists of the pairs (S,f), where the support of f is non-empty and S does not contain t_f.

Definition 2.9. For $(S,f)\in J$, we set

$$\chi_{S,f} \coloneqq \left(\prod_{m\in S}\bar{\lambda}_{m+1}\gamma_{pf(m)-p}\bar{\mu}_m\right)\left(\prod_{n\notin S}\gamma_{pf(n)}\bar{\mu}_n\right)\in C. \tag{18}$$

In particular, $\chi_{\emptyset,0}=1$, $\chi_{\emptyset,p^j\delta_n}=\gamma_{p^{j+1}}\bar{\mu}_n$ and $\chi_{\{m\},\delta_m}=\lambda_{m+1}$. We note that

$$\partial\chi_{S,f} = \sum_{n\in\operatorname{supp}(f)-S}\chi_{S\cup\{n\},f}$$

since $\partial\gamma_n\bar{\mu}_i = \bar{\lambda}_{i+1}\gamma_{n-p}\bar{\mu}_i$. Next, we construct sub-complexes of (C,∂).

Definition 2.10. If $f\colon\mathbb{N}\circlearrowleft$ has finite support, the *associated f-cube* is the sub-complex

$$(C^f,\partial)\coloneqq\left(\bigoplus_{S\subseteq\operatorname{supp}f}\mathbb{F}_p\{\chi_{S,f}\},\partial\right)\subseteq(C,\partial).$$

If $f=0$, then $C^f=\mathbb{F}_p$. Furthermore, let $Z^f=\ker\partial\cap C^f$, $B^f=\operatorname{im}\partial\cap C^f$ and $H^f=Z^f/B^f$.

Note that if $f=0$, then $Z^f=H^f=\mathbb{F}_p$. Recall the number $t_f\in\operatorname{supp}f$ chosen once and for all (whenever $\operatorname{supp}f$ is non-empty) just before Definition 2.8.

Lemma 2.11. *If* $f : \mathbb{N} \circlearrowleft$ *has finite non-empty support, then* (C^f, ∂) *is contractible so that* $H^f = 0$*. Furthermore,* $B^f = Z^f$ *is generated by the* $\partial\chi_{S,f}$ *with* $t_f \notin S \subseteq \operatorname{supp} f$*.*

Proof. For $N = |\operatorname{supp} f|$ and $k = 0, \ldots, N$, let $C^f_k \subseteq C^f$ be the span of the $\chi_{S,f}$ with $k = |S|$. From the formula $\partial\chi_{S,f} = \sum_{n \in \operatorname{supp}(f)-S} \chi_{S\cup\{n\},f}$ we see that the differential restricts to a chain complex

$$(C^f, \partial) = \left\{ C^f_0 \xrightarrow{\partial} C^f_1 \xrightarrow{\partial} \ldots \xrightarrow{\partial} C^f_{N-1} \xrightarrow{\partial} C^f_N \right\}.$$

Here the $\mathbb{F}_p$-vector space C^f_k is of dimension $\binom{N}{k}$ with basis elements $\chi_{S,f}$, where $|S| = k$. Note that the set

$$\{\partial\chi_{S,f} \mid k = |S|, t_f \notin S\}$$

is linearly independent because only $\partial\chi_{S,f}$ has a nontrivial $\chi_{\{t_f\}\cup S,f}$-coefficient. Hence the rank of $\partial : C^f_k \to C^f_{k+1}$ is at least $\binom{N-1}{k}$, and we deduce that

$$\begin{aligned} \dim H_k(C^f, \partial) &= \dim \ker\{\partial : C^f_k \to C^f_{k+1}\} - \dim \operatorname{im}\{\partial : C^f_{k-1} \to C^f_k\} \\ &\leq \binom{N}{k} - \binom{N-1}{k} - \binom{N-1}{k-1} = 0 \end{aligned}$$

and so $B^f = Z^f$ is generated by the $\partial\chi_{S,f}$ with $t_f \notin S \subseteq \operatorname{supp} f$, as claimed. The calculation works when $k = 0$ or $k = N$ (but not for $N = 0$ since then we cannot choose t_f). □

We analyze the multiplicative structure.

Definition 2.12. For functions $f, g : \mathbb{N} \circlearrowleft$ with finite support and non-empty finite sets $S, T \subseteq \mathbb{N}$ define $K_{S,T,f,g} \in \mathbb{F}_p$ by

$$K_{S,T,f,g} = \left(\prod_{s\in S} \binom{f(s) - 1 + g(s)}{f(s) - 1} \right) \left(\prod_{t\in T} \binom{f(t) + g(t) - 1}{f(t)} \right) \left(\prod_{c \notin S\cup T} \binom{f(c) + g(c)}{f(c)} \right)$$

if $(S, f), (T, g) \in J$ and $S \cap T = \emptyset$, and set $K_{S,T,f,g} = 0$ otherwise. Moreover, we define

$$\epsilon_{u,S,T,f,g} = K_{S\cup\{u\},T\cup\{t_{f+g}\},f+g} + K_{S\cup\{t_{f+g}\},T\cup\{u\},f+g}.$$

Note that when $(S, f), (T, g) \in J$ and $S \cap T = \emptyset$, each factor in the formula $K_{S,T,f,g}$ is 1 unless the index is in $\operatorname{supp} f \cap \operatorname{supp} g = \operatorname{supp}(f \cdot g)$, and so we can restrict to these factors to simplify the calculation. We will need $\epsilon_{u,S,T,f,g}$ only in the case when $u \in \operatorname{supp}(f + g)$, $u \neq t_{f+g}$ and $u \notin S \cup T$.

Lemma 2.13, a consequence of the defining relations among divided power generators of C, explains the relevance of these numbers.

Lemma 2.13. *For* $(S, f), (T, g) \in J$ *we have*

$$\chi_{S,f}\chi_{T,g} = K_{S,T,f,g} \cdot \chi_{S\cup T,f+g}$$

and if $(S, f), (T, g) \in K$*, then*

$$\partial\chi_{S,f} \cdot \partial\chi_{T,g} = \sum_{t_{f+g} \neq u \in \operatorname{supp}(f+g)-S\cup T} \epsilon_u \cdot \partial\chi_{S\cup T\cup\{u\},f+g}.$$

□

Lemma 2.14. *The multiplication gives an extra grading indexed by the generators of the commutative differential graded sub-algebra $\bigoplus_f C^f \subseteq C$. In particular, if $f, g \colon \mathbb{N} \circlearrowleft$ there is a commutative diagram*

$$\begin{array}{ccc} C^f \otimes C^g & \longrightarrow & C \otimes C \\ \downarrow & & \downarrow \\ C^{f+g} & \longrightarrow & C. \end{array}$$

Here the rows are given by the evident inclusion and the columns by multiplication. The resulting algebra inclusions

$$\bigoplus_{f \colon \mathbb{N}\circlearrowleft} C^f \subseteq C$$

and

$$\mathbb{F}_p[\bar{\mu}_i]/\bar{\mu}_i^p \subseteq C$$

induce isomorphisms of graded commutative $\mathbb{F}_p$-algebras

$$\bigoplus_{f \colon \mathbb{N}\circlearrowleft} C^f \cong \bigotimes_{i \geq 0} \left(\Lambda_{\mathbb{F}_p}(\bar{\lambda}_{i+1}) \otimes \bigotimes_{j>0} \mathbb{F}_p[\gamma_{p^j}\bar{\mu}_i]/(\gamma_{p^j}\bar{\mu}_i)^p \right)$$

and

$$C \cong \left(\bigotimes_{i \geq 0} \mathbb{F}_p[\bar{\mu}_i]/\bar{\mu}_i^p \right) \otimes \left(\bigoplus_{f \colon \mathbb{N}\circlearrowleft} C^f \right).$$

Proof. The multiplicative structure follows from Lemma 2.13, and the last two isomorphisms follow from the fact that a monomial in C does not have any $\bar{\mu}_i$-factors of the form $\chi_{S,f}$. ☐

Corollary 2.15. *As an $\mathbb{F}_p$-algebra,*

$$H^\partial \cong \bigotimes_{i \geq 0} \mathbb{F}_p[\bar{\mu}_i]/\bar{\mu}_i^p$$

and Z^∂ is the subalgebra of C generated by the $\bar{\mu}_i$ with $i \geq 0$ and the $\partial\chi_{S,f}$ with $t_f \notin S \subseteq$ supp$f$. *More explicitly, and writing $\mathrm{x}_{S,f} = \partial\chi_{S,f}$, the relation expressed in Lemma* 2.13 *gives an isomorphism*

$$Z^\partial \cong \mathbb{F}_p[\bar{\mu}_i, \mathrm{x}_{S,f}]_{i \in \mathbb{N}, (S,f) \in K}/(\bar{\mu}_i^p, \mathrm{x}_{S,f} \cdot \mathrm{x}_{T,g} - \sum_u \epsilon_u \cdot \mathrm{x}_{S \cup T \cup \{u\}, f+g}).$$

Here, $u \in$ supp$(f+g) - S \cup T$ *and $u \neq t_{f+g}$.*

3. Motivic Hochschild homology over algebraically closed fields

In this section, we work over an algebraically closed field F of exponential characteristic $\mathrm{e}(F) \neq p$. Then $\rho = 0$ since every unit is a square, and

$$\mathbb{M}_\star \cong k_*^M[\tau] \cong \mathbb{F}_p[\tau] \tag{19}$$

by [48, Corollary 4.3, p.254], where $|\tau| = (0, -1)$. From (9) and (19) we deduce

$$\mathcal{A}_\star = \begin{cases} \mathbb{F}_p[\tau, \xi_{i+1}, \tau_i]_{i \geq 0}/(\tau_i^2 - \tau\xi_{i+1}) & p = 2 \\ \mathbb{F}_p[\tau, \xi_{i+1}]_{i \geq 0} \otimes \Lambda_{\mathbb{F}_p}(\tau_i)_{i \geq 0} & p \neq 2. \end{cases} \tag{20}$$

If $p = 2$ and we invert τ in $\mathcal{A}_\star$, then ξ_i is no longer needed as a generator because $\xi_{i+1} = \tau^{-1}\tau_i^2$:

$$\pi_\star((\mathbf{M}\mathbb{F}_p \wedge \mathbf{M}\mathbb{F}_p)[\tau^{-1}]) \cong \mathcal{A}_\star[\tau^{-1}] = \begin{cases} \mathbb{F}_p[\tau^{\pm 1}, \tau_i]_{i\geq 0} & p = 2 \\ \mathbb{F}_p[\tau^{\pm 1}, \xi_{i+1}]_{i\geq 0} \otimes \Lambda_{\mathbb{F}_p}(\tau_i)_{i\geq 0} & p \neq 2. \end{cases} \tag{21}$$

Likewise, since $\mathcal{A}_\star$ is free as an $\mathbb{F}_p[\tau]$-module, taking the quotient by τ^{p-1} (for any prime p) gives an isomorphism of Hopf algebras

$$\pi_\star((\mathbf{M}\mathbb{F}_p \wedge \mathbf{M}\mathbb{F}_p)/\tau^{p-1}) \cong \mathcal{A}_\star/\tau^{p-1} = (\bigotimes_{i\geq 0} \mathbb{F}_p[\xi_{i+1}] \otimes \Lambda_{\mathbb{F}_p}(\tau_i)) \otimes \mathbb{F}_{p,\tau}. \tag{22}$$

Here $\mathbb{F}_{p,\tau}$ is shorthand for $\mathbb{F}_p[\tau]/\tau^{p-1}$. In Section 3.2, we use (22) to compute the coefficients of the mod τ^{p-1} reduction of $\mathbf{MHH}(\mathbb{F}_p)$.

3.1. **Étale motivic Hochschild homology.** We refer to [3], [17] for τ-self maps and applications towards étale hyperdescent for motivic spectra. Suppose $\mathscr{R}/p$ is a motivic E_∞ ring spectrum defined over an algebraically closed field. Then the canonical map

$$\mathscr{R}/p \to \mathscr{R}/p[\tau^{-1}] \tag{23}$$

exhibits the τ-periodization as a motivic E_∞ ring spectrum under $\mathscr{R}/p$; see [4, §12], [17, §8] for recent expositions. If $\mathscr{R}$ happens to be cellular, then so is $\mathscr{R}/p[\tau^{-1}]$. Owing to [3, Theorem 1.2], (23) is an étale localization (the ρ-completion in [3] is obsolete over algebraically closed fields, and for $p \neq 2$ the étale localization involves only the "+"-part of $\mathscr{R}/p$). We note that (23) induces an isomorphism on τ-inverted homotopy groups.

At all primes, the τ-periodic mod-p motivic Steenrod algebra agrees with the tensor product of the topological mod-p Steenrod algebra with the Laurent polynomial ring $\mathbb{F}_p[\tau^{\pm 1}]$. This observation implies that after p-completion the τ-periodic motivic stable homotopy groups are isomorphic to the classical stable homotopy groups with $\tau^{\pm 1}$ adjoined [25], [27, §4]. In this section, we prove a similar statement for motivic and topological Hochschild homology.

We calculate $\mathbf{MHH}_\star(\mathbb{F}_p)[\tau^{-1}] \cong \pi_\star(\mathbf{MHH}(\mathbb{F}_p)[\tau^{-1}])$ directly by the Tor spectral sequence, using the relations and differentials from Lemma 2.3 and Remark 2.7 and by appealing to (21) and the naturally induced equivalence of motivic spectra

$$\mathbf{MHH}(\mathbb{F}_p)[\tau^{-1}] = (\mathbf{M}\mathbb{F}_p \wedge_{\mathbf{M}\mathbb{F}_p \wedge \mathbf{M}\mathbb{F}_p} \mathbf{M}\mathbb{F}_p)[\tau^{-1}] \xrightarrow{\simeq} \mathbf{M}\mathbb{F}_p[\tau^{-1}] \wedge_{(\mathbf{M}\mathbb{F}_p \wedge \mathbf{M}\mathbb{F}_p)[\tau^{-1}]} \mathbf{M}\mathbb{F}_p[\tau^{-1}]. \tag{24}$$

Our calculation uses the classes

$$\mu_i := \sigma\tau_i,\ \lambda_i := \sigma\xi_i.$$

Lemma 3.1. *The Tor spectral sequence of* $\mathbf{MHH}(\mathbb{F}_p)[\tau^{-1}]$ *collapses at the* E^p *page and*

$$E^\infty = (\bigotimes_{i\geq 0} \mathbb{F}_p[\mu_i]/\mu_i^p)[\tau^{\pm 1}].$$

For p odd the only nonzero differentials d^r for $r > 1$ are generated by

$$d^{p-1}(\gamma_j\mu_i) \doteq \tau^{p-1}\lambda_{i+1}\gamma_{j-p}\mu_i \tag{25}$$

for all $i \geq 0$, $j \geq p$.

Proof. Lemma 2.6, Equation (21), and Equation (24) yield the E^2 page. When $p = 2$, we have

$$E^2 = (\bigotimes_{i\geq 0} \Lambda_{\mathbb{F}_2}(\mu_i))[\tau^{\pm 1}]. \tag{26}$$

Since all the μ_is have filtration degree 1, there are no nontrivial differentials, and we conclude that $E^\infty = E^2$. When p is odd, the E^2 page takes the form

$$E^2 = (\bigotimes_{i\geq 0} \Gamma_{\mathbb{F}_p}(\mu_i) \otimes \Lambda_{\mathbb{F}_p}(\lambda_{i+1}))[\tau^{\pm 1}]. \tag{27}$$

The Tor spectral sequence starts as an augmented unital $\mathbb{M}_\star[\tau^{-1}]$-Hopf algebra since (27) is flat over $\mathbb{M}_\star[\tau^{-1}]$. Arguing as in [1, §4], [2, §5], [20, §1.2], [34], [39], we'll see that the nontrivial differentials are as claimed. More precisely, since the shortest differential in the lowest total degree must go from an algebra generator (these lie in filtration powers of p) to a coalgebra primitive (these lie in filtration 1), the differentials d^r for $1 < r < p-1$ are all zero. Recall from Remark 2.7 that we established the said differential for $j = p$ integrally: $d^{p-1}(\gamma_j\mu_i) \doteq \tau^{p-1}\lambda_{i+1}$ and we move from there by induction on $j \geq p$ and the coalgebra structure in Lemma 2.6; this is, for $k \geq 0$, the calculation

$$\begin{aligned}
&\psi(d^{p-1}(\gamma_{p+k}\mu_i)) - \psi(\tau^{p-1}\lambda_{i+1}\gamma_k\mu_i)\\
&\quad= (d^{p-1} \otimes 1 + 1 \otimes d^{p-1})\psi(\gamma_{p+k}\mu_i)\\
&\qquad - \tau^{p-1}(\lambda_{i+1} \otimes 1 + 1 \otimes \lambda_{i+1})(\Sigma_{a+b=k}\gamma_a\mu_i \otimes \gamma_b\mu_i)\\
&\quad= (d^{p-1} \otimes 1)(\gamma_{p+k}\mu_i \otimes 1) + (1 \otimes d^{p-1})(1 \otimes \gamma_{p+k}\mu_i)\\
&\qquad + \tau^{p-1} \sum_{a+b=p+k;a,b>0} (\lambda_{i+1}\gamma_{a-p}\mu_i \otimes \gamma_b\mu_i + \gamma_a\mu_i \otimes \lambda_{i+1}\gamma_{b-p}\mu_i)\\
&\qquad - \tau^{p-1} \sum_{a+b=k} (\lambda_{i+1}\gamma_a\mu_i \otimes \gamma_b\mu_i + \gamma_a\mu_i \otimes \lambda_{i+1}\gamma_b\mu_i)\\
&\quad= (d^{p-1} \otimes 1)(\gamma_{p+k}\mu_i \otimes 1) + (1 \otimes d^{p-1})(1 \otimes \gamma_{p+k}\mu_i)
\end{aligned}$$

shows the difference $d^{p-1}(\gamma_{p+k}\mu_i) - \tau^{p-1}\lambda_{i+1}\gamma_k\mu_i$ is a coalgebra primitive; however, 0 is the only such element in the given degree. The remaining algebra generators on the E^p page are in filtration degree ≤ 1, and hence $E^\infty = E^p$. □

Remark 3.2. Alternatively, an appeal to rigidity for extensions of algebraically closed fields as in Remark 2.2 or [44] (in characteristic zero) reduces to considering complex numbers. Over $\mathbb{C}$, the differential (25) is forced by Bökstedt's differential $d^{p-1}(\gamma_j\mu_i) = \lambda_{i+1}\gamma_{j-p}\mu_i$ in the Tor spectral sequence for $\mathbf{THH}_*(\mathbb{F}_p)$. In the motivic case, the correction term τ^{p-1} ensures agreement of the weights.

Theorem 3.3. *There are isomorphisms*

$$\begin{aligned}
\mathbf{MHH}_\star(\mathbb{F}_p)[\tau^{-1}] &\cong \mathbb{F}_p[\tau^{\pm 1}, \mu_i]_{i\geq 0}/(\mu_i^p - \tau^{p-1}\mu_{i+1})\\
&\cong \mathbb{F}_p[\mu, \tau^{\pm 1}]\\
&\cong \mathbf{THH}_*(\mathbb{F}_p)[\tau^{\pm 1}].
\end{aligned}$$

The generator μ has bidegree $(2, 0)$.

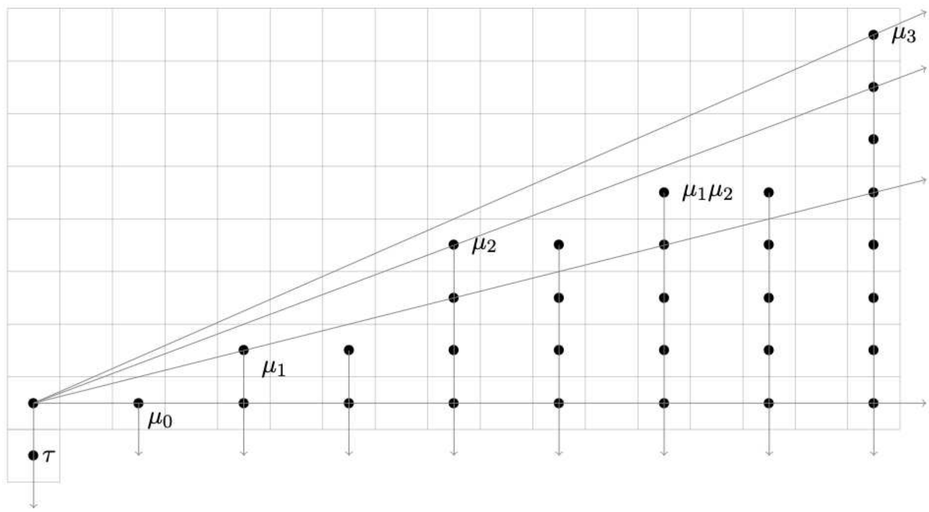

FIGURE 1. The étale motivic Hochschild homology of $\mathbb{F}_2$. The vertical lines indicate τ-multiplication, while the horizontal and diagonal lines depict powers of μ_i, $i = 0, 1, 2, 3$.

Proof. (24) shows the E^∞ page for $\mathbf{MHH}(\mathbb{F}_p)[\tau^{-1}]$ is the Laurent polynomials in τ of the E^∞ page for $\mathbf{THH}(\mathbb{F}_p)$. The result now follows from Lemma 3.1 and the multiplicative extension

$$\mu_i^p = \tau^{p-1}\mu_{i+1} \tag{28}$$

of Lemma 2.3. □

Hence all the classes $\mu_i \in \mathbf{MHH}(\mathbb{F}_p)$ are nontrivial and we may identify the τ-free part in $\mathbf{MHH}_\star(\mathbb{F}_p)$ with

$$\mathbb{F}_p[\tau, \mu_i]_{i\geq 0}/(\mu_i^p - \tau^{p-1}\mu_{i+1}). \tag{29}$$

This is depicted graphically for $p = 2$ and $p = 3$ in Figure 1 and Figure 2, respectively.

3.2. **Reduced motivic Hochschild homology.** To proceed to the next step in our strategy for calculating $\mathbf{MHH}(\mathbb{F}_p)$ over an algebraically closed field F with $\mathrm{e}(F) \neq p$, we form the cofiber of τ^n (for our calculations, it suffices to consider $n = p - 1$)

$$\Sigma^{0,n}\mathbf{M}\mathbb{F}_p \xrightarrow{\tau^n} \mathbf{M}\mathbb{F}_p \to \mathbf{M}\mathbb{F}_p/\tau^n. \tag{30}$$

We thank Markus Spitzweck for informing us that $\mathbf{M}\mathbb{F}_p/\tau^n$ is a motivic E_∞ ring spectrum for all $n \geq 1$. His argument goes as follows: $\mathbf{M}\mathbb{Z}$ is strongly periodizable and thus the mod-p homology ring $\mathbb{M}_\star$ is the homology of an E_∞ ring spectrum $C_\star\mathbb{F}_p$ in graded complexes of $\mathbb{F}_p$-vector spaces, see [47, Appendix C, Corollary C.3] for details. As noted below, $C_\star\mathbb{F}_p$ is formal, i.e., equivalent as an E_∞ ring spectrum in graded complexes of $\mathbb{F}_p$-vector spaces to the (bigraded) homology $\mathbb{M}_\star$ equipped with trivial differentials, $C_\star\mathbb{F}_p/\tau^n$ is E_∞ over $\mathbb{M}_\star$ for all $n \geq 1$. This implies the corresponding claim for $\mathbf{M}\mathbb{F}_p/\tau^n$. To show formality, consider the free E_∞ algebra $\mathcal{E}$ in graded complexes of $\mathbb{F}_p$-vector

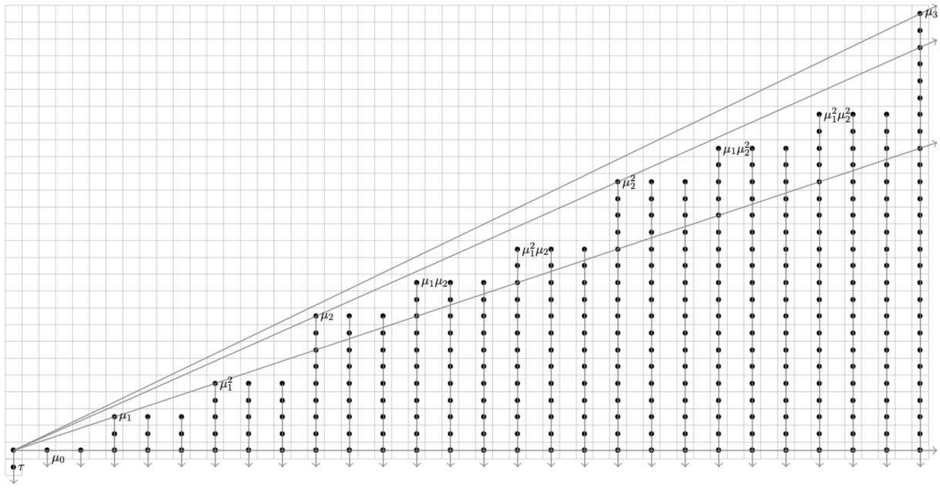

FIGURE 2. The étale motivic Hochschild homology of $\mathbb{F}_3$ depicted in the same graphical style as Figure 1.

spaces on a generator τ in bidegree $(0,1)$. Its 0-truncation, with respect to the natural t-structure on the derived category of graded abelian groups, is the formal model $\mathbb{F}_p[\tau]$. Thus $C_\star\mathbb{F}_p$ and $\mathbb{F}_p[\tau]$ are equivalent since the natural map $\mathcal{E} \to C_\star\mathbb{F}_p$ is the 0-truncation. When $n = 1$, we also refer to Gheorghe [19] for the fact that $\mathbf{M}\mathbb{F}_2 \to \mathbf{M}\mathbb{F}_2/\tau$ is a map of motivic E_∞ ring spectra.

Inserting $\mathbf{M}\mathbb{F}_p/\tau^n$ into (1) yields the derived smash product

$$\mathbf{MHH}(\mathbb{F}_p)/\tau^n \simeq \mathbf{M}\mathbb{F}_p/\tau^n \wedge_{(\mathbf{M}\mathbb{F}_p\wedge\mathbf{M}\mathbb{F}_p)/\tau^n} \mathbf{M}\mathbb{F}_p/\tau^n. \tag{31}$$

Owing to (30) and cellularity of $\mathbf{M}\mathbb{F}_p$, see Section 2.3, it follows that $\mathbf{M}\mathbb{F}_p/\tau^n$ is cellular. Thus (31) gives rise to the Tor spectral sequence

$$E^2_{h,t,w} = \mathbf{Tor}^{\mathcal{A}_\star/\tau^n}_{h,t,w}(\mathbb{M}_\star/\tau^n, \mathbb{M}_\star/\tau^n) \Rightarrow \mathbf{MHH}_{h+t,w}(\mathbb{F}_p)/\tau^n. \tag{32}$$

Recall that $\mathbb{F}_{p,\tau}$ is shorthand for $\mathbb{F}_p[\tau]/\tau^{p-1}$. Lemma 2.6 and (22) imply the Tor spectral sequence (16) for $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$ takes the form

$$\bar{E}^2_{*,\star} \cong \left(\bigotimes_{i\geq 0} \Gamma_{\mathbb{F}_p}(\sigma\tau_i) \otimes \Lambda_{\mathbb{F}_p}(\sigma\xi_{i+1})\right) \otimes \mathbb{F}_{p,\tau} \Rightarrow \mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}. \tag{33}$$

This is a first quadrant spectral sequence; the horizontal direction is the "filtration", the vertical direction is the "degree", and every term is graded by "weight." Recall that if x has filtration f_x, degree d_x and weight w_x, we write $|x| = (f_x, d_x; w_x)$ so that the differentials take the form

$$d^r : \bar{E}^r_{f,d;w} \to \bar{E}^r_{f-r,d+r-1;w}.$$

In (33), we set $\bar{\mu}_i := \sigma\tau_i$ and $\bar{\lambda}_{i+1} := \sigma\xi_{i+1}$. The bar signifies that the generators are mod-τ^{p-1} classes and should not be confused with the conjugate classes. For these classes, we note the degrees

(1) $|\bar{\lambda}_{i+1}| = (1, 2p^{i+1} - 2; p^{i+1} - 1)$,
(2) $|\gamma_{p^j}\bar{\mu}_i| = (p^j, 2p^{i+j} - p^j; p^{i+j} - p^j)$.

Thus for $x = \bar{\lambda}_{i+1}$ and $x = \gamma_{p^j}\bar{\mu}_i$ we have the congruence $w_x \equiv 0 \bmod p-1$. Hence if $x \in \bar{E}^2_{*,\star}$ in (33) has weight $w_x = -n + (p-1)m$, $0 \leq n < p-1$, then n equals x's τ-multiplicity. Another helpful bookkeeping device for our calculation is the Chow degree of x, see [5, Definition 3.1] and [26, Definition 2.1.10] for related terminology, defined by

$$c(x) = f_x + 2w_x - d_x.$$

In particular, we have

(1) $c(\bar{\lambda}_{i+1}) = 1 + 2(p^{i+1} - 1) - (2p^{i+1} - 2) = 1$
(2) $c(\gamma_{p^j}\bar{\mu}_i) = p^j + 2(p^{i+j} - p^j) - (2p^{i+j} - p^j) = 0.$

Every homogeneous class $x \in \bar{E}^2_{*,\star}$ in (33) is a monomial in the generators $\bar{\lambda}_{i+1}$ and $\gamma_{p^j}\bar{\mu}_i$. The Chow degree $c(x)$ records the number of λ_{i+1} classes in x, and the equality $0 \leq c(x) \leq f_x$ follows from the definition.

Lemma 3.4. *The Tor spectral sequence* (33) *for* $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$ *collapses at its* E^2 *page.*

Proof. For $r \geq 2$ and $x \in E^r_{*,\star}$ we note the equality of weights $w_x = w_{d^r x}$. If $x = \tau$, then $d^r\tau = 0$ since (33) is an $\mathbb{F}_p[\tau]$-algebra spectral sequence. If $x = \bar{\lambda}_{i+1}$ or $x = \gamma_{p^j}\bar{\mu}_i$, the congruence $w_{d^r x} \equiv 0 \bmod p-1$ shows the monomials in $d^r x$ are not τ-divisible. Hence, $d^r x = 0$, and we are done, or $c(d^r x) \geq 0$. It remains to note that $c(d^r x) = c(x) - 2r + 1 < 0$. □

Lemma 3.5. *There are no multiplicative extensions in the mod-*τ^{p-1} *Tor spectral sequence* (33)*.*

Proof. The Chow degree of x equals $c(x) = 2f_x + 2w_x - (d_x + f_x)$. To find a hidden extension for $g = (\gamma_{p^j}\bar{\mu}_i)^p = 0$, we search among the x's that satisfy

(1) $d_x + f_x = d_g + f_g = 2p^{i+j+1}$,
(2) $w_x = w_g = p(p^{i+j} - p^j)$,
(3) $0 < f_x < f_g = p^{j+1}$.

This rules out the existence of multiplicative extensions, since for the Chow degree, we have

$$c(x) = 2f_x + 2w_x - (d_x + f_x) = 2f_x + 2p(p^{i+j} - p^j) - 2p^{j+i+1} = 2(f_x - p^{j+1}) < 0.$$

Likewise, a hidden extension for $\bar{\lambda}^2_{i+1} = 0$ would be a class x with $|x| = (1, 4(p^{i+1} - 1), 2p^{i+1} - 2)$; by inspection, no such class exists since all possible x of filtration 1 have weight $p^j - 1$, $j \geq 0$. □

Theorem 3.6. *There is an isomorphism of graded commutative* $\mathbb{F}_{p,\tau}$*-algebras*

$$\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1} \cong \Big(\bigotimes_{i\geq 0} \Gamma_{\mathbb{F}_p}(\bar{\mu}_i) \otimes \Lambda_{\mathbb{F}_p}(\bar{\lambda}_{i+1})\Big) \otimes \mathbb{F}_{p,\tau}.$$

The bidegrees of the generators are $|\bar{\mu}_i| = (2p^i, p^i - 1)$ *and* $|\bar{\lambda}_{i+1}| = (2p^{i+1} - 1, p^{i+1} - 1)$.

Remark 3.7. The reader may recognize the answer as $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1} \cong C \otimes \mathbb{F}_{p,\tau}$ where $C = \mathbf{Tor}^{\mathcal{A}^{\mathrm{rig}}_\star}_{*,\star}(\mathbb{F}_p, \mathbb{F}_p)$ appeared in Section 2.5.

3.3. **Integral motivic Hochschild homology.** We now turn to the integral case of the Tor spectral sequence

$$E^r_{h,t,w} \Rightarrow \mathbf{MHH}_{h+t,w}(\mathbf{M}\mathbb{F}_p). \tag{34}$$

There is a natural comparison map $q\colon E^r_{\star;*} \to \bar{E}^r_{\star;*}$ to the mod-τ^{p-1} Tor spectral sequence analyzed in Section 3.2. Due to Theorem 3.6 we have the following nontrivial mod-τ^{p-1} classes and their representatives in the bar complex:

(1) $\bar{\lambda}_{i+1} \in \mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$ is the class of the permanent cycle

$$[\bar{\xi}_{i+1}] \in \bar{E}^1_{1,2p^{i+1}-2;p^{i+1}-1},$$

(2) $\gamma_j\bar{\mu}_i \in \mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$ is the class of the permanent cycle

$$[\bar{\tau}_i|\dots|\bar{\tau}_i] \in \bar{E}^1_{j,j(2p^i-1);j(p^i-1)}.$$

As before, to aid the bookkeeping, we also set

$$\lambda_{i+1} = [\xi_{i+1}] \in E^1_{1,2p^{i+1}-2;p^{i+1}-1}$$

and

$$\gamma_j\mu_i = [\tau_i|\dots|\tau_i] \in E^1_{j,j(2p^i-1);j(p^i-1)},$$

even though the $\gamma_j\mu_i$s turn out to be permanent cycles for $j < p$ only.

As already noted, when p is an odd prime $E^2 = \mathbb{F}_p[\tau] \otimes \bigotimes_{i\geq 0} \Lambda(\lambda_{i+1}) \otimes \Gamma(\mu_i)$.

Lemma 3.8. *Let p be a prime.*

(1) *For $0 < r < p$ the étale localization*

$$L^r_{\text{ét}}\colon E^r \to E^r[\tau^{-1}]$$

is an injection.

(2) *For $1 < r < p-1$, the differentials $d^r : E^r \to E^r$ are all zero.*

(3) *For all p*

$$d^{p-1}\gamma_{j+p}\mu_i \doteq \tau^{p-1}\lambda_{i+1}\gamma_j\mu_i$$

for $i, j \geq 0$ and for odd p, this generates the d^{p-1}-differential multiplicatively.

Proof. Since the dual Steenrod algebra has no τ-torsion, we have that $L^1_{\text{ét}}$ is an injection, and from the Tor-calculations we get that for odd primes p also $L^2_{\text{ét}}$ is an injection. Assume that for given $0 < r < p$ $L^r_{\text{ét}}$ is injective. For $1 < r < p-1$, we have established that the differential on $E^r[\tau^{-1}]$ is trivial, and so the differential on E^r is trivial too. Hence $L^{r+1}_{\text{ét}}$ is injective, showing that (for odd primes p) $E^2 = E^3 = \cdots = E^{p-1}$.

Finally, since for all primes p we now have $L^{p-1}_{\text{ét}}$ is an injection, the formula

$$d^{p-1}\gamma_{j+p}\mu_i \doteq \tau^{p-1}\lambda_{i+1}\gamma_j\mu_i$$

follows from the same formula in $E^{p-1}[\tau^{-1}]$. □

The case for odd and even primes p takes slightly different paths from here on. The case $p = 2$ is in many ways the simplest one but requires more care in that it turns out to be neither practical nor necessary to muddle through with the integral spectral sequence calculation: everything emanates from the torsion and τ-inverted **MHH**s together with minimal information about the integral E^1-page and an analysis of the

Bockstein homology (called “a Bockstein type complex” in Section 2.5 since it also appears in the odd primary case in a slightly different guise) giving the answer—with all multiplicative extensions—without more ado.

3.3.1. *The even case.* Let $p = 2$. Since τ is a non-zero divisor in $\mathcal{A}_\star$, multiplication by τ gives the short exact sequence

$$0 \longrightarrow E^1_{f,d;w+1} \xrightarrow{\tau} E^1_{f,d;w} \xrightarrow{q} \bar{E}^1_{f,d;w} \longrightarrow 0.$$

We recall that the mod-τ spectral sequence collapses at $\bar{E}^2$ and has no multiplicative extensions: $\mathbf{MHH}_\star(\mathbb{F}_2)/\tau \cong \bar{E}^2$. Moving on to the abutment, the τ-Bockstein on $\mathbf{MHH}_\star(\mathbb{F}_2)$ is the composite

$$\bar{\partial} \colon \mathbf{MHH}_{*+1,*}(\mathbb{F}_2)/\tau \xrightarrow{\partial} \mathbf{MHH}_{*,*+1}(\mathbb{F}_2) \xrightarrow{q} \mathbf{MHH}_{*,*+1}(\mathbb{F}_2)/\tau. \tag{35}$$

Since (35) is a derivation, we only need to know its value on the generators. These are obtained from the integral d^1-differentials analyzed in Remark 2.7 as follows. Since $\bar{\lambda}_{i+1}$ is hit by the d^1-boundary $\lambda_{i+1} = [\xi_{i+1}] \in E^1_{1,2^{i+2}-2;2^{i+1}-1}$ we get $\bar{\partial}\bar{\lambda}_{i+1} = 0$, and since $\gamma_{j+2}\bar{\mu}_i$ is hit by $\gamma_{j+2}\mu_i = [\tau_i|\dots|\tau_i] \in E^1_{j,j(2^{i+1}-1);j(2^i-1)}$ and $d^1\gamma_{j+2}\mu_i = \tau\lambda_{i+1}\gamma_j\mu_i$ we deduce Lemma 3.9.

Lemma 3.9. *The nontrivial τ-Bocksteins on $\mathbf{MHH}_\star(\mathbb{F}_2)$ are generated by*

$$\bar{\partial}\gamma_{j+2}\bar{\mu}_i = \bar{\lambda}_{i+1}\gamma_j\bar{\mu}_i$$

for all $i, j \geq 0$, i.e., $(\mathbf{MHH}_\star(\mathbb{F}_2)/\tau, \bar{\partial}) = (C, \partial)$, where (C, ∂) is the commutative differential graded algebra of Section 2.5.

Combined with Corollary 2.15, and using that the τ-free element $\mu_i \in \mathbf{MHH}_\star(\mathbb{F}_2)$ maps to $\bar{\mu}_i \in \mathbf{MHH}_\star(\mathbb{F}_2)/\tau$, we deduce the following result.

Corollary 3.10. *The Bockstein homology of $\mathbf{MHH}_\star(\mathbb{F}_2)/\tau$ is isomorphic to the graded commutative $\mathbb{F}_2$-algebra $\bigoplus_{i\geq 0}\Lambda(\bar{\mu}_i)$.*

Corollary 3.10 lets us conclude that the τ-torsion classes in $\mathbf{MHH}_\star(\mathbb{F}_2)$ are not τ-divisible. The τ-torsion in $\mathbf{MHH}_\star(\mathbb{F}_2)$ agrees with the image of $\partial \colon \mathbf{MHH}_\star(\mathbb{F}_2)/\tau \to \mathbf{MHH}_\star(\mathbb{F}_2)$ and maps injectively via $q \colon \mathbf{MHH}_\star(\mathbb{F}_2) \to \mathbf{MHH}_\star(\mathbb{F}_2)/\tau$.

There is a naturally induced commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow (\tau\text{–torsion}) & \longrightarrow & \mathbf{MHH}_\star(\mathbb{F}_2) & \longrightarrow & \mathbb{F}_2[\tau,\mu_0,\mu_1,\dots]/(\mu_i^2 - \tau\mu_{i+1}) \longrightarrow 0 \\ \cong \downarrow q & & \downarrow q & & \\ 0 \longrightarrow \operatorname{im}\bar{\partial} & \longrightarrow & \bigotimes_{i\geq 0}\Gamma(\bar{\mu}_i)\otimes\Lambda(\bar{\lambda}_{i+1}). & & \end{array}$$

More elegantly, using Corollary 2.15, we have a pullback diagram of commutative $\mathbb{F}_2[\tau]$-algebras

$$\begin{array}{ccc} \mathbf{MHH}_\star(\mathbb{F}_2) & \longrightarrow & \mathbb{F}_2[\tau,\mu_i]/\mu_i^2 - \tau\mu_{i+1} \\ \downarrow & & \downarrow \\ \mathbb{F}_2[\tau,\bar{\mu}_i,\mathrm{x}_{S,f}]/\mathcal{J} & \longrightarrow & \mathbb{F}_2[\tau,\bar{\mu}_i]/(\bar{\mu}_i^2,\tau) \end{array}$$

with indexation $i \in \mathbb{N}$, $(S, f) \in K$ (see Definition 2.8), and

$$\mathcal{I} = (\tau, \bar{\mu}_i^2, \mathrm{x}_{S,f} \cdot \mathrm{x}_{T,g} - \sum_{t_{f+g} \neq u \in \mathrm{supp}(f+g) - S \cup T} \epsilon_u \cdot \mathrm{x}_{S \cup T \cup \{u\}, f+g}).$$

Here μ_i maps to $\bar{\mu}_i$ and $\mathrm{x}_{S,f}$ maps to zero. When we finish the odd case, we'll see that by replacing 2 with p, we have the general formula.

3.3.2. *The odd case.* Let p be an odd prime. The first task uses our knowledge of d^{p-1} to calculate E^p. Consider the short exact sequence

$$0 \longrightarrow E^{p-1}_{f,d;w+p-1} \xrightarrow{\tau^{p-1}} E^{p-1}_{f,d;w} \xrightarrow{q} \bar{E}^{p-1}_{f,d;w} \longrightarrow 0$$

and the injection

$$L^{p-1}_{\text{ét}} : E^{p-1}_{f,d;w} \to E^{p-1}_{f,d;w}[\tau^{-1}].$$

Definition 3.11. For $p \leq r$, let $P(r)$ be the conjunction of the propositions $P(r)_1$, $P(r)_2$, and $P(r)_3$ defined as follows:

$P(r)_1$: $E^r_{f,d;w+p-1} \xrightarrow{\tau^{p-1}} E^r_{f,d;w} \xrightarrow{q} \bar{E}^r_{f,d;w}$ is exact,
$P(r)_2$: in $E^r_{f,d;w}$ we have $\ker L^r_{\text{ét}} = \ker \tau^{p-1}$, and
$P(r)_3$: for $p \leq j < r$ the jth differential d^j is trivial (so that $E^p = E^r$).

To simplify notation, consider the $\mathbb{F}_p$-algebra $C = \bigotimes_{i \geq 0} \Gamma_{\mathbb{F}_p}(\bar{\mu}_i) \otimes \Lambda_{\mathbb{F}_p}(\bar{\lambda}_{i+1})$ (with the above isomorphism $\bar{E}^r \cong C[\tau]/\tau^{p-1}$ for $r \geq 2$) and the derivation $\partial : C \to C$ generated by $\partial(\gamma_{j+p}\bar{\mu}_i) = \bar{\lambda}_{i+1}\gamma_j\bar{\mu}_i$. Let $B^\partial = \mathrm{im}\partial$, $Z^\partial = \ker \partial$ and $H^\partial = Z^\partial / B^\partial$.

Lemma 3.12. *The proposition $P(p)$ is true. Hence, E^p is isomorphic to $Z^\partial[\tau]/\tau^{p-1}B^\partial[\tau]$ and under this isomorphism $E^p/\ker L^p_{\text{ét}}$ is isomorphic to $H^\partial[\tau]$.*

Furthermore, the map $q : E^p \to \bar{E}^p \cong C[\tau]/\tau^{p-1}$ factors over $Z^\partial[\tau]/\tau^{p-1} \subseteq C[\tau]/\tau^{p-1}$ and the map

$$\ker\{\tau^{p-1} : E^p \to E^p\} \subseteq E^p \to \bar{E}^p \cong C[\tau]/\tau^{p-1}$$

is an injection factoring as an isomorphism $\ker\{\tau^{p-1} : E^p \to E^p\} \cong B^\partial[\tau]/\tau^{p-1}$ *followed by the injection $B^\partial[\tau]/\tau^{p-1} \subseteq C[\tau]/\tau^{p-1}$. Summing up, the resulting diagram of commutative $\mathbb{F}_p[\tau]$-algebras*

$$\begin{array}{ccc} E^p & \longrightarrow & H^\partial[\tau] \\ \downarrow & & \downarrow \\ Z^\partial[\tau]/\tau^{p-1} & \longrightarrow & H^\partial[\tau]/\tau^{p-1} \end{array}$$

is a pullback.

Proof. For odd p, the first thing to notice is that E^{p-1} is a free $\mathbb{F}_p[\tau^{p-1}]$-module and that the differential factors $d^{p-1} = \tau^{p-1}\bar{\partial}$ where $\bar{\partial}$ (aka the Bockstein) is homogeneous with respect to the τ^{p-1}-grading on E^{p-1} and $\bar{\partial}^2 = 0$. Let Q be the degree zero part of E^{p-1} (so that $E^{p-1} = Q[\tau^{p-1}]$ and $Q \subseteq E^{p-1} \to E^{p-1}/\tau^{p-1} \cong \bar{E}^{p-1} = \bar{E}^p$ is an isomorphism). If $Z^{\bar{\partial}} = \ker\{\bar{\partial} : Q \to Q\}$, then $\ker d^{p-1} = Z^{\bar{\partial}}[\tau^{p-1}]$, whereas if $B^{\bar{\partial}} = \mathrm{im}\{\bar{\partial} : Q \to Q\}$, then $\mathrm{im}\, d^{p-1} = \tau^{p-1}B^{\bar{\partial}}[\tau^{p-1}]$, and if $H^{\bar{\partial}} = Z^{\bar{\partial}}/B^{\bar{\partial}}$, then (as an $\mathbb{F}_p[\tau^{p-1}]$-module)

$$E^p = Z^{\bar{\partial}} \oplus \tau^{p-1}H^{\bar{\partial}}[\tau^{p-1}],$$

and $q\colon E^p \to \bar{E}^p$ may be identified with the composite

$$Z^{\bar\partial} \oplus \tau^{p-1}H^{\bar\partial}[\tau^{p-1}] \to Z^{\bar\partial} \subseteq Q$$

of the projection to the degree zero part, followed by the inclusion. Hence

$$\ker q = \tau^{p-1}H^{\bar\partial}[\tau^{p-1}] = \operatorname{im}\{\tau^{p-1}\colon E^p \to E^p\},$$
$$\ker\{\tau^{p-1}\colon E^p \to E^p\} = B^{\bar\partial} = \ker L^p_{\text{ét}},$$

and

$$E^p/\ker\tau^{p-1} \cong H^{\bar\partial}[\tau^{p-1}] \subseteq H^{\bar\partial}[\tau^{\pm(p-1)}] \cong E^p[\tau^{-1}].$$

Since $P(r)_3$ is vacuous in this case, we have proven $P(p)$.

The formulation with the pullback follows when writing the above out as $\mathbb{F}_p[\tau]$-algebras, so that $\ker d^{p-1} = Z^D[\tau]$ and $\operatorname{im} d^{p-1} = \tau^{p-1}B^D[\tau]$ and remembering that $\bar{E}^{p-1} \cong C[\tau]/\tau^{p-1}$. □

Lemma 3.13. *For all $r \geq p$, the proposition $P(r)$ is true. Hence,*

(1) $E^\infty = E^p$.

(2) *The algebra map from the $\mathbb{F}_p[\tau^{p-1}]$-free part to the τ^{p-1}-localization*

$$L_{\text{ét}}\colon \left[\mathbf{MHH}_\star(\mathbb{F}_p)\right]/\ker\tau^{p-1} \to \mathbf{MHH}_\star(\mathbb{F}_p)[\tau^{-1}] \cong \mathbb{F}_p[\tau^{\pm1},\mu_i]_{i\geq0}/(\mu_i^p - \tau^{p-1}\mu_{i+1})$$

is injective so that $\left[\mathbf{MHH}_\star(\mathbb{F}_p)\right]/\ker\tau^{p-1} \cong \mathbb{F}_p[\tau,\mu_i]_{i\geq0}/(\mu_i^p - \tau^{p-1}\mu_{i+1})$.

(3) *The algebra map induced by* $q\colon \mathbf{MHH}(\mathbb{F}_p) \to \mathbf{MHH}(\mathbb{F}_p)/\tau^{p-1}$

$$q\colon \left[\mathbf{MHH}_\star(\mathbb{F}_p)\right]/\operatorname{im}\tau^{p-1} \to \mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1} \cong \Big(\bigotimes_{i\geq0}\Gamma_{\mathbb{F}_p}(\bar\mu_i)\otimes\Lambda_{\mathbb{F}_p}(\bar\lambda_{i+1})\Big)\otimes\mathbb{F}_{p,\tau}$$

is injective.

(4) *The composite* $\ker\tau^{p-1} \subseteq \mathbf{MHH}_\star(\mathbb{F}_p) \to \left[\mathbf{MHH}_\star(\mathbb{F}_p)\right]/\operatorname{im}\tau^{p-1}$ *is injective.*

Proof. By Lemma 3.12 we have $P(p)$ so we only need to show that $P(r)$ implies $P(r+1)$ for all $r \geq p$. Note that if $P(r)$ and $P(r+1)_3$ are true, then $P(r+1)$ is true. Recall from Lemma 3.1 and Lemma 3.4 that the rth differentials in both the localized and reduced Tor-spectral sequences are trivial.

Assume $P(r)$ and consider $x \in E^r_{f,d;w}$. From the fact that $d^r\colon \bar{E}^r \to \bar{E}^r$ is trivial so that $0 = d^rqx = qd^rx$ we get that there is a $y \in E^r_{f-r,d+r-1;w+p-1}$ so that $P(r)_1$ implies that $d^rx = \tau^{p-1}y$. Since $d^r\colon E^r[\tau^{-1}] \to \bar{E}^r[\tau^{-1}]$ is trivial we get that $0 = d^rL^r_{\text{ét}}x = L^r_{\text{ét}}d^rx = L^r_{\text{ét}}\tau^{p-1}y = \tau^{p-1}L^r_{\text{ét}}y$ so that $0 = L^r_{\text{ét}}y$ and $P(r)_2$ implies that $0 = \tau^{p-1}y = d^rx$.

The other points then follow directly, where in the last point we have used that $\ker\tau^{p-1} = \ker L_{\text{ét}}$ gives that $\ker\tau^{p-1}\cap\operatorname{im}\tau^{p-1} = 0$. □

Summing up in the language of Lemma 3.12, we have achieved a pullback of commutative $\mathbb{F}_p[\tau]$-algebras

(36)
$$\begin{array}{ccc} E^\infty & \longrightarrow & H^\partial[\tau] \\ \downarrow & & \downarrow \\ Z^\partial[\tau]/\tau^{p-1} & \longrightarrow & H^\partial[\tau]/\tau^{p-1}. \end{array}$$

Moreover, the pullback survives to the abutment in the sense that the maps out of E^∞ are the associated graded versions of maps induced from maps of commutative ring spectra.

We now set out to analyze Z^∂ and H^∂.

3.4. **Multiplicative extensions.** From Lemma 3.13 we deduced the pullback (36) of commutative $\mathbb{F}_p[\tau]$-algebras, which, given the information of Corollary 2.15, takes the form

$$\begin{array}{ccc} E^\infty & \longrightarrow & \mathbb{F}_p[\tau,\mu_i]/\mu_i^p \\ \downarrow & & \downarrow \\ \mathbb{F}_p[\tau,\bar\mu_i,\mathsf{x}_{S,f}]/\overline{\mathcal{J}} & \longrightarrow & \mathbb{F}_p[\tau,\bar\mu_i]/(\bar\mu_i^p,\tau^{p-1}) \end{array}$$

with indexation $i \in \mathbb{N}$, $(S,f) \in K$, and

$$\overline{\mathcal{J}} = (\tau^{p-1}, \bar\mu_i^p, \mathsf{x}_{S,f}\cdot \mathsf{x}_{T,g} - \sum_{t_{f+g}\neq u \in \operatorname{supp}(f+g)-S\cup T} \epsilon_u \cdot \mathsf{x}_{S\cup T\cup\{u\}, f+g}).$$

Here μ_i maps to $\bar\mu_i$ and $\mathsf{x}_{S,f}$ maps to zero. Moreover, the pullback survives to the abutment in the sense that the maps out of E^∞ are the associated graded versions of maps induced from maps of commutative ring spectra.

In the abutment, we know that $\mu_i^p = \tau^{p-1}\mu_{i+1}$, but can there be further extensions? Since μ_i maps to $\bar\mu_i$, such an extension must be witnessed when passing from the associated graded $\mathbb{F}_p[\tau,\bar\mu_i,\mathsf{x}_{S,f}]/\overline{\mathcal{J}}$ to $\mathbf{MHH}_\star(\mathbb{F}_p)/\tau^{p-1}$, but this we have seen in the mod τ^{p-1}-calculation is not the case. In conclusion, we have shown the following result.

Theorem 3.14. *There is an isomorphism of graded commutative $\mathbb{F}_p[\tau]$-algebras*

$$\mathbf{MHH}_\star(\mathbb{F}_p) \cong \mathbb{F}_p[\tau,\mu_i,\mathsf{x}_{S,f}]_{i\in\mathbb{N},(S,f)\in K}/\mathcal{J}$$

where the indexing set K is given in Definition 2.8 *and $\mathcal{J}$ is the ideal generated by*

- $\mu_i^p - \tau^{p-1}\mu_{i+1}$,
- $\tau^{p-1}\mathsf{x}_{S,f}$, *and*
- $\mathsf{x}_{S,f}\cdot\mathsf{x}_{T,g} - \sum_u \epsilon_{u,S,T,f,g}\cdot \mathsf{x}_{S\cup T\cup\{u\},f+g}$ *where the sum runs over all elements $u \notin S\cup T$ so that $(f+g, S\cup T\cup\{u\}) \in K$, and the coefficient $\epsilon_{u,S,T,f,g}\in\mathbb{F}_p$ is given in Definition* 2.12.

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