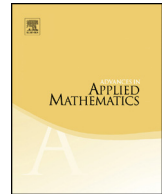




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journal homepage: www.elsevier.com/locate/yaama

Row-strict dual immaculate functions

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ARTICLE INFO

Article history:

Received 24 March 2022

Received in revised form 5 April 2023

Accepted 17 April 2023

Available online 28 April 2023

MSC:

05A05

05E05

16T30

Keywords:

Composition

Creation operator

Dual immaculate function

Hook Schur function

Hopf algebra

Pieri rule

Quasisymmetric function

Schur function

Skew Schur function

Tableau combinatorics

ABSTRACT

We define a new basis of quasisymmetric functions, the row-strict dual immaculate functions, as the generating function of a particular set of tableaux. We show that this definition gives a function that can also be obtained by applying the involution ψ to the dual immaculate functions of Berg, Bergeron, Saliola, Serrano, and Zabrocki (2014), and we establish numerous combinatorial properties for our functions. We give an equivalent formulation of our functions via Bernstein-like operators, in a similar fashion to Berg et al. (2014). We conclude the paper by defining skew dual immaculate functions and hook dual immaculate functions, and establishing combinatorial properties for them.

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1. Introduction

Quasisymmetric functions were first defined formally by Gessel [10] in relation to the theory of P -partitions, and have since grown to be a vibrant area of research in their own right, including playing a crucial role in the resolution of the Shuffle Conjecture [7]. As a natural nonsymmetric generalization of symmetric functions, one avenue of research has been to establish analogies of classical symmetric functions, for example monomial symmetric functions and chromatic symmetric functions. However an analogy with the ubiquitous Schur functions remained elusive until 2011, when the authors of [11] discovered quasisymmetric Schur functions that naturally arose from the combinatorics of nonsymmetric Macdonald polynomials. These functions became the genesis of the now flourishing area of Schur-like functions throughout algebraic combinatorics, for example [1,6,8,12,13]. Within the algebra of quasisymmetric functions, two further bases rose to attention: the dual immaculate functions [4], and the row-strict quasisymmetric Schur functions [15], the latter being quasisymmetric Schur functions via the involution ψ . In this paper we will interpolate between these two bases to yield row-strict dual immaculate functions.

More precisely, quasisymmetric Schur functions, all forms, can be defined combinatorially as the generating function of composition fillings (resp. row-strict composition fillings) where there is a requirement that the first column strictly (resp. weakly) increases, each row increases weakly (resp. strictly), and a triple rule is satisfied. The dual immaculate functions were introduced by Berg et al. [4] as the dual basis of the noncommutative symmetric immaculate functions. Combinatorially the dual immaculate functions can be viewed as the generating functions of composition fillings that satisfy just the first column and row requirements of the quasisymmetric Schur functions, omitting the triple rule.

The triple rules required to define all versions of quasisymmetric Schur functions allow those functions to retain many of the combinatorial properties of Schur functions, including an RSK-style insertion algorithm, a JDT algorithm, a Murnaghan-Nakayama rule, and Littlewood-Richardson rules. Without the triple rule, some combinatorial sim-

ilarities to Schur functions are lost, but others are gained. For example, the immaculate functions satisfy a noncommutative analogue of the Jacobi-Trudi rule.

In this paper we define *row-strict immaculate tableaux* of a given composition shape, and study their generating function. By identifying the correct descent set, we show that our combinatorial definition of the row-strict dual immaculate functions is equivalent to applying the involution ψ to the dual immaculate functions in Theorem 3.8, and can also be obtained from the Hopf algebra of noncommutative symmetric functions by suitably defined creation operators in Theorem 3.19.

We are able to quickly obtain many results from [4] by application of the involution ψ in Theorem 3.21. We also carefully construct skew row-strict dual immaculate functions and define hook dual immaculate functions, obtaining results for them in our final two sections. In this work we focus primarily on combinatorial aspects of the row-strict dual immaculate functions. We investigate 0-Hecke modules for these new functions in [17].

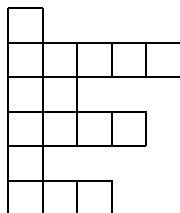
Acknowledgments. The authors would like to thank Sarah Mason for bringing to their attention an incorrect equation in the first version of the paper, the referee for thoughtful comments, and the Algebraic Combinatorics Research Community program at ICERM through which this research took place. The third author was supported in part by the National Sciences Research Council of Canada.

2. Background

In this section we introduce much of the background on quasisymmetric and non-commutative symmetric functions needed for our results. We refer the reader to [13] for additional details.

A *composition* of a positive integer n is a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ such that $\sum_i \alpha_i = n$. We write $\alpha \models n$. We sometimes denote n by $|\alpha|$ and k by $\ell(\alpha)$. The diagram of $\alpha = (\alpha_1, \dots, \alpha_k)$ is a collection of left-justified boxes with α_i boxes in row i , where row 1 is the bottom row.

Example 2.1. For $\alpha = (3, 1, 4, 2, 5, 1)$, the diagram is as follows.



Compositions of n are in bijection with subsets of $\{1, 2, \dots, n-1\}$. Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n , the corresponding set is $\text{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{k-1}\}$. For $\alpha = (3, 1, 4, 2, 5, 1)$ that is a composition of 16, $\text{set}(\alpha) = \{3, 4, 8, 10, 15\} \subseteq \{1, 2, \dots, 15\}$. Given a subset $S = \{s_1 < s_2 < \dots < s_j\}$ of $\{1, 2, \dots, n-1\}$, the

corresponding composition of n is $\text{comp}(S) = (s_1, s_2 - s_1, \dots, s_j - s_{j-1}, n - s_j)$. For $S = \{2, 3, 5, 9, 10, 14\} \subseteq \{1, 2, \dots, 15\}$, $\text{comp}(S) = (2, 1, 2, 4, 1, 4, 2)$. The composition obtained by reversing the order of the parts of α , the *reverse* of α , is $\text{rev}(\alpha) = (\alpha_k, \alpha_{k-1}, \dots, \alpha_1)$. The *complement* of a composition α , denoted α^c , is the composition obtained from α by taking the complement of the set corresponding to α . That is, $\alpha^c = \text{comp}(\text{set}(\alpha)^c)$. The *transpose* of a composition α , denoted α^t , is the composition obtained from α by taking the complement of the set corresponding to the reverse of α . That is,

$$\alpha^t = \text{comp}(\text{set}(\text{rev}(\alpha))^c).$$

For example, if $\alpha = (3, 1, 2, 4)$, $\text{rev}(\alpha) = (4, 2, 1, 3)$, $\text{set}(\text{rev}(\alpha)) = \{4, 6, 7\}$, $\text{set}(\text{rev}(\alpha))^c = \{1, 2, 3, 5, 8, 9\}$, so $\alpha^t = (1, 1, 1, 2, 3, 1, 1)$. Note that $\alpha^t = \text{rev}(\alpha)^c = \text{rev}(\alpha^c)$.

We will use several different orders on compositions. For compositions α and β , we say α precedes β in *lexicographic order*, denoted by $\alpha \leq_\ell \beta$, if either $\alpha_1 < \beta_1$ or there is a $j > 1$ such that $\alpha_j < \beta_j$ but $\alpha_i = \beta_i$, $1 \leq i \leq j - 1$. We say that a composition $\beta = (\beta_1, \dots, \beta_m)$ is a *refinement* of a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, denoted $\beta \preccurlyeq \alpha$, if each part of α can be obtained by adding consecutive parts of β . Equivalently, we say that α is a *coarsening* of β . For example, $\beta = (1, 2, 1, 1, 3, 2)$ is a refinement of $\alpha = (3, 2, 5)$. Finally, we use an order, defined in [4], where $\alpha \subset_s \beta$ if

- (1) $|\beta| = |\alpha| + s$,
- (2) $\alpha_j \leq \beta_j$, $\forall 1 \leq j \leq \ell(\alpha)$, and
- (3) $\ell(\beta) \leq \ell(\alpha) + 1$.

Note that the last two parts guarantee that $\ell(\alpha) \leq \ell(\beta) \leq \ell(\alpha) + 1$. If we have only the second condition then this is denoted $\alpha \subseteq \beta$.

A function $f \in \mathbb{Q}[[x_1, x_2, \dots]]$ is *quasisymmetric* if the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$ is the same as the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}$ for every $(\alpha_1, \alpha_2, \dots, \alpha_k)$ and $i_1 < i_2 < \dots < i_k$. The set of all quasisymmetric functions forms a Hopf algebra graded by degree, $\text{QSym} = \bigoplus_n \text{QSym}_n$, where each QSym_n is a vector space over \mathbb{Q} with bases indexed by compositions of n .

The pertinent bases for our purposes include the *monomial*, *fundamental*, *dual immaculate*, and *quasisymmetric Schur* bases. We define the monomial and fundamental bases here and defer the remaining definitions until later.

Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n , the *monomial quasisymmetric function* is

$$M_\alpha = \sum_{\substack{(i_1, i_2, \dots, i_k) \\ i_1 < i_2 < \dots < i_k}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}.$$

A second important quasisymmetric basis is the fundamental basis. Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n , the *fundamental quasisymmetric function* indexed by α is

$$F_\alpha(x_1, x_2, \dots) = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin \text{set}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Note that

$$F_\alpha = \sum_{\beta \preccurlyeq \alpha} M_\beta \quad \text{and} \quad M_\alpha = \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} F_\beta. \quad (2.1)$$

In [9] the *noncommutative symmetric functions* are defined as the algebra $\text{NSym} = \mathbb{Q}\langle \mathbf{e}_1, \mathbf{e}_2, \dots \rangle$ generated by noncommuting indeterminates \mathbf{e}_n of degree n . The set of noncommutative symmetric functions forms a graded Hopf algebra $\text{NSym} = \bigoplus_n \text{NSym}_n$ where the degree of functions in NSym_n is n . Each NSym_n has bases indexed by compositions of n .

The n th elementary noncommutative symmetric function is the indeterminate \mathbf{e}_n , where $\mathbf{e}_0 = 1$. Given a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, we define the *elementary noncommutative symmetric function* by

$$\mathbf{e}_\alpha = \mathbf{e}_{\alpha_1} \cdots \mathbf{e}_{\alpha_k}.$$

The n th complete homogeneous noncommutative symmetric function is defined by

$$\mathbf{h}_n = \sum_{(\alpha_1, \dots, \alpha_m) \models n} (-1)^{n-m} \mathbf{e}_\alpha$$

with $\mathbf{h}_0 = 1$. Then, for $\alpha = (\alpha_1, \dots, \alpha_k)$, the *complete homogeneous noncommutative symmetric function* is defined by

$$\mathbf{h}_\alpha = \mathbf{h}_{\alpha_1} \cdots \mathbf{h}_{\alpha_k}.$$

We can write \mathbf{h}_α in terms of the elementary noncommutative symmetric functions by

$$\mathbf{h}_\alpha = \sum_{\beta \preccurlyeq \alpha} (-1)^{|\alpha| - \ell(\beta)} \mathbf{e}_\beta \quad (2.2)$$

where the sum is over all β that refine α .

The *noncommutative ribbon Schur function* is defined by

$$\mathbf{r}_\alpha = \sum_{\beta \succcurlyeq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \mathbf{h}_\beta \quad (2.3)$$

where the sum is over all β that are coarsenings of α .

As Hopf algebras, NSym and QSym are dual with the pairing

$$\langle \mathbf{h}_\alpha, M_\beta \rangle = \delta_{\alpha\beta}$$

and

$$\langle \mathbf{r}_\alpha, F_\beta \rangle = \delta_{\alpha\beta}$$

where $\delta_{\alpha\beta}$ is 1 if $\alpha = \beta$ and 0 otherwise.

Recall that in Sym there is an automorphism $\omega : \text{Sym} \rightarrow \text{Sym}$ such that $\omega(s_\lambda) = s_{\lambda'}$ where λ' is the transpose of the partition λ and s_λ denotes the symmetric Schur function. In QSym we have three involutive automorphisms [13], ψ, ρ , and ω defined on the fundamental basis by

$$\psi(F_\alpha) = F_{\alpha^c} \quad (2.4)$$

$$\rho(F_\alpha) = F_{\text{rev}(\alpha)}; \text{ note that } F_{\text{rev}(\alpha)}(x_1, \dots, x_n) = F_\alpha(x_n, \dots, x_1) \quad (2.5)$$

$$\omega(F_\alpha) = F_{\alpha^t}. \quad (2.6)$$

These maps all commute and $\omega = \rho \circ \psi = \psi \circ \rho$.

Observe that more generally (2.5) implies that, for any $f \in \text{QSym}$,

$$\rho(f)(x_1, \dots, x_n) = f(x_n, \dots, x_1). \quad (2.7)$$

For completeness, we give a proof of the second statement in (2.5), which we were unable to find in the literature. Using the fact that $j \in \text{set}(\text{rev}(\alpha)) \iff n - j \in \text{set}(\alpha)$, we have

$$\begin{aligned} F_{\text{rev}(\alpha)}(x_1, \dots, x_n) &= \sum_{\substack{1 \leq i_1 \leq \dots \leq i_n \leq n \\ i_j < i_{j+1} \text{ if } j \in \text{set}(\text{rev}(\alpha))}} x_{i_1} \cdots x_{i_n} \\ &= \sum_{\substack{1 \leq i_1 \leq \dots \leq i_n \leq n \\ i_j < i_{j+1} \text{ if } n-j \in \text{set}(\alpha)}} x_{i_1} \cdots x_{i_n} \\ &= \sum_{\substack{n \geq i_n \geq \dots \geq i_1 \geq 1 \\ i_{n-j+1} > i_{n-j} \text{ if } j \in \text{set}(\alpha)}} x_{i_n} \cdots x_{i_1} \\ &= F_\alpha(x_n, \dots, x_1). \end{aligned}$$

Finally the truth of (2.7) is evident upon passing to the fundamental expansion of $f \in \text{QSym}$.

There are corresponding involutions in NSym, denoted by the same letters, and defined on the noncommutative ribbon basis by

$$\psi(\mathbf{r}_\alpha) = \mathbf{r}_{\alpha^c} \quad \psi(\mathbf{r}_\alpha \mathbf{r}_\beta) = \psi(\mathbf{r}_\alpha) \psi(\mathbf{r}_\beta) \quad (2.8)$$

$$\rho(\mathbf{r}_\alpha) = \mathbf{r}_{\text{rev}(\alpha)} \quad \rho(\mathbf{r}_\alpha \mathbf{r}_\beta) = \rho(\mathbf{r}_\beta) \rho(\mathbf{r}_\alpha) \quad (2.9)$$

$$\omega(\mathbf{r}_\alpha) = \mathbf{r}_{\alpha^t} \quad \omega(\mathbf{r}_\alpha \mathbf{r}_\beta) = \omega(\mathbf{r}_\beta) \omega(\mathbf{r}_\alpha). \quad (2.10)$$

In NSym, ρ and ω are anti-automorphisms while ψ is an automorphism. We also have that $\psi(\mathbf{h}_\alpha) = \mathbf{e}_\alpha$, $\rho(\mathbf{h}_\alpha) = \mathbf{h}_{\text{rev}(\alpha)}$ and $\omega(\mathbf{h}_\alpha) = \mathbf{e}_{\text{rev}(\alpha)}$.

Proposition 2.2. *The pairing between NSym and QSym is invariant under the map ψ . That is, for $F \in \text{QSym}$ and $\mathbf{g} \in \text{NSym}$, we have*

$$\langle \mathbf{g}, F \rangle = \langle \psi(\mathbf{g}), \psi(F) \rangle.$$

Proof. It suffices to check that the equality holds for the noncommutative ribbon basis elements $\mathbf{g} = \mathbf{r}_\alpha$ and the basis of fundamental quasisymmetric functions $F = F_\beta$, where α, β are compositions of n . But this is clear from the preceding definitions. \square

Recall from [13, Section 3.4.2], the *forgetful* map

$$\chi : \text{NSym} \longrightarrow \text{Sym}$$

satisfying $\chi(\mathbf{e}_n) = e_n$, where e_n is the n th elementary symmetric function, and similarly we will denote the n th complete homogeneous symmetric function by h_n . For a composition $\alpha \models n$, as in [13, Section 2.2], let $\tilde{\alpha}$ be the partition of n obtained by taking the parts of α in weakly decreasing order. Then

$$\chi(\mathbf{h}_\alpha) = h_{\tilde{\alpha}}, \quad \chi(\mathbf{e}_\alpha) = e_{\tilde{\alpha}}.$$

Proposition 2.3. *For $\mathbf{g} \in \text{NSym}$, $(\chi \circ \psi)(\mathbf{g}) = (\omega \circ \chi)(\mathbf{g})$.*

Proof. It suffices to verify the equality for the basis elements \mathbf{h}_α . We have

$$\chi(\psi(\mathbf{h}_\alpha)) = \chi(\mathbf{e}_\alpha) = e_{\tilde{\alpha}} = \omega(h_{\tilde{\alpha}}) = \omega(\chi(\mathbf{h}_\alpha)),$$

as claimed, where $e_{\tilde{\alpha}} = \omega(h_{\tilde{\alpha}})$ follows by the definition of ω in Sym. \square

2.1. Dual immaculate functions

The immaculate functions \mathfrak{S}_α are a basis of NSym formed by iterated creation operators [4]. Their duals in QSym form the basis consisting of dual immaculate functions, \mathfrak{S}_α^* . These functions can be defined combinatorially as the generating function for immaculate tableaux.

Definition 2.4. Given a composition α , an *immaculate tableau* of shape α is a filling, D , of the cells of the diagram of α with positive integers such that

- (1) The leftmost column entries strictly increase from bottom to top.
- (2) The row entries weakly increase from left to right.

An immaculate tableau of shape $\alpha \models n$ is *standard* if it is filled with distinct entries taken from $\{1, 2, \dots, n\}$. Given an immaculate tableau D , we form a *content monomial*, x^D , by setting the exponent of x_i to be d_i , the number of i 's in the tableau D , namely, $x^D = x_1^{d_1} x_2^{d_2} \dots x_k^{d_k}$. We call the vector (d_1, d_2, \dots) the *content* of the tableau D . In particular a standard tableau of shape $\alpha \models n$ has content equal to the composition (1^n) .

Definition 2.5. The *dual immaculate function* indexed by the composition α is

$$\mathfrak{S}_\alpha^* = \sum_D x^D$$

where the sum is over all immaculate tableaux of shape α .

We can rewrite the dual immaculate functions in terms of the fundamental basis as a sum over standard immaculate tableaux. To do this, we first standardize each immaculate tableau and define a descent set on the standard immaculate tableaux. The *reading word* of an immaculate tableau D is obtained by reading the entries of D from left to right starting with the top row. We can standardize a semi-standard tableau (repeated entries allowed) by replacing all the 1's in the reading word by $1, 2, \dots$, in reading order, then the 2's, etc.

Example 2.6. Here is an immaculate tableau of shape $\alpha = (3, 2, 4, 1, 2)$ that has reading word $6\,7\,5\,3\,4\,4\,5\,2\,2\,1\,1\,2$, and its standardization.

$$T = \begin{array}{|c|c|c|c|} \hline 6 & 7 & & \\ \hline 5 & & & \\ \hline 3 & 4 & 4 & 5 \\ \hline 2 & 2 & & \\ \hline 1 & 1 & 2 & \\ \hline \end{array} \quad S = \begin{array}{|c|c|c|c|} \hline 11 & 12 & & \\ \hline 9 & & & \\ \hline 6 & 7 & 8 & 10 \\ \hline 3 & 4 & & \\ \hline 1 & 2 & 5 & \\ \hline \end{array}$$

For a composition α , let $\text{SIT}(\alpha)$ denote the set of standard immaculate tableaux of shape α .

Definition 2.7. [4, Definition 3.20] Given a standard immaculate tableau S , the *descent set* of S , denoted $\text{Des}_{\mathfrak{S}^*}(S)$, is

$$\text{Des}_{\mathfrak{S}^*}(S) = \{i : i+1 \text{ appears strictly above } i \text{ in } S\}.$$

We refer to $\text{Des}_{\mathfrak{S}^*}(S)$ as the \mathfrak{S}^* -descent set of S , and to its associated composition $\text{comp}(\text{Des}_{\mathfrak{S}^*}(S))$ as the \mathfrak{S}^* -descent composition of S .

For the standard immaculate tableau in Example 2.6, $\text{Des}_{\mathfrak{S}^*}(S) = \{2, 5, 8, 10\}$. Then [4, Proposition 3.37]

$$\mathfrak{S}_{\alpha}^* = \sum_{S \in \text{SIT}(\alpha)} F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(S))}, \quad (2.11)$$

where the sum is over all standard immaculate tableaux of shape α .

3. Row-strict dual immaculate functions

In this section we start with a combinatorial definition of a new quasisymmetric function that we call the row-strict dual immaculate function.

Definition 3.1. Given a composition α , a *row-strict immaculate tableau* of shape α is a filling, U , of the cells of the diagram of α with positive integers such that

- (1) The leftmost column entries weakly increase from bottom to top.
- (2) The row entries strictly increase from left to right.

We now define our new function, where x^U is the content monomial of the tableau U , defined just before Definition 2.5. The content of the tableau U is also defined just before Definition 2.5.

Definition 3.2. The *row-strict dual immaculate function* indexed by α is

$$\mathcal{R}\mathfrak{S}_{\alpha}^* = \sum_U x^U$$

where the sum is over all row-strict immaculate tableaux of shape α .

We say the row-strict tableau U is standard if $x^U = x_1 \cdots x_n$. Thus standard row-strict immaculate tableaux coincide with standard immaculate tableaux.

As before, standardization provides us with a way to expand $\mathcal{R}\mathfrak{S}_{\alpha}^*$ in terms of the fundamental basis using only standard tableaux.

Definition 3.3. Given a row-strict immaculate tableau T , the *row-strict immaculate reading word* of T , denoted $\text{rw}_{\mathcal{R}\mathfrak{S}^*}(T)$, is the word obtained by reading the entries in the rows of T from right to left starting with the bottom row and moving up.

To *standardize* a row-strict immaculate tableau T , replace the 1's in T with $1, 2, \dots$, in the order they appear in $\text{rw}_{\mathcal{R}\mathfrak{S}^*}(T)$, then the 2's, etc.

Definition 3.4. The *descent set* of a standard row-strict immaculate tableau T is the set

$$\text{Des}_{\mathcal{RS}^*}(T) = \{i : i+1 \text{ is weakly below } i \text{ in } T\}.$$

We refer to $\text{Des}_{\mathcal{RS}^*}(T)$ as the \mathcal{RS}^* -*descent set* of T , and to its associated composition $\text{comp}(\text{Des}_{\mathcal{RS}^*}(T))$ as the \mathcal{RS}^* -*descent composition* of T .

Example 3.5. Consider the row-strict immaculate tableau

$$T = \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline 3 & 4 & 5 & 6 \\ \hline 2 & 5 & & \\ \hline 1 & 2 & 6 & \\ \hline \end{array}$$

The row-strict immaculate reading word of T is 6215265434 and the corresponding standardized row-strict immaculate tableau is

$$S = \begin{array}{|c|c|c|c|} \hline 6 & & & \\ \hline 4 & 5 & 8 & 10 \\ \hline 3 & 7 & & \\ \hline 1 & 2 & 9 & \\ \hline \end{array}$$

Here $\text{Des}_{\mathcal{RS}^*}(T) = \{1, 4, 6, 8\}$.

The row-strict dual immaculate functions expand positively in the fundamental basis.

Theorem 3.6. Let $\alpha \models n$. Then

$$\mathcal{RS}^*_\alpha = \sum_S F_{\text{comp}(\text{Des}_{\mathcal{RS}^*}(S))}$$

where the sum is over all standard row-strict immaculate tableaux of shape α .

Proof. Let T be a row-strict immaculate tableau of shape α . Then T standardizes to some standard row-strict immaculate tableau S . Suppose $i \in \text{Des}_{\mathcal{RS}^*}(S)$. Then $i+1$ is weakly below i in S . If i and $i+1$ are in the same row of S , then the entry of T replaced by i is strictly less than the entry replaced by $i+1$ since rows of T strictly increase. If $i+1$ is in a lower row than i , then the entry of T replaced by i must be strictly less than the entry replaced by $i+1$, else the standardization process was not followed. Thus x^T has strict increases at each position in $\text{Des}_{\mathcal{RS}^*}(S)$ and x^T is a monomial in $F_{\text{comp}(\text{Des}_{\mathcal{RS}^*}(S))}$. Thus every monomial in \mathcal{RS}^*_α appears in $\sum_S F_{\text{comp}(\text{Des}_{\mathcal{RS}^*}(S))}$.

Now let S be a standard row-strict immaculate tableau and let $x_{i_1} \cdots x_{i_n}$ with $i_1 \leq i_2 \leq \cdots \leq i_n$ be a monomial in $F_{\text{comp}(\text{Des}_{\mathcal{RS}^*}(S))}$. Create a new diagram T from S by replacing each entry k in S with i_k . If $i_k = i_{k+1}$ then $k \notin \text{Des}_{\mathcal{RS}^*}(S)$, so k must appear

strictly below $k + 1$ in S and thus each entry in a row of T is distinct and increases left to right. By construction, the first column will weakly increase from bottom to top. Thus T is a semi-standard row-strict immaculate tableau with content (i_1, \dots, i_n) , and $x_{i_1} \cdots x_{i_n}$ is a monomial in $\mathcal{R}\mathfrak{S}_\alpha^*$. \square

Example 3.7. Let

$$S = \begin{array}{|c|c|c|c|} \hline 6 & & & \\ \hline 4 & 5 & 8 & 10 \\ \hline 3 & 7 & & \\ \hline 1 & 2 & 9 & \\ \hline \end{array}$$

be a standard row-strict immaculate tableau. Then $\text{Des}_{\mathcal{R}\mathfrak{S}^*}(S) = \{1, 4, 6, 8\}$ and $x^P = x_1 x_2^2 x_3 x_4^2 x_5^2 x_6^2$ is a monomial in $F_{\text{comp}(\text{Des}_{\mathcal{R}\mathfrak{S}^*}(S))}$. We can “destandardize” S as described in the proof of Theorem 3.6 to obtain

$$T = \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline 3 & 4 & 5 & 6 \\ \hline 2 & 5 & & \\ \hline 1 & 2 & 6 & \\ \hline \end{array}.$$

For any standard immaculate tableau S , note by definition that $\text{Des}_{\mathfrak{S}^*}(S) = \text{Des}_{\mathcal{R}\mathfrak{S}^*}(S)^c$.

It will be helpful to know how the involutions ψ , ρ , and ω act on \mathfrak{S}_α^* .

Theorem 3.8. *Let α be a composition. Then*

$$\psi(\mathfrak{S}_\alpha^*) = \mathcal{R}\mathfrak{S}_\alpha^* \quad (3.1)$$

$$\rho(\mathfrak{S}_\alpha^*(x_1, \dots, x_n)) = \mathfrak{S}_\alpha^*(x_n, \dots, x_1) \quad (3.2)$$

$$\omega(\mathfrak{S}_\alpha^*(x_1, \dots, x_n)) = \mathcal{R}\mathfrak{S}_\alpha^*(x_n, \dots, x_1) \quad (3.3)$$

Proof. Let α be a composition. Recall from (2.4) that $\psi(F_\alpha) = F_{\alpha^c}$. Then

$$\begin{aligned} \psi(\mathfrak{S}_\alpha^*) &= \psi \left(\sum_S F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(S))} \right) \\ &= \sum_S \psi(F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(S))}) \\ &= \sum_S F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(S)^c)} \\ &= \sum_S F_{\text{comp}(\text{Des}_{\mathcal{R}\mathfrak{S}^*}(S))} \\ &= \mathcal{R}\mathfrak{S}_\alpha^*. \end{aligned}$$

The second equation follows from (2.7). Finally the third equation is now a consequence of the fact that $\omega = \rho \circ \psi$. \square

Corollary 3.9. *We have that $\{\mathcal{R}\mathfrak{S}_\alpha^* \mid \alpha \models n\}$ is a basis for QSym_n .*

Proof. Since $\{\mathfrak{S}_\alpha^* \mid \alpha \models n\}$ is a basis for QSym_n and ψ is an involution it follows by Theorem 3.8 that $\{\mathcal{R}\mathfrak{S}_\alpha^* \mid \alpha \models n\}$ is also a basis for QSym_n . \square

Recall from Section 2.1 that the *immaculate functions* \mathfrak{S}_β form a basis of NSym satisfying, by definition,

$$\langle \mathfrak{S}_\alpha, \mathfrak{S}_\beta^* \rangle = \delta_{\alpha\beta}.$$

Similarly, by definition, we have *row-strict immaculate functions* $\mathcal{R}\mathfrak{S}_\beta$ in NSym satisfying

$$\langle \mathcal{R}\mathfrak{S}_\alpha, \mathcal{R}\mathfrak{S}_\beta^* \rangle = \delta_{\alpha\beta}.$$

An immediate consequence of these definitions is the effect of the map ψ on \mathfrak{S}_α . Using Proposition 2.2, we have, by duality,

$$\delta_{\alpha\beta} = \langle \mathfrak{S}_\alpha, \mathfrak{S}_\beta^* \rangle = \langle \psi(\mathfrak{S}_\alpha), \psi(\mathfrak{S}_\beta^*) \rangle = \langle \psi(\mathfrak{S}_\alpha), \mathcal{R}\mathfrak{S}_\beta^* \rangle,$$

and hence $\psi(\mathfrak{S}_\alpha) = \mathcal{R}\mathfrak{S}_\alpha$.

From [4, Proposition 3.36] we have that the dual immaculate functions are monomial positive:

$$\mathfrak{S}_\alpha^* = \sum_{\beta \leq_\ell \alpha} K_{\alpha,\beta} M_\beta,$$

where $K_{\alpha,\beta}$ is the number of immaculate tableaux of shape α and content β . It follows from this expansion that $K_{\alpha,\beta} = \langle \mathbf{h}_\beta, \mathfrak{S}_\alpha^* \rangle = \langle \mathbf{e}_\beta, \mathcal{R}\mathfrak{S}_\alpha^* \rangle$. Similarly, for row-strict dual immaculate functions, we have by their definition and that of monomial quasisymmetric functions that

$$\mathcal{R}\mathfrak{S}_\alpha^* = \sum_{\beta} K_{\alpha,\beta}^* M_\beta$$

where $K_{\alpha,\beta}^*$ is the number of row-strict immaculate tableaux of shape α and content β , and $K_{\alpha,\beta}^* = \langle \mathbf{h}_\beta, \mathcal{R}\mathfrak{S}_\alpha^* \rangle = \langle \mathbf{e}_\beta, \mathfrak{S}_\alpha^* \rangle$. Note that $K_{\alpha,\beta} \neq K_{\alpha,\beta}^*$ in general, and $K_{\alpha,\alpha}^* = 0$ unless $\ell(\alpha) \geq \alpha_1$.

By contrast, it is easy to see [4, Proposition 3.36] that the transition matrix $(K_{\alpha,\beta})$ is upper unitriangular:

$$K_{\alpha,\alpha} = 1 \text{ and } K_{\alpha,\beta} \neq 0 \implies \beta \leq_\ell \alpha. \quad (3.4)$$

Example 3.10. Let $\alpha = (1, 2)$. Then the tableau definitions give the monomial expansions

$$\mathcal{R}\mathfrak{S}_{12}^* = M_{21} + M_{111}, \quad \mathfrak{S}_{12}^* = M_{12} + M_{111}.$$

As this example shows, the triangularity condition of (3.4) is false for the transition matrix $K_{\alpha,\beta}^*$, making it difficult to deduce, directly from the monomial expansion, that the row-strict dual immaculate functions form a basis.

Let $L_{\alpha,\beta}$ denote the number of standard immaculate tableaux of shape α with \mathfrak{S}^* -descent composition β and $L_{\alpha,\beta}^*$ denote the number of standard immaculate tableaux of shape α with $\mathcal{R}\mathfrak{S}^*$ -descent composition β . Given a standard immaculate tableau T , we have $\text{Des}_{\mathfrak{S}^*}(T)^c = \text{Des}_{\mathcal{R}\mathfrak{S}^*}(T)$, and so $L_{\alpha,\beta}^* = L_{\alpha,\beta^c}$.

Theorem 3.11. Fix a composition α . For any composition γ with $|\gamma| = |\alpha|$,

$$K_{\alpha,\gamma}^* = \sum_{\beta \succcurlyeq \gamma} L_{\alpha,\beta^c} = \sum_{\beta \succcurlyeq \gamma} L_{\alpha,\beta}^*.$$

Proof. We have

$$\mathcal{R}\mathfrak{S}_{\alpha}^* = \sum_{\gamma} K_{\alpha,\gamma}^* M_{\gamma},$$

and

$$\mathcal{R}\mathfrak{S}_{\alpha}^* = \sum_{T \in \text{SIT}(\alpha)} F_{\text{comp}(\text{Des}_{\mathcal{R}\mathfrak{S}^*}(T))} = \sum_{\beta} L_{\alpha,\beta}^* F_{\beta} = \sum_{\beta} L_{\alpha,\beta^c} F_{\beta}.$$

Since the monomial expansion of F_{β} is $F_{\beta} = \sum_{\gamma \preccurlyeq \beta} M_{\gamma}$, equating coefficients of M_{γ} gives

$$K_{\alpha,\gamma}^* = \sum_{\beta \succcurlyeq \gamma} L_{\alpha,\beta^c} = \sum_{\beta \succcurlyeq \gamma} L_{\alpha,\beta}^*. \quad \square$$

3.1. Creation operators and row-strict immaculate functions

In [4], the authors defined a family of operators on NSym , modelled after Bernstein's operators that were used to define the ordinary Schur functions in the Hopf algebra of symmetric functions [14, pp. 95-97, Exercise 29]. This new family of "creation operators" was then used to define the immaculate basis of NSym , and, via the pairing between NSym and its dual QSym , the dual immaculate quasisymmetric functions \mathfrak{S}_{α}^* .

In this section we define a variant of the creation operators of [4], and show how they in turn lead to a definition of the row-strict immaculate basis of NSym and our row-strict dual immaculate quasisymmetric functions $\mathcal{R}\mathfrak{S}_{\alpha}^*$.

A pair of dual Hopf algebras A and B over a field \mathbb{K} induces a pairing $\langle, \rangle : A \times B \rightarrow \mathbb{K}$. Hence for each element $F \in B$, one can define the adjoint operator $F^{\perp} : A \rightarrow A$ by

$$\langle F^\perp(a), b \rangle = \langle a, Fb \rangle.$$

Explicitly, if $\{a_\alpha\}$ and $\{b_\alpha\}$ are bases of A and B respectively so that $\langle a_\alpha, b_\beta \rangle = \delta_{\alpha\beta}$ as before, then the operator F^\perp may be computed according to the formula

$$F^\perp(g) = \sum_{\alpha} \langle g, Fb_\alpha \rangle a_\alpha. \quad (3.5)$$

As in [4], we apply this to the graded dual Hopf algebras $A = \text{NSym}$ and $B = \text{QSym}$. Let $\{F_\alpha\}_{\alpha \models n}$ be the basis of fundamental quasisymmetric functions in QSym , indexed by the compositions α of the nonnegative integer n . We will consider the linear transformation F_α^\perp of NSym that is adjoint to multiplication by F_α in QSym .

First we record the following important effect of the involution ψ on the adjoint transformation.

Proposition 3.12. *Let $F \in \text{QSym}$, $H \in \text{NSym}$. Then*

$$\psi[F^\perp(\psi(H))] = [\psi(F)]^\perp(H),$$

or equivalently,

$$\psi[F^\perp(H)] = [\psi(F)]^\perp(\psi(H)).$$

In particular, for the fundamental quasisymmetric function F_α indexed by the composition α , we have $F_\alpha^\perp(\psi(H)) = \psi[F_{\alpha^c}^\perp(H)]$ and hence

$$F_{(1^i)}^\perp(\psi(H)) = \psi[F_{(i)}^\perp(H)], \quad F_{(i)}^\perp(\psi(H)) = \psi[F_{(1^i)}^\perp(H)].$$

Proof. Let $\{a_\alpha\}_{\alpha \models n}$ and $\{b_\alpha\}_{\alpha \models n}$ be dual bases of NSym and QSym respectively, so that $\langle a_\alpha, b_\beta \rangle = \delta_{\alpha\beta}$.

From Equation (3.5) we have

$$F^\perp(\psi(H)) = \sum_{\alpha} \langle \psi(H), Fb_\alpha \rangle a_\alpha = \sum_{\alpha} \langle H, \psi(F)\psi(b_\alpha) \rangle a_\alpha$$

by Proposition 2.2, and hence

$$\psi[F^\perp(\psi(H))] = \sum_{\alpha} \langle H, \psi(F)\psi(b_\alpha) \rangle \psi(a_\alpha) = [\psi(F)]^\perp(H),$$

since again Proposition 2.2 implies that duality of bases is preserved under ψ . \square

Lemma 3.13. [4, Lemma 2.6] *For $i, j > 0$ and $f \in \text{NSym}$,*

$$F_{(1^i)}^\perp(f\mathbf{h}_j) = F_{(1^i)}^\perp(f)\mathbf{h}_j + F_{(1^{i-1})}^\perp(f)\mathbf{h}_{j-1}; \quad F_{(i)}^\perp(f\mathbf{h}_j) = \sum_{k=0}^{\min(i,j)} F_{(i-k)}^\perp(f)\mathbf{h}_{j-k}.$$

In particular we have

$$F_{(i)}^\perp(\mathbf{h}_j) = \begin{cases} 0, & i > j \\ \mathbf{h}_{j-i}, & 1 \leq i \leq j \\ \mathbf{h}_j, & i = 0; \end{cases} \quad F_{(1^i)}^\perp(\mathbf{h}_j) = \begin{cases} 0, & i > 1 \\ \mathbf{h}_{j-1}, & i = 1 \\ \mathbf{h}_j, & i = 0. \end{cases}$$

The next two definitions are made in [4].

Definition 3.14. [4, Definition 3.1] The *noncommutative Bernstein operator* \mathbb{B}_m is defined by

$$\mathbb{B}_m = \sum_{i \geq 0} (-1)^i \mathbf{h}_{m+i} F_{(1^i)}^\perp,$$

and for $\alpha \in \mathbb{Z}^m$,

$$\mathbb{B}_\alpha = \mathbb{B}_{\alpha_1} \cdots \mathbb{B}_{\alpha_m}.$$

Note that when $i = 0$, (1^0) is the empty composition and thus $F_{(1^0)}^\perp(f) = f = F_\emptyset^\perp(f)$ for all $f \in \text{NSym}$, since $F_\emptyset = 1$ in QSym . Also $F_{(1^i)}^\perp(1) = F_{(i)}^\perp(1) = \begin{cases} 0 & i > 0, \\ 1 & i = 0. \end{cases}$

While we chose duality to define immaculate functions, the following is the original definition, which was proven to be equivalent in [4].

Definition 3.15. [4, Definition 3.2] For any $\alpha \in \mathbb{Z}^m$, the *immaculate function* $\mathfrak{S}_\alpha \in \text{NSym}$ is given by

$$\mathfrak{S}_\alpha = \mathbb{B}_\alpha(1) = \mathbb{B}_{\alpha_1} \cdots \mathbb{B}_{\alpha_m}(1).$$

This definition was inspired by Bernstein's original definition in the Hopf algebra of symmetric functions for a Schur function s_α indexed by any m -tuple $\alpha \in \mathbb{Z}^m$.

As observed in [4, Example 3.3], we have

$$\mathfrak{S}_{(m)} = \mathbb{B}_m(1) = \mathbf{h}_m, \quad \mathfrak{S}_{(a,b)} = \mathbb{B}_a(\mathbf{h}_b) = \mathbf{h}_a \mathbf{h}_b - \mathbf{h}_{a+1} \mathbf{h}_{b-1}.$$

Applying ψ to Lemma 3.13, and using Proposition 3.12 and the fact that $\psi(F_\alpha) = F_{\alpha^c}$, so that $\psi(F_{(1^i)}) = F_{(i)}$ in NSym_i , we obtain

Lemma 3.16. For $i, j > 0$ and $f \in \text{NSym}$,

$$F_{(i)}^\perp(f \mathbf{e}_j) = F_{(i)}^\perp(f) \mathbf{e}_j + F_{(i-1)}^\perp(f) \mathbf{e}_{j-1}; \quad F_{(1^i)}^\perp(f \mathbf{e}_j) = \sum_{k=0}^{\min(i,j)} F_{(1^{i-k})}^\perp(f) \mathbf{e}_{j-k}.$$

In particular we have

$$F_{(1^i)}^\perp(\mathbf{e}_j) = \begin{cases} 0, & i > j \\ \mathbf{e}_{j-i}, & 1 \leq i \leq j \\ \mathbf{e}_j, & i = 0; \end{cases} \quad F_{(i)}^\perp(\mathbf{e}_j) = \begin{cases} 0, & i > 1 \\ \mathbf{e}_{j-1}, & i = 1 \\ \mathbf{e}_j, & i = 0. \end{cases}$$

Now we define new operators as follows.

Definition 3.17. Define the *noncommutative Bernstein operator* \mathbb{B}_m^{rs} by

$$\mathbb{B}_m^{rs} = \sum_{i \geq 0} (-1)^i \mathbf{e}_{m+i} F_{(i)}^\perp,$$

and for $\alpha \in \mathbb{Z}^m$,

$$\mathbb{B}_\alpha^{rs} = \mathbb{B}_{\alpha_1}^{rs} \cdots \mathbb{B}_{\alpha_m}^{rs}.$$

Note that when $i = 0$, this is the empty composition and $F_\emptyset = 1$ in QSym , and thus $F_{(0)}^\perp(f) = f = F_\emptyset^\perp(f)$ for all $f \in \text{NSym}$.

Furthermore we have the following.

Lemma 3.18. For $\alpha \in \mathbb{Z}^m$, $\psi(\mathfrak{S}_\alpha) = \mathbb{B}_\alpha^{rs}(1)$.

Proof. From the above properties, it is clear that

$$\mathbb{B}_m^{rs}(1) = \mathbf{e}_m, \quad \psi(\mathfrak{S}_{(a,b)}) = \mathbb{B}_a^{rs}(\mathbf{e}_b) = \mathbf{e}_a \mathbf{e}_b - \mathbf{e}_{a+1} \mathbf{e}_{b-1}.$$

Hence the result is true for $m \leq 2$. Let $f \in \text{NSym}$. We claim that

$$\psi(\mathbb{B}_m(f)) = \mathbb{B}_m^{rs}(\psi(f)). \quad (3.6)$$

We have

$$\begin{aligned} \psi(\mathbb{B}_m(f)) &= \psi \left[\sum_{i \geq 0} (-1)^i \mathbf{h}_{m+i} F_{(1^i)}^\perp(f) \right] = \sum_{i \geq 0} (-1)^i \mathbf{e}_{m+i} \psi[F_{(1^i)}^\perp(f)] \\ &= \sum_{i \geq 0} (-1)^i \mathbf{e}_{m+i} F_{(i)}^\perp(\psi(f)) = \mathbb{B}_m^{rs}(\psi(f)), \end{aligned}$$

where the penultimate equality is thanks to Proposition 3.12.

Since for $\alpha \in \mathbb{Z}^m$,

$$\mathbb{B}_\alpha(1) = \mathbb{B}_{\alpha_1}(f), \quad f = \mathbb{B}_{\alpha_2} \cdots \mathbb{B}_{\alpha_m}(1),$$

the result now follows by induction. \square

Theorem 3.19. *The row-strict immaculate function $\mathcal{R}\mathfrak{S}_\alpha$ can be defined as the result of applying a creation operator as follows:*

$$\mathcal{R}\mathfrak{S}_\alpha = \mathbb{B}_\alpha^{rs}(1).$$

Proof. Immediate from the preceding lemma, since we already know that $\mathcal{R}\mathfrak{S}_\alpha = \psi(\mathfrak{S}_\alpha)$. \square

Finally, just as left multiplication by \mathbf{h}_m can be expressed in terms of creation operators [4, Remark 3.6], we have the following.

Lemma 3.20. *Left multiplication by \mathbf{h}_m in NSym can be expressed as applying the operator*

$$\mathbf{h}_m = \sum_{i \geq 0} \mathbb{B}_{m+1} F_{(i)}^\perp,$$

and left multiplication by \mathbf{e}_m in NSym can be expressed as applying the operator

$$\mathbf{e}_m = \sum_{i \geq 0} \mathbb{B}_{m+1}^{rs} F_{(1^i)}^\perp.$$

Proof. Immediate from Equation (3.6). \square

3.2. Results obtained by using ψ

We can immediately obtain the row-strict analogue of many results in [4] by using the involution ψ . We list here the most pertinent for the remainder of the paper. We leave results for skew row-strict dual immaculate functions to the next section, as the combinatorial definition is not obviously equivalent.

Theorem 3.21.

(1) [4, Lemma 3.4] For $s \geq 0, m \in \mathbb{Z}$ and $f \in \text{NSym}$,

$$\begin{aligned} \mathbb{B}_m(f) \mathbf{h}_s &= \mathbb{B}_{m+1}(f) \mathbf{h}_{s-1} + \mathbb{B}_m(f \mathbf{h}_s) \\ \xLeftrightarrow{\psi} \mathbb{B}_m^{rs}(f) \mathbf{e}_s &= \mathbb{B}_{m+1}^{rs}(f) \mathbf{e}_{s-1} + \mathbb{B}_m^{rs}(f \mathbf{e}_s). \end{aligned}$$

(2) [4, Theorem 3.5] (Multiplicity-free right Pieri rule) For a composition α and $s \geq 0$,

$$\mathfrak{S}_\alpha \mathbf{h}_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta \xLeftrightarrow{\psi} \mathcal{R}\mathfrak{S}_\alpha \mathbf{e}_s = \sum_{\alpha \subset_s \beta} \mathcal{R}\mathfrak{S}_\beta.$$

- (3) [4, Proposition 3.32] (Another multiplicity-free right Pieri rule) For a composition α and $s \geq 0$,

$$\mathfrak{S}_\alpha \mathfrak{S}_{(1^s)} = \mathfrak{S}_\alpha \mathbf{e}_s = \sum_{\beta} \mathfrak{S}_\beta \xleftrightarrow{\psi} \mathcal{R}\mathfrak{S}_\alpha \mathcal{R}\mathfrak{S}_{(1^s)} = \mathcal{R}\mathfrak{S}_\alpha \mathbf{h}_s = \sum_{\beta} \mathcal{R}\mathfrak{S}_\beta,$$

where the summation ranges over compositions of β of $|\alpha| + s$ such that $\alpha_i \leq \beta_i \leq \alpha_i + 1$ and $\alpha_i = 0$ for $i > \ell(\alpha)$.

- (4) [4, Corollary 3.31] For $n \geq 0$,

$$\mathfrak{S}_{(1^n)} = \sum_{\alpha \models n} (-1)^{n-\ell(\alpha)} \mathbf{h}_\alpha = \mathbf{e}_n \xleftrightarrow{\psi} \mathcal{R}\mathfrak{S}_{(1^n)} = \sum_{\alpha \models n} (-1)^{n-\ell(\alpha)} \mathbf{e}_\alpha = \mathbf{h}_n.$$

- (5) [4, Theorem 3.27] (Jacobi-Trudi) For $\ell(\alpha) = m$,

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_m} (-1)^{\text{sgn}(\sigma)} \mathbf{h}_{(\alpha_1+\sigma_1-1, \alpha_2+\sigma_2-2, \dots, \alpha_m+\sigma_m-m)}$$

$$\xleftrightarrow{\psi}$$

$$\mathcal{R}\mathfrak{S}_\alpha = \sum_{\sigma \in S_m} (-1)^{\text{sgn}(\sigma)} \mathbf{e}_{(\alpha_1+\sigma_1-1, \alpha_2+\sigma_2-2, \dots, \alpha_m+\sigma_m-m)}$$

where S_m is the symmetric group on m elements and $(-1)^{\text{sgn}(\sigma)}$ is the sign of σ .

- (6) From [4, Lemma 2.5] and Equation (3.6),

$$F_{(1^r)}^\perp(\mathfrak{S}_{(1^n)}) = \mathfrak{S}_{(1^{n-r})}, \text{ and for } s > 1, F_{(1^s)}^\perp(\mathfrak{S}_{(1^n)}) = 0$$

which is equivalent via the map ψ to

$$F_{(1^r)}^\perp(\mathcal{R}\mathfrak{S}_{(1^n)}) = \mathcal{R}\mathfrak{S}_{(1^{n-r})}, \text{ and for } s > 1, F_{(1^s)}^\perp(\mathcal{R}\mathfrak{S}_{(1^n)}) = 0.$$

- (7) [4, Proposition 3.16 and Corollary 3.18]

$$\mathbf{h}_\beta = \sum_{\alpha \geq_\ell \beta} K_{\alpha, \beta} \mathfrak{S}_\alpha \xleftrightarrow{\psi} \mathbf{e}_\beta = \sum_{\alpha \geq_\ell \beta} K_{\alpha, \beta} \mathcal{R}\mathfrak{S}_\alpha$$

and by Theorem 3.11

$$\mathbf{h}_\beta = \sum_{\alpha} K_{\alpha, \beta}^* \mathcal{R}\mathfrak{S}_\alpha \xleftrightarrow{\psi} \mathbf{e}_\beta = \sum_{\alpha} K_{\alpha, \beta}^* \mathfrak{S}_\alpha.$$

- (8) [4, Theorem 3.25] The ribbon function \mathbf{r}_β expands positively in both immaculate bases:

$$\mathbf{r}_\beta = \sum_{\alpha \geq \ell \beta} L_{\alpha, \beta} \mathfrak{S}_\alpha \xLeftrightarrow{\psi} \mathbf{r}_{\beta^c} = \sum_{\alpha \geq \ell \beta} L_{\alpha, \beta} \mathcal{R} \mathfrak{S}_\alpha.$$

(Recall that $L_{\alpha, \beta}$ denotes the number of standard immaculate tableaux of shape α and descent composition β .)

- (9) [4, Theorem 3.38] The Schur function s_λ with $\ell(\lambda) = k$ expands into the dual immaculate and row-strict dual immaculate bases as follows:

$$s_\lambda = \sum_{\sigma \in S_k} (-1)^{\text{sgn}(\sigma)} \mathfrak{S}_{\sigma(\lambda)}^* \xLeftrightarrow{\psi} s_{\lambda'} = \sum_{\sigma \in S_k} (-1)^{\text{sgn}(\sigma)} \mathcal{R} \mathfrak{S}_{\sigma(\lambda)}^*$$

where, for λ a partition and $\sigma \in S_k$, we define $\sigma(\lambda) = (\lambda_{\sigma_1} + 1 - \sigma_1, \dots, \lambda_{\sigma_k} + k - \sigma_k)$ provided $\lambda_{\sigma_i} + i - \sigma_i > 0$ for each i . If the latter condition is not satisfied, we define $\mathfrak{S}_{\sigma(\lambda)}^* = 0 = \mathcal{R} \mathfrak{S}_{\sigma(\lambda)}^*$.

- (10) [2, Theorem 1.1] For α a composition and $c_{\alpha\beta} \geq 0$,

$$\mathfrak{S}_\alpha^* = \sum_{\beta} c_{\alpha\beta} \hat{\mathcal{S}}_\beta \xLeftrightarrow{\psi} \mathcal{R} \mathfrak{S}_\alpha^* = \sum_{\beta} c_{\alpha\beta} \mathcal{R} \hat{\mathcal{S}}_\beta,$$

where $\hat{\mathcal{S}}$ and $\mathcal{R} \hat{\mathcal{S}}$ are the Young quasisymmetric Schur and Young row-strict quasisymmetric Schur functions.

4. Skew row-strict dual immaculate functions

Following the work of Berg et al. [4], we define the poset \mathfrak{P} of immaculate tableaux. The labelled poset \mathfrak{P} is on the set of all compositions. Place an arrow from α to β if $\beta \subset \alpha$, and $|\alpha| - |\beta| = 1$, denoted by $\beta \subset_1 \alpha$. The label of m on each cover $\alpha \xrightarrow{m} \beta$ denotes the row containing the single additional box. Denote a path from α to β in \mathfrak{P} by $P = [\alpha, \beta]$.

To obtain a standard skew immaculate tableau from a path $P = [\alpha, \beta]$, for each m_i , $1 \leq i \leq k$, label the rightmost unlabeled cell in row m_i of α with $k-i+1$, see Example 4.3. In order to understand the combinatorial models for skew dual immaculate and skew row-strict dual immaculate functions we define two special types of paths.

Definition 4.1. A path $P = \{\alpha = \beta^{(0)} \xrightarrow{m_1} \beta^{(1)} \xrightarrow{m_2} \dots \xrightarrow{m_k} \beta^{(k)} = \beta\}$ in the poset \mathfrak{P} is a

- *horizontal k -strip* if $m_1 \leq m_2 \leq \dots \leq m_k$, and a
- *vertical k -strip* if $m_1 > m_2 > \dots > m_k$.

The horizontal 3-strip ((red) dotted path) and vertical 3-strip ((blue) dashed path) in Fig. 1 give rise to the following tableaux.

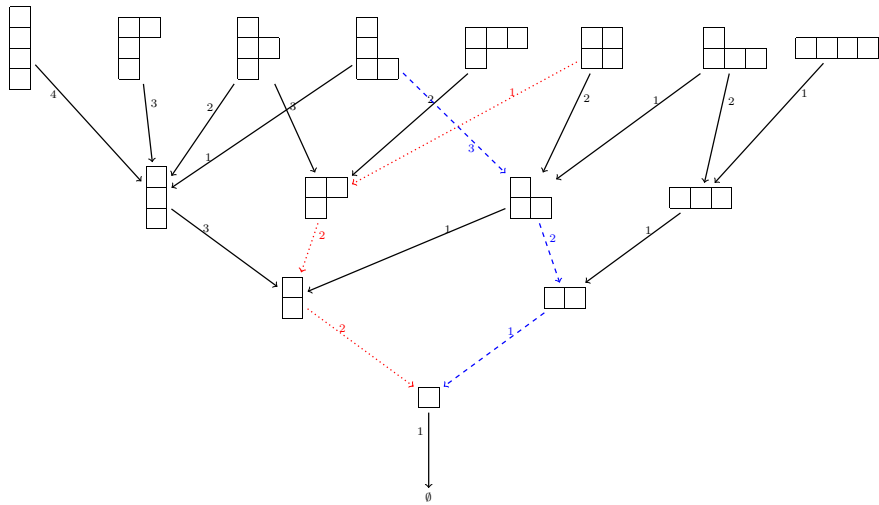
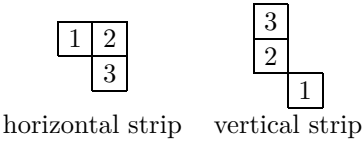


Fig. 1. The start of the poset \mathfrak{P} with edge labels. A horizontal 3-strip is shown in (red) dotted arrows and a vertical 3-strip is shown in (blue) dashed arrows. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

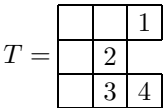


We can now make the following definition.

Definition 4.2. A *standard skew immaculate tableau* of shape α/β is a filling of the shape α/β with the distinct positive integers $\{1, 2, \dots, |\alpha/\beta|\}$, such that rows strictly increase from left to right and the labels in α/β in cells that are in the first column of α must increase from bottom to top.

For a path $P = [\alpha, \beta]$ of length k , define the *descent set* of P to be $D(P) = \{k - i : m_i > m_{i+1}\}$ and the *weak ascent set* of P to $A(P) = \{k - i : m_i \leq m_{i+1}\}$. Each such path $P = [\alpha, \beta]$ corresponds to a unique standard skew immaculate tableau T of shape α/β , and conversely. Furthermore, the descent set $D(P)$ coincides with the descent set $\text{Des}_{\mathfrak{S}^*}(T) = \{i : i + 1 \text{ appears strictly above } i \text{ in } T\}$, and similarly the ascent set $A(P)$ coincides with the descent set $\text{Des}_{\mathcal{R}\mathfrak{S}^*}(T) = \{i : i + 1 \text{ appears weakly below } i \text{ in } T\}$.

Example 4.3. For $\alpha/\beta = (3, 2, 3)/(1, 1, 2)$,



$$T = \begin{array}{|c|c|c|} \hline 4 & 7 & \\ \hline 2 & 3 & 5 \\ \hline 1 & 6 & \\ \hline \end{array}$$

$$P = (2, 3, 2) \xrightarrow{3} (2, 3, 1) \xrightarrow{1} (1, 3, 1) \xrightarrow{2} (1, 2, 1) \xrightarrow{3} (2, 1) \xrightarrow{2} (1, 1) \xrightarrow{2} (1) \xrightarrow{1} \emptyset$$

Fig. 2. The path P has $D(P) = \{1, 3, 6\}$ and $A(P) = \{2, 4, 5\}$, while $\text{Des}_{\mathfrak{S}^*}(T) = \{1, 3, 6\}$ and $\text{Des}_{\mathcal{R}\mathfrak{S}^*}(T) = \{2, 4, 5\}$.

$$T = \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline & 3 & \\ \hline & & 1 \\ \hline \end{array}$$

$$P = (3, 2, 3) \xrightarrow{3} (3, 2, 2) \xrightarrow{3} (3, 2, 1) \xrightarrow{2} (3, 1, 1) \xrightarrow{3} (3, 1) \xrightarrow{1} (2, 1)$$

Fig. 3. The standard skew immaculate tableau T and its corresponding path can be decomposed into maximal horizontal strips $(3, 2, 3) \xrightarrow{3} (3, 2, 2) \xrightarrow{3} (3, 2, 1)$, $(3, 2, 1) \xrightarrow{2} (3, 1, 1) \xrightarrow{3} (3, 1)$, and $(3, 1) \xrightarrow{1} (2, 1)$. Alternatively, decompose P into maximal vertical strips $(3, 2, 3) \xrightarrow{3} (3, 2, 2)$, $(3, 2, 2) \xrightarrow{3} (3, 2, 1) \xrightarrow{2} (3, 1, 1)$, and $(3, 1, 1) \xrightarrow{3} (3, 1) \xrightarrow{1} (2, 1)$.

is a valid standard skew immaculate tableau. It corresponds to the path $P = (3, 2, 3) \xrightarrow{1} (2, 2, 3) \xrightarrow{1} (1, 2, 3) \xrightarrow{2} (1, 1, 3) \xrightarrow{3} (1, 1, 2)$. Further, $\text{Des}_{\mathcal{R}\mathfrak{S}^*}(T) = \{1, 2, 3\}$, $D(P)$ is empty, and $A(P) = \{1, 2, 3\}$.

Given a path $P = [\alpha, \emptyset]$ corresponding to a standard immaculate tableau T , we have that $\text{Des}_{\mathfrak{S}^*}(T) = D(P)$ and $\text{Des}_{\mathcal{R}\mathfrak{S}^*}(T) = A(P)$, by comparing the definitions, and is illustrated in Fig. 2.

Note that given a skew immaculate tableau, it can be decomposed into horizontal or vertical strips in several ways. An example of decomposing a tableau into either horizontal or vertical strips is given in Fig. 3.

In [4] the poset \mathfrak{P} and horizontal strips are used to define the skew dual immaculate functions as follows.

Definition 4.4. For $\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}$ an interval in \mathfrak{P} , define the *skew dual immaculate function* to be

$$\mathfrak{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathfrak{S}_{\beta} \mathbf{h}_{\gamma}, \mathfrak{S}_{\alpha}^* \rangle M_{\gamma}.$$

This can be rewritten in terms of both the fundamental basis and the dual immaculate basis.

Proposition 4.5. [4, Propositions 3.47 and 3.48] For $\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}$ an interval in \mathfrak{P} ,

$$\mathfrak{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathfrak{S}_{\beta} \mathbf{r}_{\gamma}, \mathfrak{S}_{\alpha}^* \rangle F_{\gamma} \quad (4.1)$$

$$= \sum_{\gamma} \langle \mathfrak{S}_{\beta} \mathfrak{S}_{\gamma}, \mathfrak{S}_{\alpha}^* \rangle \mathfrak{S}_{\gamma}^* \quad (4.2)$$

$$= \sum_{P=[\beta, \alpha] \in \mathfrak{P}} F_{\text{comp}(D(P))} = \sum_{T \text{ a standard skew immaculate tableau of shape } \alpha/\beta} F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(T))}; \quad (4.3)$$

in the last line, each path P from β to α corresponds to a unique standard skew immaculate tableau T of shape α/β .

Note that the number of standard skew immaculate tableaux T of shape α/β with $\text{comp}(\text{Des}_{\mathfrak{S}^*}(T)) = \gamma$ is $\langle \mathfrak{S}_{\beta} \mathbf{r}_{\gamma}, \mathfrak{S}_{\alpha}^* \rangle$.

Definition 4.6. For $\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}$ an interval in \mathfrak{P} , define the *skew row-strict dual immaculate function* to be

$$\mathcal{R}\mathfrak{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathcal{R}\mathfrak{S}_{\beta} \mathbf{h}_{\gamma}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle M_{\gamma}.$$

We now quickly obtain the following.

Theorem 4.7. For $\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}$ an interval in \mathfrak{P} ,

$$\mathcal{R}\mathfrak{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathcal{R}\mathfrak{S}_{\beta} \mathbf{r}_{\gamma}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle F_{\gamma} \quad (4.4)$$

$$= \psi(\mathfrak{S}_{\alpha/\beta}^*) \quad (4.5)$$

$$= \sum_{\gamma} \langle \mathcal{R}\mathfrak{S}_{\beta} \mathcal{R}\mathfrak{S}_{\gamma}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle \mathcal{R}\mathfrak{S}_{\gamma}^* \quad (4.6)$$

$$= \sum_{P=[\beta, \alpha] \in \mathfrak{P}} F_{\text{comp}(A(P))} = \sum_{T \text{ a standard skew immaculate tableau of shape } \alpha/\beta} F_{\text{comp}(\text{Des}_{\mathcal{R}\mathfrak{S}^*}(T))}. \quad (4.7)$$

Proof. The first equality is immediate from Definition 4.6 by using (2.3) to expand \mathbf{h}_{γ} in terms of the ribbon basis, interchanging the order of summation, and finally using (2.1):

$$\begin{aligned} \mathcal{R}\mathfrak{S}_{\alpha/\beta}^* &= \sum_{\gamma} \langle \mathcal{R}\mathfrak{S}_{\beta} \sum_{\tau \succ \gamma} \mathbf{r}_{\tau}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle M_{\gamma} = \sum_{\tau} \langle \mathcal{R}\mathfrak{S}_{\beta} \mathbf{r}_{\tau}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle \left(\sum_{\gamma \preccurlyeq \tau} M_{\gamma} \right) \\ &= \sum_{\tau} \langle \mathcal{R}\mathfrak{S}_{\beta} \mathbf{r}_{\tau}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle F_{\tau}. \end{aligned}$$

The second line then follows by applying ψ to the first equality in Proposition 4.5, and using the invariance of the pairing under ψ , which gives

$$\psi(\mathfrak{S}_{\alpha/\beta}^*) = \sum_{\gamma} \langle \psi(\mathfrak{S}_{\beta}) \psi(\mathbf{r}_{\gamma}), \psi(\mathfrak{S}_{\alpha}^*) \rangle \psi(F_{\gamma}) = \sum_{\gamma} \langle \mathcal{R}\mathfrak{S}_{\beta}^* \mathbf{r}_{\gamma^c}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle F_{\gamma^c},$$

where we have used (3.1), (2.8) and (2.4). The last two lines are now immediate by applying ψ to the last two equations in Proposition 4.5, since $A(P)$ and $D(P)$ are complementary by definition, and each path P from β to α corresponds to a unique standard skew immaculate tableau T of shape α/β . \square

Definition 4.8. Let α and β be compositions with $\beta \subseteq \alpha$. Then a filling T of the diagram of α/β with positive integers is a *skew immaculate tableau* provided

- (1) the entries in the first column of α (if any remain in α/β) are strictly increasing from bottom to top, and
- (2) rows weakly increase from left to right.

Similarly, T is a *skew row-strict immaculate tableau* if

- (1) the entries in the first column of α (if any remain in α/β) are weakly increasing from bottom to top, and
- (2) rows strictly increase from left to right.

We now have the needed interpretation of the coefficients in Definitions 4.4 and 4.6 to rewrite $\mathfrak{S}_{\alpha/\beta}^*$ and $\mathcal{R}\mathfrak{S}_{\alpha/\beta}^*$ as generating functions of skew immaculate tableaux.

Theorem 4.9. Let α and β be compositions with $\beta \subseteq \alpha$. Then

$$\mathfrak{S}_{\alpha/\beta}^* = \sum_T x^T$$

where the sum is over all skew immaculate tableaux of shape α/β , and

$$\mathcal{R}\mathfrak{S}_{\alpha/\beta}^* = \sum_T x^T$$

where the sum is over all skew row-strict immaculate tableaux of shape α/β .

Proof. By Point (3) in Theorem 3.21, we know that for $\gamma = \gamma_1\gamma_2\cdots\gamma_k$, α can be obtained from β by a series of vertical strips of lengths $\gamma_1, \gamma_2, \dots, \gamma_k$. Thus the coefficient $\langle \mathcal{R}\mathfrak{S}_{\beta} \mathbf{h}_{\gamma}, \mathcal{R}\mathfrak{S}_{\alpha}^* \rangle$ represents the number of ways to add a sequence of vertical strips of lengths $\gamma_1, \gamma_2, \dots, \gamma_k$ from β to α , which counts the number of skew immaculate tableaux T of shape α/β such that the descent composition of T is coarser than γ , since adding

a vertical strip after another one may or may not create a descent. See Example 4.10 below. Thus

$$\langle \mathfrak{S}_\beta \mathbf{h}_\gamma, \mathfrak{S}_\alpha^* \rangle$$

is the number of skew immaculate tableaux of shape α/β of content γ and

$$\langle \mathcal{R}\mathfrak{S}_\beta \mathbf{h}_\gamma, \mathcal{R}\mathfrak{S}_\alpha^* \rangle$$

is the number of skew row-strict immaculate tableaux of shape α/β of content γ . The result now follows immediately from the definitions. \square

Example 4.10. Consider

$$T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline & 3 \\ \hline & 2 \\ \hline \end{array}$$

and corresponding path

$$P = (2, 2, 2) \xrightarrow{3} (2, 2, 1) \xrightarrow{2} (2, 1, 1) \xrightarrow{1} (1, 1, 1) \xrightarrow{3} (1, 1).$$

Note that T can be considered to be formed from vertical strips corresponding to $\gamma = (1, 3)$ or $(1, 1, 2)$, or $(1, 2, 1)$ or $(1, 1, 1, 1)$ since $\text{comp}(\text{Des}_{\mathcal{R}\mathfrak{S}^*}(T)) = (1, 3)$ and is coarser than the listed options for γ .

In analogy with the well-known Schur function identity [14, Chapter 1 Eqn. (5.9)],

$$s_\lambda(X, Y) = \sum_{\mu \subset \lambda} s_\mu(X) s_{\lambda/\mu}(Y), \quad (4.8)$$

Theorem 4.9 immediately gives us the following:

Theorem 4.11. *Suppose we have two sets of variables, X and Y , ordered so that the alphabet X precedes the alphabet Y . Then*

$$\mathfrak{S}_\alpha^*(X, Y) = \sum_{\beta \subset \alpha} \mathfrak{S}_\beta^*(X) \mathfrak{S}_{\alpha/\beta}^*(Y), \quad (4.9)$$

and

$$\mathcal{R}\mathfrak{S}_\alpha^*(X, Y) = \sum_{\beta \subset \alpha} \mathcal{R}\mathfrak{S}_\beta^*(X) \mathcal{R}\mathfrak{S}_{\alpha/\beta}^*(Y). \quad (4.10)$$

4.1. Hopf algebra approach

We consider the Hopf algebra approach to defining skew dual immaculate functions and establish that it is equivalent to the previous definition. To start, we provide a brief introduction to the necessary Hopf algebra background.

We have that NSym and QSym form dual Hopf algebras using the pairing $\langle \cdot, \cdot \rangle : \text{NSym} \otimes \text{QSym} \rightarrow \mathbb{Q}$ defined by $\langle \mathbf{h}_\alpha, M_\beta \rangle = \delta_{\alpha\beta}$ where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise.

Given dual bases $\{B_i\}_{i \in I}$ and $\{D_i\}_{i \in I}$,

$$\begin{aligned} B_i \cdot B_j &= \sum_k b_{i,j}^k B_k & \Leftrightarrow & \Delta D_k = \sum_{i,j} b_{i,j}^k D_i \otimes D_j \\ D_i \cdot D_j &= \sum_k d_{i,j}^k D_k & \Leftrightarrow & \Delta B_k = \sum_{i,j} d_{i,j}^k B_i \otimes B_j \end{aligned}$$

where \cdot is the product and Δ is the coproduct.

For the fundamental quasisymmetric functions, we have that

$$\Delta F_\alpha = \sum_{\substack{(\beta, \gamma) \text{ with} \\ \beta \cdot \gamma = \alpha \text{ or} \\ \beta \odot \gamma = \alpha}} F_\beta \otimes F_\gamma \quad (4.11)$$

where for $\beta = (\beta_1, \dots, \beta_k)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$, $\beta \cdot \gamma = (\beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_n)$ is the *concatenation* of β and γ , and $\beta \odot \gamma = (\beta_1, \dots, \beta_{k-1}, \beta_k + \gamma_1, \gamma_2, \dots, \gamma_n)$ is the *near-concatenation* of β and γ .

Following [5], we can define the coproduct $\Delta \mathfrak{S}_\alpha^*$ in terms of skew elements $\widetilde{\mathfrak{S}_{\alpha/\gamma}^*}$.

Definition 4.12. Let $\alpha \models n$. Then we define

$$\Delta \mathfrak{S}_\alpha^* = \sum_\gamma \mathfrak{S}_\gamma^* \otimes \widetilde{\mathfrak{S}_{\alpha/\gamma}^*}.$$

We show that $\widetilde{\mathfrak{S}_{\alpha/\gamma}^*} = \mathfrak{S}_{\alpha/\gamma}^*$ as described in Proposition 4.5.

Lemma 4.13.

$$\widetilde{\mathfrak{S}_{\alpha/\gamma}^*} = \mathfrak{S}_{\alpha/\gamma}^* = \sum_T F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(T))}$$

where the sum is over all standard skew immaculate tableaux T of shape α/γ .

Proof. We use the technique of [5, Proposition 3.1]. Let T be a standard skew immaculate tableaux such that $|T| = n$. For any k with $0 \leq k \leq n$, let $\mathcal{U}_k(T)$ be the standardization of the skew tableaux consisting of cells of T with entries $\{n-k+1, \dots, n\}$. Also let $\Omega_k(T)$

$$T = \begin{array}{|c|c|c|} \hline 4 & 5 & 8 \\ \hline * & * & 6 \\ \hline * & * & 2 \\ \hline * & 1 & 9 \\ \hline \end{array} \quad \Omega_4(T) = \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline * & * & & \\ \hline * & * & 2 & 3 \\ \hline * & 1 & & \\ \hline \end{array} \quad \mathfrak{U}_5(T) = \begin{array}{|c|c|c|c|} \hline * & 1 & 4 & \\ \hline * & * & 2 & 3 \\ \hline * & * & * & * \\ \hline * & * & 5 & \\ \hline \end{array}$$

Fig. 4. An example of $\Omega_{n-k}(T)$ and $\mathfrak{U}_k(T)$.

be the skew tableaux consisting of the cells of T after removing the entries $\{k+1, \dots, n\}$ as in Fig. 4.

Note that if T is a standard immaculate tableau of shape α , then $T = \Omega_{n-k}(T) \cup (\mathfrak{U}_k(T) + (n-k))$ where $\mathfrak{U}_k(T) + (n-k)$ is $\mathfrak{U}_k(T)$ with $n-k$ added to each entry. Suppose $\text{Des}_{\mathfrak{S}^*}(T) = \alpha$ with $|\alpha| = n$. Then we can rewrite (4.11) as

$$\Delta F_\alpha = \sum_{i=0}^n F_{\beta_i} \otimes F_{\gamma_i}$$

where $|\beta_i| = n-i$, $|\gamma_i| = i$, and either $\beta_i \cdot \gamma_i = \alpha$ or $\beta_i \odot \gamma_i = \alpha$. Observe that $\beta_i = \text{comp}(\text{Des}_{\mathfrak{S}^*}(\Omega_{n-i}(T)))$ and $\gamma_i = \text{comp}(\text{Des}_{\mathfrak{S}^*}(\mathfrak{U}_i(T)))$.

Then

$$\begin{aligned} \Delta \mathfrak{S}_\alpha^* &= \Delta \left(\sum_T F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(T))} \right) \\ &= \sum_T \Delta F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(T))} \\ &= \sum_T \sum_{i=0}^n F_{\beta_i} \otimes F_{\gamma_i} \end{aligned}$$

where T is a standard immaculate tableau of shape α .

Further, by Definition 4.12 we have

$$\begin{aligned} \Delta \mathfrak{S}_\alpha^* &= \sum_\delta \mathfrak{S}_\delta^* \otimes \widetilde{\mathfrak{S}_{\alpha/\delta}^*} \\ &= \sum_\delta \sum_S F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(S))} \otimes \widetilde{\mathfrak{S}_{\alpha/\delta}^*} \end{aligned}$$

where S is a standard immaculate tableau of shape δ .

For a fixed S of shape δ with $|\delta| = n-k$ for some k , there exists a standard immaculate tableau T of shape α such that $S = \Omega_{n-k}(T)$. Then $\mathfrak{U}_k(T)$ has shape α/δ . Similarly, given a standard immaculate tableau T of shape α , $T = \Omega_{n-k}(T) \cup (\mathfrak{U}_k(T) + (n-k))$ where $\Omega_{n-k}(T)$ has shape δ with $|\delta| = n-k$ and $\mathfrak{U}_k(T)$ has shape α/δ . Thus

$$\widetilde{\mathfrak{S}_{\alpha/\delta}^*} = \sum_T F_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(T))} = \mathfrak{S}_{\alpha/\delta}^*$$

where T is a standard skew immaculate tableau of shape α/δ . \square

By the above lemma and Theorem 4.7, we have the following.

Theorem 4.14. *Let α be a composition. Then*

$$\Delta \mathfrak{S}_{\alpha}^* = \sum_{\beta} \mathfrak{S}_{\beta}^* \otimes \mathfrak{S}_{\alpha/\beta}^* \quad \text{and} \quad \Delta \mathcal{R} \mathfrak{S}_{\alpha}^* = \sum_{\beta} \mathcal{R} \mathfrak{S}_{\beta}^* \otimes \mathcal{R} \mathfrak{S}_{\alpha/\beta}^*.$$

4.2. Expansions of skew Schur functions

We can also use a Hopf algebra approach to establish skew versions of Point (9) in Theorem 3.21, from where we recall that for λ a partition and $\sigma \in S_{\ell(\lambda)}$, define $\sigma(\lambda) = (\lambda_{\sigma_1} + 1 - \sigma_1, \dots, \lambda_{\sigma_k} + k - \sigma_k)$ provided $\lambda_{\sigma_i} + i - \sigma_i > 0$ for each i .

Also recall that $s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})$. If we consider compositions $\alpha \subseteq \lambda$, we can define $s_{\lambda/\alpha} = \det(h_{\lambda_i - \alpha_j - i + j})$. Note that if there exists some $\alpha_j - j = \alpha_k - k$ for some $j \neq k$, $s_{\lambda/\alpha} = 0$ since two columns of the matrix will be equal. If no such pair j, k exists, then there exists a unique permutation τ such that $\tau(\alpha) = (\alpha_{\tau_1} + 1 - \tau_1, \dots, \alpha_{\tau_k} + k - \tau_k) = \mu$ where μ is a partition. In this case,

$$s_{\lambda/\mu} = (-1)^{\text{sgn}(\tau)} s_{\lambda/\alpha}. \quad (4.12)$$

Theorem 4.15. *Let λ and μ be partitions with $\mu \subseteq \lambda$. Then*

$$s_{\lambda/\mu} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma) + \text{sgn}(\tau)} \mathfrak{S}_{\sigma(\lambda)/\tau(\mu)}^*$$

for any choice of τ such that $\tau(\mu)$ is a composition.

Proof. Recall that $\Delta(s_{\lambda}) = \sum_{\mu} s_{\lambda/\mu} \otimes s_{\mu} = \sum_{\mu} s_{\mu} \otimes s_{\lambda/\mu}$ because the Hopf algebra of symmetric functions is cocommutative. We can rewrite $\Delta(s_{\lambda})$ using Theorem 3.21, Point (9). Then

$$\begin{aligned} \Delta(s_{\lambda}) &= \Delta \left(\sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} \mathfrak{S}_{\sigma(\lambda)}^* \right) \\ &= \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} \Delta \mathfrak{S}_{\sigma(\lambda)}^* \\ &= \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} \left(\sum_{\beta} \mathfrak{S}_{\beta}^* \otimes \mathfrak{S}_{\sigma(\lambda)/\beta}^* \right) \end{aligned}$$

$$= \sum_{\beta} \mathfrak{S}_{\beta}^* \otimes \left(\sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} \mathfrak{S}_{\sigma(\lambda)/\beta}^* \right).$$

On the other hand,

$$\begin{aligned} \sum_{\mu} s_{\mu} \otimes s_{\lambda/\mu} &= \sum_{\mu} \left(\sum_{\tau \in S_{\ell(\mu)}} (-1)^{\text{sgn}(\tau)} \mathfrak{S}_{\tau(\mu)}^* \right) \otimes s_{\lambda/\mu} \\ &= \sum_{\mu} \sum_{\tau \in S_{\ell(\mu)}} (-1)^{\text{sgn}(\tau)} \left(\mathfrak{S}_{\tau(\mu)}^* \otimes s_{\lambda/\mu} \right) \\ &= \sum_{\beta} \mathfrak{S}_{\beta}^* \otimes \left(\sum_{\tau \in S_{\ell(\beta)}} (-1)^{\text{sgn}(\tau)} s_{\lambda/\tau^{-1}(\beta)} \right) \end{aligned}$$

where β is a composition and $\tau^{-1}(\beta)$ is a partition. Thus for a fixed choice of β ,

$$\sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} \mathfrak{S}_{\sigma(\lambda)/\beta}^* = \sum_{\tau \in S_{\ell(\beta)}} (-1)^{\text{sgn}(\tau)} s_{\lambda/\tau^{-1}(\beta)}.$$

Note that for each β , there is at most one $\tau \in S_{\ell(\beta)}$ such that $s_{\lambda/\tau^{-1}(\beta)} = s_{\lambda/\mu} \neq 0$ for a partition μ . Thus

$$s_{\lambda/\mu} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma) + \text{sgn}(\tau)} \mathfrak{S}_{\sigma(\lambda)/\tau(\mu)}^*$$

for any valid choice of τ . \square

Choosing τ as the identity gives the following corollary.

Corollary 4.16. *For partitions λ and μ with $\mu \subseteq \lambda$,*

$$s_{\lambda/\mu} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} \mathfrak{S}_{\sigma(\lambda)/\mu}^*.$$

Applying ψ to both sides of Theorem 4.15 gives us an expansion in terms of the row-strict dual immaculate functions.

Corollary 4.17. *For partitions λ and μ with $\mu \subseteq \lambda$ and $\tau \in S_{\ell(\mu)}$ such that $\tau(\mu)$ is a composition,*

$$s_{\lambda'/\mu'} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma) + \text{sgn}(\tau)} \mathcal{R} \mathfrak{S}_{\sigma(\lambda)/\tau(\mu)}^*.$$

5. Hook dual immaculate functions

Now that we have skew row-strict dual immaculate functions, we can define *hook dual immaculate functions* in a combinatorial manner analogous to the hook Schur functions [18] and hook quasisymmetric Schur functions [16].

Definition 5.1. Let $\mathcal{A} = \{1, 2, \dots, \ell\}$ and $\mathcal{A}' = \{1', 2', \dots, k'\}$ be two alphabets with $1 < 2 < \dots < \ell < 1' < 2' < \dots < k'$. Then a *semistandard hook immaculate tableau of shape α* is a filling of the diagram of α with entries from $\mathcal{A} \cup \mathcal{A}'$ such that

- (1) the first column increases from bottom to top with the increase strict in \mathcal{A} and weak in \mathcal{A}' , and
- (2) each row increases from left to right, weakly in \mathcal{A} and strictly in \mathcal{A}' .

Denote the set of all semistandard hook immaculate tableaux of shape α by HI_α .

The content monomial of a semistandard hook immaculate tableau T is a monomial in two alphabets, x_1, \dots, x_ℓ and y_1, \dots, y_k , where

$$z^T = \prod_{i \in \mathcal{A} \cup \mathcal{A}'} z_i^{\# \text{ of } i\text{'s in } T}$$

where $z_i = x_i$ if $i \in \mathcal{A}$ and $z_i = y_i$ if $i \in \mathcal{A}'$.

Example 5.2. Let $\alpha = (3, 1, 2, 4, 3)$. Then T , as shown below, is a semistandard hook immaculate tableau with content monomial $z^T = x_1^2 x_2 x_3^2 y_1^3 y_2 y_3 y_4^2 y_5$.

$$T = \begin{array}{|c|c|c|c|} \hline 1' & 2' & 4' & \\ \hline 1' & 3' & 4' & 5' \\ \hline 3 & 1' & & \\ \hline 2 & & & \\ \hline 1 & 1 & 3 & \\ \hline \end{array}$$

Definition 5.3. The *hook dual immaculate function* indexed by α is

$$\mathcal{H}\mathfrak{S}_\alpha^*(X, Y) = \mathcal{H}\mathfrak{S}_\alpha^*(x_1, \dots, x_\ell, y_1, \dots, y_k) = \sum_{T \in HI_\alpha} z^T.$$

It follows immediately from the definition that

$$\mathcal{H}\mathfrak{S}_\alpha^*(X, Y) = \sum_{\gamma \subseteq \alpha} \mathfrak{S}_\gamma^*(X) \mathcal{R}\mathfrak{S}_{\alpha/\gamma}^*(Y). \quad (5.1)$$

We can also expand $\mathcal{H}\mathfrak{S}_\alpha^*(X, Y)$ in terms of the super fundamental quasisymmetric functions. We use the definition in [16].

Definition 5.4. For $\alpha \models n$, the *super fundamental quasisymmetric function* indexed by α is

$$\tilde{Q}_\alpha(X, Y) = \sum_{\substack{a_1 \leq a_2 \leq \dots \leq a_n \\ a_i = a_{i+1} \in \mathcal{A} \Rightarrow i \notin \text{set}(\alpha) \\ a_i = a_{i+1} \in \mathcal{A}' \Rightarrow i \in \text{set}(\alpha)}} z_{a_1} z_{a_2} \cdots z_{a_n},$$

where $z_a = x_a$ if $a \in \mathcal{A}$ and $z_{a'} = y_{a'}$ for $a' \in \mathcal{A}'$.

Theorem 5.5. [16, Theorem 4.1] For $\alpha \models n$,

$$\tilde{Q}_\alpha(X, Y) = \sum_{i=0}^n F_\beta(X) F_\gamma(Y)$$

where $\beta \cdot \gamma = \alpha$ if $i \in \text{set}(\alpha)$ and $\beta \odot \gamma = \alpha$ if $i \notin \text{set}(\alpha)$.

As usual, we must have a standardization procedure for semistandard hook immaculate tableaux and an appropriate descent set to index the super fundamental quasisymmetric functions. To *standardize* a semistandard hook immaculate tableau T , first replace the entries of T from \mathcal{A} by scanning unprimed entries from left to right, starting with the top row, replacing 1's as they are encountered by 1, 2, ... in this reading order, followed by 2's, etc. Next continue with the entries of \mathcal{A}' by scanning from right to left starting with the bottom row. The result is denoted by $\text{stdz}(T)$.

Example 5.6. The reading order of T , as shown below, is 3, 2, 1, 1, 3, 1', 5', 4', 3', 1', 4', 2', 1', giving rise to $\text{stdz}(T)$ below.

$$T = \begin{array}{|c|c|c|} \hline 1' & 2' & 4' \\ \hline 1' & 3' & 4' & 5' \\ \hline 3 & 1' & & \\ \hline 2 & & & \\ \hline 1 & 1 & 3 \\ \hline \end{array} \quad \text{stdz}(T) = \begin{array}{|c|c|c|} \hline 8 & 9 & 12 \\ \hline 7 & 10 & 11 & 13 \\ \hline 4 & 6 & & \\ \hline 3 & & & \\ \hline 1 & 2 & 5 \\ \hline \end{array}$$

Note that the standardization of a hook immaculate tableau is a standard immaculate tableau. Recall that the descent set of a standard immaculate tableau S is $\text{Des}_{\mathfrak{S}^*}(S) = \{i : i+1 \text{ is strictly above } i \text{ in } S\}$. The descent set for $\text{stdz}(T)$ in Example 5.6 is $\text{Des}_{\mathfrak{S}^*}(\text{stdz}(T)) = \{2, 3, 5, 6, 7, 11\}$. From the definition of standardization, we note that if T is a hook immaculate tableau of shape α with $T = S \cup U$ where S is an immaculate tableau of shape β and U is a skew row-strict immaculate tableau of shape α/β , then

$$\text{Des}_{\mathfrak{S}^*}(\text{stdz}(T)) = \text{Des}_{\mathfrak{S}^*}(\text{stdz}(S)) \cup (\text{Des}_{\mathcal{R}\mathfrak{S}^*}(\text{stdz}(U)))^c + |\beta|$$

if $|\beta| + 1$ is weakly lower than $|\beta|$ in $\text{stdz}(T)$ and

$$\text{Des}_{\mathfrak{S}^*}(\text{stdz}(T)) = \text{Des}_{\mathfrak{S}^*}(\text{stdz}(S)) \cup (\text{Des}_{\mathcal{R}\mathfrak{S}^*}(\text{stdz}(U))^c + |\beta|) \cup \{|\beta|\}$$

if $|\beta| + 1$ appears strictly above $|\beta|$ in $\text{stdz}(T)$. (Here we have written $J + k$ for the subset obtained from a nonempty set of integers J by adding the integer k to each element of J . If $J = \emptyset$, then $J + k = \emptyset$.)

Theorem 5.7. *Let $\alpha \models n$. Then*

$$\mathcal{H}\mathfrak{S}_{\alpha}^*(X, Y) = \sum_S \tilde{Q}_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(S))}(X, Y)$$

where the sum is over all standard immaculate tableaux of shape α .

Proof. We show that each polynomial consists of the same monomials. Suppose $x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$ is the content monomial associated with a hook immaculate tableau T of shape α with $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_m$. Note that if $a_i = a_{i+1}$, then $i \notin \text{Des}_{\mathfrak{S}^*}(\text{stdz}(T))$ by the standardization procedure. Similarly, if $b_i = b_{i+1}$, $i + k \in \text{Des}_{\mathfrak{S}^*}(\text{stdz}(T))$, since b'_i must occur in a lower row of T than b'_{i+1} . Thus $x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$ is a monomial in $\tilde{Q}_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(\text{stdz}(T)))}(X, Y)$.

Now suppose $x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$ is a monomial in $\tilde{Q}_{\text{comp}(\text{Des}_{\mathfrak{S}^*}(S))}(X, Y)$ for some standard immaculate tableau S of shape α . We must show that there exists a hook immaculate tableau with content $a_1, \dots, a_k, b'_1, \dots, b'_m$. We do this by replacing n in S with b'_m , $n - 1$ in S with b'_{m-1} and so on. Since $b_i = b_{i+1}$ implies that $i + k \in \text{Des}_{\mathfrak{S}^*}(S)$, we have that each primed entry in a row is distinct and increasing from left to right. Similarly, if $a_i = a_{i+1}$, then $i \notin \text{Des}_{\mathfrak{S}^*}(S)$, guaranteeing that the first column is increasing bottom to top and has distinct unprimed entries. Thus the result is a hook immaculate tableau of content $x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$. \square

Berele and Regev [3] defined hook Schur functions indexed by a partition λ as

$$\mathcal{H}s_{\lambda}(X, Y) = \sum_{\mu \subseteq \lambda} s_{\mu}(X) s_{\lambda' / \mu'}(Y).$$

We have the following analogue of Theorem 3.21, Point (9).

Theorem 5.8. *Let λ be a partition. Then*

$$\mathcal{H}s_{\lambda}(X, Y) = \sum_{\tau \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\tau)} \mathcal{H}\mathfrak{S}_{\tau(\lambda)}^*(X, Y).$$

Proof. Let λ be a partition. Then by Theorem 3.21, Point (9) and Corollary 4.17,

$$\mathcal{H}s_{\lambda}(X, Y) = \sum_{\mu \subseteq \lambda} s_{\mu}(X) s_{\lambda' / \mu'}(Y)$$

$$\begin{aligned}
&= \sum_{\mu \subseteq \lambda} \left(\sum_{\sigma \in S_{\ell(\mu)}} (-1)^{\text{sgn}(\sigma)} \mathfrak{S}_{\sigma(\mu)}^*(X) s_{\lambda'/\mu'}(Y) \right) \\
&= \sum_{\mu \subseteq \lambda} \left(\sum_{\sigma \in S_{\ell(\mu)}} \mathfrak{S}_{\sigma(\mu)}^*(X) \sum_{\tau \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\tau)} \mathcal{R} \mathfrak{S}_{\tau(\lambda)/\sigma(\mu)}^*(Y) \right) \\
&= \sum_{\tau \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\tau)} \left(\sum_{\mu \subseteq \lambda} \sum_{\sigma \in S_{\ell(\mu)}} \mathfrak{S}_{\sigma(\mu)}^*(X) \mathcal{R} \mathfrak{S}_{\tau(\lambda)/\sigma(\mu)}^*(Y) \right). \quad (5.2)
\end{aligned}$$

Note that the only terms $\sigma(\mu)$ that appear in (5.2) are those such that $\sigma(\mu) = \beta$ for a composition β . We rewrite (5.2) as

$$\begin{aligned}
\mathcal{H} s_{\lambda}(X, Y) &= \sum_{\tau \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\tau)} \left(\sum_{\mu \subseteq \lambda} \sum_{\sigma \in S_{\ell(\mu)}} \mathfrak{S}_{\sigma(\mu)}^*(X) \mathcal{R} \mathfrak{S}_{\tau(\lambda)/\sigma(\mu)}^*(Y) \right) \\
&= \sum_{\tau \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\tau)} \sum_{\beta \subseteq \tau(\lambda)} \mathfrak{S}_{\beta}^*(X) \mathcal{R} \mathfrak{S}_{\tau(\lambda)/\beta}^*(Y) \\
&= \sum_{\tau \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\tau)} \mathcal{H} \mathfrak{S}_{\tau(\lambda)}^*(X, Y). \quad \square
\end{aligned}$$

References

- [1] Farid Aliniaefard, Shu Xiao Li, Stephanie van Willigenburg, Schur functions in noncommuting variables, *Adv. Math.* 406 (2022) 108536.
- [2] Edward E. Allen, Joshua Hallam, Sarah K. Mason, Dual immaculate quasisymmetric functions expand positively into Young quasisymmetric Schur functions, *J. Comb. Theory, Ser. A* 157 (2018) 70–108.
- [3] A. Berele, A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. Math.* 64 (2) (1987) 118–175.
- [4] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions, *Can. J. Math.* 66 (3) (2014) 525–565.
- [5] C. Bessenrodt, K. Luoto, S. van Willigenburg, Skew quasisymmetric Schur functions and noncommutative Schur functions, *Adv. Math.* 226 (5) (2011) 4492–4532.
- [6] John Campbell, Karen Feldman, Jennifer Light, Pavel Shuldiner, Yan Xu, A Schur-like basis of NSym defined by a Pieri rule, *Electron. J. Comb.* 21 (3) (2014) 3.41.
- [7] Erik Carlsson, Anton Mellit, A proof of the shuffle conjecture, *J. Am. Math. Soc.* 31 (3) (2018) 661–697.
- [8] Sylvie Corteel, Jim Haglund, Olya Mandelshtam, Sarah Mason, Lauren Williams, Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials, *Sel. Math. New Ser.* 28 (32) (2022).
- [9] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, Jean-Yves Thibon, Noncommutative symmetric functions, *Adv. Math.* 112 (2) (1995) 218–348.
- [10] Ira M. Gessel, Multipartite P -partitions and inner products of skew Schur functions, in: *Combinatorics and Algebra*, Boulder, Colo., 1983, in: *Contemp. Math.*, vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 289–317.
- [11] J. Haglund, K. Luoto, S. Mason, S. van Willigenburg, Quasisymmetric Schur functions, *J. Comb. Theory, Ser. A* 118 (2) (2011) 463–490.

- [12] Naihuan Jing, Yunnan Li, A lift of Schur's Q -functions to the peak algebra, *J. Comb. Theory, Ser. A* 135 (2015) 268–290.
- [13] Kurt Luoto, Stefan Mykytiuk, Stephanie van Willigenburg, An introduction to quasisymmetric Schur functions, in: *Hopf Algebras, Quasisymmetric Functions, and Young Composition Tableaux*, in: SpringerBriefs in Mathematics, Springer, New York, 2013.
- [14] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995.
- [15] Sarah Mason, Jeffrey Remmel, Row-strict quasisymmetric Schur functions, *Ann. Comb.* 18 (1) (2014) 127–148.
- [16] Sarah K. Mason, Elizabeth Niese, Quasisymmetric (k, l) -hook Schur functions, *Ann. Comb.* 22 (1) (2018) 167–199.
- [17] Elizabeth Niese, Sheila Sundaram, Stephanie van Willigenburg, Julianne Vega, Shiyun Wang, 0-Hecke modules for row-strict dual immaculate functions, *arXiv:2202.00708*, 2022.
- [18] Jeffrey B. Remmel, The combinatorics of (k, l) -hook Schur functions, in: *Combinatorics and Algebra*, Boulder, Colo., 1983, in: *Contemp. Math.*, vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 253–287.