

PERIOD DOMAINS FOR GRAVITATIONAL INSTANTONS

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ABSTRACT. Based on the uniformization theorems of gravitation instantons by Chen–Chen [4], Chen–Viaclovsky [7], Collins–Jacob–Lin [13], and Hein–Sun–Viaclovsky–Zhang [25], we prove that the period maps for the ALH^* , ALG , and ALG^* gravitational instantons are surjective. In particular, the period domains of these gravitational instantons are exactly their moduli spaces.

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1. INTRODUCTION

Gravitational instantons, introduced by Hawking [22] for his Euclidean quantum gravity theory, are defined as non-compact complete hyperKähler 4-manifolds with L^2 curvature tensors. From the viewpoint of differential geometry, gravitational instantons arise naturally as a bubbling limit of hyperKähler metrics on $K3$ surfaces [8, 17, 24]. Therefore, they can be viewed as the building blocks towards the understanding of 2-dimensional Calabi–Yau metrics. The early discovered gravitational instantons are classified by their volume growths r^4, r^3, r^2, r . Those with volume growth r^4 are called locally asymptotically Euclidean (ALE), those with volume growth r^3 are called locally asymptotically flat (ALF) and the rest two are named ALG and ALH by induction. Later, Hein [23] found two new types of gravitational

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instantons named as ALG* and ALH*. The former has volume growth r^2 , as the ALG gravitational instantons, but with a different curvature decay rate while the latter has volume growth $r^{4/3}$. Recently, Sun–Zhang [33] used the Cheeger–Fukaya–Gromov theory to prove that any non-flat gravitational instanton has a unique asymptotic cone and it must belong to one of the above six types. As a summary, there are six types of gravitational instantons in total: ALE, ALF, ALG, ALH, ALG*, and ALH*.

To further classify the gravitational instantons within each type, people are interested in the following questions:

- (1) What are the possible diffeomorphism types of the gravitational instantons within each type?
- (2) What are the possible cohomology classes of the hyperKähler triples for a fixed diffeomorphism type of gravitational instantons?
- (3) Does the cohomology classes of the hyperKähler triple uniquely determine the gravitational instantons isometrically?

The set of possible cohomology classes supporting the hyperKähler triples of gravitational instantons within a fixed diffeomorphism type is usually known as the period domain. The second question can be then rephrased as “how to characterize the period domain of gravitational instantons within a fixed diffeomorphism type?” The third question is usually known as the Torelli theorem of gravitational instantons.

Kronheimer first answered all these questions for ALE gravitational instantons [26, 27]. In which case, topologically, the underlying geometry always comes from the crepant resolution of the quotient of \mathbb{C}^2 by a finite subgroup of $\mathrm{SU}(2)$. Any triple in $H^2(X, \mathbb{R})$ can be realized as the cohomology classes of the hyperKähler triples when they do not vanish simultaneously on the (-2) -classes in $H_2(X, \mathbb{Z})$. Moreover, Kronheimer established a Torelli-type theorem for ALE gravitational instantons. The analogue theorem for ALF gravitational instantons has been established by Chen–Chen [5]. For the rest of gravitational instantons, the first question is answered by certain “uniformization theorems” (see Section 2): for any gravitational instantons of types ALG, ALH, ALG*, ALH*, up to a suitable hyperKähler rotation they can be compactified to rational elliptic surfaces by filling in a fibre with monodromy of finite order, smooth fibre, an I_k^* -fibre or an I_k -fibre respectively [4, 11, 13, 23, 25]. In particular, there are finitely many diffeomorphism types of the gravitational instantons from the classification of singular fibres of rational elliptic surfaces of Perron [32]. The Torelli-type theorems for these gravitational instantons are also established: the ALH case by Chen–Chen [6], the ALG and ALG* cases by Chen–Viaclovsky–Zhang [9] and the ALH* case by the second author with Collins and Jacob [13]. While the questions about characterizations of period domains of gravitational instantons remain open, it is observed that not all the cohomology classes can be realized as those of the hyperKähler triples of gravitational instantons – there are some obvious topological constraints: those homology classes with self-intersection -2 can be realized as holomorphic curves after a suitable hyperKähler rotation and particularly the corresponding Kähler form can not vanish on it¹. Subsequently, Chen–Viaclovsky–Zhang [9] conjectured that given a diffeomorphism type of ALG or ALG* gravitational instanton, any

¹If a (-2) class vanishes on the hyperkähler triple, then, up to a hyperkähler rotation, it can be realized a (-2) curve. We can contract it to get an orbifold. In which case, the Calabi–Yau metric should be replaced by the orbifold Calabi–Yau metric.

cohomology classes of hyperkähler triples which do not vanish simultaneously can be realized by a gravitational instanton. One can make a similar conjectural statement for the ALH^* gravitational instantons.

The goal of this manuscript is to study these conjectures. Let us outline the organization of this manuscript and, in the meanwhile, briefly explain the idea of the proof of the conjecture for the ALH^* case since the ideas for the other two cases are pretty much similar. We treat ALH^* gravitational instantons in §2 and ALG as well as ALG^* gravitational instantons in §3. In §2.1, we recall some basics about pairs (Y, D) with Y a (weak) del Pezzo surface and $D \in |-K_Y|$ smooth, and the fact that for such a pair (Y, D) the complement $Y \setminus D$ can support ALH^* gravitational instantons. In §2.2–§2.4, we construct pairs (Y, D) to realize cohomology classes in $H^2(X_r, \mathbb{C})$ of a reference ALH^* gravitational instanton X_r as the cohomology classes of the $(2, 0)$ -form Ω on $X = Y \setminus D$. We also show that any cohomology class which is positive on every holomorphic curve in X supports a Ricci-flat metric asymptotic to Calabi ansatz and thus gives an ALH^* gravitational instanton. In §2.5, we demonstrate how to use monodromy transformations to reduce all the other cases to the previous one. Finally we give a complete proof of the surjectivity of the period map in §2.6. In §3.1, we construct ALG and ALG^* pairs (Y, D) to realize cohomology classes in $H^2(X_r, \mathbb{C})$ of the complement $X_r = Y_r \setminus D_r$ of a reference ALG or ALG^* pair (Y_r, D_r) as the cohomology classes of the $(2, 0)$ -form on $X = Y \setminus D$. In §3.2, we recall some basics of ALG and ALG^* gravitational instantons, including the definition of the period maps as well as the uniformization theorem. Finally in §3.3, we prove the surjectivity of the period maps. To sum up,

Theorem 1.1. (*=Theorem 2.5 and Theorem 3.13*) *The period maps for $\text{ALH}^*/\text{ALG}/\text{ALG}^*$ gravitational instantons are all surjective.*

At the moment when this manuscript was about to be finished, Chen–Viaclovsky–Zhang had a different proof for the conjecture in the cases of ALG in the second version of their preprint [9]. On the other hand, it is conjectured that certain gauge theory moduli spaces constructed in Biquard–Boalch [3] and Cherkis–Kapustin [10] will achieve all possible periods and known as the modularity conjecture. We will refer the readers to Mazzeo–Fridrickson–Swoboda–Weiss for the progress along this line, which would eventually lead to a different proof of the surjectivity of period maps in the cases of ALG and ALG^* .

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2. PERIOD DOMAINS OF ALH^* GRAVITATIONAL INSTANTONS

2.1. Weak del Pezzo surfaces. A rational surface Y is a weak del Pezzo surface if its anti-canonical divisor $-K_Y$ is big and nef. From the classification of compact complex surfaces, one has

Proposition 2.1. *Weak del Pezzo surfaces are either blow-up of \mathbf{P}^2 at generic $b = 9 - d$ points with $1 \leq d \leq 9$ or $\mathbf{P}^1 \times \mathbf{P}^1$ or the Hirzebruch surface \mathbb{F}_2 . Here generic configuration means*

- all points are proper (no multiplicity higher than 2);
- no three points are on a line;
- no six points are on a conic;
- no cubic passes through the points with one of them being a singular point of that cubic.

From the above proposition, any holomorphic curve in a weak del Pezzo surface has self-intersection at least -2 . The self-intersection number $d = (-K_Y)^2$ is the *degree* of the weak del Pezzo surface Y . Every weak del Pezzo surface admits a smooth anti-canonical divisor. Thus, there are in total 10 deformation families of pairs consisting of a weak del Pezzo surface and a smooth anti-canonical divisor: one deformation family for each $d \neq 8$ and two for $d = 8$. Notice that the Hirzebruch surface \mathbb{F}_2 is in the deformation family of $\mathbf{P}^1 \times \mathbf{P}^1$. For notational simplicity, we shall denote the degree of $\mathbf{P}^1 \times \mathbf{P}^1$ or \mathbb{F}_2 by $d = 8'$.

To describe the period domains of ALH^* gravitational instantons, we need to compute $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$. We use the long exact sequence of the pair (Y, D) (cf. [28, §I.5.1]),

$$0 \rightarrow H^1(D, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z}) \rightarrow H_2(Y, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z}) \rightarrow 0. \quad (2.1)$$

Notice that $H_1(X, \mathbb{Z})$ is torsion and in particular $\text{rank}_{\mathbb{Z}} H_2(X, \mathbb{Z}) = 11 - d$ is determined by the degree of the weak del Pezzo surface Y . The connecting homomorphism $H_2(Y, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z})$ in (2.1) is identified with the signed intersection

$$\varphi_{[D]}: [C] \mapsto ([D] \mapsto [C] \cdot [D])$$

and we obtain a short exact

$$0 \rightarrow H^1(D, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z}) \rightarrow \ker(\varphi_{[D]}) \rightarrow 0 \quad (2.2)$$

where $\varphi_{[D]}$ denotes the signed intersection map. Via Poincaré duality, we can further identify $\ker(\varphi_{[D]})$ with $[D]^\perp$, the subgroup of $\text{Pic}(Y)$ with zero pairing with the Poincaré dual of $[D]$.

It is known that the middle cohomology group of a smooth weak del Pezzo surface is isomorphic to either $\mathbb{Z}^{1,9-d}$ or U_2 (the hyperbolic lattice of rank two). Let Y be a weak del Pezzo surface of degree $d \neq 8'$ and let $\pi: Y \rightarrow \mathbf{P}^2$ be a blow-up (at $b = 9 - d$ points) realization of Y . Denote by E_1, \dots, E_b the exceptional divisors of π and by H the hyperplane in \mathbf{P}^2 . Then the assignments $e_0 \mapsto [H]$ and $e_i \mapsto [E_i]$ (the pullbacks are omitted) give rise to an isomorphism of lattices $\mathbb{Z}^{1,b} \rightarrow H^2(Y, \mathbb{Z})$. The anti-canonical divisor of Y is linearly equivalent to $3H - E_1 - \dots - E_b$. Moreover,

$$\{H - 3E_1, E_i - E_{i+1}, i = 1, \dots, b-1\}$$

is a basis of $[D]^\perp$ with $D \in |-K_Y|$.

If Y is such that $d = 8'$, it is straightforward to check that $H^2(Y, \mathbb{Z}) \cong U_2$ under the basis $\{[\ell_1], [\ell_2]\}$ where ℓ_i 's are (parallel transport of) the rulings in $\mathbf{P}^1 \times \mathbf{P}^1$. The anti-canonical divisor is linearly equivalent to $2[\ell_1] + 2[\ell_2]$ and $[D]^\perp \cong \langle \ell_1 - \ell_2 \rangle$ for $D \in |-K_Y|$.

2.2. Constructions of (Y, D) for ALH* gravitational instantons. The purpose of this subsection is to construct reference marked log Calabi–Yau pairs coming from (weak) del Pezzo surfaces. By a *marked log Calabi–Yau pair* we mean a log Calabi–Yau pair (Y, D) together with a basis \mathcal{B} of $H_2(X, \mathbb{Z})$ with $X := Y \setminus D$, called the *distinguished basis*. We will treat the cases $1 \leq d \leq 9$ and $d = 8'$ separately.

We now construct a marked log Calabi–Yau pair $(Y_{\mathfrak{r},d}, D_{\mathfrak{r},d})$ for each $1 \leq d \leq 9$ ($d \neq 8'$) where $Y_{\mathfrak{r},d}$ is a smooth del Pezzo surface of degree d and $D_{\mathfrak{r},d}$ is a smooth anti-canonical divisor.

For $d = 9$, we simply take $Y_{\mathfrak{r},9} = \mathbf{P}^2$ and $D_{\mathfrak{r},9}$ to be a smooth elliptic curve. From the long exact sequence of compactly supported cohomology

$$0 \rightarrow H_c^1(D_{\mathfrak{r},9}, \mathbb{Z}) \rightarrow H_c^2(X_{\mathfrak{r},9}, \mathbb{Z}) \rightarrow [D_{\mathfrak{r},9}]^\perp = \{0\}, \quad (2.3)$$

we have an isomorphism

$$H_c^1(D_{\mathfrak{r},9}, \mathbb{Z}) \xrightarrow{\delta} H_c^2(X_{\mathfrak{r},9}, \mathbb{Z}) \quad (2.4)$$

which, under Poincaré duality, is identified with “taking an S^1 -bundle.” δ is also known as the Leray coboundary map.

Choose a symplectic basis $\{\alpha_{\mathfrak{r}}, \beta_{\mathfrak{r}}\}$ of $H_1(D_{\mathfrak{r},9}, \mathbb{Z}) \cong H_c^1(D_{\mathfrak{r},9}, \mathbb{Z})$ and denote their image in $H_2(X_{\mathfrak{r},9}, \mathbb{Z})$ by the same notation. Then $(Y_{\mathfrak{r},9}, D_{\mathfrak{r},9})$ and $\mathcal{B}_{\mathfrak{r},9} = \{\alpha_{\mathfrak{r}}, \beta_{\mathfrak{r}}\}$ form our reference marked log Calabi–Yau pair in degree 9.

To continue, we pick 8 distinct points $q_{\mathfrak{r},1}, \dots, q_{\mathfrak{r},8} \in D_{\mathfrak{r},9}$. For the case $d = 8$, we take $Y_{\mathfrak{r},8} = \text{Bl}_{\{q_{\mathfrak{r},1}\}} \mathbf{P}^2$ and $D_{\mathfrak{r},8}$ to be the proper transform of $D_{\mathfrak{r},9}$. Notice that $D_{\mathfrak{r},8} \in |-K_{Y_{\mathfrak{r},8}}|$ since $q_{\mathfrak{r},1}$ belongs to $D_{\mathfrak{r},9}$. Put $X_{\mathfrak{r},8} := Y_{\mathfrak{r},8} \setminus D_{\mathfrak{r},8}$ as before. Since $D_{\mathfrak{r},8} \cong D_{\mathfrak{r},9}$, we can still (and should) use $\{\alpha_{\mathfrak{r}}, \beta_{\mathfrak{r}}\}$ as our basis of $H_c^1(D_{\mathfrak{r},8}, \mathbb{Z})$. Denote their image in $H_2(X_{\mathfrak{r},8}, \mathbb{Z})$ by the same notation. Moreover, $[D_{\mathfrak{r},8}]^\perp = [3H - E_{\mathfrak{r},1}]^\perp \cong \langle H - 3E_{\mathfrak{r},1} \rangle_{\mathbb{Z}}$ ($E_{\mathfrak{r},1}$ is the exceptional divisor over $q_{\mathfrak{r},1}$). We fix once for all a lifting $\gamma_{\mathfrak{r},1} \in H_2(X_{\mathfrak{r},8}, \mathbb{Z})$ of $H - 3E_{\mathfrak{r},1}$ and therefore we achieve a distinguished basis $\mathcal{B}_{\mathfrak{r},8} = \{\alpha_{\mathfrak{r}}, \beta_{\mathfrak{r}}, \gamma_{\mathfrak{r},1}\}$ of $H_2(X_{\mathfrak{r},8}, \mathbb{Z})$.

We can construct reference marked log Calabi–Yau pairs inductively. For the degree d model $(Y_{\mathfrak{r},d}, D_{\mathfrak{r},d})$, we blow-up our degree $d + 1$ model $Y_{\mathfrak{r},d+1}$ at (the proper transform of) $q_{\mathfrak{r},b}$ and we set $D_{\mathfrak{r},d}$ to be the proper transform of $D_{\mathfrak{r},d+1}$. In the present case,

$$[D_{\mathfrak{r},d}]^\perp = \langle H - 3E_{\mathfrak{r},1}, E_{\mathfrak{r},1} - E_{\mathfrak{r},2}, \dots, E_{\mathfrak{r},b-1} - E_{\mathfrak{r},b} \rangle_{\mathbb{Z}}. \quad (\text{Recall that } b = 9 - d.) \quad (2.5)$$

Here the pullback is omitted. We may choose the liftings $\gamma_{\mathfrak{r},1}, \dots, \gamma_{\mathfrak{r},b} \in H^2(X_{\mathfrak{r},d}, \mathbb{Z})$ in a such way that they are identified with the corresponding elements in the distinguished basis $\mathcal{B}_{\mathfrak{r},d+1}$ in the degree $d + 1$ model under the blow-up $Y_{\mathfrak{r},d} \rightarrow Y_{\mathfrak{r},d+1}$. Then $(Y_{\mathfrak{r},d}, D_{\mathfrak{r},d})$ and the basis $\{\alpha_{\mathfrak{r}}, \beta_{\mathfrak{r}}, \gamma_{\mathfrak{r},1}, \dots, \gamma_{\mathfrak{r},b}\}$ of $H_2(X_{\mathfrak{r},d}, \mathbb{Z})$ give our degree d model.

For $d = 8'$, we begin with \mathbf{P}^2 and a smooth elliptic curve $E \subset \mathbf{P}^2$. Pick $p_{\mathfrak{r}}, q_{\mathfrak{r}} \in E$ such that $L := \overline{p_{\mathfrak{r}}q_{\mathfrak{r}}}$ intersects E transversally and consider the blow-up

$$\text{Bl}_{\{p_{\mathfrak{r}}, q_{\mathfrak{r}}\}} \mathbf{P}^2 \xrightarrow{\pi} \mathbf{P}^2.$$

Denote by $E_{\mathfrak{r},p}$ and $E_{\mathfrak{r},q}$ the exceptional divisors over $p_{\mathfrak{r}}$ and $q_{\mathfrak{r}}$. The proper transform \bar{L} of L becomes a (-1) curve. We can contract \bar{L} and obtain a blow-down $\rho: \text{Bl}_{\{p_{\mathfrak{r}}, q_{\mathfrak{r}}\}} \mathbf{P}^2 \rightarrow Y$ to a smooth projective surface $Y_{\mathfrak{r}}$.

$$\begin{array}{ccc} \text{Bl}_{\{p_{\mathfrak{r}}, q_{\mathfrak{r}}\}} \mathbf{P}^2 & \xrightarrow{\pi} & \mathbf{P}^2 \\ & \downarrow \rho & \\ & & Y_{\mathfrak{r}} \end{array}$$

By surface classification, we have $Y_{\mathfrak{r}} \cong \mathbf{P}^1 \times \mathbf{P}^1$ and $\rho(E_{\mathfrak{r},p})$ and $\rho(E_{\mathfrak{r},q})$ are the rulings. Put $Y_{\mathfrak{r},8'} = Y_{\mathfrak{r}}$. The proper transform \bar{E} of E projects down to a smooth elliptic curve $D_{\mathfrak{r},8'}$ and we put $X_{\mathfrak{r},8'} = Y_{\mathfrak{r},8'} \setminus D_{\mathfrak{r},8'}$. Again we fix a symplectic basis $\{\alpha_{\mathfrak{r}}, \beta_{\mathfrak{r}}\}$ of $H_1(D_{\mathfrak{r},8'}, \mathbb{Z})$ and denote their images in $H_2(X_{\mathfrak{r},8'}, \mathbb{Z})$ by the same notation. In this case, $[D_{\mathfrak{r},8'}]^\perp$ is generated by the difference of the rulings and we shall again fix once for all a lifting $\gamma_{\mathfrak{r}}$ of $\rho_*([E_q] - [E_p])$ so that $\{\alpha_{\mathfrak{r}}, \beta_{\mathfrak{r}}, \gamma_{\mathfrak{r}}\}$ is our distinguished basis.

2.3. ALH* gravitational instantons. ALH* gravitational instantons intuitively are the gravitational instantons which are asymptotics to Calabi ansatz. We first explain the construction of Calabi ansatz. Let D be an elliptic curve and $p: L \rightarrow D$ be a positive line bundle of degree d . Let $Y_{\mathcal{C}}$ be the total space of L with projection $\pi_{\mathcal{C}}: Y_{\mathcal{C}} \rightarrow D$. Let $X_{\mathcal{C}}$ be the complement of the zero section in $Y_{\mathcal{C}}$. Let h be the unique hermitian metric on L whose curvature form is ω_D with the normalization $\int_D \omega_D = 2\pi d$. If z is the coordinate on D and ξ is a local trivialization of L , we get coordinates on L via $(z, w) \mapsto (z, w\xi)$. The Calabi ansatz is then given by

$$\omega_{\mathcal{C}} = \frac{2ic}{3} \partial \bar{\partial} (-\log |\xi|_h^2)^{\frac{3}{2}}, \quad \Omega_{\mathcal{C}} = c\pi_{\mathcal{C}}^* \Omega_D \wedge \frac{dw}{w},$$

where c is any positive real number and Ω_D is a holomorphic volume form such that

$$\frac{i}{2} \int_D \frac{\Omega_D}{2\pi i} \wedge \overline{\left(\frac{\Omega_D}{2\pi i} \right)} = 2\pi d.$$

It is straightforward to check that $(\omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ is a hyperKähler triple, i.e., $2\omega_{\mathcal{C}}^2 = \Omega_{\mathcal{C}} \wedge \bar{\Omega}_{\mathcal{C}}$.

Definition 2.2. Given $d \in \mathbb{N}, \tau \in \mathfrak{h}/\text{SL}(2, \mathbb{Z}), c > 0$. An ALH* gravitational instanton (of type (d, τ, c)) is a triple (X, ω, Ω) , where X is a non-compact complete hyperKähler 4-manifold with a Kähler form ω , and a holomorphic volume form Ω such that

- (1) $2\omega^2 = \Omega \wedge \bar{\Omega}$ and
- (2) there exists a compact set $K \subseteq X$, an $\varepsilon > 0$ and a diffeomorphism $F: X_{\mathcal{C}} \cong X \setminus K$ such that

$$\|F^* \omega - \omega_{\mathcal{C}}\|_{g_{\mathcal{C}}} = O(r^{-k-\varepsilon}), \quad \|F^* \Omega - \Omega_{\mathcal{C}}\|_{g_{\mathcal{C}}} = O(r^{-k-\varepsilon}),$$

where r is the distance to a fixed point in $X_{\mathcal{C}}$.

Remark 2.3. (1) From (2.1), let $\alpha, \beta \in \text{Im}(H^1(D, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z}))$ be the image of an oriented basis of $H^1(D, \mathbb{Z})$. Then

$$\{\Omega\} := \frac{\int_{\beta} \Omega}{\int_{\alpha} \Omega} = \tau \mod \text{SL}(2, \mathbb{Z})$$

is an invariant of the ALH^* gravitational instanton.

(2) Any ALH^* gravitational instanton can be compactified to a rational elliptic surface by adding an I_d -fibre at infinity [13, 25]. From the classification of singular fibres of rational elliptic surfaces [32], one has $1 \leq d \leq 9$. We will use ALH_d^* gravitational instanton to indicate the diffeomorphism type of an ALH^* gravitational instanton, with $1 \leq d \leq 9$ or $d = 8'$. See the discussion after [12, Proposition 5.4].

It is proven that any ALH^* gravitational instanton can be compactified to a weak del Pezzo surface [14, 25] by adding a smooth anti-canonical divisor at infinity with modulus τ . It is natural to introduce the markings for ALH^* gravitational instantons.

From now on, to ease the notation, we will omit d in the subscript most of the time and only specify it when it plays a role in our discussion.

Definition 2.4. Fix a reference ALH^* gravitational instanton of type (d, τ, c) with $d \in \{1, \dots, 9\}$, $\tau \in \mathfrak{h}$, and $c > 0$ and an ambient space $X_{\mathfrak{r}}$. A quadruple (X, ω, Ω, μ) is called a *marked ALH^* gravitational instanton* of type (d, τ, c) if it satisfies

- (1) (X, ω, Ω) is an ALH^* gravitational instanton of type (d, τ, c) ;
- (2) $\mu: X_{\mathfrak{r}} \rightarrow X$ is a diffeomorphism from the complement $X_{\mathfrak{r}} := Y_{\mathfrak{r}} \setminus D_{\mathfrak{r}}$ of our marked log Calabi–Yau pair $(Y_{\mathfrak{r}}, D_{\mathfrak{r}})$.

Two marked ALH^* gravitational instantons $(X_i, \omega_i, \Omega_i, \mu_i)$ are isomorphic if there exists a diffeomorphism $f: X_2 \rightarrow X_1$ such that $f^* \omega_1 = \omega_2$, $f^* \Omega_1 = \Omega_2$ and $\mu_1^* = \mu_2^* \circ f^*$. Denote $\text{mALH}^*(d, \tau, c)$ be the set of marked ALH^* gravitational instantons of type (d, τ, c) .

Now we fixed a reference ALH^* gravitational instanton $(X_{\mathfrak{r}}, \omega_{\mathfrak{r}}, \Omega_{\mathfrak{r}})$ for (d, τ, c) as above. We define the period domain of ALH^* gravitational instanton $\mathcal{P}\Omega(d, \tau, c)$ to be the subset of $H^2(X_{\mathfrak{r}}, \mathbb{R}) \times H^2(X_{\mathfrak{r}}, \mathbb{C})$ consisting of pairs $([\omega], [\Omega])$ such that

- (1) if $[C] \in H_2(X_{\mathfrak{r}}, \mathbb{Z})$ and $[C]^2 = -2$, then $[[\omega] \cdot [C]]^2 + [[\Omega] \cdot [C]]^2 \neq 0$.
- (2) $[\omega]$ vanishes on $\text{Im}(H^1(D_{\mathfrak{r}}, \mathbb{Z}) \rightarrow H_2(X_{\mathfrak{r}}, \mathbb{Z}))$.
- (3) $\{\Omega\} = \tau \pmod{\text{SL}(2, \mathbb{Z})}$.

The period map for ALH^* gravitational instantons is then defined to be

$$\begin{aligned} \mathcal{P}(d, \tau, c): \text{mALH}^*(d, \tau, c) &\rightarrow \mathcal{P}\Omega(d, \tau, c) \\ (X, \omega, \Omega, \mu) &\mapsto (\mu^*[\omega], \mu^*[\Omega]). \end{aligned}$$

The goal of this section is to prove the following theorem.

Theorem 2.5. *For each (d, τ, c) with $d \in \{1, \dots, 9\}$, $\tau \in \mathfrak{h}$, and $c > 0$ as above, the period map $\mathcal{P}(d, \tau, c)$ is surjective.*

2.4. Period domains for holomorphic 2-forms. Adopting the construction of references log Calabi–Yau pairs in §2.1, we can achieve the following theorem regarding the surjectivity of the period map.

Theorem 2.6 (Surjectivity of the periods of the $(2, 0)$ -forms). *Given complex numbers d_1, d_2 satisfying $d_1/d_2 \in \mathfrak{h}$ (in particular, d_1 and d_2 are non-zero) and $c_i \in \mathbb{C}$, $1 \leq i \leq b = 9 - d$, let*

$$[\Omega'] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) + \sum_{i=1}^b c_i \text{PD}(\gamma_{\mathfrak{r},i}) \in H^2(X_{\mathfrak{r}}, \mathbb{C}). \quad (2.6)$$

There exists a marked log Calabi–Yau pair (Y, D) and a diffeomorphism $\mu: X_{\mathfrak{r}} \rightarrow X$ with $X = Y \setminus D$ such that

$$\mu^*[\Omega] = [\Omega'], \quad (2.7)$$

where Ω is a holomorphic 2-form on X , that is, any cohomology class in $H^2(X_{\mathfrak{r}}, \mathbb{C})$ satisfying the condition (2.6) can be realized as a cohomology class of a holomorphic 2-form on some log Calabi–Yau pair.

Proof. We will construct a marked log Calabi–Yau pair (Y, D) such that the complement $X := Y \setminus D$ supports a holomorphic 2-form realizing the class $[\Omega']$.

(a) We deal with the case $d = 9$. In this case, we have $b = 0$ and

$$[\Omega'] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}).$$

We will construct X as a complement of an elliptic curve in \mathbf{P}^2 .

Put $\tau = d_1/d_2 \in \mathfrak{h}$ and let $X := \mathbf{P}^2 \setminus D$ where D is an elliptic curve with modulus τ ;

$$\mathbb{C}/\Lambda_{\tau} \cong D \subset \mathbf{P}^2, \quad \Lambda_{\tau} = \mathbb{Z} \oplus \mathbb{Z}\tau.$$

Let Ω be a meromorphic 2-form on \mathbf{P}^2 with a simple pole along D . Notice that Ω is unique up to a constant. By the residue formula, we have

$$\int_{\delta(\alpha)} \Omega = \int_{\alpha} \text{Res } \Omega, \quad \text{and} \quad \int_{\delta(\beta)} \Omega = \int_{\beta} \text{Res } \Omega$$

where $\{\alpha, \beta\}$ is a symplectic basis of $H_1(D, \mathbb{Z})$ and δ is the connecting homomorphism in (2.4). Rescaling Ω if necessary, we may assume

$$\int_{\alpha} \text{Res } \Omega = \int_{\delta(\alpha)} \Omega = 1.$$

Then

$$\int_{\beta} \text{Res } \Omega = \int_{\delta(\beta)} \Omega \equiv \tau \pmod{\text{SL}(2, \mathbb{Z})}. \quad (2.8)$$

We can lift the congruence in (2.8) to an equality in \mathfrak{h} . Indeed, we can find a path Γ in $H^0(\mathbf{P}^2, \mathcal{O}(3))_{\text{sm}}$ (the space of smooth sections) such that α (resp. β) is the parallel transport of $\alpha_{\mathfrak{r}}$ (resp. $\beta_{\mathfrak{r}}$) along Γ since the monodromies for the family of elliptic curves in \mathbf{P}^2 generate $\text{SL}(2, \mathbb{Z})$. Consequently, Γ gives rise to a marking $\mu: X_{\mathfrak{r}} \rightarrow X$ satisfying

$$\mu^*(\text{PD}(\delta(\alpha))) = \text{PD}(\delta(\alpha_{\mathfrak{r}})) \quad \text{and} \quad \mu^*(\text{PD}(\delta(\beta))) = \text{PD}(\delta(\beta_{\mathfrak{r}})).$$

To ease the notation, we will drop $\delta(-)$ and simply write $\alpha \in H_2(X, \mathbb{Z})$ instead of $\delta(\alpha)$. Adapting our convention, the equalities are transformed into

$$\mu^*(\text{PD}(\alpha)) = \text{PD}(\alpha_{\mathfrak{r}}) \quad \text{and} \quad \mu^*(\text{PD}(\beta)) = \text{PD}(\beta_{\mathfrak{r}})$$

when the context is clear. Then the marked log Calabi–Yau pair (X, D) together with the basis $\{\alpha, \beta\} \subset H_2(X, \mathbb{Z})$ and μ is what we want. Indeed, because

$$d_2 = d_2 \int_{\alpha} \Omega = d_2 \int_X \Omega \wedge \text{PD}(\alpha) \quad \text{and} \quad d_1 = d_2 \int_{\beta} \Omega = d_2 \int_X \Omega \wedge \text{PD}(\beta),$$

we have

$$\mu^*[d_2 \Omega] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) = [\Omega']. \quad (2.9)$$

(b) We now deal with the case $d = 8$. Let

$$[\Omega'] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) + c_1 \text{PD}(\gamma_{\mathfrak{m},1}) \in H^2(X_{\mathfrak{r}}, \mathbb{C}) \quad (2.10)$$

with $d_1/d_2 \in \mathfrak{h}$ and $c_1 \in \mathbb{C}$. By the argument in (a), we can find a smooth cubic $E \subset \mathbf{P}^2$, a symplectic basis $\{\alpha, \beta\} \subset H_2(V, \mathbb{Z})$ with $V = \mathbf{P}^2 \setminus E$, and a diffeomorphism $\nu: X_{\mathfrak{r},9} \rightarrow V$ along a curve Γ such that

$$\nu^*[\Omega] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}). \quad (2.11)$$

Here Ω is a holomorphic 2-form on V with the normalization

$$\int_{\alpha} \text{Res } \Omega = d_2, \quad (2.12)$$

and the cycles $\alpha_{\mathfrak{r}}$ and $\beta_{\mathfrak{r}}$ in (2.10) are regarded as cycles in $H_2(X_{\mathfrak{r},9}, \mathbb{Z})$.

Let $p_1 \in E$ be the parallel transport of $q_{\mathfrak{r},1} \in D_{\mathfrak{r},9}$ along Γ and consider the blow-up $\pi': Y' = \text{Bl}_{p_1} \mathbf{P}^2 \rightarrow \mathbf{P}^2$. We have

$$(\pi')^* \Omega_{\mathbf{P}^2}(E) \cong \Omega_{Y'}(D')$$

where D' is the proper transform of E . The path Γ determines a marking $\nu: X_{\mathfrak{r},8} \rightarrow X'$ for $X' := Y' \setminus D'$ such that $\nu^*(\text{PD}(\alpha)) = \text{PD}(\alpha_{\mathfrak{r}})$ and $\nu^*(\text{PD}(\beta)) = \text{PD}(\beta_{\mathfrak{r}})$. Now let $\gamma_1 \in H_2(X', \mathbb{Z})$ such that $\text{PD}(\gamma_1') = (\nu^*)^{-1}(\text{PD}(\gamma_{\mathfrak{r},1}))$. We get by integration

$$\nu^*[\Omega] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) + c'_1 \text{PD}(\gamma_{\mathfrak{r},1}), \quad c'_1 = \int_{\gamma_1'} (\pi')^* \Omega \in \mathbb{C}. \quad (2.13)$$

Let $p \in E$ and Γ' be a smooth curve joining p and p_1 . Then Γ' gives rise to a diffeomorphism $\rho: X' \rightarrow X'$ which takes p_1 to p . Consider the blow-up $\pi: Y = \text{Bl}_p \mathbf{P}^2 \rightarrow \mathbf{P}^2$. The curve Γ' also gives rise to a diffeomorphism $\rho: X' \rightarrow X = Y \setminus D$ where D is the proper transform of E under π . Let $\gamma_1 = \rho_*(\gamma_1')$. We can achieve the coefficient c_1 in (2.10) by moving p around in E . Indeed, by Lemma 2.7 below, we have

$$\int_{\gamma_1} \Omega \equiv -3 \int_O^p \text{Res } \Omega \pmod{d_2 \Lambda_{\tau}}$$

where O is a flex point on $E \cong D'$ served as the additive identity element and $\Lambda_{\tau} = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\tau = d_1/d_2$. Now we choose $p \in E$ such that

$$c_1 \equiv -3 \int_O^p \text{Res } \Omega \pmod{d_2 \Lambda_{\tau}}.$$

We can lift the congruence to an equality by adding a loop in E passing through p and deforming ρ accordingly. Then the pair (Y, D) , the holomorphic top form $\pi^* \Omega$ and the diffeomorphism $\mu = \rho \circ \nu: X_{\mathfrak{r},8} \rightarrow X$ are what we are looking for, i.e.,

$$\mu^* [d_2 \pi^* \Omega] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) + c_1 \text{PD}(\gamma_{\mathfrak{m},1}) = [\Omega'].$$

This proves the theorem when $d = 8$

(c) Now let us deal with the case $d = 7$. Let

$$[\Omega'] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) + c_1 \text{PD}(\gamma_{\mathfrak{r},1}) + c_2 \text{PD}(\gamma_{\mathfrak{r},2}) \in H^2(X_{\mathfrak{r},7}, \mathbb{C}) \quad (2.14)$$

with $d_1/d_2 \in \mathfrak{h}$ and $c_1, c_2 \in \mathbb{C}$. By our discussion in (b), we can find a marked log Calabi–Yau pair (Y', D') , a symplectic basis $\{\alpha, \beta\}$ of $H_1(D', \mathbb{Z})$, and a diffeomorphism $\nu: X_{\mathfrak{r},8} \rightarrow X'$ with

$$\nu^*[\Omega] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) + c_1 \text{PD}(\gamma_{\mathfrak{r},1}). \quad (2.15)$$

Here $X' = Y' \setminus D'$. We remark that (Y', D') is constructed from a blow-up of \mathbf{P}^2 and D' is a proper transform of an elliptic curve in \mathbf{P}^2 . Let $p_2 \in D'$ be the image of $q_{\mathfrak{r},2} \in D_{\mathfrak{r},8}$ (see construction in §2.2) under ν . Consider the blow-up $\pi_2: \text{Bl}_{p_2} Y' =: \bar{Y}' \rightarrow Y'$ and denote by \bar{D}' the proper transform of D' . Let E_2 (resp. E_1) be the exceptional divisor of $\bar{Y}' \rightarrow Y'$ (resp. the pullback of the exceptional divisor of $Y' \rightarrow \mathbf{P}^2$). We obtain a diffeomorphism $\mu: X_{\mathfrak{r},7} \rightarrow \bar{X}' = \bar{Y}' \setminus \bar{D}'$. Define a homology class $\bar{\gamma}'_2 \in H_2(\bar{X}', \mathbb{Z})$ via

$$\mu^*(\text{PD}(\bar{\gamma}'_2)) = \text{PD}(\gamma_{\mathfrak{r},2}). \quad (2.16)$$

Let $p \in D'$ and Γ' be a curve in D' connecting p and p_2 . Similar to the case (b), the curve Γ' gives rise to a diffeomorphism $\rho: \bar{X}' \rightarrow X = \text{Bl}_p Y' \setminus D$ where D is the proper transform of D' and put $\gamma_2 = \rho_*(\bar{\gamma}'_2)$. Then it follows that

$$\int_{\gamma_2} \pi^* \Omega \equiv \int_{p_1}^p \text{Res} \Omega \pmod{d_2 \Lambda_{\tau}}.$$

Here we recall that $p_1 = \nu(q_{\mathfrak{r},1})$. We can achieve c_2 in (2.14) by moving p around, i.e., we can find an appropriate curve Γ' in D connecting p_1 and p such that

$$c_2 = \int_{\Gamma'} \text{Res} \Omega = \int_{\gamma_2} \Omega.$$

This completes the proof when $d = 7$. The remaining cases $1 \leq d \leq 6$ can be done by the same procedure inductively.

(d) Let us deal with the last case $d = 8'$. Let

$$[\Omega'] = d_1 \text{PD}(\alpha_{\mathfrak{r}}) + d_2 \text{PD}(\beta_{\mathfrak{r}}) + c \text{PD}(\gamma_{\mathfrak{r},1}) \in H^2(X_{\mathfrak{r},8'}, \mathbb{C})$$

with $\tau := d_1/d_2 \in \mathfrak{h}$ and $c \in \mathbb{C}$. Similar to the case $d = 9$, let $E \subset \mathbf{P}^2$ be an elliptic curve with modulus τ . Denote by $\{\alpha, \beta\}$ a symplectic basis of $H_1(E, \mathbb{Z})$. Let Ω be a meromorphic 2-form on \mathbf{P}^2 having a simple pole along E with the normalization

$$\int_{\alpha} \text{Res} \Omega = 1.$$

Then we have

$$\int_{\beta} \text{Res} \Omega \equiv \tau \pmod{\text{SL}(2, \mathbb{Z})}.$$

Now we can pick $p, q \in E$ and a curve Γ connecting them such that

$$\int_{\Gamma} \text{Res} \Omega = c/d_2. \quad (2.17)$$

Consider the blow-up $\pi: \text{Bl}_{\{p,q\}} \mathbf{P}^2 \rightarrow \mathbf{P}^2$. Let L be the line passing through p and q . Denote by E_p (resp. E_q) the exceptional divisor over p (resp. q) and by \bar{L} (resp. \bar{E}) the proper transform of L (resp. E). If it happens $p = q$ (i.e., $c \equiv 0 \pmod{\Lambda_{\tau}}$), we shall take L to be the tangent of E at p and consider the blow-up at infinitely near points p and the intersection of the proper transform of L and E . In any case, we have $\pi^* L = \bar{L} + E_p + E_q$ and \bar{L} becomes a (-1) curve. Let $\rho: \text{Bl}_{\{p,q\}} \mathbf{P}^2 \rightarrow Y$ be the blow-down of \bar{L} . When $p \neq q$, we have $Y \cong \mathbf{P}^1 \times \mathbf{P}^1$

($\ell_1 = \rho(E_p)$ and $\ell_2 = \rho(E_q)$ give the rulings). When $p = q$, we have $Y \cong \mathbb{F}_2$ and $\rho_*([E_q] - [E_p])$ represents the homology class of the unique (-2) curve. Put $D = \rho(\bar{E})$. Then $E \cong \bar{E} \cong D$. In any case, $\gamma := \rho_*([E_q] - [E_p])$ gives a homology class in the complement $X := Y \setminus D$. We can prove that

$$\int_{\gamma} \tilde{\Omega} = \frac{c}{d_2} \pmod{\Lambda_{\tau}}. \quad (2.18)$$

Here $\tilde{\Omega}$ is the unique meromorphic 2-form on Y having a simple pole along D such that $\rho^* \tilde{\Omega} = \pi^* \Omega$. As in the case (a), by choosing a path in $H^0(\mathbf{P}^2, \mathcal{O}(3))$ appropriately and deforming p, q on E suitably, we can find a diffeomorphism $\mu: X_{\tau, 8'} \rightarrow X$ such that

$$\mu^*(\text{PD}(\alpha)) = \text{PD}(\alpha_{\tau}), \quad \mu^*(\text{PD}(\beta)) = \text{PD}(\beta_{\tau}), \quad \text{and } \mu^*(\text{PD}(\gamma)) = \text{PD}(\gamma_{\tau}).$$

Then $\Omega' = d_2 \tilde{\Omega}$ is what we need. \square

Lemma 2.7. *Adapt the notation in the proof of Theorem 2.6. We have*

$$\int_{\gamma_1} \Omega \equiv -3 \int_O^p \text{Res} \Omega \pmod{d_2 \Lambda_{\tau}}$$

where O is a flex point served as the additive identity element on \bar{D} and $\Lambda_{\tau} = \mathbb{Z} \oplus \mathbb{Z} \tau$ with $\tau = d_2/d_1$. From the expression, it is independent of the choice of the flex point.

Proof. Choose a hyperplane H in \mathbf{P}^2 passing through p and intersecting D at three distinct points, say $D \cap H = \{p, s, t\}$, and transversally at p . Recall that $[\bar{D}]^{\perp} = \langle H - 3E_1 \rangle$, where E_1 is the exceptional divisor of π_1 .

Choose a smooth curve σ_1 (resp. σ_2) from p and s (resp. p and t). We may assume that the relative interior of σ_i are disjoint. By the construction in [18], we can lift the cycle $[H - 3E_1]$ to a cycle δ in X by gluing S^1 -bundles over σ_1 and σ_2 . Again the lifting is not unique; any two liftings differ by an element in $H^1(D, \mathbb{Z})$. Therefore,

$$\int_{\gamma_1} \Omega \equiv \int_{\delta} \Omega \pmod{d_2 \Lambda_{\tau}}.$$

Now,

$$\begin{aligned} \int_{\delta} \Omega &= \int_{\sigma_1} \text{Res} \Omega + \int_{\sigma_2} \text{Res} \Omega \\ &= \int_p^t \text{Res} \Omega + \int_p^s \text{Res} \Omega \\ &= \int_O^p \text{Res} \Omega + \int_O^t \text{Res} \Omega + \int_O^s \text{Res} \Omega - 3 \int_O^p \text{Res} \Omega \\ &\equiv -3 \int_O^p \text{Res} \Omega \pmod{d_2 \Lambda_{\tau}}. \end{aligned}$$

The last equation holds since p, t , and s are collinear. \square

2.5. Monodromy of the moduli of pairs. For a smooth projective surface S , denote by $\text{Hilb}^b(S)$ the Hilbert scheme of length b subscheme on S ; it is a smooth algebraic variety of dimension $2b$ equipped with a universal family $\mathcal{U} \rightarrow \text{Hilb}^b(S)$. There exists also a birational

morphism (a.k.a. Hilbert–Chow morphism)

$$\mathrm{Hilb}^b(S) \rightarrow \mathrm{Sym}^b(S), \quad p \mapsto \sum_{x \in S} \mathrm{mult}_x(p) \cdot x,$$

where the right hand side is understood as a formal sum. Note that $p \in \mathrm{Sym}^b(S)$ represents a set of unlabelled b points on S and labeling them is equivalent to choosing a preimage under the canonical surjection $S^b \rightarrow \mathrm{Sym}^b(S)$. Moreover, fixing a labeling and deforming p around gives rise to a well-defined section of $S^b \rightarrow \mathrm{Sym}^b(S)$ as long as $|\mathrm{Supp}(p)| = b$ remains constant in the deformation.

Regard \mathcal{U} as a subscheme in $S \times \mathrm{Hilb}^b(S)$ and let \mathcal{Y} be the blow-up of $S \times \mathrm{Hilb}^b(S)$ along \mathcal{U} . Then the general fiber of the family $\mathcal{Y} \rightarrow \mathrm{Hilb}^b(S)$ is the blow-up of S along distinct b points. We consider a codimension two closed subscheme

$$T := \{p \in \mathrm{Hilb}^b(S) \mid |\mathrm{Supp}(p)| \leq b-1\}.$$

We can choose a curve C in $\mathrm{Hilb}^b(S)$ such that

- C meets T transversely and smooth at p ;
- C maps isomorphically onto its image under the Hilbert–Chow morphism;
- any $q \in C \setminus \{p\}$ near p represents a set of points in almost general position.

Let $\mathcal{U} \rightarrow C$ be the pullback of the universal family $\mathcal{Y} \rightarrow \mathrm{Hilb}^b(S)$. We may regard \mathcal{U} as a (reducible) subscheme of $S \times C$. Let \mathcal{Z} be the blow-up of $S \times C$ along \mathcal{U} and $\mathcal{Z} \rightarrow C$ be the associated family. Note that \mathcal{Z} is not smooth; it acquires an ordinary double point singularity over $p \in C$.

Remark 2.8. One can construct the local model in the following way. Consider \mathbb{C}^3 with coordinate (x, y, t) . The ideals $\langle y, x - t \rangle$ and $\langle y, x + t \rangle$ give two lines in \mathbb{C}^3 whose union is defined by $\langle y, x^2 - t^2 \rangle$.

Denote by X the blow-up of \mathbb{C}^3 along the ideal $\langle y, x^2 - t^2 \rangle$;

$$X = \mathrm{Proj} \mathbb{C}[x, y, t][\xi, \eta]/\langle \xi y - \eta(x^2 - t^2) \rangle$$

where Proj is taken with respect to the \mathbb{Z} -grading on ξ, η with $\deg(\xi) = \deg(\eta) = 1$. On the affine chart $\xi \neq 0$, X is isomorphic to

$$\mathrm{Spec} \mathbb{C}[x, y, t, \eta']/\langle y - \eta'(x^2 - t^2) \rangle, \quad \eta' = \eta/\xi,$$

which is smooth, while on the affine chart $\eta \neq 0$, X is isomorphic to

$$\mathrm{Spec} \mathbb{C}[x, y, t, \xi']/\langle \xi'y - (x^2 - t^2) \rangle, \quad \xi' = \xi/\eta,$$

which is singular and has a ODP singularity. Introduce an \mathbb{Z}_2 -action on t via

$$\mu \cdot t := t^2, \quad \text{where } \mu \text{ is the generator of } \mathbb{Z}_2,$$

and denote by $s = t^2$ the \mathbb{Z}_2 -invariant coordinate. Then the quotient defines local model of a smoothing of an ordinary double point (at the origin on the affine chart $\eta \neq 0$)

$$\mathrm{Spec} \mathbb{C}[x, y, s, \xi']/\langle \xi'y - (x^2 - s) \rangle \rightarrow \mathrm{Spec} \mathbb{C}[s].$$

This is the only affine chart of the local model of $\mathcal{U} \rightarrow C$ containing the singularity of the singular fiber with p identifying with $s = 0$.

We can compute the monodromy of $\mathcal{Z} \rightarrow C$ around p by Picard–Lefschetz formula. Pick a smooth reference fibre \mathcal{Z}_q of $\mathcal{Z} \rightarrow C$ and denote by E_1, \dots, E_b the exceptional divisors in the blow-up $\mathcal{Z}_q \rightarrow S$ such that x_1 and x_2 (the images of E_1 and E_2 on S) collides when $q \mapsto p$. Then we have

$$H^2(\mathcal{Z}_q, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\langle E_1, \dots, E_b \rangle.$$

As before, the pullback is omitted. Note that $E_1 - E_2$ is a generator of the vanishing cohomology. Then Picard–Lefschetz formula says that the monodromy transformation $\varpi: H^2(\mathcal{Z}_q, \mathbb{Z}) \rightarrow H^2(\mathcal{Z}_q, \mathbb{Z})$ is given by

$$\varpi(\gamma) = \gamma + \langle \gamma, E_1 - E_2 \rangle (E_1 - E_2), \quad \gamma \in H^2(\mathcal{Z}_q, \mathbb{Z}).$$

This is on the nose the reflection on $H^2(\mathcal{Z}_q, \mathbb{Z})$ generated by the root $E_1 - E_2$.

Recall that rational elliptic surfaces are rational surfaces with an elliptic fibration structure admitting a section. We will need the following proposition.

Proposition 2.9. *Let (Y, D) be either a pair of a weak del Pezzo surface and a smooth anti-canonical divisor or a rational elliptic surface and an anti-canonical divisor with configuration II, III, IV, IV*, III*, II*, or I_k^* with $k = 0, \dots, 4$. Let C be a smooth holomorphic curve in $X := Y \setminus D$ with $[C]^2 = -2$. Then the root reflection associated with $[C]$ on $H^2(X, \mathbb{Z})$ can be realized as a monodromy transformation of some deformation of (Y, D) .*

Proof. Let Y be a weak del Pezzo surface of degree d and $D \in |-K_Y|$ be smooth; (Y, D) is a blowup of \mathbf{P}^2 along $b = 9 - d$ points on a smooth cubic in \mathbf{P}^2 . Denote by E_1, \dots, E_b the pullback of the exceptional divisors, x_1, \dots, x_b the corresponding points on \mathbf{P}^2 and by H the pullback of the hyperplane class on \mathbf{P}^2 . By [29, Lemma 2.8], C is given by

- (1) the proper transform of E_i over which there exists exactly one E_j lying over;
- (2) the proper transform of a line in \mathbf{P}^2 passing through exactly three points in $\{x_1, \dots, x_b\}$;
- (3) the proper transform of a conic in \mathbf{P}^2 passing through exactly six points in $\{x_1, \dots, x_b\}$;
- (4) the proper transform of a cubic in \mathbf{P}^2 passing through exactly eight points in $\{x_1, \dots, x_b\}$ such that one of them is the singular point of the cubic.

Case (1) occurs when $b \geq 2$, Case (2) occurs when $b \geq 3$, Case (3) occurs when $b \geq 6$ and Case (4) appears only when $b = 8$.

The root reflection from (1) can be realized by collapsing x_i and x_j . For (2), we begin with \mathbf{P}^2 and pick a line H joining x_i and x_j . Consider $F := H - E_i - E_j$ (the proper transform of H in Y). Then F is a (-1) curve on the smooth surface Y . By Castelnuovo's theorem, we can contract F to $x \in Y'$ for a smooth surface Y' . Since $x_k \notin H$, x_k is mapped to a point $x'_k \in Y'$. Note that the class $H - E_i - E_j - E_k$ is equal to $F - E_k$ in $H^2(Y, \mathbb{Z})$. Regarding E'_k as the exceptional divisor over x'_k , we see that the associated root reflection can be realized as the monodromy transformation of the degeneration by collapsing x'_k and x . The remaining cases can be treated in a similar way.

Let Y be a rational elliptic surface and D an anti-canonical divisor with configuration described in the Proposition. It is known that Y is a blow-up of the base locus of a pencil of cubics on \mathbf{P}^2 with a smooth member. Let $\pi: Y \rightarrow \mathbf{P}^1$ be the associated elliptic fibration. We also assume that D is the fiber at $\infty \in \mathbf{P}^1$. Let C be as in the proposition. Then C must be an irreducible component of a fiber of π . There exists a sequence of blow-downs (-1) curves

$$Y = Z_0 \rightarrow Z_1 \rightarrow \dots \rightarrow Z_k =: Z$$

satisfying the following properties

- the image of C in Z_{k-2} (still denoted by C) remains a (-2) curve;
- $Z_{k-2} \rightarrow Z_{k-1}$ is a contraction of a (-1) curve F with $F \cap C \neq \emptyset$, and therefore C becomes a (-1) curve in Z_{k-1} ;
- $Z_{k-1} \rightarrow Z$ is given by contracting C .

Moreover, by Castelnuovo's theorem, Z_i is a smooth projective variety for each $i = 0, \dots, k$. (Indeed, one can begin with contracting a section of π . Since any section must meet the fiber containing C and every fiber is connected, one can continue the process to reach C .) Using $\text{Hilb}^2(Z)$ the Hilbert scheme of length 2 subscheme on Z , from the discussion right before Proposition 2.9, we can find a suitable degeneration whose monodromy transformation equals the root reflection constructed from $[C]$. This completes the proof. \square

2.6. Surjectivity of period maps of ALH^* gravitational instantons. Any holomorphic curve in X is a (-2) -curve in Y by adjunction formula. We first recall a theorem of Tian–Yau [34]. We say a cohomology class $[\omega] \in H^2(X, \mathbb{R})$ satisfies the condition (\dagger) if $[\omega]$ is positive on every (-2) -curve of Y contained in X and there exists a Kähler class $[\omega_Y]$ on Y such that $[\omega] = [\omega_Y]|_X$.

Theorem 2.10. *Given $c > 0$ and $[\omega] \in H^2(X, \mathbb{R})$ satisfies the condition (\dagger) , then there exists a Ricci-flat metric ω in the given cohomology class on X with $2\omega^2 = \Omega \wedge \bar{\Omega}$, where Ω is a meromorphic volume form on Y with a simple pole along D such that $\text{Res}_D \Omega = c\Omega_D$.*

Proof of Theorem 2.5. From Theorem 2.6, there exists a del Pezzo surface Y of degree d and a smooth anti-canonical divisor D with modulus τ and a meromorphic volume form Ω on Y with a simple pole along D and $\text{Res}_D \Omega = c\Omega_D$ such that there exists a diffeomorphism $\mu: X_0 \rightarrow X$ with $\mu^*[\Omega] = [\Omega_0]$, where $X = Y \setminus D$.

If $(\mu^{-1})^*[\omega_0]$ satisfies the condition (\dagger) , then the theorem follows from Theorem 2.10 directly. Otherwise, from Proposition 2.9 and [16, Theorem 2.1] there exists a diffeomorphism $g: X \rightarrow X$ such that $g^*(\mu^{-1})^*[\omega_0]$ satisfies the condition (\dagger) . Thus, there exists a Ricci-flat metric ω in the cohomology class $g^*(\mu^{-1})^*[\omega_0]$ by Theorem 2.10. Notice that the diffeomorphism is induced by the compositions of monodromies in the moduli space of pairs (Y, D) . In particular, the Picard–Lefchetz formula implies that $g^*[\Omega] = [\Omega]$ since $[\Omega]$ vanishes on every (-2) -curve in X . Then $(X, \omega, \Omega, g^{-1} \circ \mu)$ is the marked ALH_d^* gravitational instantons realizing the given cohomology classes and this finishes the proof of the theorem. \square

3. PERIOD DOMAINS FOR ALG AND ALG^* GRAVITATIONAL INSTANTONS

Recall that rational elliptic surfaces are rational surfaces with an elliptic fibration structure². It is well known that any rational elliptic surface is a blow up of the base points of a pencil of cubics with a smooth member in \mathbf{P}^2 , i.e., given a rational elliptic surface Y and a fibre D , the pair (Y, D) can be derived from blow-ups of \mathbf{P}^2 on a possibly singular cubic \underline{D} and D contains the proper transform of \underline{D} .

²Here we use the definition that an elliptic fibration admits a section.

Definition 3.1. An *ALG pair* is a log Calabi–Yau pair (Y, D) with Y a smooth rational elliptic surface and $D \in |-K_Y|$ a divisor of type II, III, IV, IV*, III*, II*, or I₀*. An *ALG pair of type Z* is an ALG pair (Y, D) such that D is of type Z. A *marked ALG pair* is an ALG pair (Y, D) together with a basis \mathcal{B} of $H_2(X, \mathbb{Z})$. Finally, a *marked ALG pair of type Z* is a marked ALG pair of type Z with a basis \mathcal{B} of $H_2(X, \mathbb{Z})$.

Similarly, we can define the notions for ALG^* pairs. In which case, the configurations of D can be I₁*, I₂*, I₃*, and I₄*,

3.1. Constructions of (Y, D) for ALG and ALG^* gravitational instantons. In this subsection, we will give constructions of families of marked ALG and ALG^* pairs of various types and study their period maps.

We begin with a general discussion. Let (Y, D) be either an ALG pair or an ALG^* pair and $X = Y \setminus D$ be the complement. In any case, we have $H^1(D, \mathbb{C}) = 0$. Let $i: D \rightarrow Y$ be the closed embedding and $j: X \rightarrow Y$ be the open embedding. We have the short exact sequence

$$0 \rightarrow j_! j^{-1} \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow i_* i^{-1} \mathbb{Q} \rightarrow 0. \quad (3.1)$$

Taking compactly supported cohomology yields the long exact sequence

$$\cdots \rightarrow 0 = H_c^1(D, \mathbb{Q}) \rightarrow H_c^2(X, \mathbb{Q}) \rightarrow H_c^2(Y, \mathbb{Q}) \rightarrow H_c^2(D, \mathbb{Q}) \rightarrow \cdots. \quad (3.2)$$

It then follows that the homology group $H_2(X, \mathbb{Q}) \cong H_c^2(X, \mathbb{Q})$ can be identified with the kernel of the signed intersection map

$$\gamma \mapsto (\gamma \cdot D_i)_{i=1}^k \quad (3.3)$$

where $D = \sum_{i=1}^k m_i D_i$ and D_i 's are irreducible components. By the vanishing of $H^1(D, \mathbb{C})$, any $\gamma \in H_2(Y, \mathbb{C})$ satisfying $\gamma \cdot D_i = 0$ for all i can be lifted *uniquely* to $H_2(X, \mathbb{C})$.

To construct a basis \mathcal{B} , we can therefore pick any basis of

$$\{\gamma \in H_2(Y, \mathbb{Z}) \mid \gamma \cdot D_i = 0, i = 1, \dots, k\}$$

and lift it to $H_2(X, \mathbb{Z})$.

Let $\pi: (\mathcal{Y}, \mathcal{D}) \rightarrow \mathcal{M}$ be a deformation family of a marked ALG pair (Y, D) and $\mathfrak{r} \in \mathcal{M}$ be a reference pair. We denote the reference pair by $(Y_{\mathfrak{r}}, D_{\mathfrak{r}})$ and its complement by $X_{\mathfrak{r}} := Y_{\mathfrak{r}} \setminus D_{\mathfrak{r}}$. Let Ω be a section of $\pi_* \Omega_{\mathcal{Y}/\mathcal{M}}^2(\mathcal{D})$. Assuming \mathcal{M} is simply connected, we can define the period integrals of (Y, D) to be the function

$$\mathcal{M} \ni t \rightarrow \int_{(\varphi_{\Gamma})^* \gamma} \Omega_t \quad (3.4)$$

where $\varphi_{\Gamma}: X_{\mathfrak{r}} \rightarrow X$ is the diffeomorphism induced by a path Γ connecting t and \mathfrak{r} in \mathcal{M} , $\gamma \in H_2(X_{\mathfrak{r}}, \mathbb{Z})$, and Ω_t is the restriction of Ω to the fibre (Y_t, D_t) . This is well-defined since \mathcal{M} is simply connected. When $\pi_1(\mathcal{M})$ is non-trivial, the period integrals above form a local system on \mathcal{M} and in general have non-trivial monodromies.

To define period integrals in general cases, we must keep track of the trivialization, i.e., the trivialization from the curve Γ .

Definition 3.2. Let $\pi: (\mathcal{Y}, \mathcal{D}) \rightarrow \mathcal{M}$ be a deformation family of a marked ALG or ALG^* pair (Y, D) and $\mathfrak{r} \in \mathcal{M}$ be a reference pair. Let Ω be a section of $\pi_* \Omega_{\mathcal{Y}/\mathcal{M}}^2(\mathcal{D})$. For $t \in \mathcal{M}$,

the period integral is defined to be the multi-valued function

$$\mathcal{M} \ni t \mapsto \int_{(\varphi_\Gamma)_*\gamma} \Omega_t \quad (3.5)$$

where $\varphi_\Gamma: X_\tau \rightarrow X$ is the diffeomorphism induced by a path Γ connecting t and τ in \mathcal{M} , $\gamma \in H_2(X_\tau, \mathbb{Z})$, and Ω_t is the restriction of Ω to the fibre (Y_t, D_t) . Let $\mathcal{B}_\tau := \{\gamma_{\tau,1}, \dots, \gamma_{\tau,m}\}$ be a basis of $H_2(X_\tau, \mathbb{Z})$. The *period map* $\mathcal{P}_{\mathcal{B}_\tau}(\Omega)$ is a multi-vector-valued function

$$\mathcal{M} \ni t \mapsto \left(\int_{(\varphi_\Gamma)_*\gamma_{\tau,1}} \Omega_t, \dots, \int_{(\varphi_\Gamma)_*\gamma_{\tau,m}} \Omega_t \right) \in \mathbb{C}^m. \quad (3.6)$$

For simplicity, when the context is clear, we drop \mathcal{B}_τ and Ω in the notation.

Note that $\text{Im}(\mathcal{P})$ always lies in a hyperplane in \mathbb{C}^m determined by the fibre class. More precisely, let $\mathcal{B}_\tau = \{\gamma_{\tau,1}, \dots, \gamma_{\tau,m}\}$ and assume that

$$[f] = \sum_{i=1}^m a_i \gamma_{\tau,i}$$

where $[f]$ is the homology class of a fibre in the rational elliptic surface Y_τ . Then

$$\sum_{i=1}^m a_i \int_{\gamma_{\tau,i}} \Omega_t = 0.$$

That is, if (y_1, \dots, y_m) denotes the coordinates on \mathbb{C}^m , we have

$$\text{Im}(\mathcal{P}) \subset \left\{ (y_1, \dots, y_m) \mid \sum_{i=1}^m a_i y_i = 0 \right\}.$$

The main result in this subsection is the following theorem.

Theorem 3.3. *Let notation be as above. Let (Y, D) be an ALG or an ALG* pair. Then there exist a family $\pi: (\mathcal{Y}, \mathcal{D}) \rightarrow \mathcal{M}$ of deformation of (Y, D) and $\Omega \in \Omega_{\mathcal{Y}/\mathcal{M}}^2(\mathcal{D})$ such that*

$$\text{Im}(\mathcal{P}) = \left\{ (y_1, \dots, y_m) \mid \sum_{i=1}^m a_i y_i = 0 \right\}. \quad (3.7)$$

The rest of the subsection is devoted to proving Theorem 3.3. To achieve this, we will

- construct for each type a reference marked ALG or ALG* pair (Y_τ, D_τ) , i.e.,
 - a pencil of cubics in \mathbf{P}^2 giving the ALG or ALG* pair (Y_τ, D_τ) of the desired type after resolving the base locus;
 - a basis \mathcal{B}_τ of $H_2(X_\tau, \mathbb{Z})$ where $X_\tau := Y_\tau \setminus D_\tau$ is the complement;
 - a choice of a section $\Omega \in \pi_* \Omega_{\mathcal{Y}/\mathcal{M}}^2(\mathcal{D})$;
- analyze $\text{Im}(\mathcal{P})$ the image of the period map defined by the data in the first bullet and argue that the equality (3.7) holds.

To achieve these, one also needs to verify that the pencil constructed in various situations contains a smooth member. The following two classical results will be useful. First, we recall Bertini's theorem.

Theorem 3.4 (Bertini's Theorem [20, p. 137]). *Assume the ambient variety is smooth. Then general elements of a pencil are smooth away from its base locus.*

Assume that the pencil is spanned by C and D . Suppose that $C \cap D \in C_{\text{sm}} \cup D_{\text{sm}}$, that is, the intersection points $C \cap D$ are either a smooth point of C or a smooth point of D . Then the linear system $|uC + vD|$ contains a smooth member. Indeed, by Bertini's theorem, one could pick a general member A which is smooth outside the base locus. Since $C \cap D \in C_{\text{sm}} \cup D_{\text{sm}}$, we can perturb the defining equation of A to eliminate the singularities of A by adding the defining equation of C or D .

Second, we recall the adjunction formula and the residue formalism which will be crucial in our calculation of periods. Let Y be a smooth algebraic variety over \mathbb{C} and $D \in | -K_Y |$ be an anti-canonical divisor. We will be interested in the case when D is singular. More precisely, assume that $D = \sum_{i=1}^r m_i D_i \in | -K_Y |$ is anti-canonical such that each D_i is smooth but the intersections are allowed to be non-transversal (e.g. three lines meet at one point in \mathbf{P}^2).

Let $\Omega_Y^n(D)$ be the sheaf of meromorphic differentials on Y whose pole divisor is equal to D where $n = \dim_{\mathbb{C}} Y$. (This is indeed a trivial bundle owing to our assumption $D \in | -K_Y |$.) Then if $m_1 = 1$, from the adjunction formula, we have

Proposition 3.5.

$$\Omega_Y^n(D)|_{D_1} = \Omega_Y^n(D_1)|_{D_1} \otimes \mathcal{O}_Y(D - D_1)|_{D_1} \xrightarrow{\cong} \Omega_{D_1}^{n-1}((D - D_1)|_{D_1}). \quad (3.8)$$

This isomorphism is realized by the Poincaré residue.

We will construct the pencil for each singular configuration (cf. FIGURE 1 and FIGURE 2).

3.1.1. Type II. A type II fibre is a rational curve with a cusp singularity. In this case, \underline{D} is a cuspidal rational curve in \mathbf{P}^2 with the cusp at p and Y is the blow up of \mathbf{P}^2 at nine points on $\underline{D} \setminus \{p\}$. It is well-known that the cuspidal rational curve in \mathbf{P}^2 is unique up to $\text{PGL}(3)$ -action [19, p. 55]. We may assume that \underline{D} is given by the equation $\{y^2z - x^3 = 0\}$. Recall that the Cayley–Bacharach theorem states that any distinct eight points in \mathbf{P}^2 without four on a line or seven points on a non-degenerate conic would determine uniquely a pencil of cubics; in other words, any such eight points in $\underline{D} \setminus \{p\}$ determines a pencil of cubics. Since the other members in the pencil avoid p , the pencil must contain at least one smooth member and thus determines a rational elliptic surface. We also remark that $\underline{D} \setminus \{p\}$ is an affine group variety which is isomorphic to \mathbb{C} (the additive group).

To construct a marked reference ALG pair of type II, we simply pick a smooth cubic C which meets $\underline{D} \setminus \{p\}$ at nine points. Denote by $p_{\mathfrak{r},1}, \dots, p_{\mathfrak{r},9}$ the intersections $\underline{D} \cap C$. Let $Y_{\mathfrak{r}}$ be the blow-ups of \mathbf{P}^2 at those nine points and $D_{\mathfrak{r}}$ be the proper transform of \underline{D} . Then $(Y_{\mathfrak{r}}, D_{\mathfrak{r}})$ is an ALG pair of type II (cf. figure (a) in FIGURE 1). Let $E_{\mathfrak{r},1}, \dots, E_{\mathfrak{r},9}$ be exceptional divisors. Then

$$\mathcal{B}_{\mathfrak{r}} := \{H_{\mathfrak{r}} - 3E_{\mathfrak{r},1}\} \bigcup \bigcup_{i=2}^9 \{E_{\mathfrak{r},1} - E_{\mathfrak{r},i}\} \quad (3.9)$$

is a basis of $H_2(X_{\mathfrak{r}}, \mathbb{C})$. For simplicity, we denote the elements in $\mathcal{B}_{\mathfrak{r}}$ by $\gamma_{\mathfrak{r},1}, \dots, \gamma_{\mathfrak{r},9}$. The fibre class is represented by

$$3\gamma_{\mathfrak{r},1} + \sum_{j=2}^9 \gamma_{\mathfrak{r},j}.$$

Let

$$\Omega = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{y^2 z - x^3}. \quad (3.10)$$

It follows that

$$\text{Im}(\mathcal{P}) \subset \left\{ (y_1, \dots, y_9) \mid 3y_1 + \sum_{j=2}^9 y_j = 0 \right\}. \quad (3.11)$$

Under the affine coordinates $u = x/y$ and $v = z/y$, we have

$$\Omega = -\frac{du \wedge dv}{v - u^3}$$

and the residue around $\{v - u^3 = 0\}$ is $-du$.

Now we prove that the inclusion above is indeed an equality. Let (y_1, \dots, y_9) be a vector satisfying the condition $3y_1 + \sum_{j=2}^9 y_j = 0$. We will need a few computational results.

Lemma 3.6. *Let $\underline{D} = \{y^2z - x^3 = 0\} \subset \mathbf{P}^2$ and $p = [0:0:1]$ be the unique singular point on \underline{D} . Let Ω be the meromorphic two form defined in (3.10). Let $x_1, \dots, x_9 \in \underline{D} \setminus \{p\} \cong \mathbb{C}$ and Y is the blow-up of \mathbf{P}^2 at x_1, \dots, x_9 . Denote by E_i the exceptional divisor over x_i . Then*

(a) *Let H be the hyperplane class in \mathbf{P}^2 . We have*

$$\int_{H-3E_1} \Omega = -3x_1.$$

(b)

$$\int_{E_i - E_j} \Omega = x_i - x_j.$$

Proof. This follows from the residue calculations. \square

By Lemma 3.6, the vector (y_1, \dots, y_9) uniquely determines the points x_1, \dots, x_9 on \underline{D} . Moreover, the constraint $3y_1 + \sum_{j=2}^9 y_j = 0$ implies that $x_1 + \dots + x_9 = O$ in the additive group scheme $\underline{D} \setminus \{p\}$ and it turns out that this condition is sufficient by Max Noether's fundamental theorem [19, p. 61], i.e., given any 9 points $x_1, \dots, x_9 \in \underline{D} \setminus \{p\}$ (not necessarily distinct) with $x_1 + \dots + x_9 = O$, there exists a cubic C passing through all the x_i 's. According to Theorem 3.4 and the discussion after it, the pencil spanned by C and \underline{D} contains a smooth member. This shows that $Y = \text{Bl}_{\{x_1, \dots, x_9\}} \mathbf{P}^2$ is a smooth rational elliptic surface.

3.1.2. Type III. A type III fibre is a union of three smooth rational curves intersecting at a single point. In this case, \underline{D} can be three lines $L_1 \cup L_2 \cup L_3$ or $C \cup L$, where L is a line and C is a conic tangent to L at p . In the former case, each L_i contains three points of the blow up loci. In the latter case, Y is the blow up of five points on $C \setminus \{p\}$, two points on $L \setminus \{p\}$ and p then blow up a point on the exceptional curve corresponding to p avoiding the proper transform of C .

To construct a marked reference ALG pair of type III, we pick a smooth cubic C which meets $\underline{D} = L_1 \cup L_2 \cup L_3$ at nine points. Denote by $p_{\mathfrak{r},1}, \dots, p_{\mathfrak{r},9}$ the intersections $\underline{D} \cap C$ in a way such that $p_{\mathfrak{r},i} \in L_j$ if and only if $i \equiv j \pmod{3}$. Let $Y_{\mathfrak{r}}$ be the blow-ups of \mathbf{P}^2 at those nine points and $D_{\mathfrak{r}}$ be the proper transform of \underline{D} . Then $(Y_{\mathfrak{r}}, D_{\mathfrak{r}})$ is an ALG pair of type III (cf. figure (b) in FIGURE 1). Let $E_{\mathfrak{r},1}, \dots, E_{\mathfrak{r},9}$ be exceptional divisors. Then

$$\begin{aligned} \mathcal{B}_{\mathfrak{r}} := & \{H_{\mathfrak{r}} - E_{\mathfrak{r},1} - E_{\mathfrak{r},2} - E_{\mathfrak{r},3}\} \bigcup \bigcup_{i=4,7} \{E_{\mathfrak{r},1} - E_{\mathfrak{r},i}\} \\ & \bigcup \bigcup_{i=5,8} \{E_{\mathfrak{r},2} - E_{\mathfrak{r},i}\} \bigcup \bigcup_{i=6,9} \{E_{\mathfrak{r},3} - E_{\mathfrak{r},i}\} \end{aligned}$$

is a basis of $H_2(X_{\mathfrak{r}}, \mathbb{C})$. For simplicity, we denote the elements in $\mathcal{B}_{\mathfrak{r}}$ by $\gamma_{\mathfrak{r},1}, \dots, \gamma_{\mathfrak{r},7}$. The fibre class is represented by

$$3\gamma_{\mathfrak{r},1} + \sum_{j=2}^7 \gamma_{\mathfrak{r},j}.$$

Let $\underline{D} = \{xy(x+y) = 0\}$. This can be always achieved using the $\mathrm{PGL}(3, \mathbb{C})$ action. Let

$$\Omega = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{xy(x+y)}. \quad (3.12)$$

It follows that

$$\mathrm{Im}(\mathcal{P}) \subset \left\{ (y_1, \dots, y_7) \mid 3y_1 + \sum_{j=2}^7 y_j = 0 \right\}. \quad (3.13)$$

Under the affine coordinates $u = y/x$ and $v = z/x$, we have

$$\Omega = -\frac{du \wedge dv}{u(1+u)}.$$

We will need the following computational results.

Lemma 3.7. *Let $\underline{D} = \{xy(x+y) = 0\} \subset \mathbf{P}^2$ and $p = [0:0:1]$ be the unique singular point on \underline{D} . Let Ω be the meromorphic two form defined in (3.12). Let $x_1, \dots, x_9 \in \underline{D} \setminus \{p\}$ such that $x_i \in L_j$ if and only if $i \equiv j \pmod{3}$ and Y be the blow-up of \mathbf{P}^2 at x_1, \dots, x_9 . Denote by E_i the exceptional divisor over x_i as before. Let $x_i = [0:a_i:b_i]$ for $i = 1, 4, 7$, $x_i = [a_i:0:b_i]$ for $i = 2, 5, 8$, and $x_i = [a_i:-a_i:b_i]$ for $i = 3, 6, 9$. Then*

(a) *The line $\overline{x_1x_2}$ intersects $x+y=0$ at $[a_1a_2:-a_1a_2:a_1b_2-a_2b_1] \neq p$. (Note that $a_i \neq 0$ for all i by our assumption.) Let H be the hyperplane class in \mathbf{P}^2 . Then we have*

$$\int_{\gamma_1} \Omega = \int_{H-E_1-E_2-E_3} \Omega = \frac{a_1b_2-a_2b_1}{a_1a_2} - \frac{b_3}{a_3}.$$

(b) *We have*

$$\int_{E_{k+3}-E_k} \Omega = \frac{b_{k+3}}{a_{k+3}} - \frac{b_k}{a_k}, \text{ and } \int_{E_{k+6}-E_k} \Omega = \frac{b_{k+6}}{a_{k+6}} - \frac{b_k}{a_k}$$

for $k = 1, 2, 3$.

Proof. This follows from a direct calculation on residues and hence the proof is omitted. \square

By Lemma 3.7, the vector $(y_1, \dots, y_7) \in \mathbb{C}^7$ determines x_1, \dots, x_9 on $\underline{D} \setminus \{p\}$. Indeed, we can put $x_1 = p_{\mathfrak{r},1}$ and $x_2 = p_{\mathfrak{r},2}$ and the results in (a) and (b) in Lemma 3.7 would determine the location of all the rest x_i 's. The only thing we have to show is that $Y = \mathrm{Bl}_{\{x_1, \dots, x_9\}} \mathbf{P}^2$ is a rational elliptic surface, i.e., there is a smooth cubic passing through x_1, \dots, x_9 .

Again it suffices to construct a cubic passing through the points x_1, \dots, x_9 . This can be done directly. Indeed, suppose the coordinate of x_i is given as in Lemma 3.7. The cubic defined by

$$\prod_{i=1,4,7} (b_iy - a_iz) + x \cdot (ax^2 + by^2 + cz^2 + dxy + eyz + fxz) = 0 \quad (3.14)$$

passing through x_1, x_4, x_7 . Now set $y = 0$ in the above equation. We obtain

$$-a_1a_4a_7z^3 + ax^3 + cxz^2 + fx^2z = 0 = \frac{a_1a_4a_7}{a_2a_5a_8} \prod_{i=2,5,8} (b_ix - a_iz). \quad (3.15)$$

This equation uniquely determines the coefficients a, c , and f . We are left with b, d and e , i.e., the coefficient of xy^2 , x^2y , and xyz . Now set $y = -x$ in the above equation. We see that

$$\begin{aligned} & - \prod_{i=1,4,7} (b_ix + a_iz) + x \cdot (ax^2 + bx^2 + cz^2 - dx^2 - exz + fxz) = 0 \\ & = -\frac{a_1a_4a_7}{a_3a_6a_9} \prod_{i=3,6,9} (a_iz - b_ix) \end{aligned}$$

from which e and $b - d$ are uniquely determined. This shows that there exists a one parameter family of cubics passing through x_1, \dots, x_9 and therefore implies the existence of the cubic \underline{D} other than $xy(x + y)$.

3.1.3. Type IV. A type IV fibre consists of two smooth rational curves tangent at a point. In this case, \underline{D} is union of a line L and a conic Q tangent at p . Then Y is blow up of six points on $C \setminus \{p\}$ and three points on $L \setminus \{p\}$.

To construct a reference marked ALG pair of type IV, we simply fix a smooth cubic C which intersects $\underline{D} \setminus \{p\}$ at 9 distinct points. Denote by $p_{\tau,1}, p_{\tau,2}, p_{\tau,3}$ the intersection $L \cap C$ and $p_{\tau,4}, \dots, p_{\tau,9} \in Q \cap C$. Consider the blow-up $Y_{\tau} = \text{Bl}_{p_{\tau,1}, \dots, p_{\tau,9}} \mathbf{P}^2$ and D_{τ} , the proper transform of \underline{D} . Let $E_{\tau,i}$ be the exceptional divisor over $p_{\tau,i}$. In which case, we can choose

$$\mathcal{B}_{\tau} := \{H - E_{\tau,4} - E_{\tau,7} - E_{\tau,1}\} \cup \bigcup_{i=2,3} \{E_{\tau,1} - E_{\tau,i}\} \cup \bigcup_{i=5,6} \{E_{\tau,4} - E_{\tau,i}\} \cup \bigcup_{i=8,9} \{E_{\tau,7} - E_{\tau,i}\}$$

to be our basis of $H_2(X_{\tau}, \mathbb{Z})$. For simplicity, we denote the elements in \mathcal{B}_{τ} by $\gamma_{\tau,1}, \dots, \gamma_{\tau,8}$. The fibre class is represented by

$$3\gamma_{\tau,1} + \sum_{j=2}^8 \gamma_{\tau,j}.$$

We may assume $\underline{D} = \{y(x^2 + yz) = 0\}$; we can achieve this using the $\text{PGL}(3, \mathbb{C})$ action. Let

$$\Omega = \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{y(x^2 + yz)}. \quad (3.16)$$

It follows that

$$\text{Im}(\mathcal{P}) \subset \left\{ (y_1, \dots, y_8) \mid 3y_1 + \sum_{j=2}^8 y_j = 0 \right\}. \quad (3.17)$$

Under the affine coordinates $u = x/z$ and $v = y/z$, we have

$$\Omega = \frac{du \wedge dv}{v(u^2 + v)}. \quad (3.18)$$

One can easily check that

$$\text{Res}_L \Omega = \frac{du}{u^2}, \text{ and } \text{Res}_Q \Omega = \frac{du}{u^2}.$$

We need the following computational results.

Lemma 3.8. *Let $\underline{D} = \{y(x^2 + yz) = 0\} \subset \mathbf{P}^2$ and $p = [0:0:1]$ be the unique singular point on \underline{D} . Let Ω be the meromorphic two form defined in (3.16). Let $x_1, \dots, x_9 \in \underline{D} \setminus \{p\}$ such that $x_1, x_2, x_3 \in L$ and $x_4, \dots, x_9 \in Q$. Let $Y = \text{Bl}_{\{x_1, \dots, x_9\}} \mathbf{P}^2$. Denote by E_i the exceptional divisor over x_i . Let $x_i = [a_i:0:c_i]$ for $i = 1, 2, 3$ and $x_i = [a_i:b_i:c_i]$ for $i = 4, \dots, 9$. Then*

(a) *Assume that $x_4 \neq x_7$. Then the line $\overline{x_4x_7}$ intersects $y = 0$ at $[c_7b_4 - c_4b_7:0:a_4b_7 - a_7b_4] \neq p$. Let H be the hyperplane class in \mathbf{P}^2 . Then we have*

$$\int_{\gamma_1} \Omega = \int_{H - E_1 - E_4 - E_7} \Omega = \frac{a_4b_7 - a_7b_4}{c_7b_4 - c_4b_7} - \frac{c_1}{a_1}.$$

(b) *We have for $k = 1, 4, 7$*

$$\int_{E_k - E_{k+j}} \Omega = \frac{c_{k+j}}{a_{k+j}} - \frac{c_k}{a_k} \text{ for } j = 1, 2.$$

Proof. This follows from the formulae

$$\text{Res}_L \Omega = \frac{du}{u^2}, \text{ and } \text{Res}_Q \Omega = \frac{du}{u^2}$$

and the residue theorem. The proof is hence omitted. \square

By Lemma 3.8, the vector $(y_1, \dots, y_8) \in \mathbb{C}^8$ determines x_1, \dots, x_9 on $\underline{D} \setminus \{p\}$. Indeed, we can put $x_4 = p_{\mathbf{r},4}$ and $x_7 = p_{\mathbf{r},7}$ and the results in (a) and (b) in Lemma 3.8 would determine the location of all the rest x_i 's. The only thing we have to show is that $Y = \text{Bl}_{\{x_1, \dots, x_9\}} \mathbf{P}^2$ is a rational elliptic surface, i.e., there is a smooth cubic passing through x_1, \dots, x_9 .

Again it suffices to construct a cubic passing through the points x_1, \dots, x_9 . This can be done directly. Indeed, suppose the coordinate of x_i is given as in Lemma 3.8. The cubic defined by

$$\prod_{i=1,2,3} (c_i x - a_i z) + y \cdot (ax^2 + by^2 + cz^2 + dxy + eyz + fxz) = 0 \quad (3.19)$$

passing through x_1, x_2, x_3 . Note that the rational curve Q is parameterized by

$$[\alpha:\beta] \mapsto [\alpha\beta:\alpha^2:\beta^2].$$

Now set $yz = -x^2$ in the above equation. It follows that b, c, d, f and $a - e$ are uniquely determined. This shows that there exists a one parameter family of cubics passing through x_1, \dots, x_9 and therefore implies the existence of the cubic \underline{D} other than $y(x^2 + yz)$.

3.1.4. Type II^* . A type II^* fibre is the E_8 configuration. Assume that Y is a rational elliptic surface with an type II^* fibre D . The section of Y can only intersect the unique component of D with multiplicity one. One can then iteratively contracts the section, the component with multiplicity 1, 2, 3, 4, 5, 6, 4, 2 (in total nine curves) and end up with a smooth projective surface of Picard number one, that is, \mathbf{P}^2 . The only non-contracted component of D in the process has multiplicity three. In other words, any rational elliptic surface with a type II^* fibre can be realized as blow up on the base points of the cubic pencil containing a triple line which is tri-tangent to any other smooth element in the pencil. If the pencil contains a cusp curve, then the singular configuration of Y is II^*II . Otherwise, the pencil contains a nodal curve and the singular configuration of Y is II^*I_1^2 . From the long exact sequence (2.1), we have $H_2(X)$ is of rank one and generated by the fibre class of Y . Thus, the periods all vanish in

both cases. Depending on whether the pencil contains a cusp cubic or not, there are exactly two different rational elliptic surfaces with an II^* fibre. Both of them are extremal rational elliptic surfaces (rational elliptic surfaces whose relative automorphism group is finite, cf. [31]) and their singular configuration is II^*II or II^*I_1^2 . One can have an isotrivial deformation of the latter which degenerates to the former. In particular, the periods won't distinguish these two cases.

3.1.5. Type III^* . A type III^* fibre is the E_7 configuration. Assume that Y is a rational elliptic surface with a type III^* fibre D . Sections of Y must intersect a component of D with multiplicity one. One can then iteratively contracts the section, followed by the components with multiplicity 1, 2, 3, 4, 2 (six curves in total) and end up with a smooth projective surface Y' of Picard group rank four. Denote by C_1, C_2, C_3 the image of the remaining components of D with multiplicity 1, 2, 3 respectively. Recall that $\mathbf{P}^1 \times \mathbf{P}^1$ contains no curve with negative self-intersection and \mathbb{F}_2 contains a unique one such curve. We can conclude that the minimal model of Y' must be \mathbf{P}^2 since $C_1^2 = C_2^2 = -2$. Then the (-1) -curve must intersect C_1 otherwise the image of C_3 to \mathbf{P}^2 would have self-intersection 2 which is absurd. One may iteratively contracts the sections and the components with multiplicity 1, 2, 3, 4, 3, 2 (seven curves in total) from Y . Denote the resulting smooth projective surface by Y'' and the image of the remaining components of D with multiplicity i by $D_i \subseteq Y''$ for $i = 1, 2$. From the earlier discussion D_1 must intersect a (-1) -curve. We claim that Y'' then becomes the Hirzebruch surface \mathbb{F}_1 . Indeed, any irreducible curve in Y has self-intersection at least -2 , so we may exclude the possibility of \mathbb{F}_n , $n \geq 3$. Notice that $D_2^2 = 1$ and the self-intersection pairing in $\mathbf{P}^1 \times \mathbf{P}^1$ or \mathbb{F}_2 are even. So the claim is established. From $D_1^2 = 0$ and Riemann–Roch theorem, D_1 must be a fibre. In particular, D_1 intersects a (-1) -curve. Therefore, after contracting this (-1) -curve to \mathbf{P}^2 , the image of D is a union of a double line and a line.

To sum up, any rational elliptic surface with an III^* fibre can be realized as a blow-up of the base locus of the pencil spanned by a smooth cubic \underline{D} and a union of a double line M and a line N , with the double line intersecting the smooth cubic at its flex point and the other line N also passes through the flex point of the cubic.

There is another way to construct a rational elliptic surface with an III^* -fibre which is easier to calculate the periods. Consider a triple line $\underline{D} = 3L$ in \mathbf{P}^2 . Take C to be a smooth cubic which is tangent at $p \in \underline{D}$ and intersects transversally at another point $q \in \underline{D}$. Blowing up p and q yields a rational elliptic surface with an III^* fibre. Explicitly, if we denote by $[x:y:z]$ the coordinate on \mathbf{P}^2 , we can take $\underline{D}_{\text{red}} = \{x = 0\}$ and C to be the plane curve defined by

$$y^2z + x(z^2 + xy + a'yz), \quad a' \in \mathbb{C}^*.$$

In which case, $p = [0:0:1]$ and $q = [0:1:0]$. One checks that this is smooth whenever $q^3 \neq -27$. Using change of variables, the equation displayed above can be transformed into

$$y^2z + x(z^2 + axy + yz), \quad a \in \mathbb{C}^*. \quad (3.20)$$

We see that (3.20) is smooth for general a . If it happens that (3.20) is singular, we can always add a multiple of x^3 to the equation to make it smooth. In any case, we obtain a rational elliptic surface with singular fibre configuration III^* at infinity.

To obtain cycles in $H_2(X, \mathbb{Z})$, let T' be the tangent line of C at $[0:1:0]$, i.e., $T' = \{z = 0\}$. After blowing-ups, the proper transform T of T' becomes a (-1) curve and therefore it is a

section. Then $\gamma_1 := [T] - [E]$, where E is the section obtained in the last step of blow-ups of \mathbf{P}^2 at $[0:1:0]$, gives an element in $H_2(X, \mathbb{Z})$. The fibre class $[f]$ gives another element in $H_2(X, \mathbb{Z})$. One can check $\mathcal{B} := \{\gamma_1, [f]\}$ is a basis of $H_2(X, \mathbb{Z})$. As before, we shall pick a smooth cubic and a basis of the homology of its complement (after blow-ups) as above to serve our marked ALG pair of type III*. We denote the pair by (Y_r, D_r) and the basis by \mathcal{B}_r .

We now choose a section of $\Omega_{\mathbf{P}^2}(\underline{D})$

$$\Omega := \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{x^3}. \quad (3.21)$$

We shall compute the periods using the two form Ω .

Let us investigate the blow-ups over $[0:1:0]$ first. Using the affine coordinates $u := x/y$ and $v := z/y$, the form Ω is transformed into

$$\Omega = -\frac{du \wedge dv}{u^3} \quad (3.22)$$

and C is defined by

$$\{v + u(v^2 + au + v) = 0\}. \quad (3.23)$$

Now we compute the blow-up. Set $v = us$ (here s is the coordinate on \mathbf{P}^1). We then have

$$\Omega = -\frac{du \wedge ds}{u^2}. \quad (3.24)$$

Here $\{u = 0\}$ corresponds to the exceptional divisor (with multiplicity two as expected). In the meanwhile, the proper transform of C is

$$\{s + u^2 s^2 + au + us = 0\} \quad (3.25)$$

and the proper transform of T' is $\{s = 0\}$. We blow up at $(u, s) = (0, 0)$ one more time. Let $s = ut$. Then the meromorphic two form becomes

$$\Omega = -\frac{du \wedge dt}{u}.$$

The proper transform of C is

$$\{t + u^3 t^2 + a + ut = 0\} \quad (3.26)$$

and the proper transform of T' is defined by $\{t = 0\}$. Denote by E' the exceptional divisor of the second blow-up. By our convention, t serves as an affine coordinate on $E' \cong \mathbf{P}^1$. In order to achieve Y , we need one more blow up at $(u, t) = (0, -a)$, the intersection of the proper transform of C and E' . Denote by E the exceptional divisor and by T the proper transform of T' . Then $\gamma_1 := [E] - [T]$ represents a homology cycle in $X := Y \setminus D$. The cycle γ_1 together with the fibre class $[f]$ form a basis \mathcal{B} of $H_2(X, \mathbb{Z})$. One can compute

$$\int_{[E]-[T]} \Omega = \int_0^{-a} dt = -a. \quad (3.27)$$

By varying a , we have proven that

$$\text{Im}(\mathcal{P}) = \{(y_1, y_2) \mid y_2 = 0\} \quad (3.28)$$

where y_1 (resp. y_2) is the coordinate corresponding to

$$\int_{\gamma_1} \Omega, \quad (\text{resp. } \int_{[f]} \Omega \equiv 0.) \quad (3.29)$$

Let us now describe the moduli space of rational elliptic surfaces with a III^* -fibre. From the classification of the singular configuration of a rational elliptic surface Y containing a type III^* -fibre, Y must contain an I_1 -fibre unless its singular configuration is III^*III . We have the following two cases:

(a) The pencil contains a nodal curve C . Up to the $\text{PGL}(3)$ -action on \mathbf{P}^2 , we may assume that the nodal curve C is of the form $x^3 + y^3 + xyz = 0$ with a node at $p = [0:0:1]$. Indeed, if C is a nodal curve with a node at p , we can always move p to $[0:0:1]$. Let $F(x, y, z)$ be the defining equation of C and $f(x, y) = F(x, y, 1)$ be the equation of C on the affine chart $\{z \neq 0\}$. We may further use the $\text{PGL}(3)$ -action to assume that

$$f(x, y) = xy + g(x, y) \quad (3.30)$$

where $g(x, y)$ is homogeneous of degree 3. In other words, $F(x, y, z) = xyz + g(x, y)$. Now we can use the remaining symmetries to eliminate the x^2y and xy^2 terms in g as well as adjust the coefficients of x^3 and y^3 . As a result, we achieve the equation $x^3 + y^3 + xyz = 0$. It is known that there is an isomorphism

$$\mathbb{C}^* \cong C \setminus \{p\}, \quad t \mapsto [t: -t^2: 1 - t^3].$$

The flex points are located at $[1: -1: 0]$, $[\omega: -\omega^2: 0]$, and $[\omega^2: -\omega: 0]$ where ω is the primitive 3rd root of unity. Moreover, these three flex points are equivalent under the $\text{PGL}(3)$ -action. One can easily check that if $A \in \text{PGL}(3, \mathbb{C})$ leaves $x^3 + y^3 + xyz$ invariant and fixes $[1: -1: 0]$, then either $A = \text{id}$ or $A: x \mapsto y, y \mapsto x, z \mapsto z$. Then M is the tangent line of C at $[1: -1: 0]$ and N can be any line passing through $[1: -1: 0]$. In particular, by rotating N , we obtain a \mathbf{P}^1 -family of rational elliptic surfaces with a III^* -fibre. If N meets the node of C , then the rational elliptic surface contains an I_2 -fibre. If N is tangent to a smooth point of C , then the rational elliptic surface contains an II -fibre. To sum up, the moduli space of rational elliptic surfaces with singular fibres of III^* and I_1 is $\mathbb{C}^* \subset \mathbf{P}^1$. The boundary points parameterize the rational elliptic surfaces with singular configuration II^*I_1^2 and $\text{III}^*\text{I}_2\text{I}_1$.

(b) The pencil contains a cuspidal curve C . One can use the $\text{PGL}(3)$ -action to assume that $C = \{y^2z = x^3\}$ and $p = [0:0:1]$ is the cusp. It is known that C the complement $C \setminus \{p\}$ is isomorphic to \mathbb{C} as an additive group via

$$\mathbb{C} \cong C \setminus \{p\}, \quad t \mapsto [t: 1: t^3]$$

and $C \setminus \{p\}$ admits a unique flex point. (Recall that the group law on $C \setminus \{p\}$ is defined in the same manner as the one defined on elliptic curves. For P and Q on $C \setminus \{p\}$, $P + Q$ is the point $R \in C \setminus \{p\}$ such that P, Q , and R are colinear.) Let M be the tangent of C at the flex point and N be a line passing through it. As in the previous case, rotating N gives rise to a \mathbf{P}^1 -family of rational elliptic surfaces. If N passes through smooth points of C , we obtain a rational elliptic surface with singular configuration III^*II I_1 . Moreover, any such two lines determine the same rational elliptic surface. When N passes through the cusp of C , then the resulting rational elliptic surface has the singular configuration III^*III . When N is also tangent to C , the corresponding rational elliptic surface has the singular configuration II^*II . As a summary, the parameter space of rational elliptic surface with a type III^* fibre is a \mathbf{P}^1 .

The generic point of \mathbf{P}^1 parametrizes those with singular configuration III^*I_1^3 which admits degenerations to $\text{III}^*\text{II I}_1$ and to II^*II .

In particular, there are three rational elliptic surfaces with trivial periods, with singular configuration III^*III , $\text{III}^*\text{II I}_1$, and $\text{III}^*\text{I}_2\text{I}_1$.

3.1.6. Type IV^* . A type IV^* is the E_6 configuration.

To construct the model, we consider $\underline{D} = \{x^3 = 0\}$. Let C be a cubic of the form

$$yz(y - z) + x(cx\bar{y} + xz + dyz) \text{ with } c, d \in \mathbb{C}.$$

Then C intersects $\underline{D}_{\text{red}}$ at $[0:0:1]$, $[0:1:1]$, $[0:1:0]$ with all multiplicity one. Moreover, one can check that for any $c, d \in \mathbb{C}$, the linear system spanned by C and \underline{D} contains a smooth member. As before, we pick a smooth cubic C (with constants c_r and d_r in the equation) to build our marked reference ALG pair of type IV^* .

Recall that in the present situation, one can achieve a desired pair (Y_r, D_r) by blowing up at those intersections $C \cap \underline{D}$ (we blow up three times at each intersection point and there are nine blow-ups needed in total). Denote by $E_{r,0}$ and $E_{r,\infty}$ the exceptional divisor over $[0:0:1]$ and $[0:1:0]$ from the last (the third) blow-up. Let $T'_{r,0}$ (resp. $T'_{r,\infty}$) be the tangent line of C at $[0:0:1]$ (resp. $[0:1:0]$) and $T_{r,0}$ (resp. $T_{r,\infty}$) be the proper transform on Y . It is easy to check that

$$\mathcal{B}_r := \{[E_{r,0}] - [T_{r,0}], [E_{r,\infty}] - [T_{r,\infty}], [f]\} \quad (3.31)$$

is a basis of $H_2(X_r, \mathbb{Z})$ where $X_r = Y_r \setminus D_r$ as before.

Like in the previous case, we take

$$\Omega = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{x^3}. \quad (3.32)$$

For arbitrary $c, d \in \mathbb{C}$, we can compute (as in the case of type III^*)

$$\int_{[E_0] - [T_0]} \Omega = -c, \text{ and } \int_{[E_\infty] - [T_\infty]} \Omega = -d.$$

This shows that

$$\text{Im}(\mathcal{P}) = \{(y_1, y_2, y_3) \mid y_3 = 0\}. \quad (3.33)$$

3.1.7. Type I_0^* . A type I_0^* fibre is the D_4 configuration. We begin with $\underline{D} = \{x^2y = 0\}$ and consider a cubic C intersecting with \underline{D} at three distinct points on each irreducible component. Using the $\text{PGL}(3, \mathbb{C})$ -action on \mathbf{P}^2 , we may assume that

- $C \cap \{x = 0\} = \{[0:1:0], [0:1:1], [0:1:a]\}$;
- $C \cap \{y = 0\} = \{[1:0:0], [1:0:b], [1:0:c]\}$ with $b, c \in \mathbb{C}^*$ distinct;

It follows that C is defined by the following equation

$$z(y - z)(ay - z) + x(dy^2 - (b + c)z^2 + fxy + bcxz + hyz) = 0.$$

We may as well assume that $f = 0$ by adding the defining equation of \underline{D} . To summarize, we may assume that C is given by

$$z(y - z)(ay - z) + x(dy^2 - (b + c)z^2 + bcxz + hyz) = 0.$$

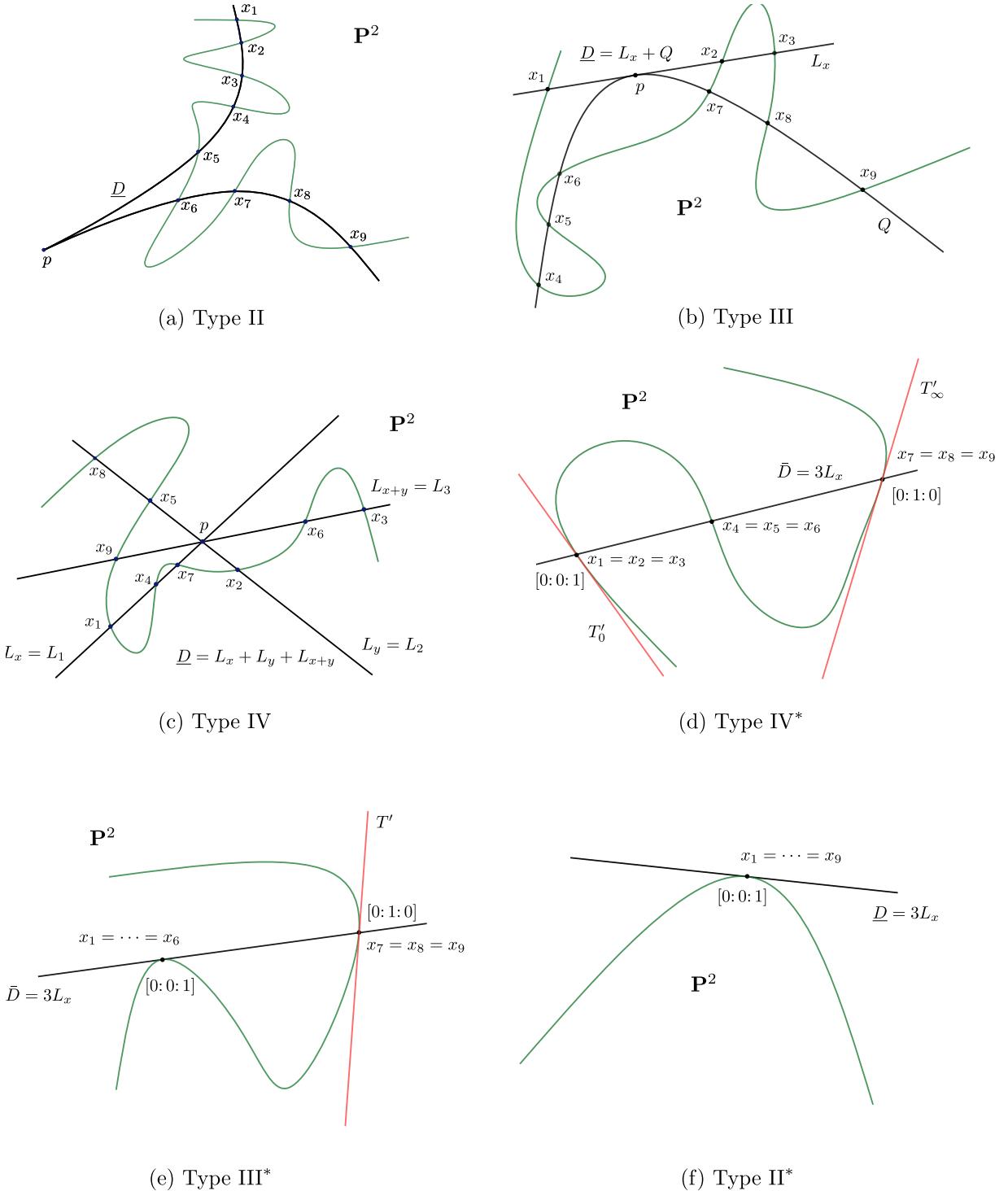


FIGURE 1. The constructions of ALG pairs of type II, III, IV, IV*, III*, and II*. The black lines stand for the singular divisor \underline{D} , the green curves stand for the cubic C in the corresponding pencil, and the red lines are the image of the (-1) curves under $Y \rightarrow \mathbf{P}^2$.

Picking a smooth cubic C of the form above and blowing-up at those points yield a ALG pair (Y, D) of type I_0^* . We need to construct a basis of $H_2(X, \mathbb{Z})$, i.e., we need to construct five cycles lying in $[D]^\perp$ under the identification.

- Consider the tangent of C at $[0:1:0]$. Explicitly, it is given by

$$dx + az = 0$$

which intersects $\{y = 0\}$ at $[a:0:-d]$. Denote by L_1 its proper transform on Y ;

- Consider the tangent of C at $[0:1:1]$. Explicitly, it is given by

$$(d + h - b - c)x - (1 - a)y + (1 - a)z = 0.$$

It intersects $\{y = 0\}$ at

$$[1 - a:0:b + c - d - h].$$

Denote by L_2 its proper transform on Y ;

- Consider the tangent of C at $[0:1:a]$. Explicitly, it is given by

$$(-a^2(b + c) + d + ah)x - a(1 - a)(z - ay) = 0$$

It intersects $\{y = 0\}$ at

$$[a(1 - a):0:-a^2(b + c) + d + ah].$$

Denote by L_2 its proper transform on Y ;

- Let E_7 be the exceptional divisor over $[1:0:b]$ in the second blow-up;
- Let E_8 be the exceptional divisor over $[1:0:c]$;
- Let E_9 be the exceptional divisor over $[1:0:0]$.

Now we can construct our cycles via

- (1) $\gamma_1 = L_1 - E_9$;
- (2) $\gamma_2 = L_2 - E_9$;
- (3) $\gamma_3 = L_3 - E_9$;
- (4) $\gamma_4 = E_9 - E_7$;
- (5) $\gamma_5 = E_9 - E_8$.

One can easily check that each of them lies in $[D]^\perp$ and they form a basis of $H_2(X, \mathbb{Z})$. Also the fibre class $[f]$ is given by $\sum_{i=1}^5 [\gamma_i]$. In the present case, we take

$$\Omega = \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{x^2y}.$$

It is also straightforward to check (parallel to the computation in previous sections) that the set of period vectors

$$\left\{ (y_1, \dots, y_5) \mid y_i := \int_{[\gamma_i]} \Omega, \quad i = 1, \dots, 5 \right\}$$

is equal to $\{(y_1, \dots, y_5) \mid \sum_{i=1}^5 y_i = 0\}$. Indeed, one can check that

$$\int_{\gamma_1} \Omega = \frac{-d}{a}, \quad \int_{\gamma_2} \Omega = \frac{b + c - d - h}{1 - a}, \quad \int_{\gamma_3} \Omega = \frac{-a^2(b + c) + d + ah}{a(1 - a)}, \quad \int_{\gamma_4} \Omega = -b, \quad \text{and} \quad \int_{\gamma_5} \Omega = -c.$$

This proves the surjectivity of the period map in this case.

3.1.8. **Type I_1^* .** A type I_1^* fibre is the D_6 configuration. We begin with $\underline{D} = \{x^2y = 0\}$ and consider a cubic C intersecting with \underline{D} at three distinct points on $\{y = 0\}$ but tangent to $\{x = 0\}$. Using the $\mathrm{PGL}(3, \mathbb{C})$ -action on \mathbf{P}^2 , we may assume that

- $C \cap \{x = 0\} = \{[0:1:0], [0:1:a]\}$ and C is tangent to $\{x = 0\}$ at $[0:1:0]$.
- $C \cap \{y = 0\} = \{[1:0:0], [1:0:b], [1:0:c]\}$ with $b \neq c$;

It follows that C is defined by the following equation

$$z^2(ay - z) + x(dy^2 + (b + c)z^2 + fxy - bcxz + hyz) = 0.$$

We may as well assume that $f = 0$ by adding the defining equation of \underline{D} . To summarize, we may assume that C is given by

$$z^2(ay - z) + x(dy^2 + (b + c)z^2 - bcxz + hyz) = 0.$$

Picking a smooth cubic C of the form above and blowing-up at those points yield an ALG pair (Y, D) of type I_1^* . Note that C is smooth implies $d \neq 0$. We need to construct a basis of $H_2(X, \mathbb{Z})$, i.e., we need to construct four cycles lying in $[D]^\perp$ under the identification.

- Consider the tangent of C at $[0:1:a]$. Explicitly, it is given by

$$(a^2(b + c) + d + ah)x - a^2(z - ay) = 0.$$

which intersects $\{y = 0\}$ at

$$[a^2:0:a^2(b + c) + d + ah].$$

Denote by L its proper transform on Y ;

- Consider a conic passing through $[1:0:0]$ and tangent to $\{x = 0\}$ at $[0:1:0]$ such that the intersection at $[0:1:0]$ with C has multiplicity 3. Explicitly, when $d \neq 0$, we could take for example

$$az^2 + dxy - abxz = 0.$$

It intersects $\{y = 0\}$ at

$$[1:0:0] \text{ and } [1:0:b].$$

Denote by Q its proper transform on Y ; Q is a (-1) curve on Y and therefore it is a section.

- Let E_7 be the exceptional divisor over $[1:0:b]$;
- Let E_8 be the exceptional divisor over $[1:0:c]$;

Let E_8 be the exceptional divisor over $[1:0:c]$ and E_9 be the exceptional divisor over $[1:0:0]$. Now we can construct our cycles via

- (1) $\gamma_1 = L - E_9 \sim H - E_5 - E_6 - E_9$;
- (2) $\gamma_2 = Q - E_4 \sim 2H - (E_1 + \dots + E_4) - E_7 - E_9$;
- (3) $\gamma_3 = E_7 - E_9$;
- (4) $\gamma_4 = E_8 - E_9$;

One can easily check that each of them lies in $[D]^\perp$ and they form a basis of $H_2(X, \mathbb{Z})$. Also the fibre class $[f]$ is given by $[\gamma_1] + [\gamma_2] - [\gamma_4]$. In the present case, we take

$$\Omega = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{x^2 y}.$$

It is also straightforward to check (parallel to the computation in previous sections) that the set of period vectors

$$\left\{ (y_1, \dots, y_4) \mid y_i := \int_{[\gamma_i]} \Omega, i = 1, \dots, 4 \right\}$$

is equal to $\{(y_1, \dots, y_4) \mid y_1 + y_2 - y_4 = 0\}$. More accurately, one can check

$$\int_{\gamma_1} \Omega = \frac{a^2(b+c) + d + ah}{a^2}, \quad \int_{\gamma_2} \Omega = -\frac{d}{a^2} - \frac{h}{a} - b, \quad \text{and} \quad \int_{\gamma_4} \Omega = c.$$

This proves the surjectivity of the period map in this case.

3.1.9. Type I_2^* . A type I_2^* fibre is the D_7 configuration. We begin with $\underline{D} = \{x^2y = 0\}$ and consider a cubic C intersecting with \underline{D} at three distinct points on $\{y = 0\}$ but tangent to $\{x = 0\}$. Moreover, we require that C passes through the unique singularity in $\underline{D}_{\text{red}}$. Using the $\text{PGL}(3, \mathbb{C})$ -action on \mathbf{P}^2 , we may assume that

- $C \cap \{x = 0\} = \{[0:1:0], [0:0:1]\}$ and C is tangent to $\{x = 0\}$ at $[0:1:0]$.
- $C \cap \{y = 0\} = \{[1:0:0], [1:0:b], [0:0:1]\}$;

It follows that C is defined by the following equation

$$z^2y + x(cy^2 + dz^2 + exy - bdxz + gyz) = 0.$$

We may as well assume that $e = 0$ as before by adding the defining equation of \underline{D} .

$$z^2y + x(cy^2 + dz^2 - bdxz + gyz) = 0.$$

Choosing a smooth cubic C of the form above and blowing-up at those points yield an ALG pair (Y, D) of type I_2^* . Now we need to construct a basis of $H_2(X, \mathbb{Z})$, i.e., we need to construct four cycles lying in $[D]^\perp$ under the identification $[D]^\perp \subset H^2(Y, \mathbb{Z}) \cong H_2(Y, \mathbb{Z})$.

- Consider a conic tangent to $\{x = 0\}$ at $[0:1:0]$ and meeting C at $[0:1:0]$ with multiplicity four and passing through $[1:0:0]$. Explicitly, it is given by

$$z^2 + cxy + gxz = 0.$$

which intersects $\{y = 0\}$ at

$$[1:0:0] \text{ and } [1:0:-g].$$

One checks the conic intersects C at $[0:1:0]$ with multiplicity four. Indeed, we can solve $x \sim z^2$ in the local ring at $[0:1:0]$. Denote by Q its proper transform on Y ;

- Consider the tangent of C at $[0:0:1]$. Explicitly, we have

$$y + dx = 0.$$

Denote by L the proper transform on Y .

Let E_8 be the exceptional divisor over $[1:0:b]$ and E_9 be the exceptional divisor over $[1:0:0]$. Now we can construct our cycles via

- (1) $\gamma_1 = L - E_7 \sim H - E_5 - E_6 - E_7$;
- (2) $\gamma_2 = Q - E_9 \sim 2H - (E_1 + \dots + E_4) - 2E_9$;
- (3) $\gamma_3 = E_8 - E_9$;

One can easily check that each of them lies in $[D]^\perp$ and they form a basis of $H_2(X, \mathbb{Z})$. Also the fibre class $[f]$ is given by $[\gamma_1] + [\gamma_2] - [\gamma_3]$. In the present case, we take

$$\Omega = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{x^2 y}.$$

Lemma 3.9. *We have*

$$\int_{[\gamma_1]} \Omega = -(b + g), \quad \int_{[\gamma_2]} \Omega = g, \quad \text{and} \quad \int_{[\gamma_3]} \Omega = -b.$$

Proof. Using the affine coordinates $u = x/z$ and $v = y/z$, we may re-write

$$\Omega = \frac{du \wedge dv}{u^2 v}$$

and the equation of C and the tangent line are given by

$$v + du - bdu^2 + guv + cuv^2 \text{ and } v + du.$$

Let $(u, v = \alpha u)$ be the coordinates on (an affine chart of) the blow-up (α is the affine coordinate on the exceptional divisor). Then Ω is transformed into

$$\Omega = \frac{du \wedge d\alpha}{u^2 \alpha}$$

and the proper transform of C and the tangent line are given by

$$\alpha + d - bdu + g\alpha u + c\alpha^2 u^2 \text{ and } \alpha + d.$$

Now we have to blow-up at $u = 0$ and $\alpha = -d$. Let $\alpha' = \alpha + d$. The equations above become

$$\alpha' - bdu + g(\alpha' - d)u + c(\alpha' - d)^2 u^2 \text{ and } \alpha'.$$

Moreover, we have

$$\Omega = \frac{du \wedge d\alpha'}{u^2(\alpha' - d)}.$$

Now we perform the blow-up via $\alpha' = \alpha'$ and $u = \alpha' \beta$.

$$\Omega = \frac{d\beta \wedge d\alpha'}{\alpha' \beta^2(\alpha' - d)}.$$

Taking the residue around $\{\alpha' = 0\}$, we obtain a one-form

$$-\frac{1}{d} \frac{d\beta}{\beta^2}$$

on \mathbf{P}^1 with a double pole at $\beta = 0$. The proper transform of C becomes

$$1 - (bd + gd)\beta + \text{higher order terms.}$$

Therefore, we have

$$\int_{[\gamma_1]} \Omega = -(b + g).$$

The other cases are similar. \square

It is also straightforward to check (parallel to the computation in previous sections) that the set of period vectors

$$\left\{ (y_1, y_2, y_3) \mid y_i := \int_{[\gamma_i]} \Omega, i = 1, 2, 3 \right\}$$

is equal to $\{(y_1, y_2, y_3) \mid y_1 + y_2 - y_3 = 0\}$. This proves the surjectivity of the period map in this case.

3.1.10. Type I_3^* . A type I_3^* fibre is the D_7 configuration. We again begin with $\underline{D} = \{x^2y = 0\}$ and consider a cubic C intersecting with \underline{D} as follows.

- $C \cap \{x = 0\} = \{[0:1:0], [0:0:1]\}$ and C is tangent to $\{x = 0\}$ at $[0:1:0]$.
- $C \cap \{y = 0\} = \{[1:0:0], [0:0:1]\}$ and C is tangent to $\{y = 0\}$ at $[0:0:1]$;

It follows that C is defined by the following equation

$$z^2y + x(ay^2 + dxz + eyz) = 0.$$

Picking a smooth cubic C of the form above and blowing-up at those points yield a ALG pair (Y, D) of type I_3^* . We need to construct a basis of $H_2(X, \mathbb{Z})$, i.e., we need to construct two cycles lying in $[D]^\perp$ under the identification.

- Consider a conic intersecting C at $[0:1:0]$ with multiplicity four. Explicitly, we may pick

$$axy + z^2 + exz = 0.$$

which intersects $\{y = 0\}$ at $[1:0:0]$ and $[1:0:-e]$.

Denote by Q_1 its proper transform on Y ;

- Consider another conic intersecting C at $[0:0:1]$ with multiplicity greater than or equal to four and passing through $[0:1:0]$. Explicitly, we can take

$$dx^2 + zy + exy = 0.$$

Denote by Q_2 its proper transform on Y ;

- Let L be the proper transform of the line connecting $[1:0:0]$ and $[0:0:1]$.

Let E_9 be the exceptional divisor over $[1:0:0]$. Now we can construct our cycles via

- (1) $\gamma_1 = Q_1 - E_9 \sim 2H - (E_1 + \dots + E_4) - 2E_9$;
- (2) $\gamma_2 = Q_2 - L \sim 2H - (E_5 + \dots + E_8) - E_1 - (H - E_1 - E_9)$;

One can easily check that each of them lies in $[D]^\perp$ and they form a basis of $H_2(X, \mathbb{Z})$. Also the fibre class $[f]$ is given by $[\gamma_1] + [\gamma_2]$. In the present case, we take

$$\Omega = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{x^2 y}.$$

It is also straightforward to check (parallel to the computation in previous sections) that the set of period vectors

$$\left\{ (y_1, y_2) \mid y_i := \int_{[\gamma_i]} \Omega, i = 1, 2 \right\}$$

is equal to the set $\{(y_1, y_2) \mid y_1 + y_2 = 0\}$. This proves the surjectivity of the period map in this case.

3.1.11. Type I_4^* . A type I_4^* fibre is the D_8 configuration. Suppose (Y, D) is an ALG* pair of type I_4^* . One easily checks that $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$ is generated by the homology class of a fibre, which is represented by a holomorphic curve. Consequently, similar to the case II^* , the period map must be constant (in fact the zero map). To make the treatment comprehensive, we will outline the construction of an ALG* pair of type I_4^* .

We begin with a line and a conic tangent at a point p . Again we may assume that the line is given by $\{x = 0\}$ and $p = [0: 0: 1]$. Consider a smooth cubic which intersects the conic with multiplicity five. In which case, the cubic intersects both the line and the conic at a point other than p . Explicitly, we can take for instance the conic to be

$$\{xz - y^2 = 0\}$$

and the cubic to be

$$\{xz^2 - zy^2 - yx^2 = 0\}.$$

We can blow-up all the intersection points (nine points in total) to achieve a rational elliptic surface with an I_4^* configuration.

Remark 3.10. Let (Y, D) be an ALG or ALG* pair. In each case above, our construction gives a deformation family $\pi: (\mathcal{Y}, \mathcal{D}) \rightarrow \mathcal{M}$ of (Y, D) together with a section of $\pi_* \Omega_{\mathcal{Y}/\mathcal{M}}^2(\mathcal{D})$ with a fixed normalization; it is the pullback of a fixed section

$$\Omega \in H^0(\mathbf{P}^2, \Omega_{\mathbf{P}^2}^2(\underline{D})) = H^0(Y, \Omega_Y^2(D)). \quad (3.34)$$

This normalization will become essential in the later subsection when we discuss the ALG or ALG* gravitational instantons, i.e., when metrics are involved.

3.2. ALG and ALG* gravitational instantons. We will start with introducing the models for ALG and ALG* gravitational instantons. A model for ALG gravitational instantons is determined by (β, τ, L, R) , where $R, L > 0$ and β, τ is chosen from Table 3.1 below.

∞	I_0^*	II	III	IV	IV^*	III^*	II^*
β	$1/2$	$1/6$	$1/4$	$1/3$	$2/3$	$3/4$	$5/6$
τ	any	$e^{2\pi i/3}$	i	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	$e^{2\pi i/3}$

TABLE 3.1

For each triple (β, τ, L) chosen, denote by $X_{mod} = X_{mod}(\beta, \tau, L, R)$ the complex manifold $\{(u, v) \in \mathbb{C} \oplus \mathbb{C} \mid \text{Arg}(u) \in [0, 2\pi\beta] \text{ and } |u| > R\} / \sim$,

where the equivalence relation is given by

$$\begin{aligned} (u, v) &\sim (u, v + (m + n\tau)), \text{ for } (m, n) \in \mathbb{Z}^2, \\ (u, v) &\sim (e^{2\pi i\beta} u, e^{-2\pi i\beta} v). \end{aligned}$$

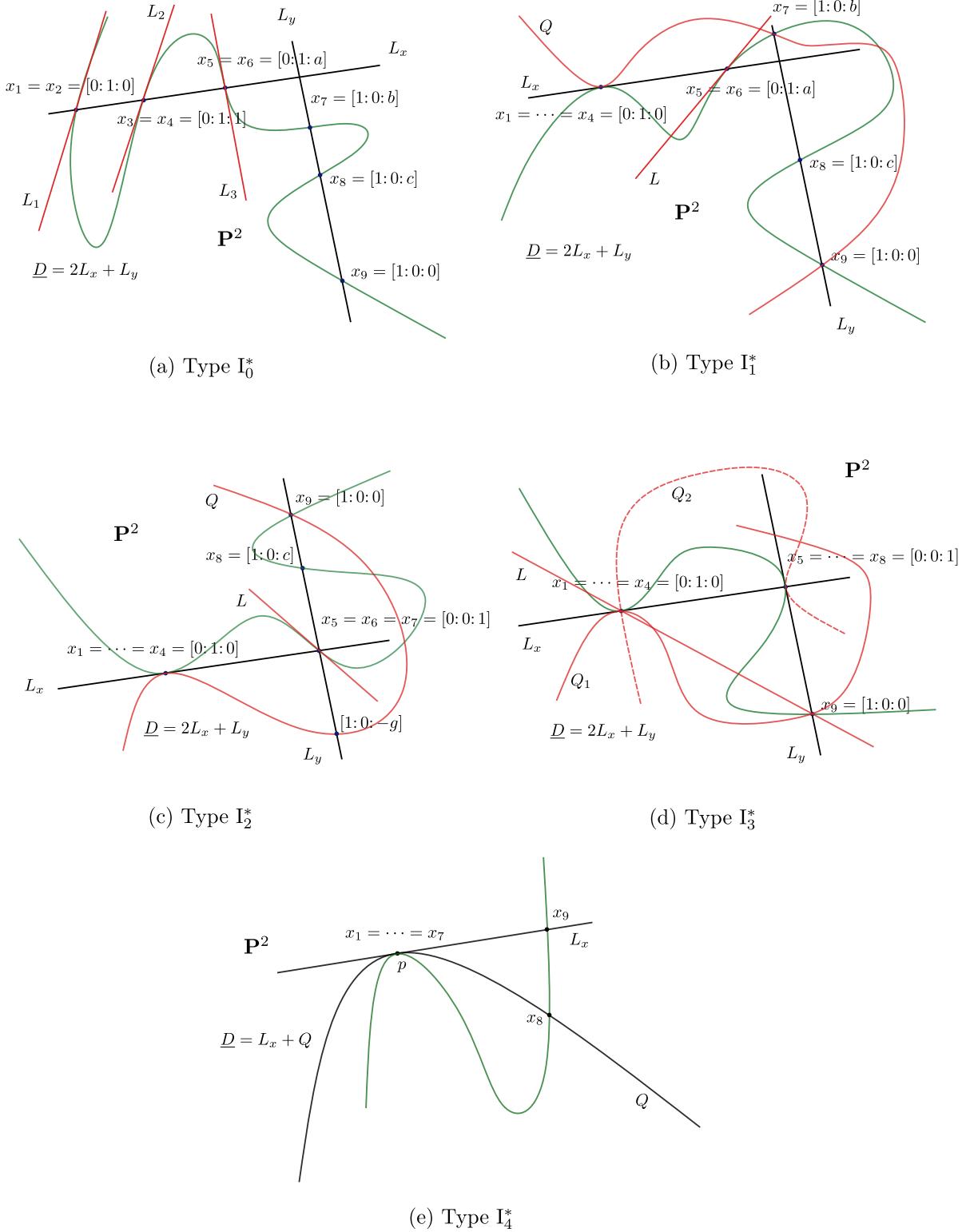


FIGURE 2. The constructions of ALG* pairs of type I₁^{*} to I₄^{*} and ALG pair of type I₀^{*}. Here again the black lines stand for the singular divisor \underline{D} , the green lines stand for the cubic C in the corresponding pencil, and the red lines (solid/dashed) indicate the cycles we use to construct (-1) curves on Y .

The hyperKähler triple is given by

$$\omega_{mod} = \frac{\sqrt{-1}L^2}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v}), \quad \Omega_{mod} = L^2 du \wedge dv.$$

By definition, there is a natural elliptic fibration structure $u^{\frac{1}{\beta}}: X_{mod} \rightarrow \{z \mid |z| > R^{\frac{1}{\beta}}\} \subseteq \mathbb{C}$ with torus fibres of area $L^2 \operatorname{Im}(\tau)$. Moreover, one can fill in the fibre at infinity to partially compactify X_{mod} and the singular configuration of the fibre at infinity is described in Table 3.1 and always with monodromy of finite order.

On the other hand, a model of ALG^* gravitational instanton is determined by $\nu \in \mathbb{N}$ and $R, \varepsilon > 0$. For each pair (ν, ε) , we denote model by $X_{mod}^* = X_{mod}^*(\nu, \varepsilon, R)$ as a complex manifold is given by

$$\{(u, v) \in \mathbb{C} \oplus \mathbb{C} \mid u \neq 0, |u| < R\} / \sim,$$

where the equivalence equation is given by

$$\begin{aligned} (u, v) &\sim (u^2, uv) \\ (u, v) &\sim (u, v + m + n \frac{\nu}{\pi i} \log u), \text{ for } (m, n) \in \mathbb{Z}^2. \end{aligned}$$

The Ricci-flat metric and the corresponding holomorphic volume form is given by

$$\begin{aligned} \omega_{mod}^* &= i \frac{\nu |\log|u||}{\pi \varepsilon} \frac{du \wedge d\bar{u}}{|u|^4} + \frac{i}{2} \frac{\pi \varepsilon}{\nu |\log|u||} \left(dv - \frac{1}{i} \frac{\operatorname{Im}(v) du}{u |\log|u||} \right) \wedge \overline{\left(dv - \frac{1}{i} \frac{\operatorname{Im}(v) du}{u |\log|u||} \right)}, \\ \Omega_{mod}^* &= u^{-2} du \wedge dv. \end{aligned}$$

There is also a natural elliptic fibration structure $u^{\frac{1}{2}}: X_{mod}^* \rightarrow \{0 < |u|^{\frac{1}{2}} < R^{\frac{1}{2}}\} \subseteq \mathbb{C}$ and one can extends the fibration over the puncture by adding an \mathbb{I}_ν^* -fibre.

Definition 3.11. We say that a gravitational instanton (X, ω, Ω) is of type $\operatorname{ALG}(\beta, \tau, L)$ (or ALG for simplicity) if the Calabi ansatz $(X_C, \omega_C, \Omega_C)$ in Definition 2.2 is replaced by

$$(X_{mod}(\beta, \tau, L, R), \omega_{mod}, \Omega_{mod}) \text{ for some } R > 0.$$

We also define marked ALG gravitational instanton similar to Definition 2.4. We will denote the set of marked $\operatorname{ALG}(\beta, \tau, L)$ gravitational instantons by $\operatorname{mALG}(\beta, \tau, L)$. We will define (marked) $\operatorname{ALG}^*(\nu, \varepsilon)$ gravitational instantons and $\operatorname{mALG}^*(\nu, \varepsilon)$ similarly.

Remark 3.12. Here the definition seems different from the one in [9] a priori. However, the definitions of ALG gravitational instantons are equivalent from [23, (3.10)] and different type of D corresponds to different value of choice of β in [9], which is a discrete parameter. The definitions of ALG^* gravitational instantons are equivalent by [7, Proposition 2.3].

With the above definition, there are natural invariants of the ALG gravitational instantons given by the cohomology classes of the hyperKähler triple. The set of possible cohomology classes are called the period domain of the gravitational instantons. In the cases of ALG and ALG^* gravitational instantons, the period domains are described by Chen–Viaclovsky–Zhang [9]. For the ALG case, we first fixed β, τ, L and a reference $\operatorname{ALG}(\beta, \tau, L)$ gravitational instanton $(X_0, \omega_0, \Omega_0)$. The period domain $\mathcal{P}\Omega(\beta, \tau, L)$ is a subset of $H^2(X_0, \mathbb{R}) \times H^2(X_0, \mathbb{C})$ consisting of pairs $([\omega], [\Omega])$ satisfying the following conditions:

- (1) if $[C] \in H_2(X_0, \mathbb{Z})$, $[C]^2 = -2$, then $|[\omega] \cdot [C]|^2 + |[\Omega] \cdot [C]|^2 \neq 0$;

- (2) $[\Omega] \cdot [F] = 0$, where $[F] \in H_2(X_0, \mathbb{Z})$ is the homology class of the elliptic fibre;
- (3) $[\omega] \cdot [F] = L^2 \operatorname{Im}(\tau)$.

The period domain $\mathcal{P}\Omega(\nu, \varepsilon)$ of $\operatorname{ALG}^*(\nu, \varepsilon)$ gravitational instantons are defined similarly except the last condition is replaced by $[\omega] \cdot [F] = \varepsilon$. Then the period map of marked $\operatorname{ALG}(\beta, \tau, L)$ gravitational instanton is defined by

$$\mathcal{P}(\beta, \tau, L): \operatorname{mALG}(\beta, \tau, L) \rightarrow \mathcal{P}\Omega(\beta, \tau, L) \quad (3.35)$$

$$(X, \omega, \Omega, \alpha) \mapsto (\alpha^*[\omega], \alpha^*[\Omega]). \quad (3.36)$$

We define the period map $\mathcal{P}(\nu, \varepsilon)$ for $\operatorname{ALG}^*(\nu, \varepsilon)$ gravitational instantons similarly.

The goal of the section is to prove the surjectivity of the period maps of ALG and ALG^* gravitational instantons, conjectured by Chen–Viaclovsky–Zhang [9, Conjecture 7.8].

Theorem 3.13. *The period maps $\mathcal{P}(\beta, \tau, L)$ and $\mathcal{P}(\nu, \varepsilon)$ are surjective.*

Similar to the ALH^* gravitational instantons, we have the following uniformization results for ALG and ALG^* gravitational instantons.

Theorem 3.14 ([4, Theorem 1.2] and [7, Theorem 1.5]). *Any ALG (or ALG^*) gravitational instanton can be compactified to a rational elliptic surfaces by adding a singular fibre of finite monodromy (or of type I_ν^*).*

Remark 3.15. From the Persson’s classification of singular configurations in rational elliptic surfaces [32], one can only have $\nu \leq 4$ for ALG^* gravitational instantons [7].

Therefore, we will follow the method similar to the proof of the surjectivity of the period map for ALH^* gravitational instantons to prove Theorem 3.13. We already proved the surjectivity of the $(2, 0)$ -form for rational elliptic surfaces with a prescribed fibre with finite monodromy or of type I_ν^* in Theorem 3.3, and we will later prove that every cohomology class of the complement of the prescribed fibre in the rational elliptic surface can support a Ricci-flat metric up to monodromy (see Theorem 3.16 and Lemma 3.17).

3.3. Surjectivity of the period maps for ALG and ALG^* gravitational instantons. We next modify a theorem of Hein [23, Theorem 1.3]. Let Y be a rational elliptic surface with a fibre D of finite order monodromy. Denote by $X = Y \setminus D$ and by $p: X \rightarrow B \cong \mathbb{C}$ the restriction of the elliptic fibration structure from Y . We fix a holomorphic coordinate u on a neighborhood of the base such that the singular fibre D is located at $u = 0$. Finally, let $U_r = \{u \in B \mid |u| < r\}$.

Theorem 3.16. *Let ω_0 be any Kähler metric on $X = Y \setminus D$ such that $\int_X \omega_0^2 < \infty$ ³. Given $\alpha > 0$, there exists a Ricci-flat metric ω such that $[\omega] = [\omega_0]$ and $\omega^2 = \alpha \Omega \wedge \bar{\Omega}$ for a fixed meromorphic volume form Ω with a simple pole along D . Moreover, one has*

$$\|\nabla^k(\omega - \omega_{\operatorname{mod}})\|_{g_{\operatorname{mod}}} \lesssim O(r^{-k-2})$$

for any $k \in \mathbb{N}$.

Proof. From [23, Eq. (3.25)], Hein constructed a background Kähler form ω_a on X such that

- (1) $[\omega_a] = [\omega_0] \in H^2(X, \mathbb{R})$.

³For our purpose, we will only take those Kähler forms on Y and restrict to X .

(2) There exists $0 < r_1 < r_2$ such that

- $\omega_a = \omega_0$ in $U_{r_2}^c$.
- $\omega_a = T^* \omega_{sf, \varepsilon}(\alpha)$ on U_{r_1} , where T is the fibrewise translation by a holomorphic section over U_{r_1} . In particular, $\omega_a^2 = \alpha \Omega \wedge \bar{\Omega}$ on $p^{-1}(U_{r_1})$.

We will modify ω_a such that it satisfies the integrability condition

$$\int_X \omega_a^2 - \alpha \Omega \wedge \bar{\Omega} = 0.$$

For $0 < r < s < r_1$, we define $\beta_{r,s}$ to be a 2-form on B such that $\beta_{r,s} = \frac{1}{2} \chi(|u|) f(|u|) du \wedge d\bar{u}$, where $\chi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a cut-off function with $\chi(t) \in [0, 1]$ such that $\chi(t) = 1$ on $U_s \setminus U_r$ and

$$f(t) = \begin{cases} \frac{\nu |\log t|}{2\pi \varepsilon t^4}, & \text{if } D \text{ is of type } I_\nu^*, \nu > 0, \\ 1/t^4, & \text{if } D \text{ is of finite monodromy.} \end{cases}$$

By a direct calculation, we have $\omega_a \pm \beta_{r,s}$ which is again a Kähler form. Notice that by another straightforward calculation, $\int_X \omega_a \wedge \beta_{r,s} \rightarrow \infty$ as $r \rightarrow 0$. Thus we have

$$\begin{aligned} \int_X (\omega_a + \beta_{r,s})^2 - \alpha \Omega \wedge \bar{\Omega} &\rightarrow \infty, \text{ for } r \rightarrow 0, \\ \int_X (\omega_a^2 - \beta_{r,s})^2 - \alpha \Omega \wedge \bar{\Omega} &\rightarrow -\infty, \text{ for } r \rightarrow 0. \end{aligned}$$

Then there exists $t' \in [-1, 1]$ such that $\omega_a + t' \beta_{r,s}$ achieves the integrability condition for some r by intermediate value theorem.

With the integrability condition, the existence of the Ricci-flat metric in the same cohomology class (actually in the same Bott–Chern cohomology class) is guaranteed by [34, Theorem 1.1]. Then [23, Proposition 2.9] provides the decay to the model metrics

$$\|\nabla^k(\omega - \omega_{mod})\|_{g_{mod}} \lesssim O(r^{-k-\mathfrak{n}})$$

for any $k \in \mathbb{N}$. Here \mathfrak{n} can be taken to be 2 if D is of type II, III, IV, or I_ν^* and the theorem is proved. For the case when D is of the type II^* , III^* , or IV^* , [7, Proposition 5.1] showed that there exists a gravitational instanton with hyperKähler triple of the same cohomology class and the required asymptotic. \square

To prove the surjectivity of the period map (Theorem 3.13), we also need the following lemma.

Lemma 3.17. *Given $[\omega] \in H^2(X, \mathbb{R})$ such that $[\omega]$ is positive on every holomorphic curve in X , then there exists a Kähler class $[\omega_Y] \in H^2(Y, \mathbb{R})$ such that $[\omega_Y]|_X = [\omega]$.*

Proof. From the dual of (3.2), any two liftings of $[\omega]$ in $H^2(Y, \mathbb{R})$ are differed by a linear combination of $\text{PD}([D_i])$. Recall that a cohomology class $[\omega_Y] \in H^2(Y, \mathbb{R})$ is Kähler if it is positive on every holomorphic curve in Y by [15, Theorem 0.1]. Holomorphic curves in Y are either those that avoid D , those that has positive intersection with D or the components of D . Choose any lifting $[\omega'_Y] \in H^2(Y, \mathbb{R})$ of $[\omega]$ which is positive on the holomorphic curves of the first kind.

For the case D is not of type IV, the dual intersection complex of D is a tree. We choose a root and label the components of D with respect to the partial ordering given by the distance to the root, say D_1, \dots, D_n . In the case when D is of type IV, we will simply choose arbitrary labeling. Then we can inductive solve a_i such that $([\omega'_Y] + \sum_i a_i \text{PD}([D_i])).[D_j] = \varepsilon > 0$ for $j = 1, \dots, n-1$. Since $[\omega'_Y].[D] > 0$, we have $([\omega'_Y] + \sum_i a_i \text{PD}([D_i])).[D_n] > 0$ by choosing ε small enough.

From [2, Proposition 6.2], the cone of effective curves of Y is the convex hull of a set of extremal rays given by rational curves and possibly $[D]$, accumulating at most to $\mathbb{R}_+[D]$. Since $[\omega'_Y] + \sum_i a_i \text{PD}([D_i])$ is positive on $[D]$, we have $[\omega_Y] = [\omega'_Y] + \sum_i a_i \text{PD}([D_i]) + t \text{PD}([D])$ is also positive on the curves of the second kind for $t \gg 0$. Thus, $[\omega_Y]$ is a Kähler class we are looking for. \square

Proof of Theorem 3.13. The proof is similar to the proof of Theorem 2.5, where Theorem 3.3, and Theorem 3.16 are the replacements for Theorem 2.6, and Theorem 2.10 [34]. \square

Finally, we comment on a Torelli theorem of the pairs (Y, D) . It is known that the periods of the holomorphic $(2, 0)$ -form on $X = Y \setminus D$ determined the isomorphism class of the pair (Y, D) when D is an I_k -fibre [21] and when D is smooth [1] (see also Appendix A). However, it seems that there is less study when D has components with multiplicities. Here we take the advantage of the Torelli theorem of gravitational instantons of ALG or ALG* [9] and give an optimal result when D is not reduced.

Proposition 3.18. *Assume that D is of type II, III, IV or I_ν^* with $\nu \in \{0, 1, 2, 3, 4\}$. Let (Y_1, D) and (Y_2, D) be two pairs of rational elliptic surfaces with prescribed singular fibres. Let Ω_i be the meromorphic $(2, 0)$ -form on Y_i with a simple pole along D with the residue of Ω_i being fixed and there exists a diffeomorphism $f: X_2 \rightarrow X_1$ such that $f^*\Omega_1, \Omega_2$ have the same periods on X . For $\nu = 0$, we further require that the limits of j -invariants are the same. Then there exists an isomorphism $(Y_1, D) \cong (Y_2, D)$ as pairs.*

Proof. From Theorem 3.16 and Lemma 3.17, there exists Ricci-flat metrics ω_i on X_i such that $f^*[\omega_1] = [\omega_2]$. Then by Torelli theorem of ALG (or ALG*) gravitational instantons [9], one may modify the diffeomorphism f such that $f^*\omega_1 = \omega_2$ and $f^*\Omega_1 = \Omega_2$. In particular, f is a biholomorphism and thus Y_1 and Y_2 are birational to each other. Therefore, there exist a compact complex surface Y and birational morphisms $f_1: Y \rightarrow Y_1$ and $f_2: Y \rightarrow Y_2$ such that f_i are compositions of sequences of simple blow-ups. Since $Y_1 \setminus X_1 \cong Y_2 \setminus X_2$ both biholomorphic to D , f_1 and f_2 must undo each other. In other words, $f: X_2 \rightarrow X_1$ can be extended to a biholomorphism $Y_2 \rightarrow Y_1$, sending the one boundary divisor isomorphically to another. \square

Remark 3.19. (1) Here the condition fixing the residue of Ω_i is the substitution of the normalization condition in [18, p. 22].
(2) The injectivity of the period map is only true when the metric is asymptotic to the model of order 2 when D is of type II*, III*, or IV* [6, 7] and thus the argument of the proof for Proposition 3.18 breaks down in these cases. This is because that there are isotrivial degenerations of rational elliptic surfaces with such prescribed fibres.

APPENDIX A. TORELLI THEOREM FOR LOG CALABI-YAU SURFACES

The following Torelli theorem is implicitly hidden in the work of [18, 30] and is known to experts. However, the authors cannot find the exact statement in the literature and so we include the proof here to make the article self-contained.

Theorem A.1. *Consider two pairs consisting of a weak del Pezzo surface⁴ and a smooth anti-canonical divisor (Y, D) and (Y', D') . Assume that there exists a deformation family of pairs $(\mathcal{Y}, \mathcal{D}) \rightarrow B$ such that both (Y, D) and (Y', D') are fibres. Denote by $\mu: H^2(X, \mathbb{C}) \rightarrow H^2(X', \mathbb{C})$ the isomorphism via some parallel transport, where $X = Y \setminus D$ and $X' = Y' \setminus D'$. If there exist meromorphic volume forms Ω on Y and Ω' on Y' , with simple poles along D and D' (respectively) such that $\mu([\Omega]) = [\Omega']$, then there exists an isomorphism of pairs $f: (Y', D') \cong (Y, D)$ ⁵*

First we review some lattice theory. Denote by $\mathbb{Z}^{1,n}$ the lattice generated by h, e_1, \dots, e_n with the pairing $h^2 = 1$, $h \cdot e_i = 0$, and $e_i \cdot e_j = -\delta_{ij}$. Set $f = 3h - \sum_i e_i$, $\alpha_0 = e_0 - e_1 - e_2 - e_3$, and $\alpha_i = e_i - e_{i+1}$. Let $\mathbb{L}_n \subseteq \mathbb{Z}^{1,n}$ be the sublattice generated by α_i 's. If Y is a blow-up of \mathbf{P}^2 at n points, then $\text{Pic}(Y) \cong \mathbb{Z}^{1,n}$. If D is a smooth irreducible anti-canonical divisor of Y and $\Lambda(Y, D)$ denotes the sublattice of $\text{Pic}(Y)$ with zero pairing with $[D]$, then $\Lambda(Y, D) \cong \mathbb{L}_n$.

Consider the data (Y, D) , Ω , and a homology class $\delta \in H_2(Y, \mathbb{Z})$ such that $\delta \cdot D = 0$. From the long exact sequence (2.1), we can find a representative $\tilde{\delta}$ of δ contained in X and thus $\int_{\tilde{\delta}} \Omega$ is defined. Again from (2.1) and the residue theorem, we have

$$\int_{\delta} \Omega := \int_{\tilde{\delta}} \Omega \in \mathbb{C}$$

is well-defined. In particular, the complex structure of D is determined by $[\Omega]|_{\text{Im}(H^1(D, \mathbb{Z}))}$ from the residue formula. The meromorphic volume form Ω then determines the period map

$$\varphi_{\Omega}: \Lambda(Y, D) \cong H_2(X, \mathbb{Z}) / \text{Im} H^1(D, \mathbb{Z}) \rightarrow D \cong \text{Pic}^0(D),$$

which is similar to the period map of K3 surfaces. Notice that φ_{Ω} is independent of the \mathbb{C}^* -scaling of Ω . On the other hand, one can have another notion of period $\varphi_{(Y, D)}$ in algebraic geometry similar to the one introduced in Gross-Hacking-Keel [21],

$$\begin{aligned} \varphi_{(Y, D)}: \Lambda(Y, D) &\rightarrow \text{Pic}^0(D) \\ L &\mapsto L|_D. \end{aligned}$$

Lemma A.2. *The two notions of periods coincide, i.e., $\varphi_{\Omega} = \varphi_{(Y, D)}$.*

Proof. We first consider the case when Y is obtained by blowing up of a smooth cubic \underline{D} at distinct points x_1, \dots, x_b . Let $Y \rightarrow \mathbf{P}^2$ be the blow-up and E_i the exceptional divisors. Denote by D the proper transform of \underline{D} and H the pullback of the hyperplane class on \mathbf{P}^2 . Then $\Lambda(Y, D)$ is spanned by elements of the form $E_i - E_j$ for and $H - E_i - E_j - E_k$. It is easy to see that both φ_{Ω} and $\varphi_{(Y, D)}$ are linear and thus it suffices to prove the two coincide on the generators. Let $p = E_i \cap D$, $q = E_j \cap D$. Then $\varphi_{(Y, D)}(E_j - E_i) = \mathcal{O}_D(q - p)$. On the other hand, one can find a smooth curve γ in D connecting q and p and denote by $C_{\gamma} \subseteq X$ the S^1 -bundle over γ . One can glue C_{γ} into the complement in $E_j - E_i$ of small discs around

⁴More generally it is true for successive blow-ups of \mathbf{P}^2 on a smooth irreducible anti-canonical divisor.

⁵In general, f^* and μ may differ by reflection of (-2) -curves and might not coincide.

p, q to obtain a 2-cycle in X which is homologous to $E_j - E_i$. Denote the 2-cycle by C_{ji} . Then one has

$$\varphi_\Omega(E_j - E_i) = \int_{C_{ji}} \Omega = \int_\gamma dz = q - p = \varphi_{(Y,D)}(E_j - E_i),$$

where the second equality comes from the residue and the last equality holds via the identification $D \cong \text{Pic}^0(D)$. Now we consider the case Y is successive blow up (possibly infinitely near) points on \mathbf{P}^2 at the smooth cubic. Notice that given a family of pairs (Y_t, D_t) of successive blow up (possibly infinitely near) points on \mathbf{P}^2 at the smooth cubic over a parameter space T , the two periods φ_Ω and $\varphi_{(Y,D)}$ are both continuous with respect to $t \in T$. Since one can take T such that generic points correspond to blow up of \mathbf{P}^2 at distinct points on a smooth cubic, this proves the lemma for the case when Y is a blow up of \mathbf{P}^2 . When $Y \cong \mathbf{P}^1 \times \mathbf{P}^1$, $\Lambda(Y, D)$ is generated by $F_1 - F_2$, where F_i are the fibres of different rulings. By choosing the generic fibre representative, we may assume that F_i intersect D at p_i, q_i . Choose smooth curves γ_p (and γ_q) connecting p_1, p_2 (and q_1, q_2 respectively) without intersection. Then the proof is reduced to the above case. When $Y \cong \mathbb{F}_2$, then $\Lambda(Y, D)$ is generated by the unique (-2) -curve and both periods simply vanish. \square

Proof of Theorem A.1. We will first prove the case when Y is not isomorphic $\mathbf{P}^1 \times \mathbf{P}^1$ nor \mathbb{F}_2 . The marking μ and the period $\varphi_{(Y,D)}$ determines a homomorphism $\mathbb{L}_n \rightarrow \text{Pic}^0(D)$. Corollary 4.4 [30] implies that it uniquely determines a homomorphism $\mathbb{Z}^{1,n} \rightarrow \text{Pic}(D)$. Theorem 6.4 [30] says that such a homomorphism recovers the blow up loci of $Y \rightarrow \mathbf{P}^2$ up to the Weyl group action and thus uniquely determines the pair (Y, D) up to isomorphism.

Now we will consider the case $Y = Y' = \mathbf{P}^1 \times \mathbf{P}^1$ and $[F_1], [F_2]$ are the homology classes of two rulings. Assume that D, D' are smooth anti-canonical divisors in $Y = Y' = \mathbf{P}^1 \times \mathbf{P}^1$ such that

$$\varphi_{(Y,D)}([F_1] - [F_2]) = \varphi_{(Y',D')}([F_1] - [F_2]).$$

From the group law on elliptic curves D and D' , there exist $p \in Y$, $p' \in Y'$ such that $\varphi_{(\tilde{Y}, \tilde{D})} = \varphi_{(\tilde{Y}', \tilde{D}')}}$, where $\tilde{Y} = \text{Bl}_p Y$, $\tilde{Y}' = \text{Bl}_{p'} Y'$ and \tilde{D}, \tilde{D}' are the corresponding proper transforms. Notice that $\tilde{Y} \cong \tilde{Y}'$ are isomorphic to del Pezzo surface of degree 7. From the previous part of the proof, we have the isomorphism of the pairs $(\tilde{Y}, \tilde{D}) \cong (\tilde{Y}', \tilde{D}')$. In a del Pezzo surface of degree 7 there are three (-1) -curves and exactly one of them intersects the other two. Therefore, such (-1) -curve in \tilde{Y} is identified with a corresponding (-1) -curve in \tilde{Y}' via the isomorphism of the pairs $(\tilde{Y}, \tilde{D}) \cong (\tilde{Y}', \tilde{D}')$. Blowing down such (-1) -curves leads to the isomorphism of the pair $(Y, D) \cong (Y', D')$. The proof of the case \mathbb{F}_2 is similar. \square

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