On the Range Assignment in Wireless Sensor Networks for Minimizing the Coverage-Connectivity Cost

SAJAL K. DAS, Department of Computer Science, Missouri University of Science and Technology, USA
RAFAŁ KAPELKO, Department of Fundamentals of Computer Science, Wrocław University of Science and Technology, Poland

This article deals with reliable and unreliable mobile sensors having identical sensing radius r, communication radius R, provided that $r \leq R$ and initially randomly deployed *on the plane* by dropping them from an aircraft according to general random process. The sensors have to move from their initial random positions to the final destinations to provide greedy path k_1-coverage simultaneously with k_2-connectivity. In particular, we are interested in assigning the sensing radius r and communication radius R to minimize *the time required* and *the energy consumption* of transportation cost for sensors to provide the desired k_1-coverage with k_2-connectivity. We prove that for both of these optimization problems, the optimal solution is to assign the sensing radius equal to $r = k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ and the communication radius $R = k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$, where $||\mathbf{E}[\mathbb{S}]||$ is the characteristic of general random process according to which the sensors are deployed. When $r < k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ or $R < k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$, and sensors are reliable, we discover and explain *the sharp increase* in the time required and the energy consumption in transportation cost to ensure the desired k_1-coverage with k_2-connectivity.

CCS Concepts: • **Information systems** → **Sensor networks**; • **Mathematics of computing** → **Probability and statistics**; • **Networks** → **Network performance analysis**; • **Theory of computation** → **Design and analysis of algorithms;**

Additional Key Words and Phrases: Sensors, coverage, random process, time, energy

ACM Reference format:
Sajal K. Das and Rafał Kapelko. 2021. On the Range Assignment in Wireless Sensor Networks for Minimizing the Coverage-Connectivity Cost. *ACM Trans. Sen. Netw.* 17, 4, Article 46 (August 2021), 48 pages.
https://doi.org/10.1145/3457408

46

1 INTRODUCTION

A **wireless sensor network (WSN)** typically consists of a large number of sensor nodes deployed either randomly or according to some predefined statistical distribution over a geographical region of interest. There exists a wide variety of applications of WSNs, such as environmental monitoring (e.g., pollution, earthquake or seismic activities), wildlife habitat monitoring, structural health

This work of S. K. Das was partially supported by NSF grants CNS-1850851, OAC-1725755, OAC-2104078, and SCC-1952045. Rafał Kapelko supported by Polish National Science Center (NCN) grant 2019/33/B/ST6/02988.
Authors' addresses: S. K. Das, Department of Computer Science, 315 Computer Science Bldg., Missouri University of Science and Technology, Rolla, MO 65409; email: sdas@mst.edu; R. Kapelko, Department of Fundamentals of Computer Science, Wrocław University of Science and Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland; email: rafal.kapelko@pwr.edu.pl.

1550-4859/2021/08-ART46 $15.00
https://doi.org/10.1145/3457408

monitoring, border security and surveillance, intrusion detection, health care, diagnostics in industrial process control, and so on. In many of these applications, the environment could be hostile and/or the terrain could be difficult to reach, implying that manual deployment of sensors might not be possible. In such situations, sensor nodes are often deployed randomly or sprinkled from an aircraft, and they may remain unattended for months or years without any battery replenishment.

In WSNs, a fundamental problem to study is the *sensing coverage* [25]. However, due to limited resources (CPU, memory, battery, signal processing, sensing and wireless communication capabilities), a sensor node can sense only a small region. Therefore, the objective is to design optimal node deployment strategies such that each point in the entire monitoring field is sensed (or covered) by at least one sensor. There exist different notions of sensing coverage, such as *blanket coverage* (static deployment of sensors that maximizes the target detection rate in the sensing field); *barrier coverage* (deployment of sensors that minimizes the probability of undetected intrusion through obstacles or barriers); *sweep coverage* (move sensor nodes to balance the cost, such as maximizing event detection rate and minimizing number of missed detection), among others.

Alongside sensing coverage, another fundamental problem in WSNs is *connectivity*. Since a sensor node has limited wireless capability constrained by the antenna size, the sensor can directly communicate with only those that are within its radio communication radius. Thus, for any wide area deployment, the sensors typically form a multihop network that supports various operations, such as routing of sensed data to a sink (base station) or between far-off sensors including fusion or aggregation en route. Now, for random deployment, a challenging problem is to guarantee that the underlying network topology is connected. A high degree of connectivity (from graph theoretic viewpoint) provides higher reliability of the network against node or link failures. In WSNs, the coverage and connectivity issues are often tackled together. However, finding an optimal node deployment strategy that maximize coverage (i.e., how well each point in the region is covered by sensors) yet maintaining high connectivity (i.e., how well the sensors are connected) is challenging. In this article, the mobile sensors are deployed on the plane by dropping them from an aircraft according to general random process. We give the optimal sensor movement that maximize coverage together with connectivity for both *reliable* and *unreliable* sensors. We also present further insights including sensor deployment in three-dimensional (3D). Our solution can be widely used in applications, such as intruder detection and border security.

Depending on the applications, sensor deployments may be static or mobile. The WSN monitoring region could be one-dimensional (e.g., border security between two countries, or highway traffic), two-dimensional (e.g., agricultural field), or three-dimensional (e.g., air pollution monitoring, structural health of a building, or underwater sensing for oceanographic data collection). The sensor network could be homogeneous in terms of identical sensing and/or communication radius, or heterogeneous with non-uniform sensing and/or communication radii in which the sensing/communication range is irregular (e.g., surveillance with directional antennas of different capacities, multipath and shadowing effects).

In this study, we focus on path k_1-coverage (i.e., every point on the path is within a sensing range of at least k_1 sensors) simultaneously with k_2-connectivity (i.e., every point on the path is within a communication range of at least k_2 sensors) that n sensors are deployed on the plane by dropping them from an aircraft according to a general random process. As mentioned, random deployment of sensors is not unrealistic, because there are situations in which it is dangerous or even impossible for a human to deploy sensors in deterministic patterns. Moreover, due to wind, geographic terrain and other factors, random deployment may be the only option.

The aim of this article is to analyze the optimal sensor movement to ensure greedy path k_1-coverage together with k_2-connectivity. We consider binary sensing and communication disc model, i.e., the sensing area of a sensor is a circular disk of radius r and its communication area is also a circular disk of radius R, provided that $r \leq R$. Thus, a sensor placed at location x can sense any point at a distance of at most r and can communicate any point at a distance of at most R.

The sensor can be *reliable* or *unreliable*. *Reliable* sensors can move, sense, and communicate. The motivation for investigating *unreliable* sensors follows from some realistic situations (e.g., some sensors may fail after deployment on the plane). Thus, we assume that each sensor with some fixed probability $1 - p$ independently from other sensors is *unreliable* (not active), i.e., it cannot move, sense, and communicate anymore.

Specifically, we investigate *two optimization problems* to provide path k_1-coverage simultaneously with k_2-connectivity on the plane and are interested in assigning the sensing radius r and communication radius R to minimize

- the maximum displacement to the fixed power $a > 0$ of n sensors (the *time*),
- the sum of movement to the power $a > 0$ of the individual sensors (the *energy*).

Energy consumption and time-efficient reallocation of mobile sensors are the fundamental issues in WSNs. Mobile sensors consume much more energy during the movement than that during the communication or sensing process. Thus how to schedule mobile sensors to minimize the time and energy to provide the required k_1-coverage together with k_2-connectivity has great significance. Our solution can be widely used in border surveillance and securing buildings or a city. Sensor barriers are used to detect intruders illegally crossing the protected area. The random deployment according to general random process may be the only option for military surveillance or wild animals. Moreover, there are situations in which premature uncontrolled crashes of sensors are common. Our 3D network design is also useful for real-world applications such as underwater sensor networks. In realistic deployments, the tradeoff between coverage-connectivity and time or energy is very important to study. Moreover, the parameter a in the exponents can represent various conditions of the line, such as friction, lubrication, and so on, which may affect the movement of sensors.

For the optimization problems in WSN involving reliable sensors, we develop in Section 3 novel statistical analysis of the moments for general random processes. In the analysis of unreliable on the plane in Section 5.2, we combine results from unreliable sensors on the line (see Section 5.1) and for reliable sensors on the plane (see Section 4) to get results for unreliable sensors on the plane.

1.1 Contributions of This Paper

Fix $k_1, k_2 \geq 1$. Assume that n mobile *reliable* or *unreliable* sensors with identical sensing radius r and communication radius R, provided that $r \leq R$, are initially randomly deployed on the plane according to general random process (see Definition 2, as well as Definition 1 in Section 3.1). The sensors have to move to the final destinations to ensure greedy path k_1-coverage simultaneously with k_2-connectivity (for reliable sensors see Definitions 3–4 in Section 4.1 and for unreliable sensors see Definition 5 in Section 5.2, as well Assumption 12 in Section 5).

The objective is to assign the sensing radius r and the communication radius R so as to minimize *the time required* and *the energy consumption* of transportation cost for sensors to provide the desired k_1-coverage simultaneously with k_2-connectivity. To this aim, we make the following four novel theoretical contributions.

(1) For both the optimization problems as defined above, the optimal solution is obtained when the sensing radius $r = k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ and communication radius $R = k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$, where $||\mathbf{E}[\mathbb{S}]||$ is the expected distance of general a-random process, i.e., the characteristic of random process according to which the sensors are deployed.

(2) Let $\varepsilon > 0$ be an arbitrary small constant independent on the number of sensors n and $||\mathbf{E}[\mathbb{S}]||$. We discover a sharp decrease $\Omega(n^{\frac{a}{2}})$ in both the maximum displacement to the power $a > 0$ of n sensors (**time** required) and the sum of movement to the power $a > 0$ of the individual sensors (**energy** consumption): When r increases from $k_1(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ to $k_1\frac{||\mathbf{E}[\mathbb{S}]||}{2}$, and when R increases from $k_2(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ to $k_2\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ (see Tables 1 and 2 for a summary).

(3) For unreliable sensors, both the time required and energy consumption remain asymptotically the same when r and R increases within the same range as in step (2), as shown in Theorem 18 (Section 5.2). We design and analysis of four *novel optimal randomized* Algorithms 1–4 to provide the desired k_1-coverage simultaneously with k_2-connectivity. Although they are *simple*, the asymptotic probabilistic analysis is *challenging*. Our protocols are based on a novel mathematical theory of moments for general random processes.

The rest of the article is organized as follows. Section 2 summarizes some related works. Section 3 analyzes the moments for random processes, the results of which are used to derive theorems pertinent to the range assignment problems in WSNs. Section 4 derives the main results on the sensing and communication radii that minimize k_1-coverage simultaneously with k_2-connectivity cost in terms of time and energy. Section 5 analyzes k_1-coverage and k_2-connectivity cost in terms of time and energy when the sensors are unreliable. Section 6 presents further insights including exact formulas, variable sensing and communication radii, sensor deployment in higher dimension, other trajectories and real-life sensor deployment. The numerical results are discussed in Section 7. The final section offers conclusions.

2 RELATED WORK

The coverage problem in sensor networks has been extensively studied in the literature [2, 5, 7–9, 12, 15, 19, 22, 25, 28, 38–40, 42, 43, 46, 52, 55, 57, 59]. Two notions of probabilistic barrier coverage in a belt region, namely weak and strong barrier coverage, was introduced in Reference [42]. The barrier coverage of airdropped wireless sensors is studied in Reference [49]. It is assumed that along each line, sensors are to be evenly distributed. Because of mechanical inaccuracy, wind, terrain constraints, and other environment factors, the sensors will be scattered around the deployment line with some random offsets. In this article, the authors model the offsets as normally distributed random variables. In Reference [59], the authors provided a comprehensive survey on the optimized node placement in wireless sensor networks, while the authors in Reference [25] presented and compared several state-of-the-art algorithms and techniques to address the integrated coverage-connectivity issues in WSNs. The optimal movement of mobile sensors to the fence (perimeter) of a region delimited by a simple polygon to detect intruders, was investigated in Reference [9]. The barrier coverage in a mobile survivability-heterogeneous wireless sensor network is studied in Reference [55]. In Reference [3] is addressed the problem of k-coverage in 3D WSNs, where each point in a 3D field is covered by at least k sensors simultaneously. The authors of Reference [40] introduced a new architecture of barrier, called event-driven partial barrier, which is able to monitor any movements of objects in the event-driven environment. In Reference [15] is addressed three optimization problems to achieve weak barrier coverage in WSNs to minimize the number of sensors moved, the average distance as well as the maximum distance moved by the sensors. The authors of Reference [60] focused on the k-coverage problem, which requires a selection of a minimum subset of nodes among the deployed ones such that each point in the target region is covered by at least k nodes. The target coverage problem in mobile sensor networks where all the targets need to be covered by sensors continuously is studied in Reference [21]. The goal is to minimize the moving distance of sensors to cover all targets in the surveillance region, which is in the Euclidean space. It is assumed that initially all the sensors are located at k base stations. In Reference [48] is proposed developed a fully autonomous system that controls drones to provide high- quality unobstructed coverage of targets from appropriate viewpoints based on a novel Oriented Line Segment Target Model. In Reference [29], the authors present a complete solution to the minimum-cost barrier coverage problem. The cost here can be any performance measurement and is usually defined as the resource consumed or occupied by the sensor barriers. The proposed PUSH-PULL-IMPROVE algorithm, is the first one that provides a distributed solution to the minimum-cost barrier coverage problem in asynchronous wireless sensor networks. The authors of Reference [17] proposed a taxonomy for classifying coverage protocols in WSNs. In Reference [23], the authors investigated the cooperative sweep coverage problem with mobile sensors to periodically cover all positions of interest in the surveillance region, while the authors of Reference [53] addressed the coverage control problem for a network of heterogeneous mobile sensors with bound position measurement errors on a circle.

Connectivity has been the subject of extensive interest (e.g., see References [1, 4, 6, 18, 24, 27, 30, 58, 62]). In Reference [14] the availability of nodes, the sensor coverage, and the connectivity have been discussed on network lifetime. The authors of Reference [58] present the design and analysis of novel protocols that can dynamically configure a network to achieve guaranteed degrees of coverage and connectivity. In Reference [4], the authors investigate the critical density for percolation in coverage and connectivity in 3D WSNs, as well as the corresponding critical network degree. The proposed approach is based on Baxter's factorization of the Ornstein-Zernike equation and the pair-connectedness theory. The critical sensor density for partial connectivity of a large area sensor network was studied in Reference [11], assuming that sensor deployment follows the Poisson distribution. The quality of connectivity of a wireless network that has a realistic number of nodes is characterized in Reference [13]. In Reference [10], the authors classify and summarize the state-of-the-art algorithms and techniques that address the connectivity-coverage issues in the wireless sensor networks. In Reference [27], the authors assume that the sensors are deployed uniformly at random in a 3D Field of Interest. It is considered the case when the sensors have only directional sensing capability and may have heterogeneity in terms of the sensing range, communication range, and/or probability of being alive. For such 3D heterogeneous directional WSNs, the authors derive probabilistic expressions for k-coverage and m-connectivity that are useful to optimize the cost of random deployment. The authors of Reference [5] investigate connectivity based on the degree of sensing coverage by studying k-covered WSNs, where every location in the field is simultaneously covered (or sensed) by at least k sensors (property known as k-coverage, where k is the degree of coverage). The model called the Reuleaux Triangle, to characterize k-coverage with the help of Helly's Theorem and the analysis of the intersection of sensing disks of k sensors were proposed. In Reference [6], the authors focus on the connectivity and k-coverage issues in 3D WSNs, where each point is covered by at least k sensors (the maximum value of k is called the coverage degree). The Reuleaux tetrahedron model to characterize k-coverage of a 3D field was proposed to investigate the corresponding minimum sensor spatial density. The family of problems whose goal is to design a network with maximal connectedness subject to a fixed budget constraint is investigated in Reference [61]. In Reference [44], the connectivity of an uncertain random graph with respect to edges is discussed.

Unreliable sensors has been studied in sensor networks. The problem of optimally placing unreliable sensors in a one-dimensional environment is considered in Reference [19]. In wireless sensor networks, the effect of a high rate of node failure in wireless sensor networks on network connectivity was investigated in Reference [50]; the authors provide a formal analysis that establishes the relationship between node density, network size, failure probability, and network connectivity. The unreliable sensor network with n nodes, arranged in a grid over a square region of unit area is investigated in Reference [51]; here the authors give the necessary and sufficient conditions for the random grid network to cover the unit square region as well as ensure that the active nodes are connected.

There is also interest in the statistical community for investigating the absolute moments and moments around the mean of some random variables [16, 20, 31, 32, 36, 41, 56]. Recurrence relations for integrals that involve the density of multivariate normal distributions are developed in Reference [31]. In Reference [32], the expected absolute difference of the arrival times to the integer power between two identical and independent Poisson processes is represented as the combination of the Pochhammer polynomials. Some inequalities for absolute moments of independent random variables, using the representation in terms of the characteristic function, is presented in Reference [56]. Moreover, the lower bound of the probability that a binomial random variable exceeds its expectation is analyzed in Reference [16].

In this article, we present a novel mathematical theory of moments for general random processes on the plane. As an application to sensor networks, the time required ($\mathbf{Time_a}(n,r,R)$, $\mathbf{Time_{a,p}}(n,r,R)$) and the energy consumption ($\mathbf{Energy_a}(n,r,R)$, $\mathbf{Energy_{a,p}}(n,r,R)$) of the transportation cost to the power $a > 0$ for reliable and unreliable sensors from initial random position according to general random process to anchor points on the plane are analyzed (see Definition 4 in Section 4 for reliable sensors and Definition 5 in Section 5.2 for unreliable sensors).

We remark that our work is related to the series of papers [20, 33, 34, 36, 37, 39] dealing with reliable sensors. In References [38, 39], the **Energy** metric was analyzed for uniformly distributed random sensors in the unit interval for barrier coverage and in the higher dimension for area coverage. The works in References [33, 34, 37] deal with **Time** and **Energy** respectively for coverage (1-coverage) with interference, where the sensors are deployed according to the arrival times of Poisson process with arrival rate $\lambda > 0$. It is worth pointing out the above mentioned papers treat only the very special case when random sensors obey the beta (uniform distribution) and gamma distributions (Poisson process) and when the sensors are only *reliable*, i.e., it can move, sense and communicate with probability 1. Thus, it is natural to extend the previous works and analyze the sensor deployment according to general random process *on the plane.*

Our investigation of greedy path k_1-coverage simultaneously with k_2-connectivity for *unreliable* sensors is inspired [19], where the authors consider the problem of optimal disk-coverage in a one-dimensional environment by unreliable sensors, under a probabilistic failure model. It is assumed that sensors can fail independently and with the same probability. The aim is to minimize, in expectation, the largest distance between a point in the environment and an active sensor. It is worth pointing out the mentioned paper [19] consider the equispaced placement and random placement according to uniform distribution of n unreliable sensors in the unit interval. and when the sensors cannot move. Thus, it is natural to extend the previous work and investigate time required and energy consumption for transportation cost of sensors to ensure greedy path k_1-coverage together with k_2-connectivity when the sensors are deployed according to general random process on the plane and can fail independently and with the same probability p.

The novelty of our work in this article current article lies in the investigation of greedy path k_1-coverage simultaneously with k_2-connectivity for both *reliable* and *unreliable* sensors and in the derivation of closed form asymptotic formulas for both **Time** and **Energy** without using any specific density function (gamma and beta) for a wide class of distributions. Although there are studies that consider the problem of coverage and connectivity simultaneously, none of them so far derived the closed form asymptotic formulas for **Time** and **Energy** for *reliable* and *unreliable* sensors.

3 ANALYSIS OF THE MOMENTS FOR GENERAL RANDOM PROCESSES

In this section, we present an analytical combinatorial approach to probabilistic analysis of moments for random processes. The obtained new result about moments are pertinent *to the range*

Reference	Deployment distribution	Time	Energy	2D	3D	unreliable sensors	Homogenous sensors	Heterogeneous sensors	Coverage with connectivity
This paper	General random process	✓	✓	✓	✓	✓	✓	✓	✓
[49]	Gaussian distribution			✓			✓		
[34]	Poisson process	✓					✓		
[37]	Poisson process		✓	✓	✓		✓		✓
[21]	Evenly distributed		✓	✓			✓		
[29]	Evenly distributed		✓	✓			✓	✓	
[6]	Evenly distributed		✓		✓		✓		✓
[50]	Evenly distributed			✓		✓	✓		
[51]	Evenly distributed				✓	✓	✓		✓
[4]	Random deployment		✓		✓		✓		✓
[38]	Random deployment		✓	✓	✓		✓		
[19]	Random deployment	✓				✓			

assignment problem in WSNs. In particular, the random vector $\mathbb{S}_j$ given by Equation (6) represents the position of jth sensor on the plane in Section 4.

3.1 The Model, Assumptions, and Preliminaries

Now we will introduce two new definitions of (m, α, β)-property and general a-random process on the plane. We also recall the notations and some known special inequalities to be used in the sequel. Definition 1 together with Definition 2 allow us to obtain the novel results about the moments for general random processes for a wide class of distributions, without using specific density function, just moment equations. The formal definition of (m, α, β)-property is as follows.

Definition 1 ((m, α, β)-Property). Fix $\alpha, \beta > 0$. Let m be an even positive integer. Consider two sequences $\{\tau_i\}_{i\geq 1}$, $\{\xi_i\}_{i\geq 1}$ of positive, absolutely continuous random variables. Assume independence between sequences $\{\tau_i\}_{i\geq 1}$, $\{\xi_i\}_{i\geq 1}$; and assume that

$$\forall_{i\geq 1} \quad \left(\mathbf{E}\left[\tau_i\right] = \alpha, \;\; \mathbf{E}\left[\tau_i^p\right] \leq C_{1,m}, \;\; p \in \{2, 3, \ldots m\}\right), \tag{1}$$

$$\forall_{i_t, t\in \mathbf{N}\setminus\{0\}, \;\; p_t\in \mathbf{N}, \;\; 2\leq p_1+p_2+\cdots+p_l\leq m} \quad \mathbf{E}\left[\tau_{i_1}^{p_1}\tau_{i_2}^{p_2}\ldots\tau_{i_l}^{p_l}\right] = \mathbf{E}\left[\tau_{i_1}^{p_1}\right]\mathbf{E}\left[\tau_{i_2}^{p_2}\right]\ldots\mathbf{E}\left[\tau_{i_l}^{p_l}\right]. \tag{2}$$

$$\forall_{i\geq 1} \quad \left(\mathbf{E}\left[\xi_i\right] = \beta, \;\; \mathbf{E}\left[\xi_i^p\right] \leq C_{2,m}, \;\; p \in \{2, 3, \ldots m\}\right), \tag{3}$$

$$\forall_{i_t, t\in \mathbf{N}\setminus\{0\}, \;\; p_t\in \mathbf{N}, \;\; 2\leq p_1+p_2+\cdots+p_l\leq m} \quad \mathbf{E}\left[\xi_{i_1}^{p_1}\xi_{i_2}^{p_2}\ldots\xi_{i_l}^{p_l}\right] = \mathbf{E}\left[\xi_{i_1}^{p_1}\right]\mathbf{E}\left[\xi_{i_2}^{p_2}\right]\ldots\mathbf{E}\left[\xi_{i_l}^{p_l}\right]. \tag{4}$$

The random vector $\mathbb{V}_j$ with (m, α, β)-property is defined by the following formula:

$$\mathbb{V}_j := \left(\sum_{i=1}^{j}\tau_i, \sum_{i=1}^{j}\xi_i\right). \tag{5}$$

Note that complicated Assumption (2) is *weaker* than *independence of random variables* $\{\tau_i\}_{i\geq 1}$. To observe this, consider the case when $m = 2$. Then Equation (2) is indeed only pairwise independence of random variables $\{\tau_i\}_{i\geq 1}$. It is well known that pairwise independence does not imply independence (see Reference [54]). Hence, Assumption (4) is also *weaker* than *independence of random variables* $\{\tau_i\}_{i\geq 1}$.

Let us define the general two-dimensional a-random process as follows.

Definition 2 (General a-Random Process on the Plane). Let $\lambda > 0$ be parameter. Fix $a, \alpha, \beta > 0$. Let m be the smallest even integer greater than or equal to a. Assume that, random vector $\mathbb{V}_j$ has the (m, α, β)-property. The general a-random process is finite random process $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$ defined by the formula

$$\mathbb{S}_j := \frac{\mathbb{V}_j}{\lambda} \quad \text{for} \quad j \in \{1, 2, \ldots, n\}. \tag{6}$$

Let $||\mathbf{E}[\mathbb{S}_1]||$ be the Euclidean norm of vector $\mathbb{S}_1$. We call the expected vector of general a-random process as

$$\mathbf{E}[\mathbb{S}] = \mathbf{E}[\mathbb{S}_1] = \left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) \tag{7}$$

and the expected distance of general a-random process as

$$||\mathbf{E}[\mathbb{S}]|| = ||\mathbf{E}[\mathbb{S}_1]|| = \frac{\sqrt{\alpha^2+\beta^2}}{\lambda}. \tag{8}$$

From Equations (1), (3), (5), and (6) for $j := 1$, it is clear that $\mathbf{E}[\mathbb{S}_1] = (\frac{\alpha}{\lambda}, \frac{\beta}{\lambda})$. Note that the random variable $\mathbb{S}_j$ *represents the position of jth sensor on the plane* (see Section 4). Equations (1), (3), (5), (6), and (7) yields

$$\mathbf{E}[\mathbb{S}_j] = j\mathbf{E}[\mathbb{S}]. \tag{9}$$

In this study, we will provide *asymptotic probabilistic analysis* for *the range assignment in wireless mobile sensor networks*. Hence let us recall the Landau asymptotic notations:

- $f(n) = O(g(n))$ if there exists a constant $C_1 > 0$ and integer N such that $|f(n)| \le C_1|g(n)|$ for all $n > N$;
- $f(n) = \Omega(g(n))$ if there exists a constant $C_2 > 0$ and integer N such that $|f(n)| \ge C_2|g(n)|$ for all $n > N$;
- $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

We will also apply Jensen's inequality for expectations. If f is a convex function and X is random variable, then

$$f(\mathbf{E}[X]) \le \mathbf{E}[f(X)], \tag{10}$$

provided the expectations exists (see Reference [47, Proposition 3.1.2]).

Finally, the following elementary inequality will also be useful. Fix $a > 0$. Let $x, y \in \mathbb{R}_+ \cup \{0\}$. Then

$$(x+y)^a \le \max\left(2^{a-1}, 1\right)(x^a + y^a). \tag{11}$$

3.2 The mth Central Moment for Special Random Variable

In this subsection, we derive closed analytical formula for the expected mth moment around the mean for the random variable $X_j = \sum_{i=1}^{j} \tau_i$, assuming that Equations (1) and (2) hold and m is a fixed positive even integer.

Namely, we prove Theorem 1. Notice that Theorem 1 for random variable X_j is helpful *in deriving the main results* for random vector $\mathbb{V}_j$ in the next subsection (see Theorem 2) and thus *necessary* in analysis of *greedy path k_1-coverage simultaneously with k_2-connectivity in WSNs*. Moreover, to the best of our knowledge, the closed analytical asymptotic formula in Theorem 1 present *new statistical properties* of random variable X_j.

To illustrate the asymptotic closed formula in Theorem 1, we consider the special case when $m = 2$. The analysis of the second central moment for random variable X_j is easy and the asymptotic formula in Theorem 1 for $m := 2$ holds as *identity*.

Applying Equation (2) for $m := 2$, we get $\mathbf{E}[\tau_{i_1}\tau_{i_2}] = \mathbf{E}[\tau_{i_1}]\mathbf{E}[\tau_{i_2}]$. From Equation (1), it follows that $\mathbf{E}[\tau_{i_1}] = \mathbf{E}[\tau_{i_2}] = \alpha$. Thus,

$$\mathbf{E}[(\tau_{i_1} - \alpha)\,(\tau_{i_2} - \alpha)] = \mathbf{E}[(\tau_{i_1} - \alpha)]\mathbf{E}[(\tau_{i_2} - \alpha)] = 0.$$

Therefore, we conclude

$$\begin{aligned}\mathbf{E}\left(\left(X_j - \mathbf{E}(X_j)\right)^2\right) &= \mathbf{E}\left(\left(X_j - j\alpha\right)^2\right) = \mathbf{E}\left((\tau_1-\alpha)^2 + (\tau_2-\alpha)^2 + \cdots + \left(\tau_j - \alpha\right)^2\right)\\ &= j\mathbf{E}\left((\tau_1-\alpha)^2\right) + \sum_{1\le\tau_{i_1}\ne\tau_{i_2}\le j} \mathbf{E}[(\tau_{i_1}-\alpha)]\mathbf{E}[(\tau_{i_2}-\alpha)] = \mathbf{E}\left((\tau_1-\alpha)^2\right)j = \mathbf{Var}\,[\tau_1]\,j.\end{aligned}$$

Hence, identity $\mathbf{E}((X_j - \mathbf{E}(X_j))^2) = \mathbf{Var}\,[\tau_1]\,j$ confirms the closed analytical formula in Theorem 1 for $m := 2$.

THEOREM 1. *Let us fix an even positive integer m. Consider the sequence $\{\tau_i\}_{i\ge1}$ of positive, absolutely continuous random variables. Assume that Equations (1) and (2) hold in Definition 2. Let*

$X_j := \sum_{i=1}^{j} \tau_i$. *Then the following identity is valid:*

$$\mathbf{E}\left[\left(X_j - \mathbf{E}[X_j]\right)^m\right] = \frac{m!\,(\mathbf{Var}\,[\tau_1])^{\frac{m}{2}}}{2^{\frac{m}{2}}\left(\frac{m}{2}\right)!} j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right).$$

It is worthwhile to mention that, the asymptotic analysis in the proof of Theorem 1 lies in combinatoric. The technique is somewhat similar to the proof of Theorem 2 in Reference [35].

3.3 Expected Distance to the Power m for Special Random Vectors

In this subsection, we derive closed analytical formula for the expected distance to the power m between random vector $\mathbb{V}_j$ with (m, α, β)-property and its mean $\mathbf{E}[\mathbb{V}_j]$, provided that m is a fixed positive even integer.

It will be seen later in Section 4 that *the random position of the sensor* $\mathbb{S}_j$ on the plane is *determined by the random vector* $\frac{\mathbb{V}_j}{\lambda}$, where λ is positive real parameter.

We are now ready to present Theorem 2. We note that Theorem 2 is *crucial* to explain the *sharp increase* in the time required and in the energy consumption for transportation cost of sensors to ensure the desired coverage (see Section 4). Moreover, if we restrict the sensor displacements to specific random variables, then Corollary 3 is useful in deriving the exact formulas for *the minimal time required and energy consumption* of transportation cost for sensors to provide the desired k_1-coverage with k_2-connectivity (see Section 6.1 for details).

Theorem 2. *Let m be an even positive integer. Let $\mathbb{V}_j = (X_j, Y_j)$ be the random vector with (m, α, β)-property. Then the following identity is valid*

$$\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m\right] = \left(\frac{\left(\frac{m}{2}\right)!}{2^{\frac{m}{2}}} \sum_{i=0}^{\frac{m}{2}} \binom{2i}{i}\binom{m-2i}{\frac{m}{2}-i} (\mathbf{Var}\,[\tau_1])^i\,(\mathbf{Var}\,[\xi_1])^{\frac{m}{2}-i}\right) j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right),$$

where $||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||$ is the Euclidean distance between $\mathbb{V}_j$ and $\mathbf{E}[\mathbb{V}_j]$.

Finally, we give a simpler expression for the $\mathbf{E}[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m]$ when $\mathbf{Var}[\tau_1] = \mathbf{Var}\,[\xi_1]$.

Corollary 3. *Let m be an even positive integer. Let $\mathbb{V}_j = (X_j, Y_j)$ be the random vector with (m, α, β)-property. Assume that $\mathbf{Var}\,[\tau_1] = \mathbf{Var}\,[\xi_1]$. Then the following identity is valid*

$$\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m\right] = \left(\frac{m}{2}\right)!\,(2\mathbf{Var}\,[\tau_1])^{\frac{m}{2}}\, j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right),$$

where $||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||$ is the Euclidean distance between $\mathbb{V}_j$ and $\mathbf{E}[\mathbb{V}_j]$.

Proof. Corollary 3 follows immediately from Theorem 2 for $\mathbf{Var}\,[\tau_1] = \mathbf{Var}\,[\xi_1]$ and the following identity $\sum_{i=0}^{\frac{m}{2}} \binom{2i}{i}\binom{m-2i}{\frac{m}{2}-i} = 2^m$ (see Reference [26, Identity (5.39)]). □

3.4 Expected Distance to the Power a for General a-random Process on the Plane

In this subsection, we extend Theorem 1 from Section 3.3 to real-valued exponents. Namely, we combine together the obtained earlier result for $\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m\right]$, where m is an even integer with Jensen's inequality (see Equation (10)), as well as Equations (6) and (8) to get the new result for $\mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]||^a\right]$, where a is positive real.

It will be seen later in Section 4 that Theorems 4–6 proved in this subsection are *crucial* in the analysis of Algorithms 1 and 2 and thus in *deriving the main results of this article for reliable sensors* (see Tables 1 and 2).

The following theorem is about the power cost when the sensor $\mathbb{S}_j$ moves to the position $\mathbf{E}[\mathbb{S}_j]$.

Theorem 4. *Consider the random variable $\mathbb{S}_j$ as in Definition 2. Then,*

$$\mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]||^a\right] = \begin{cases} \Theta\left(j^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 2 \\ O\left(j^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,2) \end{cases}.$$

The next result about the expected distance to the power a between random vector $\mathbb{S}_j$ and $\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]$ support our earlier Theorem 4.

Theorem 5. *Consider the random variable $\mathbb{S}_j$ as in Definition 2. Then,*

$$\mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a\right] = \begin{cases} \Theta\left(j^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 2 \\ O\left(j^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,2) \end{cases}.$$

The next theorem is about the power cost when the sensor $\mathbb{S}_j$ moves to the position $(1-\varepsilon)\mathbf{E}[\mathbb{S}_j]+O(1)\mathbf{E}[\mathbb{S}]$. The proof of Theorem 6 is analogous to that of Theorem 5.

Theorem 6. *Fix $\varepsilon > 0$ arbitrary small constant independent on j and $||\mathbf{E}[\mathbb{S}]||$. Consider the random variable $\mathbb{S}_j$ as in Definition 2. Fix $a > 0$. Then*

$$\mathbf{E}\left[||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a\right] = \Theta\left(j^a\right) ||\mathbf{E}[\mathbb{S}]||^a.$$

4 k_1-COVERAGE AND k_2-CONNECTIVITY IN SENSOR NETWORKS

In this section, we formally define k_1-coverage and k_2-connectivity problem and then formulate two optimization problem: time required and energy consumption for this problem. We also propose two optimal algorithms for minimizing the time required and the energy consumption of the transportation cost to the power $a > 0$ as a function of the sensing radius r, communication radius R to provide k_1-coverage simultaneously with k_2-connectivity.

4.1 Problem Formulation

Recall that in this study we investigate sensing and communication binary disc model, i.e., the sensing area of a sensor is a circular disk of radius r and the communication area of a sensor is a circular disk of radius R, provided that $r \leq R$.

Throughout this subsection, $\varepsilon > 0$ is arbitrary small constant independent on the number of sensors n and on the expected vector of general a-random process $\mathbf{E}[\mathbb{S}]$ (see Definition 2 in Section 3.1).

Let us now formulate the movement requirement for providing the k_1-coverage simultaneously with k_2-connectivity.

Definition 3. Let us fix the positive integers k_1, k_2 and move the sensors from their initial random positions $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$ on the plane to the final destination $\mathbb{P}_1, \mathbb{P}_2, \ldots \mathbb{P}_n$ on the plane such that every point on the path connecting points $(0,0)$, $\mathbb{P}_1, \mathbb{P}_2, \ldots \mathbb{P}_n$ is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors.

Figure 1 illustrates our initial random placement according to general random process. In Section 4, we restrict our analysis to the greedy path k_1-coverage with k_2-connectivity, i.e., the points $\mathbb{P}_1, \mathbb{P}_2, \ldots \mathbb{P}_n$ are situated on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$. Obviously, when the distances

$$||\mathbb{P}_1 - (0,0)||, ||\mathbb{P}_2 - \mathbb{P}_1||, \ldots, ||\mathbb{P}_n - \mathbb{P}_{n-1}||$$

are fixed the maximal distance $\max_{i\in\{1,2,\ldots,n\}} ||\mathbb{P}_i - (0,0)||$ is maximized when the points $\mathbb{P}_1, \mathbb{P}_2, \ldots \mathbb{P}_n$ are situated on the line. It is also well known that border surveillance for intrusion

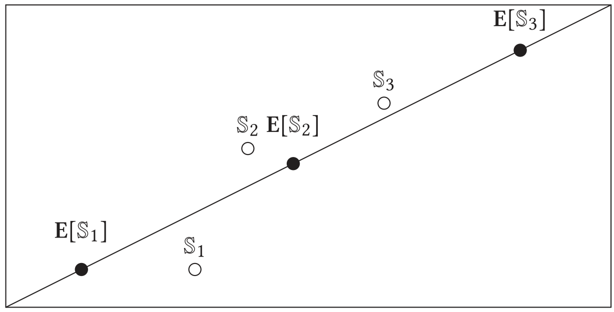

Fig. 1. The positions of three mobile sensors $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3$ on the plane according to general random process.

detection is an important application of sensor networks. Hence, it is natural to maximize the protected line, i.e., the length from the origin $(0,0)$ to the point (sensor) $\mathbb{P}_n$ when $||\mathbb{P}_n - (0,0)|| = \max_{i\in\{1,2,\ldots,n\}} ||\mathbb{P}_i - (0,0)||$, i.e., the sensor $\mathbb{P}_n$ is the rightmost sensor.

Therefore, we consider the greedy strategy. The *others greedy strategies* when the points $\mathbb{P}_1, \mathbb{P}_2, \ldots \mathbb{P}_n$ are situated on the others lines will be discussed in Section 6.4. Namely, it will be explain that analyzed *time required* and *energy consumption* is *minimized* when the points $\mathbb{P}_1, \mathbb{P}_2, \ldots \mathbb{P}_n$ are on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$.

We are now ready to formulate two optimization problems: time required and energy consumption for transportation cost of sensors to ensure greedy path k_1-coverage together with k_2-connectivity.

Definition 4. Fix $a > 0$. Let $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$ be the initial locations of n sensors with identical sensing radius r and communication radius R on the plane $[0,\infty) \times [0,\infty)$ according to general a-random process. Assume that (x_j, y_j) is the final destination of sensor $\mathbb{S}_j$ ($j \in \{1,2,\ldots n\}$) on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$ such that every point on the line connecting points $(0,0)$, (x_1,y_1), (x_2,y_2), $\ldots$ (x_n,y_n) is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors. We are interested in asymptotic (in large number of sensors n) for

$$\mathbf{Time_a}(n,r,R) = \max_{1\le j\le n} \mathbf{E}\left[||\mathbb{S}_j - (x_j,y_j)||^a\right],$$

$$\mathbf{Energy_a}(n,r,R) = \sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - (x_j,y_j)||^a\right].$$

Tables 1 and 2 summarize the results proved in the next subsection. It is discovered that $\Omega\left(n^{\frac{a}{2}}\right)$ sharply declines for both $\mathbf{Time_a}(n,r,R)$, and $\mathbf{Energy_a}(n,r,R)$ for all exponents $a > 0$ when the sensing radius r increases from $k_1(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ to $k_1\frac{||\mathbf{E}[\mathbb{S}]||}{2}$; and when the communication radius R increases from $k_2(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ to $k_2\frac{||\mathbf{E}[\mathbb{S}]||}{2}$.

Finally, we give a simple Lemma 7 about a one-dimensional scenario that will help us to find relationship between sensing radius r and communication radius R on the plane in the analysis of Algorithms 1 and 2 in the next subsection.

Obviously, the sensor with one-dimensional sensing radius r_1 and communication radius R_1 placed at location x on the line $[0,\infty)$ can sense any point at distance at most r_1 either to the left or right of x and can communicate any point at distance at most R_1 either to the left or right of x.

Lemma 7. *Consider n sensors $w_{1,n} \le w_{2,n} \le \cdots \le w_{n,n}$ with identical one-dimensional sensing radius $r_1 = k_1\frac{d}{2}$ and one-dimensional communication radius $R_1 = k_2\frac{d}{2}$ on the line $[0,\infty)$. Let*

Table 1. Time Required $\mathbf{Time_a}(n,r,R)$ of the Transportation Cost to the Power $a>0$ to Ensure Greedy Path k_1-coverage Together with k_2-connectivity as a Function of Sensing Radius r, Communication Radius R and the Expected Distance of General a-random Process $||\mathbf{E}[\mathbb{S}]||$ Provided That $\varepsilon>0$

	Sensing radius r	Communication radius R	$\mathbf{Time_a}(n,r,R)$	Algorithm
(a)	$r=k_1\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta\left(n^{\frac{a}{2}}\right)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a\geq 2$; $O\left(n^{\frac{a}{2}}\right)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a\in(0,2)$	1
(b)	$r=k_1(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta(n^a)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a>0$	2
(c)	$r=k_1\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta(n^a)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a>0$	2
(d)	$r=k_1(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta(n^a)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a>0$	2

Table 2. Energy Consumption $\mathbf{Energy_a}(n,r,R)$ and of the Transportation Cost to the Power $a>0$ to Ensure Greedy Path k_1-coverage Together with k_2-connectivity as a Function of Sensing Radius r, Communication Radius R and the Expected Distance of General a-random Process $||\mathbf{E}[\mathbb{S}]||$ Provided That $\varepsilon>0$

	Sensing radius r	Communication radius R	$\mathbf{Energy_a}(n,r,R)$	Algorithm
(a)	$r=k_1\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta\left(n^{\frac{a}{2}+1}\right)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a\geq 2$; $O\left(n^{\frac{a}{2}+1}\right)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a\in(0,2)$	1
(b)	$r=k_1(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta\left(n^{a+1}\right)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a>0$	2
(c)	$r=k_1\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta\left(n^{a+1}\right)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a>0$	2
(d)	$r=k_1(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$	$R=k_2(1-\varepsilon)\frac{\|\|\mathbf{E}[\mathbb{S}]\|\|}{2}$,	$\Theta\left(n^{a+1}\right)\|\|\mathbf{E}[\mathbb{S}]\|\|^a$ if $a>0$	2

$k=\max(k_1,k_2)$ *and* $n\geq k$. *Assume that the sensors occupy the following positions*

$$
\begin{aligned}
w_{j,n} &= j\frac{d}{2} \text{ if } j\in\{1,2,\dots,k\},\\
w_{j,n} &= R+(j-k)d \text{ if } j\in\{k+1,k+2,\dots,\min(n,n+2-k)\},\\
w_{j,n} &= R+(n+2-2k)d+(j-(n+2-k))\frac{d}{2} \text{ if } j\in\{n+3-k,\dots,n\} \text{ and } k\geq 3.
\end{aligned}
$$

Then every point from the origin to the last sensors is within the sensing radius of at least k_1 *sensors and the communication radius of at least* k_2 *sensors.*

Proof. Assume that the sensors have identical one-dimensional sensing radius r_1 equal to $k_1\frac{d}{2}$ and one-dimensional communication radius R_1 equal to $k_2\frac{d}{2}$. Let $k=\max(k_1,k_2)$. Notice that the point $p_k=k\frac{d}{2}$ is the position of kth sensor.

Therefore, every point in the interval $[0,p_k]$ is in the sensing radius of at least k_1 sensors and in the communication radius of at least k_2 sensors. As observed from Figure 2, every point in the interval $[p_k,p_{\min(n,n+2-k)-1}]$

- is in the sensing range of q_{l_1} sensors on the left and q_{r_1} sensors on the right and $q_{l_1}+q_{r_1}\geq k_1$,
- is in the communication range of q_{l_2} sensors on the left and q_{r_2} sensors on the right and $q_{l_2}+q_{r_2}\geq k_2$.

The length of interval $[p_{\min(n,n+2-k)-1}],p_n]$ is equal to $(k+1)\frac{d}{2}$ for $k\neq 2$ and is equal to d for $k=2$. Therefore, every point in the interval $[p_k,p_{\min(n,n+2-k)-1}]$ is within the sensing radius of at

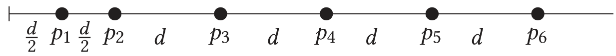

Fig. 2. Illustration of Lemma 7 for $k_1 = 1$, $k_2 = 2$, $r_1 = \frac{d}{2}$, $R_1 = d$, and $n = 6$.

least k_1 sensors and the communication radius of at least $k = k_2$ sensors. This completes the proof of lemma. □

4.2 Analysis of Algorithms 1 and 2

In this subsection, we *minimize* the *time required* and the *energy consumption* of the transportation cost to the power $a > 0$ as a function of the *sensing radius r*, *communication radius R* and large number of sensors n to provide k_1-coverage simultaneously with k_2-connectivity (see Definition 4, as well as Definition 3).

Namely, we present two *asymptotically optimal algorithms.* It is worth pointing out that Algorithms 1 and 2 are very simple but the asymptotic analysis in Theorems 8 and 9 is challenging. In the proof of Theorems 8 and 9 we apply the statistical results from Section 3.2 and Section 3.3 about central moments special random variables and the distance to the power for special random vectors and Lemma 7 from Section 4.1.

We are now ready to formulate the main results in this subsection.

THEOREM 8. *Assume that n sensors $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_n$ with identical sensing radius r and identical communication radius R are initially randomly placed according to general a-random process. Let assumption (a) in Tables 1 and 2 about r and R holds. Fix $k = \max(k_1, k_2)$.*[1] *Then* $\mathbf{Time_a}(n, r, R)$ *and* $\mathbf{Energy_a}(n, r, R)$ *of Algorithm 1 is respectively,*

$$\mathbf{Time_a}(n, r, R) = \begin{cases} \Theta\left(n^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{if } a \geq 2 \\ O\left(n^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{if } a \in (0, 2) \end{cases},$$

$$\mathbf{Energy_a}(n, r, R) = \begin{cases} \Theta\left(n^{\frac{a}{2}+1}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{if } a \geq 2 \\ O\left(n^{\frac{a}{2}+1}\right) ||\mathbf{E}[\mathbb{S}]||^a & \text{if } a \in (0, 2) \end{cases}.$$

PROOF. Fix $k = \max(k_1, k_2)$. Assume that the n sensors on the plane have identical sensing radius $r = k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ and communication radius $R = k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$. First, observe that sensors at the final positions $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ after Algorithm 1 lie on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$. Observe that

$$||(x_1, y_1) - (0, 0)|| = \frac{||\mathbf{E}\,[\mathbb{S}]\,||}{2},$$

$$||(x_j, y_j) - (x_{j-1}, y_{j-1})|| = \frac{||\mathbf{E}\,[\mathbb{S}]\,||}{2} \text{ if } j \in \{2, \ldots, k\} \text{ (see steps 2–4 of Algorithm 1)},$$

$$||(x_j, y_j) - (x_{j-1}, y_{j-1})|| = ||\mathbf{E}\,[\mathbb{S}]\,|| \text{ if } j \in \{k+1, k+2, \ldots, \min(n, n+2-k)\} \text{ (see steps 5–7 of Algorithm 1)},$$

$$||(x_j, y_j) - (x_{j-1}, y_{j-1})|| = \frac{||\mathbf{E}\,[\mathbb{S}]\,||}{2} \text{ if } j \in \{n+3-k, \ldots, n\} \text{ and } k \geq 3 \text{ (see steps 8–11 of Algorithm 1)}.$$

Therefore, we can apply Lemma 7 for $d := ||\mathbf{E}\,[\mathbb{S}]\,||$ and deduce that every point on the line connecting points $(0, 0), (x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors. Hence, Algorithm 1 is *correct.*

[1] Note that in this study $\max(r, R) = R$ and $\max(k_1, k_2) = k_2$.

We now estimate $\mathbf{Time_a}(n,r,R)$ and $\mathbf{Energy_a}(n,r,R)$ of Algorithm 1. Recall that $j\mathbf{E}[\mathbb{S}] = \mathbf{E}[\mathbb{S}_j]$ (see Equation (9)). Hence

$$\mathbb{S}_j - (x_j, y_j) = \mathbb{S}_j - \frac{\mathbf{E}\left[\mathbb{S}_j\right]}{2} \text{ if } j \in \{1,\dots,k\} \text{ (see steps 2–4 of Algorithm 1)}, \tag{12}$$

$$\mathbb{S}_j - (x_j, y_j) = \mathbb{S}_j - \mathbf{E}\left[\mathbb{S}_j\right] - k\frac{\mathbf{E}\left[\mathbb{S}\right]}{2} \text{ if } j \in \{k+1, k+2, \dots, \min(n, n+2-k)\} \tag{13}$$

(see steps 5–7 of Algorithm 1),

$$\mathbb{S}_j - (x_j, y_j) = \mathbb{S}_j - \mathbf{E}\left[\mathbb{S}_j\right] - k\frac{\mathbf{E}\left[\mathbb{S}\right]}{2} - (j - (n+2-k))\frac{\mathbf{E}\left[\mathbb{S}\right]}{2} \text{ if } j \in \{n+3-k, \dots, n\} \text{ and } k \geq 3 \tag{14}$$

(see steps 8–11 of Algorithm 1).

We are now ready to apply Theorem 4, Theorem 5 and Theorem 6 to evaluate separately Equations (12), (13), and (14).

Case of Equation (12)

Passing to the expectations and using Theorem 6 with $\varepsilon = \frac{1}{2}$ and $O(1) := 0$, we get

$$\mathbf{E}\left[\left\|\mathbb{S}_j - (x_j, y_j)\right\|^a\right] = \mathbf{E}\left[\left\|\mathbb{S}_j - \frac{1}{2}\mathbf{E}[\mathbb{S}_j]\right\|^a\right] = \Theta\left(j^a\right)||\mathbf{E}[\mathbb{S}]||^a.$$

Since $j \in \{2, \dots, k\}$ and k is fixed, we have

$$\mathbf{E}\left[\left\|\mathbb{S}_j - (x_j, y_j)\right\|^a\right] = O(1)||\mathbf{E}[\mathbb{S}]||^a \text{ if } j \in \{2, \dots, k\}, \quad a > 0.$$

Case of Equation (13)

Since $k = O(1)$, we can apply Theorem 5 with $O(1) := \frac{k}{2}$ and get

$$\mathbf{E}\left[\left\|\mathbb{S}_j - (x_j, y_j)\right\|^a\right] = \begin{cases} \Theta\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 2, \\ O\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,2), \end{cases} \quad \text{provided that } j \in \{k+1, k+2, \dots, \min(n, n+2-k)\}.$$

Case of Equation (14)

Observe that $2 \leq k + j - (n+2-k) \leq 2k - 2 = O(1)$. Therefore, we can apply Theorem 5 with $O(1) := -\frac{1}{2}\left(k + j - (n+2-k)\right)$ and get

$$\mathbf{E}\left[\left\|\mathbb{S}_j - (x_j, y_j)\right\|^a\right] = \begin{cases} \Theta\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 2, \\ O\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,2), \end{cases} \quad \text{provided that } j \in \{n+3-k, \dots, n\} \text{ and } k \geq 3.$$

Putting together the Estimations Cases of Equation (12), Case of Equation (13), Case of Equation (14), we have

$$\mathbf{Time_a}(n,r,R) = \max_{1 \leq j \leq n} \mathbf{E}\left[\left\|\mathbb{S}_j - (x_j, y_j)\right\|^a\right] = \begin{cases} \Theta\left(n^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 2 \\ O\left(n^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,2) \end{cases}.$$

of Algorithm 1.

Combining together Estimations: Case of Equation (63), Case of Equation (64), Case of Equation (65) and the well-known identity $\sum_{j=1}^{n} j^{\frac{a}{2}} = \Theta(n^{\frac{a}{2}+1})$, when $a > 0$ we get

$$\mathbf{Energy_a}(n,r,R) = \sum_{j=1}^{n} \mathbf{E}\left[\left\|\mathbb{S}_j - (x_j, y_j)\right\|^a\right] = \begin{cases} \Theta\left(n^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 2, \\ O\left(n^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,2) \end{cases}$$

of Algorithm 1.

This completes the proof of Theorem 8. □

ALGORITHM 1: k-Moving sensors to the anchor points

Require: The initial locations $\mathbb{S}_1, \mathbb{S}_2, \dots \mathbb{S}_n$ of the n sensors with identical sensing radius r and communication radius R on the plane $[0, \infty) \times [0, \infty)$ according to general a-random process. Let assumption (a) in Tables 1 and 2 holds.

Ensure: The final positions $(x_1, y_1), (x_2, y_2), \dots\ (x_n, y_n)$ of the n sensors on the plane $[0, \infty) \times [0, \infty)$ such that every point on the path connecting points $(0, 0), (x_1, y_1), (x_2, y_2), \dots\ (x_n, y_n)$ is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors.

$k := \max(k_1, k_2)$;
for $j = 1$ **to** k **do**
  move the sensor $\mathbb{S}_j$ to the position $(x_j, y_j) = j\frac{\mathbf{E}[\mathbb{S}]}{2}$;
end for
for $j = k + 1$ **to** $\min(n, n + 2 - k)$ **do**
  move the sensor $\mathbb{S}_j$ to the position $(x_j, y_j) = k\frac{\mathbf{E}[\mathbb{S}]}{2} + (j - k)\mathbf{E}[\mathbb{S}]$;
end for
if $k \geq 3$ **then**
  for $j = n + 3 - k$ **to** n **do**
    move the sensor $\mathbb{S}_j$ to the position $(x_j, y_j) = k\frac{\mathbf{E}[\mathbb{S}]}{2} + (n + 2 - 2k)\mathbf{E}[\mathbb{S}] + (j - (n + 2 - k))\frac{\mathbf{E}[\mathbb{S}]}{2}$;
  end for
end if

ALGORITHM 2: (ε, k)-Moving sensors to the anchor points

Require: The initial locations $\mathbb{S}_1, \mathbb{S}_2, \dots \mathbb{S}_n$ of the n sensors with identical sensing radius r and communication radius R on the plane $[0, \infty) \times [0, \infty)$ according to general a-random process. Let assumption (b) or (c) or (d) in Tables 1 and 2 holds.

Ensure: The final positions $(x_1, y_1), (x_2, y_2), \dots\ (x_n, y_n)$ of the n sensors on the plane $[0, \infty) \times [0, \infty)$ such that every point on the path connecting points $(0, 0), (x_1, y_1), (x_2, y_2), \dots\ (x_n, y_n)$ is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors.

$k := \max(k_1, k_2)$;
for $j = 1$ **to** k **do**
  move the sensor $\mathbb{S}_j$ to the position $(x_j, y_j) = j(1 - \varepsilon)\frac{\mathbf{E}[\mathbb{S}]}{2}$;
end for
for $j = k + 1$ **to** $\min(n, n + 2 - k)$ **do**
  move the sensor $\mathbb{S}_j$ to the position $(x_j, y_j) = k(1 - \varepsilon)\frac{\mathbf{E}[\mathbb{S}]}{2} + (j - k)(1 - \varepsilon)\mathbf{E}[\mathbb{S}]$;
end for
if $k \geq 3$ **then**
  for $j = n + 3 - k$ **to** n **do**
    move the sensor $\mathbb{S}_j$ to the position $(x_j, y_j) = k(1 - \varepsilon)\frac{\mathbf{E}[\mathbb{S}]}{2} + (n + 2 - 2k)(1 - \varepsilon)\mathbf{E}[\mathbb{S}] + (j - (n + 2 - k))(1 - \varepsilon)\frac{\mathbf{E}[\mathbb{S}]}{2}$;
  end for
end if

In the next theorem, we analyze Algorithm 2. The proof of Theorem 9 is analogous to that of Theorem 8.

THEOREM 9. *Fix $\epsilon > 0$ independent on $||\mathbf{E}[\mathbb{S}]||$ and n. Assume that n sensors $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_n$ with identical sensing radius r and identical communication radius R are initially randomly placed according to general a-random process. Let assumption (b) or (c) or (d) in Tables 1 and 2 about r and R holds. Fix $k = \max(k_1, k_2)$. Then $\mathbf{Time_a}(n, r, R)$ and $\mathbf{Energy_a}(n, r, R)$ of Algorithm 2 is respectively:*

$$\mathbf{Time_a}(n, r, R) = \Theta\left(n^a\right) ||\mathbf{E}[\mathbb{S}]||^a \quad \text{if } a > 0,$$
$$\mathbf{Energy_a}(n, r, R) = \Theta\left(n^{a+1}\right) ||\mathbf{E}[\mathbb{S}]||^a \quad \text{if } a > 0.$$

4.3 Optimality of Algorithms 1 and 2

In this subsection, we investigate optimality of Algorithms 1 and 2. Namely, we prove that algorithms analyzed in the previous subsection minimize the desired costs, i.e., the time required and the energy consumption of the transportation cost to the power $a > 0$.

First, we must define a optimality metric. We assume that *any algorithm reallocate random sensors to the anchor points.* Namely the jth sensors $\mathbb{S}_j$ on the plane is moved to the position $\mathbb{Q}_j$ on the plane and the anchor position $\mathbb{Q}_j$ does not depend on the random vector $\mathbb{S}_j$.

The optimality of Algorithm 1 when $a \geq 2$ follows directly from Theorem 10. Algorithm 1 indeed minimize the required time and the energy consumption of transportation cost in reallocation of sensors.

We can prove the following general reallocation theorem.

THEOREM 10. *Fix $a \geq 2$. Let $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$ be the initial locations of n according to general a-random process. Assume that $\mathbb{Q}_1, \mathbb{Q}_2, \ldots \mathbb{Q}_n$ is the final location. Then*

$$\max_{1 \leq j \leq n} \mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^a\right] = \Omega\left(n^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a,$$
$$\sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^a\right] = \Omega\left(n^{\frac{a}{2}+1}\right) ||\mathbf{E}[\mathbb{S}]||^a.$$

PROOF. Fix $a \geq 2$. First, observe that

$$||\mathbb{S}_j - \mathbb{Q}_j||^2 = ||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + \mathbf{E}[\mathbb{S}_j] - \mathbb{Q}_j||^2.$$

Since $\mathbf{E}[\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]] = 0$ and the anchor position $\mathbb{Q}_j$ does not depend on the random vector $\mathbb{S}_j$, we have

$$\mathbf{E}[||\mathbb{S}_j - \mathbb{Q}_j||^2] = \mathbf{E}[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j||^2] + \mathbf{E}[||\mathbf{E}[\mathbb{S}_j] - \mathbb{Q}_j||^2].$$

Therefore

$$\mathbf{E}[||\mathbb{S}_j - \mathbb{Q}_j||^2] \geq \mathbf{E}[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j||^2].$$

Using Theorem 4 for $a := 2$, we get

$$\mathbf{E}[||\mathbb{S}_j - \mathbb{Q}_j||^2] = \Theta(j) ||\mathbf{E}[\mathbb{S}]||^2.$$

Applying Jensen's inequality (see Equation (10)) for $X := ||\mathbb{S}_j - \mathbb{Q}_j||^2$ and $f(x) := x^{\frac{a}{2}}$, we get

$$\mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^a\right] \geq \left(\mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^2\right]\right)^{\frac{a}{2}} = \left(\Theta(j) ||\mathbf{E}[\mathbb{S}]||^2\right)^{\frac{a}{2}}.$$

Hence

$$\mathbf{E}[||\mathbb{S}_j - \mathbb{Q}_j||^a] = \Omega\left(j^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a.$$

Finally, using the well-known identity $\sum_{j=1}^{n} j^{\frac{a}{2}} = \Theta(n^{\frac{a}{2}+1})$ we have

$$\max_{1\le j\le n} \mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^a\right] = \Omega\left(n^{\frac{a}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^a,$$

$$\sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^a\right] = \Omega\left(n^{\frac{a}{2}+1}\right) ||\mathbf{E}[\mathbb{S}]||^a.$$

This completes the proof of Theorem 10. □

The situation is more subtle with optimality of Algorithm 2. Let us recall that the final destination of sensors in Algorithm 2 is on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$. Therefore, we must restate the general reallocation in this case.

The next theorem states that Algorithm 2 minimize the time required and the energy consumption of transportation cost in reallocation of sensors on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$.

Theorem 11. *Fix $a \ge 2$. Fix $\varepsilon > 0$ independent on $||\mathbf{E}[\mathbb{S}]||$ and n. Assume that n sensors $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_n$ with identical sensing radius r and identical communication radius R are initially randomly placed according to general a-random process. Let $r = k_1(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ or $R = k_2(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$. Then*

$$\mathbf{Time_a}(n, r, R) = \Omega\left(n^a\right) ||\mathbf{E}[\mathbb{S}]||^a,$$

$$\mathbf{Energy_a}(n, r, R) = \Omega\left(n^{a+1}\right) ||\mathbf{E}[\mathbb{S}]||^a.$$

5 k_1-COVERAGE AND k_2-CONNECTIVITY BY UNRELIABLE SENSORS

After deployment from an aircraft, a mobile sensor on the plane may fail with a certain probability implying each sensor, with some fixed probability independently from other sensors, cannot move, sense and communicate. In this section, we analyze k_1-coverage simultaneously with k_2-connectivity for unreliable sensors. The assumptions about our model are the followings.

Assumption 12 (Unreliable Sensors). *Fix $p \in (0, 1)$ independent on the number of sensors n. We assume that each sensor with probability $1-p$ independently from the others is unreliable (not active), i.e., it cannot move, sense, and communicate anymore.*

Observe that in Assumption 12 each sensor with probability p independently from the others sensors can move, sense and communicate. Obviously when $p = 1$ the sensors can move, sense and communicate and thus are reliable.

We break this section into two subsections. First, the closed analytical formulas are designed for the time required and the energy consumption of the transportation cost to the power $a > 0$ of Algorithm 3 to achieve 1-coverage simultaneously with 1-connectivity for unreliable sensors on the line $[0, \infty)$. Second, we analyze optimal Algorithm 4 for minimizing the time required and the energy consumption of the transportation cost to the power $a > 0$ as a function of the sensing radius r, communication radius R to provide k_1-coverage simultaneously with k_2-connectivity when the sensors are unreliable on the plane. Let us recall that analysis of unreliable sensors on the plane combine results from unreliable sensors on the line and for reliable sensors on the plane.

5.1 Unreliable Sensors on the Line

In this subsection, we give closed analytical formulas for the maximum of expected displacement to the positive integer power t and the sum of expected movement to the positive integer power t to achieve 1-coverage simultaneously with 1-connectivity for equispaced placement of n unreliable sensors on the line $[0, \infty)$ (see Theorem 13 together with Theorem 15 and Theorem 16).

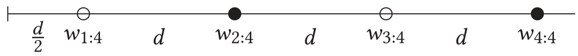

Fig. 3. Four unreliable mobile sensors $w_{1:4}, w_{2:4}, w_{3:4}, w_{4:4}$ on the line.

It will be seen later in this subsection that the mentioned closed analytical formulas for unreliable *sensors only in a very special case* (equispaced placement for 1-coverage with 1-connectivity on the line) *are sufficient to get* tight bounds for the maximum displacement to the real power $a > 0$ and the sum of movement to the real power $a > 0$ to achieve k_1-coverage simultaneously with k_2-connectivity on the plane.

In this subsection, we assume that the n sensors with identical one-dimensional sensing radius $r_1 = \frac{d}{2}$ and one-dimensional communication radius $R_1 = \frac{d}{2}$ occupy the equidistance points $\frac{d}{2} + (i-1)d$ for $i = 1, 2, \ldots n$. Let us recall that the sensor with one-dimensional sensing radius r_1 and communication radius R_1 placed at location x on the line $[0, \infty)$ can sense any point at distance at most r_1 either to the left or right of x and can communicate any point at distance at most R_1 either to the left or right of x. It is easy to see that the deployment of n *reliable sensors* that can sense, communicate, and move with $r_1 = R_1 = \frac{d}{2}$ at the equidistance points $\frac{d}{2} + (i-1)d$ for $i = 1, 2, \ldots n$. ensures the desired 1-coverage with 1-connectivity without any additional movement.

However, the perfectly reliable configuration of n sensors is possible but it is very rare events. Observe that, in our model the probability of perfectly reliable configuration is equal to p^n. Since p is fixed, we see that p^n, which is exponentially small for large n. Therefore, in most cases the sensors have to move to achieve the final location such that, every point from the origin to the last active sensors is within the sensing range and communication range of exactly one sensor (see Algorithm 3).

Although Algorithm 3 is simple, the asymptotic analysis is non-trivial. Figure 3 illustrates four unreliable sensors $w_{1:4}, w_{2:4}, w_{3:4}, w_{4:4}$. Let the black dots represent reliable (active) sensors and white dots represent unreliable sensors. In this example the sensor $w_{2,4}$ moves left-to-right to the position $\frac{d}{2}$ and the sensor $w_{4,4}$ moves to the position $\frac{d}{2} + d$.

ALGORITHM 3: Moving unreliable sensors on the line

Require: The initial locations $w_{1,n} \leq w_{2,n} \leq \cdots \leq w_{n,n}$ of the n sensors with identical one-dimensional sensing radius $r_1 = \frac{d}{2}$ and one-dimensional communication radius $R_1 = \frac{d}{2}$ on the line $[0, \infty)$ at the equidistance points , i.e., $w_{i,n} = \frac{d}{2} + (i-1)d$ for $i = 1, 2, \ldots, n$.

Ensure: The final positions of n sensors on the $[0, \infty)$ such that, every point from the origin to the last active sensors is within the sensing range of at least one sensor and the communication range of at least 1 sensors.

```
z := d/2;
for j = 1 to n do
    if sensor w_{j,n} is active then
        move w_{j,n} left-to-right to z;
        z := z + d;
    else
        do nothing;
    end if
end for
```

We now prove the following exact asymptotic result about the expected displacement to the integer power t for nth unreliable sensor on the line $[0, \infty)$.

THEOREM 13. *Fix t positive integer. Let $|mw_{n,n}|$ be the movement of sensor $w_{n,n}$ in Algorithm 3. Then*

$$\mathbf{E}\left[|mw_{n,n}|^t\right] = d^t p\left((1-p)^t n^t + O(n^{t-1})\right).$$

PROOF. Assume that random variable X_n denotes the number of *unreliable sensors* in the set of sensors of cardinality n. Let us recall that failures of n sensors are random and independent with probability $1-p$. Therefore, random variable X_n obeys the binomial distribution with parameters n and $1-p$. Hence

$$\Pr[X_n = l] = \binom{n}{l}(1-p)^l p^{n-l}, \quad \text{for } \ l \in \{0,1,2,\ldots,n\}. \tag{15}$$

Obviously, the sensor $w_{n,n}$ moves only when it is active. Therefore

$$\mathbf{E}[|mw_{n,n}|^t] = \mathbf{E}[|mw_{n,n}|^t \, |w_{n,n} \ \text{ is active}]p. \tag{16}$$

We now make the following *important observation.*

The movement of sensor $w_{n,n}$ is equal to ld conditional on the event that the sensor $w_{n,n}$ is active and the number of unreliable sensors is l. (Figure 3 illustrates this observation for $n=4$ and $l=2$. In this case, the movement pf sensors $w_{4,4}$ is equal to $2d$). Hence,

$$\mathbf{E}\left[\left(|mw_{n,n}|^t \,\middle|\, w_{n,n} \ \text{ is active}\right)\middle| X_{n-1} = l\right] = (ld)^t. \tag{17}$$

Putting together Equations (15), (16), and (17), we have

$$\mathbf{E}\left[|mw_{n,n}|^t\right] = p\sum_{l=0}^{n-1}\binom{n-1}{l}(1-p)^l p^{n-1-l}(ld)^t = pd^t\sum_{l=0}^{n-1}\binom{n-1}{l}(1-p)^l p^{n-1-l} l^t. \tag{18}$$

Equation (18) give indeed closed expression for the expected movement to the integer power of sensor $w_{n,n}$. However, the resulting formula is difficult to obtain the desired asymptotic result, the main result of Theorem 13. We now apply Stirling number of the second kind technology to Equation (18) to provide asymptotic analysis and thus to prove Theorem 13. We use the following notations for the rising factorial [26]

$$l^{\underline{l_1}} = \begin{cases} 1 & \text{for } l_1 = 0 \\ l(l-1)\ldots(l-l_1+1) & \text{for } l_1 \geq 1 \end{cases}.$$

Observe that

$$\begin{aligned}\sum_{l=0}^{n-1}\binom{n-1}{l}(1-p)^l p^{n-1-l} l^{\underline{l_1}} &= \sum_{l=l+1}^{n-1}\binom{n-1}{l}(1-p)^l p^{n-1-l} l^{\underline{l_1}} \\ &= (n-1)^{\underline{l_1}}(1-p)^{l_1}\sum_{l=l_1}^{n-1}\binom{n-1-l_1}{l-l_1}(1-p)^{l-l_1}p^{n-1-l_1-(l-l_1)} \\ &= (n-1)^{\underline{l_1}}(1-p)^{l_1} = n^{l_1}(1-p)^{l_1} + O\left(n^{l_1-1}\right). \end{aligned} \tag{19}$$

Let $\left\{ t \atop l_1 \right\}$ be the Stirling numbers of the first kind, which are defined for all integer numbers such that $0 \leq l_1 \leq t$.

The following basic formula involving Stirling numbers of the second and rising factorial is known

$$l^t = \sum_{l_1=0}^{t}\left\{ t \atop l_1 \right\} l^{\underline{l_1}} \tag{20}$$

(see Identity (6.10) in Reference [26]).

Putting together Equations (18), (19), and (20), as well as $\left\{ {t \atop t} \right\} = 1$, we have

$$\mathbf{E}\left[|mw_{n,n}|^t\right] = d^t p \left(n^t(1-p)^t + O(n^{t-1})\right).$$

This completes the proof of Theorem 13. □

Combining together the result of Theorem 13 for positive integer t with Jensen's inequality for expectations (Equation (10)), we prove the following asymptotic result about the expected displacement to the power $a > 0$ for nth unreliable sensor. The proof of Theorem 14 is analogous to that of Theorem 4.

THEOREM 14. *Fix $a > 0$. Let $|mw_{n,n}|$ be the movement of sensor $w_{n,n}$ in Algorithm 3. Then*

$$\mathbf{E}\left[|mw_{n,n}|^a\right] = \begin{cases} d^a \Theta(n^a) & \text{if } a \geq 1 \\ d^a O(n^a) & \text{if } a \in (0,1) \end{cases}.$$

We now prove that the maximal expected movement to the power a achieves the latest sensor.

THEOREM 15. *Fix $a > 0$. Let $\left|mw_{j,n}\right|$ be the movement of sensor $w_{j,n}$ in Algorithm 3. Then*

$$\max_{1 \leq j \leq n} \mathbf{E}\left[\left|mw_{j,n}\right|^a\right] = \mathbf{E}\left[|mw_{n,n}|^a\right].$$

PROOF. Fix $j \in \{1, 2, \ldots, n\}$. As in the proof of Theorem 13, we observe that the movement of sensor $w_{j,n}$ is equal to $j_l d$ conditional on the sensor $w_{j,n}$ is active and the number of unreliable sensors in the interval $[0, w_{j,n}]$ is equal to j_l. (See important observation in the proof of Theorem 13).

Hence

$$\left|mw_{j,n}\right|^a \leq |mw_{n,n}|^a, \tag{21}$$

provided that both $w_{j,n}$ and $w_{n,n}$ are active (reliable).

Let A denotes the event that both sensors $w_{j,n}$ and $w_{n,n}$ are active. From Equation (21), we have

$$\mathbf{E}[\left|mw_{j,n}\right|^a |A] \leq \mathbf{E}[|mw_{n,n}|^a |A]. \tag{22}$$

Let B denotes the event that the sensor $w_{j,n}$ is active (reliable) and $w_{n,n}$ is not active (unreliable). Let C denotes the event that the sensor $w_{j,n}$ is not active (unreliable) and $w_{n,n}$ is active (reliable). Let us recall that in our model each sensor with probability p independently from the others sensors is active (see Assumption 12). Hence

$$\Pr B = \Pr C = p(1-p).$$

Observe that

$$\mathbf{E}[\left|mw_{j,n}\right|^a |B] \leq \mathbf{E}[|mw_{n,n}|^a |C].$$

Therefore

$$\mathbf{E}[\left|mw_{j,n}\right|^a |B \cup C] \leq \mathbf{E}[|mw_{n,n}|^a |B \cup C]. \tag{23}$$

Putting together Equations (22) and (23), we have

$$\max_{1 \leq j \leq n} \mathbf{E}\left[\left|mw_{j,n}\right|^a\right] = \mathbf{E}\left[|mw_{n,n}|^a\right].$$

This completes the proof of Theorem 15. □

We now prove the following exact asymptotic result about the sum of expected movement to the integer power t for unreliable sensors.

THEOREM 16. *Fix t positive integer. Let $|mw_{j,n}|$ be the movement of sensor $w_{j,n}$ in Algorithm 3. Then*

$$\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|^t\right] = d^t\left(\frac{(1-p)^t p}{t+1} n^{t+1} + O\left(n^t\right)\right). \tag{24}$$

PROOF. First, we define

$$\mathrm{Cost}(t,n) := \mathbf{E}\left[\sum_{j=1}^{n}\left[\left|mw_{j,n}\right|^t\right]\right] = \sum_{j=1}^{n}\mathbf{E}\left[\left|mw_{j,n}\right|^t\right].$$

Let Y_n be the set of active sensors in the set of n unreliable sensors. We consider the number of active sensor, namely the random variable $|Y_n|$ and define two costs:

$$\mathrm{Cost}(t,n)_i := \mathbf{E}\left[\sum_{w_{j,n}\in Y_n}\left|mw_{j,n}\right|^t \middle| |Y_n| = i\right]$$

and

$$\mathrm{Dost}(t,n)_i := \mathbf{E}\left[\sum_{w_{j,n}\in Y_n}\left(\left|mw_{j,n}\right| + d\right)^t \middle| |Y_n| = i\right].$$

Observe that

$$\mathrm{Cost}(t,n) = \sum_{i=0}^{n}\mathrm{Cost}(t,n)_i \Pr[|Y_n| = i].$$

Let us recall that failures of n sensors are random and independent with probability $1-p$. Therefore $\Pr[|Y_n| = i] = p^i(1-p)^{n-i}$. Hence

$$\sum_{i=0}^{n}\mathbf{E}\left[\sum_{w_{j,n}\in Y_n} d \middle| |Y_n| = i\right]\Pr[|Y_n| = i] = \sum_{i=0}^{n}\mathbf{E}\left[di\binom{n}{i}\right]\Pr[|Y_n| = i] = \sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}di$$
$$= \sum_{i=1}^{n}\binom{n-1}{i-1}p^i(1-p)^{n-i}dn = dpn\sum_{i=1}^{n}\binom{n-1}{i-1}p^{i-1}(1-p)^{n-1-(i-1)} = dpn. \tag{25}$$

We now make the following *crucial observation.*

- When the sensor $w_{1:n}$ is active with probability p then the $\mathrm{Cost}(t,n)$ for n sensors is reduced to the cost $\mathrm{Cost}(t,n-1)$ for $n-1$ sensors.
- If the sensor $w_{1:n}$ is not active with probability $1-p$, then each active sensor in the set of $n-1$ sensors has to move additional distance d.

Therefore

$$\mathrm{Cost}(t,n) = p\mathrm{Cost}(t,n-1) + (1-p)\sum_{i=0}^{n-1}\mathrm{Dost}(t,n-1)_i \Pr[|Y_{n-1}| = i]. \tag{26}$$

We are now ready to prove the main asymptotic result in Theorem 16, namely Equation (24).

The proof of Equation (24) for $\mathrm{Cost}(t,n)$ will be done by induction. For $t = 1$, we directly calculate

$$\sum_{i=0}^{n-1}\mathrm{Dost}(1,n-1)_i \Pr[|Y_{n-1}| = i] = \mathrm{Cost}(1,n-1) + \sum_{i=0}^{n-1}\mathbf{E}\left[\sum_{w_{j,n-1}\in Y_{n-1}} d \middle| |Y_{n-1}| = i\right]\Pr[|Y_{n-1}| = i]. \tag{27}$$

Putting together Equation (26) for $t := 1$ and Equations (27) and (25) for $n := n-1$, we have

$$\text{Cost}(1,n) = \text{Cost}(1,n-1) + d(1-p)p(n-1).$$

Hence, by telescoping, as well as Formula $\text{Cost}(1,0) = 0$ we get

$$\text{Cost}(1,n) = \sum_{j=1}^{n} (\text{Cost}(1,j) - \text{Cost}(1,j-1)) = \sum_{j=1}^{n} d(1-p)p(j-1) = d\left((1-p)p\frac{n^2}{2} + O(n)\right).$$

Assume that the Equation (24) for $n := n-1$ and thus for $\text{Cost}(t,n-1)$ holds for the numbers $1,2,\ldots,t$. Putting together the binomial theorem for $(mw_{j,n-1}+d)^{t+1}$, inductive assumption, and Equation (25) for $d := d^{t+1}$ and $n := n-1$, we have

$$\begin{aligned}\sum_{i=0}^{n-1}&\text{Dost}(t+1,n-1)_i \Pr[|Y_{n-1}| = i] = \sum_{l=1}^{t+1}\binom{t+1}{l} d^{t+1-l}\text{Cost}(l,n-1) + d^{t+1}p(n-1)\\ &= \text{Cost}(t+1,n-1) + d^{t+1}p(n-1) + \sum_{l=1}^{t}\binom{t+1}{l} d^{t+1-l}\left(d^l\frac{(1-p)^l p}{t+1}(n-1)^{l+1} + O\left((n-1)^l\right)\right)\\ &= \text{Cost}(t+1,n-1) + d^{t+1}\left((t+1)\frac{(1-p)^t p}{t+1}n^{t+1} + O\left(n^t\right)\right). \qquad (28)\end{aligned}$$

Putting together Equations (28) and (26) for $t := t+1$ leads to

$$\text{Cost}(t+1,n) = \text{Cost}(t+1,n-1) + d^{t+1}\left((1-p)^{t+1}pn^{t+1} + O\left(n^t\right)\right).$$

Hence, by telescoping, as well as Formula $\text{Cost}(t+1,0) = 0$ we have

$$\begin{aligned}\text{Cost}(t+1,n) &= \sum_{j=1}^{n}(\text{Cost}(t+1,j) - \text{Cost}(t+1,j-1)) = d^{t+1}\sum_{j=1}^{n}\left((1-p)^{t+1}pj^{t+1} + O\left(j^t\right)\right)\\ &= d^{t+1}\left(\frac{(1-p)^{t+1}p}{t+2}n^{t+2} + O\left(n^{t+1}\right)\right).\end{aligned}$$

This gives the claimed Equation (24) for $t := t+1$ and thus for $\text{Cost}(t+1,n)$. □

Putting together the result of Theorem 16 for positive integer t with Jensen's inequality for expectations (Equation (10)), discrete Hölder inequality we prove the following asymptotic result about the sum of expected movement to the real power $a > 0$ for unreliable sensors.

THEOREM 17. *Fix $a > 0$. Let $\left|mw_{j,n}\right|$ be the movement of sensor $w_{j,n}$ in Algorithm 3. Then*

$$\sum_{j=1}^{n}\mathbf{E}\left[\left|mw_{j,n}\right|^t\right] = \begin{cases} d^a\Theta(n^{a+1}) & \text{if } a \geq 1\\ d^a O(n^{a+1}) & \text{if } a \in (0,1)\end{cases}.$$

5.2 Unreliable Sensors on the Plane

In this subsection, we study the k_1-coverage simultaneously with k_2-connectivity for unreliable sensors on the plane. Namely, we minimize the time required and the energy consumption of the transportation cost to the power $a > 0$ of Algorithm 4 as a function of the sensing radius r, communication radius R and large number of sensors n. The precise formulation of our optimization problems is as follows.

Definition 5. Fix $a > 0$. Let $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$ be the initial locations of n unreliable sensors with identical sensing radius r and communication radius R on the plane $[0,\infty) \times [0,\infty)$ according to general a-random process. Assume that $m(\mathbb{S}_j)$ is the movement of sensor $\mathbb{S}_j$ in Algorithm 4. We are interested in asymptotic (in large number of sensors n) for

$$\mathbf{Energy}_{\mathbf{a,p}}(n, r, R) = \sum_{j=1}^{n} \mathbf{E}\left[\left\|m\left(\mathbb{S}_j\right)\right\|^a\right],$$

$$\mathbf{Time}_{\mathbf{a,p}}(n, r, R) = \max_{1 \le j \le n} \mathbf{E}\left[\left\|m\left(\mathbb{S}_j\right)\right\|^a\right].$$

We present Algorithm 4 in two phases. In the first phase (see steps 1–5 in initialization), we apply Algorithm 1 or Algorithm 2 from Section 4.2 to the sensors on the plane. We deduce that every point on the line connecting points $(0, 0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors. Therefore, the first phase reduces the transportation cost to the power $a > 0$ on the plane to the transportation cost to the power $a > 0$ on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$.

Then, in the second phase (see steps 6–16), we provide necessary additional movement to assure that, every point on the path connecting the origin $(0, 0)$ and active sensors is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors. Namely, apply Theorem 14, as well as Theorem 15 and Theorem 17 on the line $[0, \infty)$ for unreliable sensors on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$.

Hence, our Algorithm 4 is *correct*.

ALGORITHM 4: Moving unreliable sensors on the plane

Require: The initial locations $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$ of the n **unreliable** sensors with identical sensing radius r and communication radius R on the plane $[0, \infty) \times [0, \infty)$ according to general a-random process.

Ensure: The final positions of n sensors such that, every point on the path connecting the origin $(0, 0)$ and active sensors is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors.

initialization

1: **if** $\frac{r}{k_1} = \frac{R}{k_2} = \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ **then**
2:     apply Algorithm 1 for the sensors $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$;
3: **else**
4:     Apply Algorithm 2 for the sensors $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$;
5: **end if**

end initialization

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be the location of the n sensors after Algorithm 1 or Algorithm 2 from Section 4.2.

6: $i = 1$;
7: $(x, y) := (x_1, y_1)$;
8: **for** $j = 1$ **to** n **do**
9:     **if** sensor $\mathbb{S}_j$ is active **then**
10:         move $\mathbb{S}_j$ to (x, y);
11:         $i := i + 1$;
12:         $(x, y) := (x_i, y_i)$;
13:     **else**
14:         do nothing;
15:     **end if**
16: **end for**

Moreover in some situations the sensors are often deployed randomly or sprinkled from an aircraft. After the deployment some sensors may become unreliable, i.e., it cannot move, sense and communicate. Hence, the proposed Algorithm 4 seems to be of practical importance.

We are now ready to formulate the main results for unreliable sensors on the plane.

THEOREM 18. *Fix $\varepsilon > 0$ independent on n and $||\mathbf{E}[\mathbb{S}]||$. Assume that*

$$\frac{r}{k_1}, \frac{R}{k_2} \in \left\{(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}, \frac{||\mathbf{E}[\mathbb{S}]||}{2}\right\}.$$

Then the following asymptotic identities are valid:

$$\mathbf{Time}_{\mathbf{a},\mathbf{p}}(n,r,R) = \begin{cases} \Theta\left(n^a\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 1 \\ O\left(n^a\right)||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,1) \end{cases}, \tag{29}$$

$$\mathbf{Energy}_{\mathbf{a},\mathbf{p}}(n,r,R) = \begin{cases} \Theta(n^{a+1})||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 1 \\ O(n^{a+1})||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,1) \end{cases}, \tag{30}$$

PROOF. We now estimate $\mathbf{Time}_{\mathbf{a},\mathbf{p}}(n,r,R)$ and $\mathbf{Energy}_{\mathbf{a},\mathbf{p}}(n,r,R)$ of Algorithm 4. Fix $a > 0$. Let $m_1(\mathbb{S}_j)$ be the movement of sensor $\mathbb{S}_j$ in the first phase of Algorithm 4, $m_2(\mathbb{S}_j)$ be the movement of sensor $\mathbb{S}_j$ in the second phase of Algorithm 4 and $m(\mathbb{S}_j)$ be the movement of sensor $\mathbb{S}_j$ in Algorithm 4. We know that in two-phases Algorithm 4 each sensor moves in the first phase and then moves additionally in the second phase. Therefore,

$$\left\|m(\mathbb{S}_j)\right\| = \left\|m_1(\mathbb{S}_j)\right\| + \left\|m_2(\mathbb{S}_j)\right\|. \tag{31}$$

Now, we define the time required and the energy consumption of the transportation cost to the power $a > 0$ in the second phase of Algorithm 4 by the following formulas:

$$\mathbf{Time}(2) = \max_{1\leq j\leq n} \mathbf{E}\left[\left\|m(\mathbb{S}_j)\right\|^a\right],$$

$$\mathbf{Energy}(2) = \sum_{j=1}^{n} \mathbf{E}\left[\left\|m(\mathbb{S}_j)\right\|^a\right].$$

First, we evaluate $\mathbf{Time}(2)$ and $\mathbf{Energy}(2)$. Let us recall that (x_1,y_1), (x_2,y_2), $\dots$, (x_n,y_n) is the location of the n sensors after the first phase of Algorithm 4 and on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$ (see steps 1–5 in initialization). Since the first phase in currently analyzed Algorithm 4 is exactly Algorithm 1 or Algorithm 2 from Section 4.2, we have

$$||(x_1,y_1)-(0,0)|| = (1-\Delta)\frac{||\mathbf{E}\left[\mathbb{S}\right]||}{2}, \tag{32}$$

$$||(x_j,y_j)-(x_{j-1},y_{j-1})|| = (1-\Delta)\frac{||\mathbf{E}\left[\mathbb{S}\right]||}{l} \text{ if } j \in \{2,\dots,n\}, \tag{33}$$

provided that $\Delta \in \{0,\varepsilon\}$ and $l \in \{1,2\}$ (see the proof of Theorem 8 and Theorem 9 from Section 4.2).

We know from the analysis of unreliable sensors on the line in Section 5.1 in that the movement of active sensor sensor w is equal to $d_1 + d_2 + \dots d_l$, provided that l is the number of unreliable sensors in the interval $[0,w]$, and d_j is the distance between two consecutive sensors. (See the important observation after Equation (16) in the proof of Theorem 13.)

From Equation (32) and Equation (33), we have

$$(1-\varepsilon)\frac{||\mathbf{E}\left[\mathbb{S}\right]||}{2} \leq ||(x_j,y_j)-(x_{j-1},y_{j-1})|| \leq ||\mathbf{E}\left[\mathbb{S}\right]||, \text{ provided that } j \in \{1,\dots,n\}.$$

Therefore, we can apply Theorem 14, as well as Theorem 15 and Theorem 17 on the line $[0,\infty)$ for unreliable sensors on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$. Namely,

- we upper bound $\mathbf{Time}(2)$ and $\mathbf{Energy}(2)$ by the maximal expected displacement to the fixed power $a > 0$ of n sensors and the sum of expected movement to the power $a > 0$ of the individual sensors of Algorithm 3 for $d := ||\mathbf{E}[\mathbb{S}]||$,
- we lower bound $\mathbf{Time}(2)$ and $\mathbf{Energy}(2)$ by the maximal expected displacement to the fixed power $a > 0$ of n sensors and the sum of expected movement to the power $a > 0$ of the individual sensors of Algorithm 3 for $d := \frac{1-\epsilon}{2}||\mathbf{E}[\mathbb{S}]||$.

Hence,

$$\mathbf{Time}(2) = \begin{cases} \Theta(n^a)\,||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 1 \\ O(n^a)\,||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,1) \end{cases}, \tag{34}$$

$$\mathbf{Energy}(2) = \begin{cases} \Theta(n^{a+1})||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \geq 1 \\ O(n^{a+1})||\mathbf{E}[\mathbb{S}]||^a & \text{when } a \in (0,1) \end{cases}. \tag{35}$$

Putting together Inequality (11) for $x := ||m_1(\mathbb{S}_j)||$ and $y := ||m_1(\mathbb{S}_j)||$, as well as Equation (31), we have

$$\left\|m(\mathbb{S}_j)\right\|^a \leq \max(2^{a-1},1)\left(\left\|m_1(\mathbb{S}_j)\right\|^a + \left\|m_2(\mathbb{S}_j)\right\|^a\right),$$

$$\left\|m_2(\mathbb{S}_j)\right\|^a \leq \left\|m(\mathbb{S}_j)\right\|^a.$$

Hence, passing to the expectations lead to

$$\mathbf{Time_{a,p}}(n,r,R) \leq \max(2^{a-1},1)\left(\mathbf{Time_a}(n,r,R) + \mathbf{Time}(2)\right), \tag{36}$$

$$\mathbf{Energy_{a,p}}(n,r,R) \leq \max(2^{a-1},1)\left(\mathbf{Energy_a}(n,r,R) + \mathbf{Energy}(2)\right), \tag{37}$$

$$\mathbf{Time}(2) \leq \mathbf{Time_{a,p}}(n,r,R), \tag{38}$$

$$\mathbf{Energy}(2) \leq \mathbf{Energy_{a,p}}(n,r,R). \tag{39}$$

Putting all together (34)–(39) and Theorems 8 and 9, we obtain Equations (29)–(30). This completes the proof of Theorem 18. □

6 EXTENSIONS

In this study, n mobile sensors $\mathbb{S}_1, \mathbb{S}_2, \ldots \mathbb{S}_n$ are initially randomly deployed on the plane $[0,\infty) \times [0,\infty)$ according to general process. It is assumed that

$$||\mathbf{E}[\mathbb{S}_{j+1}] - \mathbf{E}[\mathbb{S}_j]|| = ||\mathbf{E}[\mathbb{S}_1]|| = ||\mathbf{E}[\mathbb{S}]|| \text{ for } j = 1,2\ldots,n-1,$$

where $||\mathbf{E}[\mathbb{S}]||$ is the expected distance of general a-random process according to which sensors are deployed (see Definition 2 in Section 3.1 and Figure 1 in Section 4).

For both the optimization problems: *time required* and *energy consumption* for transportation cost of sensors to ensure *greedy path k_1-coverage simultaneously with k_2-connectivity* (see Definition 4, as well as Definition 3 in Section 4) the optimal solution is Algorithm 5. (Notice that Algorithm 5 is indeed Algorithm 1 analyzed in Section 4.2. To verify this fact it is enough to apply substitution $j\mathbf{E}[\mathbb{S}] = \mathbf{E}[\mathbb{S}_j]$ (see Equation (9) in Algorithm 1). Then, to attain the k_1-coverage together with k_2-connectivity is for the sensors to assign the sensing radius $r = k_1\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ and communication radius $r = k_2\frac{||\mathbf{E}[\mathbb{S}]||}{2}$.

However, the presented Algorithm 5 is simple it does not give the intuitions about the optimal solution for the general random process. The next Algorithm 6 is both simple and intuitive. We are able to prove the following remark.

ALGORITHM 5: optimal solution

$k = \max(k_1, k_2)$;
for $j = 1$ **to** k **do**
  move the sensor $\mathbb{S}_j$ to the position $\frac{\mathbf{E}[\mathbb{S}_j]}{2}$;
end for
for $j = k + 1$ **to** $\min(n, n + 2 - k)$ **do**
  move the sensor $\mathbb{S}_j$ to the position $\mathbf{E}[\mathbb{S}_j] - k\frac{\mathbf{E}[\mathbb{S}]}{2}$;
end for
if $k \geq 3$ **then**
  for $j = n + 3 - k$ **to** n **do**
    move the sensor $\mathbb{S}_j$ to the position $\mathbf{E}[\mathbb{S}_j] - k\frac{\mathbf{E}[\mathbb{S}]}{2} - (j - (n + 2 - k))\frac{\mathbf{E}[\mathbb{S}]}{2}$;
  end for
end if

Remark 1. If Algorithm 6 is executed for $r = k_1\frac{||\mathbf{E}[\mathbb{S}]||}{2}$ and $R = k_2\frac{||\mathbf{E}[\mathbb{S}]||}{2}$, then every point on the line connecting points $\mathbf{E}[\mathbb{S}_k]$ and $\mathbf{E}[\mathbb{S}_{n-k}]$ is within the range sensing range of k_1 sensors and the communication range of k_2 sensors, provided that $k = \max(k_1, k_2)$.

Proof. The proof of Remark 1 is analogous to that of Theorem 8 and even simpler. We know that sensors at the final positions $\mathbf{E}[\mathbb{S}_1], \mathbf{E}[\mathbb{S}_2], \ldots \mathbf{E}[\mathbb{S}_n]$ after Algorithm 6 lie on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$; and $||\mathbf{E}[\mathbb{S}_{j+1}] - \mathbf{E}[\mathbb{S}_j]|| = ||\mathbf{E}[\mathbb{S}]||$, provided that $j \in \{1, 2, \ldots, n\}$.

Let $k = \max(k_1, k_2)$. Observe that every point $\mathbb{P}$ on the line connecting the points $\mathbf{E}[\mathbb{S}_k]$ and $\mathbf{E}[\mathbb{S}_{n-k}]$ is

- in the sensing range of q_{l_1} sensors in the interval connecting the points $\mathbf{E}[\mathbb{S}_1]$ and $\mathbb{P}$; and q_{r_1} sensors in the interval connecting the points $\mathbb{P}$ and $\mathbf{E}[\mathbb{S}_n]$ and $q_{l_1} + q_{r_1} \geq k_1$,
- in the communication range of q_{l_2} sensors in the interval connecting the points $\mathbf{E}[\mathbb{S}_1]$ and $\mathbb{P}$; and q_{r_2} sensors in the interval connecting the points $\mathbb{P}$ and $\mathbf{E}[\mathbb{S}_n]$ and $q_{l_2} + q_{r_2} \geq k_2$.

This completes the proof of Remark 1. □

ALGORITHM 6: Simplified version of optimal solution

$k = \max(k_1, k_2)$;
for $j = 1$ **to** n **do**
  move the sensor $\mathbb{S}_j$ to the position $\mathbf{E}[\mathbb{S}_j]$;
end for

6.1 Exact Formulas

Fix an even positive integer m. Let us recall that

$$\max_{1 \leq j \leq n} j^{\frac{m}{2}} = n^{\frac{m}{2}}, \quad \sum_{j=1}^{n} j^{\frac{m}{2}} = \frac{1}{\frac{m}{2} + 1} n^{\frac{m}{2}+1} + O\left(n^{\frac{m}{2}}\right). \tag{40}$$

Our theoretical results in the previous sections are for general random process including uniform, exponential and others distributions. If we restrict to specific random variable, then we can give exact formulas for

$$\mathbf{Time_m} = \max_{1 \leq j \leq n} \mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]||^m\right] \quad \text{and} \quad \mathbf{Energy_m} = \sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]||^m\right]$$

for Algorithm 6.

However, the exact Equations (42), (43), (45), (46), (47), and (48) are for Algorithm 6 it is possible to apply the similar combinatorial arguments as in the proof of Theorem 1 and Theorem 2 to validate Equations (42), (43), (45), and (46) for Algorithm 5 and Equation (47) and (48) for properly modified version of Algorithm 5 to the sensors on the $(-\infty, \infty)$.

(a) The uniform distribution.

Let $\{\tau_i\}_{i\ge 1}$ and $\{\xi_i\}_{i\ge 1}$ be identically distributed uniform random variables over the interval $[0, 2]$ and let $\mathbb{V}_j = (\sum_{i=1}^{j} \tau_i, \sum_{i=1}^{j} \xi_i)$. Assume that Assumptions (2) and (4) in Definition 1 hold. Notice that $\mathbf{E}[\tau_i] = 1$ and $\mathbf{Var}[\tau_i] = \frac{1}{3}$ (see Reference [45]). Hence, Assumptions (1) and (3) in Definition 1 hold for $\alpha := 1$ and $\beta := 1$. Observe that

$$\mathbf{E}[\mathbb{V}_j] = \left(\sum_{i=1}^{j} \mathbf{E}[\tau_i], \sum_{i=1}^{j} \mathbf{E}[\xi_i]\right) = (j, j). \tag{41}$$

Combining Corollary 3, Equation (41), as well as $\mathbf{Var}[\tau_1] = \frac{1}{3}$ lead to

$$\mathbf{E}\left[||\mathbb{V}_j - (j, j)||^m\right] = \left(\frac{m}{2}\right)! \left(\frac{2}{3}\right)^{\frac{m}{2}} j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right).$$

Let us recall that the position of jth sensor is determined by the random variable $\mathbb{S}_j = \frac{\mathbb{V}_j}{\lambda}$ for $j \in \{1, 2, \dots, n\}$. Hence

$$\mathbf{E}[\mathbb{S}_j] = \frac{(j, j)}{\lambda}, \quad ||\mathbf{E}[\mathbb{S}]||^m = \frac{2^{\frac{m}{2}}}{\lambda^m}$$

(see Definition 2 for $\alpha := 1$ and $\beta := 1$ in Section 3.1).

Using this, we have

$$\mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]||^m\right] = \frac{\left(\frac{m}{2}\right)!}{(3)^{\frac{m}{2}}} j^{\frac{m}{2}} \, ||\mathbf{E}[\mathbb{S}]||^m + O\left(j^{\frac{m}{2}-1}\right) ||\mathbf{E}[\mathbb{S}]||^m.$$

Applying Equation (40), we have

$$\mathbf{Time_m} = \frac{\left(\frac{m}{2}\right)!}{(3)^{\frac{m}{2}}} n^{\frac{m}{2}} \, ||\mathbf{E}[\mathbb{S}]||^m + O\left(n^{\frac{m}{2}-1}\right) ||\mathbf{E}[\mathbb{S}]||^m, \tag{42}$$

$$\mathbf{Energy_m} = \frac{\left(\frac{m}{2}\right)!}{(3)^{\frac{m}{2}} \left(\frac{m}{2}+1\right)} n^{\frac{m}{2}+1} \, ||\mathbf{E}[\mathbb{S}]||^m + O\left(n^{\frac{m}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^m. \tag{43}$$

(b) The exponential distribution.

Let $\{\tau_i\}_{i\ge 1}, \{\tau_i\}_{i\ge 1}$ be identically distributed exponential random variables with rate equal to 1 (parameter $\lambda = 1$) and let $\mathbb{V}_j = (\sum_{i=1}^{j} \tau_i, \sum_{i=1}^{j} \xi_i)$. Assume that Assumptions (2) and (4) in Definition 1 hold. We know that $\mathbf{E}[\tau_i] = 1$ and $\mathbf{Var}[\tau_i] = 1$ (see Reference [45]). Hence, Assumptions (1) and (3) in Definition 1 hold for $\alpha := 1$ and $\beta := 1$. Notice that

$$\mathbf{E}[\mathbb{V}_j] = \left(\sum_{i=1}^{j} \mathbf{E}[\tau_i], \sum_{i=1}^{j} \mathbf{E}[\xi_i]\right) = (j, j). \tag{44}$$

Applying Corollary 3, Equation (44), as well as $\mathbf{Var}[\tau_1] = 1$ we have

$$\mathbf{E}\left[||\mathbb{V}_j - (j, j)||^m\right] = \left(\frac{m}{2}\right)! 2^{\frac{m}{2}} j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right).$$

Since $\lambda = 1$ the position of jth sensor is determined by the random variable $\mathbb{S}_j = \mathbb{V}_j$. Hence,

$$\mathbf{E}[\mathbb{S}_j] = (j, j), \quad ||\mathbf{E}[\mathbb{S}]||^m = 2^{\frac{m}{2}}$$

(see Definition 2 for $\lambda := 1$, $\alpha := 1$ and $\beta := 1$ in Section 3.1).

Using this, we have

$$\mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]||^m\right] = \left(\frac{m}{2}\right)! j^{\frac{m}{2}} ||\mathbf{E}[\mathbb{S}]||^m + O\left(j^{\frac{m}{2}-1}\right) ||\mathbf{E}[\mathbb{S}]||^m.$$

Applying Equation (40), we get

$$\mathbf{Time_m} = \left(\frac{m}{2}\right)! n^{\frac{m}{2}} ||\mathbf{E}[\mathbb{S}]||^m + O\left(n^{\frac{m}{2}-1}\right) ||\mathbf{E}[\mathbb{S}]||^m, \tag{45}$$

$$\mathbf{Energy_m} = \frac{\left(\frac{m}{2}\right)!}{\left(\frac{m}{2}+1\right)} n^{\frac{m}{2}+1} ||\mathbf{E}[\mathbb{S}]||^m + O\left(n^{\frac{m}{2}}\right) ||\mathbf{E}[\mathbb{S}]||^m. \tag{46}$$

(c) The Gaussian distribution.
Let $\{\tau_i\}_{i\geq 1}$ be identically distributed normal random variables with $\mathbf{E}[\tau_i] = 1$ and $\mathbf{Var}[\tau_i] = \sigma^2 > 0$, and let $X_j := \sum_{i=1}^{j} \tau_i$. Assume that Assumption (2) in Definition 1 holds. Let us recall that the theoretical analysis in this article consider random sensors on the plane $[0, \infty) \times [0, \infty)$ displaced according to general random process, This random placement uses normal distribution on the $(-\infty, \infty)$. Therefore, we must restate and get the result on the $(-\infty, \infty)$; but our analysis is even simpler. First, observe that the main Theorem 1 in Section 3.2 is valid regardless of the assumption about positivity of random variables. Putting together Theorem 1, $\mathbf{Var}[\tau_i] = \sigma^2 > 0$, and $\mathbf{E}[X_j] = j$, we have

$$\mathbf{E}\left[\left(X_j - j\right)^m\right] = \frac{m!\sigma^m}{(2)^{\frac{m}{2}}\left(\frac{m}{2}\right)!} j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right).$$

In our random placement of sensors on the $(-\infty, \infty)$ the position of jth sensor is determined by the random variable $\mathbb{S}_j = \frac{X_j}{\lambda}$ for $j \in \{1, 2, \ldots, n\}$. Hence

$$\mathbf{E}\left[S_j\right] = \frac{j}{\lambda}.$$

Therefore

$$|\mathbf{E}[S]|^m = |\mathbf{E}[S_1]|^m = \left|\frac{\mathbf{E}[\tau_1]}{\lambda}\right|^m = \frac{1}{\lambda^m}.$$

Using this, we have

$$\mathbf{E}\left[\left(S_j - \mathbf{E}\left[S_j\right]\right)^m\right] = \frac{m!\sigma^m}{(2)^{\frac{m}{2}}\left(\frac{m}{2}\right)!} j^{\frac{m}{2}} |\mathbf{E}[S]|^m + O\left(j^{\frac{m}{2}-1}\right) |\mathbf{E}[S]|^m.$$

Applying Equation (40), we have

$$\mathbf{Time_m} = \frac{m!\sigma^m}{(2)^{\frac{m}{2}}\left(\frac{m}{2}\right)!} n^{\frac{m}{2}} |\mathbf{E}[\mathbb{S}]|^m + O\left(n^{\frac{m}{2}-1}\right) |\mathbf{E}[\mathbb{S}]|^m, \tag{47}$$

$$\mathbf{Energy_m} = \frac{m!\sigma^a}{(2)^{\frac{m}{2}}\left(\frac{m}{2}+1\right)!} n^{\frac{m}{2}+1} |\mathbf{E}[\mathbb{S}]|^m + O\left(n^{\frac{m}{2}}\right) |\mathbf{E}[\mathbb{S}]|^m. \tag{48}$$

6.2 Variable Sensing and Communication Radii

In the above derivations, it is assumed that the sensors have identical sensing radius r and communication radius R. However, this approach is not limited to homogeneous setting only, and the proposed *methodology can also handle sensors with variable sensing and communication radii.* Let $r_1, r_2, \dots r_n$ be the sensing radii of n sensors and let $R_1, R_2, \dots R_n$ be the communication radii of n sensors in a heterogeneous mobile sensor network. Let $r_{\min} = \min(r_1, r_2, \dots r_n,)$. and let $R_{\min} = \min(R_1, R_2, \dots R_n,)$. Then, the optimal solution for the time required and the energy consumption of transportation cost for sensors to provide the desired greedy path k_1-coverage simultaneously with k_2-connectivity is to choose the characteristic of general random process $||\mathbf{E}[\mathbb{S}]|| = 2\min(\frac{r_{\min}}{k_1}, \frac{R_{\min}}{k_2})$ and execute optimal Algorithm 5. Of course, the tradeoffs arising among the parameters $r_1, r_2, \dots, r_n, R_1, R_2, \dots, R_n$, and $||\mathbf{E}[\mathbb{S}]||$ to provide the desired greedy path k_1-coverage together with k_2-connectivity need further theoretical studies, as well as experimental evaluation.

6.3 Sensor Deployment in Three-dimensional Space

The *proposed theory* for sensors on the plane *can be extended* to the cases where the sensors are dropped in three-dimensional region as well as in higher dimensions. Let us consider three dimensional case. We can similarly to (m, α, β)-property in two dimension (see Definition 1) define $(m, \alpha, \beta, \gamma)$-property in three dimension. Then the position of the sensor $\mathbb{S}_j$ is determined by the vector $\frac{1}{\lambda}(X_j, Y_j, Z_j)$, where the vector (X_j, Y_j, Z_j) has $(m, \alpha, \beta, \gamma)$-property. However, it is natural extension our two-dimensional investigation to higher dimension the *crucial in analysis* of the *threshold phenomena* closed analytical formula in three dimension for $\mathbf{E}[||(X_j, Y_j, Z_j) - \mathbf{E}[(X_j, Y_j, Z_j)]||^m]$ is even complicated than closed analytical formula in two dimension for $\mathbf{E}[||(X_j, Y_j) - \mathbf{E}[(X_j, Y_j)]||^m]$ in Theorem 2 from Section 3.3.

6.4 Other Trajectories

This subsection discusses other greedy strategies. We assume that n mobile sensors $\mathbb{S}_1, \mathbb{S}_2, \dots \mathbb{S}_n$ initially randomly deployed on the plane $[0, \infty) \times [0, \infty)$ according to general process move to the final destination $\mathbb{P}_1, \mathbb{P}_2, \dots \mathbb{P}_n$ situated on the other lines not only passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$ (see Figure 4). We explain that the time required and the energy consumption is minimized when the points $\mathbb{P}_1, \mathbb{P}_2, \dots \mathbb{P}_n$ are on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$.

We consider general straight line with gradient $M_\lambda \mathbf{E}[\mathbb{S}]$ and intercept $c = 0$, where $M_\lambda = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{bmatrix}$ is the rotation matrix.

Let $z_j \geq 0$. In the first strategy the sensor $\mathbb{S}_j$ moves to the position $z_j \mathbf{E}[\mathbb{S}]$ on the line with gradient $\mathbf{E}[\mathbb{S}]$ and intercept $c = 0$ and in the second the sensor $\mathbb{S}_j$ moves to the position $z_j M_\gamma \mathbf{E}[\mathbb{S}]$ on the line with gradient $z M_\gamma \mathbf{E}[\mathbb{S}]$ and intercept $c = 0$ We now compare these movements.

The direct calculation for vectors, as well as $\mathbf{E}[\mathbb{S}_j] = j\mathbf{E}[\mathbb{S}]$ (see Equation (9)) lead to

$$\mathbf{E}\left[||\mathbb{S}_j - z_j M_\gamma \mathbf{E}[\mathbb{S}]||^2\right] = \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^2\right] + 2jz_j(1 - \cos(\gamma))||\mathbf{E}[\mathbb{S}]||^2.$$

Since $(1 - \cos(\gamma))||\mathbf{E}[\mathbb{S}]||^2 \geq 0$ and $z_j \geq 0$, we have

$$\mathbf{E}\left[||\mathbb{S}_j - z_j M_\gamma \mathbf{E}[\mathbb{S}]||^2\right] \geq \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^2\right].$$

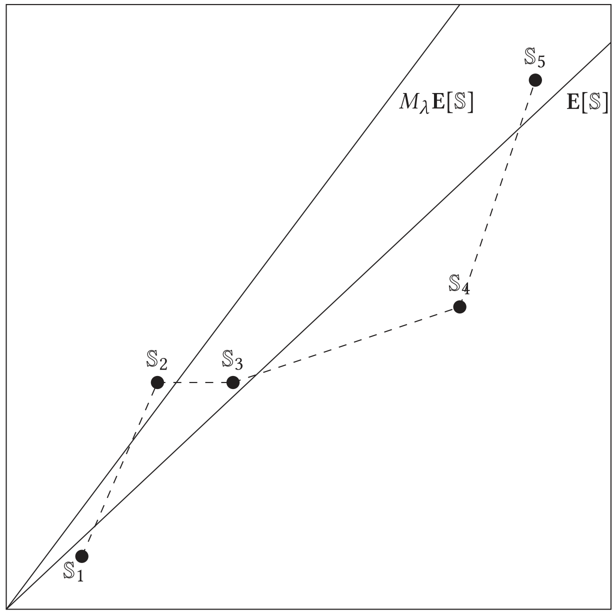

Fig. 4. Five mobile sensors $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_5$ dropped from the aircraft on the plane.

Therefore,

$$\max_{j \in \{1,2,\ldots,n\}} \mathbf{E}\left[||\mathbb{S}_j - z_j M_\gamma \mathbf{E}[\mathbb{S}]||^2\right] \geq \max_{j \in \{1,2,\ldots,n\}} \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^2\right],$$

$$\sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - z_j M_\gamma \mathbf{E}[\mathbb{S}]||^2\right] \geq \sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^2\right].$$

Hence, on the line not passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$ the time required and the energy of the transportation cost to the power 2 of transportation cost in reallocation of sensors to provide the desired k_1-coverage together with k_2-connectivity are both minimized.

However, when $a \geq 2$ and we know the asymptotic of the expected cost for large j. Namely

$$\mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^2\right] = \Theta(j)||\mathbf{E}[\mathbb{S}]||^2, \quad \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^a\right] = \Theta\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^2 \tag{49}$$

the much stronger result is possible. (Notice that the mentioned expected costs (Equation (49)) are the expected movement of sensors when the time required and the energy consumption of transportation cost for sensors to provide the desired k_1-coverage with k_2-connectivity is minimized). Combining Equation (49) with Jensen's inequality for expectations (Equation (10)) we can prove the following upper bounds

$$\max_{j \in \{1,2,\ldots,n\}} \mathbf{E}\left[||\mathbb{S}_j - z_j M_\gamma \mathbf{E}[\mathbb{S}]||^a\right] \geq \max_{j \in \{1,2,\ldots,n\}} \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^a\right],$$

$$\sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - z_j M_\gamma \mathbf{E}[\mathbb{S}]||^a\right] \geq \sum_{j=1}^{n} \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^a\right].$$

The presented argument is valid for other expected costs for large j. Namely

$$\mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^2\right] = \Theta\left(j^2\right)||\mathbf{E}[\mathbb{S}]||^2, \quad \mathbf{E}\left[||\mathbb{S}_j - z_j \mathbf{E}[\mathbb{S}]||^a\right] = \Theta\left(j^a\right)||\mathbf{E}[\mathbb{S}]||^a.$$

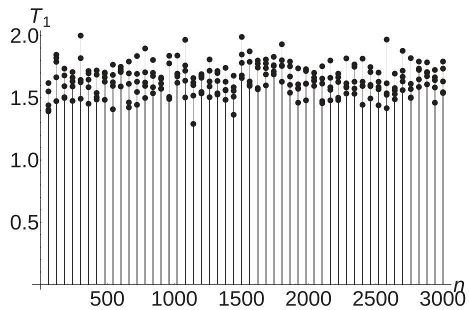

Fig. 5. $\mathbf{Time_1}(n,r,R) = \Theta(1)$ of Algorithm 1 for $k = k_1 = k_2 = 2$.

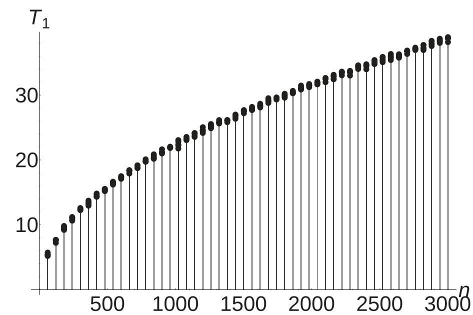

Fig. 6. $\mathbf{Time_1}(n,r,R) = \Theta(n^{\frac{1}{2}})$ of Algorithm 2 for $\varepsilon = 0.5$ and $k = k_1 = k_2 = 2$.

6.5 Real-life Sensor Deployment

It is worth mentioning that our experimental evaluations in the next Section 7 are restricted to some specific random variables. It is assumed that the sensors are randomly displaced according to the evaluated distributions. *However,* the methodology *proposed in this article is also* applicable *for the real-life sensor deployment.* When the sensors $\mathbb{S}_1, \mathbb{S}_2, \dots, \mathbb{S}_n$ are distributed according to some unknown distribution, we have to estimate $\mathbf{E}[\mathbb{S}_1], \mathbf{E}[\mathbb{S}_2], \dots \mathbf{E}[\mathbb{S}_n]$ and execute optimal Algorithm 6. The estimation of $\mathbf{E}[\mathbb{S}_1], \mathbf{E}[\mathbb{S}_2], \dots\ \mathbf{E}[\mathbb{S}_n]$ is even reduced to the estimation $\mathbf{E}[\mathbb{S}] = \mathbf{E}[\mathbb{S}_1]$ when $\mathbf{E}[\mathbb{S}_j] = j\mathbf{E}[\mathbb{S}_1]$. Then, we assign the sensing radius $r = k_1\frac{\mathbf{E}[\mathbb{S}]}{2}$ and communication radius $R = k_2\frac{\mathbf{E}[\mathbb{S}]}{2}$ to optimize the energy consumption and the time required in movement for greedy path k_1-coverage simultaneously with k_2-connectivity.

Hence, it will be interesting to provide experiments considering some realistic settings. However, this experimental evaluation may be expensive due to the large realistic data that would be required for reliable estimation.

7 NUMERICAL RESULTS

In this section, we provide a set of experiments to illustrate Theorem 8 and Theorem 9 for reliable sensors (see Section 4.2); Theorem 14 together with Theorem 15 and Theorem 17 for unreliable sensors (see Section 5.1). While the theoretical results are for general random process, in the experiments we have restricted to specific random variable. We evaluate both time and energy.

7.1 Evaluation of $\mathbf{Time_1}(n,r,R)$.

For the case of time, we evaluate 2-*coverage* together 2-*connectivity* we choose to experiments two sequences $\{g_i\}_{i\ge1}$, $\{h_i\}_{i\ge1}$ of exponential distribution with parameter $\lambda = \sqrt{n}$. Assume independence between sequences $\{\tau_i\}_{i\ge1}$, $\{\xi_i\}_{i\ge1}$; and additionally assume that, random variables $\{\tau_i\}_{i\ge1}$ and $\{\xi_i\}_{i\ge1}$ are independent. Then, in Definition 1 properties (2) and (4) hold for all m integer greater than 2. Notice that $\mathbf{E}[g_i] = \mathbf{E}[h_i] = \frac{1}{\lambda}$ for $i \ge 1$ (see Reference [47]). In this experimental evaluation, the position of the jth sensor random variable $\mathbb{S}_j$ is defined by the formula $\mathbb{S}_j = (\sum_{i=1}^{j} g_i, \sum_{i=1}^{j} h_i)$. Hence, in Definition 1 we have $\tau_i = g_i\lambda$, $\xi_i = h_i\lambda$ and $\alpha = \beta = 1$. Therefore, the expected distance of this specified random process is given by the formula $||\mathbf{E}\left[\mathbb{S}\right]|| = \frac{\sqrt{2}}{\lambda} = \frac{\sqrt{2}}{\sqrt{n}}$ (see Equation (8)).

Figure 5 depicts experimental $\mathbf{Time_1}(n,r,R)$ of Algorithm 1 (Algorithm 2 when $\varepsilon = 0$) for 2-*coverage* together with 2-*connectivity* when $r = R = 2\frac{||\mathbf{E}[\mathbb{S}]||}{2} = \frac{\sqrt{2}}{\sqrt{n}}$. In this case, we conduct Algorithm 7 for $\varepsilon = 0$.

ALGORITHM 7:

$n := 1$
while n $\leq$ 50 **do**
  for $j = 1$ **to** 100 **do**
    Generate $\mathbb{S}_1 = (X_1(j), Y_1(j)), \mathbb{S}_2 = (X_2(j), Y_2(j)), \ldots, \mathbb{S}_{60n} = (X_{60n}(j), Y_{60n}(j))$ random points on the plane $[0, \infty) \times [0, \infty)$ such that $\forall_{i \in \{1,2,\ldots,60n\}}$, ($X_i(j)$ and $Y_i(j)$ are the sum of i independent and identically distributed exponential random variables with parameter $\lambda = \sqrt{60n}$; $X_i(j)$ and $Y_i(j)$ are independent);
    Calculate $\mathbf{T_1}(60n, j) = \max_{1 \leq t \leq 60n} \|\mathbb{S}_t - (x_t, y_t)\|$ according to Algorithm 2 for $k = 2$, $\mathbf{E}[\mathbb{S}] = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ and ε;
  end for
  for $l = 1$ **to** 5 **do**
    Calculate the average $\mathbf{T_1}(60n) = \frac{1}{20}\sum_{v=1}^{20} \mathbf{T_1}(60n, v + (l-1)20)$;
    Insert the points $\mathbf{T_1}(60n)$ into the chart;
  end for
  $n := n + 1$
end while

In Figure 6, the black points represent numerical results of conducted Algorithm 7 for $\varepsilon = 0.5$. Notice that the experimental $\mathbf{Time_1}(n, r, R)$ of Algorithm 2 is in $\Theta(n^{\frac{1}{2}})$. Hence, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(n^{\frac{1}{2}})$ (see $\mathbf{Time_a}(n, r, R)$ in Theorem 9 for $a := 1$ and $||\mathbf{E}[\mathbb{S}]|| = \frac{\sqrt{2}}{\sqrt{n}}$).

In Figure 5, the black points represent numerical results of conducted Algorithm 7 for $\varepsilon = 0$. Notice that the experimental $\mathbf{T_1}(n, r, R)$ of Algorithm 1 is in $\Theta(1)$. Hence, the carried out experiments confirm very well the obtained theoretical upper bound $O(1)$ (see $\mathbf{Time_a}(n, r, R)$ in Theorem 8 for $a := 1$ and $||\mathbf{E}[\mathbb{S}]|| = \frac{\sqrt{2}}{\sqrt{n}}$).

Figure 6 depicts the experimental $\mathbf{Time_1}(n, r, R)$ of Algorithm 2 for 2-*coverage* together with 2-*connectivity* when $\varepsilon = 0.5$ considering the parameters $r = \frac{\sqrt{2}}{2\sqrt{n}}$ or $R = \frac{\sqrt{2}}{2\sqrt{n}}$. In this case, we conduct Algorithm 7 for $\varepsilon = 0.5$.

It is worth pointing out that Figures 5 and 6 together illustrates the sharp decline from $\Theta(n^{\frac{1}{2}})$ to $\Theta(1)$ in $\mathbf{Time_1}(n, r, R)$ for 2-*coverage* together with 2-*connectivity* when r increases from $\frac{\sqrt{2}}{2\sqrt{n}}$ to $\frac{\sqrt{2}}{\sqrt{n}}$ or R increases from $\frac{\sqrt{2}}{2\sqrt{n}}$ to $\frac{\sqrt{2}}{\sqrt{n}}$.

7.2 Evaluation of $\mathbf{Energy_2}(n, r, R)$

For the case of energy, we evaluate 1-coverage together with 1-connectivity we choose to experiments two sequences $\{g_i\}_{i\geq 1}$, $\{h_i\}_{i\geq 1}$ of uniform distribution over the interval $[0, \frac{2}{\sqrt{n}}]$. Assume independence between sequences $\{\tau_i\}_{i\geq 1}$, $\{\xi_i\}_{i\geq 1}$; and additionally assume that, random variables $\{\tau_i\}_{i\geq 1}$ and $\{\xi_i\}_{i\geq 1}$ are independent. Then, in Definition 1 properties (2) and (4) hold for all m integer greater than 2. Notice that $\mathbf{E}[g_i] = \mathbf{E}[h_i] = \frac{1}{\sqrt{n}}$ for $i \geq 1$ (see Reference [47]). In this experimental evaluation, the position of the jth sensor random variable $\mathbb{S}_j$ is defined by the formula $\mathbb{S}_j = (\sum_{i=1}^{j} g_i, \sum_{i=1}^{j} h_i)$. Hence, in Definition 1 we have $\tau_i = g_i\sqrt{n}$, $\xi_i = h_i\sqrt{n}$, $\alpha = \beta = 1$ and in Definition 2 it is $\lambda = \sqrt{n}$. Therefore, the expected distance of of this specified random process is given by the formula $||\mathbf{E}[\mathbb{S}]|| = \frac{\sqrt{2}}{\lambda} = \frac{\sqrt{2}}{\sqrt{n}}$ (see Equation (8)).

Figure 7 depicts experimental $\mathbf{Energy}_2(n, r, R)$ of Algorithm 1 (Algorithm 2 when $\varepsilon = 0$) for 1-*coverage* together with 1-*connectivity* when $r = R = \frac{||\mathbf{E}[\mathbb{S}]||}{2} = \frac{\sqrt{2}}{2\sqrt{n}}$. In this case, we conduct Algorithm 8 for $\varepsilon = 0$.

ALGORITHM 8:

$n := 1$
while n ≤ 50 **do**
  for $j = 1$ **to** 100 **do**
    Generate $\mathbb{S}_1 = (X_1(j), Y_1(j))$, $\mathbb{S}_2 = (X_2(j), Y_2(j))$, ..., $\mathbb{S}_{60n} = (X_{60n}(j), Y_{60n}(j))$ random points on the plane $[0, \infty) \times [0, \infty)$ such that $\forall_{i \in \{1,2,...,60n\}}$, ($X_i(j)$ and $Y_i(j)$ are the sum of i independent and identically distributed of uniform distribution over the interval $[0, \frac{2}{\sqrt{n}}]$; $X_i(j)$ and $Y_i(j)$ are independent);
    Calculate $\mathbf{E}_2(60n, j) = \sum_{t=1}^{60n} ||\mathbb{S}_t - (x_t, y_t)||^2$ according to Algorithm 2 for $k = 1$, $\mathbf{E}[\mathbb{S}] = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ and ε;
  end for
  for $l = 1$ **to** 5 **do**
    Calculate the average $\mathbf{E}_2(60n) = \frac{1}{20} \sum_{v=1}^{20} \mathbf{E}_2(60n, v + (l-1)20)$;
    Insert the points $\mathbf{E}_2(60n)$ into the chart;
  end for
  $n := n + 1$
end while

In Figure 7, the black points represent numerical results of conducted Algorithm 8 for $\varepsilon = 0$. The additional line $\{(n, \frac{1}{3}n), 1 \leq n \leq 3000\}$ is the leading term in theoretical estimation. Applying Corollary 3 for $\mathbb{V}_j = (\sum_{i=1}^{j} g_i \sqrt{n}, \sum_{i=1}^{j} h_i \sqrt{n})$, where $\mathbf{var}[g_i] = \mathbf{var}[h_i] = \frac{1}{3n}$ and omitting the technical details, we can get the leading term. Hence,

$$\sum_{t=1}^{n} ||\mathbb{S}_t - (x_t, y_t)||^2 = \frac{1}{3}n + O(1) \quad \text{according to Algorithm 1 for } k = 1 \text{ and } \mathbf{E}[\mathbb{S}] = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right).$$

Figure 8 depicts the experimental $\mathbf{Energy}_2(n, r, R)$ of Algorithm 2 for 1-*coverage* together with 1-*connectivity* when $\varepsilon = 0.5$ considering the parameters $r = \frac{\sqrt{2}}{4\sqrt{n}}$ or $R = \frac{\sqrt{2}}{4\sqrt{n}}$. In this case, we conduct Algorithm 8 for $\varepsilon = 0.5$.

In Figure 8, the black points represent numerical results of conducted Algorithm 8 for $\varepsilon = 0.5$. Notice that the experimental $\mathbf{Energy}_2(n, r, R)$ of Algorithm 2 is in $\Theta(n^2)$. Hence, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(n^2)$ (see $\mathbf{Energy}_2(n, r, R)$ in Theorem 9 for $a := 2$ and $||\mathbf{E}[\mathbb{S}]|| = \frac{\sqrt{2}}{\sqrt{n}}$).

It is worth pointing out that Figures 7 and 8 together illustrates the sharp decline from $\Theta(n^2)$ to $\Theta(n)$ in $\mathbf{Energy}_2(n, r, R)$ for 1-*coverage* together with 1-*connectivity* when r increases from $\frac{\sqrt{2}}{2\sqrt{n}}$ to $\frac{\sqrt{2}}{4\sqrt{n}}$ or R increases from $\frac{\sqrt{2}}{2\sqrt{n}}$ to $\frac{\sqrt{2}}{4\sqrt{n}}$.

7.3 Evaluation of Time for Unreliable Sensors

We evaluate the maximum displacement to the power $a = 1$ (**Time**) of unreliable sensors in Algorithm 3. In the experimental evaluation, we choose reliability parameter $p = \frac{1}{2}$, i.e., each sensor with probability $p = \frac{1}{2}$ independently from the other sensor is active (see Assumption 12).

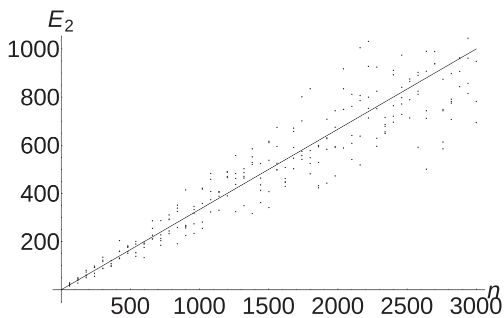

Fig. 7. $\mathbf{Energy}_2(n, r, R) = \frac{1}{3}n$ of Algorithm 1 for $k_1 = k_2 = k = 1$.

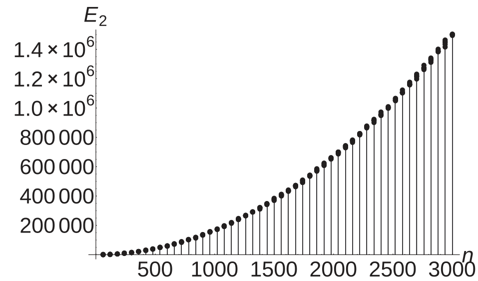

Fig. 8. $\mathbf{Energy}_2(n, r, R) = \Theta\left(n^2\right)$ of Algorithm 2 for $\varepsilon = 0.5$ and $k = k_1 = k_2 = 1$.

ALGORITHM 9:

$n := 1$
while n ≤ 50 **do**
  for $j = 1$ **to** 50 **do**
    Generate $X_1(j), X_2(j), \ldots, X_{60n}(j)$ unreliable sensors at the equidistance points, i.e., $X_i(j) = \frac{d}{2} + (i-1)d$ for $i = 1, 2, \ldots, 60n$ such that $\forall_{i \in \{1,2,\ldots,60n\}}$, ($X_i(j)$ with probability $\frac{1}{2}$ independently from others sensors is realiable, i.e., it can move, sense and communicate);
    Calculate $\mathbf{T}_{\mathbf{a},\frac{1}{2}}(60n, j)$ the maximum of the reliable sensor's displacements to the power a in Algorithm 3;
  end for
  for $l = 1$ **to** 5 **do**
    Calculate the average $\mathbf{T}_{\mathbf{a},\frac{1}{2}}(60n) = \frac{1}{10}\sum_{v=1}^{10} \mathbf{T}_{\mathbf{a},\frac{1}{2}}(60n, v + (l-1)10)$;
    Insert the points $\mathbf{T}_{\mathbf{a},\frac{1}{2}}(60n)$ into the chart;
  end for
  $n := n + 1$
end while

Figures 9 and 10 depict experimental maximum displacement to the power 1 of Algorithm 3 when $d = 1$ and $d = \frac{1}{n}$. In this case, we conduct Algorithm 9 for parameters $a = 1$, $d = 1$ and parameters $a = 1$, $d = \frac{1}{n}$. In Figures 9 and 10 the black dots represents numerical results of conducted Algorithm 9 for parameters $a = 1$, $d = 1$ and parameters $a = 1$, $d = \frac{1}{n}$.

Notice that the experimental maximum displacement to the power 1 of Algorithm 3 for $d = 1$ is in $\Theta(n)$. Hence, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(n)$ (see Theorem 14 together with Theorem 15 for $a := 1$ and $d := 1$).

Observe that the experimental maximum displacement to the power 1 of Algorithm 3 for $d = \frac{1}{n}$ is in $\Theta(1)$. Hence, as in the previous experiment, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(1)$ (see Theorem 14 together with Theorem 15 for $a := 1$ and $d := \frac{1}{n}$).

Notice that the experimental maximum displacement to the power 2 of Algorithm 3 for $d = 1$ is in $\Theta\left(n^2\right)$. Hence, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(n)$ (see Theorem 14 together with Theorem 15 for $a := 2$ and $d := 1$).

Observe that the experimental maximum displacement to the power 2 of Algorithm 3 for $d = \frac{1}{n}$ is in $\Theta(1)$. Hence, as in the previous experiment, the carried out experiments confirm very well

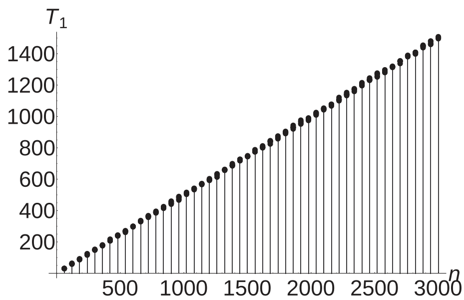

Fig. 9. Maximum displacement to the power 1 of Algorithm 3 for $d = 1$.

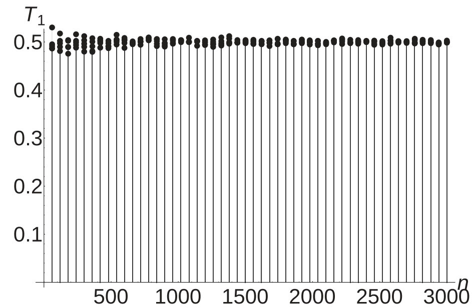

Fig. 10. Maximum displacement to the power 1 of Algorithm 3 for $d = \frac{1}{n}$.

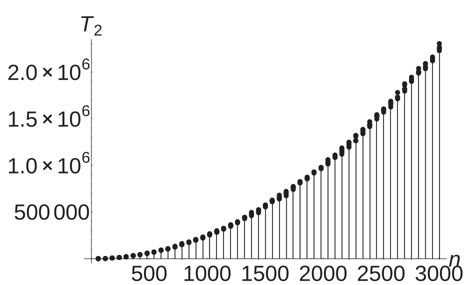

Fig. 11. Maximum displacement to the power 2 of Algorithm 3 for $d = 1$.

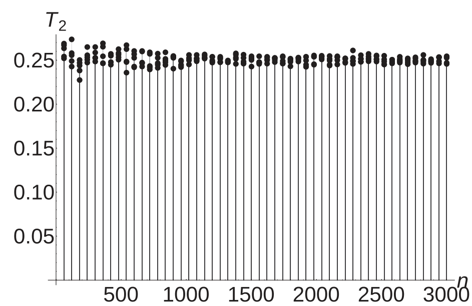

Fig. 12. Maximum displacement to the power 2 of Algorithm 3 for $d = \frac{1}{n}$.

the obtained theoretical tight bound $\Theta(1)$ (see Theorem 14 together with Theorem 15 for $a := 2$ and $d := \frac{1}{n}$).

Figures 11 and 12 depict experimental maximum displacement to the power 2 of Algorithm 3 when $d = 1$ and $d = \frac{1}{n}$. In this case, we conduct Algorithm 9 for parameters $a = 2, d = 1$ and parameters $a = 2, d = \frac{1}{n}$. In Figures 11 and 12 the black dots represent numerical results of Algorithm 9 for parameters $a = 2, d = 1$ and parameters $a = 2, d = \frac{1}{n}$.

7.4 Evaluation of Energy for Unreliable Sensors

We evaluate the sum of displacement to the power $a = 1$ (**Energy**) of unreliable sensors in Algorithm 3. As in the previous subsection, in the experimental evaluation we choose reliability parameter $p = \frac{1}{2}$, i.e., each sensor with probability $p = \frac{1}{2}$ independently from the other sensor is active (see Assumption 12).

Figures 13 and 14 depict experimental sum of displacement to the power 1 of Algorithm 3 when $d = \frac{1}{n}$ and $d = \frac{1}{n^{3/2}}$. In this case, we conduct Algorithm 10 for parameters $a = 1, d = \frac{1}{n}$ and parameters $a = 1, d = \frac{1}{n^{3/2}}$. In Figures 13 and 14 the black dots represent numerical results of Algorithm 10 for parameters $a = 1, d = \frac{1}{n}$ and parameters $a = 1, d = \frac{1}{n^{3/2}}$.

The experimental sum of displacement to the power 1 of Algorithm 3 for $d = \frac{1}{n}$ is in $\Theta(n)$. Hence, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(n)$ (see Theorem 17 for $a := 1$ and $d := \frac{1}{n}$).

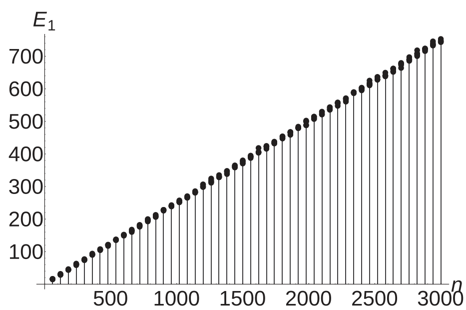

Fig. 13. Sum of displacement to the power 1 of Algorithm 3 for $d = \frac{1}{n}$.

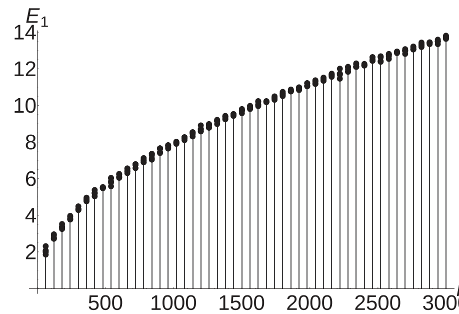

Fig. 14. Sum of displacement to the power 1 of Algorithm 3 for $d = \frac{1}{n^{3/2}}$.

Observe that the experimental maximum displacement to the power 1 of Algorithm 3 for $d = \frac{1}{n^{3/2}}$ is in $\Theta(\sqrt{n})$. Hence, as in the previous experiment, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(\sqrt{n})$ (see Theorem 17 for $a := 1$ and $d := \frac{1}{n^{3/2}}$).

ALGORITHM 10:

$n := 1$
while n $\leq$ 50 **do**
  for $j = 1$ **to** 50 **do**
    Generate $X_1(j), X_2(j), \ldots, X_{60n}(j)$ unreliable sensors at the equidistance points, i.e., $X_i(j) = \frac{d}{2} + (i-1)d$ for $i = 1, 2, \ldots, 60n$ such that $\forall_{i \in \{1,2,\ldots,60n\}}$, ($X_i(j)$ with probability $\frac{1}{2}$ independently from others sensors is reliable, i.e., it can move, sense and communicate);
    Calculate $\mathbf{E}_{\mathbf{a},\frac{1}{2}}(60n, j)$ the sum of displacement to the power a of reliable sensors in Algorithm 3;
  end for
  for $l = 1$ **to** 5 **do**
    Calculate the average $\mathbf{E}_{\mathbf{a},\frac{1}{2}}(60n) = \frac{1}{10}\sum_{v=1}^{10} \mathbf{E}_{\mathbf{a},\frac{1}{2}}(60n, v + (l-1)10)$;
    Insert the points $\mathbf{E}_{\mathbf{a},\frac{1}{2}}(60n)$ into the chart;
  end for
  $n := n + 1$
end while

Figures 15 and 16 depict experimental sum of displacement to the power 2 of Algorithm 3 when $d = \frac{1}{n}$ and $d = \frac{1}{n^{3/2}}$. In this case, we conduct Algorithm 10 for parameters $a = 2$, $d = \frac{1}{n}$ and parameters $a = 2$, $d = \frac{1}{n^{3/2}}$. In Figures 15 and 16 the black dots represent numerical results of Algorithm 10 for parameters $a = 2$, $d = \frac{1}{n}$ and parameters $a = 2$, $d = \frac{1}{n^{3/2}}$.

The experimental sum of displacement to the power 2 of Algorithm 3 for $d = \frac{1}{n}$ is in $\Theta(n)$. Hence, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(n)$ (see Theorem 17 for $a := 2$ and $d := \frac{1}{n}$).

Observe that the experimental maximum displacement to the power 2 of Algorithm 3 for $d = \frac{1}{n^{3/2}}$ is in $\Theta(1)$. Hence, as in the previous experiment, the carried out experiments confirm very well the obtained theoretical tight bound $\Theta(1)$ (see Theorem 17 for $a := 2$ and $d := \frac{1}{n^{3/2}}$).

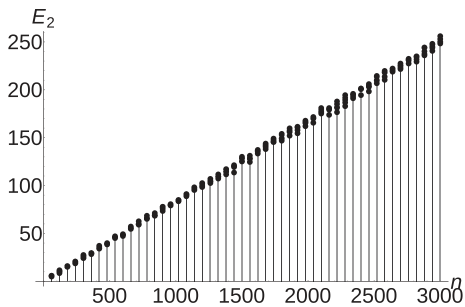

Fig. 15. Sum of displacement to the power 2 of Algorithm 3 for $d = \frac{1}{n}$.

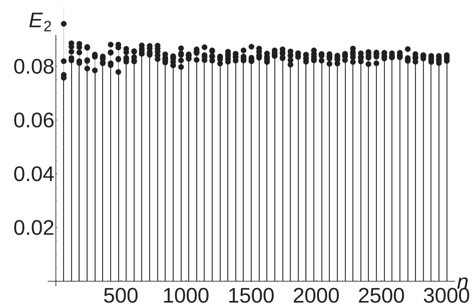

Fig. 16. Sum of displacement to the power 2 of Algorithm 3 for $d = \frac{1}{n^{3/2}}$.

8 CONCLUSION

In this article, we addressed the fundamental problem of the range assignment in wireless mobile sensors networks in which n mobile sensors with identical sensing radius r and communication radius R, provided that $r \leq R$ are initially randomly deployed on the plane by dropping them from an aircraft according to general random process. To this end, we minimized *the time required* and *the energy consumption* of transportation cost for sensors as the function of sensing radius r and communication radius R to provide the desired *greedy* k_1-coverage simultaneously with k_2-connectivity. We proved that for both of these optimization problems, the optimal solution is to assign the sensing radius equal to $r = k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ and the communication radius equal to $R = k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$, where $||\mathbf{E}[\mathbb{S}]||$ is the distance of general random process according to which the sensors are deployed. We also discovered and explained *the sharp increase*, i.e., *the threshold phenomena* in *the time required* and *the energy consumption* in transportation cost to ensure the desired k_1-coverage together with k_2-connectivity when $r < k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ or $R < k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$.

We further analyzed the desired k_1-coverage together with k_2-connectivity for unreliable sensors. For unreliable sensors both *the time required* and *the energy consumption* in transportation cost to ensure the desired k_1-coverage together with k_2-connectivity remains asymptotically the same when r is below or equal to $k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ or R is below or equal to $k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$. While we have discussed the applicability of our approach to sensors having variable sensing and communication radii an open problem for future study is the range assignment in heterogeneous wireless mobile sensors networks. Additionally, it would be interesting to explore the range assignment problem for sensors with variable sensing and communication radii when some sensors are unreliable or even fail with some fixed probability.

APPENDICES

APPENDIX A

Proof. (Theorem 1) Fix an even positive integer m. Assume that $j > \frac{m}{2}$.

First, combining together Equation (1), multinomial theorem, as well as Equation (2), we deduce that

$$\mathbf{E}\left[(X_j - \mathbf{E}[X_j]|^m\right] = \mathbf{E}\left[(X_j - j\alpha)^m\right] = \sum_B \frac{m!}{(l_1)!(l_2)!\dots(l_j)!}\mathbf{E}\left[\prod_{i=1}^{j}(\tau_i - \alpha)^{l_i}\right]$$
$$= \sum_B \frac{m!}{(l_1)!(l_2)!\dots(l_j)!}\prod_{i=1}^{j}\mathbf{E}\left[(\tau_i - \alpha)^{l_i}\right],$$

where

$$B = \{(l_1, l_2, \dots l_j) \in \mathbb{N}^j : l_1 + l_2 + \cdots + l_j = m\}.$$

Applying Equation (1), we get $\mathbf{E}[(\tau_i - \alpha)] = \mathbf{E}[\tau_i] - \alpha = 0$. Hence, we have a simpler expression,

$$\mathbf{E}\left[(X_j - \mathbf{E}[X_j]|^m\right] = \sum_{B_1} \frac{m!}{(l_1)!(l_2)!\dots(l_j)!} \prod_{i=1}^{j} \mathbf{E}\left[(\tau_i - \alpha)^{l_i}\right], \tag{50}$$

where

$$B_1 = \{(l_1, l_2, \dots l_j) \in \mathbb{N}^j : l_1 + l_2 + \cdots + l_j = m, \quad l_i \neq 1 \text{ for } i = 1, 2, \dots, j\}.$$

Observe that

$$B_1 = B_2 \cup B_3 \text{ and } B_2 \cap B_3 = \emptyset, \tag{51}$$

$$B_2 = \{(l_1, l_2, \dots l_j) \in \mathbb{N}^j : l_1 + l_2 + \cdots + l_j = m, \quad l_i \neq 1, \quad l_i \in \{0, 2\} \text{ for } i = 1, 2, \dots, j\},$$

$$B_3 = \{(l_1, l_2, \dots l_j) \in \mathbb{N}^j : l_1 + l_2 + \cdots + l_j = m, \quad l_i \neq 1 \text{ for } i = 1, 2, \dots, j, \quad \exists i\, (l_i \neq 2)\},$$

$$|B_2| = \binom{j}{\frac{m}{2}}, \quad |B_3| = O\left(j^{\frac{m}{2}-1}\right). \tag{52}$$

Notice that

$$\mathbf{E}\left[(\tau_i - \alpha)^2\right] = \mathbf{Var}[\tau_1]. \tag{53}$$

We now make the *important observation* that **the sum** (Equation (50)) **is equal to the sum** of Equations (54) and (55).

The second sum (Equation (55)) is negligible. Thus, the asymptotics of Equation (50) depends on the expression given by the first sum (Equation (54)).

Together, Equations (52) and (53) imply

$$\begin{aligned}\sum_{B_2} \frac{m!}{(l_1)!(l_2)!\dots(l_j)!} \prod_{i=1}^{j} \mathbf{E}\left[(\tau_i - \alpha)^{l_i}\right] &= \sum_{B_2} \frac{m!\,(\mathbf{Var}[\tau_1])^{\frac{m}{2}}}{2^{\frac{m}{2}}} \\ &= \frac{m!\,(\mathbf{Var}[\tau_1])^{\frac{m}{2}}}{2^{\frac{m}{2}}}|B_2| = \frac{m!\,(\mathbf{Var}[\tau_1])^{\frac{m}{2}}}{2^{\frac{m}{2}}\left(\frac{m}{2}\right)!} j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right).\end{aligned} \tag{54}$$

Using Equation (1) in Definition 1, we have

$$\begin{aligned}\left|\mathbf{E}\left[(\tau_i - \alpha)^{l_i}\right]\right| &\leq \mathbf{E}\left[|\tau_i - \alpha|^{l_i}\right] \leq \mathbf{E}\left[(|\tau_i| + \alpha)^{l_i}\right] = \sum_{t=0}^{l_i} \binom{l_i}{t} \mathbf{E}\left[|\tau_i|^t\right] \alpha^{l_i - t} = \sum_{t=0}^{l_i} \binom{l_i}{t} \mathbf{E}\left[\tau_i^t\right] \alpha^{l_i - t} \\ &\leq C_{1,m} \sum_{t=0}^{l_i} \binom{l_i}{t} \alpha^{l_i - t} = C_{1,m}\,(\alpha + 1)^{l_i}.\end{aligned}$$

Hence, by Equation (52), as well as formula $l_1 + l_2 + \cdots + l_j = m$, we get

$$\begin{aligned}\left|\sum_{B_3} \frac{m!}{(l_1)!(l_2)!\dots(l_j)!} \prod_{i=1}^{j} \mathbf{E}\left[(\tau_i - \alpha)^{l_i}\right]\right| &\leq \sum_{B_3} m! \prod_{i=1}^{j} \left|\mathbf{E}\left[(\tau_i - \alpha)^{l_i}\right]\right| \leq \sum_{B_3} m! C_{1,m}^m (\alpha + 1)^m \\ &\leq m! C_{1,m}^m (\alpha + 1)^m |B_3| = O\left(j^{\frac{m}{2}-1}\right).\end{aligned} \tag{55}$$

Finally, combining Equations (50), (51), (54), and (55) completes the proof of Theorem 1. □

APPENDIX B

Proof. (Theorem 2) As a first step, we combine together binomial theorem with independence of random variables X_j, Y_j and get

$$\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m\right] = \mathbf{E}\left(\left(X_j - \mathbf{E}[X_j]\right)^2 + \left(Y_j - \mathbf{E}[Y_j]\right)^2\right)^{\frac{m}{2}} = \mathbf{E}\left(\sum_{i=0}^{\frac{m}{2}} \binom{\frac{m}{2}}{i} \left(X_j - \mathbf{E}[X_j]\right)^{2i} \left(Y_j - \mathbf{E}[Y_j]\right)^{m-2i}\right)$$
$$= \sum_{i=0}^{\frac{m}{2}} \binom{\frac{m}{2}}{i} \mathbf{E}\left(\left(X_j - \mathbf{E}[X_j]\right)^{2i}\right) \mathbf{E}\left(\left(Y_j - \mathbf{E}[Y_j]\right)^{m-2i}\right). \tag{56}$$

Hence, estimating the expected distance to the power m between random vector $\mathbb{V}_j$ with (m, α, β)-property and its mean $\mathbf{E}[\mathbb{V}_j]$ is reduced to estimating the expected mth moments around the mean for the random variables X_j and Y_j. Due to properties (3) and (4) in Definition 1 for the sequence $\{\tau_i\}_{i\geq 1}$ Theorem 1 also holds for random variable $Y_j = \sum_{i=1}^{j} \xi_i$. Therefore, applying Theorem 1 for random variable $X_j = \sum_{i=1}^{j} \tau_i$ and $m := 2i$, as well as for random variable $Y_j = \sum_{i=1}^{j} \xi_i$ and $m := m - 2i$ we get

$$\mathbf{E}\left(\left(X_j - \mathbf{E}[X_j]\right)^{2i}\right) = \frac{(2i)!\,(\mathbf{Var}\,[\tau_1])^i}{2^i\,(i)!} j^i + O\left(j^{i-1}\right). \tag{57}$$

$$\mathbf{E}\left(\left(Y_j - \mathbf{E}[Y_j]\right)^{m-2i}\right) = \frac{(m-2i)!\,(\mathbf{Var}\,[\xi_1])^{\frac{m}{2}-i}}{2^{\frac{m}{2}-i}\left(\frac{m}{2}-i\right)!} j^{\frac{m}{2}-i} + O\left(j^{\frac{m}{2}-i-1}\right). \tag{58}$$

By substituting Equations (56) and (57) into (58), we get

$$\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m\right] = \left(\frac{\left(\frac{m}{2}\right)!}{2^{\frac{m}{2}}} \sum_{i=0}^{\frac{m}{2}} \binom{2i}{i} \binom{m-2i}{\frac{m}{2}-i} (\mathbf{Var}\,[\tau_1])^i\,(\mathbf{Var}\,[\xi_1])^{\frac{m}{2}-i}\right) j^{\frac{m}{2}} + O\left(j^{\frac{m}{2}-1}\right).$$

This completes the proof of Theorem 2. □

APPENDIX C

Proof. (Theorem 4) Putting together Equations (6) and (8), we have

$$\mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]||^a\right] = \frac{\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a\right]}{\left(\sqrt{\alpha^2+\beta^2}\right)^a} ||\mathbf{E}[\mathbb{S}]||^a. \tag{59}$$

From Theorem 2, we have for m positive even integer

$$\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m\right] = \Theta\left(j^{\frac{m}{2}}\right). \tag{60}$$

First, let us prove *the upper bound.* Let m be the smallest even integer greater than or equal to $a > 0$. Applying Jensen's inequality (see Equation (10)) for $X := ||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a$ and $f(x) := x^{\frac{m}{a}}$, we get

$$\left(\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a\right]\right)^{\frac{m}{a}} \leq \mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^m\right].$$

Using Equation (60), we get $\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a\right] \leq \left(\Theta\left(j^{\frac{m}{2}}\right)\right)^{\frac{a}{m}} = \Theta\left(j^{\frac{a}{2}}\right)$. Hence

$$\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a\right] = O\left(j^{\frac{a}{2}}\right) \quad \text{when} \quad a > 0. \tag{61}$$

This proves the upper bound.

Next, we prove *the lower bound.* Assume $a \geq 2$. Using Jensen's inequality for $X := ||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^2$ and $f(x) := x^{\frac{a}{2}}$,

$$\left(\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^2\right]\right)^{\frac{a}{2}} \leq \mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a\right].$$

Using Equation (60), we get $\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a\right] \geq \left(\Theta\left(j^2\right)\right)^{\frac{a}{2}} = \Theta\left(j^{\frac{a}{2}}\right)$. Hence

$$\mathbf{E}\left[||\mathbb{V}_j - \mathbf{E}[\mathbb{V}_j]||^a\right] = \Omega\left(j^{\frac{a}{2}}\right) \quad \text{when} \quad a \geq 2. \tag{62}$$

This proves the lower bound.

Finally combining Equations (59), (61), and (62), as well as the asymptotic $\left(\sqrt{\alpha^2 + \beta^2}\right)^a = \Theta(1)$ completes the proof of Theorem 4. □

APPENDIX D

Proof. (Theorem 5) There are two cases to consider.

The upper bound.

Fix $a > 0$. We argue as follows. Combining together the triangle inequality for vectors $\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]$ and $O(1)\mathbf{E}[\mathbb{S}]$, as well as elementary Inequality (11) for $x := \left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]\right\|$ and $y := ||O(1)\mathbf{E}[\mathbb{S}]||$ we have

$$\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right\|^a \leq \max\left(2^{a-1}, 1\right)\left(\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]\right\|^a + |O(1)|^a||\mathbf{E}[\mathbb{S}]||^a\right).$$

Passing to the expectations, we get

$$\mathbf{E}\left[\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right\|^a\right] \leq \max\left(2^{a-1}, 1\right)\left(\mathbf{E}\left[\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]\right\|^a\right] + |O(1)|^a||\mathbf{E}[\mathbb{S}]||^a\right).$$

Applying Theorem 4, we have

$$\mathbf{E}\left[\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right\|^a\right] = O\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a, \quad a > 0.$$

This proves the upper bound.

The lower bound.

Fix $a \geq 2$. We argue as follows. Combining together the triangle inequality for vectors $\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]$ and $O(1)\mathbf{E}[\mathbb{S}]$, as well as elementary Inequality (11) for $x := \left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]\right\|$ and $y := ||-O(1)\mathbf{E}[\mathbb{S}]||$ we have

$$\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]\right\|^a \leq \max\left(2^{a-1}, 1\right)\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right\|^a + \max\left(2^{a-1}, 1\right)|-O(1)|^a||\mathbf{E}[\mathbb{S}]||^a.$$

Passing to the expectations, we get

$$\frac{\mathbf{E}\left[\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j]\right\|^a\right]}{\max\left(2^{a-1}, 1\right)} \leq \left(\mathbf{E}\left[\left\|\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right\|^a\right] + |-O(1)|^a||\mathbf{E}[\mathbb{S}]||^a\right).$$

Applying Theorem 4, we have $\mathbf{E}[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a] = \Omega(j^{\frac{a}{2}})||\mathbf{E}[\mathbb{S}]||^a$, $a \geq 2$. This is enough to prove the desired lower bound and completes the proof of Theorem 5. □

APPENDIX E

Proof. (Theorem 6) The proof of Theorem 6 is analogous to that of Theorem 5.

First, we prove *the upper bound.* Combining together the triangle inequality for vectors $\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]$ and $\varepsilon\mathbf{E}[\mathbb{S}_j]$, as well as Inequality (11) for $x := ||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||$ and $y := ||\varepsilon\mathbf{E}[\mathbb{S}_j]||$ we get

$$||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a \leq \max\left(2^{a-1}, 1\right)\left(||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + (O(1) - 1)\,\mathbf{E}[\mathbb{S}]||^a + ||\varepsilon\mathbf{E}[\mathbb{S}_j]||^a\right).$$

Passing to the expectations and applying Theorem 5, Identity $O(1) - 1 = O(1)$, as well as Equation (9) we have

$$\frac{\mathbf{E}\left[||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a\right]}{\max\left(2^{a-1},1\right)} \leq \mathbf{E}\left[||\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a\right] + \mathbf{E}\left[||\varepsilon\mathbf{E}[\mathbb{S}_j]||^a\right]$$
$$= O\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a + \varepsilon^a j^a ||\mathbf{E}[\mathbb{S}]||^a.$$

Hence

$$\mathbf{E}\left[||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a\right] = O\left(j^a\right)||\mathbf{E}[\mathbb{S}]||^a.$$

This proves the upper bound.

To prove *the lower bound*, we combine the triangle inequality for vectors $\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]$ and $-\left(\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right)$, and Inequality (11) for $x := ||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||$ and $y := ||-\left(\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right)||$. Then, we get

$$||\varepsilon\mathbf{E}[\mathbb{S}_j]||^a \leq \max\left(2^{a-1},1\right)||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a + \max\left(2^{a-1},1\right)||-\left(\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right)||^a.$$

Passing to the expectations, we have

$$\mathbf{E}\left[||\varepsilon\mathbf{E}[\mathbb{S}_j]||^a\right] \leq \max\left(2^{a-1},1\right)\mathbf{E}\left[||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a\right]$$
$$+ \max\left(2^{a-1},1\right)\mathbf{E}\left[||-\left(\mathbb{S}_j - \mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]\right)||^a\right].$$

Hence, applying Theorem 5, as well as Equation (9) we have

$$\mathbf{E}\left[||\mathbb{S}_j - (1-\varepsilon)\mathbf{E}[\mathbb{S}_j] + O(1)\mathbf{E}[\mathbb{S}]||^a\right] = \frac{\varepsilon^a j^a ||\mathbf{E}[\mathbb{S}]||^a}{\max\left(2^{a-1},1\right)} - O\left(j^{\frac{a}{2}}\right)||\mathbf{E}[\mathbb{S}]||^a = \Omega\left(j^a\right)||\mathbf{E}[\mathbb{S}]||^a.$$

This proves the lower bound and completes proof of Theorem 6. □

APPENDIX F

Proof. (Theorem 9) The proof of Theorem 9 is analogous to that of Theorem 8.

Fix $k = \max(k_1, k_2)$ Assume that the n sensors on the plane have identical sensing radius $r = k_1 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$ and communication radius $R = k_2 \frac{||\mathbf{E}[\mathbb{S}]||}{2}$. First, observe that sensors at the final positions $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ after Algorithm 2 lie on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$. Observe that

$$||(x_1, y_1) - (0,0)|| = \frac{(1-\varepsilon)||\mathbf{E}[\mathbb{S}]||}{2},$$
$$||(x_j, y_j) - (x_{j-1}, y_{j-1})|| = \frac{(1-\varepsilon)||\mathbf{E}[\mathbb{S}]||}{2} \text{ if } j \in \{2, \ldots, k\} \text{ (see steps 2–4 of Algorithm 2)},$$
$$||(x_j, y_j) - (x_{j-1}, y_{j-1})|| = ||(1-\varepsilon)\mathbf{E}[\mathbb{S}]|| \text{ if } j \in \{k+1, k+2, \ldots, \min(n, n+2-k)\} \text{ (see steps 5–7 of Algorithm 2)},$$
$$||(x_j, y_j) - (x_{j-1}, y_{j-1})|| = \frac{||(1-\varepsilon)\mathbf{E}[\mathbb{S}]||}{2} \text{ if } j \in \{n+3-k, \ldots, n\} \text{ and } k \geq 3 \text{ (see steps 8–12 of Algorithm 2)}.$$

Therefore, we can apply Lemma 7 for $d := ||(1-\varepsilon)\mathbf{E}[\mathbb{S}]||$ and deduce that every point on the line connecting points $(0,0), (x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ is within the sensing range of at least k_1 sensors and the communication range of at least k_2 sensors. Hence, Algorithm 2 is correct.

We now estimate $\mathbf{Time_a}(n, r, R)$ and $\mathbf{Energy_a}(n, r, R)$ of Algorithm 2.

Let us recall that $j\mathbf{E}[\mathbb{S}] = \mathbf{E}[\mathbb{S}_j]$ (see Equation (9)). Hence

$$\mathbb{S}_j - (x_j, y_j) = \mathbb{S}_j - \frac{(1-\varepsilon)\mathbf{E}\left[\mathbb{S}_j\right]}{2} \text{ if } j \in \{1, \ldots, k\} \text{ (see steps 2–4 of Algorithm 2)}, \tag{63}$$

$$\mathbb{S}_j - (x_j, y_j) = \mathbb{S}_j - (1-\varepsilon)\mathbf{E}\left[\mathbb{S}_j\right] - k\frac{(1-\varepsilon)\mathbf{E}\left[\mathbb{S}\right]}{2} \text{ if } j \in \{k+1, k+2, \ldots, \min(n, n+2-k)\} \tag{64}$$

(see steps 5–7 of Algorithm 2),

$$\mathbb{S}_j - (x_j, y_j) = \mathbb{S}_j - (1-\varepsilon)\mathbf{E}\left[\mathbb{S}_j\right] - k(1-\varepsilon)\frac{\mathbf{E}\left[\mathbb{S}\right]}{2} - (j-(n+2-k))(1-\varepsilon)\frac{\mathbf{E}\left[\mathbb{S}\right]}{2}$$
$$\text{if } j \in \{n+3-k, \ldots, n\} \text{ and } k \geq 3 \tag{65}$$

(see steps 8–12 of Algorithm 2).

We are now ready to apply Theorem 6 and evaluate separately Equations (63), (64), and (65).

Case of Equation (63)

Passing to the expectations and using Theorem 6 with $\varepsilon := \frac{1+\varepsilon}{2}$ and $O(1) := 0$, we get

$$\mathbf{E}\left[\left|\left|\mathbb{S}_j - (x_j, y_j)\right|\right|^a\right] = \Theta\left(j^a\right) ||\mathbf{E}[\mathbb{S}]||^a \text{ if } j \in \{2, \ldots, k\}.$$

Case of Equation (64)

Since $k = O(1)$, we can apply Theorem 6 with $O(1) := \frac{k(1-\varepsilon)}{2}$ and get

$$\mathbf{E}\left[\left|\left|\mathbb{S}_j - (x_j, y_j)\right|\right|^a\right] = \Theta\left(j^a\right) \text{ provided that } j \in \{k+1, k+2, \ldots, \min(n, n+2-k)\}.$$

Case of Equation (65)

Observe that $2 \leq k + j - (n+2-k) \leq 2k - 2 = O(1)$. Therefore, we can apply Theorem 6 with $O(1) := -\frac{(1-\varepsilon)}{2}(k + j - (n+2-k))$ and and get

$$\mathbf{E}\left[\left|\left|\mathbb{S}_j - (x_j, y_j)\right|\right|^a\right] = \Theta\left(j^a\right) \text{ provided that } j \in \{n+3-k, \ldots, n\} \text{ and } k \geq 3.$$

Combining together Estimations: Case of Equation (63), Case of Equation (64), Case of Equation (65), we have

$$\mathbf{Time_a}(n, r, R) = \max_{1 \leq j \leq n} \mathbf{E}\left[\left|\left|\mathbb{S}_j - (x_j, y_j)\right|\right|^a\right] = \Theta\left(n^a\right)$$

of Algorithm 2.

Putting together Estimations: Case of Equation (63), Case of Equation (64), Case of Equation (65) and the well-known identity $\sum_{j=1}^{n} j^{\frac{a}{2}} = \Theta(n^{\frac{a}{2}+1})$, when $a > 0$ we have

$$\mathbf{Energy_a}(n, r, R) = \sum_{j=1}^{n} \mathbf{E}\left[\left|\left|\mathbb{S}_j - (x_j, y_j)\right|\right|^a\right] = \Theta\left(n^{a+1}\right).$$

of Algorithm 2.

This completes the proof of Theorem 9. □

APPENDIX G

Proof. (Theorem 11) There are two cases to consider

Case 1: $r = k_1(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$

Assume that sensor $\mathbb{S}_j$ moves to the position $\mathbb{Q}_j$, which lie on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0, 0)$. provided that $j \in \{1, 2, \ldots, n\}$. Let $\mathbb{Q}_0 = (0.0)$. We look at the points $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3, \ldots \mathbb{Q}_{k_1}$ and choose the point $P_0 = \frac{\mathbb{Q}_{k_1} - (0,0)}{2}$.

Point P_0 is in the middle of interval connecting point $(0, 0)$ and $\mathbb{Q}_{k_1}$. If P_0 is not in sensing range of $\mathbb{Q}_{k_1}$, then it is in the sensing range of at most $k_1 - 1$ sensors. Therefore, P_0 must be in the sensing range of sensor $\mathbb{Q}_{k_1}$, i.e.,

$$||\mathbb{Q}_{k_1} - P_0|| \le \frac{k_1}{2}||\mathbf{E}[\mathbb{S}]||(1-\varepsilon).$$

Therefore, $||\mathbb{Q}_{k_1}|| \le ||k_1\mathbf{E}[\mathbb{S}]||(1-\varepsilon)$. Since $\mathbb{Q}_{k_1}$ lies on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$, we have

$$Q_{k_1} = k_1\mathbf{E}[\mathbb{S}](1-\Delta_1), \tag{66}$$

provided that $\varepsilon \le \Delta_1 \le 1$

Let j be the positive integer and $j+1 \le \frac{n}{k_1}$. We now look at the points $\mathbb{Q}_{k_1 j}, \mathbb{Q}_{k_1 j+1}, \ldots \mathbb{Q}_{k_1 j+k_1}$ and choose the point

$$P_j = \frac{\mathbb{Q}_{k_1 j+k_1} - \mathbb{Q}_{k_1 j}}{2}.$$

Point P_j is in the middle of interval connecting point $\mathbb{Q}_{k_1 j}$ and $\mathbb{Q}_{k_1 j+k_1}$. If P_j is not in the sensing range of $\mathbb{Q}_{k_1 j+k_1}$, then it is not in the sensing range of $\mathbb{Q}_{k_1 j}$ and thus is in the sensing range of at most $k_1 - 1$ sensors. Therefore, P_j must be in the sensing range of the sensor $\mathbb{Q}_{k_1 j+k_1}$, i.e.,

$$||\mathbb{Q}_{k_1 j+k_1} - P_j|| \le \frac{k_1}{2}||\mathbf{E}[\mathbb{S}]||(1-\varepsilon).$$

Therefore

$$||\mathbb{Q}_{k_1 j+k_1} - \mathbb{Q}_{k_1 j}|| \le k_1\mathbf{E}[\mathbb{S}](1-\varepsilon). \tag{67}$$

Putting together Equations (66) and (67), as well as the fact that $\mathbb{Q}_{k_1 j}$ lies on the line passing through the points $\mathbf{E}[\mathbb{S}]$ and $(0,0)$ we have

$$\mathbb{Q}_{k_1 j} = \sum_{i=1}^{j} k_1\mathbf{E}[\mathbb{S}](1-\Delta_i),$$

provided that $\varepsilon \le \Delta_i \le 1$. Using this, as well as Equation (9), we have

$$\mathbb{S}_{k_1 j} - \mathbb{Q}_{k_1 j} = \mathbb{S}_{k_1 j} - \mathbf{E}[\mathbb{S}_{k_1 j}] + \mathbf{E}[\mathbb{S}_{k_1 j}]\left(\frac{\Delta_1 + \Delta_2 + \ldots \Delta_j}{j}\right).$$

Since $\epsilon \le \Delta_i \le 1$ and $0 < \epsilon \le 1$, we have

$$\varepsilon \le \varepsilon_j = \frac{\Delta_1 + \Delta_2 + \ldots \Delta_j}{j} \le 1.$$

Clearly ε_j depends on j but $1 \ge \varepsilon_j \ge \varepsilon$. Hence, we can apply the similar arguments to that as in the proof of Theorem 6 to deduce that

$$\mathbf{E}\left[\left|\left|\mathbb{S}_{k_1 j} - \mathbb{Q}_{k_1 j}\right|\right|\right]^a = (k_1)^a\Theta\left(j^a\right)||\mathbf{E}[\mathbb{S}]||^a.$$

Since $\sum_{j=1}^{\frac{n}{k_1}}(k_1 j)^a = \Theta\left(n^{a+1}\right)$, when $a > 0$ and k_1 is fixed we have

$$\max_{1\le j\le n}\mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^a\right] \ge \max_{1\le j\le \frac{n}{k_1}}\mathbf{E}\left[||\mathbb{S}_{k_1 j} - \mathbb{Q}_{k_1 j}||^a\right] = \Theta\left(n^a\right)||\mathbf{E}[\mathbb{S}]||^a.$$

Therefore,

$$\mathbf{Time_a}(n,r,R) = \Omega\left(n^a\right)||\mathbf{E}[\mathbb{S}]||^a,$$

$$\sum_{j=1}^{n}\mathbf{E}\left[||\mathbb{S}_j - \mathbb{Q}_j||^a\right] \ge \sum_{j=1}^{\frac{n}{k_1}}\mathbf{E}\left[||\mathbb{S}_{k_1 j} - \mathbb{Q}_{k_1 j}||^a\right] = \Theta\left(n^{a+1}\right)||\mathbf{E}[\mathbb{S}]||^a.$$

Hence

$$\mathbf{Energy_a}(n,r,R) = \Omega\left(n^{a+1}\right)||\mathbf{E}[\mathbb{S}]||^a.$$

Case 2 $R = k_2(1-\varepsilon)\frac{||\mathbf{E}[\mathbb{S}]||}{2}$

The proof of **Case** 2 is analogous to that of **Case** 1. Namely, we apply the same arguments as in Case 1 for $k_1 := k_2$, and *the sensing radius* := *the communication radius.*

This completes the proof of Theorem 11. □

APPENDIX H

Proof. (Theorem 14) The proof of the theorem is analogous to the proof of Theorem 4.

From Theorem 13, we have for t positive integer

$$\mathbf{E}\left[|mw_{n,n}|^t\right] = d^t\Theta\left(n^t\right). \tag{68}$$

First, let us prove *the upper bound.* Let t be the smallest integer greater than or equal to $a > 0$. Applying Jensen's inequality (see Equation (10)) for $X := mw_{n,n}$ and $f(x) := x^{\frac{t}{a}}$, we get

$$(\mathbf{E}\,[|mw_{n,n}|^a])^{\frac{t}{a}} \leq \mathbf{E}\left[|mw_{n,n}|^t\right].$$

Therefore,

$$\mathbf{E}\,[|mw_{n,n}|^a] \leq \left(d^t\Theta\left(n^t\right)\right)^{\frac{a}{t}} = d^a\Theta\,(n^a)\,.$$

Hence

$$\mathbf{E}\,[|mw_{n,n}|^a] = d^aO\,(t^a) \quad \text{when} \quad a > 0. \tag{69}$$

This proves the upper bound.

Next, we prove *the lower bound.* Assume that $a \geq 1$. Using Jensen's inequality for $X := mw^1_{n,n}$ and $f(x) := x^a$, we have

$$\left(\mathbf{E}\left[mw^1_{n,n}\right]\right)^a \leq \mathbf{E}\,[|mw_{n,n}|^a]\,.$$

Applying Equation (68) for $t := 1$ lead to

$$\mathbf{E}\,[|mw_{n,n}|^a] \geq (d\Theta\,(n))^a = d^a\Theta\,(n^a)\,.$$

Hence

$$\mathbf{E}\,[|mw_{n,n}|^a] = d^a\Omega\,(n^a) \quad \text{when} \quad a \geq 1.$$

This proves the upper bound and completes the proof of Theorem 14. □

APPENDIX I

Proof. (Theorem 17) From Theorem 16, we have for t positive integer

$$\sum_{j=1}^{n}\mathbf{E}\left[\left|mw_{j,n}\right|^t\right] = d^t\Theta\left(n^{t+1}\right). \tag{70}$$

There are two cases to consider.

Case the upper bound

Let k be the smallest integer greater than real $a > 0$. We use discrete Hölder inequality with parameters $\frac{t}{a}$ and $\frac{t}{t-a}$ and get

$$\sum_{j=1}^{n}\mathbf{E}\left[\left|mw_{j,n}\right|^a\right] \leq \left(\sum_{j=1}^{n}\left(\mathbf{E}\left[\left|mw_{j,n}\right|^a\right]\right)^{\frac{t}{a}}\right)^{\frac{a}{t}}\left(\sum_{j=1}^{n}1\right)^{\frac{t-a}{t}} = \left(\sum_{j=1}^{n}\left(\mathbf{E}\left[\left|mw_{j,n}\right|^a\right]\right)^{\frac{t}{a}}\right)^{\frac{a}{t}} n^{\frac{t-a}{t}}. \tag{71}$$

Next, we use Jensen's inequality (see Equation (10)) for $f(x) := x^{\frac{t}{a}}$ and $X =: [|mw_{j,n}|^a]$ and get

$$\left(\mathbf{E}\left[\left|mw_{j,n}\right|^a\right]\right)^{\frac{t}{a}} \leq \mathbf{E}\left[\left|mw_{j,n}\right|^a\right]. \tag{72}$$

Combining together Equations (71) and (72), as well as Equation (70), we deduce that

$$\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|^{a}\right] \le \left(\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|^{t}\right]\right)^{\frac{a}{t}} n^{\frac{t-a}{t}} = \left(d^{t}\Theta\left(n^{t+1}\right)\right)^{\frac{a}{t}} n^{\frac{t-a}{t}} = d^{a}\Theta\left(n^{a+1}\right).$$

Hence,

$$\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|^{a}\right] = d^{a}O\left(n^{a+1}\right), \text{ when } a > 0.$$

This is enough to prove the lower bound.

Case the lower bound

Fix $a \ge 1$. We use discrete Hölder inequality with parameters a and $\frac{a}{a-1}$ and get

$$\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|\right] \le \left(\sum_{j=1}^{n}\left(\mathbf{E}\left[\left|mw_{j,n}\right|\right]\right)^{a}\right)^{\frac{1}{a}} \left(\sum_{j=1}^{n} 1\right)^{\frac{a-1}{a}} = \left(\sum_{j=1}^{n}\left(\mathbf{E}\left[\left|mw_{j,n}\right|\right]\right)^{a}\right)^{\frac{1}{a}} n^{\frac{a-1}{a}}. \tag{73}$$

Next, we use Jensen's inequality (see Equation (10)) for $f(x) := x^{a}$ and $X =: [|mw_{j,n}|]$ and get

$$\left(\mathbf{E}\left[\left|mw_{j,n}\right|\right]\right)^{a} \le \mathbf{E}\left[\left|mw_{j,n}\right|^{a}\right]. \tag{74}$$

Combining together Equations (73) and (74), as well as Equation (70), for $t := 1$ we deduce that

$$\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|^{a}\right] \ge \left(\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|\right]\right)^{a} n^{-a+1} = \left(d\Theta\left(n^{2}\right)\right)^{a} n^{-a+1} = d^{a}\Theta\left(n^{a+1}\right).$$

Hence,

$$\sum_{j=1}^{n} \mathbf{E}\left[\left|mw_{j,n}\right|^{a}\right] = d^{a}\Omega\left(n^{a+1}\right), \text{ when } a \ge 1.$$

This is enough to prove the lower bound and completes the proof of Theorem 17. □

REFERENCES

[1] A. A. Abbasi, M. Younis, and K. Akkaya. 2009. Movement-assisted connectivity restoration in wireless sensor and actor networks. *IEEE Trans. Parallel Distrib. Syst.* 20, 9 (2009), 1366–1379.

[2] F. Abbasi, A. Mesbahi, and J. Mohammadpour Velni. 2019. A new voronoi-based blanket coverage control method for moving sensor networks. *IEEE Trans. Contr. Syst. Technol.* 27, 1 (2019), 409–417.

[3] H. M. Ammari. 2016. 3D-kCov-ComFor: An energy-efficient framework for composite forwarding in three-dimensional duty-cycled k-covered wireless sensor networks. *ACM Trans. Sens. Netw.* 12, 4 (2016).

[4] H. M. Ammari and S. K. Das. 2009. Critical density for coverage and connectivity in three-dimensional wireless sensor networks using continuum percolation. *IEEE Trans. Parallel Distrib. Syst.* 20, 6 (2009), 872–885.

[5] H. M. Ammari and S. K. Das. 2009. Fault tolerance measures for large-scale wireless sensor networks. *ACM Trans. Auton. Adapt. Syst.* 4, 1 (2009).

[6] H. M. Ammari and S. K. Das. 2010. A study of k-coverage and measures of connectivity in 3D wireless sensor networks. *IEEE Trans. Comput.* 59, 2 (2010), 243–257.

[7] H. M. Ammari and S. K. Das. 2012. Centralized and clustered k-coverage protocols for wireless sensor networks. *IEEE Trans. Comput.* 61, 1 (2012), 118–133.

[8] Ch. Anagnostopoulos, S. Hadjiefthymiades, and K. Kolomvatsos. 2016. Accurate, dynamic, and distributed localization of phenomena for mobile sensor networks. *ACM Trans. Sen. Netw.* 12, 2 (2016).

[9] B. Bhattacharya, M. Burmester, Y. Hu, E. Kranakis, Q. Shi, and A. Wiese. 2009. Optimal movement of mobile sensors for barrier coverage of a planar region. *Theor. Comput. Sci.* 410, 52 (2009), 5515–5528.

[10] A Boukerche and P. Sun. 2018. Connectivity and coverage based protocols for wireless sensor networks. *Ad Hoc Netw.* 80 (2018), 54–69.

[11] H. Cai, X. Jia, and M. Sha. 2011. Critical sensor density for partial connectivity in large area wireless sensor networks. *ACM Trans. Sen. Netw.* 7, 4 (Feb. 2011).

[12] J. Czyzowicz, E. Kranakis, D. Krizanc, L. Narayanan, and J. Opatrny. 2020. Optimal online and offline algorithms for robot-assisted restoration of barrier coverage. *Discr. Appl. Math.* 285 (2020), 650–662.

[13] S. Dasgupta, G. Mao, and B. Anderson. 2015. A new measure of wireless network connectivity. *IEEE Trans. Mobile Comput.* 14, 9 (2015), 1765–1779.

[14] I. Dietrich and F. Dressler. 2009. On the lifetime of wireless sensor networks. *ACM Trans. Sen. Netw.* 5, 1 (2009).

[15] S. Dobrev, E. Kranakis, D. Krizanc, M. Lafond, J. Maňuch, L. Narayanan, J. Opatrny, S. Shende, and L. Stacho. 2020. Weak coverage of a rectangular barrier. *Algorithmica* 82 (2020), 721–746.

[16] B. Doerr. 2018. An elementary analysis of the probability that a binomial random variable exceeds its expectation. *Stat. Probab. Lett.* 139 (2018), 67–74.

[17] R. Elhabyan, W. Shi, and M. St-Hilaire. 2019. Coverage protocols for wireless sensor networks: Review and future directions. *J. Commun. Netw.* 21, 1 (2019), 45–60.

[18] A. Eslami, M. Nekoui, H. Pishro-Nik, and F. Fekri. 2013. Results on finite wireless sensor networks: Connectivity and coverage. *ACM Trans. Sen. Netw.* 9, 4 (Jul. 2013).

[19] P. Frasca, F. Garin, B. Gerencsér, and Hendrickx J. M. 2015. Optimal one-dimensional coverage by unreliable sensors. *SIAM J. Contr. Optimiz.* 53, 5 (2015), 3120–3140.

[20] M. Fuchs, L. Kao, and W. Wu. 2020. On binomial and poisson sums arising from the displacement of randomly placed sensors. *Taiwan. J. Math.* 24, 6 (2020), 1353–1382.

[21] X. Gao, Z. Chen, F. Wu, and G. Chen. 2017. Energy efficient algorithms for k -sink minimum movement target coverage problem in mobile sensor network. *IEEE/ACM Trans. Netw.* 25, 6 (2017), 3616–3627.

[22] X. Gao, J. Fan, F. Wu, and G. Chen. 2018. Approximation algorithms for sweep coverage problem with multiple mobile sensors. *IEEE/ACM Trans. Netw.* 26, 2 (2018), 990–1003.

[23] X. Gao, J. Fan, F. Wu, and G. Chen. 2020. Cooperative sweep coverage problem with mobile sensors. *IEEE Trans. Mobile Comput.* (2020), 1–1.

[24] A. Ghosh, A. Chattopadhyay, A. Arora, and A. Kumar. 2017. Measurement based as-you-go deployment of two-connected wireless relay networks. *ACM Trans. Sen. Netw.* 13, 3 (2017).

[25] A. Ghosh and S. K. Das. 2008. Coverage and connectivity issues in wireless sensor networks: A survey. *Perv. Mobile Comput.* 4 (2008), 303–334.

[26] R. Graham, D. Knuth, and O. Patashnik. 1994. *Concrete Mathematics A Foundation for Computer Science.* Addison-Wesley, Reading, MA.

[27] H. P. Gupta, S. V. Rao, and T. Venkatesh. 2016. Analysis of stochastic coverage and connectivity in three-dimensional heterogeneous directional wireless sensor networks. *Perv. Mobile Comput.* 29 (2016), 38–56.

[28] S. Guruacharya, H. Tabassum, and E. Hossain. 2016. Integral approximations for coverage probability. *IEEE Wireless Commun. Lett.* 5, 1 (2016), 24–27.

[29] J. He and H. Shi. 2012. Constructing sensor barriers with minimum cost in wireless sensor networks. *J. Parallel Distrib. Comput.* 72, 12 (2012), 1654–1663.

[30] P. Hoyingcharoen and W. Teerapabkajorndet. 2019. Expected probabilistic detection and sink connectivity in wireless sensor networks. *IEEE Sens. J.* 19, 12 (2019), 4480–4493.

[31] R. Kan and C. Robotti. 2017. On moments of folded and truncated multivariate normal distributions. *J. Comput. Graph. Stat.* 26, 4 (2017), 930–934.

[32] R. Kapelko. 2018. On the moment distance of poisson processes. *Communications in Statistics - Theory and Methods* 47, 24 (2018), 6052–6063.

[33] R. Kapelko. 2018. On the maximum movement of random sensors for coverage and interference on a line. In *Proceedings of the 19th International Conference on Distributed Computing and Networking.* ACM, 36:1–36:10.

[34] R. Kapelko. 2018. On the maximum movement to the power of random sensors for coverage and interference. *Perv. Mobile Comput.* 51 (2018), 174–192.

[35] R. Kapelko. 2018. On the moments of absolute difference between random processes with application to sensor network. *IEEE Sens. Lett.* 2, 4 (Dec. 2018), 1–4.

[36] R. Kapelko. 2019. Asymptotic formula for sum of moment mean deviation for order statistics from uniform distribution. *Discr. Math. Algor. Appl.* 11 (Apr. 2019), 1–23.

[37] R. Kapelko. 2020. On the energy in displacement of random sensors for interference and connectivity. In *Proceedings of the 21st International Conference on Distributed Computing and Networking.* ACM, 1–10.

[38] R. Kapelko and E. Kranakis. 2016. On the displacement for covering a d-dimensional cube with randomly placed sensors. *Ad Hoc Netw.* 40 (2016), 37–45.

[39] R. Kapelko and E. Kranakis. 2016. On the displacement for covering a unit interval with randomly placed sensors. *Inform. Process. Lett.* 116 (2016), 710–717.

[40] H. Kim, H. Oh, P. Bellavista, and J. Ben-Othman. 2017. Constructing event-driven partial barriers with resilience in wireless mobile sensor networks. *J. Netw. Comput. Appl.* 82 (2017), 77–92.

[41] E. Kranakis. 2014. On the event distance of poisson processes with applications to sensors. *Discr. Appl. Math.* 179 (2014), 152–162.

[42] S. Kumar, T. H. Lai, and A. Arora. 2007. Barrier coverage with wireless sensors. *Wireless Netw.* 13, 6 (Dec. 2007), 817–834.

[43] F. Li, J Luo, S. Xin, and Y. He. 2016. Autonomous deployment of wireless sensor networks for optimal coverage with directional sensing model. *Comput. Netw.* 108 (2016), 120–132.

[44] H. Li and H. Zhang. 2020. On the edge-connectivity of an uncertain random graph. *IEEE Access* 8 (2020), 59126–59134.

[45] M. Mitzenmacher and E. Upfal. 2005. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Cambridge University Press, New York, NY.

[46] S. M. Mohamed, H. S. Hamza, and I. A. Saroit. 2017. Coverage in mobile wireless sensor networks (M-WSN): A survey. *Comput. Commun.* 110 (2017), 133–150.

[47] S. M. Ross. 2002. *Probability Models for Computer Science.* Academic Press.

[48] A. Saeed, A. Abdelkader, M. Khan, A. Neishaboori, K. A. Harras, and A. Mohamed. 2019. On realistic target coverage by autonomous drones. *ACM Trans. Sen. Netw.* 15, 3 (2019).

[49] A. Saipulla, C. Westphal, B. Liu, and J. Wang. 2009. Barrier coverage of line-based deployed wireless sensor networks. In *Proceedings of the IEEE International Conference on Computer Communications (INFOCOM'09).* IEEE, 127–135.

[50] S. Sastry, T. Radeva, J. Chen, and J. Welch. 2013. Reliable networks with unreliable sensors. *Perv. Mobile Comput.* 9, 2 (2013), 311–323.

[51] S. Shakkottai, R. Srikant, and N. Shroff. 2005. Unreliable sensor grids: coverage, connectivity and diameter. *Ad Hoc Netw.* 3, 6 (2005), 702–716.

[52] T. Shi, S. Cheng, J. Li, H. Gao, and Z. Cai. 2019. Dominating sets construction in rf-based battery-free sensor networks with full coverage guarantee. *ACM Trans. Sen. Netw.* 15, 4 (2019).

[53] Ch. Song, L. Liu, G. Feng, Y. Fan, and Sh. Xu. 2020. Coverage control for heterogeneous mobile sensor networks with bounded position measurement errors. *Automatica* 120 (2020), 109118.

[54] J. Svante. 1988. Some pairwise independent sequences for which the central limit theorem fails. *Stochastics* 23, 4 (1988), 439–448.

[55] J. Tian, X. Liang, and G. Wang. 2016. Deployment and reallocation in mobile survivability-heterogeneous wireless sensor networks for barrier coverage. *Ad Hoc Netw.* 36 (2016), 321–331.

[56] N. G. Ushakov. 2011. Some inequalities for absolute moments. *Stat. Probab. Lett.* 81, 12 (2011), 2011–2015.

[57] W. Wu, Z. Zhang, W. Lee, and D. Du. 2020. *Optimal Coverage in Wireless Sensor Networks.* Springer, Cham.

[58] G. Xing, X. Wang, Y. Zhang, Ch. Lu, R. Pless, and Ch. Gill. 2005. Integrated coverage and connectivity configuration for energy conservation in sensor networks. *ACM Trans. Sen. Netw.* 1, 1 (2005), 36–72.

[59] M. Younis and K. Akkaya. 2008. Strategies and techniques for node placement in wireless sensor networks: A survey. *Ad Hoc Netw.* 6, 4 (2008), 621–655.

[60] J. Yu, S. Wan, X. Cheng, and D. Yu. 2017. Coverage contribution area based k -coverage for wireless sensor networks. *IEEE Trans. Vehic. Technol.* 66, 9 (2017), 8510–8523.

[61] Ch. Zhou, A. Mazumder, A. Das, K. Basu, N. Matin-Moghaddam, S. Mehrani, and A. Sen. 2019. Relay node placement under budget constraint. *Perv. Mobile Comput.* 53 (2019), 1–12.

[62] Z. Zhou, S. R. Das, and H. Gupta. 2009. Variable radii connected sensor cover in sensor networks. *ACM Trans. Sen. Netw.* 5, 1 (Feb. 2009).

Received May 2020; revised October 2020; accepted March 2021