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Desingularization of binomial varieties using toric Artin stacks

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Abstract
We show how the notion of fantastacks can be used to effectively desingularize binomial varieties defined over algebraically closed fields. In contrast to a desingularization via blow-ups in smooth centers, we drastically reduce the number of steps and the number of charts appearing along the process. Furthermore, we discuss how our considerations extend to a partial simultaneous normal crossings desingularization of finitely many binomial hypersurfaces.

Keywords Binomials · Resolution of singularities · Toric stacks

Mathematics Subject Classification 13F56 · 14M25 · 14B05 · 14J17 · 14D23

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1 Introduction

Motivated by recent works with focus on stacks in the service of resolution of singularities over fields of characteristic zero [1–4, 27, 28], we investigate the case of binomials defined over algebraically closed fields of arbitrary characteristic from the perspective of stacks, and with a view towards efficient computations of resolutions and of p-adic and motivic integrals. Classical resolution approaches are discussed in the recent [13].

We work over an algebraically closed field K so as to keep technicalities at a minimum.

1.1 Key ingredient: fantastacks

Building on the results of [18, 19, 31], the key new ingredient for our approach is the notion of fantastacks by Geraschenko and Satriano [15], which are examples of toric Artin stacks, and a birational reinterpretation of the toric Cox construction. Up to isomorphism, a fantastack $\mathcal{F}_{\Sigma,\beta}$ can be constructed from a fan Σ on the lattice $N \simeq \mathbb{Z}^m$ and a homomorphism of lattices $\beta\colon \mathbb{Z}^n \to N$ with certain properties. For a more detailed discussion of fantastacks we refer to Sect. 2.

1.2 Setup: binomials in toric varieties

A binomial in affine space $\operatorname{Spec} K[x_1, \ldots, x_m]$ always has the form

$$f \quad = \quad x_1^{C_1} \ldots x_m^{C_m} \quad \left(x_1^{A_1} \ldots x_r^{A_r} \quad - \quad \lambda y_1^{B_1} \cdots y_s^{B_s} \right), \tag{1.1}$$

where $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_s\}$ are disjoint subsets of $\{x_1, \ldots x_m\}$ and $\lambda \in K^\times$. The *monomial factor* $x_1^{C_1} \ldots x_m^{C_m}$ will be of secondary interest and the *purely binomial factor* $x_1^{A_1} \ldots x_r^{A_r} - \lambda y_1^{B_1} \ldots y_s^{B_s}$ is the main factor to be addressed.

Note that the purely binomial factor is nonreduced if and only if A_i, B_i are all divisible by the characteristic of K, as K is assumed algebraically closed.

In Sect. 3.1 we extend the notion of binomials from affine space to the generality of smooth toric varieties and stacks.

Tevelev [32, Definition 1.3] defined *schön subvarieties* of toric varieties, which we do not reproduce here, as our special situation is simpler. In Sect. 3.2 we specialize to the case of binomials, in the generality of toric stacks. Our usage is slightly relaxed since we allow monomial factors: in affine charts, a binomial (1.1) is *schön* if either all $A_i = 0$ or all $B_j = 0$.

1.3 Arrangements of binomials and binomial subschemes

Given a finite collection of binomials, locally represented by $f_1, \ldots, f_a$, they together define an arrangement of subschemes as the zero-set $X := V(f_1 \cdots f_a)$ of their product. The arrangement is said to be schön if each f_i is schön.

If all f_i are schön pure binomials, the binomial ideal $\mathcal{I} = \langle f_1, \ldots, f_a \rangle$ they generate defines a schön subscheme in the sense of [32, Definition 1.3]. The situation is a bit more involved when the monomial factors are not trivial, see Sect. 6.

1.4 Simple arrangements and problematic primes

In Sect. 3.3 we recall the notion of a simple arrangement of smooth subschemes, and upgrade it to stacks.

In characteristic 0, a schön arrangement of binomials on a toric stack is automatically a simple arrangement. This is not the case in positive characteristic, a phenomenon also described in Sect. 3.3, Observation 3.3 — there is a finite set of prime characteristics $\mathcal{E}$, computable from the collection of exponents of the binomials f_i, outside of which the binomials form a simple arrangement.

1.5 Subdivisions and modifications associated to binomials

Consider again a binomial (1.1) in affine space. The Newton polyhedron of f defines a singular subdivision Σ_f of the fan Σ_0 of affine space, and a corresponding singular toric modification of affine space. It is shown in [18], [19, Section 3], [31, Section 6] that any smooth subdivision Σ of Σ_f results in a resolution of $\{f = 0\}$; however such a smooth subdivision is typically computationally expensive. Our goal here is precisely to avoid such subdivision.

Similarly, for a collection $f_1, \ldots, f_a \in K[x_1, \ldots, x_m]$ of binomials we obtain a subdivision of Σ_0 into a fan Σ_f of at most 2^a cones with $< \binom{m+a}{m-1}$ edges, independently of the exponents and coefficients of f_i — it is the subdivision dual to the Newton polyhedron of the product $f_1 \cdots f_a$. By the above references any smooth subdivision of Σ_f resolves all the f_j. Here we provide a stack-theoretic resolution requiring no subdivision of Σ_f at all.

1.6 Main result on arrangements of binomials

Our main point is that the fantastack associated to the fan Σ_f, or any subdivision Σ thereof, gives such a simultaneous resolution of f_i immediately.

Theorem A (See Theorem 5.4) *Let K be an algebraically closed field of arbitrary characteristic $p \geq 0$. Let $f_1, \ldots, f_a \in K[x] = K[x_1, \ldots, x_m]$ be finitely many binomials, where $a, m \in \mathbb{Z}_+$ with $m \geq 2$. Let $\mathcal{E} \subset \mathbb{Z}$ be the set of problematic primes associated to the exponents of the pure binomial factors of $f_1, \ldots, f_a$.*

Let Σ be any subdivision of the dual fan Σ_f of the Newton polyhedron of the product $f_1 \cdots f_a$, inducing the morphism of smooth toric stacks $\rho\colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$, and let $X := V(f_1 \cdots f_a) \subset \mathbb{A}^m$.

Then the reduced preimage $\rho^{-1}(X)_{\mathrm{red}} \subset \mathcal{F}_{\Sigma,\beta}$ is a schön arrangement of smooth binomials.

If furthermore $p \notin \mathcal{E}$, then $\rho^{-1}(X)_{\mathrm{red}}$ induces a simple arrangement on $\mathcal{F}_{\Sigma,\beta}$.

1.7 Remarks

(1) The main reason for our restriction to algebraically closed fields is that, as far as we know, there exists no detailed reference on the theory of fantastacks over an arbitrary base. Note that the treatment of [15] applies in any characteristic, though the classification result, not used here, is only stated in characteristic 0,[1]

(2) For $a = 1$, we obtain a desingularization of the binomial hypersurface defined by f_1, while for $a = 2$, assuming $p \notin \mathcal{E}$, Theorem A provides us with a simultaneous normal crossings desingularization of the binomial hypersurfaces given by f_1 and f_2.

(3) Notice that the ambient space in Theorem A is an Artin stack and not a scheme. However, any triangulation of its fan replaces it by a Deligne–Mumford stack; this is achieved, without adding rays, using star subdivisions $\Sigma' \to \Sigma$ at all the rays of Σ. The resulting stack $\mathcal{F}_{\Sigma',\beta}$ is a Deligne–Mumford open substack of $\mathcal{F}_{\Sigma,\beta}$, and requires no additional computation.
We also note that any smooth subdivision provides a scheme where the result holds; this recovers earlier known results, such as those of [18], but also loses the efficiency of the current method.

(4) In general, when $a \geq 3$ and $p \notin \mathcal{E}$, Theorem A does not give a simultaneous normal crossings desingularization, as the resulting arrangement of hypersurfaces is not in normal crossings position in general, see Example 5.3. This scenario is present also in earlier work which uses smooth blowup centers. To resolve a simple arrangement and make it normal crossings one may use well–known procedures requiring at most m blowups, see [21, 26]. We recall this in Theorem 5.6.
The situation where $p \in \mathcal{E}$ is quite interesting. One expects to have an elegant procedure for a simultaneous normal crossings desingularization. We do not pursue this question here.

(5) While we do not give a detailed complexity analysis, we note that the geometric complexity encoded in the number of cones of the fan, in particular the number of charts and divisors, is bounded solely in terms of the ambient dimension and the number a of binomials, and not depending on the exponent or coefficients of the generators. For future discussion we will say that such process is of *combinatorially bounded complexity*.

1.8 Main result on binomial varieties

In the context of desingularization of binomial schemes using fantastacks, we prove the following result. Following earlier sources, an ideal $\mathcal{I} \subset k[x_1, \ldots, x_m]$ is *binomial* if it is generated by binomials and a subscheme $X \subset \mathbb{A}^m$ is *binomial* if its ideal is binomial. The subscheme is *purely binomial* if it has no component or embedded component supported

[1] According to [15, Remark 1.2] their restriction to algebraically closed fields is made in order to avoid technicalities.

on the boundary divisor $V(x_1 \cdots x_m)$. We denote by X^{pure} the *purely binomial part* of a binomial subscheme X, namely the closure of its intersection with the torus.

Throughout the paper we denote the reduced scheme underlying a scheme X by X_{red}, and call it *the reduction* of X for brevity.

For describing our result we introduce the following somewhat ad-hoc terminology: a subscheme X of a smooth scheme V is said to be *in simple normal position* if the components of its reduction X_{red} are smooth, and at any geometric point p of V there are local parameters $x_i \in \hat{\mathcal{O}}_{V,p}$ such that X_{red} is defined by a monomial ideal in x_i. This is the same as saying that, in these coordinates, X_{red} is the union of coordinate subspaces, of arbitrary dimensions, not necessarily meeting transversely. As an example, two or more coordinate lines $L_1, L_2, \ldots$ in $\mathbb{A}^n$, $n \geq 3$, are in simple normal position even though they do not meet transversely. To see the latter, observe that $L_1 \cap L_2$ is the origin, which is of codimension $n < 2(n-1)$. The same phenomenon holds for coordinate subspaces of dimension $< n/2$.

Theorem B (See Theorem 6.1) *Let K be an algebraically closed field and $m \in \mathbb{Z}_{\geq 2}$. Let $X \subset \mathbb{A}^m$ be a binomial subscheme. Denote by Σ_0 the fan in $\mathbb{R}^m$ with unique maximal cone $\mathbb{R}^m_{\geq 0}$. Let $f = \{f_1, \ldots, f_a\}$ be binomial generators of $\mathcal{I}_X$. Let Σ_f denote again the dual fan of the Newton polyhedron of $f_1 \cdots f_a$, with resulting toric stack $\mathcal{F}_{\Sigma_f,\beta}$.*

(1) (See [18, 19, 31]) *The proper transform of X^{pure} in $\mathcal{F}_{\Sigma_f,\beta}$ is a schön purely binomial subscheme, in particular its reduction is smooth.*

(2) *There exists a further subdivision Σ of Σ_0, of combinatorially bounded complexity, with the following property: If $\beta\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is the homomorphism determined by the matrix $M \in \mathbb{Z}^{m\times n}$, whose columns are the primitive generators for the rays of the fan Σ, then*

$$\rho^{-1}(X)_{\text{red}} \subset \mathcal{F}_{\Sigma,\beta}$$

is in simple normal position on $\mathcal{F}_{\Sigma,\beta}$, where we denote by $\rho\colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$ the morphism of toric stacks induced by β.

Once again we note that in (2) the work of Li Li [26] allows for a further sequence, of at most m blowups, such that the total transform of X_{red} is a simple normal crossings divisor. We recall this in Proposition 6.2.

1.9 Prior work

Let us mention that there exist several results on the desingularization of toric and binomial varieties, e.g., [5–7, 18, 23], where the assumptions slightly vary compared to those stated here. In contrast to blow-ups in smooth centers, we need to apply the fantastack construction only once in Theorem A and it is covered by at most 2^a affine charts with $< \binom{m+a}{m-1}$ coordinates in total. This number is significantly smaller than that obtained using blow-ups in smooth centers once the exponents appearing in $f_1, \ldots, f_a$ are large (see Remark 5.5). The primary reason is that we do not require a smooth subdivison of the dual fan.

1.10 Computational motivation

Besides our interest in the problem on its own, another motivation for this project is coming from the explicit computation of p-adic integrals, whose data are given by binomials. For example, the following task appears in the context of computing local subring zeta functions:

determine the integral

$$\int_B |x_1|_p^{s-3}|x_2|_p^{s-2}|x_3|_p^{s-1}d\mu,$$

where

- $|.|_p$ is the p-adic absolute value on $\mathbb{Z}_p$; we denote the corresponding valuation $v_p(.)$;
- $d\mu$ is an additive Haar measure; and
- $B \subset \mathbb{Z}_p^6$ is the subset determined by the inequalities

$$\begin{array}{ll} v_p(x_1) \leq v_p(x_4^2 - x_2x_4), & v_p(x_1) \leq v_p(x_4x_5 - x_4x_6), \\ v_p(x_2) \leq v_p(x_6^2), & v_p(x_1x_2) \leq v_p(x_2x_5^2 - x_4x_6^2). \end{array}$$

A possible approach to determine such an integral is to first perform blow-ups so that all data appearing becomes monomial, and then to compute the monomial integrals in every chart. Evidently, it is desirable to keep the number of final charts as small as possible in this context. Notice that even though we are in the p-adic setting and not working over an algebraically closed field, the techniques discussed in this article can be applied at least for all but finitely many primes p, where the problematic set of primes depends on the precise data given. For more details, we refer to Sect. 5.1.

If one follows the method of Theorem A, one has to be careful since the outcome is a toric Artin stack and some extra work is required.

For algebraically closed fields of characteristic zero, Satriano and Usatine initiated an investigation for a method to study stringy Hodge numbers of a singular variety using motivic integration for Artin stacks in [29, 30]. To address p-adic integration, an analogous framework for p-adic integration on Artin stacks needs to be developed. Alternatively, one could efficiently replace the Artin stack by a Deligne–Mumford stack, as described in Remark 1.7(3) above, so that one could apply existing results on motivic [35, 36] as well as p-adic integration [20, Section 2] for them.

Since our focus is on Theorems A and B, we do not go into further details here.

1.11 Summary

Let us briefly summarize the content of the paper. After recalling basics on fantastacks (Sect. 2), we treat the case of a single binomial (Sect. 4). Then, we turn to the case of finitely many binomials and how the techniques extend to give Theorem A (Sect. 5). Finally, we prove Theorem B in full generality (Sect. 6).

2 Basics on fantastacks

Fantastacks are the key tool for our main theorem. Therefore, we begin by recalling their definition following the work of Geraschenko and Satriano on toric stacks [15], where fantastacks are discussed as first examples. For more details on the theory of toric stacks, we also refer to [16]. Let us point out that there exist earlier works on the notion of toric stacks by Lafforgue [25], Borisov, Chen, Smith [8], Fantechi, Mann, Nironi [12], and Tyomkin [33], each of them focusing on different aspects. The introduction of [15] provides a brief overview on them as well as a discussion how [15] unifies them. There is a related treatment by Gillam and Molcho [17] which allows for Deligne–Mumford stacks which are not global quotients but glued from such. We do not need this generality in this paper.

Definition 2.1 ([15, Definition 1.1]) Let X be a normal toric variety with torus T_0 and let G be a subgroup of T_0. The Artin stack $[X/G]$ equipped with the action by the dense torus $T := T_0/G$ is called a *toric stack*.

For a lattice L, we denote by $L^* = \mathrm{Hom}_{\mathrm{gp}}(L, \mathbb{Z})$ the dual of L as a finitely generated abelian group. If $\beta\colon L \to N$ is a homomorphism of lattices, we denote by $\beta^*\colon N^* \to L^*$ the corresponding dual.

Definition 2.2 ([15, Definitions 2.4 and 2.5])

(1) A *stacky fan* is a pair (Σ, β) consisting of a fan Σ on a lattice L as well as a homorphism of lattices $\beta\colon L \to N$ such that β has a finite cokernel.
(2) Let (Σ, β) be a stacky fan. Let X_Σ be the toric variety associated to Σ and let $T_\beta\colon T_L \to T_N$ be the homomorphism of tori induced by $\beta^*\colon N^* \to L^*$. Set $G_\beta := \ker(T_\beta)$. The toric stack $\mathcal{X}_{\Sigma,\beta}$ associated to this data is defined as $[X_\Sigma/G_\beta]$, where its torus is $T_N \cong T_L/G_\beta$.

As explained in [15, after Definition 2.5], one can construct to every toric stack $[X/G]$ a stacky fan (Σ, β) such that $\mathcal{X}_{\Sigma,\beta} = [X/G]$. Moreover, after these explanations, Geraschenko and Satriano give explicit examples of toric stacks. In particular, if Σ is a fan on a lattice N and if we choose $L = N$ and $\beta = \mathrm{id}_N$, then $\mathcal{X}_{\Sigma,\beta} = X_\Sigma$, where X_Σ is the toric variety associated to Σ, see [15, Example 2.6].

We denote by $(e_1, \ldots, e_n)$ the standard basis of $\mathbb{Z}^n$.

Definition 2.3 ([15, Definition 4.1]) Let Σ be a fan on a lattice N. Let $\beta\colon \mathbb{Z}^n \to N$ be a homomorphism so that

- the cokernel of β is finite,
- every ray of Σ contains some $\beta(e_i)$, and
- every $\beta(e_i)$ is contained in the support of Σ.

For a cone $\sigma \in \Sigma$, define $\hat{\sigma} := \mathrm{cone}(\{e_i \mid \beta(e_i) \in \sigma\})$. Let $\widehat{\Sigma}$ be the fan on $\mathbb{Z}^n$ generated by all these $\hat{\sigma}$. Finally, one defines $\mathcal{F}_{\Sigma,\beta} := \mathcal{X}_{\widehat{\Sigma},\beta}$. A toric stack which is isomorphic to some $\mathcal{F}_{\Sigma,\beta}$, is called a *fantastack*.

By [15, Remark 4.2], one has

$$X_{\widehat{\Sigma}} = \mathbb{A}^n \backslash V(J_\Sigma), \quad \text{where } J_\Sigma := \langle \prod_{\beta(e_i)\notin\sigma} x_i \mid \sigma \in \Sigma \text{ maximal cone} \rangle. \tag{2.1}$$

In particular, $X_{\widehat{\Sigma}}$ (and thus also $\mathcal{F}_{\Sigma,\beta} = [X_{\widehat{\Sigma}}/G_\beta]$) is covered by the charts $D_+(\prod_{\beta(e_i)\notin\sigma} x_i)$, where σ runs through all maximal cones of Σ. Here, $D_+(h) := \mathbb{A}^n \backslash V(h)$ is the notation for the standard open set.

Remark 2.4 In our setting, we will always have that $N = \mathbb{Z}^m$, the support of Σ is equal to $\mathbb{R}^m_{\geq 0}$, $n \geq m$, and $\beta\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is determined by a matrix $M \in \mathbb{Z}^{m\times n}_{\geq 0}$ of full rank m — in fact it will be surjective on lattices. In particular, β induces a morphism of toric stacks $\rho_\beta\colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$ determined by the morphism of stacky fans represented by the diagram

$$\begin{array}{ccc} \Sigma & \longrightarrow & \Sigma_0 \\ \mathbb{Z}^n & \xrightarrow{\beta} & \mathbb{Z}^m \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\ \mathbb{Z}^m & \xrightarrow{\mathrm{id}} & \mathbb{Z}^m \end{array}$$

where Σ_0 is the fan on $\mathbb{Z}^m$ providing the toric variety $\mathbb{A}^m$. For more details on morphisms of toric stacks and the connection to morphisms of stacky fans, we refer to [15, Section 3]. Let us only mention that the rows of M determine ρ_β on the level of rings.

For example, if $M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \end{pmatrix}$, then ρ_β arises from the restriction to the open subset $\mathbb{A}^3 \backslash V(J_\Sigma)$ (as in (2.1)) of the map $\mathbb{A}^3 = \mathrm{Spec}(K[z_1, z_2, z_3]) \to \mathbb{A}^2 = \mathrm{Spec}(K[x_1, x_2])$, which is given by $x_1 = z_1 z_2^2 z_3^3$, having exponents $(1, 2, 3)$ and $x_2 = z_2 z_3^5$ having exponents $(0, 1, 5)$.

3 Binomial hypersurfaces: generalities

3.1 Setup: binomials in the torus and in affine space

We follow the notation and terminology of Sect. 1.2, in particular (1.1).

By restriction, the description (1.1) discussed above applies to any open subset $U \subset \mathrm{Spec}\, K[x_1, \ldots, x_m]$.

Inside the maximal torus $\mathrm{Spec}\, K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, a binomial is of the form

$$x_1^{A_1} \cdots x_r^{A_r} y_1^{-B_1} \cdots y_s^{-B_s} = \lambda,$$

a coset of the codimension-1 subgroup $x_1^{A_1} \cdots x_r^{A_r} y_1^{-B_1} \cdots y_s^{-B_s} = 1$. Its reduction is therefore always smooth, as K is assumed algebraically closed and hence there has to exist an exponent in its equation which is not divisible by the characteristic of the ground field K.

3.2 Schön binomials and arrangements

A binomial in affine space is said to be *schön* if either all $A_i = 0$ or all $B_j = 0$. For a pure binomial this is in agreement with [32, Definition 1.3] — note however that we impose no condition on the monomial factor, as it is an arrangement of smooth toric divisors. This means that the reduction of the purely binomial factor is smooth, and meets the toric divisor transversally. We use this condition to define a schön binomial in a smooth toric variety or stack:

Definition 3.1 A hypersurface on a smooth toric stack $\mathcal{X}$ is *binomial* if on each affine chart it is given by an equation of the form (1.1).

Moreover, a binomial hypersurface is *schön* if such equations can further be chosen so that either all $A_i = 0$ or all $B_j = 0$.

Note that being schön still means that the reduction of the purely binomial factor is smooth, and meets the toric divisor transversally.

Again in the torus, a collection of binomials $f_1, \ldots, f_a$ thus defines an arrangement of codimension-1 cosets as the zero-set $X := V(f_1 \cdots f_a)$ of their product. Denote by $\mathcal{I} := \langle f_1, \ldots, f_a \rangle$ the ideal they generate and by $V(\mathcal{I})$ the scheme it defines. The scheme $V(\mathcal{I})$, when nonempty, is itself a coset of a subgroup of the torus, of possibly higher codimension — the intersection of the stabilizers of f_i.

As we assume that the base field is algebraically closed, it follows that the reduction $W := V(\mathcal{I})_{\mathrm{red}}$ of such coset $V(\mathcal{I})$ is again smooth:

First note that W, as any variety over an algebraically closed field K, is generically smooth over K. Let $U \subset W$ be the maximal smooth open dense subset. We claim that $U = W$ and

argue by contradiction. Let $x \in W \backslash U$ and $y \in U$ be closed points. Translation by $y - x$ is an automorphism of W, hence it must preserve U, but sends the singular point x to the smooth point y, a contradiction.

In affine space, assume first that f_i are pure binomials. Then the scheme $X = V(f_1 \cdots f_a)$ is an arrangement of schön subschemes in the sense of [32, Definition 1.3] if and only if each binomial f_i is schön. As above, we generalize this and say that an arrangement $X = V(f_1 \cdots f_a)$ of arbitrary binomials, not necessarily pure, on a smooth toric stack is schön if each f_i is schön.

If the binomials f_i are schön and have trivial monomial factor, the binomial ideal $\mathcal{I} = \langle f_1, \ldots, f_a \rangle$ they generate defines a schön subscheme in the sense of [32, Definition 1.3]. The situation is a bit more involved when the monomial factors are not trivial, see Sect. 6.

3.3 Simple arrangements and problematic primes

Definition 3.2 Given a smooth Artin stack $\mathcal{A}$ and a substack $Y \subseteq \mathcal{A}$, we say that Y is a *simple arrangement on* $\mathcal{A}$, if the following conditions hold for the intersection lattice $\mathcal{S} = \{S_i\}_{i \in I}$ of Y:

(1) S_i is smooth and non-empty,
(2) S_i and S_j *meet cleanly.* This is defined to mean that their scheme-theoretic intersection is smooth and their tangent spaces fulfill $T(S_i) \cap T(S_j) = T(S_i \cap S_j)$, and
(3) $S_i \cap S_j = \varnothing$ or $S_i \cap S_j$ is a non-empty, disjoint union of elements $S_\ell \in \mathcal{S}$,

for every $i, j \in I$.

In characteristic 0, a schön arrangement of binomials on a toric stack is automatically a simple arrangement. This is not the case in positive characteristic. The precise condition is easily computed by taking derivatives:

Observation 3.3 As in [31, Section 6] it suffices to check that each S_i is (absolutely) reduced. Thus, consider a set of distinct binomials $f_1, \ldots, f_a$, of the form $f_j = \prod_{i=1}^m x_i^{D_{ij}} - \lambda_j$ with $D_{ij} \in \mathbb{Z}$, vanishing in codimension k. For the common vanishing locus to be smooth, the matrix $\bar{D}_{ij} \in M_{m,a}(\mathbb{F}_p)$ obtained by reducing the integers D_{ij} modulo p must be of rank k. To see this, first note that the schön condition forces the variables x_i that appear to be nonzero, hence we may work in the torus. We need to compute the rank of the derivative matrix with respect to all x_i, and since the variables are invertible we may instead use the logarithmic derivatives $x_i \partial / \partial x_i$. Note that $x_i \partial f_j / \partial x_i = D_{ij} \prod_{i=1}^m x_i^{D_{ij}}$. Factoring out the invertible monomials, it remains to compute the rank of the matrix $\bar{D}_{ij}$ as claimed. Putting these conditions together, we require that for each subset $\{f_j, j \in I\}$ of the given binomials, vanishing in codimension k_I, the corresponding matrix $(\bar{D}_{ij})_{i \in \{1,\ldots,m\}, j \in I} \in M_{m,|I|}(\mathbb{F}_p)$ must be of rank k_I.

Note that given an integer matrix D describing a collection of pure binomials in the torus, there are only finitely many primes where the requisite minors $\det(\bar{D}_{ij})_{i \in \{1,\ldots,m\}, j \in I}$ above vanish. We denote this collection of *problematic* primes $\mathcal{E}$. Thus the schön arrangement defined by f_i fails to be simple if and only if $p \in \mathcal{E}$ is problematic.

3.4 Subdivisions and modifications associated to binomials

Consider again a binomial $f = x_1^{C_1}\cdots x_m^{C_m}\left(x_1^{A_1}\cdots x_r^{A_r} - \lambda y_1^{B_1}\cdots y_s^{B_s}\right)$ in affine space. The fan Σ_0 of affine space has a single maximal cone $\sigma = \mathbb{R}^m_{\geq 0}$. We denote by ξ_i the coordinates in $\mathbb{R}^m$ corresponding to x_i and η_j those corresponding to y_j. The binomial thus defines a hyperplane $h_f(\xi_i, \eta_j) = \sum A_i\xi_i - \sum B_j\eta_j = 0$ which splits σ into a fan Σ_f of two cones; note that the monomial factor $x_1^{C_1}\cdots x_m^{C_m}$ does not affect this subdivision. In modern terms, the hyperplane $h_f(\xi_i, \eta_j)$ is the tropicalization of the binomial f.[2] As in Sect. 1.5, any smooth subdivision of Σ_f results in a resolution of $\{f = 0\}$, and here we avoid these subdivisions using stacks.

A collection $f_1, \ldots, f_a$ of binomials provides a collection of hyperplanes $h_{f_j} = 0$. Inductively, each hyperplane subdivides any cone into at most two subcones, hence together they divide σ into a fan Σ_f of at most 2^a cones. The subdivision has $< \binom{m+a}{m-1}$ edges, as an edge is determined by the intersection of $(m-1)$ hyperplanes chosen from the union of the m coordinate hyperplanes and h_{f_j}. Note that these bounds depend only on the ambient dimension and the number of hypersurfaces, and not on the exponents appearing.

For a subdivision $\Sigma \to \Sigma_0$ we write M for the matrix whose columns are the primitive generators of the rays of Σ. It defines a homomorphism $\beta : \mathbb{Z}^n \to \mathbb{Z}^m$, resulting in a fantastack $\mathcal{F}_{\Sigma,\beta}$ and a morphism of toric stacks $\rho = \rho_\beta \colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$. We show that this stack theoretic modification desingularizes the f_i simultaneously.

4 Resolving a single binomial hypersurface

In this section, we prove Theorems A and B for a single binomial hypersurface, where they coincide. Let K be an algebraically closed field and let f be a binomial with coefficients in K. In the hypersurface case, the stacky fan (Σ, β) for the fantastack construction will arise from the dual fan of the Newton polyhedron of f. In particular, we have $N = \mathbb{Z}^m$, for some $m \in \mathbb{Z}_{\geq 2}$. Furthermore, a monomial factor dividing f will not have an effect. Therefore, we assume without loss of generality that f is a pure binomial,

$$f = x^A - \lambda y^B = x_1^{A_1}\cdots x_s^{A_s} - \lambda y_1^{B_1}\cdots y_r^{B_r} \in K[x, y],$$

where $\lambda \in K\backslash\{0\}$. Set $m := s + r$.

Recall [24], [9, Section 18.2] that the *Newton polyhedron* $\Delta(g; z)$ of a polynomial $g \in K[z] = K[z_1, \ldots, z_m]$ is defined as follows: If we have

$$g = \sum_{C \in \mathbb{Z}^m_{\geq 0}} \lambda_C z_1^{C_1}\cdots z_m^{C_m}$$

with coefficients $\lambda_C \in K$, then $\Delta(g; z)$ is the smallest closed convex subset containing all points of the set

$$\{C + v \mid v \in \mathbb{R}^m_{\geq 0} \text{ and } C \in \mathbb{Z}^m_{\geq 0} \text{ is such that } \lambda_C \neq 0\}.$$

In particular, $\Delta(f; x, y)$ is the smallest closed convex subset $\Delta \subseteq \mathbb{R}^m_{\geq 0}$ containing the points $(A, 0), (0, B) \in \mathbb{Z}^m_{\geq 0}$ and which is stable under translations by the non-negative orthant $\mathbb{R}^m_{\geq 0}$, i.e., $\Delta + \mathbb{R}^m_{\geq 0} = \Delta$. When there is no confusion possible, we sometimes write $\Delta(f)$ instead of $\Delta(f; x, y)$. We will use the symbol Σ_f for the *dual fan* of the Newton polyhedron $\Delta(f)$,

[2] We thank the referee for pointing this out. Note that the works [31] and [11] predate this notion!

see [28, 34], which is the fan in $\mathbb{R}^m$, whose cones are determined by the normal vectors of the facets of $\Delta(f)$. (For some explicit examples and pictures, we refer to the examples later on.) Observe that $e_1, \dots, e_m$ appear among the normal vectors of $\Delta(f)$ and the support of the dual fan Σ_f is equal to $\mathbb{R}^m_{\geq 0}$.

Let us introduce the homorphism $\beta\colon \mathbb{Z}^n \to \mathbb{Z}^m$, which we use for the construction of the fantastack $\mathcal{F}_{\Sigma,\beta}$, where Σ is a subdivision of Σ_f.

Definition 4.1 Let $f = x^A - \lambda y^B \in K[x, y]$ be a binomial as above, and let Σ be any subdivision of the dual fan Σ_f of the Newton polyhedron of f; note that $\Sigma = \Sigma_f$ is allowed. Let $e_1, \dots, e_m, v_{m+1}, \dots, v_n \in \mathbb{Z}^m_{\geq 0}$ be primitive generators for the rays of Σ. Notice that the normal vectors of the facets of the Newton polyhedron $\Delta(f; x, y)$ are among them. Let M_Σ be the $m \times n$ matrix whose columns are the primitive generators

$$M_\Sigma = \begin{pmatrix} e_1 & \cdots & e_m & v_{m+1} & \cdots & v_n \end{pmatrix} = \begin{pmatrix} E_m & v_{m+1} & \cdots & v_n \end{pmatrix} \in \mathbb{Z}^{m\times n}_{\geq 0},$$

where $E_m \in \mathbb{Z}^{m\times m}$ is the unit matrix. We define $\beta_\Sigma\colon \mathbb{Z}^n \to \mathbb{Z}^m$ to be the homorphism determined by the matrix M_Σ. If the reference to Σ is clear from the context, we sometimes just write β instead of β_Σ.

Since M_Σ has full rank, β_Σ is surjective. In particular, its cokernel is finite. Furthermore, by construction, every ray of Σ contains some $\beta_\Sigma(e_i)$ and every $\beta_\Sigma(e_i)$ is contained in the support of Σ. Therefore, (Σ, β) fulfills all hypotheses of Definition 2.3. Recall that β induces a morphism of toric stacks $\rho_\beta\colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$ (Remark 2.4).

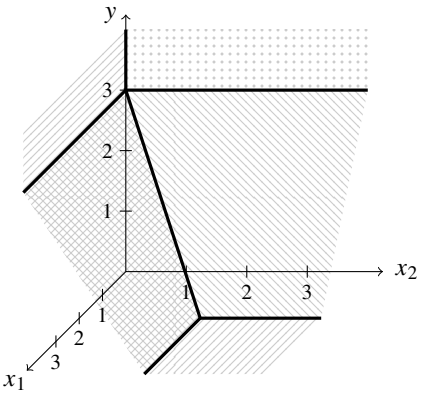
In the given situation, Theorems A and B boil down to the following statement.

Proposition 4.2 *Let K be an algebraically closed field and let f be a binomial with coefficients in K. Let Σ be any subdivision of the dual fan of the Newton polyhedron of f. Let $\mathcal{F}_{\Sigma,\beta}$ be the fantastack associated to the stacky fan $(\Sigma, \beta) = (\Sigma, \beta_\Sigma)$ and let $\rho_\beta\colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$ be the morphism induced by β.*

If we set $X := V(f) \subset \mathbb{A}^m$, then $\rho_\beta^{-1}(X)_{\mathrm{red}}$ is a schön binomial hypersurface (Definition 3.1) and a simple normal crossing divisor on $\mathcal{F}_{\Sigma,\beta}$.

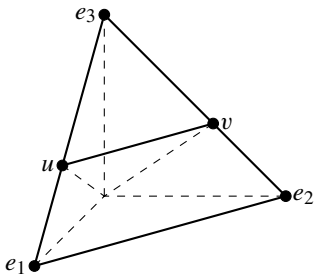
Let us first discuss this in an example to explain our intuition. The proof is given in Sect. 4.1.

Example 4.3 Consider the binomial $f = x_1^2x_2^2 - y^3$. The Newton polyhedron looks as follows:



Here we highlighted the 1-skeleton and the facets of the polyhedron, and the polyhedron itself forms the solid bounded by the resulting 5 facets and closer to the observer.

The normal vectors of its facets are e_1, e_2, e_3, v, w, where $u = (3, 0, 2)^T$ and $v = (0, 3, 2)^T$. Therefore, the dual fan of the Newton polyhedron looks as follows



Here we highlighted a transverse slice of the fan. The plane spanned by the vectors u and v is precisely the plane $2\xi_1 + 2\xi_2 = 3\eta$ described in the introduction and appearing in the paper [18].

Take $\Sigma = \Sigma_f$ to be the dual fan of the Newton polyhedron itself, without further modification. The columns of the matrix $M := M_\Sigma$ are the normal vectors above,

$$M = \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}.$$

Note that the kernel of M has the basis $((-3, 0, -2, 1, 0)^T, (0, -3, -2, 0, 1)^T)$. The rows of M determine the morphism $\rho\colon \mathbb{A}^5 \to \mathbb{A}^3$ given by

$$x_1 = x_1' z_1^3,\ x_2 = x_2' z_2^3,\ y = y' z_1^2 z_2^2,$$

where x_1', x_2', y', z_1, z_2 are the variables of $\mathbb{A}^5$. We observe that the binomial f becomes

$$f = z_1^6 z_2^6 (x_1'^2 x_2'^2 - y'^3).$$

In the construction of $\mathcal{F}_{\Sigma,\beta}$, we have to take the quotient by $G_\beta = \ker(T_\beta)$, where $T_\beta\colon T_{\mathbb{Z}^n} \to T_{\mathbb{Z}^m}$ is the homomorphism induced by β. In this example, $G_\beta \simeq \mathbb{G}_m^2$ acts via

$$(t_1, t_2) \cdot (x_1', x_2', y', z_1, z_2) := (t_1^{-3} x_1',\ t_2^{-3} x_2',\ t_1^{-2} t_2^{-2} y',\ t_1 z_1,\ t_2 z_2).$$

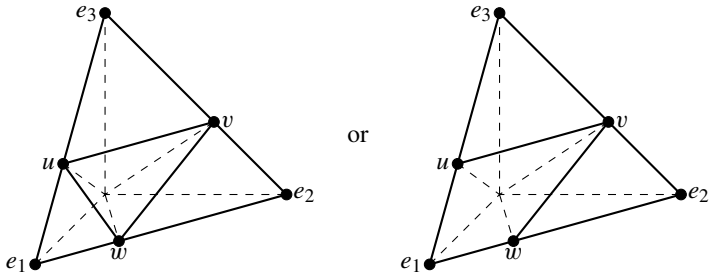
Indeed, writing

$$(a_{x_1}, a_{x_2}, a_y, a_{z_1}, a_{z_2}) := (-3, 0, -2, 1, 0) \quad \text{and} \quad (b_{x_1}, b_{x_2}, b_y, b_{z_1}, b_{z_2}) := (0, -3, -2, 1, 0)$$

for the basis of $\ker(M)$, we have that $t_1^{-3} x_1' = t_1^{a_{x_1}} t_2^{b_{x_1}} x_1'$ etc. Observe that the original variables (x_1, x_2, y) are stable under the action, by construction.

By (2.1), $\mathcal{F}_{\Sigma,\beta}$ is covered by the two charts $D_+(x_1' x_2')$ and $D_+(y')$. In both charts — i.e., if $x_1' x_2' \neq 0$ or if $y' \neq 0$ — the polynomial $x_1'^2 x_2'^2 - y'^3 = 0$ defines a smooth hypersurface.

Let us consider a further subdivision, in which we have the additional ray $w = (2, 1, 0)^T$, e.g.,



In the matrix M, we have to add the extra column w. Analogous to above, one may determine the corresponding morphism of toric stacks and verify that the preimage of the binomial defines a simple normal crossings divisor. Notice that there are more maximal cones compared to above and hence, we have to consider more charts. On the other hand the fan on the left, which is a triangulation of the fan on the right, is not much more complex: the chart on the right corresponding to $\langle e_1, u, v, w\rangle$ is defined by $x_2'y'$ being invertible. On the left we have its two open substacks where either $x_1'x_2'y'$ or $z_2'x_2'y'$ is invertible.

Remark 4.4 Recall that by (2.1), we are working in an open subset of $\mathbb{A}^5$ in the example. If one is only interested in the transform of $X = V(x_1^2x_2^2 - y^3)$, then it is sufficient to consider one of the charts $D_+(x_1'x_2')$ and $D_+(y')$ since the transform is contained in the intersection of both.

On the other hand, if one needs to keep track how the ambient space behaves, e.g. for integration, one needs all of its charts.

4.1 Proof of Proposition 4.2

Let f be a binomial. If the preimage of f provides a simple normal crossing divisor, necessarily of the form $x^C(x^A - \lambda)$, when applying the fantastack construction for Σ_f, then any further subdivison Σ of Σ_f cannot change this. Therefore, it suffices to consider the case $\Sigma = \Sigma_f$.

As before, we assume without loss of generality that $f = x^A - \lambda y^B$, where $A = (A_1, \ldots, A_s)$ and $B = (B_1, \ldots, B_r)$. If $A = 0$ or $B = 0$, then $V(f)$ is smooth and there is nothing to prove. Hence, we assume $A \in \mathbb{Z}_+^s$ and $B \in \mathbb{Z}_+^r$ in the following. We prove the result by giving an explicit description of the matrix $M = M_\Sigma$: Without loss of generality, we assume that all entries of A and B are non-zero. For every $i \in \{1, \ldots, s\}$, we define

$$D(A_i) \in \mathbb{Z}^{r\times r}$$

to be the diagonal matrix with all entries to be A_i. For every $\ell \in \{1, \ldots, s\}$, we introduce

$$R_\ell(B) \in \mathbb{Z}^{s\times r}$$

to be the matrix, whose ℓ-th row is equal to B and all other entries are zero. For example, if $s = 3$, $r = 4$, $\ell = 2$, we have

$$R_2(B) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote by $E_\alpha \in \mathbb{Z}^{\alpha\times\alpha}$ the unit matrix and by $0^{\alpha\times\beta} \in \mathbb{Z}^{\alpha\times\beta}$ we mean the $\alpha \times \beta$ matrix with all entries zero, where $\alpha, \beta \in \mathbb{Z}_+$. Using this notation, we claim that the facets of the Newton polyhedron $\Delta(f) = \Delta(f; x, y)$ provide the matrix

$$M' = \begin{pmatrix} E_s & 0^{s\times r} & R_1(B) & R_2(B) & \cdots & R_s(B) \\ 0^{r\times s} & E_r & D(A_1) & D(A_2) & \cdots & D(A_s) \end{pmatrix} \in \mathbb{Z}^{m\times n},$$

where we recall that $m = r + s$ and we set $n := r + s + rs$. In general the columns of M' might not be primitive. Dividing each column by the greatest common divisors of its entries we obtain the desired matrix M.[3]

Let us explain this: For $m = 2$, the statement can be easily verified since $\Delta(f)$ has only three facets. Thus, assume $m \geq 3$. Since we have $\Delta(f) + \mathbb{R}^m_{\geq 0} = \Delta(f)$, the Newton polyhedron $\Delta(f)$ has unbounded facets parallel to the coordinate hyperplanes of $\mathbb{R}^m$. In other words, all unit vectors $e_1, \ldots, e_m$ appear as normal vectors. Further, notice that the vertices of $\Delta(f)$ are $(A, 0)$ and $(0, B)$. The columns of M, not coming from E_{s+r}, are of the form

$$v_{i,j} := B_{j-s}e_i + A_i e_j, \quad \text{for } i \in \{1, \ldots, s\},\ j \in \{s+1, \ldots, s+r\}.$$

Fix such i, j and let $k \in \{1, \ldots, m\}\backslash\{i, j\}$ be any other element in $\{1, \ldots, m\}$ different from i and j. Then, the following equalities hold

$$(A, 0) \cdot v_{i,j} = (0, B) \cdot v_{i,j} = ((A, 0) + e_k) \cdot v_{i,j} = ((0, B) + e_k) \cdot v_{i,j} = A_i B_{j-s}. \quad (4.1)$$

Note that the face of $\Delta(f)$ defined by $v_{i,j}$ is

$$\mathcal{F}_{i,j} := \{w \in \Delta(f) \mid w \cdot v_{i,j} = A_i B_{j-s}\}.$$

Let $\mathfrak{S}$ be the segment determined by $(A, 0)$ and $(0, B)$. The latter can be described as

$$\mathfrak{S} = \{\varepsilon(A, 0) + (1 - \varepsilon)(0, B) \mid \varepsilon \in [0; 1] \subset \mathbb{R}\}.$$

By (4.1), we have

$$w \cdot v_{i,j} = A_i B_{j-s}, \quad \text{for every } w \in \mathfrak{S}.$$

In other words, $\mathfrak{S} \subseteq \mathcal{F}_{i,j}$. Since further (4.1) holds for all $k \in \{1, \ldots, m\}\backslash\{i, j\}$, we obtain that $\mathcal{F}_{i,j}$ has dimension $m - 1$ and thus is a facet.

It remains to prove that we determined all facets of $\Delta(f)$. Suppose there exists some $v = (v_1, \ldots, v_m) \in \mathbb{R}^m_{\geq 0}$, which is a normal vector of $\Delta(f)$ different from those discussed before. Let $\alpha := \#\{k \in \{1, \ldots, m\} \mid v_k \neq 0\} \geq 2$ and let $v_{k_1}, \ldots, v_{k_\alpha}$ be the non-zero entries of v, where $k_1 < k_2 < \ldots < k_\alpha$. If $\alpha = m$, then v determines a compact face of $\Delta(f)$. The only compact faces are the two vertices and the segment connecting them. Since $m \geq 3$, none of them is a facet. Hence, suppose that $\alpha < m$. For $\alpha \geq 3$, consider the projection $\pi_v \colon \mathbb{R}^m \to \mathbb{R}^\alpha$, $(w_1, \ldots, w_m) \mapsto (w_{k_1}, \ldots, w_{k_\alpha})$. Then, $\pi_v(v)$ determines a compact face of $\pi_v(\Delta(f))$ and the latter has to be the segment connecting the two vertices. Since $\alpha \geq 3$ and since all entries of A and B are non-zero, the preimage of the compact face along π_v does not correspond to a facet of $\Delta(f)$. Finally, if $\alpha = 2$, the equality $(A, 0) \cdot v = (0, B) \cdot v$ implies $\frac{v_{k_1}}{v_{k_2}} = \frac{B_{k_2-s}}{A_{k_1}}$ and thus v is a multiple of v_{k_1,k_2}.

Next, let us describe the columns of M. For $i \in \{1, \ldots, m\}$, the i-th column is the i-th unit vector of $\mathbb{Z}^m$. Fix $k \in \{1, \ldots, s\}$, $\ell \in \{1, \ldots, r\}$ and set $i := i_{k,\ell} := m + (k - 1)r + \ell$. The non-zero entries of the i-th column of M' are B_ℓ and A_k. Let $d_{k,\ell}$ be the greatest common

[3] We note however that M' will also give a fantastack resolution, obtained by a suitable root stack of ρ, see [28].

divisor of B_ℓ and A_k. Define $B_{k,\ell} := B_\ell/d_{k,\ell}$ and $A_{k,\ell} := A_k/d_{k,\ell} \in \mathbb{Z}_{\geq 0}$. Then, the non-zero entries of the i-th column of M are $B_{k,\ell}$ and $A_{k,\ell}$.

The matrix M induces $\beta = \beta_f \colon \mathbb{Z}^n \to \mathbb{Z}^m$ and thus the homomorphism $\rho_\beta \colon \mathbb{A}^n \to \mathbb{A}^m$, which is given by

$$x_i = x_i' z_{i,1}^{B_{i,1}} z_{i,2}^{B_{i,2}} \cdots z_{i,r}^{B_{i,r}} = x_i' \prod_{j=1}^{r} z_{i,j}^{B_{i,j}}, \quad \text{for } i \in \{1, \ldots, s\},$$
$$y_j = y_j' z_{1,j}^{A_{1,j}} z_{2,j}^{A_{2,j}} \cdots z_{s,j}^{A_{s,j}} = y_j' \prod_{i=1}^{s} z_{i,j}^{A_{i,j}}, \text{ for } j \in \{1, \ldots, r\},$$

where $x_1', \ldots, x_s', y_1', \ldots, y_r', z_{1,1}, \ldots, z_{1,r}, z_{2,1}, \ldots, z_{2,r}, \ldots z_{s,r}$ are the variables of $\mathbb{A}^n$. Note that the variable $z_{k,\ell}$ corresponds to the $i_{k,\ell}$-th column of M with $i_{k,\ell} = m+(k-1)r+\ell$. We obtain

$$\begin{aligned} x^A - \lambda y^B &= \prod_{i=1}^{s} x_i^{A_i} - \lambda \prod_{j=1}^{r} y_j^{B_j} = \\ &= \prod_{i=1}^{s} \Big(x_i'^{A_i} \prod_{j=1}^{r} z_{i,j}^{A_i B_{i,j}} \Big) - \lambda \prod_{j=1}^{r} \Big(y_j'^{B_j} \prod_{i=1}^{s} z_{i,j}^{A_{i,j} B_j} \Big) = \\ &= \Big(\prod_{i=1}^{s} \prod_{j=1}^{r} z_{i,j}^{d_{i,j} A_{i,j} B_{i,j}} \Big) \cdot (x'^A - \lambda y'^B). \end{aligned}$$

The fantastack $\mathcal{F}_{\Sigma_f,\beta}$ is covered by the two charts $D_+(\prod_{i=1}^s x_i')$ and $D_+(\prod_{j=1}^r y_j')$. Since $x'^A - \lambda y'^B$ is a schön smooth binomial in both charts, this ends the proof. □

Remark 4.5 Let f be a product of binomials, $f = \prod_{i=1}^d (x^A - \lambda_i y^B)$, for some $\lambda_i \in K \setminus \{0\}$, where the exponents A, B appearing are *the same* for each factor. Then, the same procedure as for $V(x^A - \lambda y^B)$ transforms $V(f)$ into a simple normal crossings divisor. Notice that, for $\lambda \neq \lambda'$, $V(x^A - \lambda y^B)$ and $V(x^A - \lambda' y^B)$ are disjoint if $x_i \neq 0$ for all i (or $y_j \neq 0$ for all j).

5 Partial simultaneous resolution of finitely many binomials

In Remark 4.5, we have seen that we can simultaneously resolve the singularities of finitely many binomial varieties $V(x^A - \lambda_i y^B)$ via a single step using fantastacks, where $\lambda_i \in K$ are pairwise different and the pairs of exponents (A, B) are the same for all i. Our next goal is to extend this result without restrictions on the exponents. In particular, we prove Theorem A in full generality, for finitely many binomials.

Observation 5.1 Let $f_1, \ldots, f_a \in K[x] = K[x_1, \ldots, x_m]$ be finitely many binomials.

(1) For every $i \in \{1, \ldots, a\}$, we determined a matrix $M(f_i) \in \mathbb{Z}^{m \times n_i}$ via the normal vectors of the facets of $\Delta(f_i)$ in the previous section, which induced a transformation $\rho_i \colon \mathbb{A}^{n_i} \to \mathbb{A}^m$ resolving the singularities of $V(f_i)$. In general, it is not sufficient to consider the transformation induced by the matrix whose set of columns coincides with the union of the normal vectors of $\Delta(f_i; x)$, for $i \in \{1, \ldots, a\}$, see Example 5.2(2).
(2) The task to simultaneously monomialize $V(f_1), \ldots, V(f_a)$ is equivalent to the monomialization of the hypersurface determined by their product $V(f_1 \cdots f_a)$.

(3) From the definition of the Minkowski sum, it follows directly that the Newton polyhedron of the product $f_1 \cdots f_a$ is the Minkowski sum of the Newton poyhedra for f_i,

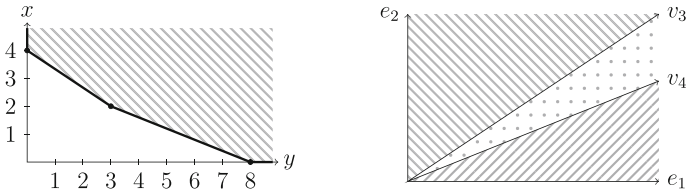
$$\Delta(f_1 \cdots f_a; x) = \Delta(f_1; x) + \cdots + \Delta(f_a; x),$$

where "+" is the Minkowski sum. In particular, any vertex v of $\Delta(f_1 \cdots f_a)$ can be decomposed as $v = v_1 + \cdots + v_a$, where $v_i \in \Delta(f_i)$, for $i \in \{1, \ldots, a\}$. Note that the converse of the last statement is not true, i.e., not every sum of vertices $v_1 + \cdots + v_a$ is also a vertex of $\Delta(f_1 \cdots f_a)$, see Example 5.2(1).

Let us discuss some examples with two binomials, where we write $f = f_1$ and $g = f_2$.

Example 5.2

(1) Consider the binomials $f = x^2 - y^3$ and $g = x^2 - y^5$. We have $fg = x^4 - x^2(y^3 + y^5) - y^8$. Its Newton polyhedron has three vertices $(4, 0), (2, 3), (0, 8)$. Note that $(2, 5) = (2, 0) + (0, 5)$ is a sum of two vertices of $\Delta(f)$ and $\Delta(g)$, but it is not a vertex of $\Delta(fg)$. The normal vectors of the facets of $\Delta(fg)$ are $e_1, e_2, v_3 := (3, 2)^T, v_4 := (5, 2)^T$, which is precisely the union of the sets of normal vectors of the facets of $\Delta(f)$ and $\Delta(g)$.



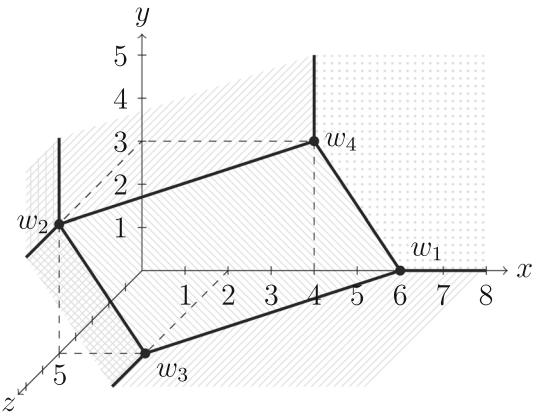
Newton polyhedron (left) and its dual fan (right).
On the right hand side, the rays are marked with their corresponding generating vectors e_1, e_2, v_3, v_4 .

The resulting matrix is $M = \begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 2 \end{pmatrix}$, which provides the transformation $\rho\colon \mathbb{A}^4 \to \mathbb{A}^2$ determined by $x = x' z_1^3 z_2^5$ and $y = y' z_1^2 z_2^2$. Thus f and g become

$$f = x'^2 z_1^6 z_2^{10} - y'^3 z_1^6 z_2^6 = z_1^6 z_2^6 (x'^2 z_2^4 - y'^3),$$
$$g = x'^2 z_1^6 z_2^{10} - y'^4 z_1^{10} z_2^{10} = z_1^6 z_2^{10} (x'^2 - y'^4 z_1^4)$$

By (2.1), $\mathcal{F}_{\Sigma,\beta}$, which is determined by the dual fan of $\Delta(fg)$ and the matrix M, can be covered by three charts: $D_+(y'z_1)$, $D_+(x'y')$, and $D_+(x'z_2)$. We observe that in each of the charts, $V(f)$ and $V(g)$ are simultaneously resolved.

(2) Let $f = x^2 - y^3$ and $g = x^4 - z^5$. The picture of the Newton polyhedron $\Delta(fg; x, y, z)$ is as follows:



Using `polymake`, we may determine all relevant data of $\Delta(fg)$. The vertices are: $w_1 := (6, 0, 0)$, $w_2 := (0, 3, 5)$, $w_3 := (2, 0, 5)$, $w_4 := (4, 3, 0)$. The rays of the dual fan are e_1, e_2, e_3, $v_1 := (3, 2, 0)^T$, $v_2 := (5, 0, 4)^T$, $v_3 := (15, 10, 12)^T$. Observe that the vector v_3, which corresponds to the compact facet of $\Delta(fg)$, is neither a normal vector of $\Delta(f)$ nor of $\Delta(g)$. The resulting matrix is

$$M = \begin{pmatrix} 1 & 0 & 0 & 3 & 5 & 15 \\ 0 & 1 & 0 & 2 & 0 & 10 \\ 0 & 0 & 1 & 0 & 4 & 12 \end{pmatrix}$$

and this determines the transformation $\mathbb{A}^6 \to \mathbb{A}^3$ given by

$$x = x' z_1^3 z_2^5 z_3^{15},\ y = y' z_1^2 z_3^{10},\ z = z' z_2^4 z_3^{12}.$$

We obtain

$$\begin{aligned} f &= x'^2 z_1^6 z_2^{10} z_3^{30} - y'^3 z_1^6 z_3^{30} = z_1^6 z_3^{30} (x'^2 z_2^{10} - y'^3), \\ g &= x'^4 z_1^{12} z_2^{20} z_3^{60} - z'^5 z_2^{20} z_3^{60} = z_2^{20} z_3^{60} (x'^4 z_1^{12} - z'^5). \end{aligned}$$

Furthermore, we have (again using `polymake`)

normal vector of facet	vertices of the facet
e_1	w_2
e_2	w_1, w_3
e_3	w_1, w_4
v_1	w_2, w_3
v_2	w_2, w_4
v_3	w_1, w_2, w_3, w_4

and from this we determine the rays of the maximal cones of the dual fan: (Recall that the maximal cones correspond to the vertices of the Newton polyhedron.)

vertex	rays of corresponding maximal cone	corresponding chart
w_1	e_2, e_3, v_3	$x'z_1z_2 \neq 0$
w_2	e_1, v_1, v_2, v_3	$y'z' \neq 0$
w_3	e_2, v_1, v_3	$x'z'z_2 \neq 0$
w_4	e_3, v_2, v_3	$x'y'z_1 \neq 0$

Therefore, the corresponding fantastack $\mathcal{F}_{\Sigma,\beta}$ can be covered by four charts and as one can verify $V(f) = V(z_1^6 z_3^{30}(x'^2 z_2^{10} - y'^3))$ and $V(g) = V(z_2^{20} z_3^{60}(x'^4 z_1^{12} - z'^5))$ are simultaneously resolved in each of them.

(3) Let $f = x^2 - y^3z^5$ and $g = x^4 - zw^3$. Using `polymake`, we obtain the matrix

$$M(fg) := \left(\begin{array}{cccc|cc|cc|cc} 1&0&0&0&3&5&1&3&3&5\\ 0&1&0&0&2&0&0&0&2&0\\ 0&0&1&0&0&2&4&0&0&2\\ 0&0&0&1&0&0&0&4&4&6 \end{array}\right)$$

(The reason for the separating lines between the columns will become clear soon.) Let us study how to it can be constructed by hand from the data of a single binomial. If we consider only one of them, we get the matrices

$$M(f) := \left(\begin{array}{cccc|cc} 1&0&0&0&3&5\\ 0&1&0&0&2&0\\ 0&0&1&0&0&2\\ 0&0&0&1&0&0 \end{array}\right) \qquad M(g) := \left(\begin{array}{cccc|cc} 1&0&0&0&1&3\\ 0&1&0&0&0&0\\ 0&0&1&0&4&0\\ 0&0&0&1&0&4 \end{array}\right).$$

The rays which have to be added for the Newton polyhedron of the product fg arise from the intersection of the 3-dimensional cones of the dual fans, which intersect the relative interior of $\mathbb{R}^4_{\geq 0}$. Since we are working with binomials, there is only one such cone for each binomial. More precisely, the rays of the respective cones are

$$(u_1, u_2, u_3) := \left(\begin{pmatrix}3\\2\\0\\0\end{pmatrix}, \begin{pmatrix}5\\0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) \quad (t_1, t_2, t_3) := \left(\begin{pmatrix}1\\0\\4\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\\4\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)$$

(Take the vectors different from the unit vectors and if the corresponding matrix has a zero row, add unit vectors to the set so that there are no zero rows afterwards.)

A generator of a new ray, which we are looking for, is a vector r, for which there are *non-negative integers* a, b, c, d, e, f such that

$$r = au_1 + bu_2 + cu_3 = dt_1 + et_2 + ft_3.$$

It is tempting to determine a basis of $\ker \begin{pmatrix} u_1 & u_2 & u_3 & -t_1 & -t_2 & -t_3 \end{pmatrix}$, but this does not provide the correct vectors since the kernel has dimension two and there are non-trivial choices for the basis.

Thus, we have to consider the kernels $\ker \begin{pmatrix} u_i & u_j & -t_k & -t_\ell \end{pmatrix}$, for varying i, j, k, ℓ. Since the dimension of this kernel is one, the basis is unique up to multiplication by a non-zero constant. (Note that this becomes more delicate if we work with three binomials

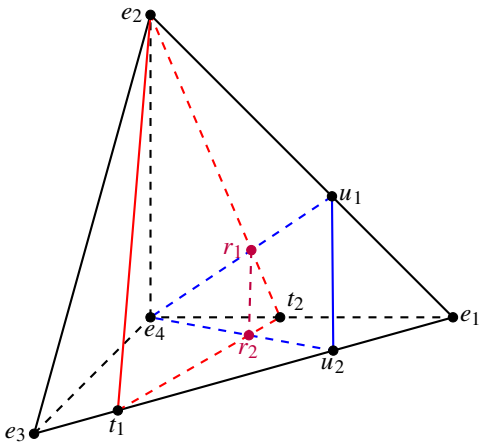
or more, as we have to repeat this process taking the newly added rays into account.) Some of the choices for i, j, k, ℓ cannot provide new rays. For example, in

$$\ker\begin{pmatrix} u_2 & u_3 & -t_2 & -t_3 \end{pmatrix} = \ker\begin{pmatrix} 5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{pmatrix},$$

the third row is telling us that $a = 0$ and thus, the generator for a potential new ray can be written as $r = au_1 + bu_2 + cu_3 = cu_3$, which provides the ray generated by u_3. A computation shows that for the remaining cases, we have:

$$\begin{aligned} \ker\begin{pmatrix} u_1 & u_3 & -t_2 & -t_3 \end{pmatrix} &= \mathrm{Span}((1,4,1,2)^T), \\ \ker\begin{pmatrix} u_2 & u_3 & -t_1 & -t_2 \end{pmatrix} &= \mathrm{Span}((2,12,1,3)^T), \\ \ker\begin{pmatrix} u_1 & u_2 & -t_1 & -t_3 \end{pmatrix} &= \mathrm{Span}((-3,2,1,-6)^T). \end{aligned}$$

The first two kernels provide the additional rays generated by $r_1 := u_1 + 4u_3 = t_2 + 2t_3 = (3,2,0,4)^T$ and $r_2 := 2u_2 + 12u_3 = t_1 + 3t_2 = (10,0,4,12)^T = 2(5,0,2,6)^T$ respectively. On the other hand, the third vector $(-3,2,1,-6)^T$ does not provide a new ray since the signs of the entries varies. In conclusion, we obtain the matrix $M(fg)$. This situation can also be seen in earlier work: the transverse slice of the dual fan is the linear subspace ℓ introduced before [19, Lemma 5, p. 1830] and before [18, Theorem 3.1], or H_ℓ introduced in [31, Section 6.1, p. 77].



In general, if we consider more than two binomials, then each binomial is resolved after applying the fantastack construction, but they do not necessarily form a normal crossings divisor. The reason for this is that we may have singularities in the torus. Let us illustrate this for simple examples.

Example 5.3

(1) If we have $f_1 := x - 1$, $f_2 := y - 1$, $f_3 := xy - 1$, then the fantastack construction does not provide new information since the Newton polyhedron of $f_1 f_2 f_3$ is the whole

orthant $\mathbb{R}^2_{\geq 0}$. On the other hand, $V(f_1 f_2 f_3)$ does not define a simple normal crossing divisor. This can be seen by introducing new coordinates $\widetilde{x} := x - 1$, $\widetilde{y} := y - 1$, which provides $f_1 = \widetilde{x}$, $f_2 = \widetilde{y}$, $f_3 = \widetilde{x} + \widetilde{y} + \widetilde{x}\widetilde{y}$. By blowing up the origin, we will simultaneously resolve $V(f_1)$, $V(f_2)$, $V(f_3)$.

(2) Let K be an algebraically closed field of characteristic $p > 0$. Consider the binomials $f_1 := x - 1$, $f_2 := y - 1$, $f_3 := xy^p - 1$. Analogous to (1), we introduce the new variables $\widetilde{x} := x - 1$, $\widetilde{y} := y - 1$ and obtain $f_1 = \widetilde{x}$, $f_2 = \widetilde{y}$, $f_3 = \widetilde{x} + \widetilde{y}^p + \widetilde{x}\widetilde{y}^p$. Notice that the scheme theoretic intersection of $V(f_1)$ and $V(f_3)$ is not reduced. In particular, the intersection lattice of $V(f_1 f_2 f_3)$ does not provide a simple arrangement.

Let us restate and prove Theorem A. It generalizes Proposition 4.2 and is the best result that could be obtained towards simultaneous normal crossings resolution of finitely many binomial varieties using fantastacks.

Theorem 5.4 *Let K be an algebraically closed field of arbitrary characteristic $p \geq 0$. Let $f_1, \ldots, f_a \in K[x] = K[x_1, \ldots, x_m]$ be finitely many binomials, where $a, m \in \mathbb{Z}_+$ with $m \geq 2$. Let $\mathcal{E} \subset \mathbb{Z}$ be the set of problematic primes associated to the exponents of the pure binomial factors of $f_1, \ldots, f_a$.*

Let Σ be a subdivision of the dual fan of the Newton polyhedron of the product $f_1 \cdots f_a$ and let $\beta \colon \mathbb{Z}^n \to \mathbb{Z}^m$ be the homomorphism determined by the matrix $M \in \mathbb{Z}^{m \times n}$, whose columns are the primitive generators for the rays of the fan Σ. Let $\rho \colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$ be the morphism of toric stacks induced by β. Set $X := V(f_1 \cdots f_a) \subset \mathbb{A}^m$.

Then the reduced preimage $\rho^{-1}(X)_{\mathrm{red}} \subset \mathcal{F}_{\Sigma,\beta}$ is a schön arrangement of smooth binomials. If furthermore $p \notin \mathcal{E}$, then $\rho^{-1}(X)_{\mathrm{red}}$ induces a simple arrangement on $\mathcal{F}_{\Sigma,\beta}$.

Proof Set $X_i := V(f_i)$, for $i \in \{1, \ldots, a\}$. Since Σ is a refinement of the normal fan Σ_{f_i}. Proposition 4.2 implies that $\rho^{-1}(X_i)_{\mathrm{red}}$ is a schön binomial hypersurface on $\mathcal{F}_{\Sigma,\beta}$.

Let $(y) = (y_1, \ldots, y_n)$, $n \geq m$, be coordinates in a chart of $\mathcal{F}_{\Sigma,\beta}$. By the schön condition, each f_i is of the form

$$y^{A(i)}(1 + \lambda_i y^{B(i)}),$$

for some units $\lambda_i \neq 0$ and $A(i), B(i) \in \mathbb{Z}^n_{\geq 0}$. Thus, the only part not being simple normal crossing is the product $\prod_{i=1}^{a}(1 + \lambda_i y^{B(i)})$, which provides a schön arrangement of smooth binomials. □

Remark 5.5 In the situation of the theorem, we have that $\mathcal{F}_{\Sigma,\beta}$ is covered by at most 2^a affine charts since the number of vertices of $\Delta(f_1 \cdots f_a)$ (and thus the number of maximal cones in Σ) is $\leq 2^a$ by Observation 5.1(3).

In general, 2^a is strictly smaller than the number of affine charts obtained by blow-ups in smooth centers. For example, a possible desingularization of $x_1x_2 - x_3x_4$ via blow-ups in smooth centers could be to choose the closed point as the center and hence creates 4 charts. This difference becomes even bigger if we consider binomials with large exponents. For further explicit examples, we refer to [13, Examples 8.4 – 8.11], where different methods for the local monomialization of a single binomial are studied, implemented, and compared. A local monomialization is a local variant of desingularization, where one does not necessarily demand the centers to be chosen globally.

If $\mathrm{char}(K) \notin \mathcal{E}$, then the remaining simultaneous desingularization is obtained by applying the following theorem. A more general variant of this result was proven in [26, Theorem 1.3].

Theorem 5.6 ([21, Theorem 1.1]) *Let X_0 be an open subset of a non-singular algebraic variety X such that $X\backslash X_0$ can be decomposed as a union $\bigcup_{i\in I} S_i$ of closed irreducible subvarieties with the properties*

(i) *S_i is smooth,*
(ii) *S_i and S_j meet cleanly, and*
(iii) *$S_i \cap S_j = \varnothing$ or a disjoint union of S_ℓ.*

The set $\mathcal{S} = \{S_i\}_i$ is a poset. Let k be the rank of $\mathcal{S}$. Then there exists a sequence of well-defined blow-ups

$$\mathrm{B}\ell_{\mathcal{S}}(X) \to \mathrm{B}\ell_{\mathcal{S}_{\leq k-1}}(X) \to \ldots \to \mathrm{B}\ell_{\mathcal{S}_{\leq 0}}(X) \to X,$$

where $\mathrm{B}\ell_{\mathcal{S}_{\leq 0}}(X) \to X$ is the blow-up of X with center all S_i of rank 0 and $\mathrm{B}\ell_{\mathcal{S}_{\leq r}}(X) \to \mathrm{B}\ell_{\mathcal{S}_{\leq r-1}}(X)$ is the blow-up of $\mathrm{B}\ell_{\mathcal{S}_{\leq r-1}}(X)$ with center the proper transform of all S_j of rank r, for $r \geq 1$. We have

(1) *$\mathrm{B}\ell_{\mathcal{S}}(X)$ is smooth,*
(2) *$\mathrm{B}\ell_{\mathcal{S}}(X)\backslash X_0 = \bigcup_{i\in I} \widetilde{S}_i$ is a normal crossings divisor, and*
(3) *$\widetilde{S}_{i_1} \cap \ldots \cap \widetilde{S}_{i_n}$ is non-empty if and only if $S_{i_1}, \ldots, S_{i_n}$ form a chain in the poset $\mathcal{S}$. Hence, $\widetilde{S}_i$ and $\widetilde{S}_j$ meet if and only if S_i and S_j are comparable.*

As explained in [26, Introduction/after Theorem 1.2], this is a generalization of De Concini and Procesi's wonderful models of subspace arrangements [10].

We need a variant of the theorem for algebraic stacks. We note the following direct consequence of smoothness:

Proposition 5.7 *Let $X_0 \subset X$ be as in the theorem, and $\pi : Y \to X$ a smooth morphism. Set $Y_0 := \pi^{-1}(X_0)$, $S_i^Y := \pi^{-1}(S_i)$ and $\mathcal{S}^Y = \{S_i^Y\}$. Then S_i^Y satisfy properties (i)-(iii), and, setting $\mathrm{B}\ell_{\mathcal{S}^Y_{\leq r}}(Y)$ as in the theorem, we have that*

$$\mathrm{B}\ell_{\mathcal{S}^Y_{\leq r}}(Y) = Y \times_X \mathrm{B}\ell_{\mathcal{S}_{\leq r}}(X),$$

and the sequence

$$\mathrm{B}\ell_{\mathcal{S}^Y}(Y) \to \mathrm{B}\ell_{\mathcal{S}^Y_{\leq k-1}}(Y) \to \ldots \to \mathrm{B}\ell_{\mathcal{S}^Y_{\leq 0}}(Y) \to Y$$

satisfies the conclusions (1)-(3) of the theorem.

Given an algebraic stack $\mathcal{X}$, an open $\mathcal{X}_0 \subset \mathcal{X}$ with complement $\bigcup_i S_i^{\mathcal{X}}$ satisfying (i)–(iii) of the theorem, and any smooth presentation by schemes $Y_1 \rightrightarrows Y_0$, we obtain pullback subschemes $S_i^{Y_1}$ and $S_i^{Y_0}$ such that $S_i^{Y_1}$ are the preimage of $S_i^{Y_0}$ under either projection $Y_1 \to Y_0$. It follows that we obtain smooth groupoids

$$\mathrm{B}\ell_{\mathcal{S}^{Y_1}_{\leq r}}(Y_1) \rightrightarrows \mathrm{B}\ell_{\mathcal{S}^{Y_0}_{\leq r}}(Y_0)$$

with quotient stacks $\mathrm{B}\ell_{\mathcal{S}^{\mathcal{X}}_{\leq r}}(\mathcal{X})$ satisfying the conclusion of the theorem. We therefore obtain:

Corollary 5.8 *The theorem applies as stated to algebraic stacks.*

5.1 What happens in mixed characteristics

While the theory of toric Artin stacks is only established over algebraically closed fields, we are interested in applications such as those discussed in Sect. 1.10, and the methods provided here can be applied and sometimes work. We discuss here what works without change and what requires further work.

We consider a collection of binomials $f_1, \dots, f_a$ over the ring of integers $\mathcal{O}_K$ of a finite extension K of $\mathbb{Q}_p$.

First, the formation of $\rho : \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$ depends only on the combinatorial data of the exponents appearing in f_i, and works without change over an arbitrary base, but with weaker outcomes.

Second, the treatment of singularities here requires working with binomials of the form $x^A - \lambda x^B$ where λ *is a unit*. Without this assumption the procedure still leads to equations of the form $x^C(x^A - \lambda)$ (or $x^C(1 - \lambda x^B)$), but if $v(\lambda) > 0$ the first of these is no longer schön. We expect that Kato's theory of toric singularities in mixed characteristics [22] will provide a way to treat these cases; another approach, suggested by the referee, is to systematically work with toric schemes over valuation rings — this is consistent with the work [20]. This will have to be addressed elsewhere.

Third, when $f = x^A - \lambda y^B$ is a pure binomial and the residue characteristic p divides all the exponents, the hypersurface $f = 0$ in the torus is not smooth over $\mathcal{O}_K$. Over a perfect field its reduction is smooth, and in general its reduction is regular, being the orbit of a smooth group scheme.

Fourth, even when p does not divide the exponents of individual binomials, if $p \in \mathcal{E}$ the binomials do not meet cleanly. As stated earlier, this situation is interesting already over an algebraically closed field of characteristic p.

Finally, if all λ_i are units and $p \notin \mathcal{E}$ then the procedure does provide an arrangement of schön binomials meeting cleanly, in which case the procedure for resolving the arrangement in Theorem 5.6 works without change.

6 Binomial ideals

The purpose of this section is to generalize Proposition 4.2 to resolution of singularities for binomial subschemes $X \subset \mathbb{A}^m$ (Theorem B). As we neglect the question of functoriality, we can work with a fixed system of generators for the binomial ideal defining X.

Note that the situation is a bit simpler than in the simultaneous setup in the previous section. For example, in both cases of Example 5.3, the ideal generated by f_1, f_2, f_3 is equal to $\langle x - 1, y - 1\rangle$ and hence, the corresponding variety is smooth.

Theorem 6.1 (Theorem B) *Let K be an algebraically closed field and $m \in \mathbb{Z}_{\geq 2}$. Let $X \subset \mathbb{A}^m$ be a binomial subscheme. Denote by Σ_0 the fan in $\mathbb{R}^m$ with unique maximal cone $\mathbb{R}^m_{\geq 0}$. Let $f = \{f_1, \dots, f_a\}$ be binomial generators of $\mathcal{I}_X$.*

(1) (See [18, 19, 31]) *The proper transform of X^{pure} in $\mathcal{F}_{\Sigma_f,\beta}$ is a schön purely binomial subscheme, in particular its reduction is smooth.*

(2) *There exists a further subdivision Σ of Σ_0, of combinatorially bounded complexity, with the following property: If $\beta\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is the homomorphism determined by the matrix $M \in \mathbb{Z}^{m\times n}$, whose columns are the primitive generators for the rays of the fan Σ, then*

$$\rho^{-1}(X)_{\text{red}} \subset \mathcal{F}_{\Sigma,\beta}$$

is in simple normal position on $\mathcal{F}_{\Sigma,\beta}$ (see Sect. 1.8), where we denote by $\rho\colon \mathcal{F}_{\Sigma,\beta} \to \mathbb{A}^m$ the morphism of toric stacks induced by β.

Proof Without loss of generality, we assume that, for every $i \in \{1,\dots,a\}$, f_i is not contained in the ideal generated by $\{f_1,\dots,f_a\}\backslash\{f_i\}$. Note that some f_i might already be monomials.

As before, $\Sigma_f = \Sigma_{f_1\cdots f_a}$ is the dual fan of the Newton polyhedron of the product $f_1\cdots f_a$ and let $\rho_{\Sigma_f}\colon \mathcal{F}_{\Sigma_f,\beta} \to \mathbb{A}^m$ be the morphism of toric stacks as in Theorem A. In particular, in every chart, $\rho_{\Sigma_f}^{-1}(f_i)$ is of the form $y^{C(i)}(1-\epsilon_i y^{A(i)})$ and $f_i' := 1-\epsilon_i y^{A(i)}$ is schön by Theorem A, where (y) is a system of local coordinates and ϵ_i is either a unit or zero. (For this, recall Remark 4.5 and use that every binomial is a product of reduced binomials – possibly with multiplicities – since K is algebraically closed).

We claim that in each chart $X' = V(f_1',\dots f_a')$ is a schön purely binomial subscheme. Indeed the variables y_j appearing in the f_i' are all invertible on X', and so $X' = X'_y \times \mathbb{A}^{m'}$ is the product of a coset of a subgroup of the torus in these parameters with the affine space with the remaining coordinates x_l. Its reduction is thus smooth and it meets $V(\prod x_l)$ transversely. This gives part (1).

To address Part (2), we need to modify the monomial factors which on each chart are of the form $y^{C(1)},\dots,y^{C(a)}$. We note that the locally principal ideals given on charts as $\langle y^{C(1)}\rangle,\dots,\langle y^{C(a)}\rangle$ are well-defined globally, since $\langle f_i\rangle = \langle f_i'\rangle\langle y^{C(i)}\rangle$.

We define monomial ideals

$$G_{i,j} := \left\langle y^{C(i)}, y^{C(j)}\right\rangle,$$

for $i,j \in \{1,\dots,a\}$ with $i < j$. Let Σ_G be the dual fan of the Newton polyhedron of $\prod_{i<j} G_{i,j}$, or, equivalently, of the product of binomials $\prod_{i<j}\left(y^{C(i)} - \lambda_{ij}y^{C(j)}\right)$ with $\lambda_{ij} \neq 0,1$. Let Σ the induced subdivision of Σ_f. Equivalently, this is the fan of the Newton polyhedron of $\prod_{i<j} G_{i,j} \times \prod_i \langle f_i\rangle$, or equivalently, of the product of binomials $\prod_{i<j}\left(y^{C(i)} - \lambda_{ij}y^{C(j)}\right) \times \prod_i f_i$.

Applying the fantastack construction for Σ, we achieve that for every subset $J \subseteq \{1,\dots,a\}$ the ideal $\langle y^{C(j)} \mid j \in J\rangle$ is principalized in every chart. The latter has the consequence that the monomial factors are totally ordered: in each chart, after a possible relabeling of the generators (and abusing notation), we have $y^{C(i)} \mid y^{C(i+1)}$ for every $i \in \{1,\dots,a-1\}$.

Let M_i be the reduced monomial obtain from $y^{C(i)}$. Set $N_1 := M_1$ and $N_i := M_i/M_{i-1}$ for $i \in \{2,\dots,a\}$. Then the reduced preimage of X in the given chart decomposes as

$$V(N_1) \cup V(N_2, f_1') \cup V(N_3, f_1', f_2') \cup \cdots \cup V(N_a, f_1', \dots, f_{a-1}') \cup V(f_1',\dots,f_a'),$$

where we write $V(I)$ for the scheme determined by the vanishing locus of the ideal generated by the elements of I. One checkes that this is in normal crossing position, as required.

Notice once again that the number of cones of the fan and therefore the number of charts and divisors, does only depend on the ambient dimension and the number a of binomial generators, but not on the explicit exponents of the binomials. □

By applying a combinatorial sequence of blowups, as for example in [26], we may obtain a divisor with simple normal crossings:

Proposition 6.2 *There exists a sequence of at most m blowups $L : Y \to \mathcal{F}_{\Sigma,\beta}$ such that $L^{-1}\rho^{-1}(X)$ is a simple normal crossings divisor.*

Proof The closures S_I of strata of a scheme in simple normal position form a building set in the sense of [26, Section 2.1]. Let G_c be the union of the S_I of codimension c. By the

theorem this applies to $\rho^{-1}(X)$. By [26, Proposition 5.3] blowing up, in decreasing order of codimension, the successive proper transforms of G_c one obtains the result. □

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