

Colding-Minicozzi Entropy and Complexity of Submanifolds

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Introduction

Intuitively, a sea urchin is more complicated than a billiard ball. However, it is not so easy to see how to formalize and quantify this distinction, i.e., what constitute good measures of the complexity of geometric objects. There are many different approaches, but on a high level, satisfactory answers should all involve something with the following properties:

1. Respects the geometric symmetries;
2. Collections of objects, taken together, should be more complex than the constituent elements;
3. The geometrically “simplest” elements in natural subclasses should be extrema of the quantity.

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In addition to these general properties, it proves helpful to add a more technical and specific property:

4. Monotonic along a geometric heat flow.

This is justified by the fact that such flows tend to simplify the geometry of the objects being evolved.

In this article we focus on the geometry of submanifolds of Euclidean space and the measure of complexity we introduce will be associated with the mean curvature flow. One aspect we highlight is topological. On the one hand, mean curvature flow, like many geometric flows, should simplify the topology along with the geometry. On the other hand, we observe that, in many cases submanifolds with low complexity in the sense we discuss must be topologically simple.

Mean Curvature Flow

Surface tension is the force that causes the surface of a liquid to behave like a stretched elastic membrane and drives the interface to minimize its surface area. The force induced by surface tension on the interface

is mathematically characterized by its mean curvature, which is the sum of the principle curvatures of the geometric surface corresponding to the interface.

This force also gives rise to a potent method for studying geometric and topological properties of submanifolds, especially embedded hypersurfaces. Indeed, by considering the (negative) gradient flow for area one obtains a dynamical process in which submanifolds of Euclidean space, e.g., curves or surfaces, continuously reduce their areas by deforming in the direction of steepest descent. A consequence of the first variation of area formula is that this gradient flow corresponds to the *mean curvature flow*. This is the flow in which points of the evolving submanifolds move with (normal) velocity determined by the mean curvature of the submanifold they lie in. That is,

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{H}$$

where $\mathbf{H}$ is the mean curvature of the appropriate submanifold in the flow and $\mathbf{x}$ is the position vector. In general, when e_i is an orthonormal frame of the tangent space of a submanifold, its mean curvature satisfies $\mathbf{H} = \sum_i (D_{e_i} e_i)^\perp$ where $\perp$ means the component orthogonal to the submanifold. For instance, the sphere of radius R in $\mathbb{R}^{n+1}$ centered at the origin satisfies $\mathbf{H} = -nR^{-2}\mathbf{x}$.

There are several equivalent characterizations of mean curvature flow. For example, a one-parameter family of n -dimensional submanifolds $t \mapsto \Sigma_t \subset \mathbb{R}^N$ is a mean curvature flow if and only if the coordinate functions, $\mathbf{x}^i$, of $\mathbb{R}^N$, $i = 1, \dots, N$, restricted to the submanifolds Σ_t , satisfy the heat equation

$$\frac{\partial \mathbf{x}^i}{\partial t} = \Delta_{\Sigma_t} \mathbf{x}^i.$$

Despite appearances, these equations are nonlinear as both the $\mathbf{x}^i$ and Laplacian, Δ_{Σ_t} , depend on Σ_t .

A direct computation shows that the function $Q = |\mathbf{x}|^2 + 2nt$ satisfies the same heat equation on the evolving submanifolds as the $\mathbf{x}^i$. If Q is bounded initially, then, by the parabolic maximum principle, it remains bounded with the same bound as long as the flow exists. Moreover, this is only a finite amount of time because $Q \geq 2nt$ would otherwise grow without bound. Thus, the flow of an initially *closed*, i.e.,

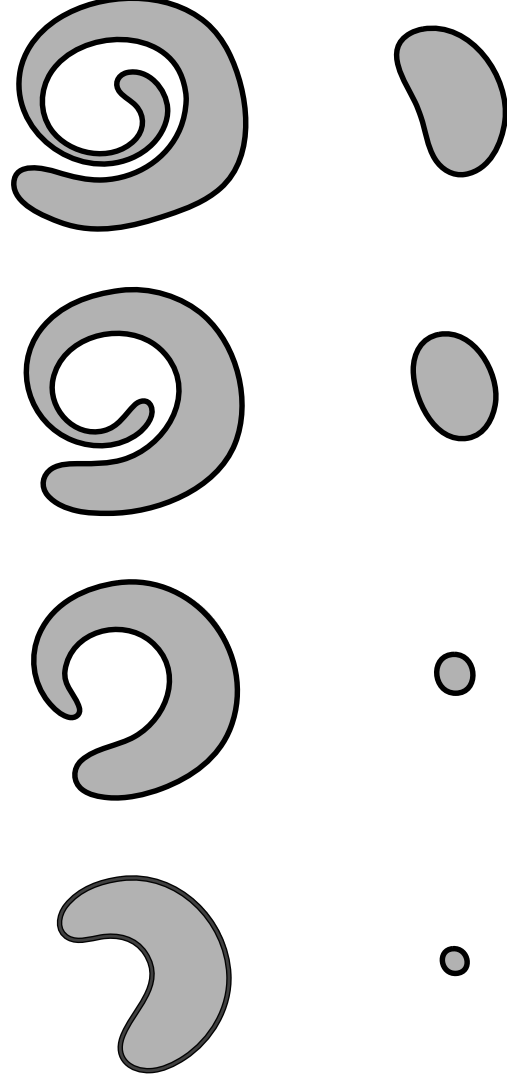


Figure 1: The evolution of a simple closed curve under curve shortening flow, i.e., mean curvature flow of curves in the plane. Here a non-convex region becomes convex before disappearing in a round point.

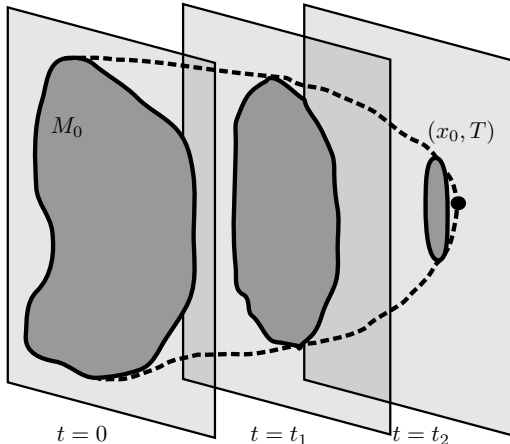


Figure 2: A smooth flow that disappears at $t = T$.

compact and without boundary, submanifold must become singular in finite time, either by becoming extinct or by forming other singularities.

When the initial submanifold is a hypersurface bounding a compact convex region, a more sophisticated application of the maximum principle shows this condition is preserved. Moreover, the celebrated results of Gage-Hamilton [GH86] and Huisken [Hui84] imply that the shapes of the evolving hypersurfaces approach the shape of a round sphere rapidly as the flow shrinks to a point. When $n \geq 2$ and the initial hypersurface is not convex, it is possible for the flow to form a singularity without disappearing – the prototypical example of this is the formation of a neck-pinch. In the latter case, it is possible to continue the flow using a weak formulation of the evolution.

Colding-Minicozzi Entropy

We now introduce a quantity that quantifies the simplifying process provided by the mean curvature flow of closed convex hypersurfaces. As we will see, this quantity also satisfies the properties laid out in the introduction and so provides a plausible measure of complexity of submanifolds. It was introduced by

Colding-Minicozzi in [CM12] where they named it *entropy*. Its monotonicity relies on a property of mean curvature flow first observed by Huisken [Hui90].

Recall that the heat kernel of $\mathbb{R}^n$ has the profile of a time-varying Gaussian

$$K_n(t, \mathbf{x}) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}} \quad \text{for } t > 0.$$

Let $t \mapsto \Sigma_t \subset \mathbb{R}^N$ be an n -dimensional mean curvature flow with starting time $t = 0$ and extend K_n to $\mathbb{R}^N$ by the same formula. For $\mathbf{x}_0 \in \mathbb{R}^N$ and $t_0 > 0$, the time derivative of the integral of $K_n(t_0 - t, \cdot - \mathbf{x}_0)$ over Σ_t is nonpositive. This is a consequence of the Huisken monotonicity formula [Hui90] whose proof uses the fact that the restriction of the coordinate functions of the ambient Euclidean space satisfy the heat equation on the evolving submanifolds as well as convexity properties of K_n .

Translating the flow $t \mapsto \Sigma_t$ by a vector in space-time, $(s_0, \mathbf{y}_0)$, produces a new mean curvature flow $t \mapsto \Sigma_{t-s_0} + \mathbf{y}_0$ of the same shape. The same holds for parabolic rescalings of the flow, i.e., the flow $t \mapsto \rho \Sigma_{\rho^{-2}t}$ for $\rho > 0$. Integrating the kernels $K_n(t_0 - t, \cdot - \mathbf{x}_0)$ over these transformed flows essentially corresponds to changing t_0 and $\mathbf{x}_0$ in the integrals over the original flow. Here for a submanifold, Σ , $\rho \Sigma + \mathbf{x}_0$ denotes the submanifold obtained by dilating Σ by factor ρ followed by translation by vector $\mathbf{x}_0$.

One way to unify this family of quantities and also account for the symmetries of the mean curvature flow is the approach taken by Colding-Minicozzi [CM12]. They define the *entropy*, $\lambda[\Sigma]$, of an n -dimensional submanifold $\Sigma \subset \mathbb{R}^N$ by

$$\lambda[\Sigma] = \sup_{\mathbf{x}_0, t_0} \int_{\Sigma} K_n(t_0, \mathbf{x} - \mathbf{x}_0) d\text{vol}_{\Sigma}(\mathbf{x}).$$

Thus, λ is invariant under rigid motions and dilations of Σ and, by the Huisken monotonicity formula, is monotone decreasing along the mean curvature flow.

It is convenient at times to express the entropy in terms of the Gaussian integral centered at the origin with scale 1

$$F[\Sigma] = \int_{\Sigma} K_n(1, \mathbf{x}) d\text{vol}_{\Sigma} = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\Sigma} e^{-\frac{|\mathbf{x}|^2}{4}} d\text{vol}_{\Sigma}.$$

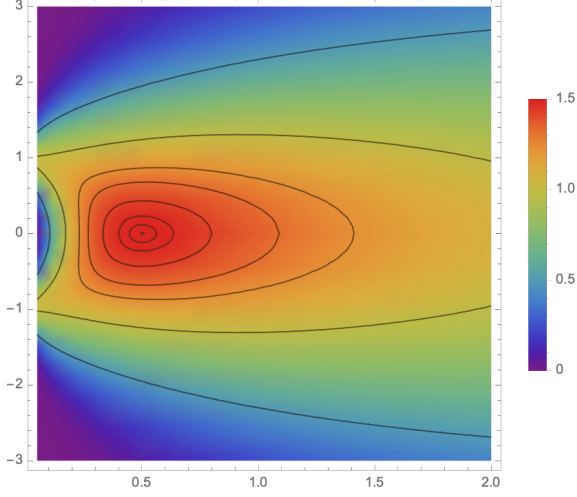


Figure 3: A plot of the integrals of $K_1(t_0, \cdot - \mathbf{x}_0)$ over the unit circle. The horizontal axis is t_0 and the vertical axis represents $\mathbf{x}_0 = y\mathbf{e}_2$. Note $\lambda[\mathbb{S}^1] \approx 1.52$.

Namely, by a change of variables,

$$\lambda[\Sigma] = \sup_{\mathbf{y}, \rho} F[\rho\Sigma + \mathbf{y}]$$

as $(\mathbf{y}, \rho)$ ranges over all points of $\mathbb{R}^N \times (0, \infty)$.

Fix a singularity of a mean curvature flow. Rescaling the flow parabolically about the singularity, which we may assume, by translation, is at the space-time origin, and simultaneously reparametrizing time gives a new flow, called the rescaled mean curvature flow, which satisfies the equation

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{H} - \frac{\mathbf{x}^\perp}{2}.$$

The static solutions of this flow satisfy $\mathbf{H} - \frac{\mathbf{x}^\perp}{2} = \mathbf{0}$ and describe the limiting behavior of the flow. A submanifold, Σ , satisfying this limiting equation is called a *self-shrinker*, because $t \mapsto \sqrt{-t}\Sigma$ is a solution of mean curvature flow. The simplest non-trivial example of a self-shrinker is the round sphere of radius $\sqrt{2n}$ in $\mathbb{R}^{n+1}$,

$$\sqrt{2n}\mathbb{S}^n = \{x_1^2 + \dots + x_{n+1}^2 = 2n\}.$$

For the flow of convex hypersurfaces discussed in the previous section, this is exactly the self-shrinker one obtains at the unique terminal singularity.

As entropy is invariant under translations and dilations, it is also monotone decreasing along the rescaled mean curvature flow. Thus, the entropy of the initial submanifold bounds the entropy of the limiting self-shrinkers associated to the singularity.

Starting from a hypersurface that bounds a compact convex region, the flow becomes round as it reaches its extinction time. Monotonicity implies the entropy of the initial hypersurface is larger than or equal to that of a round hypersphere. Hence, round hyperspheres minimize entropy among all closed convex hypersurfaces. In the following sections, we study the minimizers of entropy in more general settings.

Entropy Minimizers

We now explore the extrema of entropy in two different classes of submanifolds.

General submanifolds

For $\Sigma \subset \mathbb{R}^N$ an n -dimensional submanifold, one characterization of the tangent space at $p \in \Sigma$ is via

$$\lim_{\rho \rightarrow \infty} \rho(\Sigma - p) = T_p \Sigma$$

where the convergence may be interpreted in various senses. Geometrically this means that if we zoom in on smaller and smaller scales about $p \in \Sigma$, then the zoomed in surface becomes closer and closer to the tangent plane.

An immediate consequence of this is that

$$\lambda[\Sigma] \geq \limsup_{\rho \rightarrow \infty} F[\rho(\Sigma - p)] = F[T_p \Sigma].$$

A straightforward calculus exercise shows us that $F[T_p \Sigma] = 1$. Likewise, for any n -plane, $P \subset \mathbb{R}^N$,

$$\lambda[P] = F[P - q] = 1$$

where q is a point on P . In other words,

$$\lambda[\Sigma] \geq \lambda[P] = 1$$

where P is an n -plane. That is, planes are both the geometrically simplest elements and absolute minima of λ within the class of submanifolds. A related observation is that in the larger class of immersed submanifolds, any submanifold that is not embedded has entropy at least 2.

Closed hypersurfaces

When restricting to the smaller class of *closed hypersurfaces*, i.e., those that are compact and without boundary and of codimension one, the geometrically simplest elements become the round hyperspheres. In this case, one may, with more work, verify that

$$\lambda[\mathbb{S}^n] = F[\sqrt{2n}\mathbb{S}^n].$$

It was computed by Stone that $\lambda[\mathbb{S}^1] = \sqrt{\frac{2\pi}{e}} < 2$, $\lambda[\mathbb{S}^2] = \frac{4}{e}$ and, more generally,

$$\lambda[\mathbb{S}^1] > \lambda[\mathbb{S}^2] > \lambda[\mathbb{S}^3] > \dots > \lim_{k \rightarrow \infty} \lambda[\mathbb{S}^k] = \sqrt{2}.$$

With this in mind, Colding-Ilmanan-Minicozzi-White conjectured in [CIMW13] that round hyperspheres minimize entropy among closed hypersurfaces. Precisely, if $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$.

Some immediate evidence for the conjecture was provided by earlier work of Gage-Hamilton [GH86] and Grayson [Gra87]. Specifically, they showed that curve shortening flow – i.e., one dimensional mean curvature flow – evolves a closed simple curve into a round point in finite time. Thus, the reasoning used for convex hypersurfaces also shows the conjecture of Colding et al., [CIMW13], for simple closed curves. If one tries to reproduce this argument in higher dimensions one encounters several difficulties. The first is that there are many more singularity models. Another, more subtle, issue is that unlike what happens with curve shortening flow, a singularity may form before the flow disappears.

Additional evidence for the general conjecture was provided by work of White [Whi05], which in turn built on earlier work of Brakke and Allard. In particular, one may conclude from these results that any singularity model has entropy a definite amount

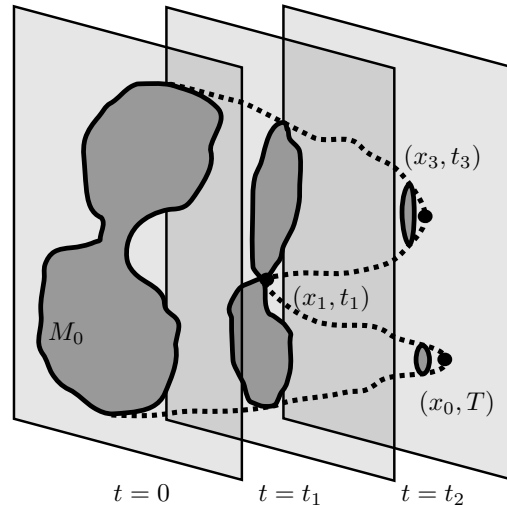


Figure 4: A schematic picture of a non-terminal singularity at $t = t_1$. The (weak) flow becomes extinct at $t = T$.

above 1 and, because the flow of any closed hypersurface forms a singularity in finite time, the same is true of any closed hypersurface. Hence, there is an $\epsilon = \epsilon(n) > 0$, so that when $\Sigma \subset \mathbb{R}^{n+1}$ is closed,

$$\lambda[\Sigma] \geq 1 + \epsilon.$$

This shows that a quantitative difference in the properties of entropy appears when restricting to the class of closed hypersurfaces.

The singularity models of mean curvature flow are the non-flat self-shrinkers. Hence, the optimal inequality is a consequence of the following more ambitious conjecture of Colding et al., [CIMW13]: If $\Sigma \subset \mathbb{R}^{n+1}$ is a non-flat self-shrinker, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$. In the same paper, they prove this is true for *closed* self-shrinkers. This stronger conjecture was resolved in [BW17] for $n = 2$, but remains open for $n \geq 3$.

At first glance, one may think that the result of Colding et al., [CIMW13], on the entropy of closed self-shrinkers suffices to prove their initial conjecture. However, there is, *a priori*, no reason for flow of closed hypersurfaces to develop a closed singular-

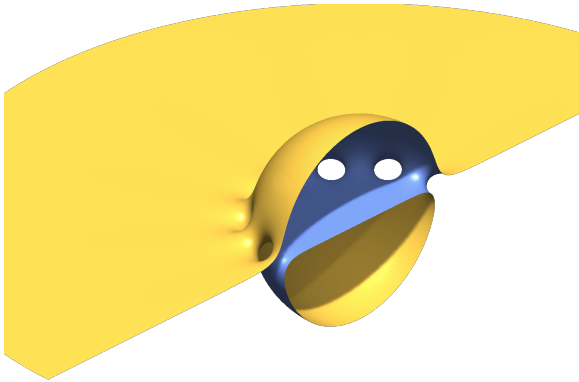


Figure 5: A numerical example of an exotic self-shrinker of positive genus and one end. Used with permission of Mario Schulz.

ity model. The example of a neck-pinch – where the first singularity is the non-compact cylinder – illustrates one possible complication. There are even more bizarre possibilities. For instance, the first singularity could be non-compact and asymptotic to a cone. Such exotic singularities can exist – we refer to the paper of Chopp [Cho94] which contains pictures of many numerically computed examples – and there are now several rigorous constructions; for instance, by Kapouleas-Kleene-Møller, X.H. Nguyen and Buzano-H. Nguyen-Schulz.

Nevertheless, restricting attention to singularities where the flow disappears rules out some of least well understood behavior. Crucially, the class of terminal singularities are more stable in a dynamical sense than general singularities and stable singularities tend to be more rigid as observed in [CM12]. Terminal singularities may be hidden behind earlier exotic singularities and so one must genuinely work with a weak flow that exists through singularities. This is technically quite challenging and involves deep results in PDEs and geometric measure theory.

With careful work, the authors established in [BW16] the conjecture in low dimensions

Theorem. *If $2 \leq n \leq 6$, and $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$.*

The bound $n \leq 6$ comes from the use of the regu-

larity theory for stable minimal hypersurfaces. J. Zhu [Zhu20] later relaxed this requirement and extended our argument to prove the result for $n \geq 7$.

Rigidity of Minimizers

For a sharp inequality, it is natural to study when it is saturated, i.e., when equality is achieved. This is the question of *rigidity* of an inequality. Within the two classes considered above there is rigidity for the corresponding optimal inequalities.

For closed hypersurfaces, the proof of the sharp inequality also shows the corresponding rigidity: If $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface with $\lambda[\Sigma] = \lambda[\mathbb{S}^n]$, then Σ is a round hypersphere. Somewhat surprisingly, rigidity in the general case proved slightly tricky. Using a rather subtle mean curvature flow construction, L. Chen [Che21] showed that if $\Sigma \subset \mathbb{R}^N$ is a properly embedded n -dimensional submanifold with $\lambda[\Sigma] = 1$, then Σ is an n -dimensional plane. A more elementary argument was later given by the first-named author when $n = 2$. One should also be able to use the rigidity properties of the Gaussian isoperimetric inequality of Sudakov-Tsirel'son and Borell to obtain this result for hypersurfaces.

Stability of Entropy Minimizers

A more robust form of rigidity is the question of *stability*, that is, whether almost minimizers are close to minimizers. This is particularly relevant, because well-behaved measures of complexity should have the feature that when the measured value of an object is near a minimum, the object is simple in a qualitative or quantitative sense. We discuss two different perspectives on this question for closed hypersurfaces with entropy near that of the round sphere.

Topological stability

One qualitative measurement of the complexity of a submanifold is in terms of its topology. It is now known that, in many dimensions, closed hypersurfaces with entropy below a certain threshold must be, topologically, as simple as possible – i.e., isotopic to the standard sphere. Of course all closed simple

curves in the plane are isotopic to the round circle without additional assumptions; this can be proven in many ways, though an appealing approach is to use curve shortening flow.

For surfaces in $\mathbb{R}^3$, the authors showed the only singularity models with entropy below that of the round circle is the shrinking round sphere [BW17] – note that an elementary calculation shows that the entropy of the round circle is the same as that of the corresponding cylinder in $\mathbb{R}^3$. Combining this with the monotonicity of entropy shows that any closed surface with entropy less than that of a circle evolves smoothly under mean curvature flow until it disappears in a round point – mirroring the behavior of the curve shortening flow. Hence, the flow provides a smooth isotopy between the initial surface and the round two-sphere and so:

Theorem. *If Σ is a closed surface in $\mathbb{R}^3$ with $\lambda[\Sigma] \leq \lambda[\mathbb{S}^1] = \lambda[\mathbb{S}^1 \times \mathbb{R}]$, then Σ has genus 0.*

That is, the entropy provides some measure of the topological complexity of closed surfaces.

In higher dimensions, the classification of low-entropy singularity models is highly incomplete, and a major challenge is to understand cone-like singularities. Nevertheless, the above theorem may be extended up to dimension five.

Theorem. *For $3 \leq n \leq 5$, if $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface with $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}]$, then Σ is smoothly isotopic to $\mathbb{S}^n$.*

When $n = 3$, the authors utilized the theory of self-expanders to show that all regular time slices of the weak mean curvature flow starting from such an initial hypersurface are in the same isotopy class [BW22]. The $n = 3$ case was also independently established by Chodosh-Choi-Mantoulidis-Schulze [CCMS21], who showed that generic one-sided perturbations of the initial hypersurface evolve smoothly under mean curvature flow until disappearing in a round point. Very recently, this second approach was extended by Chodosh-Mantoulidis-Schulze [CMS23] to dimensions $n = 4, 5$.

Finally, one may wonder if entropy can detect the presence of more complex topological structure. The

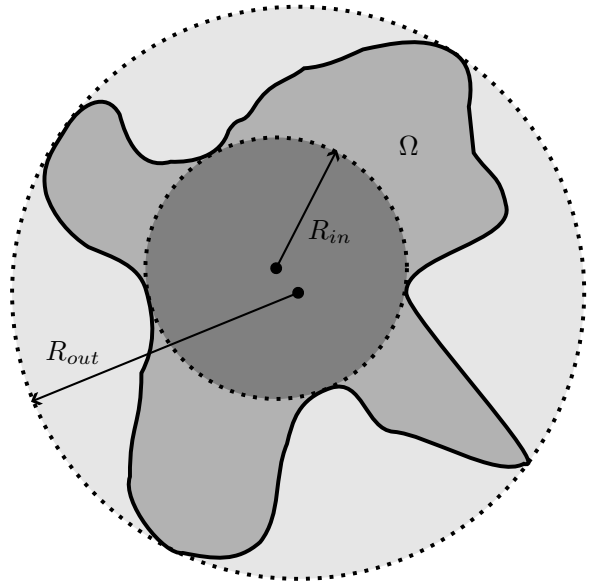


Figure 6: The in- and outradius of Ω .

expectations in this direction are quite modest. Indeed, consider a collection of distant unit spheres joined by very thin necks. These configurations can have arbitrary positive genus, but still have entropy a little bit above that of the circle. The fact that the increase is small comes from the fact that the Gaussian decays rapidly at large distances.

Stability as sets

More quantitatively, one may study the stability in terms of closeness as sets. This is inspired by the classical Bonnesen inequality which relates the isoperimetric defect of a region in the plane to the difference between its inradius and outradius. Here, for a compact set $\Omega \subset \mathbb{R}^{n+1}$, the inradius, $R_{in}(\Omega)$, is the radius of the largest ball contained in Ω and outradius, $R_{out}(\Omega)$, is the radius of the smallest ball containing Ω . Clearly, balls are the only sets where these are equal and the difference of $R_{out}(\Omega)$ and $R_{in}(\Omega)$ is small precisely when Ω is close as a set to a ball.

Unlike the Bonnesen inequality, which is a purely two-dimensional phenomenon, there is stability as sets in all dimensions for compact regions whose

boundaries nearly minimize entropy:

Theorem. *For any $\alpha > 0$, there is $\beta = \beta(\alpha) > 0$ such that, if $\Omega \subset \mathbb{R}^{n+1}$ is a compact region with $\Sigma = \partial\Omega$ a closed hypersurface and $\lambda[\Sigma] < \lambda[\mathbb{S}^n] + \beta$, then*

$$1 \leq \frac{R_{out}(\Omega)}{R_{in}(\Omega)} < 1 + \alpha.$$

Using mean curvature flow, the result was proved by the authors in dimension 2 in [BW18] and generalized to all dimensions by S. Wang [Wan20].

Further Developments and Questions

Finally, we discuss various generalizations of the Colding-Minicozzi entropy and their use as measures of geometric complexity.

Higher codimension

As we previously observed, for the class of n -dimensional submanifolds of $\mathbb{R}^N$, the entropy minimizers are the affine n -planes. However, the situation for other interesting classes of submanifolds remains largely unexplored.

As in the codimension-one case, it is a consequence of the Brakke regularity theorem, that non-flat n -dimensional self-shrinkers in $\mathbb{R}^N$ have entropy at least $1 + \epsilon(n, N)$. Moreover, properties of mean curvature flow of submanifolds imply that the flow of closed submanifolds must form a finite time singularity. Hence, any closed submanifold must have entropy at least $1 + \epsilon$. It is very plausible that the least entropy is achieved by a round n -sphere lying in an affine $(n+1)$ -plane. However, there has been very little progress towards proving this. This is due, in part, to our limited understanding of stable minimal submanifolds in higher codimension.

Some compelling evidence that entropy measures complexity in this setting is provided by work of Colding-Minicozzi [CM20]. They show that bounds on a self-shrinker's entropy provide effective bounds on its codimension. This means there is an estimate for ϵ depending only on the dimension n of the submanifold and not on the ambient dimension N .

Another interesting direction is to analyze closed entropy minimizers in other classes of higher codimension submanifolds. Of particular interest is the study of the question for closed Lagrangian submanifolds. This is because this condition interacts well with the mean curvature flow and spheres cannot be minimizers in this class.

Other ambient spaces

It is also desirable to generalize Colding-Minicozzi entropy to submanifolds of other ambient Riemannian manifolds.

One extension, to n -dimensional submanifolds of the round sphere, $\mathbb{S}^N$, was obtained by J. Zhu in his thesis. Essentially, he observed that the Colding-Minicozzi entropy of a submanifold of $\mathbb{S}^N$, thought of as a submanifold in $\mathbb{R}^{N+1}$ via the usual embedding of $\mathbb{S}^N$ into $\mathbb{R}^{N+1}$ remained monotone under mean curvature flow in $\mathbb{S}^N$. He concluded from this explicit lower bounds on the areas of topologically non-trivial minimal hypersurfaces in $\mathbb{S}^{n+1}$.

In another direction, the first-named author used the heat kernel on hyperbolic space, $\mathbb{H}^N$, to define a notion of entropy for submanifolds of hyperbolic space [Ber21]. This has subsequently been extended to the more general setting of Cartan-Hadamard manifolds – i.e., complete simply connected manifolds of non-positive sectional curvature. Many of the existing results in Euclidean space discussed above carry over in some fashion to this setting. However, some new phenomena appear. For instance, rigidity of the ambient geometry holds inside certain entropy minimizers. There is also an interesting connection between the entropy of minimal submanifolds of $\mathbb{H}^N$ and a quantity associated to the asymptotic geometry of the submanifold.

In addition to finding further applications, it would be appealing to find a broader framework for these measures of complexity. At present the proposed generalizations are largely *ad hoc*.

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