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DENSITY VERSIONS OF THE BINARY GOLDBACH PROBLEM

ALI ALSETRI AND XUANCHENG SHAO

Abstract. Let $\delta > 1/2$. We prove that if A is a subset of the primes such that the relative density of A in every reduced residue class is at least δ, then almost all even integers can be written as the sum of two primes in A. The constant $1/2$ in the statement is best possible. Moreover we give an example to show that for any $\varepsilon > 0$ there exists a subset of the primes with relative density at least $1-\varepsilon$ such that $A+A$ misses a positive proportion of even integers.

1. Introduction

Let $\mathcal{P}$ be the set of all primes and let $A \subset \mathcal{P}$ be a subset. This paper studies the representation of even integers as sums of two primes belonging to A. The famous Goldbach conjecture, which remains wide open, states that every even integer $n \geq 4$ can be written as the sum of two primes. The ternary version which concerns representing odd integers as sums of three primes has been much more tractable. Vinogradov [21] proved in 1937 that every sufficiently large odd integer is a sum of three primes (see also [2, Chapter 26]). This is now known to hold for all odd integers at least 7, thanks to work of Helfgott [8].

Returning to the binary Goldbach problem, Estermann [3] showed in 1938 that almost every even integer can be written as the sum of two primes. More precisely if $E(N)$ denotes the set of even integers $n \leq N$ which cannot be written as the sum of two primes, then

$$\frac{E(N)}{N} \ll_A (\log N)^{-A}$$

for every $A > 0$. A power saving for the error term was first obtained by Montgomery and Vaughan [13], who showed that

$$\frac{E(N)}{N} \ll N^{-\delta}$$

for some positive constant $\delta > 0$. Since then there have been a series of improvements on the precise value of δ, leading to the current record of $\delta = 0.28$ due to Pintz [14].

In this paper we study Goldbach-type problems with primes restricted to subsets of $\mathcal{P}$. For a subset $A \subset \mathcal{P}$, the relative lower density of A in $\mathcal{P}$ is defined by

$$\underline{\delta}(A) = \liminf_{N\to\infty} \frac{|A \cap [1,N]|}{|\mathcal{P} \cap [1,N]|}.$$

In recent years density versions of Vinogradov's three primes theorem have been obtained [9, 17, 18]. For example, in [17] it was proved that if $\underline{\delta}(A) > 5/8$ then all sufficiently large odd positive integers can be written as a sum of three primes in A. See also [12, 11, 20, 6] for results with (special) sparse subsets of primes.

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Motivated by these results, we seek to obtain a density version of the almost all binary Goldbach problem. The binary problem for small positive density subsets of primes has been studied in [16, 1, 10]. In particular, Matomäki [10] proved that if $\underline{\delta}(A) = \alpha$ for some positive constant $\alpha > 0$, then the sumset $A + A := \{p_1 + p_2 : p_1, p_2 \in A\}$ has positive lower density in the integers. Moreover, the lower density of $A + A$ is at least

$$(e^{-\gamma} - o(1))\frac{\alpha}{\log\log(1/\alpha)},$$

where γ is the Euler-Mascheroni constant and $o(1)$ denotes a quantity that tends to 0 as $\alpha \to 0$. See also [7] for related results with A the set of almost twin primes.

We seek conditions on $A \subset \mathcal{P}$ which guarantee $A + A$ contains almost all even integers, or equivalently, $A + A$ has density $1/2$ in the integers. Specifically we ask whether there exists a positive constant $\alpha < 1$, such that if $\underline{\delta}(A) \geq \alpha$ then $A + A$ contains almost all even integers. We show that, unlike the ternary case, such an α does not exist.

Theorem 1.1. *For any $\varepsilon > 0$ there exists a subset $A \subset \mathcal{P}$ with $\underline{\delta}(A) > 1 - \varepsilon$, such that a positive proportion of the even positive integers cannot be written as a sum of two primes in A.*

However the situation changes if we impose additional local assumptions about the set A. For a reduced residue class $b \pmod W$, we define the relative lower density of A in primes within this residue class by

$$\underline{\delta}(A; W, b) = \liminf_{N\to\infty} \frac{|A \cap \{1 \leq n \leq N : n \equiv b \pmod W\}|}{|\mathcal{P} \cap \{1 \leq n \leq N : n \equiv b \pmod W\}|}.$$

Theorem 1.2. *Let $A \subset \mathcal{P}$ be a subset such that*

$$\inf_{W,b} \underline{\delta}(A; W, b) > 1/2,$$

where the infimum is taken over all reduced residue classes $b \pmod W$. Then almost all even positive integers N can be written as $N = p_1 + p_2$ with $p_1, p_2 \in A$.

Remark 1.3. The constant $1/2$ is sharp. For any $\alpha > 2$ we may define

$$A = \{p \in \mathcal{P} : p \in [1, N_1] \cup [\alpha N_1, N_2] \cup [\alpha N_2, N_3] \cdots \},$$

where $N_1 < N_2 < N_3 < \cdots$ is a rapidly increasing sequence. Then $\underline{\delta}(A; W, b) = 1/\alpha$ for all reduced residue classes $b \pmod W$ and $A + A$ misses a positive proportion of even integers.

This result is proved using a variant of the Fourier analytic transference principle from additive combinatorics. This technique originated from the work of Green [4] who developed it to establish Roth's theorem in primes. Variants of the transference principle have been developed suitable for different problems. See [15] for a survey. For a variant suitable for additive problems involving dense subsets of the primes, see [10, Section 6] or [1]. For an almost-all version of the transference principle, see [22].

This article is organized as follows. In Section 2 we study the binary Goldbach problem in the local setting, leading to the proof of Theorem 1.1. In Section 3 we develop an almost-all variant of the transference principle. In Section 4 we use this transference principle to prove Theorem 1.2.

2. Local results

For a positive integer m, we write $\mathbb{Z}_m^*$ for the set of reduced residue classes modulo m. In this section we will prove Theroem 1.1 by studying the binary problem in the local setting of a cyclic group. We will first prove the following Theorem 2.1 which is an independent result and will not be needed in the subsequent proofs in our paper, however as we will show Theorem 1.1 is essentially a consequence of the observation that Theorem 2.1 is sharp.

Theorem 2.1. *Let m be an odd squarefree positive integer and let $A, B \subset \mathbb{Z}_m^*$ be subsets. Assume that*

$$|A| + |B| > \varphi(m) \left(2 - \prod_{p|m} \frac{p-2}{p-1} \right).$$

Then $A + B = \mathbb{Z}_m$.

Proof. Let $n \in \mathbb{Z}_m$ be arbitrary. Let

$$X = \{x \in \mathbb{Z}_m^* : n - x \in \mathbb{Z}_m^*\}.$$

Then $x \in X$ if and only if $x \not\equiv 0, n \pmod p$ for every $p \mid m$, and hence

$$|X| \geq \prod_{p|m} (p-2).$$

It follows that the number of $x \in X$ such that $x \in A$ and $n - x \in B$ is at least

$$|X| - |\mathbb{Z}_m^* \setminus A| - |\mathbb{Z}_m^* \setminus B| \geq |A| + |B| - 2\varphi(m) + \prod_{p|m} (p-2) > 0.$$

Pick any such x. Then $n = x + (n - x) \in A + B$, as desired. $\square$

The lower bound for $|A|+|B|$ is sharp. Let $m = p_1 p_2 \cdots p_s$, where $p_1, \cdots, p_s$ are distinct odd primes. Define

$$A = \bigcup_{i=1}^{s} \{a \in \mathbb{Z}_m^* : a \pmod{p_i} \in X_i\}, \quad B = \bigcup_{i=1}^{s} \{a \in \mathbb{Z}_m^* : a \pmod{p_i} \in Y_i\},$$

where $X_i = Y_i = \{1\}$ for $1 \leq i \leq s-1$, and

$$X_s = \{1, 2, \cdots, x\}, \quad Y_s = \{1, 2, \cdots, y\}.$$

for some $1 \leq x, y < p_s$. Then $1 \notin A + B$ if $x + y \leq p_s$ and

$$|A| = \varphi(m) \left(1 - \frac{p_s - 1 - x}{p_s - 1} \prod_{1 \leq i \leq s-1} \frac{p_i - 2}{p_i - 1} \right), \quad |B| = \varphi(m) \left(1 - \frac{p_s - 1 - y}{p_s - 1} \prod_{1 \leq i \leq s-1} \frac{p_i - 2}{p_i - 1} \right).$$

Hence if we choose x, y such that $x + y = p_s$ then

$$|A| + |B| = \varphi(m) \left(2 - \prod_{p|m} \frac{p-2}{p-1} \right).$$

Moreover, if we choose $x = y = (p_s - 1)/2$, then $A = B$ and we obtain $A \subset \mathbb{Z}_m^*$ with

$$|A| = \varphi(m) \left(1 - \frac{1}{2} \prod_{1 \leq i \leq s-1} \frac{p_i - 2}{p_i - 1} \right)$$

such that $A + A \neq \mathbb{Z}_m$. Since the infinite product $\prod_p \frac{p-2}{p-1}$ diverges to 0 we can suppose, for any $\varepsilon > 0$, that $|A| > \varphi(m)(1-\varepsilon)$. Now let A' be the set of all primes which are congruent to some $a \in A$. By Dirichlet's theorem on primes in arithmetic progressions, $\underline{\delta}(A') > 1 - \varepsilon$. Since $A + A \neq \mathbb{Z}_m$ it follows that $A' + A'$ does not contain any of the even integers in some fixed residue class modulo m and so we immediately arrive at Theorem 1.1.

3. A transference principle

We work in a cyclic group $\mathbb{Z}_N$. We adopt the normalization corresponding to the probability measure on the physical side $\mathbb{Z}_N$ and to the counting measure on the frequency side $\widehat{\mathbb{Z}_N}$. Thus, the Fourier transform of a function $f : \mathbb{Z}_N \to \mathbb{C}$ is defined by

$$\widehat{f}(r) = \mathbb{E}_{n \in \mathbb{Z}_N} f(n) e_N(-rn)$$

for $r \in \mathbb{Z}_N$. For $p, q > 0$, the norms $\|\widehat{f}\|_p$ and $\|f\|_q$ are normalized as follows:

$$\|\widehat{f}\|_p = \Big(\sum_{r \in \mathbb{Z}_N} |\widehat{f}(r)|^p\Big)^{1/p}, \quad \|f\|_q = \Big(\mathbb{E}_{n \in \mathbb{Z}_N} |f(n)|^q\Big)^{1/q}.$$

For two functions $f_1, f_2 : \mathbb{Z}_N \to \mathbb{C}$, their convolution $f_1 * f_2$ is defined by

$$f_1 * f_2(n) = \mathbb{E}_{n_1 \in \mathbb{Z}_N} f_1(n_1) f_2(n - n_1).$$

Proposition 3.1. *For $i \in \{1, 2\}$, let $f_i, \nu_i : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$ be functions such that $f_i(n) \leq \nu_i(n)$ for every $n \in \mathbb{Z}_N$. Let $\delta_i = \mathbb{E}_{n \in \mathbb{Z}_N} f_i(n)$. Let $\delta, \eta > 0$. Suppose that the following conditions hold.*

(1) $\delta_1 + \delta_2 \geq 1 + \delta$ for some $\delta > 0$.
(2) Each f_i satisfies a mean value estimate in the sense that $\|\widehat{f_i}\|_p \leq M$ for some $p \in (2, 4)$ and $M \geq 1$.
(3) Each ν_i has Fourier decay in the sense that $\|\widehat{\nu_i - 1}\|_\infty \leq c(\delta, \eta, p, M)$ for some sufficiently small constant $c(\delta, \eta, p, M) > 0$.

Then $f_1 * f_2(n) \geq \delta^3/1000$ for all but at most ηN values of $n \in \mathbb{Z}_N$.

To prove Proposition 3.1, first we construct in Lemma 3.2 decompositions $f_i = g_i + h_i$ for each $i \in \{1, 2\}$, such that g_i is essentially 1-bounded and h_i is Fourier uniform in the sense that $\|\widehat{h_i}\|_\infty = o(1)$. Then we show in Lemma 3.3 that $g_1 * g_2(n) \gg_\delta 1$ for all $n \in \mathbb{Z}_N$ using hypothesis (1) about the sizes of δ_1, δ_2. Finally we show in Lemma 3.4 that $f_1 * f_2(n) \gg_\delta 1$ for almost all n, using a standard Fourier analytic argument.

We now turn to the details. Let $\varepsilon > 0$ be a small constant to be chosen later in terms of δ, η, p, M.

Lemma 3.2. *Let the notations and assumptions be as above. For each $i \in \{1, 2\}$, we may construct an approximant $g_i : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$ of f_i with the following properties:*

(1) $\mathbb{E}_{n \in \mathbb{Z}_N} g_i(n) = \delta_i$.
(2) $\|g_i\|_\infty \leq 1 + \delta/10$.
(3) $\|\widehat{f_i} - \widehat{g_i}\|_\infty \leq \varepsilon$.
(4) $\|\widehat{g_i}\|_p \leq M$.

The statement of the lemma is analogous to [17, Lemma 4.2] (where we caution that the functions and the Fourier transforms are normalized differently), and the proof follows the same arguments as in [5, Proposition 5.1]. For completeness, we include a full proof.

Proof. For convenience, we drop the dependence on i, writing $f = f_i$, $g = g_i$, $\nu = \nu_i$. Define the large spectrum of f to be

$$R = \{r \in \mathbb{Z}_N : |\widehat{f}(r)| \geq \varepsilon\}.$$

From the mean value estimate $\|\widehat{f}\|_p \leq M$ it follows that

$$\varepsilon^p |R| \leq \sum_{r \in \mathbb{Z}_N} |\widehat{f}(r)|^p \leq M^p,$$

and hence $|R| \leq (M/\varepsilon)^p$. Define the Bohr set

$$B = \{x \in \mathbb{Z}_N : |e_N(xr) - 1| \leq \varepsilon \text{ for each } r \in R\}.$$

By the pigeonhole principle (see [19, Lemma 4.20]), it follows that $|B| \geq (c\varepsilon)^{|R|} N$ for some absolute constant $c > 0$ and thus $|B| \gg_{\varepsilon,p,M} N$. Now define the approximant $g : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$ by

$$g(n) = \mathbb{E}_{b_1, b_2 \in B} f(n + b_1 - b_2).$$

We will verify that g satisfies the four desired properties.

Property (1) follows trivially by the definition of g. To verify property (2), note that for each $n \in \mathbb{Z}_N$ we have

$$g(n) \leq \mathbb{E}_{b_1, b_2 \in B} \nu(n + b_1 - b_2) = \sum_{r \in \mathbb{Z}_N} \widehat{\nu}(r) e_N(-rn) \left|\mathbb{E}_{b \in B} e_N(rb)\right|^2.$$

By the Fourier decay property of ν, we may replace $\widehat{\nu}(r)$ above by $1_{r=0}$ at the cost of an error at most $c = c(\delta, \eta, p, M)$ for some sufficiently small constant $c(\delta, \eta, p, M) > 0$. It follows that

$$g(n) \leq 1 + O\Big(c \sum_{r \in \mathbb{Z}_N} |\mathbb{E}_{b \in B} e_N(rb)|^2\Big) = 1 + O\Big(\frac{cN}{|B|}\Big) = 1 + O_{\varepsilon,p,M}(c).$$

Since ε is chosen in terms of δ, η, p, M, we may ensure that $g(n) \leq 1 + \delta/10$ by choosing c sufficiently small in terms of δ, η, p, M.

To verify property (3), note that for each $r \in \mathbb{Z}_N$ we have

$$|\widehat{f}(r) - \widehat{g}(r)| = |\widehat{f}(r)| \left(1 - |\mathbb{E}_{b \in B} e_N(rb)|^2\right) \leq 2|\widehat{f}(r)| \mathbb{E}_{b \in B} |1 - e_N(rb)|.$$

We divide into two cases according to whether $r \in R$ or not. If $r \in R$, then $|e_N(rb) - 1| \leq \varepsilon$ for each $b \in B$ by the definition of the Bohr set B, and hence

$$|\widehat{f}(r) - \widehat{g}(r)| \leq 2\varepsilon |\widehat{f}(r)| \leq 4\varepsilon,$$

using the trivial bound

$$|\widehat{f}(r)| \leq \mathbb{E}_{n \in \mathbb{Z}_N} f(n) \leq \mathbb{E}_{n \in \mathbb{Z}_N} \nu(n) = \widehat{\nu}(0) \leq 2 \tag{3.1}$$

(say). If $r \notin R$, then $|\widehat{f}(r)| \leq \varepsilon$ by the definition of the large spectrum R, and hence $|\widehat{f}(r) - \widehat{g}(r)| \leq 4\varepsilon$. This verifies property (3) (after replacing ε in our argument by $\varepsilon/4$).

Finally, property (4) follows easily from the fact

$$|\widehat{g}(r)| = |\widehat{f}(r)| \cdot |\mathbb{E}_{b \in B} e_N(rb)|^2 \leq |\widehat{f}(r)| \tag{3.2}$$

and the mean value estimate for f. □

Lemma 3.3. *Let the notations and assumptions be as above, and let g_1, g_2 be the approximants constructed in Lemma 3.2. Then $g_1 * g_2(n) \geq \delta^3/200$ for every $n \in \mathbb{Z}_N$.*

Proof. For $i \in \{1,2\}$, define A_i to be the essential support of g_i:

$$A_i = \{n \in \mathbb{Z}_N : g_i(n) \geq \delta/10\}.$$

Since $\|g_i\|_\infty \leq 1 + \delta/10$, we have

$$\delta_i N = \sum_{n \in \mathbb{Z}_N} g_i(n) \leq \tfrac{1}{10}\delta N + \left(1 + \tfrac{\delta}{10}\right)|A_i| \leq \tfrac{1}{5}\delta N + |A_i|.$$

Hence $|A_i| \geq (\delta_i - \delta/5)N$. Thus for every $n \in \mathbb{Z}_N$,

$$|A_1 \cap (n - A_2)| \geq |A_1| + |A_2| - N \geq \left(\delta_1 + \delta_2 - 1 - \tfrac{2}{5}\delta\right)N \geq \tfrac{1}{2}\delta N,$$

and hence

$$g_1 * g_2(n) \geq N^{-1}\left(\tfrac{\delta}{10}\right)^2 |A_1 \cap (n - A_2)| \geq \tfrac{1}{200}\delta^3.$$

This completes the proof. □

Lemma 3.4. *Let the notations and assumptions be as above, and let g_1, g_2 be the approximants constructed in Lemma 3.2. Let E be the exceptional set defined as*

$$E = \{n \in \mathbb{Z}_N : f_1 * f_2(n) \leq \delta^3/1000\},$$

Then $|E| \leq \eta N$.

Proof. Let $\alpha = |E|/N$. Consider the inner product $I = \langle g_1 * g_2 - f_1 * f_2, 1_E\rangle$. On the one hand, we have

$$I = \frac{1}{N}\sum_{n \in E}(g_1 * g_2(n) - f_1 * f_2(n)) \gg \delta^3 \alpha$$

by Lemma 3.3 and the definition of E. On the other hand, by Plancherel's identity and Cauchy-Schwarz inequality we have

$$I = \langle \widehat{g_1}\widehat{g_2} - \widehat{f_1}\widehat{f_2}, \widehat{1_E}\rangle \leq \|\widehat{1_E}\|_2 \cdot \|\widehat{g_1}\widehat{g_2} - \widehat{f_1}\widehat{f_2}\|_2.$$

Clearly $\|\widehat{1_E}\|_2 = \|1_E\|_2 = \alpha^{1/2}$. Thus the two inequalities above together imply that

$$\alpha \ll \delta^{-6}\|\widehat{g_1}\widehat{g_2} - \widehat{f_1}\widehat{f_2}\|_2^2.$$

Note that for every $r \in \mathbb{Z}_N$ we have

$$|\widehat{g_1}(r)\widehat{g_2}(r) - \widehat{f_1}(r)\widehat{f_2}(r)| \leq |\widehat{g_1}(r)| \cdot |\widehat{g_2}(r) - \widehat{f_2}(r)| + |\widehat{f_2}(r)| \cdot |\widehat{g_1}(r) - \widehat{f_1}(r)| \leq 4\varepsilon,$$

where we used the property that $\|\widehat{f_i} - \widehat{g_i}\|_\infty \leq \varepsilon$ and the trivial bounds $|\widehat{f_i}(r)| \leq 2$ and $|\widehat{g_i}(r)| \leq 2$ (see (3.1) and (3.2)). Moreover by Cauchy-Schwarz, condition (4) of Lemma 3.2 and the mean value estimate (2) of Proposition 3.1, we have

$$\|\widehat{g_1}\widehat{g_2} - \widehat{f_1}\widehat{f_2}\|_{p/2} \leq \|\widehat{g_1}\widehat{g_2}\|_{p/2} + \|\widehat{f_1}\widehat{f_2}\|_{p/2} \leq \|\widehat{g_1}\|_p\|\widehat{g_2}\|_p + \|\widehat{f_1}\|_p\|\widehat{f_2}\|_p \leq 2M^2.$$

Since $p \in (2,4)$, it follows that

$$\|\widehat{g_1}\widehat{g_2} - \widehat{f_1}\widehat{f_2}\|_2 \leq \|\widehat{g_1}\widehat{g_2} - \widehat{f_1}\widehat{f_2}\|_\infty^{1-p/4} \cdot \|\widehat{g_1}\widehat{g_2} - \widehat{f_1}\widehat{f_2}\|_{p/2}^{p/4} \ll_{p,M} \varepsilon^{1-p/4}$$

and hence

$$\alpha \ll_{p,M} \delta^{-6}\varepsilon^{2-p/2}.$$

Thus we may ensure that $\alpha \leq \eta$ by choosing $\varepsilon = (c\delta^6\eta)^{2/(4-p)}$, where $c = c(p,M) > 0$ is a sufficiently small constant. □

As mentioned previously, the proof of Proposition 3.1 is completed by combining Lemmas 3.2, 3.3 and 3.4.

4. Binary Goldbach for dense subsets of primes

In this section we prove Theorem 1.2. Let $A \subset \mathcal{P}$ be a subset satisfying the assumptions in Theorem 1.2. Choose $\delta \in (0, 1/2)$ such that $\underline{\delta}(A; W, b) \geq 1/2 + \delta$ for every reduced residue class $b \pmod{W}$. Let E be the exceptional set consisting of those even positive integers that cannot be written as a sum of two primes in A. It suffices to show that

$$\lim_{M\to\infty} \frac{|E \cap [(1-\delta^{10})M, M]|}{M} = 0.$$

Suppose, for the purpose of contradiction, that there exists a constant $\eta > 0$ such that

$$|E \cap [(1-\delta^{10})M, M]| \geq \eta M \tag{4.1}$$

for each $M = M_i$ in an infinite increasing sequence $M_1 < M_2 < \cdots$ of positive integers. Set $W = \prod_{p\leq z} p$, where z is a constant sufficiently large in terms of δ, η. Recall that

$$\underline{\delta}(A; W, b) = \liminf_{M\to\infty} \frac{|A_{W,b} \cap [1, M]|}{|\mathcal{P}_{W,b} \cap [1, M]|},$$

where, for a set $A \subset \mathbb{Z}$ and a residue class $b \pmod{W}$, the notation $A_{W,b}$ is defined by

$$A_{W,b} = \{n \in A : n \equiv b \pmod{W}\}.$$

Thus, for some large value $M = M_i$, we have

$$\frac{|A_{W,b} \cap [1, M]|}{|\mathcal{P}_{W,b} \cap [1, M]|} \geq \frac{1}{2} + \frac{\delta}{2}$$

for each $b \in \mathbb{Z}_W^*$. Fix this value of M for the remainder of the proof, and let $N = \lfloor M/W \rfloor$.

By the pigeonhole principle, there exists an even residue class $r \pmod{W}$ such that

$$|E_{W,r} \cap [(1-\delta^{10})M, M]| \geq \frac{\eta M}{W} \geq \eta N. \tag{4.2}$$

By the Chinese remainder theorem we can choose $b_1, b_2 \in \mathbb{Z}_W^*$ with $r = b_1 + b_2$. For $i \in \{1, 2\}$, define $f_i, \nu_i : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$ (naturally identifying $\mathbb{Z}_N$ with $\{1, 2, \cdots, N\}$) by

$$\nu_i(n) = \begin{cases} \frac{\varphi(W)}{W} \log(Wn + b_i) & \text{if } Wn + b_i \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_i(n) = \begin{cases} \frac{\varphi(W)}{W} \log(Wn + b_i) & \text{if } Wn + b_i \in A \text{ and } Wn + b_i \leq (1-\delta^5)M, \\ 0 & \text{otherwise.} \end{cases}$$

We will show that f_i, ν_i satisfy the assumptions of Proposition 3.1. Note that $\frac{\nu_i}{N}$ is the same as $\lambda_{b_i,W,N}$ in the notation of Green's paper [4] which we will appeal to. Clearly $0 \leq f_i(n) \leq \nu_i(n)$ for every n. The mean value estimates $\|\widehat{f_i}\|_p = O(1)$ follows from [4, Lemma 6.6] with exponent $p = 3$ (say). The Fourier decay property of ν_i follows from [4, Lemma 6.2], once z is chosen large enough in terms of δ, η. Now note that the average of $f_i(n)$ is

$$\delta_i = \mathbb{E}_{n\in\mathbb{Z}_N} f_i(n) = \frac{\varphi(W)}{NW} \sum_{p\in A_{W,b_i}\cap[1,(1-\delta^5)M]} \log p.$$

By restricting the sum over p above to $p \in A_{W,b_i} \cap [M(\log M)^{-10}, (1-\delta^5)M]$ and noting that

$$|A_{W,b_i} \cap [M(\log M)^{-10}, (1-\delta^5)M]| \geq |A_{W,b_i} \cap [1, M]| - M(\log M)^{-10} - \frac{\delta^4 M}{\varphi(W)\log M}$$
$$\geq \left(\frac{1}{2} + \frac{\delta}{3}\right) \frac{M}{\varphi(W)\log M},$$

we deduce that

$$\delta_i \geq \frac{\varphi(W)}{NW}\left(\frac{1}{2} + \frac{\delta}{3}\right)\frac{M}{\varphi(W)\log M}(\log M - 10\log\log M) \geq \frac{1}{2} + \frac{\delta}{4}.$$

Hence, we may apply Proposition 3.1 to the functions f_i, ν_i (with δ, η replaced by $\delta/2, \eta/2$, respectively) to conclude that

$$f_1 * f_2(n) \gg \delta^3$$

for all but at most $\eta N/2$ values of $n \in \mathbb{Z}_N$. In view of (4.2), there exists $m \in E_{W,r} \cap [(1-\delta^{10})M, M]$ such that $(m - b_1 - b_2)/W$ (naturally viewed as an element in $\mathbb{Z}_N$) is in the support of $f_1 * f_2$. By the definition of f_1, f_2, this implies that we can write

$$\frac{m - b_1 - b_2}{W} \equiv n_1 + n_2 \pmod{N}$$

for some positive integers n_1, n_2 with $Wn_i + b_i \in A$ and $Wn_i + b_i \leq (1-\delta^5)M$. The congruence above can be rewritten as

$$m \equiv (Wn_1 + b_1) + (Wn_2 + b_2) \pmod{WN}.$$

Since $m \in [(1-\delta^{10})M, M]$ and $(Wn_1 + b_1) + (Wn_2 + b_2) \leq (2 - 2\delta^5)M$, the congruence above must be an equality in the integers:

$$m = (Wn_1 + b_1) + (Wn_2 + b_2).$$

This implies that m can be written as the sum of two primes in A, contradicting $m \in E$.

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Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506, USA
Email address: alialsetri@uky.edu

Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506, USA
Email address: xuancheng.shao@uky.edu