Research Article
Special Issue: In honor of Joel Spruck

Jacob Bernstein*

Rigidity properties of Colding–Minicozzi entropies

https://doi.org/10.1515/ans-2022-0082
Received February 25, 2023; accepted June 12, 2023; published online March 1, 2024

Abstract: We show certain rigidity for minimizers of generalized Colding–Minicozzi entropies. The proofs are elementary and work even in situations where the generalized entropies are not monotone along mean curvature flow.

Keywords: Colding–Minicozzi entropy; mean curvature flow; rigidity

1991 Mathematics Subject Classification: 53A07; 53A10

1 Introduction

We use an expansion of the volume of a submanifold in small geodesic balls as in Ref. [1] to show some rigidity phenomena for natural generalizations of the Colding–Minicozzi entropy [2], [3]. In particular, the arguments work even when the quantities are not monotone along mean curvature flow.

In order to define the generalized entropies we begin by setting

$$K_{n,\kappa}(t,r) = \begin{cases} \kappa^n K_n(\kappa^2 t, \kappa r) & \kappa, t > 0, r \geq 0 \\ (4\pi t)^{-n/2} e^{-\frac{r^2}{4t}} & \kappa = 0, t > 0, r \geq 0 \end{cases} \tag{1.1}$$

where $n \geq 1$, $\kappa \geq 0$ and K_n are explicit (if complicated) functions used in Ref. [4] to study the heat kernel on hyperbolic space. For instance,

$$K_3(t,r) = (4\pi t)^{-\frac{3}{2}} \frac{r}{\sinh(r)} e^{-t-\frac{r^2}{4t}}.$$

The other K_n are determined recursively – see Ref. [4] for details. In general,

$$H_{n,\kappa}(t,x;t_0,x_0) = K_{n,\kappa}(t-t_0, \operatorname{dist}_g(x,x_0))$$

Dedicated to Joel Spruck in honor of his retirement.

The author was partially supported by the DMS-2203132 and the Institute for Advanced Study with funding provided by the Charles Simonyi Endowment.

***Corresponding author: Jacob Bernstein**, Department of Mathematics, Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, USA, E-mail: bernstein@math.jhu.edu. https://orcid.org/0000-0003-2774-9030

is the heat kernel on (M, g) with singularity at x_0 and time t_0 precisely when (M, g) is a simply connected space form of constant curvature $-\kappa^2$. Following [2], [3], [5], let

$$\Phi_{n,\kappa}^{t_0,x_0}(t,x) = K_{n,\kappa}(t_0 - t, \mathrm{dist}_g(x, x_0))$$

and for $\Sigma \subset M$, an n-dimensional submanifold, define *the Colding–Minicozzi* κ*-entropy of* Σ *in* (M, g) to be

$$\lambda_g^\kappa[\Sigma] = \sup_{x_0 \in M, \tau > 0} \int_\Sigma \Phi_{n,\kappa}^{0,x_0}(-\tau, \cdot) \mathrm{d}V = \sup_{x_0 \in M, \tau > 0} \int_\Sigma \Phi_{n,\kappa}^{\tau,x_0}(0, \cdot) \mathrm{d}V.$$

When $\kappa = 0$ and $(M, g) = \left(\mathbb{R}^{n+k}, g_{\mathbb{R}}\right)$ is Euclidean space, this is the usual Colding–Minicozzi entropy, $\lambda[\Sigma]$, of Σ from Ref. [3]. When $\kappa = 1$ and $(M, g) = \left(\mathbb{H}^{n+k}, g_{\mathbb{H}}\right)$ is hyperbolic space, it is the entropy in hyperbolic space, $\lambda_{\mathbb{H}}[\Sigma]$, from Ref. [5].

In Ref. [2, Theorem 1], it is shown that if (M, g) is an $(n + k)$-dimensional Cartan–Hadamard manifold with $\sec_g \leq -\kappa_0^2$ and $0 \leq \kappa \leq \kappa_0$, then, for any mean curvature flow of closed submanifolds, $t \in [t_1, t_2] \mapsto \Sigma_t \subset M$,

$$\lambda_g^\kappa[\Sigma_{t_1}] \geq \lambda_g^\kappa[\Sigma_{t_2}].$$

This generalizes and unifies the monotonicity properties of the entropy of Refs. [3], [5]. Monotonicity also holds for non-closed flow under appropriate hypotheses.

It follows readily from the definition that for any n-dimensional submanifold $\Sigma \subset M$, one has $\lambda_g^\kappa[\Sigma] \geq 1$ – see Ref. [2, Proposition 6.3]. We seek to understand what can be said about Σ in the case of equality – i.e., when Σ minimizes a Colding–Minicozzi κ-entropy. The monotonicity of entropy can be used to answer this and related problems – see Refs. [6]–[8]. However, the question makes perfectly good sense in arbitrary Riemannian manifolds where monotonicity may not hold. With this in mind, we establish some rigidity properties that hold without using monotonicity.

Theorem 1.1. *Let* (M, g) *be a* $(2 + k)$*-dimensional Riemannian manifold with* $\sec_g \leq -\kappa^2$*. If* Σ *is a proper surface and* $\lambda_g^\kappa[\Sigma] = 1$*, then* Σ *is totally umbilic. In particular, if* (M, g) *is Euclidean space and* $\kappa = 0$*, then* Σ *is an affine two-plane.*

Remark 1.2. The result in Euclidean space follows from earlier work of L. Chen [7] who was also able to obtain the same result for all dimensions and co-dimensions and also for incomplete surfaces. However, his argument uses a fairly sophisticated mean curvature flow construction. Alternatively, it should also be possible to obtain rigidity for hypersurfaces that are boundaries of nice enough subsets of $\mathbb{R}^{n+1}$ using the Gaussian isoperimetric inequality [9], [10].

The techniques also allow us to show rigidity for minimal hypersurfaces in Einstein manifolds with appropriate Einstein constant – unlike the preceding theorem this is a setting where one may not have monotonicity of the entropy.

Theorem 1.3. *Let* (M, g) *be an n-dimensional Riemannian manifold satisfying*

$$Ric_g = -(n-1)\kappa^2 g.$$

If Σ *is a proper minimal hypersurface with* $\lambda_g^\kappa[\Sigma] = 1$*, then* Σ *is totally geodesic.*

Finally, we obtain a result for certain closed surfaces with $\lambda_g^0[\Sigma] = 1$ without any assumptions on the ambient manifold.

Theorem 1.4. *Let* (M, g) *be a* $(2 + k)$*-dimensional Riemannian manifold. If* Σ *is a closed, connected, orientable surface satisfying* $\lambda_g^0[\Sigma] = 1$*, then the genus of* Σ *satisfies* $\mathrm{gen}(\Sigma) \leq 1$*. Moreover, if* $\mathrm{gen}(\Sigma) = 1$*, then* Σ *is totally geodesic.*

Remark 1.5. This result is sharp in a certain sense as can be seen by considering a totally geodesic flat two-torus inside of a higher dimensional flat torus.

We note that similar, but stronger, rigidity results for conformal volume of submanifolds of the sphere were observed by Bryant in Ref. [11]. One difference between Ref. [11] and the current paper is that it is possible to arbitrarily change the mean curvature at a point with a conformal transformation – i.e., with a symmetry of the conformal volume. For Colding–Minicozzi entropies this cannot be done with the natural symmetries. However, flowing by mean curvature flow seems to play a similar role.

2 Small-time asymptotics of Gaussian κ-densities of submanifolds

Let (M,g) be a Riemannian manifold and $\Sigma \subset M$ a proper n-dimensional submanifold. We obtain the small time asymptotics of the (localized) pairing of the kernel $\Phi^{0,x_0}_{n,\kappa}(-t,\cdot)$ with any n-dimensional submanifold of M when $x_0 \in \Sigma$.

Proposition 2.1. *Let (M,g) be a $(n+k)$-dimensional Riemannian manifold and $\Sigma^n \subset M$ a n-dimensional submanifold. For $x_0 \in \Sigma$, if $\mathcal{B}^g_{2R}(x_0)$ is proper in M and $\Sigma_{2R} = \mathcal{B}^g_{2R}(x_0) \cap \Sigma$ is proper in $\mathcal{B}^g_{2R}(x_0)$, then, for, any $\kappa \geq 0$, the following asymptotic expansion holds:*

$$\int_{\Sigma_R} \Phi^{0,x_0}_{n,\kappa}(t,\cdot)dV = 1 - \frac{t}{3}\left(\frac{1}{2}|\mathbf{A}^g_\Sigma(x_0)|^2 + \frac{1}{4}|\mathbf{H}^g_\Sigma(x_0)|^2 - R^g_\Sigma(x_0) - n(n-1)\kappa^2\right) + O\left((-t)^{\frac{3}{2}}\right), t \to 0^-.$$

where here $\mathbf{A}^g_\Sigma$ is the second fundamental form of Σ, $\mathbf{H}^g_\Sigma = \mathrm{tr}_g \mathbf{A}^g_\Sigma$ is the mean curvature vector of Σ and R^g_Σ is the scalar curvature of Σ.

In order to prove this, we will need a pair of elementary lemmata.

Lemma 2.2. *For any $R > 0$ and $n \geq 1, k \geq 0$ integers one has*

$$\int_0^R K_{n,0}(t,\rho)\rho^{n+k-1}\mathrm{d}\rho = \frac{(4\pi t)^{k/2}}{|\mathbb{S}^{n+k-1}|_{\mathbb{R}}} + O\left(e^{-\frac{R^2}{4t}}\right), t \to 0^+.$$

There are also constants $C = C(R,n)$ so, for $0 < t \leq \frac{1}{2(n-1)}R$,

$$\int_R^\infty K_{n,1}(t,\rho)\sinh^{n-1}(\rho)d\rho \leq Ce^{-\frac{R^2}{16t}}.$$

Proof. Observe that, for all $k \geq 0$,

$$\int_0^\infty K_{n,0}(t,\rho)\rho^{n+k-1}\mathrm{d}\rho = (4\pi t)^{k/2}\int_0^\infty K_{n+k,0}(t,\rho)\rho^{n+k-1}\mathrm{d}\rho = \frac{(4\pi t)^{k/2}}{|\mathbb{S}^{n+k-1}|_{\mathbb{R}}}.$$

While for any $R > 0$ and for small times we have

$$\int_R^\infty K_{n,0}(t,\rho)\rho^{n+k-1}\mathrm{d}\rho = O\left(e^{-\frac{R^2}{4t}}\right), t \to 0^+.$$

The first claim is an immediate consequence.

For the second claim we observe that [4, Theorem 3.1] yields

$$K_{n,1}(t,\rho) \le C(1+\rho+t)^{\frac{1}{2}n-\frac{3}{2}}(1+\rho)e^{-\frac{1}{2}(n-1)^2 t-\frac{1}{2}(n-1)\rho}K_{n,0}(t,\rho).$$

One readily checks that, for $\rho \ge R > 0$ and $\frac{R}{2(n-1)} \ge t$ that,

$$\begin{aligned} K_{n,1}(t,\rho)\sinh^{n-1}(\rho) &\le C'(1+\rho+t)^{\frac{1}{2}n-\frac{3}{2}}(1+\rho)K_{n,0}(t,\rho-t(n-1)) \\ &\le C''R^{-\frac{1}{2}(n-1)}(\rho-t(n-1))^{n-1}K_{n,0}(t,\rho-t(n-1)). \end{aligned}$$

where $C'' = C''(R,n)$. It follows that, for such R and t,

$$\begin{aligned} \int_R^\infty K_{n,1}(t,\rho)\sinh^{n-1}(\rho)\mathrm{d}\rho &\le C''R^{-\frac{1}{2}(n-1)}\int_{R-t(n-1)}^\infty u^{n-1}K_{n,0}(t,u)\mathrm{d}u \\ &\le C''R^{-\frac{1}{2}(n-1)}\int_{\frac{R}{2}}^\infty u^{n-1}K_{n,0}(t,u)\mathrm{d}u \le C'''e^{-\frac{R^2}{16t}}. \end{aligned}$$

Here $C''' = C'''(R,n)$ and we used the first computation of the proof. □

We also need information about the leading order asymptotics of $K_{n,\kappa}$ near the space-time origin – the expansions was established in Ref. [4], but we needed some more information about the relationship between certain coefficients.

Lemma 2.3. *Fix $\kappa > 0$. There is a constant $C_n > 0$ so that, for $0 \le t \le \kappa^{-2}$ and $0 \le \rho \le \kappa^{-1}$,*

$$|K_{n,\kappa}(t,\rho) - \left(1+\kappa^2 a_n t + \kappa^2 b_n \rho^2\right)K_{n,0}(t,\rho)| \le C_n\kappa^4(t+\rho^2)^2K_{n,0}(t,\rho)$$

where a_n, b_n satisfy

$$a_n + 2nb_n = -\frac{1}{3}n(n-1). \tag{2.1}$$

Proof. With out loss of generality we may assume $\kappa = 1$, as the general result immediately follows from this case and the definition of $K_{n,\kappa}$. The existence of the a_n, b_n and C_n follow from Ref. [4]. To conclude the proof we observe that for, all $t > 0$,

$$|\mathbb{S}^{n-1}|_{\mathbb{R}}\int_0^\infty K_{n,1}(t,\rho)\sinh^{n-1}(\rho)\mathrm{d}\rho = 1.$$

Using the second estimate of Lemma 2.2 with $R = 1$ and the expansion $\sinh^{n-1}(\rho) = \rho + \frac{n-1}{6}\rho^3 + O(\rho^5)$, we obtain

$$\begin{aligned} |\mathbb{S}^{n-1}|_{\mathbb{R}}^{-1} &= \int_0^1 K_{n,1}(t,\rho)\sinh^{n-1}(\rho)\mathrm{d}\rho + O(t^2), t\to 0^+ \\ &= \int_0^1 K_{n,0}(t,\rho)\Big(1+a_nt+b_n\rho^2+\frac{n-1}{6}\rho^2\Big)\rho^{n-1}\mathrm{d}\rho + O(t^2), t\to 0^+ \\ &= |\mathbb{S}^{n-1}|_{\mathbb{R}}^{-1}\Big(1+\Big(a_n+2nb_n+\frac{1}{3}n(n-1)\Big)t\Big) + O(t^2), t\to 0^+. \end{aligned}$$

where we used that

$$|\mathbb{S}^{n+k+1}|_{\mathbb{R}} = \frac{2\pi}{n+k}|\mathbb{S}^{n+k-1}|_{\mathbb{R}}. \tag{2.2}$$

Hence, $a_n + 2nb_n = -\frac{1}{3}n(n-1)$. □

Proof of Proposition 2.1. For the fixed point $x_0 \in \Sigma$, choose $R \geq R_0 > 0$ small enough so that $\mathcal{B}^g_{R_0}(x_0)$ is geodesically convex. Up to shrinking R_0, we may assume that the expansion of Theorem A.1 holds for $|\mathcal{B}^g_s(x_0) \cap \Sigma|_g$ for $0 < s < R_0$. As Σ_R is proper in $\mathcal{B}^g_{2R}(\Sigma)$, we have $|\Sigma_R|_g$ finite. Hence, by the pointwise estimates on $K_{n,\kappa}$ that follow from Ref. [4, Theorem 3.1], we have, for $-t$ sufficiently small,

$$\int\limits_{\Sigma_R \setminus \mathcal{B}^g_{R_0}(x_0)} \Phi^{0,x_0}_{n,\kappa}(-t,\cdot)\mathrm{d}V \leq C|\Sigma_R|_g e^{-\frac{R_0^2}{-16t}} = O\left((-t)^{\frac{3}{2}}\right).$$

where here $C = C(n,\kappa)$.

Hence, it is enough to prove the result for $\Sigma_{R_0} = \Sigma \cap \mathcal{B}^g_{R_0}(x_0)$. Using the co-area formula we have, with $\rho(q) = \mathrm{dist}_g(q, x_0)$,

$$\begin{aligned}
\int\limits_{\Sigma_{R_0}} \Phi^{0,x_0}_{n,\kappa}(t,\cdot)\mathrm{d}V &= \int\limits_0^{R_0} \int\limits_{\partial\mathcal{B}^g_s(x_0)} \Phi^{0,x_0}_{n,\kappa}(t,\cdot)\frac{1}{|\nabla_\Sigma \rho|}\mathrm{d}V\mathrm{d}s \\
&= \int\limits_0^{R_0} K_{n,\kappa}(-t,s) \int\limits_{\partial\mathcal{B}^g_s(x_0)} \frac{1}{|\nabla_\Sigma \rho|}\mathrm{d}V\mathrm{d}s \\
&= \int\limits_0^{R_0} K_{n,\kappa}(-t,s)\frac{\mathrm{d}}{\mathrm{d}s}|\mathcal{B}^g_s(x_0) \cap \Sigma|\mathrm{d}s \\
&= K_{n,\kappa}(-t,R_0)|\Sigma_{R_0}|_g - \int\limits_0^{\infty} \partial_r K_{n,\kappa}(-t,s)|\mathcal{B}^g_s(x_0) \cap \Sigma|\mathrm{d}s.
\end{aligned}$$

Appealing again to Ref. [4, Theorem 3.1], we have

$$K_{n,\kappa}(-t,R_0)|\Sigma_{R_0}|_g \leq C|\Sigma_R|_g e^{-\frac{R_0^2}{-16t}} = O\left((-t)^{\frac{3}{2}}\right).$$

Moreover, by direct computation we have

$$\partial_r K_{n,0}(-t,r) = -\frac{r}{2t}K_{n,0}(-t,r) = -2\pi r K_{n+2,0}(-t,r)$$

and, for $\kappa > 0$, the generalized Millison identity (e.g., Refs. [2], [4]) give

$$\partial_r K_{n,\kappa}(-t,r) = -2\pi e^{-n\kappa^2 t}\kappa^{-1}\sinh(\kappa r)K_{n+2,\kappa r}(-t,r).$$

Hence, when $\kappa = 0$, we may use Theorem A.1 and Lemma 2.2 to obtain

$$\begin{aligned}
\int\limits_{\Sigma_{R_0}} \Phi^{0,x_0}_{n,\kappa}(-t,\cdot)\mathrm{d}V &= 2\pi \int\limits_0^{R_0} K_{n+2,0}(-t,s)s|\mathcal{B}^g_s(x_0) \cap \Sigma|\mathrm{d}s + O\left((-t)^{\frac{3}{2}}\right) \\
&= 2\pi|B^n_1|_{\mathbb{R}} \int\limits_0^{R_0} K_{n+2,0}(-t,s)\left(s^n + As^{n+1} + O(s^{n+2})\right)\mathrm{d}s + O\left((-t)^{\frac{3}{2}}\right)
\end{aligned}$$

$$= |\mathbb{S}^{n+1}|_{\mathbb{R}}\left(|\mathbb{S}^{n+1}|_{\mathbb{R}}^{-1} + 4\pi|\mathbb{S}^{n+3}|_{\mathbb{R}}^{-1}A(-t)\right) + O\left((-t)^{\frac{3}{2}}\right)$$
$$= 1 - 2(n+2)At + O\left((-t)^{\frac{3}{2}}\right).$$

where we used (2.2) and the coefficient A from Theorem A.1 is

$$A = \frac{1}{6(n+2)}\left(\frac{1}{2}|\mathbf{A}_{\Sigma}^{g}(p)|_g^2 + \frac{1}{4}|\mathbf{H}_{\Sigma}^{g}(p)|_g^2 - R_{\Sigma}^{g}(p)\right).$$

The expansion in the $\kappa = 0$ case follows.

When $\kappa > 0$ the same reasoning yields

$$\int_{\Sigma_{R_0}} \Phi_{n,\kappa}^{0,x_0}(-t,\cdot)\mathrm{d}V = 2\pi e^{-n\kappa^2 t}\kappa^{-1}\int_0^{R_0} K_{n+2,\kappa}(-t,s)\sinh(\kappa s)|\mathcal{B}_s^{g}(x_0)\cap\Sigma|_g \mathrm{d}s$$
$$= 2\pi|B_1^n|_{\mathbb{R}}e^{-n\kappa^2 t}\int_0^{R_0} K_{n+2,\kappa}(-t,s)\left(s + \frac{\kappa^2}{6}s^3\right)(s^n + As^{n+2} + O(s^{n+3}))\mathrm{d}s$$
$$= |\mathbb{S}^{n+1}|_{\mathbb{R}}e^{-n\kappa^2 t}\int_0^{R_0} K_{n+2,\kappa}(-t,s)\left(1 + As^2 + \frac{\kappa^2}{6}s^2 + O(s^3)\right)s^{n+1}\mathrm{d}s.$$

We now apply Lemma 2.2 ignoring terms of order $O((-t)^{\frac{3}{2}})$ to obtain

$$|\mathbb{S}^{n+1}|_{\mathbb{R}}e^{-n\kappa^2 t}\int_0^{R_0} K_{n+2,0}(-t,s)\left(1 + As^2 + \kappa^2\left(\frac{s^2}{6} + b_{n+2}s^2 - a_{n+2}t\right)\right)\mathrm{d}s$$
$$= e^{-n\kappa^2 t}\left(1 - t\left(2(n+2)A + \kappa^2\left(a_{n+2} + 2(n+2)b_{n+2} + \frac{n+2}{3}\right)\right)\right)$$
$$= 1 - t\left(2(n+2)A + \kappa^2\left(a_{n+2} + 2(n+2)b_{n+2} + \frac{n+2}{3} + n\right)\right)$$
$$= 1 - t\left(2(n+2)A - \frac{1}{3}n(n-1)\kappa^2\right).$$

where the last equality uses Lemma 2.3. The result is immediate. □

3 Rigidity

We are now able to prove the main rigidity results of the paper.

Proof of Theorem 1.1. It follows from the Gauss equations and the hypothesis $\sec_g \le -\kappa^2$ that, for any $x_0 \in \Sigma$,

$$-R_{\Sigma}^{g}(x_0) \ge n(n-1)\kappa^2 + |\mathbf{A}_{\Sigma}^{g}|^2 - |\mathbf{H}_{\Sigma}^{g}|^2.$$

Hence, Proposition 2.1 and $\lambda_g^{\kappa}[\Sigma] = 1$ together imply that, for any $x_0 \in \Sigma$,

$$0 \ge \frac{1}{2}|\mathbf{A}_{\Sigma}^{g}(x_0)|^2 + \frac{1}{4}|\mathbf{H}_{\Sigma}^{g}(x_0)|^2 - n(n-1)\kappa^2 - R_{\Sigma}^{g}(x_0)$$
$$\ge \frac{3}{2}|\mathbf{A}_{\Sigma}^{g}(x_0)|^2 - \frac{3}{4}|\mathbf{H}_{\Sigma}^{g}(x_0)|^2 = \frac{3}{2}|\mathring{\mathbf{A}}_{\Sigma}^{g}(x_0)|^2.$$

where the last equality used that Σ was two dimensional and $\mathring{\mathbf{A}}_{\Sigma}^{g} = \mathbf{A}_{\Sigma}^{g} - \frac{1}{2}\mathbf{H}_{\Sigma}^{g}g_{\Sigma}$ is the trace-free part of the second fundamental form of Σ. As x_0 was arbitrary, we conclude that $\mathring{\mathbf{A}}_{\Sigma}^{g}$ vanishes and so Σ is totally umbilic.

To conclude the proof we observe that if the ambient space is Euclidean and $\kappa = 0$, then, as Σ is proper, it must be a collection of affine two-planes and round two-spheres contained in three-dimensional affine subspaces. However, one can readily compute that any such two-sphere has entropy strictly larger than 1. Likewise, if there is more than one affine two-plane the entropy is also strictly larger than 1 and so the claim follows. □

We again use the Gauss equations to obtain the rigidity of Theorem 1.3.

Proof of Theorem 1.3. The Gauss equations imply that if Σ is an n-dimensional hypersurface in M, then

$$-R^g_\Sigma = -R_g + 2Ric_g(\mathbf{n}, \mathbf{n}) + |\mathbf{A}^g_\Sigma|^2 - |\mathbf{H}^g_\Sigma|^2.$$

The Einstein condition on (M, g) gives $-R_g = (n+1)n\kappa^2$ and so

$$-R^g_\Sigma = n(n-1)\kappa^2 + |\mathbf{A}^g_\Sigma|^2 - |\mathbf{H}^g_\Sigma|^2.$$

Hence, the hypotheses that $\lambda^\kappa_g[\Sigma] = 1$ together with Proposition 2.1 implies that, for any $x_0 \in \Sigma$,

$$\begin{aligned} 0 &\geq \frac{1}{2}|\mathbf{A}^g_\Sigma(x_0)|^2 + \frac{1}{4}|\mathbf{H}^g_\Sigma(x_0)|^2 - n(n-1)\kappa^2 - R^g_\Sigma(x_0) \\ &= \frac{3}{2}|\mathbf{A}^g_\Sigma(x_0)|^2 - \frac{3}{4}|\mathbf{H}^g_\Sigma(x_0)|^2 = \frac{3}{2}|\mathbf{A}^g_\Sigma(x_0)|^2. \end{aligned}$$

Here the last equality used that Σ was minimal. Hence, as x_0 was arbitrary, Σ is totally geodesic. □

Finally, we use the Gauss–Bonnet theorem to prove Theorem 1.4.

Proof of Theorem 1.4. As above, $\lambda^0_g[\Sigma] = 1$ and Proposition 2.1 together imply that

$$0 \geq \frac{1}{2}|\mathbf{A}^g_\Sigma|^2 + \frac{1}{4}|\mathbf{H}^g_\Sigma|^2 - R^g_\Sigma$$

for all points on Σ. As Σ is closed, we may integrate this inequality over Σ to obtain

$$\int_\Sigma R^g_\Sigma \mathrm{d}V \geq \frac{1}{2}\int_\Sigma |\mathbf{A}^g_\Sigma|^2 + \frac{1}{2}|\mathbf{H}^g_\Sigma|^2 \mathrm{d}V \geq 0.$$

As Σ is closed and orientable, the Gauss–Bonnet theorem implies

$$8\pi(1 - \mathrm{gen}(\Sigma)) \geq \frac{1}{2}\int_\Sigma |\mathbf{A}^g_\Sigma|^2 + \frac{1}{2}|\mathbf{H}^g_\Sigma|^2 \mathrm{d}A \geq 0.$$

Hence, $\mathrm{gen}(\Sigma) \leq 1$ and if $\mathrm{gen}(\Sigma) = 1$, then Σ is totally geodesic. □

Research ethics: Not applicable.
Author contributions: The author has accepted responsibility for the entire content of this manuscript and approved its submission.
Competing interests: The author states no conflict of interest.
Research funding: The author was partially supported by the DMS-2203132 and the Institute for Advanced Study with funding provided by the Charles Simonyi Endowment.
Data availability: Not applicable.

Appendix A. Generalized Karp–Pinksy expansion

We record here an asymptotic expansion of the volume of a submanifolds inside a geodesic ball in some Riemannian manifold. This generalizes a result of Karp and Pinsky [1] who treated the Euclidean case. Such an expansion is shown for submanifolds of hyperbolic space in Ref. [12] – see also Refs. [13], [14].

Theorem A.1. *Let (M, g) be a Riemannian manifold and $\Sigma \subset M$ a submanifold of dimension n. For $p \in \Sigma$, one has the following expansion for $R > 0$ small*

$$|\mathcal{B}_R^g(p) \cap \Sigma|_g = |B_1^n|_{\mathbb{R}} R^n + A|B_1^n|_{\mathbb{R}} R^{n+2} + O(R^{n+3}),$$

here $|\cdot|_g$ denote the g-volume and $|B_1^n|_{\mathbb{R}}$ is the Euclidean volume of the ball and

$$A = \frac{1}{6(n+2)}\left(\frac{1}{2}|\mathbf{A}_\Sigma^g(p)|_g^2 + \frac{1}{4}|\mathbf{H}_\Sigma^g(p)|_g^2 - R_\Sigma^g(p)\right),$$

where $\mathbf{A}_\Sigma^g$ is the second fundamental form, $\mathbf{H}_\Sigma = \mathrm{tr}_g \mathbf{A}_\Sigma^g$ is the mean curvature vector and R_Σ^g is the the scalar curvature of Σ with its induced metric.

We will prove this by isometrically embedding M into a large Euclidean space. First, we fix notation and suppose that M is $(n+K)$-dimensional and $i: M \to \mathbb{R}^{n+K+N} = \mathbb{R}_{\mathbf{x}}^{n+K} \times \mathbb{R}_{\mathbf{y}}^N$ is an isometric embedding with $i(p) = \mathbf{0}$ and so $di_p: T_pM \to \mathbb{R}_{\mathbf{x}}^n \times \{0\}$. In particular, if we set $M' = i(M)$, then $\mathbf{0} \in M'$ and $\mathbb{R}^{n+K} \times \{\mathbf{0}\} = \{\mathbf{y} = \mathbf{0}\} = T_\mathbf{0}M'$ Let g' be the induced metric on M' – i.e., so $i \times g' = g$.

We now define two maps from a neighborhood of $\mathbf{0}$ in $\mathbb{R}^{n+K}$ to M'. For the first, observe that, near $\mathbf{0}$, M' is the graph of a function $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \dots, u_N(\mathbf{x}))$. Hence, the first map, $G_{M'}$, may be defined in a neighborhood, U, of $\mathbf{0}$ by

$$G_{M'}: U \to M', \mathbf{x} \mapsto (\mathbf{x}, \mathbf{u}(\mathbf{x})).$$

Note the hypotheses on M' ensure $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{u}(\mathbf{0}) = \mathbf{0}$. Hence,

$$u_\alpha(\mathbf{x}) = \frac{1}{2}\sum_{i,j=1}^{n+K} C_\alpha^{ij} x_i x_j + O(|\mathbf{x}|^3)$$

where we can readily identify

$$C_\alpha^{ij} = \mathbf{e}_{n+k+\alpha} \cdot \mathbf{A}_{M'}^{\mathbb{R}}|_\mathbf{0}(\mathbf{e}_i, \mathbf{e}_j)$$

where $\mathbf{A}_{M'}^{\mathbb{R}}$ is the second fundamental form of M'. Up to shrinking U, we may also define a second map based on the exponential map of g'.

$$E: U \to M', \mathbf{x} \mapsto \exp_\mathbf{0}^{g'}(\mathbf{x}) = i(\exp_p^g(\mathbf{x}))$$

where here we think of $\mathbf{x}$ as an element of $T_\mathbf{0}M'$ and also identify it in the natural way with an element of T_pM via the isomorphism $di_p: T_pM \to T_\mathbf{0}M'$.

Let Σ' be the n-dimensional submanifold of M' so $i(\Sigma') = \Sigma$. Write $\mathbb{R}^{n+K} = \mathbb{R}_{\mathbf{w}}^n \times \mathbb{R}_{\mathbf{z}}^K$. We have $\mathbf{0} \in \Sigma'$ and, up to rotating the $\mathbb{R}^{n+K}$ factor, may assume $T_\mathbf{0}\Sigma' = \mathbb{R}^n \times \{\mathbf{0}\} = \{\mathbf{z}, \mathbf{y} = \mathbf{0}\} \subset \mathbb{R}^{n+K+N}$. Hence, near $\mathbf{0}$ we can express Σ' as the graph of a function $\mathbf{v}(\mathbf{w}, \mathbf{z}(\mathbf{w}) = (v_1(\mathbf{w}), \dots, v_{K+N}(\mathbf{w}))$ and define a map $G_{\Sigma'}$ in a neighborhood, V, of $\mathbf{0}$

$$G_{\Sigma'}: V \to \Sigma', \mathbf{w} \mapsto (\mathbf{w}, \mathbf{v}(\mathbf{w})).$$

Note the hypotheses on Σ' ensure $\mathbf{v}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{v}(\mathbf{0}) = \mathbf{0}$. In particular, we have

$$v_\alpha(\mathbf{w}) = \frac{1}{2}\sum_{i,j=1}^{n} A_\alpha^{ij} w_i w_j + O(|\mathbf{w}|^3),$$

where we can readily identify the coefficients as

$$A_\alpha^{ij} = \mathbf{e}_{n+\alpha} \cdot \mathbf{A}^{\mathbb{R}}_{\Sigma'}|_{\mathbf{0}}(\mathbf{e}_i, \mathbf{e}_j)$$

where $\mathbf{A}^{\mathbb{R}}_{\Sigma'}$ is the second fundamental form of Σ'.

Lemma A.2. *With notation as above, we have the asymptotic expansion*

$$|E(G_{M'}^{-1}(\mathbf{x}))|^2 = |\mathbf{x}|^2 + \frac{1}{3}Q'(\mathbf{x}) + O(|\mathbf{x}|^5)$$

where Q is a homogeneous degree four polynomial of the form

$$Q'(\mathbf{x}) = \sum_{\alpha=1}^{N}\left(\sum_{i,k=1}^{n} C_\alpha^{ik} x_i x_k\right)^2 + \sum_{i,j,k,l=1}^{n} H^{ijkl} x_i x_j x_l x_k$$

where the H^{ijkl} satisfy $H^{iiii} = 0, 1 \leq i \leq n$ and, for $i \neq j$,

$$H^{ijij} + H^{ijji} + H^{jiji} + H^{jiij} + H^{iijj} + H^{jjii} = 0.$$

Proof. The expansion of the metric in geodesic normal coordinates yields

$$g_{ij}^E = (E^* g')|_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} - \frac{1}{3}\sum_{k,l=1}^{n} R_{ikjl} x_k x_l + O(|\mathbf{x}|^3),$$

where here $R_{ijkl} = Riem|_p(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l)$. are the coefficients the Riemann curvature tensor at p. Likewise,

$$g_{ij}^G = \left(G_{M'}^* g'\right)|_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} + \sum_{k,l=1}^{n} B^{ijkl} x_k x_l + O(|\mathbf{x}|^3)$$

where, by Ref. [1, pg. 89], we have

$$B^{ijkl} = \sum_{\alpha=1}^{N} C_\alpha^{il} C_\alpha^{jk}.$$

As $E(\mathbf{0}) = E_{M'}(\mathbf{0})$ and $DE(\mathbf{0}) = DG_{M'}(\mathbf{0})$ and $(E^* g')_{ij} - \left(G^* g_\Sigma\right)_{ij} = O(|\mathbf{x}|^2)$ it readily follows that

$$E^{-1}(G_{M'}(\mathbf{x})) = (x_1 + K_1(\mathbf{x})), \dots, x_n + K_n(\mathbf{x})) + O(|\mathbf{x}|^4)$$

where K_i are cubic homogeneous polynomials. In fact, for the metrics to agree to second order, one must have, for $1 \leq i, j \leq n$,

$$\partial_j K_i(\mathbf{x}) + \partial_i K_j(\mathbf{x}) = \sum_{k,l=1}^{n}\left(B^{ijkl} x_k x_l + \frac{1}{3} R_{ikjl} x_k x_l\right).$$

Using $K_i(\mathbf{x}) = \frac{1}{3}\sum_{j=1}^{n} x_j \partial_j K_i(\mathbf{x})$, we obtain

$$\begin{aligned}
|E^{-1}(G_{M'}(\mathbf{x}))|^2 - |\mathbf{x}|^2 &= 2\sum_{i=1}^{n} x_i K_i(\mathbf{x}) + O(|\mathbf{x}|^5)\\
&= \frac{2}{3}\sum_{i,j=1}^{n} x_i x_j \partial_j K_i(\mathbf{x}) + O(|\mathbf{x}|^5)\\
&= \frac{1}{3}\sum_{i,j=1}^{n} x_i x_j (\partial_i K_j(\mathbf{x}) + \partial_j K_i(\mathbf{x})) + O(|\mathbf{x}|^5)
\end{aligned}$$

$$= \frac{1}{3} \sum_{i,j,k,l=1}^{n} \left(B^{ijkl} + \frac{1}{3} R_{ikjl} \right) x_i x_j x_k x_l + O(|\mathbf{x}|^5).$$

To conclude we observe

$$\sum_{i,j,k,l=1}^{n} B^{ijkl} x_i x_j x_k x_l = \sum_{\alpha=1}^{N} \left(\sum_{j,l=1}^{n} C_{\alpha}^{ik} x_i x_j \right)^2.$$

While if we set

$$H^{ikjl} = \frac{1}{3} R_{ikjl},$$

then the algebraic symmetries of the Riemann curvature tensor imply that, for $1 \leq i \leq n$, $H^{iiii} = H^{iijj} = H^{jjii} = 0$ and, for $i \neq j$,

$$\begin{aligned} 0 &= \frac{1}{3} \left(R_{ijij} + R_{ijji} + R_{jiij} + R_{jiji} \right) = H^{ijij} + H^{ijji} + H^{jiij} + H^{jiji} \\ &= H^{ijij} + H^{ijji} + H^{jiji} + H^{jiij} + H^{iijj} + H^{jjii}. \end{aligned}$$

The claim follows. ☐

We are now ready to prove the extension of the Karp–Pinsky estimate.

Proof. Continuing with the notation from above, let $\Sigma'' = G_{M'}^{-1}(\Sigma') \subset \mathbb{R}^{n+K} \times \{0\}$. Up to shrinking U, this is a submanifold in $\mathbb{R}^{n+K}$ through $\mathbf{0}$ and tangent to $\mathbb{R}^n \times \{0\}$ at that point. In particular, up to shrinking V, we can express Σ'' as a graph of a function $\mathbf{z}(\mathbf{w}) = (z_1(\mathbf{w}), \ldots, z_K(\mathbf{w}))$ and can define a map

$$G_{\Sigma''} : V \to \Sigma'', \mathbf{w} \mapsto (\mathbf{w}, \mathbf{z}(\mathbf{w})).$$

Note the hypotheses on Σ' ensure $\mathbf{v}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{v}(\mathbf{0}) = \mathbf{0}$. In particular, we have

$$z_{\alpha}(\mathbf{w}) = \frac{1}{2} \sum_{i,j=1}^{n} \hat{A}_{\alpha}^{ij} w_i w_j + O(|\mathbf{w}|^3),$$

where we can readily identify these terms with

$$\hat{A}_{\alpha}^{ij} = \mathbf{e}_{n+\alpha} \cdot \mathbf{A}_{\Sigma''}^{\mathbb{R}}|_{\mathbf{0}}(\mathbf{e}_i, \mathbf{e}_j).$$

As $G_{\Sigma'} = G_{M'} \circ G_{\Sigma''}$, the identification of $T_p M$ and $T_{\mathbf{0}} M'$ implies

$$A_{\alpha}^{ij} = \hat{A}_{\alpha}^{ij} = \mathbf{e}_{\alpha+n} \cdot \mathbf{A}_{\Sigma'}^{T}|_{\mathbf{0}}(\mathbf{e}_i, \mathbf{e}_j) = g(\mathbf{e}_{\alpha+n}, \mathbf{A}_{\Sigma}^{g}|_p(\mathbf{e}_i, \mathbf{e}_j)), 1 \leq \alpha \leq K,$$

where here $\mathbf{A}_{\Sigma}^{g}$ is the second fundamental form of Σ in (M, g) and $\mathbf{A}_{\Sigma'}^{T}$ the projection of the second fundamental form of Σ' onto TM'. Likewise,

$$A_{\alpha+K}^{ij} = C_{\alpha}^{ij} = \mathbf{e}_{\alpha+n+K} \cdot \mathbf{A}_{\Sigma'}^{N}|_{\mathbf{0}}(\mathbf{e}_i, \mathbf{e}_j), 1 \leq \alpha \leq N,$$

where here $\mathbf{A}_{\Sigma'}^{N}$ is the projection of the second fundamental form of Σ' onto NM'.

Arguing as in Ref. [1], there is a homogeneous quartic polynomial Q so

$$|G_{\Sigma''}(\mathbf{w})|^2 = |\mathbf{w}|^2 + \frac{1}{4} Q(\mathbf{w}) + O(|\mathbf{w}|^5)$$

where here

$$Q(\mathbf{w}) = \sum_{i,j,k,l=1}^{n} \sum_{\alpha=1}^{K} \hat{A}_{\alpha}^{ik} \hat{A}_{\alpha}^{jl} w_i w_j w_k w_l = \sum_{i,j,k,l=1}^{n} \sum_{\alpha=1}^{K} A_{\alpha}^{ik} A_{\alpha}^{jl} w_i w_j w_k w_l.$$

By Lemma A.2, if $r(q)$ is the geodesic distance in M' between $\mathbf{q}$ and $\mathbf{0}$, then

$$(r(G_{M'}(\mathbf{x})))^2 = |\mathbf{x}|^2 + \frac{1}{3}Q'(\mathbf{x}) + O(|\mathbf{x}|^5).$$

Combining the two expansions yields,

$$(r(G_{\Sigma'}(\mathbf{w})))^2 = |\mathbf{w}|^2 + \frac{1}{3}Q'((\mathbf{w},\mathbf{0})) + \frac{1}{4}Q(\mathbf{w}) + O(|\mathbf{w}|^5).$$

In particular, when $R > 0$ is sufficiently small there is a function h so that

$$G_{\Sigma'}^{-1}(\mathcal{B}_R^{g'}(p)) = \{\mathbf{w}\colon h(\mathbf{w}) \le r\}$$

where here h satisfies $h(\mathbf{0}) = 0$, and, for $\mathbf{w} \neq \mathbf{0}$, one has

$$h(\mathbf{w}) = |\mathbf{w}| - \frac{1}{6}Q'((\hat{\mathbf{w}},\mathbf{0}))|\mathbf{w}|^3 - \frac{1}{8}Q(\hat{\mathbf{w}})|\mathbf{w}|^3 + O(|\mathbf{w}|^4)$$

where $\hat{\mathbf{w}} = |\mathbf{w}|^{-1}\mathbf{w}$.

Following the argument of Karp–Pinksy directly, one obtains, for R small,

$$|\Sigma \cap \mathcal{B}_R^g(p)|_g = |B_1^n|_{\mathbb{R}} R^n + A|B_1^n|_{\mathbb{R}} R^{n+2} + O(R^{n+3}),$$

where the subleading coefficient is given as

$$A = \frac{1}{2|B_1^n|_{\mathbb{R}}}\left(\frac{|B_1^n|_{\mathbb{R}}}{n+2}\sum_{\alpha=1}^{N+K}\sum_{i,k=1}^{n}|A_\alpha^{ik}|^2 - \frac{1}{3}\int_{\mathbb{S}^{n-1}} Q' - \frac{1}{4}\int_{\mathbb{S}^{n-1}} Q\right).$$

It is shown in Ref. [1, pg. 90] that

$$\int_{\mathbb{S}^{n-1}} Q = \frac{|B_1^n|_{\mathbb{R}}}{n+2}\sum_{\alpha=1}^{K}\left(2\sum_{i,j=1}^{n}|A_\alpha^{ij}|^2 + \left(\sum_{i=1}^{n}A_\alpha^{ii}\right)^2\right).$$

The exact same computation and the properties of the H^{ijkl} also imply that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} Q' &= \frac{|B_1^n|_{\mathbb{R}}}{n+2}\sum_{\alpha=1}^{N}\left(2\sum_{i,j=1}^{n}|C_\alpha^{ij}|^2 + \left(\sum_{i=1}^{n}C_\alpha^{ii}\right)^2\right)\\ &= \frac{|B_1^n|_{\mathbb{R}}}{n+2}\sum_{\alpha=1}^{N}\left(2\sum_{i,j=1}^{n}|A_{\alpha+K}^{ij}|^2 + \left(\sum_{i=1}^{n}A_{\alpha+K}^{ii}\right)^2\right).\end{aligned}$$

Finally, using the identification of the A_α^{ij} with the geometric data we have,

$$\sum_{\alpha=1}^{N+K}\sum_{i,k=1}^{n}|A_\alpha^{ik}|^2 = |\mathbf{A}_{\Sigma'}^{\mathbb{R}}(\mathbf{0})|^2$$

Likewise, using $G_{M'}^* g' = g_{\mathbb{R}} + O(|\mathbf{x}|^2)$, yields

$$\sum_{\alpha=1}^{K}\sum_{i,j=1}^{n}|A_\alpha^{ij}|^2 = |\mathbf{A}_\Sigma^g(p)|^2 \text{ and } \sum_{\alpha=1}^{K}\left(\sum_{i=1}^{n}A_\alpha^{ii}\right)^2 = |\mathbf{H}_\Sigma^g(p)|^2$$

and

$$\sum_{\alpha=1}^{N}\sum_{i,j=1}^{n}|A_{\alpha+K}^{ij}|^2 = |\mathbf{A}_{\Sigma'}^N(\mathbf{0})|^2 \text{ and } \sum_{\alpha=1}^{N}\left(\sum_{i=1}^{n}A_{\alpha+K}^{ii}\right)^2 = |\mathbf{H}_{\Sigma'}^N(\mathbf{0})|^2.$$

It is clear that at $i(p) = \mathbf{0}$

$$|\mathbf{A}^{\mathbb{R}}_{\Sigma'}(\mathbf{0})|^2 = |\mathbf{A}^{g}_{\Sigma}(p)|^2 + |\mathbf{A}^{N}_{\Sigma'}(\mathbf{0})|^2 \text{ and } |\mathbf{H}^{\mathbb{R}}_{\Sigma'}(\mathbf{0})|^2 = |\mathbf{H}^{g}_{\Sigma}(p)|^2 + |\mathbf{H}^{N}_{\Sigma'}(\mathbf{0})|^2.$$

The Gauss equations imply

$$-R^{g}_{\Sigma}(p) = -R^{\mathbb{R}}_{\Sigma'}(\mathbf{0}) = |\mathbf{A}^{\mathbb{R}}_{\Sigma'}(\mathbf{0})|^2 - |\mathbf{H}^{\mathbb{R}}_{\Sigma'}(\mathbf{0})|^2.$$

Hence, the coefficient A can be expressed as

$$\begin{aligned} A &= \frac{1}{2(n+2)}\left(|\mathbf{A}^{\mathbb{R}}_{\Sigma}|^2 - \frac{2}{3}|\mathbf{A}^{N}_{\Sigma'}|^2 - \frac{1}{3}|\mathbf{H}^{N}_{\Sigma'}|^2 - \frac{1}{2}|\mathbf{A}^{g}_{\Sigma}|^2 - \frac{1}{4}|\mathbf{H}^{g}_{\Sigma}|^2\right) \\ &= \frac{1}{2(n+2)}\left(|\mathbf{A}^{g}_{\Sigma}|^2 + \frac{1}{3}|\mathbf{A}^{N}_{\Sigma'}|^2 - \frac{1}{3}|\mathbf{H}^{N}_{\Sigma'}|^2 - \frac{1}{2}|\mathbf{A}^{g}_{\Sigma}|^2 - \frac{1}{4}|\mathbf{H}^{g}_{\Sigma}|^2\right) \\ &= \frac{1}{2(n+2)}\left(\frac{1}{3}|\mathbf{A}^{\mathbb{R}}_{\Sigma'}|^2 - \frac{1}{3}|\mathbf{H}^{\mathbb{R}}_{\Sigma'}|^2 + \frac{1}{6}|\mathbf{A}^{g}_{\Sigma}|^2 + \frac{1}{12}|\mathbf{H}^{g}_{\Sigma}|^2\right) \\ &= \frac{1}{6(n+2)}\left(-R^{g}_{\Sigma} + \frac{1}{2}|\mathbf{A}^{g}_{\Sigma}|^2 + \frac{1}{4}|\mathbf{H}^{g}_{\Sigma}|^2\right). \end{aligned}$$

This concludes the proof. □

References

[1] L. Karp and M. Pinsky, "Volume of a small extrinsic ball in a submanifold," *Bull. Lond. Math. Soc.*, vol. 21, no. 1, pp. 87–92, 1989.
[2] J. Bernstein and A. Bhattacharya, "Colding–Minicozzi entropies in Cartan–Hadamard manifolds," Preprint, 2022.
[3] T. H. Colding and W. P. Minicozzi, II, "Generic mean curvature flow I; generic singularities," *Ann. Math.*, vol. 175, no. 2, pp. 755–833, 2012.
[4] E. B. Davies and N. Mandouvalos, "Heat kernel bounds on hyperbolic space and Kleinian groups," *Proc. Lond. Math. Soc.*, vol. s3-57, no. 1, pp. 182–208, 1988.
[5] J. Bernstein, "Colding Minicozzi entropy in hyperbolic space," *Nonlinear Anal.*, vol. 210, p. 112401, 2021.
[6] J. Bernstein and L. Wang, "A sharp lower bound for the entropy of closed hypersurfaces up to dimension six," *Invent. Math.*, vol. 206, no. 3, pp. 601–627, 2016.
[7] L. Chen, "Rigidity and stability of submanifolds with entropy close to one," *Geometriae Dedicata*, vol. 215, no. 1, pp. 133–145, 2021.
[8] J. Zhu, "On the entropy of closed hypersurfaces and singular self-shrinkers," *J. Differ. Geom.*, vol. 114, no. 3, pp. 551–593, 2020.
[9] C. Borell, "The Ehrhard inequality," *C. R. Math.*, vol. 337, no. 10, pp. 663–666, 2003.
[10] V. N. Sudakov and B. S. Tsirel'son, "Extremal properties of half-spaces for spherically invariant measures," *J. Sov. Math.*, vol. 9, no. 1, pp. 9–18, 1978.
[11] R. L. Bryant, "Surfaces in conformal geometry," in *Proceedings of Symposia in Pure Mathematics*, vol. 48, R. Wells, Ed., Providence, Rhode Island, American Mathematical Society, 1988, pp. 227–240.
[12] F. J. Carreras and A. M. Naveira, "On the volume of a small extrinsic ball in a hypersurface of the hyperbolic space," *Math. Scand.*, vol. 83, no. 2, p. 220, 1998.
[13] A. Gray and L. Vanhecke, "Riemannian geometry as determined by the volumes of small geodesic balls," *Acta Math.*, vol. 142, pp. 157–198, 1979.
[14] A. Gray, "The volume of a small geodesic ball of a Riemannian manifold," *Mich. Math. J.*, vol. 20, no. 4, pp. 329–344, 1974.