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ORIGINAL PAPER

Minimal surfaces and Colding-Minicozzi entropy in complex hyperbolic space

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Abstract
We study notions of asymptotic regularity for a class of minimal submanifolds of complex hyperbolic space that includes minimal Lagrangian submanifolds. As an application, we show a relationship between an appropriate formulation of Colding-Minicozzi entropy and a quantity we call the CR-volume that is computed from the asymptotic geometry of such submanifolds.

Keywords Complex hyperbolic · Minimal surfaces · Lagrangian

Mathematics Subject Classification 53A10 · 53C42 · 53D12 · 32Q45 · 53E10

1 Introduction

In [3], the authors generalized the Colding-Minicozzi entropy [11] to submanifolds of Cartan-Hadamard manifolds. In this article, we study the specific case where the ambient manifold is complex hyperbolic space, $\mathbb{CH}^{n+1}$. Recall, this is the $(2n+2)$-dimensional complete simply-connected Kähler-Einstein manifold with metric $g_{\mathbb{CH}}$ which has constant negative holomorphic curvature; we adopt the convention that the sectional curvatures lie in $[-4, -1]$. This leads to a relationship between the entropy of certain minimal submanifolds (e.g., minimal Lagrangian submanifolds) and a quantity we call the CR-volume, which is associated to their asymptotic geometry. This quantity is an analog, in the context of CR-geometry, of the conformal volume of Li and Yau [24]; it is also related to the visual volume of Gromov [16]—see Definition 2.4 in Sects. 2.4 and 4.

Our study requires appropriate compactifications of $\mathbb{CH}^{n+1}$; we describe two choices in Sect. 3. Both lead to well-defined and equivalent notions of *ideal boundary*, $\partial_\infty \mathbb{CH}^{n+1}$.

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Moreover, $\partial_\infty \mathbb{CH}^{n+1}$ comes equipped with a corresponding CR-structure modeled on $\mathbb{S}^{2n+1}$ viewed as the boundary of the unit complex ball $\mathbb{B}^{n+1}_{\mathbb{C}}$. Let $\Sigma \subset \mathbb{CH}^{n+1}$ be a smooth proper minimal submanifold. Such a Σ is necessarily non-compact, and therefore a natural assumption is that its asymptotic geometry can be modeled, via the compactifications of $\mathbb{CH}^{n+1}$, on a submanifold, $\partial_\infty \Sigma \subset \partial_\infty \mathbb{CH}^{n+1}$—see Sect. 3.4 for a detailed discussion. One may then impose additional conditions on the asymptotic regularity and geometry of Σ. For instance, Σ is *asymptotically horizontal* if $\partial_\infty \Sigma$ is a horizontal submanifold of $\partial_\infty \mathbb{CH}^{n+1}$ and *asymptotically Legendrian* when $\partial_\infty \Sigma$ is Legendrian—i.e., horizontal and of maximal dimension. Indeed, a Lagrangian submanifold of sufficient asymptotic regularity is asymptotically Legendrian—see Lemma 3.2. We refer to Definitions 3.1 and 3.3 in Sect. 3.1 for specifics.

Our first result is that relatively weak assumptions on the asymptotic regularity of certain minimal submanifolds imply stronger asymptotic regularity.

Theorem 1.1 *Suppose that $\Sigma \subset \mathbb{CH}^{n+1}$ is an m-dimensional minimal submanifold. If Σ is weakly C^2-asymptotically regular and weakly asymptotically horizontal, then it is C^1-asymptotically regular and strongly horizontal.*

Using this, we obtain a relationship between the generalized Colding-Minicozzi entropy of these submanifolds and the CR-volume of their asymptotic boundaries.

Definition 1.2 (*κ-entropy* [3]) Suppose that Σ is an m-dimensional submanifold of the $(m+k)$-dimensional Cartan-Hadamard manifold (M, g). Let

$$\Phi^{t_0,x_0}_{m,\kappa}(t,x) = K_{m,\kappa}(t_0 - t, \mathrm{dist}_g(x, x_0))$$

where $K_{m,\kappa}$ are functions defined in [3, pg 2] following [5, 11] and come from the heat kernel of $\mathbb{H}^m$. The Colding-Minicozzi κ-entropy of Σ in (M, g) is given by

$$\lambda^\kappa_g[\Sigma] = \sup_{x_0 \in M, \tau > 0} \int_\Sigma \Phi^{0,x_0}_{m,\kappa}(-\tau, x) dVol_\Sigma(x) = \sup_{x_0 \in M, \tau > 0} \int_\Sigma \Phi^{\tau,x_0}_{m,\kappa}(0, x) dVol_\Sigma(x).$$

When $\kappa = 0$ and $(M, g) = (\mathbb{R}^{m+k}, g_\mathbb{R})$ is Euclidean space, this is the usual Colding-Minicozzi entropy, $\lambda[\Sigma]$, of Σ. When $\kappa = 1$ and $(M, g) = (\mathbb{H}^{m+k}, g_\mathbb{H})$ is hyperbolic space, this is the entropy in hyperbolic space, $\lambda_\mathbb{H}[\Sigma]$, introduced in [5]. By [3], λ^κ_g is monotone non-increasing along reasonable mean curvature flows in a Cartan-Hadamard manifold with sectional curvatures bounded above by $-\kappa^2$. As $\mathbb{CH}^{n+1}$ has sectional curvatures in $[-4, -1]$, we make the following definition.

Definition 1.3 *(Complex hyperbolic entropy)* The Colding-Minicozzi entropy of an m-dimensional submanifold $\Sigma \subset \mathbb{CH}^{n+1}$ is given by

$$\lambda_{\mathbb{CH}}[\Sigma] = \lambda^1_{g_{\mathbb{CH}}}[\Sigma] = \sup_{x_0 \in \mathbb{CH}^{n+1}, \tau > 0} \int_\Sigma \Phi^{\tau,x_0}_{m,1}(0, x) dV_\Sigma(x)$$

where $\lambda^1_{g_{\mathbb{CH}}}$ is the κ-entropy on $\mathbb{CH}^{n+1}$ corresponding to an upper bound on sectional curvatures of -1.

Theorem 1.4 *If $\Sigma \subset \mathbb{CH}^{n+1}$ is an m-dimensional submanifold that is weakly C^1-asymptotically regular and weakly asymptotically horizontal, then*

$$|\mathbb{S}^{m-1}|_\mathbb{R} \lambda_{\mathbb{CH}}[\Sigma] \geq \lambda_{CR}[\partial_\infty \Sigma].$$

If Σ is also weakly C^2-asymptotically regular and minimal, then equality holds.

Remark 1.1 In [5, Theorem 1.5], an analogous result for appropriate submanifolds of hyperbolic space was obtained: If Σ is an m-dimensional submanifold of $\mathbb{H}^{m+k}$ that is regular up to the ideal boundary, then there is an inequality relating, $\lambda_c[\partial_\infty \Sigma]$, the conformal volume of the ideal boundary of Σ and $\lambda_{\mathbb{H}}[\Sigma]$, the Colding-Minicozzi entropy of Σ in hyperbolic space. This was applied in [29, 30] to show a type of topological uniqueness for certain minimal hypersurfaces in $\mathbb{H}^{n+1}$.

We conclude by discussing the applicability of Theorem 1.4. The most basic examples are the totally geodesic Lagrangian submanifolds, $\Sigma = \Sigma_{p,L}$, that, for any $p \in \mathbb{CH}^{n+1}$ and Lagrangian subspace $L \subset T_p\mathbb{CH}^{n+1}$, are uniquely determined by $L = T_p\Sigma$. These Σ are C^∞-asymptotically regular, strongly horizontal with $\partial_\infty \Sigma$ corresponding, modulo a CR-automorphism, to a totally geodesic Legendrian sphere $\mathbb{S}^n \subset \mathbb{S}^{2n+1}$. They also satisfy $\lambda_{\mathbb{CH}}[\Sigma] = 1$.

The fact that each $\Sigma_{p,L}$ is intrinsically, $\mathbb{H}^{n+1}$, the hyperbolic space with curvature -1, yields more examples. Indeed, work of Anderson [1, 2], Hardt-Lin [20], Lin [22], and Tonegawa [28], give many minimal hypersurfaces in $\mathbb{H}^{n+1}$ with good asymptotic regularity. This is done by solving an asymptotic Plateau problem for any C^2-regular hypersurface in $\partial_\infty \mathbb{H}^{n+1}$; note that, as observed in [28], higher asymptotic regularity is a subtle issue. See the survey of Coskunuzer [13] for a thorough overview. Embedding these into $\Sigma_{p,L}$ yields weakly C^2-asymptotically regular and asymptotically horizontal n-dimensional minimal submanifolds in $\mathbb{CH}^{n+1}$.

In another direction, in [12, Theorem 1], the authors describe a family of rotationally symmetric minimal Lagrangian submanifolds of $\mathbb{CH}^{n+1}$ that are topologically $\mathbb{R} \times \mathbb{S}^n$. One verifies that, when n is even, they are C^∞-asymptotically regular while, when n is odd, they are C^{n+1}-asymptotically regular but not C^{n+2}. In all dimensions, these submanifolds are weakly C^∞-asymptotically regular and strongly horizontal. Moreover, their ideal boundary corresponds to a pair of totally geodesic Legendrian spheres $\mathbb{S}^n$ contained in $\mathbb{S}^{2n+1}$. There seem to be few other constructions of smooth minimal Lagrangian submanifolds in $\mathbb{CH}^{n+1}$—see however [23]. If one allows (interior) singularities then, by taking a cone, any minimal Legendrian in $\mathbb{S}^{2n+1}$ gives rise to a (singular) minimal Lagrangian submanifold of $\mathbb{CH}^{n+1}$—see Corollary 3.7. There are many examples of such Γ in $\mathbb{S}^{2n+1}$—for instance, the so-called Clifford tori [17, Ex. 3.16]—see also [19]. In $\mathbb{S}^5$ there is a particularly rich collection of examples, including those of higher genus—see [10, 18, 25]. Note that Theorem 1.4 holds for minimal submanifolds with interior singularities.

Finally, for any submanifold $\Gamma \subset \partial_\infty \mathbb{CH}^{n+1}$, and in particular any horizontal submanifold, it follows from [6, Theorem 4.3] that there is a (singular) area-minimizing current asymptotic to Γ in a certain, weak, sense. To the authors' knowledge, additional asymptotic regularity has not been established in $\mathbb{CH}^{n+1}$ for the solutions of [6], especially for those of high codimension-cf. [21], which shows asymptotic regularity for high-codimension minimizers in $\mathbb{H}^{n+1}$. It would be interesting to produce solutions with sufficient asymptotic regularity in this fashion.

2 Geometric background

We introduce needed geometric background and computations. In particular, we recall the contact, Sasaki, and CR structures that arise by viewing $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$ as the boundary of the unit ball $\mathbb{B}^{2n+2} \simeq \mathbb{B}^{n+1}_{\mathbb{C}}$. Related structures exist on the ideal boundary of complex hyperbolic space. Our main sources are [7, 27].

2.1 Basic constructs on Euclidean space

Let us first introduce notation for various structures on $\mathbb{R}^{2n+2} \simeq \mathbb{R}^{n+1}_{\mathbf{x}} \times \mathbb{R}^{n+1}_{\mathbf{y}} \simeq \mathbb{C}^{n+1}_{\mathbf{z}}$. Here we make the identification of $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with $\mathbb{C}^{n+1}$ via $\mathbf{z} = \mathbf{x} + i\mathbf{y}$. In particular, the Euclidean coordinates given by $x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}$ correspond to the holomorphic coordinates $z_1 = x_1 + iy_1, \dots, z_{n+1} = x_{n+1} + iy_{n+1}$.

Denote the usual Euclidean Riemannian metric and symplectic form by

$$g_{\mathbb{R}} = \sum_{j=1}^{n+1} \left(dx_j^2 + dy_j^2\right) \text{ and } \omega_{\mathbb{R}} = \sum_{j=1}^{n+1} dx_j \wedge dy_j.$$

Let $J_{\mathbb{R}}$ be the associated almost complex structure on $\mathbb{R}^{2n+2}$ defined by

$$\omega_{\mathbb{R}}(X, Y) = g_{\mathbb{R}}(X, J_{\mathbb{R}}(Y)), \text{ i.e., } J_{\mathbb{R}}\left(\frac{\partial}{\partial x_j}\right) = -\frac{\partial}{\partial y_j} \text{ and } J_{\mathbb{R}}\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}.$$

In this convention the complexification of $J_{\mathbb{R}}$ satisfies $J_{\mathbb{R}}\left(\frac{\partial}{\partial z_j}\right) = -i\frac{\partial}{\partial z_j}$. Likewise,

$$dx_j \circ J_{\mathbb{R}} = dy_j \text{ and } dy_j \circ J_{\mathbb{R}} = -dx_j.$$

Let $r : \mathbb{R}^{2n+2} \to \mathbb{R}$ be the radial function defined by

$$r^2 = x_1^2 + y_1^2 + \cdots + x_{n+1}^2 + y_{n+1}^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 = |\mathbf{z}|^2.$$

One readily computes that

$$dr = r^{-1}\left(x_1 dx_1 + y_1 dy_1 + \cdots + x_{n+1} dx_{n+1} + y_{n+1} dy_{n+1}\right)$$

is a smooth one-form on $\mathbb{R}^{2n+2} \setminus \{0\}$. Set

$$\theta = r^{-1} dr \circ J_{\mathbb{R}} = r^{-2}\left(x_1 dy_1 - y_1 dx_1 + \cdots + x_{n+1} dy_{n+1} - y_{n+1} dy_{n+1}\right).$$

Define a symmetric (0, 2) tensor field on $\mathbb{R}^{2n+2} \setminus \{0\}$ by,

$$\eta = \frac{1}{r^2} \sum_{j,k=1, j\neq k}^{n+1} \left((x_j x_k + y_j y_k)(dx_j dx_k + dy_j dy_k) + 2x_j y_k (dx_j dy_k - dy_j dx_k)\right).$$

One readily computes that

$$g_{\mathbb{R}} = dr^2 + r^2\left(\theta^2 + \eta\right) \text{ and } \omega_{\mathbb{R}} = \frac{1}{2} r^2 d\theta + r dr \wedge \theta.$$

Hence, for $X, Y \in T_p\mathbb{R}^{2n+2}$,

$$-\frac{1}{2} d\theta(X, J_{\mathbb{R}}(Y)) = -\frac{1}{r^2}\omega_{\mathbb{R}}(X, J_{\mathbb{R}}(Y)) + \frac{1}{r}(dr \wedge \theta)(X, J_{\mathbb{R}}(Y)) = \eta(X, Y). \quad (2.1)$$

It is convenient to introduce vector fields

$$\mathbf{X} = \sum_{j=1}^{n+1}\left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right) \text{ and } \mathbf{T} = \sum_{j=1}^{n+1}\left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\right) = -J_{\mathbb{R}}(\mathbf{X})$$

where $\mathbf{X}$ is the position vector and $\mathbf{T}$ is a Killing vector field. They satisfy

$$dr(X) = \frac{1}{r} g_{\mathbb{R}}(X, \mathbf{X}) \text{ and } \theta(X) = \frac{1}{r^2} g_{\mathbb{R}}(X, \mathbf{T}).$$

One readily checks that if X is any vector field on $\mathbb{R}^{2n+2}$ and $\nabla^{\mathbb{R}}$ is the Levi-Civita connection of $g_{\mathbb{R}}$, then, because $J_{\mathbb{R}}$ is $\nabla^{\mathbb{R}}$ parallel, one has

$$\nabla^{\mathbb{R}}_X \mathbf{X} = X \text{ and } \nabla^{\mathbb{R}}_X \mathbf{T} = \nabla^{\mathbb{R}}_X (-J_{\mathbb{R}}(\mathbf{X})) = -J_{\mathbb{R}}(\nabla^{\mathbb{R}}_X \mathbf{X}) = -J_{\mathbb{R}}(X).$$

2.2 Contact, Sasaki and CR geometry of $\mathbb{S}^{2n+1}$

By thinking of $\mathbb{S}^{2n+1}$ as the boundary of the ball $\mathbb{B}^{2n+2}$ we may endow it with a natural Sasaki structure. This comes along with associated CR and contact structures. In order to understand the boundary behavior of complex hyperbolic space it will be helpful to understand the interaction of these structures as well as their symmetries.

To that end, let $\hat{\theta}$ be the pullback of θ to $\mathbb{S}^{2n+1}$. This is readily seen to be a contact form on $\mathbb{S}^{2n+1}$ with Reeb vector field $\hat{\mathbf{T}}$, the restriction of the tangential vector field $\mathbf{T}$. Denote by $\mathcal{H} \subset T\mathbb{S}^{2n+1}$ the contact distribution associated to $\hat{\theta}$, that is, the vector bundle over $\mathbb{S}^{2n+1}$ satisfying, for each $p \in \mathbb{S}^{2n+1}$,

$$\mathcal{H}_p = \ker \hat{\theta}_p = \left\{ X \in T_p\mathbb{S}^{2n+1} : \hat{\theta}_p(X) = 0 \right\} \subset T_p\mathbb{S}^{2n+1}. \tag{2.2}$$

Let $g_{\mathbb{S}}$ denote the round metric on $\mathbb{S}^{2n+1}$ induced from $g_{\mathbb{R}}$. It is clear that $\mathcal{H}$ is $g_{\mathbb{S}}$ orthogonal to $\hat{\mathbf{T}}$. It follows that $J_{\mathcal{H}} = J_{\mathbb{R}}|_{\mathcal{H}}$ is a bundle automorphism of $\mathcal{H}$. We may extend this to a bundle map $J_{\mathbb{S}} : T\mathbb{S}^{2n+1} \to T\mathbb{S}^{2n+1}$ by setting $J_{\mathbb{S}}(\hat{\mathbf{T}}) = \mathbf{0}$.

The triple $(J_{\mathbb{S}}, \hat{\mathbf{T}}, \hat{\theta})$ is, in the language of [7], an *almost contact structure* on $\mathbb{S}^{2n+1}$. In fact, together with the metric $g_{\mathbb{S}}$ this is an *almost contact metric structure* and this almost contact metric structure is also *Sasakian*. Indeed, by [7, Theorem 6.3], if $\nabla^{\mathbb{S}}$ is the Levi-Civita connection of $g_{\mathbb{S}}$, then it suffices to check

$$(\nabla^{\mathbb{S}}_X J_{\mathbb{S}})Y = g_{\mathbb{S}}(X, Y)\hat{\mathbf{T}} - \hat{\theta}(Y)X,$$

holds for $X, Y \in T_p\mathbb{S}^{n+1}$. This follows from $\nabla^{\mathbb{R}}_X J_{\mathbb{R}} = 0$. Hence, for $X \in T_p\mathbb{S}^{2n+1}$,

$$\nabla^{\mathbb{S}}_X \hat{\mathbf{T}} = -J_{\mathbb{S}}(X).$$

In a similar vein, by using the identification $\mathbb{S}^{2n+1} \simeq \partial\mathbb{B}^{n+1}_{\mathbb{C}}$ where

$$\mathbb{B}^{n+1}_{\mathbb{C}} = \left\{ (z_1, \ldots, z_{n+1}) : |z_1|^2 + \ldots + |z_{n+1}|^2 < 1 \right\} \subset \mathbb{C}^{n+1},$$

is the unit complex ball, one may interpret $(\mathcal{H}, J_{\mathbb{S}})$ as a CR-structure. In this case, $\hat{\theta}$ is a pseudo-convex pseudo-hermitian form, as the *Levi form*, $L_{\hat{\theta}}$, is a positive definite inner product on $\mathcal{H}$—see [14]. Indeed, for $X, Y \in \mathcal{H}_p$, it is given[1] by

$$L_{\hat{\theta}}(X, Y) = -\frac{1}{2}(d\hat{\theta})(X, J_{\mathbb{S}}(Y)) = -\frac{1}{2}d\theta(X, J_{\mathbb{R}}(Y)) = \eta(X, Y)$$

where we used that X and Y are in the kernel of dr and θ. Hence, $L_{\hat{\theta}} = \hat{\eta}$ where $\hat{\eta}$ is the pullback of η to $\mathbb{S}^{2n+1}$. Moreover, the *Webster metric* associated to this data on $\mathbb{S}^{2n+1}$ recovers the standard metric, i.e.,

$$\hat{\theta}^2 + L_{\hat{\theta}} = g_{\mathbb{S}}.$$

1 we use the convention that $d\beta(X, Y) = X(\beta(Y)) - Y(\beta(X)) - \beta([X, Y])$, though $d\beta(X, Y) = \frac{1}{2}(X(\beta(Y)) - Y(\beta(X)) - \beta([X, Y]))$ is also common in the literature.

Given a C^1 function defined on $\mathbb{S}^{2n+1}$ we denote

$$\nabla^{\mathcal{H}} f = \nabla^{\mathbb{S}} f - g_{\mathbb{S}}(\nabla^{\mathbb{S}} f, \hat{\mathbf{T}})\hat{\mathbf{T}}$$

where $\mathcal{H}$ is the tangential component of the gradient.

2.3 Complex automorphisms of unit ball in $\mathbb{C}^{n+1}$ and CR-automorphisms of $\mathbb{S}^{2n+1}$

We denote by $Aut_{\mathbb{C}}(\mathbb{B}^{n+1}_{\mathbb{C}})$ the set of biholomorphic automorphisms of the unit disk. We refer to [26] and [15] for properties of these automorphisms, but summarize some of the needed facts.

First of all, we observe that for any $A \in \mathbf{U}(n+1)$, a unitary matrix, the map

$$\Phi_A : \mathbf{z} \mapsto A \cdot \mathbf{z}$$

is an element of $Aut_{\mathbb{C}}(\mathbb{B}^{n+1}_{\mathbb{C}})$. Secondly, for any fixed $\mathbf{b} \in \mathbb{B}^{n+1}_{\mathbb{C}}$ there is an element $\Phi_{\mathbf{b}} \in Aut_{\mathbb{C}}(\mathbb{B}^{n+1}_{\mathbb{C}})$ given by

$$\Phi_{\mathbf{b}} : \mathbf{z} \mapsto \sqrt{1-|\mathbf{b}|^2}\frac{\mathbf{z}}{1+\bar{\mathbf{b}}\cdot\mathbf{z}} + \frac{1}{1+\sqrt{1-|\mathbf{b}|^2}}\left(1+\frac{\sqrt{1-|\mathbf{b}|^2}}{1+\bar{\mathbf{b}}\cdot\mathbf{z}}\right)\mathbf{b}.$$

This map can also be expressed as

$$\Phi_{\mathbf{b}}(\mathbf{z}) = \sqrt{1-|\mathbf{b}|^2}\frac{\mathbf{z}+\mathbf{b}}{1+\bar{\mathbf{b}}\cdot\mathbf{z}} + \frac{1}{1+\sqrt{1-|\mathbf{b}|^2}}\frac{|\mathbf{b}|^2+\bar{\mathbf{b}}\cdot\mathbf{z}}{1+\bar{\mathbf{b}}\cdot\mathbf{z}}\mathbf{b}.$$

Using the identification $\mathbb{B}^{n+1}_{\mathbb{C}} \simeq \mathbb{B}^{2n+2}$ we may think of $\Phi_{\mathbf{b}}$ as a $J_{\mathbb{R}}$-biholomorphism of $\mathbb{B}^{2n+2}$, i.e., satisfying

$$J_{\mathbb{R}} \circ D_p\Phi = D_p\Phi \circ J_{\mathbb{R}},$$

and identify $Aut_{\mathbb{C}}(\mathbb{B}^{n+1}_{\mathbb{C}})$ with $Aut_J(\mathbb{B}^{2n+2})$, the $J_{\mathbb{R}}$-biholomorphisms of the ball.

Proposition 2.1 *For $\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2 \in \mathbb{B}^{n+1}_{\mathbb{C}}$, the map $\Phi_{\mathbf{b}}$ satisfies:*

(1) There is a unique extension to a $J_{\mathbb{R}}$-biholomorpism $\bar{\Phi}_{\mathbf{b}} : \bar{\mathbb{B}}^{2n+2} \to \bar{\mathbb{B}}^{2n+2}$.
(2) On $\partial\mathbb{B}^{2n+2}$,

$$\bar{\Phi}^*_{\mathbf{b}} dr = W_{\mathbf{b}} dr \text{ and } \bar{\Phi}^*_{\mathbf{b}}\theta = W_{\mathbf{b}}\theta$$

where $W_{\mathbf{b}} \in C^\infty(\bar{\mathbb{B}}^{2n+2})$ is given by

$$W_{\mathbf{b}}(\mathbf{x},\mathbf{y}) = \frac{1-|\mathbf{b}_1|^2-|\mathbf{b}_2|^2}{(1+\mathbf{b}_1\cdot\mathbf{x}+\mathbf{b}_2\cdot\mathbf{y})^2+(\mathbf{b}_2\cdot\mathbf{x}-\mathbf{b}_1\cdot\mathbf{y})^2}.$$

(3) On $\partial\mathbb{B}^{2n+2}$,

$$\bar{\Phi}^*_{\mathbf{b}} d\theta = W_{\mathbf{b}} d\theta + 2(W_{\mathbf{b}} - W^2_{\mathbf{b}}) r dr \wedge \theta + dW_{\mathbf{b}} \wedge \theta + dr \wedge (dW_{\mathbf{b}} \circ J_{\mathbb{R}}).$$

Proof The existence of a smooth extension of $\Phi_{\mathbf{b}}$ follows from its formula and it is clear that the extended map is $J_{\mathbb{R}}$-holomorphic and smoothly invertible. A straightforward computation gives

$$|\bar{\Phi}_{\mathbf{b}}(\mathbf{z})|^2 = 1 - \frac{(1-|\mathbf{b}|^2)(1-|\mathbf{z}|^2)}{|1+\bar{\mathbf{b}}\cdot\mathbf{z}|^2}. \tag{2.3}$$

With $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, it is convenient to write

$$W_{\mathbf{b}}(\mathbf{z}) = (1 - |\mathbf{b}|^2)|1 + \bar{\mathbf{b}} \cdot \mathbf{z}|^{-2} = W_{\mathbf{b}}(\mathbf{x}, \mathbf{y}).$$

By inspection, $W_{\mathbf{b}}$ is a smooth and non-zero function on $\bar{\mathbb{B}}^{2n+2}$. We write (2.3) as

$$r^2 \circ \bar{\Phi}_{\mathbf{b}} = 1 - (1 - r^2) W_{\mathbf{b}}.$$

Hence,

$$\begin{aligned}\left(2r \circ \bar{\Phi}_{\mathbf{b}}\right) \bar{\Phi}_{\mathbf{b}}^* dr &= \bar{\Phi}_{\mathbf{b}}^*(2rdr) = \bar{\Phi}_{\mathbf{b}}^* d(r^2) = d(r^2 \circ \bar{\Phi}_{\mathbf{b}}) \\ &= d(1 - (1 - r^2) W_{\mathbf{b}}) = -(1 - r^2) dW_{\mathbf{b}} + 2W_{\mathbf{b}} r dr.\end{aligned}$$

Combining this with the $J_{\mathbb{R}}$-holomorphicity of $\bar{\Phi}_{\mathbf{b}}$ yields

$$\bar{\Phi}_{\mathbf{b}}^*(r^2\theta) = \bar{\Phi}_{\mathbf{b}}^* \left(rdr \circ J_{\mathbb{R}}\right) = (r \circ \bar{\Phi}_{\mathbf{b}})(\bar{\Phi}_{\mathbf{b}}^* dr) \circ J_{\mathbb{R}} = W_{\mathbf{b}} r^2 \theta - \frac{1 - r^2}{2} dW_{\mathbf{b}} \circ J_{\mathbb{R}}.$$

Combining the above computations yields the second claim. We further compute,

$$\begin{aligned}\bar{\Phi}_{\mathbf{b}}^* r^2 d\theta &= \bar{\Phi}_{\mathbf{b}}^* \left(d(r^2\theta) - 2rdr \wedge \theta\right) = d(\bar{\Phi}_{\mathbf{b}}^* r^2 \theta) - 2\bar{\Phi}_{\mathbf{b}}^* dr \wedge \bar{\Phi}_{\mathbf{b}}^*(r\theta) \\ &= dW_{\mathbf{b}} \wedge r^2\theta + W_{\mathbf{b}} d(r^2\theta) + rdr \wedge dW_{\mathbf{b}} \circ J_{\mathbb{R}} - \frac{1 - r^2}{2} d(dW_{\mathbf{b}} \circ J_{\mathbb{R}}) \\ &\quad - 2W_{\mathbf{b}}^2 \frac{r^2}{r^2 \circ \bar{\Phi}_{\mathbf{b}}} rdr \wedge \theta + \frac{r(1 - r^2)}{r^2 \circ \bar{\Phi}_{\mathbf{b}}} W_{\mathbf{b}} \left(dW_{\mathbf{b}} \wedge r\theta + dr \wedge dW_{\mathbf{b}} \circ J_{\mathbb{R}}\right) \\ &\quad - \frac{(1 - r^2)^2}{2r^2 \circ \bar{\Phi}_{\mathbf{b}}} dW_{\mathbf{b}} \wedge dW_{\mathbf{b}} \circ J_{\mathbb{R}}.\end{aligned}$$

On $\partial\mathbb{B}^{2n+2}$, this simplifies to

$$\bar{\Phi}_{\mathbf{b}}^* d\theta = W_{\mathbf{b}} d\theta + 2(W_{\mathbf{b}} - W_{\mathbf{b}}^2) dr \wedge \theta + dW_{\mathbf{b}} \wedge \theta + dr \wedge (dW_{\mathbf{b}} \circ J_{\mathbb{R}}),$$

which verifies the third claim.

□

Every element $\Phi \in Aut_{\mathbb{C}}(\mathbb{B}_{\mathbb{C}}^{n+1})$ satisfies

$$\Phi = \Phi_A \circ \Phi_{\mathbf{b}} \tag{2.4}$$

for some $A \in \mathbf{U}(n+1)$ and $\mathbf{b} \in \mathbb{B}_{\mathbb{C}}^{n+1}$. It follows from (2.4) and Proposition 2.1 that every element $\Phi \in Aut_J(\mathbb{B}^{2n+2})$ extends smoothly to a map $\bar{\Phi} : \bar{\mathbb{B}}^{2n+2} \to \bar{\mathbb{B}}^{2n+2}$. We write $Aut_J(\bar{\mathbb{B}}^{2n+2})$ for the group of extended maps. The maps $\bar{\Phi} \in Aut_J(\bar{\mathbb{B}}^{2n+2})$ have the additional property that $\Psi = \bar{\Phi}|_{\mathbb{S}^{2n+1}}$ is a diffeomorphism of $\mathbb{S}^{2n+1}$ to itself, which is a *CR-automorphism* of $\mathbb{S}^{2n+1}$, that is,

$$D_p\Psi(\mathcal{H}_p) = \mathcal{H}_{\Psi(p)} \text{ and on } \mathcal{H}_p, J_{\mathbb{S}} \circ D_p\Psi = D_p\Psi \circ J_{\mathbb{S}}.$$

Let us denote the set of such maps by $Aut_{CR}(\mathbb{S}^{2n+1})$. For $\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2 \in \mathbb{B}_{\mathbb{C}}^{n+1}$, let $\Psi_{\mathbf{b}} \in Aut_{CR}(\mathbb{S}^{2n+1})$ be, the restriction of $\bar{\Phi}_{\mathbf{b}} \in Aut_J(\mathbb{B}^{2n+2})$. For $A \in \mathbf{U}(n+1)$ define $\Psi_A \in Aut_{CR}(\mathbb{S}^{2n+1})$ in the same manner.

Proposition 2.2 *The following properties of $Aut_{CR}(\mathbb{S}^{2n+1})$ hold:*

(1) Every element of $Aut_{CR}(\mathbb{S}^{2n+1})$ is the restriction of a unique element of $Aut_J(\bar{\mathbb{B}}^{2n+2})$ and in particular, for every $\Psi \in Aut_{CR}(\mathbb{S}^{2n+1})$ there is an $A \in \mathbf{U}(n+1)$ and $\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2$ such that $\Psi = \Psi_A \circ \Psi_{\mathbf{b}}$.

(2) The elements $\Psi \in Aut_{CR}(\mathbb{S}^{2n+1})$ are contactomorphisms, in fact for $\Psi = \Psi_A \circ \Psi_{\mathbf{b}}$ one has $\Psi^* \hat{\theta} = W_{\mathbf{b}} \hat{\theta}$.

(3) An element $\Psi = \Psi_A \circ \Psi_{\mathbf{b}} \in Aut_{CR}(\mathbb{S}^{2n+1})$ acts on the metric $g_{\mathbb{S}}$ by

$$\begin{aligned}\Psi^* g_{\mathbb{S}} &= W_{\mathbf{b}} \left(g_{\mathbb{S}} + \hat{\omega}_{\mathbf{b}} \cdot \hat{\theta} + \hat{\theta} \cdot \hat{\omega}_{\mathbf{b}} + \partial_r \log W_{\mathbf{b}} \hat{\theta}^2 \right) \\ &= W_{\mathbf{b}} \left(\hat{\eta} + \hat{\omega}_{\mathbf{b}} \cdot \hat{\theta} + \hat{\theta} \cdot \hat{\omega}_{\mathbf{b}} + \left(W_{\mathbf{b}} + \frac{1}{4} |\nabla^{\mathcal{H}} \log W_{\mathbf{b}}|^2_{\mathbb{S}} \right) \hat{\theta}^2 \right)\end{aligned}$$

where $\hat{\omega}_{\mathbf{b}}$ is a one form on $\mathbb{S}^{2n+1}$ given by

$$\hat{\omega}_{\mathbf{b}} = \frac{1}{2} d \log W_{\mathbf{b}} \circ J_{\mathbb{S}} = -\frac{1}{2} g_{\mathbb{S}}(J_{\mathbb{S}}(\nabla^{\mathcal{H}} W_{\mathbf{b}}), \cdot).$$

Remark 2.3 The deformed metrics $g^{\mathbf{b}}_{\mathbb{S}} = \Psi^* g_{\mathbb{S}}$ are special cases of more general deformations of contact Riemannian manifolds studied by Tanno in [27, Section 9] where they were called *gauge transformations of contact Riemannian structures*.

Proof The first claim is a standard fact—see [8, Lemma 1.1] and [9]. The second follows from Proposition 2.1 and the fact that $\Psi_A^* \hat{\theta} = \hat{\theta}$.

By (2.1), on $\mathbb{S}^{2n+1} = \partial \mathbb{B}^{2n+2}$, one has

$$\begin{aligned}\Psi_{\mathbf{b}}^* \eta = W_{\mathbf{b}} \eta + (W_{\mathbf{b}} - W_{\mathbf{b}}^2)(dr^2 + \theta^2) + \frac{1}{2} d W_{\mathbf{b}} \cdot dr \\ + \frac{1}{2} \theta \cdot (d W_{\mathbf{b}} \circ J_{\mathbb{R}}) + \frac{1}{2} dr \cdot d W_{\mathbf{b}} + \frac{1}{2} (d W_{\mathbf{b}} \circ J_{\mathbb{R}}) \cdot \theta.\end{aligned}$$

Hence, if $\hat{i} : \mathbb{S}^{2n+1} \to \mathbb{R}^{2n+1}$ is the usual inclusion, then $\hat{\eta} = \hat{i}^* \eta$ and

$$\begin{aligned}\bar{\Phi}_{\mathbf{b}}^* \hat{\eta} &= W_{\mathbf{b}} \hat{\eta} + (W_{\mathbf{b}} - W_{\mathbf{b}}^2) \hat{\theta}^2 + \frac{1}{2} \hat{\theta} \cdot \hat{i}^* (d W_{\mathbf{b}} \circ J_{\mathbb{R}}) + \frac{1}{2} \hat{i}^* (d W_{\mathbf{b}} \circ J_{\mathbb{R}}) \cdot \hat{\theta} \\ &= W_{\mathbf{b}} \hat{\eta} + (W_{\mathbf{b}} - W_{\mathbf{b}}^2 + \frac{\partial W_{\mathbf{b}}}{\partial r}) \hat{\theta}^2 + \frac{1}{2} \hat{\theta} \cdot d \hat{W}_{\mathbf{b}} \circ J_{\mathbb{S}}) + \frac{1}{2} (d \hat{W}_{\mathbf{b}} \circ J_{\mathbb{S}}) \cdot \hat{\theta}\end{aligned}$$

where $\hat{W}_{\mathbf{b}} = W_{\mathbf{b}} \circ \hat{i}$. As $g_{\mathbb{S}} = \hat{\theta}^2 + \hat{\eta}$, it follows that

$$\Psi_{\mathbf{b}}^* g_{\mathbb{S}} = W_{\mathbf{b}} \left(g_{\mathbb{S}} + \hat{\omega}_{\mathbf{b}} \cdot \hat{\theta} + \hat{\theta} \cdot \hat{\omega}_{\mathbf{b}} + \partial_r \log W_{\mathbf{b}} \hat{\theta}^2 \right).$$

A straightforward computation shows that the following identity holds on $\mathbb{S}^{2n+1}$

$$\partial_r W_{\mathbf{b}} = W_{\mathbf{b}}^2 - W_{\mathbf{b}} + \frac{1}{4} \frac{|\nabla^{\mathcal{H}} W_{\mathbf{b}}|^2}{W_{\mathbf{b}}}.$$

The final claim then follows from the above computations, the first claim, and the fact that Ψ_A is a $g_{\mathbb{S}}$-isometry. □

2.4 CR-volume

We introduce a notion of CR-volume for horizontal and Legendrian submanifolds of $\mathbb{S}^{2n+1}$—this functional is further studied in [4]. An m-dimensional submanifold, $\Gamma \subset \mathbb{S}^{2n+1}$, is *horizontal* if $T_p\Gamma \subset \mathcal{H}_p$ for all $p \in \Gamma$, where $\mathcal{H}_p$ is as defined in (2.2). Properties of contact manifolds ensure that $m \leq n$. When $m = n$, Γ is *Legendrian*.

Definition 2.4 The *CR-volume* of $\Gamma \subset \mathbb{S}^{2n+1}$, an m-dimensional horizontal submanifold, is:

$$\lambda_{CR}[\Gamma] = \sup_{\Psi \in Aut_{CR}(\mathbb{S}^{2n+1})} |\Psi(\Gamma)|_{\mathbb{S}}.$$

Observe that elements $\Psi \in Aut_{CR}(\mathbb{S}^{2n+1})$ can be factored as $\Psi = \Psi_A \circ \Psi_{\mathbf{b}}$ where $A \in \mathbf{U}(n+1)$ and $\mathbf{b} \in \mathbb{B}^{n+1}_{\mathbb{C}}$. Using the facts that Ψ_A is an isometry of $g_{\mathbb{S}}$, Γ is horizontal, along with Proposition 2.2 we obtain

$$\begin{aligned}\lambda_{CR}[\Gamma] &= \sup_{\mathbf{b}\in\mathbb{B}^{n+1}_{\mathbb{C}}} |\Psi_{\mathbf{b}}(\Gamma)|_{\mathbb{S}} = \sup_{\mathbf{b}\in\mathbb{B}^{n+1}_{\mathbb{C}}} \int_\Gamma W_{\mathbf{b}}^{\frac{m}{2}}(p) dV_\Gamma(p) \\ &= \sup_{\mathbf{b}\in\mathbb{B}^{n+1}_{\mathbb{C}}} \int_\Gamma \frac{(1-|\mathbf{b}|^2)^{\frac{m}{2}}}{|1+\bar{\mathbf{b}}\cdot \mathbf{z}(p)|^m} dV_\Gamma(p).\end{aligned}$$

Here we used that $g^{\mathbf{b}}_{\mathbb{S}}|_{\mathcal{H}} = W_{\mathbf{b}} g_{\mathbb{S}}|_{\mathcal{H}}$.

3 Complex hyperbolic space and its compactifications

In this section, we establish some properties of complex hyperbolic space, $\mathbb{CH}^{n+1}$, and its submanifolds. Recall, $\mathbb{CH}^{n+1}$ is the simply connected (complex) space form with constant holomorphic sectional curvature. We think of it as a $(2n+2)$-dimensional Kähler-Einstein manifold with Riemannian metric $g_{\mathbb{CH}}$ and the integrable (almost) complex structure $J_{\mathbb{CH}}$. We refer to [15] for background on this space, however, unlike in [15], we adopt the convention that the sectional curvatures of $g_{\mathbb{CH}}$ lie between -4 and -1.

3.1 Bergman metric and the Bergman compactification of submanifolds

A standard model of complex hyperbolic space is the *Bergman model*. Here the underlying manifold is $\mathbb{B}^{2n+2} \simeq \mathbb{B}^{n+1}_{\mathbb{C}}$ with metric

$$\begin{aligned} g_B &= \frac{1}{1-r^2}\sum_{j=1}^{n+1} dz_j d\bar{z}_j + \frac{1}{(1-r^2)^2}\sum_{j,k=1}^{n+1} z_j\bar{z}_k dz_k d\bar{z}_j \\ &= \frac{1}{(1-r^2)^2}dr^2 + \frac{r^2}{(1-r^2)^2}\theta^2 + \frac{r^2}{1-r^2}\eta. \end{aligned}$$

One of the advantages of this model is that in it the complex structure, J_B, satisfies $J_B = J_{\mathbb{R}}$. In particular, $Aut^+_{g_B}(\mathbb{B}^{2n+2})$, the orientation preserving isometries of g_B, is identified with $Aut_J(\mathbb{B}^{2n+2})$ and the corresponding symplectic form is

$$\omega_B = \frac{r^2}{1-r^2}\frac{1}{2}d\theta + \frac{1}{(1-r^2)^2} r dr \wedge \theta = \frac{1}{1-r^2}\omega_{\mathbb{R}} + \frac{r^2}{(1-r^2)^2} r dr\wedge\theta.$$

For any $p \in \mathbb{CH}^{n+1}$ there is a diffeomorphism $\Upsilon_p : \mathbb{CH}^{n+1} \to \mathbb{B}^{2n+2}$ satisfying $\Upsilon_p(p) = \mathbf{0}$ and $\Upsilon_p^* g_B = g_{\mathbb{CH}}$. Moreover, this map is a biholomorphism and is unique up to post-composition with an element of $\mathbf{U}(n+1)$. By making the identification with $\bar{\mathbb{B}}^{2n+2}$, this leads naturally to a compactification, $\overline{\mathbb{CH}}^{n+1}$, of complex hyperbolic space, which we call a *Bergman compactification*. Observe, in this case, the *ideal boundary* $\partial_\infty \mathbb{CH}^{n+1}$ is identified with $\mathbb{S}^{2n+1} = \partial \mathbb{B}^{2n+2}$. Different choices of p and the corresponding Υ_p give different compactifications, but they induce equivalent structures as manifolds with boundary. They also endow $\partial_\infty \mathbb{CH}^{n+1}$ with a canonical CR-structure, though only an equivalence class of Sasaki structures.

We use Bergman compactifications to define asymptotic properties of submanifolds $\Sigma \subset \mathbb{CH}^{n+1}$. To that end, for an m-dimensional submanifold $\Sigma \subset \mathbb{CH}^{n+1}$, a point $p \in \mathbb{CH}^{n+1}$, and a Bergman compactification $\Upsilon_p : \mathbb{CH}^{n+1} \to \mathbb{B}^{2n+2}$, let

$$\Sigma_p = \overline{\Upsilon_p(\Sigma)} \subset \bar{\mathbb{B}}^{2n+2}.$$

We call Σ_p a *Bergman compactification* of Σ.

Definition 3.1 Suppose that Σ and Σ_p are as above:

(1) For any $l \geq 1$, Σ is *C^l-asymptotically regular* if Σ_p is a C^l-regular submanifold with boundary and Σ_p meets $\partial\mathbb{B}^{2n+2}$ transversally;
(2) If, in addition, $\partial\Sigma_p$ is a horizontal submanifold relative to the usual contact structure on $\mathbb{S}^{2n+1} = \partial\mathbb{B}^{2n+2}$, i.e., for all $q \in \partial\Sigma_p$, $T_q\partial\Sigma_p \subset \mathcal{H}_q = \ker\hat{\theta}_q$, then Σ is *asymptotically horizontal*;
(3) If, in addition, for all $q \in \partial\Sigma_p$, $T_q\Sigma_p$ is orthogonal to $\mathbf{T}$, i.e., $T_q\Sigma_p \subset \ker\theta_q$, then Σ is *strongly asymptotically horizontal*.

Note (2) and (3) can only hold when $m \leq n+1$. When $m = n+1$ the term *horizontal* is replaced by *Legendrian*.

If $\Sigma_{p'}$ is another choice of a Bergman compactification of Σ, then $\Sigma_p = \bar{\Phi}(\Sigma_{p'})$ for some $\bar{\Phi} \in Aut_J(\bar{\mathbb{B}}^{2n+2})$. It follows from Propositions 2.1 and 2.2 that Definition 3.1 is independent of choices. In particular, we may think of a C^l-asymptotically regular submanifold Σ, as having a well defined C^l-regular *ideal boundary*, $\partial_\infty\Sigma \subset \partial_\infty\mathbb{CH}^{n+1}$. The submanifold Σ is asymptotically horizontal precisely when $\partial_\infty\Sigma$ satisfies this property with respect to the natural contact structure of $\partial_\infty\mathbb{CH}^{n+1}$.

The notion of being asymptomatically horizontal is natural as it automatically holds for isotropic submanifolds of sufficient asymptotic regularity.

Lemma 3.2 *Let $\Sigma \subset \mathbb{CH}^{n+1}$ be a C^1-asymptotically regular m-dimensional submanifold. If $m \geq 2$ and Σ is isotropic, then it is asymptotically horizontal.*

Proof Let Σ_p be a Bergman compactification of Σ. Clearly, Σ is isotropic if and only if Σ_p is isotropic. Hence, for $X, Y \in T_q\Sigma_p$, one has $\omega_B(X, Y) = g_B(X, J_B(Y)) = 0$. Let us denote by $\mathbf{T}^\top$, the tangential component, with respect to $g_\mathbb{R}$, of $\mathbf{T}$ along Σ_p. Using $J_\mathbb{R} = J_B$ we have,

$$g_B(\mathbf{T}^\top, J_\mathbb{R}(\mathbf{X}^\top)) = 0.$$

As Σ_p is C^1 up to the boundary and meets $\partial\mathbb{B}^{2n+2}$ transversally,

$$\mathbf{X}^\top = \alpha\mathbf{X} + \gamma\mathbf{T} + \mathbf{v}$$

where $\mathbf{v}$ is $g_\mathbb{R}$-orthogonal to $\mathbf{X}$ and $\mathbf{T}$. Likewise, one has

$$\mathbf{T}^\top = \beta\mathbf{T} + \delta\mathbf{X} + \mathbf{w}$$

where $\mathbf{w}$ is $g_\mathbb{R}$-orthogonal to $\mathbf{X}$ and $\mathbf{T}$. On the boundary, we have

$$\beta = |\mathbf{T}^\top|^2 \text{ and } \alpha = |\mathbf{X}^\top|^2 \neq 0$$

while

$$\gamma = g_{\mathbb{R}}(\mathbf{X}^\top, \mathbf{T}) = g_{\mathbb{R}}(\mathbf{X}^\top, \mathbf{T}^\top) = g_{\mathbb{R}}(\mathbf{X}, \mathbf{T}^\top) = \delta.$$

As $\mathbf{w}$ and $J_{\mathbb{R}}(\mathbf{v})$ are orthogonal to $\mathbf{T}$, one computes

$$\begin{aligned} 0 &= g_B(\mathbf{T}^\top, J_{\mathbb{R}}(\mathbf{X}^\top)) = g_B(\beta\mathbf{T} + \delta\mathbf{X} + \mathbf{w}, -\alpha\mathbf{T} + \gamma\mathbf{X} + J_{\mathbb{R}}(\mathbf{v})) \\ &= (-\alpha\beta + \gamma\delta)\frac{r^2}{(1-r^2)^2} + \frac{r^2}{1-r^2}\eta(\mathbf{w}, J_{\mathbb{R}}(\mathbf{v})). \end{aligned}$$

Near the boundary, this gives the expansion:

$$0 = \frac{g_{\mathbb{R}}(\mathbf{X}^\top, \mathbf{T}^\top)^2 - |\mathbf{T}^\top|^2|\mathbf{X}^\top|^2}{4(1-r)^2} + o((1-r)^{-2}).$$

The Cauchy-Schwarz inequality and the fact that $\mathbf{X}^\top \neq 0$ on the boundary, implies $\mathbf{T}^\top = b\mathbf{X}^\top$. As $\mathbf{X}^\top$ is $g_{\mathbb{R}}$-orthogonal to $\partial\Sigma_p \subset \mathbb{S}^{2n+1}$, the same is true of $\mathbf{T}^\top$. Hence, $\partial\Sigma_p$ is orthogonal to $\mathbf{T}$ and so it is horizontal. □

3.2 Modified Bergman compactification

While the Bergman compactification is well-adapted to the complex geometry of $\mathbb{CH}^{n+1}$, it seems less satisfactory for studying the asymptotic regularity of minimal submanifolds; this is apparent in the examples of [12]. Therefore, it is convenient to introduce a related compactification, which possesses certain computational features that make it similar to the usual conformal compactification of hyperbolic space.

To begin, consider the radial function $s : \bar{\mathbb{B}}^{2n+2} \to [0, 1]$ defined by

$$s = \frac{r}{1 + \sqrt{1-r^2}} \text{ or, equivalently, by } r = \frac{2s}{1+s^2}.$$

Observe that r extends smoothly with derivative zero to $s = 1$ while s extends only as a $\frac{1}{2}$-Hölder continuous function to $r = 1$. Using

$$1 - s^2 = 2\left(1 - \frac{1}{1+\sqrt{1-r^2}}\right) = 2\frac{\sqrt{1-r^2}}{1+\sqrt{1-r^2}} \text{ and } 1 - r^2 = \frac{(1-s^2)^2}{(1+s^2)^2},$$

one computes that

$$g_B = \frac{4}{(1-s^2)^2}(ds^2 + s^2\theta^2 + s^2\eta) + \frac{16s^4}{(1-s^2)^4}\theta^2 = g_P + \frac{16s^4}{(1-s^2)^4}\theta^2$$

where g_P is the Poincaré metric on $\mathbb{B}^{2n+2}$ of constant curvature -1. Likewise,

$$\omega_B = \frac{4s^2}{(1-s^2)^2}\frac{1}{2}d\theta + \frac{4(1+s^2)}{(1-s^2)^3}sds \wedge \theta.$$

Now consider the diffeomorphism $\mathcal{S} : \mathbb{B}^{2n+2} \to \mathbb{B}^{2n+2}$ given by

$$\mathcal{S} : \mathbf{z} \mapsto \frac{\mathbf{z}}{1 + \sqrt{1 - |\mathbf{z}|^2}} \text{ with inverse } \mathcal{S}^{-1} : \mathbf{z} \mapsto \frac{2\mathbf{z}}{1 + |\mathbf{z}|^2}.$$

This map extends to a $\frac{1}{2}$-Hölder continuous, but not smooth, homeomorphism, $\bar{\mathcal{S}}$, from the closed ball $\bar{\mathbb{B}}^{2n+2}$ to itself. Clearly, $s(p) = r(\mathcal{S}(p))$ and so

$$\mathcal{S}^* dr = ds = \frac{1}{\sqrt{1-r^2}(1+\sqrt{1-r^2})} dr, (\mathcal{S}^{-1})^* ds = dr = \frac{2(1-s^2)}{(1+s^2)^2} ds. \text{ and}$$

$$\mathcal{S}^*\theta = \theta \text{ and } \mathcal{S}^*\eta = \eta.$$

Hence, if we define a metric on $\mathbb{B}^{2n+2}$ by

$$g_{\tilde{B}} = (\mathcal{S}^{-1})^* g_B = \frac{4}{(1-r^2)^2} g_E + \frac{16r^4}{(1-r^2)^4}\theta^2 = g_P + \frac{16r^4}{(1-r^2)^4}\theta^2,$$

then $\mathcal{S}^* g_{\tilde{B}} = g_B$. Here $g_{\tilde{B}}$ is the *modified Bergman metric* which is obtained from the Poincaré metric in a particularly simple manner. The corresponding symplectic form, $\omega_{\tilde{B}}$ satisfies $\mathcal{S}^*\omega_{\tilde{B}} = \omega_B$ and the compatible almost complex structure is

$$J_{\tilde{B}} = J_{\mathbb{R}} + 2r\left(\frac{1}{1-r^2}\mathbf{X}\otimes r\theta - \frac{1}{1+r^2}\mathbf{T}\otimes dr\right).$$

A consequence is that $Aut^+_{g_{\tilde{B}}}(\mathbb{B}^{2n+2})$, the orientation preserving isometries of $g_{\tilde{B}}$ are not holomorphic with respect to the usual complex structure of the ball. However, as every element $\tilde{\Phi} \in Aut^+_{g_{\tilde{B}}}(\mathbb{B}^{2n+2})$ is of the form

$$\tilde{\Phi} = \mathcal{S} \circ \Phi \circ \mathcal{S}^{-1}$$

for a unique $\Phi \in Aut^+_{g_B}(\mathbb{B}^{2n+2})$, it follows that $\tilde{\Phi}$ extends smoothly to $\bar{\mathbb{B}}^{2n+2}$ and induces the same map in $Aut_{CR}(\mathbb{S}^{2n+1})$ as $\bar{\Phi}$.

We now use $\mathcal{S}$ to define a modified form of the Bergman compactification. Fix $p \in \mathbb{CH}^{n+1}$ and let $\Upsilon_p : \mathbb{CH}^{n+1} \to \mathbb{B}^{2n+2}$ be the corresponding choice of Bergman compactification. Let $\tilde{\Upsilon}_p : \mathbb{CH}^{n+1} \to \mathbb{B}^{2n+2}$ be the map $\tilde{\Upsilon}_p = \mathcal{S} \circ \Upsilon_p$ so $\tilde{\Upsilon}^*_p \tilde{g}_B = g_{\mathbb{CH}}$ and $\tilde{\Upsilon}_p(p) = \mathbf{0}$. For $\Sigma \subset \mathbb{CH}^{n+1}$, an m-dimensional submanifold, set

$$\tilde{\Sigma}_p = \bar{\mathcal{S}}(\Sigma_p) = \overline{\tilde{\Upsilon}_p(\Sigma)} \subset \bar{\mathbb{B}}^{2n+2},$$

which we call a *modified Bergman compactification* of Σ.

Definition 3.3 Suppose that Σ and $\tilde{\Sigma}_p$ are as above:

(1) For any $l \geq 1$, Σ is *weakly C^l-asymptotically regular* if $\tilde{\Sigma}_p$ is a C^l-regular submanifold with boundary in $\mathbb{B}^{2n+2}$ that meets $\partial\mathbb{B}^{2n+2}$ transversally;
(2) If, in addition to (1), $\tilde{\Sigma}_p$ meets $\partial\mathbb{B}^{2n+2}$ orthogonally then Σ is *asymptotically orthogonal*;
(3) If, in addition to (1), $\partial\tilde{\Sigma}_p$ is a horizontal submanifold of $\mathbb{S}^{2n+1} = \partial\mathbb{B}^{2n+2}$, then Σ is *weakly asymptotically horizontal*.

For (3) to hold, $m \leq n+1$ and when $m = n+1$ the term *horizontal* in (3) is replaced by *Legendrian*.

In the above definition, the independence of items (1) and (3) from the choice of p follow from the properties of $Aut^+_{g_{\tilde{B}}}(\mathbb{B}^{2n+2})$. To establish the independence of item (2) and to relate Definitions 3.3–3.1 we use the following result.

Lemma 3.4 *Let $\Sigma \subset \mathbb{CH}^{n+1}$ be an m-dimensional submanifold and fix an $l \geq 1$.*

(1) If Σ is C^l-asymptotically regular, then Σ is weakly C^l-asymptotically regular and asymptotically orthogonal.
(2) Conversely, if Σ is weakly C^l-asymptotically regular and asymptotically orthogonal, then Σ is C^1-asymptotically regular.
(3) If Σ is C^1-asymptotically regular, then Σ is asymptotically horizontal if and only if Σ is weakly asymptotically horizontal.
(4) If Σ is weakly C^2-asymptotically regular and asymptotically orthogonal, then Σ is strongly asymptotically horizontal if and only if Σ is weakly asymptotically horizontal and, for a modified Bergman compactification, $\tilde{\Sigma}_p$,

$$g_{\mathbb{R}}(\mathbf{T}, \mathbf{A}^{\mathbb{R}}_{\tilde{\Sigma}_p}(\mathbf{X}^\top, \mathbf{X}^\top)) = 0 \textit{ along } \partial\tilde{\Sigma}_p.$$

Remark 3.5 We observe that the converse direction of (2) must be genuinely weaker when $l \geq 2$ as can be seen by the examples of [12].

Proof Let $\Sigma_p = \overline{\Upsilon_p(\Sigma)} \subset \bar{\mathbb{B}}^{2n+2}$ be a Bergman compactification of Σ. The fact that Σ is C^l-asymptotically regular means the following: Σ is a C^l- submanifold with boundary, $\Gamma = \partial\Sigma_p \subset \partial\mathbb{B}^{2n+2}$ is a C^l-submanifold, and Σ_p meets $\partial\mathbb{B}^{2n+2}$ transversally along Γ. Hence, there is a parametrization of Σ_p in a neighborhood of Γ given by a C^l map

$$\mathbf{F} : (1-\epsilon, 1] \times \Gamma \to \Sigma_p \subset \bar{\mathbb{B}}^{2n+2}$$

with the property that $|\mathbf{F}(\rho, q)| = \rho$. By Taylor's theorem, we may write

$$\mathbf{F}(\rho, q) = \mathbf{X}(q) + \sum_{i=1}^{l}(1-\rho)^i \mathbf{a}_i(q) + (1-\rho)^l \mathbf{f}(\rho, q)$$

where $\mathbf{f}(1, q) = \mathbf{0}$. The properties of Σ_p ensure $g_{\mathbb{R}}(\mathbf{a}_1(q), \mathbf{X}(q)) < 0$. Thus,

$$\begin{aligned}\mathcal{S}(\mathbf{F}(\rho, q)) &= \frac{1}{1+\sqrt{1-\rho^2}}\mathbf{F}(\rho, q)\\ &= \frac{1}{1+\sqrt{1-\rho^2}}\mathbf{X}(q) + \sum_{i=1}^{l}\frac{(1-\rho)^i}{1+\sqrt{1-\rho^2}}\mathbf{a}_i(q) + \frac{(1-\rho)^l}{1+\sqrt{1-\rho^2}}\mathbf{f}(\rho, q).\end{aligned}$$

Consider the related map and its expansion

$$\begin{aligned}\tilde{\mathbf{F}}(\sigma, q) &= \mathcal{S}\left(\mathbf{F}\left(\frac{2\sigma}{1+\sigma^2}, q\right)\right)\\ &= \frac{1+\sigma^2}{2}\mathbf{X}(q) + \frac{1}{2}\sum_{i=1}^{l}(1-\sigma)^{2i}(1+\sigma^2)^{i-1}\mathbf{a}_i(q) + \frac{1}{2}(1-\sigma)^{2l}\tilde{\mathbf{f}}(\sigma, q).\end{aligned}$$

It follows that, up to shrinking ϵ,

$$\tilde{\mathbf{F}} : (1-\epsilon, 1] \times \Gamma \to \tilde{\Sigma}_p \subset \bar{\mathbb{B}}^{2n+2}$$

is a C^l-embedding and so parametrizes $\tilde{\Sigma}_p = \bar{\mathcal{S}}(\Sigma_p)$ in a neighborhood of $\tilde{\Gamma} = \Gamma$. In particular, $\tilde{\Sigma}_p$ is a C^l-regular submanifold with boundary. Moreover,

$$\partial_\sigma \tilde{\mathbf{F}}(1, q) = \mathbf{X}(q)$$

and so we can conclude that $\tilde{\Sigma}_p$ meets $\partial\mathbb{B}^{2n+2}$ orthogonally. This verifies (1).

In the converse direction, let $\tilde{\Sigma}_p$ be the appropriate modified Bergman compactification of Σ. The hypotheses ensure that $\tilde{\Sigma}_p$ is a C^l-regular submanifold with boundary that meets $\partial\mathbb{B}^{2n+2}$ orthogonally. This means that there is a parametrization of $\tilde{\Sigma}_p$ in a neighborhood of Γ by a C^l-embedding

$$\tilde{\mathbf{G}} : (1-\epsilon, 1] \times \Gamma \to \tilde{\Sigma}_p \subset \bar{\mathbb{B}}^{2n+2}$$

with the property that $|\tilde{\mathbf{G}}(\sigma, q)| = \sigma$ and $\partial_\sigma \tilde{\mathbf{G}}(1, q) = \mathbf{X}(q)$. By Taylor's theorem,

$$\tilde{\mathbf{G}}(\sigma, q) = \sigma \mathbf{X}(q) + \frac{1}{2}(1-\sigma)^2 \mathbf{b}_2(q) + (1-\sigma)^2 \mathbf{g}(\sigma, q)$$

where $\mathbf{g}(1, q) = \mathbf{0}$ and we set $\mathbf{b}_2(q) = \mathbf{0}$ when $l = 1$. As $|\tilde{\mathbf{G}}(\sigma, q)| = \sigma$,

$$g_{\mathbb{R}}(\mathbf{b}_2(q), \mathbf{X}(q)) = 0.$$

It follows from the definition that

$$\begin{aligned} \mathcal{S}^{-1}(\tilde{\mathbf{G}}(\sigma, q)) &= \frac{2}{1+\sigma^2} \tilde{\mathbf{G}}(\sigma, q) \\ &= \frac{2\sigma}{1+\sigma^2} \mathbf{X}(q) + \left(1 - \frac{2\sigma}{1+\sigma^2}\right) \mathbf{b}_2(q) + 2\left(1 - \frac{2\sigma}{1+\sigma^2}\right) \mathbf{g}(\sigma, q) \end{aligned}$$

which is not an immersion at $\sigma = 1$. Consider instead the expansion of the map

$$\begin{aligned} \mathbf{G}(\rho, q) &= \mathcal{S}^{-1}\left(\tilde{\mathbf{G}}\left(\frac{\rho}{1+\sqrt{1-\rho^2}}, q\right)\right),) \\ &= \rho \mathbf{X}(q) + (1-\rho)\left(\mathbf{b}_2(q) + \tilde{\mathbf{g}}(\rho, q)\right) \end{aligned}$$

where $\tilde{\mathbf{g}}(1, q) = \mathbf{0}$. Up to shrinking ϵ, this is readily checked to be a C^1 embedding on $(1-\epsilon, 1] \times \Gamma$ and so Σ_p is a C^1-regular submanifold with boundary. Moreover, $g_{\mathbb{R}}(\partial_\rho \mathbf{G}(1, q), \mathbf{X}(q)) = 1$ and so Σ_p meets $\partial\mathbb{B}^{2n+2}$ transversally. This verifies (2). In addition, combined with (1) it also shows that being asymptotically orthogonal is independent of the choice of modified Bergman compactification. Note that, when $l \geq 2$, unless $\mathbf{g}$ has appropriate parity at $\sigma = 1$, there can be a loss of regularity.

Having established (1) and (2), (3) is an immediate consequence of the definitions. Finally, by what has already been shown, the hypotheses of (4) imply that Σ is C^1-asymptotically regular. Using the parameterization, $\mathbf{F}$, from above with $l = 1$, we see Σ is strongly asymptotically horizontal, if and only if $g_{\mathbb{R}}(\mathbf{a}_1(q), \mathbf{T}) = 0$ for all $q \in \Gamma$. It is not hard to see that this is equivalent to

$$g_{\mathbb{R}}(\partial_\sigma^2 \tilde{\mathbf{F}}(1, q), \mathbf{T}) = 0,$$

which can be readily checked to be equivalent to the geometric condition on $\tilde{\Sigma}_p$. □

3.3 Second fundamental form and mean curvature in $\mathbb{CH}^{n+1}$

On $\mathbb{B}^{2n+2}$, let $h = g_{\tilde{B}}$, be the modified Bergman metric, $g = g_P$, be the Poincaré metric, and set $\tau = \frac{4s^2}{(1-s^2)^2}\theta$. Here and in the following subsections we abuse notation and use s instead of r to emphasize that we are working with the modified Bergman metric. As such, h is, in the sense of Appendix B.1, a rank one deformation of g by τ and $\tau(X) = g(X, \mathbf{T})$. We specialize the computations of Appendix B.2 to this case.

First observe that because $g_{\mathbb{R}}$ and $g = g_P$ are conformal one has

$$\mathbf{T}^{\hat{N}} = \mathbf{T} - \frac{1+|\mathbf{T}|_g^2}{1+|\mathbf{T}^\top|_g^2}\mathbf{T}^\top = \mathbf{T} - \frac{(1+s)^2}{(1-s)^2+4|\mathbf{T}^\top|_{\mathbb{R}}^2}\mathbf{T}^\top$$

where we used the fact that $\mathbf{T}^N = \mathbf{T}^\perp$, i.e., the orthogonal component of $\mathbf{T}$ with respect to Σ is the same for $g_{\mathbb{R}}$ and g. Likewise, for a vector field $\mathbf{V}$ along Σ

$$\mathbf{V}^{\tilde{N}} = \mathbf{V}^N - g(\mathbf{V}^N, \mathbf{T})\frac{\mathbf{T}^N}{|\mathbf{T}^N|_g^2} = \mathbf{V}^\perp - g_{\mathbb{R}}(\mathbf{V}^\perp, \mathbf{T})\frac{\mathbf{T}^\perp}{|\mathbf{T}^\perp|_{\mathbb{R}}^2}.$$

Proposition 3.6 *Let $\Sigma \subset \mathbb{B}^{2n+2}$ be an m-dimensional submanifold. The mean curvature of Σ in $h = g_{\tilde{B}}$ and in $g_{\mathbb{R}}$ are related by:*

$$\begin{aligned}(\mathbf{H}_\Sigma^h)^{\tilde{N}} &= \frac{(1-s^2)^2}{4}(\mathbf{H}_\Sigma^{\mathbb{R}})^{\tilde{N}} - \frac{1-s^2}{2}(m+1)\mathbf{X}^{\tilde{N}} \\ &\quad + \frac{(1-s^2)^2}{(1-s^2)^2+4|\mathbf{T}^\top|_{\mathbb{R}}^2}\left(\frac{1-s^2}{2}\mathbf{X} - \mathbf{A}_\Sigma^{\mathbb{R}}(\mathbf{T}^\top, \mathbf{T}^\top) - 2J_{\mathbb{R}}(\mathbf{T}^\top)\right)^{\tilde{N}}\end{aligned}$$

and

$$\begin{aligned}g_{\mathbb{R}}(\mathbf{H}_\Sigma^h, \mathbf{T}^{\hat{N}}) &= \frac{(1-s^2)^2}{4}g_{\mathbb{R}}(\mathbf{H}_\Sigma^{\mathbb{R}}, \mathbf{T}) + \frac{1-s^2}{2}\left(m+1+\frac{2(1-s^2)}{1+s^2}\right)g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X}) \\ &\quad + \frac{(1-s^2)^2}{(1-s^2)^2+4|\mathbf{T}^\top|_{\mathbb{R}}^2}\left(\frac{1-s^2}{2}g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X}) - g_{\mathbb{R}}(\mathbf{A}_\Sigma^{\mathbb{R}}(\mathbf{T}^\top, \mathbf{T}^\top), \mathbf{T})\right).\end{aligned}$$

Proof Using $\nabla_X^{\mathbb{R}}\mathbf{T} = -J_{\mathbb{R}}(X)$ and the formula for the connection of a conformally changed metric, one has

$$\begin{aligned}\nabla_Z^g\mathbf{T} &= -J_{\mathbb{R}}(Z) - Z\cdot\log(1-s^2)\mathbf{T} - \mathbf{T}\cdot\log(1-s^2) + g_{\mathbb{R}}(Z,\mathbf{T})\nabla^{\mathbb{R}}\log(1-s^2) \\ &= -J_{\mathbb{R}}(Z) + 2g_{\mathbb{R}}(Z,\mathbf{X})\frac{\mathbf{T}}{1-s^2} - 2g_{\mathbb{R}}(Z,\mathbf{T})\frac{\mathbf{X}}{1-s^2}.\end{aligned}$$

Using $g(X, J_{\mathbb{R}}(Y)) = -g(J_{\mathbb{R}}(X), Y)$ and $J_{\mathbb{R}}(\mathbf{T}) = \mathbf{X}$ this yields,

$$\begin{aligned}g(\nabla_Z^g\mathbf{T}, Y) &= g(Z, J_{\mathbb{R}}(Y)) + g_P\left(\frac{2g_{\mathbb{R}}(Z,\mathbf{X})}{1-s^2}\mathbf{T} - \frac{2g_{\mathbb{R}}(Z,\mathbf{T})}{1-s^2}\mathbf{X}, Y\right) \\ &= g(Z, J_{\mathbb{R}}(Y)) + g_P\left(-\frac{2g_{\mathbb{R}}(Y,\mathbf{X})}{1-s^2}\mathbf{T} + \frac{2g_{\mathbb{R}}(Y,\mathbf{T})}{1-s^2}\mathbf{X}, Z\right).\end{aligned}$$

Hence, the tensor field, $\mathbf{a}$, from Proposition B.1 satisfies

$$\mathbf{a}(\mathbf{T}^\top) = -\nabla_{\mathbf{T}^\top}^g\mathbf{T} = J_{\mathbb{R}}(\mathbf{T}^\top) - 2g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X})\frac{\mathbf{T}}{1-s^2} + 2|\mathbf{T}^\top|_{\mathbb{R}}^2\frac{\mathbf{X}}{1-s^2}.$$

By Corollary B.2 and $(1-s^2)^2|\mathbf{T}^\top|_g^2 = 4|\mathbf{T}^\top|_{\mathbb{R}}^2$, it follows that

$$(\mathbf{H}_\Sigma^h)^{\tilde{N}} = (\mathbf{H}_\Sigma^g)^{\tilde{N}} - \frac{(\mathbf{A}_\Sigma^g(\mathbf{T}^\top, \mathbf{T}^\top))^{\tilde{N}} + 2(J_{\mathbb{R}}(\mathbf{T}^\top))^{\tilde{N}} + (1-s^2)|\mathbf{T}^\top|_g^2\mathbf{X}^{\tilde{N}}}{1+|\mathbf{T}^\top|_g^2}.$$

The formula for the conformal change of the second fundamental form yields

$$\begin{aligned} \mathbf{A}^g_\Sigma(X,Y) &= \mathbf{A}^{\mathbb{R}}_\Sigma(X,Y) - \frac{2g_{\mathbb{R}}(X,Y)\mathbf{X}^\perp}{1-s^2}, \\ \mathbf{H}^g_\Sigma &= \frac{(1-s^2)^2}{4}\mathbf{H}^{\mathbb{R}}_\Sigma - \frac{m(1-s^2)\mathbf{X}^\perp}{2}. \end{aligned} \tag{3.1}$$

Hence,

$$\begin{aligned} (\mathbf{H}^h_\Sigma)^{\tilde{N}} &= \frac{(1-s^2)^2}{4}(\mathbf{H}^{\mathbb{R}}_\Sigma)^{\tilde{N}} - \frac{m(1-s^2)\mathbf{X}^{\tilde{N}}}{2} \\ &\quad - \frac{(\mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{T}^\top,\mathbf{T}^\top))^{\tilde{N}} + 2(J_{\mathbb{R}}(\mathbf{T}^\top))^{\tilde{N}} + \frac{1}{2}(1-s^2)|\mathbf{T}^\top|^2_g\mathbf{X}^{\tilde{N}}}{1+|\mathbf{T}^\top|^2_g} \\ &= \frac{(1-s^2)^2}{4}(\mathbf{H}^{\mathbb{R}}_\Sigma)^{\tilde{N}} - \frac{(1-s^2)}{2}(m+1)\mathbf{X}^{\tilde{N}} \\ &\quad + \frac{1}{1+|\mathbf{T}^\top|^2_g}\left(\frac{(1-s^2)}{2}\mathbf{X} - \mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{T}^\top,\mathbf{T}^\top) - 2J_{\mathbb{R}}(\mathbf{T}^\top)\right)^{\tilde{N}}. \end{aligned}$$

Using $(1-s^2)^2|\mathbf{T}^\top|^2_g = 4|\mathbf{T}^\top|^2_{\mathbb{R}}$ again, yields the first formula.

Corollary B.2, (3.1) and $g_{\mathbb{R}}(\mathbf{X}^\top,\mathbf{T}) = -g_{\mathbb{R}}(\mathbf{X},\mathbf{T}^\perp)$ imply

$$\begin{aligned} g(\mathbf{H}^h_\Sigma,\mathbf{T}^{\hat{N}}) &= g(\mathbf{H}^g_\Sigma,\mathbf{T}) - \frac{g(\mathbf{A}^g_\Sigma(\mathbf{T}^\top,\mathbf{T}^\top),\mathbf{T})}{1+|\mathbf{T}^\top|^2_g} + \frac{\frac{4g_{\mathbb{R}}(\mathbf{T}^\top,\mathbf{X})}{1-s^2}|\mathbf{T}|^2_g - 2g(J_{\mathbb{R}}(\mathbf{T}^\top),\mathbf{T})}{1+|\mathbf{T}|^2_g} \\ &= g_{\mathbb{R}}(\mathbf{H}^{\mathbb{R}}_\Sigma,\mathbf{T}) - \frac{g(\mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{T}^\top,\mathbf{T}^\top),\mathbf{T})}{1+|\mathbf{T}^\top|^2_g} - \frac{2mg_{\mathbb{R}}(\mathbf{X}^\perp,\mathbf{T})}{1-s^2} \\ &\quad - \frac{2|\mathbf{T}^\top|^2_g g_{\mathbb{R}}(\mathbf{T}^\top,\mathbf{X})}{(1-s^2)(1+|\mathbf{T}^\top|^2_g)} + \frac{4g_{\mathbb{R}}(\mathbf{T}^\top,\mathbf{X})|\mathbf{T}|^2_g}{(1-s^2)(1+|\mathbf{T}|^2_g)} + \frac{2g(\mathbf{T}^\top,\mathbf{X})}{1+|\mathbf{T}|^2_g} \\ &= g_{\mathbb{R}}(\mathbf{H}^{\mathbb{R}}_\Sigma,\mathbf{T}) - \frac{g(\mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{T}^\top,\mathbf{T}^\top),\mathbf{T})}{1+|\mathbf{T}^\top|^2_g} + \frac{2(m+1)g_{\mathbb{R}}(\mathbf{T}^\top,\mathbf{X})}{1-s^2} \\ &\quad + \frac{2g_{\mathbb{R}}(\mathbf{T}^\top,\mathbf{X})}{(1-s^2)(1+|\mathbf{T}^\top|^2_g)} + \frac{4g_{\mathbb{R}}(\mathbf{T}^\top,\mathbf{X})}{1+s^2}. \end{aligned}$$

The second formula follows. □

Corollary 3.7 *If $\Gamma \subset \mathbb{S}^{2n+1}$ is minimal in $g_{\mathbb{S}}$ and horizontal and Σ is the (Euclidean) cone over Γ with vertex $\mathbf{0}$ restricted to $\mathbb{B}^{2n+2}$, then $\Sigma \setminus \{\mathbf{0}\}$ is minimal in $h = g_{\tilde{B}}$ and isotropic with respect to $\omega_{\tilde{B}}$.*

Proof As Γ is minimal in $g_{\mathbb{S}}$, $\mathbf{H}^{\mathbb{R}}_\Sigma = \mathbf{0}$ on $\Sigma\setminus\{\mathbf{0}\}$. Likewise, as Γ is horizontal, $\mathbf{T}^\top = \mathbf{0}$ on $\Sigma \setminus \{\mathbf{0}\}$ and so $\mathbf{H}^h_\Sigma = \mathbf{0}$. This also implies that $\Sigma \setminus \{\mathbf{0}\}$ is isotropic. □

3.4 Asymptotic behavior of minimal submanifolds in $\mathbb{CH}^{n+1}$

We use Proposition 3.6 to show an improvement of boundary regularity for minimal submanifolds of the modified Bergman metric. That is, we prove Theorem 1.1.

We first show a preliminary partial result.

Lemma 3.8 *Let* $\Sigma \subset \bar{\mathbb{B}}^{2n+2}$ *be an* m*-dimensional* C^2*-regular submanifold with boundary that meets* $\partial\mathbb{B}^{2n+2}$ *transversely along* $\partial\Sigma \subset \partial\mathbb{B}^{2n+2}$*. If* $\mathbf{T}^\top = \mathbf{0}$ *on* $\partial\Sigma$*, then, near* $\partial\Sigma$*,*

$$\mathbf{T}^\top = (1-s)\frac{1}{|\mathbf{X}^\top|^2}\left((J_{\mathbb{R}}(\mathbf{X}^\top))^\top - \mathbf{S}^\Sigma_{\mathbf{T}^\perp}(\mathbf{X}^\top)\right).$$

Moreover, along $\partial\Sigma$*, we may write*

$$\mathbf{X}^\top = |\mathbf{X}^\top|^2_{\mathbb{R}}\mathbf{X} + Z = |\mathbf{X}^\top|^2_{\mathbb{R}}\mathbf{X} + J_{\mathbb{S}}(Z_1) + Z_2,$$

where Z *is normal to* $\mathbf{X}$*,* $\partial\Sigma$ *and* $\mathbf{T}$*,* Z_1 *is tangent to* $\partial\Sigma$*,* Z_2 *and* $J_{\mathbb{S}}(X_2)$ *are normal to* $\partial\tilde{\Sigma}$*,* $\mathbf{X}$*, and* $\mathbf{T}$*. Using this decomposition we obtain, along* $\partial\Sigma$*,*

$$(J_{\mathbb{R}}(\mathbf{X}^\top))^\top = -Z_1 \text{ and } \mathbf{S}^\Sigma_{\mathbf{T}^\perp}(\mathbf{X}^\top) = g_{\mathbb{R}}(\mathbf{T}, \mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{X}^\top, \mathbf{X}^\top))\frac{\mathbf{X}^\top}{|\mathbf{X}^\top|^2_{\mathbb{R}}} + Z_1.$$

Proof The hypotheses on Σ ensure that $\partial\Sigma$ is horizontal and we can write

$$\mathbf{T}^\top = (1-s)\mathbf{v}$$

for a C^2 vector field $\mathbf{v}$ tangent to Σ that extends in a C^1 fashion to the boundary. We compute that on the boundary

$$\nabla^\Sigma_{\mathbf{X}^\top}\mathbf{T}^\top = (\nabla^{\mathbb{R}}_{\mathbf{X}^\top}(\mathbf{T} - \mathbf{T}^\perp))^\top = -(J_{\mathbb{R}}(\mathbf{X}^\top))^\top + S^\Sigma_{\mathbf{T}^\perp}(\mathbf{X}^\top).$$

As $\mathbf{X}^\top \cdot s = |\mathbf{X}^\top|^2$ on the boundary we verify that

$$\mathbf{v} = -\frac{1}{|\mathbf{X}^\top|^2}\left(-(J_{\mathbb{R}}(\mathbf{X}^\top))^\top + S^\Sigma_{\mathbf{T}^\perp}(\mathbf{X}^\top)\right).$$

On $\partial\Sigma$ we may write

$$\mathbf{X}^\top = |\mathbf{X}^\top|^2\mathbf{X} + Z = |\mathbf{X}^\top|^2\mathbf{X} + J_{\mathbb{S}}(Z_1) + Z_2$$

where Z is tangent to $\mathbb{S}^{2n+1} = \partial\mathbb{B}^{2n+2}$ and thus orthogonal to $\partial\Sigma$. The hypotheses on Σ ensure that $\mathbf{X}^\top$, and, thus Z, are orthogonal to $\mathbf{T}$ and $\partial\Sigma$. Hence, we may choose Z_1 to be tangent to $\partial\Sigma$ and so Z_2 and $J_{\mathbb{S}}(Z_2)$ are orthogonal to $\partial\Sigma$. When $\partial\Sigma$ is Legendrian, Z_2 vanishes. Clearly, along $\partial\Sigma$,

$$(J_{\mathbb{R}}(\mathbf{X}^\top))^\top = (-|\mathbf{X}^\top|^2\mathbf{T} + J_{\mathbb{S}}(J_{\mathbb{S}}(Z_1) + Z_2))^\top = -Z_1.$$

For Y tangent to $\partial\Sigma$, the hypotheses on Σ ensure

$$\begin{aligned} g_{\mathbb{R}}(\mathbf{T}, \mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{X}^\top, Y)) &= g_{\mathbb{R}}(\mathbf{T}, \nabla^{\mathbb{R}}_Y\mathbf{X}^\top) = g_{\mathbb{R}}(\mathbf{T}, \nabla^{\mathbb{R}}_Y\left(|\mathbf{X}^\top|^2_{\mathbb{R}}\mathbf{X} + Z\right)) \\ &= g_{\mathbb{R}}(\mathbf{T}, \nabla^{\mathbb{R}}_Y Z) = Y \cdot g_{\mathbb{R}}(\mathbf{T}, Z) - g_{\mathbb{R}}(\nabla^{\mathbb{R}}_Y\mathbf{T}, Z) \\ &= g_{\mathbb{R}}(J_{\mathbb{R}}(Y), Z) = -g_{\mathbb{R}}(J_{\mathbb{S}}(Z), Y) = g_{\mathbb{R}}(Z_1, Y). \end{aligned}$$

Hence, along $\partial\Sigma$, we have

$$\mathbf{S}^\Sigma_{\mathbf{T}^\perp}(\mathbf{X}^\top) = \mathbf{S}^\Sigma_{\mathbf{T}}(\mathbf{X}^\top) = g_{\mathbb{R}}(\mathbf{T}, \mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{X}^\top, \mathbf{X}^\top))\frac{\mathbf{X}^\top}{|\mathbf{X}^\top|^2_{\mathbb{R}}} + Z_1.$$

□

We now prove Theorem 1.1

Proof of Theorem 1.1 Let us choose a modified Bergman compactification, $\tilde{\Sigma}$, of Σ. The hypotheses ensure that $\tilde{\Sigma}$ is C^2 up to $\partial\mathbb{B}^{2n+2}$, meets the boundary transversally, has $\partial\tilde{\Sigma}$ horizontal, and is minimal with respect to $g_{\tilde{B}}$. For simplicity, we will write Σ instead of $\tilde{\Sigma}$ for the remainder of the proof.

Our hypotheses ensure that near the boundary

$$\mathbf{T}^\top = g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X}) \frac{\mathbf{X}^\top}{|\mathbf{X}^\top|^2_{\mathbb{R}}} + \mathbf{V} \tag{3.2}$$

where $\mathbf{V}$ is tangent to Σ, orthogonal to $\mathbf{X}^\top$, and vanishes along the boundary. This is well defined as the hypotheses ensure that $\mathbf{X}^\top \neq 0$ along the boundary.

Using the minimality of Σ along with the fact that Σ is C^2 up to the boundary, the second formula of Proposition 3.6 can be rewritten as

$$0 = ((1-s^2)^2 + 4|\mathbf{T}^\top|^2_{\mathbb{R}}) \left(\frac{1-s^2}{2} g_{\mathbb{R}}(\mathbf{H}^{\mathbb{R}}_\Sigma, \mathbf{T}) + \left(m + 1 + \frac{2(1-s^2)}{1+s^2} \right) g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X}) \right)$$
$$+ 2(1-s^2) \left(\frac{1-s^2}{2} g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X}) - g_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{T}^\top, \mathbf{T}^\top), \mathbf{T}) \right).$$

From this expression one obtains that, near $\partial\Sigma$,

$$4(m+1)|\mathbf{T}^\top|^2_{\mathbb{R}} g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X}) = O(1-s).$$

Hence, on $\partial\Sigma$, either $\mathbf{T}^\top = \mathbf{0}$ or $g_{\mathbb{R}}(\mathbf{T}^\top, \mathbf{X}) = 0$. In the latter case, (3.2) implies $\mathbf{T}^\top = \mathbf{0}$ on $\partial\Sigma$ and, so in either case there is a vector field, $\mathbf{v}$, tangent to Σ with

$$\mathbf{T}^\top = (1-s)\mathbf{v}.$$

From the first formula of Proposition 3.6, it follows that near $\partial\Sigma$, one has

$$\mathbf{0} = \frac{2}{1+|\mathbf{v}|^2_{\mathbb{R}}} (J_{\mathbb{R}}(\mathbf{v}))^{\tilde{N}} + \left(m + 1 - \frac{1}{1+|\mathbf{v}|^2_{\mathbb{R}}} \right) \mathbf{X}^{\tilde{N}} + o(1).$$

Hence, along the boundary

$$\mathbf{0} = 2(J_{\mathbb{R}}(\mathbf{v}))^{\tilde{N}} + \left((m+1)|\mathbf{v}|^2_{\mathbb{R}} + m\right) \mathbf{X}^{\tilde{N}}.$$

As we have shown $\mathbf{T}^\perp = \mathbf{T}$ on $\partial\Sigma$, we may appeal to Lemma 3.8 to see

$$\mathbf{X}^{\tilde{N}} = \mathbf{X}^\perp = \mathbf{X} - \mathbf{X}^\top = |\mathbf{X}^\perp|^2_{\mathbb{R}}\mathbf{X} - Z = |\mathbf{X}^\perp|^2_{\mathbb{R}}\mathbf{X} - J_{\mathbb{S}}(Z_1) - Z_2$$

where Z, Z_1 and Z_2 are as in the statement of Lemma 3.8. By Lemma 3.8, for appropriate β, on $\partial\Sigma$ one has

$$\mathbf{v} = -\frac{2}{|\mathbf{X}^\top|^2} Z_1 + \beta\mathbf{X}^\top = -\frac{2}{|\mathbf{X}^\top|^2} Z_1 + \beta(|\mathbf{X}^\top|^2\mathbf{X} + J_{\mathbb{S}}(Z_1) + Z_2).$$

Hence,

$$(J_{\mathbb{R}}(\mathbf{v}))^{\tilde{N}} = \left(-\frac{2}{|\mathbf{X}^\top|^2} J_{\mathbb{S}}(Z_1) + \beta(|\mathbf{X}^\top|^2\mathbf{T} - Z_1 + J_{\mathbb{S}}(Z_2)) \right)^{\tilde{N}}$$
$$= -\frac{2}{|\mathbf{X}^\top|^2} (J_{\mathbb{S}}(Z_1))^\perp + \beta(J_{\mathbb{S}}(Z_2))^\perp.$$

Plugging this into the previous identity we obtain

$$\mathbf{0} = -\frac{4}{|\mathbf{X}^\top|^2}(J_{\mathbb{S}}(Z_1))^\perp + 2\beta(J_{\mathbb{S}}(Z_2))^\perp + \left((m+1)|\mathbf{v}|^2 + m\right)\mathbf{X}^\perp.$$

As $J_{\mathbb{S}}(Z_2)$ is, by construction, orthogonal to $\partial\Sigma$, $\mathbf{X}^\perp$, $\mathbf{X}^\top$, and $\mathbf{T}$, one has

$$J_{\mathbb{S}}(Z_2)^{\tilde{N}} = (J_{\mathbb{S}}(Z_2))^\perp = J_{\mathbb{S}}(Z_2).$$

Moreover, as $J_{\mathbb{S}}(Z_1)$ is orthogonal to $\partial\Sigma$,

$$(J_{\mathbb{S}}(Z_1))^\perp = J_{\mathbb{S}}(Z_1) - g_{\mathbb{R}}(J_{\mathbb{S}}(Z_1), \mathbf{X}^\top)\frac{\mathbf{X}^\top}{|\mathbf{X}^\top|^2},$$

which immediately implies that $J_{\mathbb{S}}(Z_2)$ is also orthogonal to $(J_{\mathbb{S}}(Z_1))^\perp$. Hence,

$$\beta J_{\mathbb{S}}(Z_2) = \mathbf{0} = -\frac{4}{|\mathbf{X}^\top|^2}(J_{\mathbb{S}}(Z_1))^\perp + \left((m+1)|\mathbf{v}|^2 + m\right)\mathbf{X}^\perp.$$

This implies $\beta J_{\mathbb{S}}(Z_2)$ vanishes and

$$\begin{aligned} 0 &= -\frac{4}{|\mathbf{X}^\top|^2}|J_{\mathbb{S}}(Z_1)^\perp|^2 + \left((m+1)|\mathbf{v}|^2 + m\right) g_{\mathbb{R}}(\mathbf{X}^\perp, J_{\mathbb{S}}(Z_1)) \\ &= -\frac{4}{|\mathbf{X}^\top|^2}|J_{\mathbb{S}}(Z_1)^\perp|^2 - \left((m+1)|\mathbf{v}|^2 + m\right)|J_{\mathbb{S}}(Z_1)|^2. \end{aligned}$$

It follows that $Z_1 = J_{\mathbb{S}}(Z_1) = (J_{\mathbb{S}}(Z_1))^\perp = \mathbf{0}$. Hence,

$$\mathbf{0} = \left((m+1)|\mathbf{v}|^2 + m\right)\mathbf{X}^\perp$$

and so $\mathbf{X}^\perp = 0$. In particular, as a submanifold of $\mathbb{CH}^{n+1}$, Σ is asymptotically orthogonal. Moreover, along $\partial\Sigma$, one has a function α so

$$\mathbf{v} = -\frac{1}{|\mathbf{X}^\top|^2}\mathbf{S}^\Sigma_{\mathbf{T}^\perp}(\mathbf{X}^\top) = -\mathbf{S}^\Sigma_{\mathbf{T}}(\mathbf{X}) = g_{\mathbb{R}}(\mathbf{T}, \mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{X}^\top, \mathbf{X}^\top))\mathbf{X}^\top = \alpha\mathbf{X}^\top.$$

As Σ meets $\partial\mathbb{B}^{2n+2}$ orthogonally, for X, Y tangent to $\partial\Sigma$,

$$\mathbf{A}^{\mathbb{S}}_{\partial\Sigma}(X, Y) = (\nabla^{\mathbb{R}}_X Y)^\perp = \mathbf{A}^{\mathbb{R}}_\Sigma(X, Y).$$

Moreover, as $\mathbf{T}^\top = 0$, for such X, Y,

$$\begin{aligned} g_{\mathbb{R}}(\mathbf{T}, \mathbf{A}^{\mathbb{R}}_\Sigma(X, Y)) &= g_{\mathbb{R}}(\mathbf{T}, \nabla^{\mathbb{R}}_X Y) = X \cdot g_{\mathbb{R}}(\mathbf{T}, Y) - g_{\mathbb{R}}(\nabla^{\mathbb{R}}_X \mathbf{T}, Y) \\ &= g_{\mathbb{R}}(J_{\mathbb{R}}(X), Y) = 0. \end{aligned}$$

Hence, along the boundary,

$$\mathbf{H}^{\mathbb{R}}_\Sigma = \mathbf{H}^{\mathbb{S}}_{\partial\Sigma} + \mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{X}^\top, \mathbf{X}^\top) \text{ and } g_{\mathbb{R}}(\mathbf{H}^{\mathbb{R}}_{\partial\Sigma}, \mathbf{T}) = g_{\mathbb{S}}(\mathbf{H}^{\mathbb{S}}_{\partial\Sigma}, \hat{\mathbf{T}}) = 0.$$

It follows that,

$$g_{\mathbb{R}}(\mathbf{H}^{\mathbb{R}}_\Sigma, \mathbf{T}) = g_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{X}^\top, \mathbf{X}^\top), \mathbf{T}) = g_{\mathbb{R}}(\mathbf{v}, \mathbf{X}) = \alpha.$$

Using $\mathbf{T}^\top = (1-s)\mathbf{v}$, the second formula of Proposition 3.6 reduces, on $\partial\Sigma$, to:

$$\begin{aligned} 0 &= (1 + |\mathbf{v}|^2_{\mathbb{R}})(g_{\mathbb{R}}(\mathbf{H}^{\mathbb{R}}_\Sigma, \mathbf{T}) + (m+1)g_{\mathbb{R}}(\mathbf{v}, \mathbf{X})) + g_{\mathbb{R}}(\mathbf{v}, \mathbf{X}) - g_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_\Sigma(\mathbf{v}, \mathbf{v}), \mathbf{T}) \\ &= (1+\alpha^2)(\alpha + (m+1)\alpha)) + \alpha - \alpha^3 = (m+1)\alpha^3 + (m+3)\alpha. \end{aligned}$$

The only real root is $\alpha = 0$. Hence, $\mathbf{v} = \mathbf{0}$ and $g_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top}), \mathbf{T}) = 0$ along $\partial\Sigma$. It follows from Lemma 3.4 that Σ, thought of as a submanifold of $\mathbb{CH}^{n+1}$, is C^1-asymptotically regular and strongly asymptotically horizontal. □

While we do not use it elsewhere in this paper, there is finer information about the boundary geometry of the compactified surface.

Corollary 3.9 *Suppose that $\Sigma \subset \mathbb{CH}^{n+1}$ is an m-dimensional minimal submanifold. If Σ is weakly C^3-asymptotically regular and weakly asymptotically horizontal, and $\tilde{\Sigma} \subset \bar{\mathbb{B}}^{2n+2}$ is a modified Bergman compactification of Σ, then, on $\partial\tilde{\Sigma}$,*

(1) $g_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_{\tilde{\Sigma}}(X, Y), \mathbf{T}) = 0$ for any tangent vectors X, Y to Σ.
(2) $\mathbf{A}^{\mathbb{R}}_{\tilde{\Sigma}}(\mathbf{X}^{\top}, Y) = \frac{1}{m+1}\mathbf{H}^{\mathbb{S}}_{\partial\tilde{\Sigma}} g_{\mathbb{R}}(\mathbf{X}^{\top}, Y)$ for any tangent vector Y to Σ.

Proof The hypotheses ensure that $\tilde{\Sigma}$ is C^3 up to $\partial\mathbb{B}^{2n+2}$ and meets it transversally, has $\partial\tilde{\Sigma}$ horizontal and is minimal with respect to $g_{\tilde{B}}$. For simplicity, we will write Σ instead of $\tilde{\Sigma}$ for the remainder of the proof. Observe, that by Theorem 1.1, Σ meets $\partial\mathbb{B}^{2n+2}$ orthogonally and there is a vector field, $\mathbf{w}$, tangent to Σ so

$$\mathbf{T}^{\top} = (1-s)^2\mathbf{w}.$$

Hence, on $\partial\Sigma$, $\mathbf{S}^{\Sigma}_{\mathbf{T}}(X) = \mathbf{0}$ which implies (1) holds.

The first formula of Proposition 3.6 implies that near $\partial\Sigma$,

$$\mathbf{0} = (1-s)(\mathbf{H}^{\mathbb{R}}_{\Sigma})^{\tilde{N}} - 2(1-s)(J_{\mathbb{R}}(\mathbf{w}))^{\tilde{N}} - m\mathbf{X}^{\tilde{N}} + o((1-s)).$$

To go further it is helpful to observe that

$$\begin{aligned}\mathbf{X}^{\tilde{N}} &= \mathbf{X}^{\perp} - g_{\mathbb{R}}(\mathbf{X}, \mathbf{T}^{\perp})\frac{\mathbf{T}^{\perp}}{|\mathbf{T}^{\perp}|^2_{\mathbb{R}}} = \mathbf{X}^{\perp} + g_{\mathbb{R}}(\mathbf{X}, \mathbf{T}^{\top})\frac{\mathbf{T}^{\perp}}{|\mathbf{T}^{\perp}|^2_{\mathbb{R}}} \\ &= \mathbf{X}^{\perp} + O((1-s)^2).\end{aligned}$$

As $(\nabla^{\mathbb{R}}_{\mathbf{X}^{\top}}\mathbf{X}^{\perp})^{\perp} = -\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top})$, it follows that near the boundary

$$\mathbf{X}^{\tilde{N}} = (1-s)(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top}))^{\tilde{N}} + o(1-s).$$

Hence, we conclude that

$$\mathbf{0} = \left(\mathbf{H}^{\mathbb{R}}_{\Sigma} - 2J_{\mathbb{R}}(\mathbf{w}) - m\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top})\right)^{\tilde{N}}.$$

From these computations and item (1), this means that, on $\partial\Sigma$,

$$2(J_{\mathbb{R}}(\mathbf{w}))^{\tilde{N}} = \left((1-m)\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top}) + \mathbf{H}^{\mathbb{S}}_{\partial\Sigma}\right)^{\tilde{N}} = (1-m)\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top}) + \mathbf{H}^{\mathbb{S}}_{\partial\Sigma}.$$

To complete the proof we need to compute $\mathbf{w}$ in terms of the geometry of Σ. To that end, observe that

$$\nabla^{\mathbb{R}}_{\mathbf{X}^{\top}}\mathbf{X}^{\top} = \mathbf{X}^{\top} + \mathbf{S}^{\Sigma}_{\mathbf{X}^{\perp}}(\mathbf{X}^{\top}) + \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top}).$$

Hence,

$$\begin{aligned}\nabla^{\Sigma}_{\mathbf{X}^{\top}}(J_{\mathbb{R}}(\mathbf{X}^{\top}))^{\top} &= \left(\nabla^{\mathbb{R}}_{\mathbf{X}^{\top}}\left(J_{\mathbb{R}}(\mathbf{X}^{\top}) - (J_{\mathbb{R}}(\mathbf{X}^{\top}))^{\perp}\right)\right)^{\top} \\ &= (J_{\mathbb{R}}(\mathbf{X}^{\top} + \mathbf{S}^{\Sigma}_{\mathbf{X}^{\perp}}(\mathbf{X}^{\top}) + \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^{\top}, \mathbf{X}^{\top})))^{\top} + \mathbf{S}^{\Sigma}_{(J_{\mathbb{R}}(\mathbf{X}^{\top}))^{\perp}}(\mathbf{X}^{\top}).\end{aligned}$$

The properties already established about the boundary geometry of Σ yield, on $\partial\Sigma$,

$$\nabla^{\Sigma}_{\mathbf{X}^\top}(J_{\mathbb{R}}(\mathbf{X}^\top))^\top = J_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)) - \mathbf{S}^{\Sigma}_{\mathbf{T}^\perp}(\mathbf{X}^\top) = J_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)).$$

We further compute that

$$\begin{aligned} g_{\mathbb{R}}(\nabla^{\mathbb{R}}_{\mathbf{X}^\top}(\mathbf{S}^{\Sigma}_{\mathbf{T}^\perp}(\mathbf{X}^\top)), Y) &= \mathbf{X}^\top \cdot g((\mathbf{S}^{\Sigma}_{\mathbf{T}^\perp}(\mathbf{X}^\top), Y) - g((\mathbf{S}^{\Sigma}_{\mathbf{T}^\perp}(\mathbf{X}^\top), \nabla^{\mathbb{R}}_{\mathbf{X}^\top} Y) \\ &= \mathbf{X}^\top \cdot g_{\mathbb{R}}(\mathbf{T}^\perp, \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top, Y)) - g((\mathbf{S}^{\Sigma}_{\mathbf{T}^\perp}(\mathbf{X}^\top), \nabla^{\mathbb{R}}_{\mathbf{X}^\top} Y) \\ &= g_{\mathbb{R}}(-(J_{\mathbb{R}}(\mathbf{X}^\top))^\perp - \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{T}^\top), \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top, Y)) \\ &\quad + g_{\mathbb{R}}(\mathbf{T}^\perp, (\nabla^{\perp}_{\mathbf{X}^\top}\mathbf{A}^{\mathbb{R}}_{\Sigma})(\mathbf{X}^\top, Y) + g_{\mathbb{R}}(\mathbf{T}^\perp, \mathbf{A}^{\mathbb{R}}_{\Sigma}(\nabla^{\Sigma}_{\mathbf{X}^\top}\mathbf{X}^\top, Y)) \end{aligned}$$

The Codazzi equations imply

$$\begin{aligned} & g_{\mathbb{R}}(\mathbf{T}^\perp, (\nabla^{\perp}_{\mathbf{X}^\top}\mathbf{A}^{\mathbb{R}}_{\Sigma})(\mathbf{X}^\top, Y) = g_{\mathbb{R}}(\mathbf{T}^\perp, (\nabla^{\perp}_{Y}\mathbf{A}^{\mathbb{R}}_{\Sigma})(\mathbf{X}^\top, \mathbf{X}^\top)) \\ &\quad = Y \cdot g_{\mathbb{R}}(\mathbf{T}^\perp, \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)) + g_{\mathbb{R}}((J_{\mathbb{R}}(Y))^\perp + \mathbf{A}^{\mathbb{R}}_{\Sigma}(Y,\mathbf{T}^\top), \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)). \end{aligned}$$

On the boundary, as $\mathbf{T}^\top = \mathbf{0}$, $\mathbf{X}^\top = \mathbf{X}$ and item (1) holds, this simplifies to

$$\begin{aligned} g_{\mathbb{R}}(\nabla^{\mathbb{R}}_{\mathbf{X}^\top}(\mathbf{S}^{\Sigma}_{\mathbf{T}^\perp}(\mathbf{X}^\top)), Y) &= -g_{\mathbb{R}}(J_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top), Y) + Y \cdot g_{\mathbb{R}}(\mathbf{T}^\perp, \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)) \\ &= -g_{\mathbb{R}}(J_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top), Y) + \mathbf{X}^\top \cdot g_{\mathbb{R}}(\mathbf{T}^\perp, \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)) g_{\mathbb{R}}(Y, \mathbf{X}) \end{aligned}$$

where we used $\mathbf{X}^\top = \mathbf{X}$ on $\partial\Sigma$.

Hence, on the boundary,

$$\mathbf{w} = \frac{1}{2}\nabla^{\Sigma}_{\mathbf{X}^\top}\nabla^{\Sigma}_{\mathbf{X}^\top}\mathbf{T}^\top = -J_{\mathbb{R}}(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)) + \frac{\mathbf{X}^\top}{2} \cdot g_{\mathbb{R}}(\mathbf{T}^\perp, \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top))\mathbf{X}$$

and so

$$(J_{\mathbb{R}}(\mathbf{w}))^{\tilde{N}} = \left(\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top) - \frac{\mathbf{X}^\top}{2} \cdot g_{\mathbb{R}}(\mathbf{T}^\perp, \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top))\mathbf{T}\right)^{\tilde{N}} = \mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top)$$

where we used item (1). Hence,

$$2\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top) = (1-m)\mathbf{A}^{\mathbb{R}}_{\Sigma}(\mathbf{X}^\top,\mathbf{X}^\top) + \mathbf{H}^{\mathbb{S}}_{\partial\Sigma},$$

which yields item (2) for $Y = \mathbf{X}^\top$. When Y is orthogonal to $\mathbf{X}^\top$ is immediate. □

4 Colding-Minicozzi entropy in $\mathbb{CH}^{n+1}$

In this section, we prove Theorem 1.4. To do so we first introduce some notation. Let $\Upsilon, \Upsilon' : \mathbb{CH}^{n+1} \to \mathbb{B}^{2n+2}$ be two choices of Bergman compactifications satisfying $\Upsilon(p_0) = \Upsilon'(p_0') = \mathbf{0}$. By construction, there is an element $\Phi \in Aut^+_{g_B}(\mathbb{B}^{2n+2})$ satisfying $\Upsilon' = \Phi \circ \Upsilon$. Moreover, when $p_0 = p_0'$, this element Φ is also an isometry of $g_{\mathbb{R}}$. As a consequence, Υ endows $\partial_\infty \mathbb{CH}^{n+1}$ with a well-defined Riemannian metric, obtained by pulling back $g_{\mathbb{S}}$ from $\mathbb{S}^{2n+1} = \partial\mathbb{B}^{2n+2}$. While this metric depends on p_0, it is otherwise independent of the choices of Υ and so we denote it by $g^{p_0}_{\partial_\infty\mathbb{CH}}$. Clearly, $g^{p_0}_{\partial_\infty\mathbb{CH}}$ and $g^{q_0}_{\partial_\infty\mathbb{CH}}$ are related by an element of $Aut_{CR}(\mathbb{S}^{2n+1})$ for different choices of distinguished points p_0 and q_0. Therefore any geometric quantity defined on $\mathbb{S}^{2n+1}$ that is $Aut_{CR}(\mathbb{S}^{2n+1})$ invariant is well-defined on $\partial_\infty\mathbb{CH}^{n+1}$. Furthermore, as $\bar{S}$ acts as identity on $\mathbb{S}^{2n+1}$ one may also use the modified Bergman compactification to the same effect.

Fix an m-dimensional C^1 submanifold $\Gamma \subset \partial_\infty \mathbb{CH}^n$ and let $\Upsilon(\Gamma) \subset \mathbb{S}^{2n+1}$ be the submanifold associated to Γ by the Bergman compactification, Υ. Set

$$Vol_{\partial_\infty \mathbb{CH}}(\Gamma, p_0) = |\Upsilon(\Gamma)|_{\mathbb{S}}.$$

When p_0' is a different choice of a distinguished point, then, by the above discussion there is an element, $\Psi \in \mathrm{Aut}_{CR}(\mathbb{S}^{2n+1})$ such that

$$Vol_{\partial_\infty \mathbb{CH}}(\Gamma, p_0') = |\Psi'(\Gamma)|_{\mathbb{S}} = |\Psi(\Upsilon(\Gamma))|_{\mathbb{S}}.$$

Hence, the *CR-volume* of a horizontal submanifold, $\Gamma \subset \partial_\infty \mathbb{CH}^{n+1}$, defined by

$$\lambda_{CR}[\Gamma] = \sup_{\Psi \in \mathrm{Aut}_{CR}(\mathbb{S}^{2n+1})} |\Psi(\Upsilon(\Gamma))|_{\mathbb{S}} = \sup_{p_0 \in \mathbb{CH}^{n+1}} Vol_{\partial_\infty \mathbb{CH}^n}(\Gamma, p_0)$$

is a well-defined quantity independent of choices. Moreover, using modified Bergman compactifications yields the same value.

We now begin the proof of Theorem 1.4 and first record a basic relationship between geodesic balls of the modified Bergman metric and the Euclidean metric.

Lemma 4.1 *Setting $s(R) = \tanh \frac{R}{2}$, it follows that*

$$B_R^{g_{\tilde{B}}}(\mathbf{0}) = B_{s(R)}^{\mathbb{R}}(\mathbf{0}).$$

Proof Suppose that ℓ is a line segment in $\mathbb{B}^{2n+2}$ with one endpoint through $\mathbf{0}$. Proposition 3.6 implies ℓ is also a geodesic of $g_{\tilde{B}}$. The length with respect to $g_{\tilde{B}}$ is

$$R = \int_0^s \frac{2}{1-t^2} dt = \ln\left(\frac{1+s}{1-s}\right).$$

As geodesics are always minimizing in Cartan-Hadamard spaces,

$$s(R) = \frac{e^R - 1}{e^R + 1} = \tanh \frac{R}{2}.$$

□

It is also useful to clarify the relationship between $Vol_{\partial_\infty \mathbb{CH}}(\Gamma; p_0)$ and the geometry of submanifolds asymptotic to Γ for varying degrees of asymptotic regularity.

Lemma 4.2 *Let $\Sigma \subset \mathbb{CH}^{n+1}$ be an m-dimensional submanifold that is weakly C^1-asymptotically regular and weakly asymptotically horizontal. For any $p_0 \in \mathbb{CH}^{n+1}$,*

$$Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0) \leq \liminf_{r \to \infty} \frac{Vol_{\mathbb{CH}}(\Sigma \cap \partial B_r^{\mathbb{CH}}(p_0))}{\sinh^{m-1}(r)}$$

where $B_r^{\mathbb{CH}}(p_0)$ is a ball of radius r centered at p_0 in $\mathbb{CH}^{n+1}$. If Σ is C^1-asymptotically regular and strongly asymptotically horizontal, then

$$Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0) = \lim_{r \to \infty} \frac{Vol_{\mathbb{CH}}(\Sigma \cap \partial B_r^{\mathbb{CH}}(p_0))}{\sinh^{m-1}(r)}.$$

Proof Let $\tilde{\Upsilon}_{p_0} : \mathbb{CH}^{n+1} \to \mathbb{B}^{2n+2}$ be a modified Bergman compactification sending p_0 to $\mathbf{0}$. By Lemma 4.1, one has

$$\tilde{\Upsilon}_{p_0}(\partial B_r^{\mathbb{CH}}(p_0)) = \partial B_r^{g_{\tilde{B}}}(0) = \partial B_s(0)$$

where $r = \ln\left(\frac{1+s}{1-s}\right)$.

Set $\Sigma_r = \Sigma \cap \partial B_r^{\mathbb{CH}}(p_0)$ and let $\Sigma' = \tilde{\Upsilon}_{p_0}(\Sigma)$. Clearly, $\tilde{\Upsilon}_{p_0}(\Sigma_r) = \Sigma' \cap \partial B_s(0) = \Sigma'_s$. Let g_r be the metric on Σ_r induced from $g_{\mathbb{CH}}$, g'_s be the metric induced on Σ'_s from $g_{\mathbb{R}}$, and g''_s be the metric on Σ'_s induced from $g_{\tilde{B}}$. From the form of $g_{\tilde{B}}$ and properties of Σ' near the boundary, that follow from Σ being weakly C^1-asymptotically regular, one has

$$g''_s = \frac{4}{(1-s^2)^2} g'_s + \frac{4s^4}{(1-s^2)^4} (i^*_{\Sigma'_s}\theta)^2 \geq \frac{4}{(1-s^2)^2} g'_s, \tag{4.1}$$

where the inequality is in the sense of symmetric bilinear forms.

Hence,

$$\tilde{\Upsilon}^*_{p_0} g'_s \leq \left(\frac{e^r - 1}{e^r + 1}\right)^2 \sinh^{-2}(r) \tilde{\Upsilon}^*_{p_0} g''_s = \frac{1}{(1+\cosh(r))^2} g_r.$$

In particular, as Σ'_s is $(m-1)$-dimensional, one has

$$|\Sigma'_s|_{\mathbb{R}} \leq \frac{|\Sigma_r|_{\mathbb{CH}}}{(1+\cosh(r))^{m-1}}.$$

The definition of weakly C^1-regular asymptotic boundary ensures that

$$\lim_{s\to 1} |\Sigma'_s|_{\mathbb{R}} = |\partial\Sigma'|_{\mathbb{R}} = |\partial\Sigma'|_{\mathbb{S}} = Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0).$$

As $s \to 1$, $r \to \infty$, and so, using $\lim_{r\to\infty} \frac{\sinh(r)}{1+\cosh(r)} = 1$, we conclude

$$Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0) \leq \liminf_{r\to\infty} \frac{|\Sigma_r|_{\mathbb{CH}}}{(1+\cosh(r))^{m-1}}.$$

To see the second claim, observe that Lemma 3.4 implies that the new hypotheses on Σ encompasses the previous ones and ensures that Σ' has the property that $\mathbf{T}^\top$ on Σ'_s is of size $O((1-s)^2)$. It follows that in this case, (4.1) satisfies

$$g''_s = \frac{4}{(1-s^2)^2} g'_s + \frac{4s^4}{(1-s^2)^4} (i^*_{\Sigma'_s}\theta)^2 \leq \frac{4}{(1-s^2)^2} g'_s + C g'_s$$

for some constant $C > 0$. Hence, up to increasing C

$$(1 + Ce^{-2r}) \tilde{\Upsilon}^*_{p_0} g'_s \geq \frac{1 - Ce^{-2r}}{(1+\cosh(r))^2} g_r.$$

This means that up to increasing C,

$$|\Sigma'_s|_{\mathbb{R}} \geq \frac{(1 - Ce^{-2r})|\Sigma_r|_{\mathbb{CH}}}{(1+\cosh(r))^{m-1}}.$$

The second claim then follows as before by taking $s \to 1$ and $r \to \infty$. □

Theorem 1.4 is a consequence of Theorem 1.1 and the following proposition:

Proposition 4.3 *Let $\Sigma \subset \mathbb{CH}^{n+1}$ be an m-dimensional submanifold that is weakly C^1-asymptotically regular and weakly horizontal. Then*

$$\liminf_{t\to-\infty} \int_\Sigma \Phi^{t_0,p_0}_{m,1}(t,p) dVol_\Sigma(p) \geq \frac{Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Gamma; p_0)}{|\mathbb{S}^{m-1}|_{\mathbb{R}}}. \tag{4.2}$$

If Σ is C^1-asymptotically regular and strongly asymptotically horizontal, then

$$\lim_{t\to-\infty}\int_\Sigma \Phi_{m,1}^{t_0,p_0}(t,p)dVol_\Sigma(p)=\frac{Vol_{\partial_\infty\mathbb{CH}}(\partial_\infty\Gamma;p_0)}{|\mathbb{S}^{m-1}|_{\mathbb{R}}}. \tag{4.3}$$

Proof Let $\Sigma_r = \Sigma\cap\partial B_r^{\mathbb{CH}}(p_0)$. Observe that, by the definition of being weakly C^1-asymptotically regular there is an $R_0>0$ such that, for $r\geq R_0$, Σ meets $\partial B_r^{\mathbb{CH}}(p_0)$ transversally and so Σ_r is a smooth $(m-1)$-dimensional submanifold of $\partial B_r^{\mathbb{CH}}(p_0)$.

By Lemma 4.2, for any $\epsilon>0$, there is an $R_\epsilon>R_0$ so, for $r>R_\epsilon$,

$$(1-\epsilon)Vol_{\partial_\infty\mathbb{CH}}(\partial_\infty\Sigma;p_0)\leq\frac{|\Sigma_r|_{\mathbb{CH}}}{\sinh^{m-1}(r)}.$$

As $|\nabla_{\mathbb{CH}}\rho|\leq 1$, using the co-area formula, for $R>R_\epsilon$, one has

$$\begin{aligned}(1-\epsilon)Vol_{\partial_\infty\mathbb{CH}}(\partial_\infty\Sigma;p_0)&\int_R^\infty K_{m,1}(t_0-t,r)\sinh^{m-1}(r)dr\\ &\leq\int_{\Sigma\setminus B_R^{\mathbb{CH}}(p_0)}\Phi_{m,1}^{t_0,p_0}(t,p)dVol_\Sigma(p).\end{aligned}$$

As $K_{m,1}(t,r)=K_m(t,r)$, it follows from the proof of [5, Proposition 4.2] that

$$\lim_{t\to-\infty}\int_R^\infty K_{m,1}(t_0-t,r)\sinh^{m-1}(r)dr=|\mathbb{S}^{m-1}|_{\mathbb{R}}^{-1}.$$

Hence,

$$\begin{aligned}(1-\epsilon)\frac{Vol_{\partial_\infty\mathbb{CH}}(\partial_\infty\Sigma;p_0)}{|\mathbb{S}^{m-1}|_{\mathbb{R}}}&\leq\liminf_{t\to-\infty}\int_{\Sigma\setminus B_R^{\mathbb{CH}}(p_0)}\Phi_{m,1}^{t_0,p_0}(t,p)dVol_\Sigma(p)\\ &=\liminf_{t\to-\infty}\int_\Sigma\Phi_{m,1}^{t_0,p_0}(t,p)dVol_\Sigma(p)\end{aligned}$$

where the second equality again uses (4.2). Sending $\epsilon\to 0$ yields,

$$\frac{Vol_{\partial_\infty\mathbb{CH}}(\partial_\infty\Sigma;p_0)}{|\mathbb{S}^{m-1}|_{\mathbb{R}}}\leq\liminf_{t\to-\infty}\int_\Sigma\Phi_{m,1}^{t_0,p_0}(t,p)dVol_\Sigma(p),$$

which verifies the first claim.

Suppose Σ is C^1-asymptotically regular and strongly asymptotically horizontal. By Lemma 3.4, in the modified Bergman compactification, the compactified surface meets the ideal boundary orthogonally. It follows that if $\rho=\mathrm{dist}_{\mathbb{CH}}(p,p_0)$, then

$$\lim_{p\to\infty}|\nabla_\Sigma\rho|=1.$$

In particular, there is an $R'_\epsilon>0$ sufficiently large such that for $p\in\Sigma\setminus B_{R'_\epsilon}^{\mathbb{CH}}(p_0)$,

$$1\leq|\nabla_\Sigma\rho|^{-1}\leq 1+\epsilon.$$

Appealing to Lemma 4.2, up to increasing $R'_\epsilon>0$ one has for $r>R'_\epsilon$,

$$\frac{|\Sigma_r|_{\mathbb{CH}}}{\sinh^{m-1}(r)}\leq(1+\epsilon)Vol_{\partial_\infty\mathbb{CH}}(\partial_\infty\Sigma;p_0).$$

Hence, for $R>R'_\epsilon$, one has

$$\int_{\Sigma\setminus B_R^{\mathbb{CH}}(p_0)}\Phi_{m,1}^{t_0,p_0}(t,p)dVol_\Sigma(p)$$

$$\leq (1+\epsilon)^2 Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0) \int_R^\infty K_{m,1}(t_0 - t, r) \sinh^{m-1}(r) dr.$$

Arguing as above,

$$\limsup_{t \to -\infty} \int_\Sigma \Phi_{m,1}^{t_0, p_0}(t, p) dVol_\Sigma(p) \leq \frac{Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0)}{|\mathbb{S}^{m-1}|_\mathbb{R}}.$$

This proves that

$$\lim_{t \to -\infty} \int_\Sigma \Phi_{m,1}^{t_0, p_0}(t, p) dVol_\Sigma(p) = \frac{Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0)}{|\mathbb{S}^{m-1}|_\mathbb{R}}$$

verifying the second claim. □

We may now prove Theorem 1.4.

Proof of Theorem 1.4 By definition, for any fixed $p_0 \in \mathbb{CH}^{n+1}$,

$$\lambda_{\mathbb{CH}}[\Sigma] \geq \limsup_{t \to -\infty} \int_\Sigma \Phi_{m,1}^{0, p_0}(t, p) dVol_\Sigma(p).$$

Hence, Proposition 4.3 implies that

$$\lambda_{\mathbb{CH}}[\Sigma] \geq \frac{Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0)}{|\mathbb{S}^{m-1}|_\mathbb{R}}.$$

Taking the supremum over $p_0 \in \mathbb{CH}^{n+1}$ and using $\Gamma = \partial_\infty \Sigma$ yields,

$$\lambda_{\mathbb{CH}}[\Sigma] \geq \frac{\lambda_{CR}[\Gamma]}{|\mathbb{S}^{m-1}|_\mathbb{R}}.$$

This proves the first claim.

To see the second claim, observe that if Σ is weakly C^2-asymptotically regular, weakly asymptotically horizontal and minimal, then we may apply Theorem 1.1 to see that Σ is C^1-asymptotically regular and strongly asymptotically horizontal. In particular, Proposition 4.3 implies

$$\lim_{t \to -\infty} \int_\Sigma \Phi_{m,1}^{0, p_0}(t, p) dVol_\Sigma(p) = \frac{Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Sigma; p_0)}{|\mathbb{S}^{m-1}|_\mathbb{R}}.$$

Furthermore, Σ may be thought of as a static solution of mean curvature flow that, by Lemma 4.2, and the co-area formula has exponential volume growth. Hence, by [3, Theorem 1.1], for all $\tau > 0$, one has

$$\begin{aligned} \int_\Sigma \Phi_{m,1}^{0, p_0}(-\tau, p) dVol_\Sigma(p) &\leq \lim_{t \to -\infty} \int_\Sigma \Phi_{m,1}^{0, p_0}(t - \tau, p) dVol_\Sigma(p) \\ &= \frac{Vol_{\partial_\infty \mathbb{CH}}(\partial_\infty \Gamma; p_0)}{|\mathbb{S}^{m-1}|_\mathbb{R}} \leq \frac{\lambda_{CR}[\Gamma]}{|\mathbb{S}^{m-1}|_\mathbb{R}}. \end{aligned}$$

Taking the supremum over $\tau > 0$ and $p_0 \in \mathbb{CH}^{n+1}$ yields

$$\lambda_{\mathbb{CH}}[\Sigma] \leq \frac{\lambda_{CR}[\Gamma]}{|\mathbb{S}^{m-1}|_\mathbb{R}}.$$

Combined with the first claim this completes the proof. □

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Declaration

Conflict of interest The authors have no conflicts of interest.

Appendix A. Huisken monotonicity in $\mathbb{CH}^{n+1}$

In $\mathbb{CH}^{n+1}$, the monotonicity formula from [3] has a particularly simple form. We first record without proof the computation of the Hessian of a radial function.

Lemma A.1 *Let $\rho(p) = \mathrm{dist}_{\mathbb{CH}}(p, p_0)$ and suppose that $F : [0, \infty) \to \mathbb{R}$ is a C^2 function. If $f(p) = F(\rho(p))$ on $\mathbb{CH}^{n+1} \setminus \{p_0\}$, then*

$$\nabla^2_{\mathbb{CH}} f = \coth(\rho) F'(\rho) g_{\mathbb{CH}} + \left(F''(\rho) - \coth(\rho) F'(\rho)\right) d\rho^2 + \\ + \tanh(\rho) F'(\rho)(d\rho \circ J_{\mathbb{CH}})^2.$$

We may now specialize some of the conclusions of [3] to $\mathbb{CH}^{n+1}$.

Proposition A.2 *Suppose that $\{\Sigma_t\}_{t\in[0,T)}$ is a mean curvature flow in $\mathbb{CH}^{n+1}$ of m-dimensional submanifolds with exponential volume growth. For any $t_0 > 0$, $p_0 \in \mathbb{CH}^{n+1}$, and $t \in (0, \min\{t_0, T\})$ one has*

$$\frac{d}{dt}\int_{\Sigma_t} \Phi^{t_0,p_0}_{m,1} dV_{\Sigma_t} = -\int_{\Sigma_t} \left(\left| \frac{\nabla^\perp_{\Sigma_t} \Phi^{t_0,p_0}_{m,1}}{\Phi^{t_0,p_0}_{m,1}} - \mathbf{H}_{\Sigma_t} \right|^2 + Q^{t_0,p_0}_{m,1}(t, x, N_x\Sigma_t) \right) \Phi^{t_0,p_0}_{m,1} dV_{\Sigma_t}.$$

Here, if we set $k_{m,1}(t,r) = \log K_{m,1}(t,r)$ and $\rho(x) = \mathrm{dist}_{\mathbb{CH}}(x, p_0)$, then

$$Q^{t_0,p_0}_{m,1}(t, x, N_x\Sigma_t) = \left(k''_{m,1}(t,\rho) - \coth(\rho) k'_{m,1}(t,\rho)\right) |\nabla^\perp_\Sigma \rho|^2 \\ - \tanh(\rho) |(J_{\mathbb{CH}}(\nabla_{\mathbb{CH}}\rho))^\top|^2 k'_{m,1}(t,\rho) \geq 0.$$

Moreover, this inequality is strict somewhere unless Σ_t is an isotropic cone over p_0.

Proof Set $k = 2n + 2 - m$ and let $E_1, \ldots, E_k$ be an orthonormal basis of $N_x\Sigma_t$. By [3, Proposition 5.1] to obtain the above formulas, it is enough to compute

$$Q^{t_0,p_0}_{m,1}(t, x, N_x\Sigma_t) = \sum_{i=1}^k \nabla^2_{\mathbb{CH}} \Phi^{t_0,p_0}_{m,1}(E_i, E_i) \\ + ((m-1)\coth(\rho) - \Delta_{\mathbb{CH}}\rho)\, \partial_\rho \log \Phi^{t_0,p_0}_{m,1}.$$

By Lemma A.1,

$$\Delta_{\mathbb{CH}}\rho = (2n+1)\coth(\rho) + \tanh(\rho)$$

while, using $\log \Phi^{t_0,p_0}_{m,1}(t,x) = k_{m,1}(t_0 - t, \mathrm{dist}_{\mathbb{CH}}(x, p_0))$, one has

$$\sum_{i=1}^{k} \nabla^2_{\mathbb{CH}} \Phi^{t_0,p_0}_{m,1}(E_i, E_i) = k \coth(\rho) k'_{m,1} + (k''_{m,1} - \coth(\rho) k'_{m,1}) |\nabla^{\perp}_{\Sigma} \rho|^2 + \tanh(\rho) k'_{m,1} |(J_{\mathbb{CH}}(\nabla_{\mathbb{CH}} \rho))^{\perp}|^2.$$

The formula for $Q^{t_0,p_0}_{m,1}$ follows from this. The inequality is an immediate consequence of properties of $K_{m,1}$ – see [3, Section 5]. The strictness of the inequality unless Σ_t is a geodesic cone over p_0 follows from [3, Lemma 5.3]. Finally, if Σ is a geodesic cone, then $\nabla_{\mathbb{CH}}\rho$ is tangent to Σ_t. If the inequality is not strict, then $J_{\mathbb{CH}}(\nabla_{\mathbb{CH}}\rho)$ is orthogonal to Σ_t. If $\Upsilon : \mathbb{CH}^{n+1} \to \mathbb{B}^{2n+2}$ is a Bergman compactification with $\Upsilon(p_0) = \mathbf{0}$, then this is equivalent to $\Upsilon(\Sigma_t)$, which is a Euclidean cone over $\mathbf{0}$, being orthogonal to $\mathbf{T}$. This only occurs when the cone is isotropic. □

Appendix B. Geometric computations

B.1. Rank one deformation

Let (M, g) be a Riemannian manifold, τ a smooth one-form and α a smooth function. When $|\tau|^2_g \alpha > -1$, the Riemannian metric

$$h = g + \alpha \tau \otimes \tau, \tag{B.1}$$

is called a *rank one deformation of* g. Let ∇^g and ∇^h denote the Levi-Civita connections of g and h and let $C(X, Y)$ be the $(1, 2)$ tensor field

$$C(X, Y) = \nabla^h_X Y - \nabla^g_X Y.$$

Using the Koszul formula we compute that

$$h(C(X, Y), Z) = g(C(X, Y), Z) + \alpha \tau(C(X, Y)) \tau(Z) = c(X, Y, Z)$$

where $c(X, Y, Z)$ satisfies

$$c(X, Y, Z) = \frac{1}{2}(\nabla^g_X \alpha\tau)(Y)\tau(Z) + \frac{1}{2}(\nabla^g_Y \alpha\tau)(X)\tau(Z) + \frac{1}{2}\alpha \nabla_X \tau(Z)\tau(Y) + \frac{1}{2}\alpha \nabla_Y \tau(Z)\tau(X) - \frac{1}{2}(\nabla^g_Z \alpha\tau)(X)\tau(Y) - \frac{1}{2}\alpha \nabla^g_Z \tau(Y)\tau(X).$$

Let $\mathbf{T}$ be the vector field satisfying

$$g(\mathbf{T}, Z) = \tau(Z).$$

Choose orthonormal vectors $E_1, \ldots, E_n$ orthogonal to $\mathbf{T}$ so $\{E_1, \ldots, E_n, \mathbf{T}\}$ spans T_pM. It follows that

$$C(X, Y) = \frac{|\mathbf{T}|^{-2}_g}{1 + \alpha |\mathbf{T}|^2_g} c(X, Y, \mathbf{T})\mathbf{T} + \sum_{i=1}^{n} c(X, Y, E_i) E_i$$

where c is a $(0, 3)$ tensor field symmetric in the first two entries given by

$$c(X, Y, \mathbf{T}) = \frac{1}{2}\big(g(\nabla^g_X \alpha \mathbf{T}, Y)|\mathbf{T}|^2_g + g(\nabla^g_Y \alpha \mathbf{T}, X)|\mathbf{T}|^2_g + \alpha g(\nabla^g_X \mathbf{T}, \mathbf{T}) g(\mathbf{T}, Y)$$

$$+ \alpha g(\nabla^g_Y \mathbf{T}, \mathbf{T}) g(\mathbf{T}, X) - g(\nabla^g_{\mathbf{T}} \alpha \mathbf{T}, X) g(\mathbf{T}, Y) - \alpha g(\nabla^g_{\mathbf{T}} \mathbf{T}, Y) g(\mathbf{T}, X)\big)$$

and

$$c(X, Y, E_i) = \frac{1}{2}\alpha g(\nabla^g_X \mathbf{T}, E_i) g(\mathbf{T}, Y) + \frac{1}{2}\alpha g(\nabla^g_Y \mathbf{T}, E_i) g(\mathbf{T}, X) - \frac{1}{2} g(\nabla^g_{E_i} \alpha \mathbf{T}, X) g(\mathbf{T}, Y) - \frac{1}{2} g(\nabla^g_{E_i} \alpha \mathbf{T}, Y) g(\mathbf{T}, X).$$

B.2. Second fundamental form

We compare the second fundamental forms computed for certain metrics, g and h, that are related by a rank one deformation. Let $\Sigma \subset M$ be a submanifold. Observe that for X and Y tangent to Σ one has

$$h(X, Y) = g(X, Y) + \tau(X)\tau(Y) = g(X, Y) + g(\mathbf{T}, X) g(\mathbf{T}, Y),$$
$$h_\Sigma(X, Y) = g_\Sigma(X, Y) + g_\Sigma(\mathbf{T}^\top, X) g_\Sigma(\mathbf{T}^\top, Y)$$

where $\mathbf{T}^\top$ is the g-tangential component of $\mathbf{T}$. In particular, h_Σ is a rank one deformation of g_Σ. Denote by $\nabla^{\Sigma,g}$ and $\nabla^{\Sigma,h}$ the induced connections on Σ.

Let $\mathbf{T}^N$ be the g-normal component of $\mathbf{T}$ to Σ and let $\mathbf{T}^{\hat{N}}$ be the h-normal component. We have

$$\mathbf{T}^{\hat{N}} = \mathbf{T}^N - \frac{|\mathbf{T}^N|_g^2}{1 + |\mathbf{T}^\top|_g^2} \mathbf{T}^\top = \mathbf{T} - \frac{1 + |\mathbf{T}|_g^2}{1 + |\mathbf{T}^\top|_g^2} \mathbf{T}^\top.$$

Now choose $N_1, \ldots, N_l$ to be g-unit length tangent vectors that are g-orthogonal to $\mathbf{T}$ and Σ. These are also of h-unit length and are h-orthogonal to $\mathbf{T}$ and Σ. Likewise, let $E_1, \ldots, E_k$ be g-unit length vectors tangent to Σ and g-orthogonal to $\mathbf{T}$. For a vector $\mathbf{V} \in T_p M$ with $p \in \Sigma$, let us denote

$$\mathbf{V}^{\tilde{N}} = \mathbf{V}^N - g(\mathbf{V}^N, \mathbf{T}) \frac{\mathbf{T}^N}{|\mathbf{T}^N|_g^2} = \sum_{i=1}^{l} g(\mathbf{V}, N_i) N_i.$$

So $\mathbf{V}^{\tilde{N}}$ is both h and g-orthogonal to Σ and $\mathbf{T}$. In particular,

$$g(\mathbf{V}^{\tilde{N}}, \mathbf{T}^{\hat{N}}) = g(\mathbf{V}^N, \mathbf{T}^{\hat{N}}) = g(\mathbf{V}^{\tilde{N}}, \mathbf{T}) = 0.$$

Proposition B.1 *Let (M, g) be a Riemannian manifold and $\tau = g(\mathbf{T}, \cdot)$ a smooth one form. Suppose that there is a $(1,1)$-tensor field, $\mathbf{a}$, satisfying*

$$\nabla^g_Z \mathbf{T} = -\mathbf{a}(Z) \text{ and } g(\mathbf{a}(X), Y) = -g(X, \mathbf{a}(Y)).$$

If h is the rank one deformation of g by τ and $\Sigma \subset M$ a submanifold, then one has the following relationship between the second fundamental forms of Σ,

$$(\mathbf{A}^h_\Sigma(X, Y))^{\tilde{N}} = (\mathbf{A}^g_\Sigma(X, Y))^{\tilde{N}} - g(\mathbf{T}, Y)(\mathbf{a}(X))^{\tilde{N}} - g(\mathbf{T}, X)(\mathbf{a}(Y))^{\tilde{N}} \text{ and}$$

$$g(\mathbf{A}^h_\Sigma(X, Y), \mathbf{T}^{\hat{N}}) = g(\mathbf{A}^g_\Sigma(X, Y), \mathbf{T}) - \frac{1 + |\mathbf{T}^\top|_g^2}{1 + |\mathbf{T}|_g^2} \big(g(\mathbf{T}^{\hat{N}}, \mathbf{a}(X)) g(\mathbf{T}, Y) + g(\mathbf{T}^{\hat{N}}, \mathbf{a}(Y)) g(\mathbf{T}, X)\big).$$

Proof The computations of Section B.1 yield, for X and Y tangent to Σ, and N_j g-orthogonal to Σ and $\mathbf{T}$,

$$\begin{aligned} h(\nabla^h_X Y, N_j) &= g(\nabla^g_X Y, N_j) + c(X, Y, N_j) \\ &= g(\mathbf{A}^g_\Sigma(X, Y), N_j) + \frac{1}{2} g(\nabla^g_X \mathbf{T}, N_j) g(\mathbf{T}, Y) + \frac{1}{2} g(\nabla^g_Y \mathbf{T}, N_j) g(\mathbf{T}, X) \\ &\quad - \frac{1}{2} g(\nabla^g_{N_j} \mathbf{T}, X) g(\mathbf{T}, Y) - \frac{1}{2} g(\nabla^g_{N_j} \mathbf{T}, Y) g(\mathbf{T}, X). \end{aligned}$$

The additional hypothesis on $\mathbf{T}$ and $\mathbf{a}$ imply

$$h(\nabla^h_X Y, N_j) = g(\mathbf{A}^g_\Sigma(X, Y), N_j) - g(\mathbf{a}(X), N_j) g(\mathbf{T}, Y) - g(\mathbf{a}(Y), N_j) g(\mathbf{T}, X).$$

This immediately yields the first formula.

It directly follows from Section B.1 that

$$\begin{aligned} h(\nabla^h_X Y, \mathbf{T}) &= (1 + |\mathbf{T}|^2_g) g(\nabla^g_X Y, \mathbf{T}) + c(X, Y, \mathbf{T}) \\ &= (1 + |\mathbf{T}|^2_g)\big(g(\nabla^{\Sigma, g}_X Y, \mathbf{T}^\top) + g(\mathbf{A}^g_\Sigma(X, Y), \mathbf{T}\big) + c(X, Y, \mathbf{T}) \end{aligned}$$

where, the properties of $\mathbf{a}$ and $\mathbf{T}$, yield

$$\begin{aligned} c(X, Y, \mathbf{T}) &= \frac{1}{2}\big(g(\nabla^g_X \mathbf{T}, Y)|\mathbf{T}|^2_g + g(\nabla^g_Y \mathbf{T}, X)|\mathbf{T}|^2_g + g(\nabla^g_X \mathbf{T}, \mathbf{T}) g(\mathbf{T}, Y) \\ &\quad + g(\nabla^g_Y \mathbf{T}, \mathbf{T}) g(\mathbf{T}, X) - g(\nabla^g_{\mathbf{T}} \mathbf{T}, X) g(\mathbf{T}, Y) - g(\nabla^g_{\mathbf{T}} \mathbf{T}, Y) g(\mathbf{T}, X)\big) \\ &= g(\mathbf{a}(X), \mathbf{T}) g(\mathbf{T}, Y) + g(\mathbf{a}(Y), \mathbf{T}) g(\mathbf{T}, X). \end{aligned}$$

Likewise, treating h_Σ as a rank one deformation of g_Σ one has

$$\begin{aligned} h(\nabla^{\Sigma, h}_X Y, \mathbf{T}^\top) &= (1 + |\mathbf{T}^\top|^2_g) g(\nabla^{\Sigma, g}_X Y, \mathbf{T}^\top) + c_\Sigma(X, Y, \mathbf{T}^\top) \\ &= (1 + |\mathbf{T}^\top|^2_g) g(\nabla^{\Sigma, g}_X Y, \mathbf{T}) + c_\Sigma(X, Y, \mathbf{T}^\top). \end{aligned}$$

As $(\nabla^g_V W)^\top = \nabla^{g, \Sigma}_V W$, for tangential V, W, the definition of c_Σ gives

$$\begin{aligned} c_\Sigma(X, Y, \mathbf{T}^\top) &= \frac{1}{2}\Big(\big(g(\nabla^g_X \mathbf{T}^\top, Y) + g(\nabla^g_Y \mathbf{T}^\top, X)\big)|\mathbf{T}^\top|^2_g + g(\nabla^g_X \mathbf{T}^\top, \mathbf{T}^\top) g(\mathbf{T}^\top, Y) \\ &\quad + g(\nabla^g_Y \mathbf{T}^\top, \mathbf{T}^\top) g(\mathbf{T}^\top, X) \\ &\quad - g(\nabla^g_{\mathbf{T}^\top} \mathbf{T}^\top, X) g(\mathbf{T}^\top, Y) - g(\nabla^g_{\mathbf{T}^\top} \mathbf{T}^\top, Y) g(\mathbf{T}^\top, X)\Big). \end{aligned}$$

For tangent vectors V, W one has

$$g(\nabla^g_V \mathbf{T}^\top, W) = g(\nabla^g_V (\mathbf{T} - \mathbf{T}^N), W) = g(\nabla^g_V \mathbf{T}, W) + g(\mathbf{T}, \mathbf{A}^g_\Sigma(V, W)).$$

Hence, the properties of $\mathbf{a}$ and $\mathbf{T}$, yield

$$\begin{aligned} c_\Sigma(X, Y, \mathbf{T}^\top) &= \frac{1}{2}\Big(\big(g(\nabla^g_X \mathbf{T}, Y) + g(\nabla^g_Y \mathbf{T}, X)\big)|\mathbf{T}^\top|^2_g + g(\nabla^g_X \mathbf{T}, \mathbf{T}^\top) g(\mathbf{T}, Y) \\ &\quad + g(\nabla^g_Y \mathbf{T}, \mathbf{T}^\top) g(\mathbf{T}, X) - g(\nabla^g_{\mathbf{T}^\top} \mathbf{T}, X) g(\mathbf{T}, Y) - g(\nabla^g_{\mathbf{T}^\top} \mathbf{T}, Y) g(\mathbf{T}, X)\Big) \\ &\quad + g(\mathbf{T}, \mathbf{A}^g_\Sigma(X, Y))|\mathbf{T}^\top|^2_g \\ &= g(\mathbf{a}(X), \mathbf{T}^\top) g(\mathbf{T}, Y) + g(\mathbf{a}(Y), \mathbf{T}^\top) g(\mathbf{T}, X) + g(\mathbf{T}, \mathbf{A}^g_\Sigma(X, Y))|\mathbf{T}^\top|^2_g. \end{aligned}$$

Hence, the definition of $\mathbf{T}^{\hat{N}}$ yields

$$h(\nabla^h_X Y, \mathbf{T}^{\hat{N}}) = h(\nabla^h_X Y, \mathbf{T}) - \frac{1+|\mathbf{T}|^2_g}{1+|\mathbf{T}^\top|^2_g} h(\nabla^{\Sigma,h}_X Y, \mathbf{T}^\top)$$
$$= \frac{1+|\mathbf{T}|^2_g}{1+|\mathbf{T}^\top|^2_g} g(\mathbf{T}, \mathbf{A}^g_\Sigma(X,Y)) - g(\mathbf{a}(X), \mathbf{T}^{\hat{N}}) g(\mathbf{T}, Y) - g(\mathbf{a}(Y), \mathbf{T}^{\hat{N}}) g(\mathbf{T}, X).$$

As $\mathbf{T}^{\hat{N}}$ is h-orthogonal to Σ, the formula for h in terms of g yields

$$h(\nabla^h_X Y, \mathbf{T}^{\hat{N}}) = h(\mathbf{A}^h_\Sigma(X,Y), \mathbf{T}^{\hat{N}}) = \frac{1+|\mathbf{T}|^2_g}{1+|\mathbf{T}^\top|^2_g} g(\mathbf{A}^h_\Sigma(X,Y), \mathbf{T}^{\hat{N}}).$$

Combining this with the previous computation yields the second formula. □

Corollary B.2 *The relationship between the mean curvatures is given by*

$$(\mathbf{H}^h_\Sigma)^{\tilde{N}} = (\mathbf{H}^g_\Sigma)^{\tilde{N}} - \frac{(\mathbf{A}^g_\Sigma(\mathbf{T}^\top, \mathbf{T}^\top))^{\tilde{N}} + 2(\mathbf{a}(\mathbf{T}^\top))^{\tilde{N}}}{1+|\mathbf{T}^\top|^2_g} \text{ and}$$

$$g(\mathbf{H}^h_\Sigma, \mathbf{T}^{\hat{N}}) = g(\mathbf{H}^g_\Sigma, \mathbf{T}) - \frac{g(\mathbf{A}^g_\Sigma(\mathbf{T}^\top, \mathbf{T}^\top), \mathbf{T})}{1+|\mathbf{T}^\top|^2_g} - \frac{2g(\mathbf{a}(\mathbf{T}^\top), \mathbf{T})}{1+|\mathbf{T}|^2_g}.$$

Proof At a point, $p \in \Sigma$, where $\mathbf{T}^\top(p) = \mathbf{0}$, we immediately see that $\mathbf{H}^h_\Sigma(p) = \mathbf{H}^g_\Sigma(p)$. This verifies both equations in this case.

When $\mathbf{T}^\top(p) \neq 0$, we may choose a g-orthonormal basis of $T_p\Sigma$ of the form $E_1' = |\mathbf{T}^\top|^{-1}_g \mathbf{T}^\top, E_2, \dots, E_k$. Clearly, $g(\mathbf{T}, E_j) = 0$ for $2 \leq j \leq k$. Set

$$\beta = \frac{1}{1+|\mathbf{T}^\top|^2_g + \sqrt{1+|\mathbf{T}^\top|^2_g}} \text{ and observe } 2\beta - |\mathbf{T}^\top|^2_g \beta^2 = \frac{1}{1+|\mathbf{T}^\top|^2_g}.$$

We obtain an h-orthonormal basis, $E_1, \dots, E_k$ by taking

$$E_1 = \frac{\mathbf{T}^\top}{|\mathbf{T}^\top|_g \sqrt{1+|\mathbf{T}^\top|^2_g}} = (1-\beta|\mathbf{T}^\top|^2_g) E_1'.$$

Using this h-orthonormal basis and Proposition B.1 one obtains

$$(\mathbf{H}^h_\Sigma)^{\tilde{N}} = (\mathbf{H}^g_\Sigma)^{\tilde{N}} - (2\beta - |\mathbf{T}^\top|^2_g \beta^2)(\mathbf{A}^g_\Sigma(\mathbf{T}^\top, \mathbf{T}^\top))^{\tilde{N}} - \frac{2(\mathbf{a}(\mathbf{T}^\top))^{\tilde{N}}}{1+|\mathbf{T}^\top|^2_g}$$
$$= (\mathbf{H}^g_\Sigma)^{\tilde{N}} - \frac{(\mathbf{A}^g_\Sigma(\mathbf{T}^\top, \mathbf{T}^\top))^{\tilde{N}}}{1+|\mathbf{T}^\top|^2_g} - \frac{2(\mathbf{a}(\mathbf{T}^\top))^{\tilde{N}}}{1+|\mathbf{T}^\top|^2}.$$

This gives the first equation.

In the same manner, one obtains

$$g(\mathbf{H}^h_\Sigma, \mathbf{T}^{\hat{N}}) = g(\mathbf{H}^h_\Sigma, \mathbf{T}) - (2\beta - |\mathbf{T}^\top|^2_g \beta^2) g(\mathbf{A}^g_\Sigma(\mathbf{T}^\top, \mathbf{T}^\top), \mathbf{T}) - \frac{2g(\mathbf{T}^{\hat{N}}, \mathbf{a}(\mathbf{T}^\top))}{1+|\mathbf{T}|^2_g}$$
$$= g(\mathbf{H}^g_\Sigma, \mathbf{T}) - \frac{g(\mathbf{A}^g_\Sigma(\mathbf{T}^\top, \mathbf{T}^\top), \mathbf{T})}{1+|\mathbf{T}^\top|^2_g} - \frac{2g(\mathbf{T}, \mathbf{a}(\mathbf{T}^\top))}{1+|\mathbf{T}|^2_g}.$$

Here the second equality used the anti-symmetry of **a** to see that

$$g(\mathbf{T}^{\hat{N}}, \mathbf{a}(\mathbf{T}^{\top})) = g(\mathbf{T}, \mathbf{a}(\mathbf{T}^{\top})) - \frac{1 + |\mathbf{T}|_g^2}{1 + |\mathbf{T}^{\top}|_g^2} g(\mathbf{T}^{\top}, \mathbf{a}(\mathbf{T}^{\top})) = g(\mathbf{T}, \mathbf{a}(\mathbf{T}^{\top})).$$

□

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