Strong *CP* and flavor in multi-Higgs theories

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We introduce a class of multi-Higgs doublet extensions of the Standard Model that solve the strong *CP* problem with profound consequences for the flavor sector. The Yukawa matrices are constrained to have many zero entries by a "Higgs-flavor" symmetry, $G_{\rm HF}$, that acts on Higgs and quark fields. The violation of both *CP* and $G_{\rm HF}$ occurs in the Higgs mass matrix so that, for certain choices of $G_{\rm HF}$ charges, the strong *CP* parameter $\bar{\theta}$ is zero at tree level. Radiative corrections to $\bar{\theta}$ are computed in this class of theories. They vanish in realistic two-Higgs doublet models with $G_{\rm HF} = \mathbb{Z}_3$. We also construct realistic three-Higgs models with $G_{\rm HF} = {\rm U}(1)$, where the one-loop results for $\bar{\theta}$ are model-dependent. Requiring $\bar{\theta} < 10^{-10}$ has important implications for the flavor problem by constraining the Yukawa coupling and Higgs mass matrices. Contributions to $\bar{\theta}$ from higher-dimension operators are computed at one loop and can also be sufficiently small, although the hierarchy problem of this class of theories is worse than in the Standard Model.

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I. INTRODUCTION

The violation of parity (P) and its combination with charge conjugation (*CP*) in the strong interactions, described by the parameter $\bar{\theta}$, are constrained by the experimental limit on the neutron electric dipole moment (nEDM): $\bar{\theta} \leq 10^{-10}$ [1]. Hence, $\bar{\theta}$ is the smallest dimensionless parameter of the Standard Model (SM). This small value is particularly puzzling because of the lack of theoretical understanding of it, and because the dimensionless parameter describing *CP* violation in the weak interactions is order unity. Within the SM, the parameter $\bar{\theta}$ has two sources. Firstly, the Lagrangian of quantum chromodynamics (QCD), restricted just by Lorentz and gauge invariance, contains a *CP*-odd term, $\theta_{\rm QCD} g_s^2/(32\pi^2) G_a^{\mu\nu} \tilde{G}_{a,\mu\nu}$, where $G_a^{\mu\nu}$ is the gluon field strength tensor, g_s is the strong coupling constant, and $\tilde{G}_{a,\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} G_a^{\alpha\beta}$. Moreover, $\bar{\theta}$ receives a contribution through the chiral transformation needed to diagonalize the quarks' Yukawa matrices y^u and y^d,

$$\bar{\theta} = \theta_{\rm QCD} + \arg\det(y^u y^d). \tag{1}$$

The two contributions to $\bar{\theta}$ arise from very different physics and have no reason to cancel.

Solutions based on an underlying *CP* or P symmetry at high energies that is softly or spontaneously broken have profound consequences for the flavor sector of the SM. In these theories, the bare parameter $\theta_{\rm QCD}$ in Eq. (1) is forced to be zero by either *CP* or P, and one must explain why $\arg\det(y^u y^d)$ is so small even though the order unity phase of the observed *CP* violation in weak interactions also arises from the quark mass matrices.

In the case of *CP*, Nelson and Barr [2,3] introduced N_Q extra generations of heavy vectorlike quarks, Q, that mix with the SM quarks, q, through the interactions with new scalars, so that the Yukawa interactions are now described by square $3 + N_Q$ dimensional matrices. Careful model building arranges for the Qq block of these matrices to vanish, while all *CP*-violating parameters appear in the qQ block. The observed *CP* violation of the weak interactions arises from the mixing between Q and q. As well as being enlarged, the Yukawa matrices involve more parameters than in the SM. Different approaches with spontaneous *CP* breaking have been proposed that make use of the non-renormalization theorems of supersymmetry [4–7].

There is another scheme based on soft or spontaneous breaking of *CP* symmetry—it predates the Nelson-Barr framework, but has not been widely studied and appears to be largely forgotten. Instead of adding additional quark generations with *CP* violation in the mixing with SM quarks, it augments the SM by having more than one copy of the Higgs doublet, $\phi \to \phi_\alpha$. *CP* violation appears only in

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the Higgs mass matrix. The quark and Higgs fields transform under a flavor symmetry in such a way that the Yukawa matrices are highly restricted: each entry in the quark mass matrices is either zero or arises from a single Higgs field. Hypercharge gauge symmetry forces all Yukawa interactions of the up (down) quarks to involve ϕ_α (ϕ_α^*). If the determinant of the up quark mass matrix is proportional to a single product of ϕ_a fields, $\Pi(\phi_\alpha)$, and if the same product appears in the determinant of the down quark masses, $\Pi(\phi_\alpha^*)$, then $\arg\det y_u y_d = 0$ by a cancellation between up and down sectors.

The hierarchy problem of the SM will not be made significantly worse if the effective field theory with multiple Higgs doublets is valid over a small energy interval. As in Nelson-Barr theories, one must ensure that the radiative contributions to $\bar{\theta}$ are sufficiently small.

Georgi constructed the first model of this type in 1978 [8]. A three-generation $SU(5)$ grand unified theory, with flavor group $\mathbb{Z}_{16}$, was constructed in 1979 by Mohapatra and Wyler [9], and an extension of the SM at the weak scale was constructed by Glashow in 2001 [10]. In light of modern data, only Glashow's model is phenomenologically viable, and we will show that it does not convincingly solve the strong CP problem.

In this letter we provide a general framework for multi-Higgs extensions of the SM that solve the strong CP problem at tree level by a combination of a flavor symmetry that strongly restricts the form of Yukawa interactions, together with soft or spontaneous breaking of this symmetry and of CP in the Higgs mass matrix.

II. FRAMEWORK

The Standard Model is minimally extended by increasing the number of Higgs doublets from 1 to N, all with the same quantum numbers. There are no additional fermions and no extension to the gauge group. All but one of the Higgs doublets acquires a mass of order M, and below M the minimal SM is the effective field theory (EFT) describing the model. The lower bound on M arises from limits on flavor changing neutral currents and from direct searches for scalar mesons at the LHC, and is typically expected to be about a TeV. We assume that the heavy Higgs bosons obey these mass bounds. There is no upper bound, although the hierarchy problem of the SM becomes more severe as M is increased. The UV completion of the multi-Higgs theory occurs at a scale Λ. For the hierarchy problem of the multi-Higgs theory to be mild, Λ should not be too far above M.

In this theory, the Yukawa interactions are described by $\mathcal{L}_Y = x^\alpha_{ij} q_i \bar{u}_j \phi_\alpha + \tilde{x}^\alpha_{ij} q_i \bar{d}_j \phi^\dagger_\alpha$, where q_i are quark doublets and $\bar{d}_j, \bar{u}_j$ are singlet antiquarks, all left-handed Weyl fields, and the Higgs doublets are ϕ_α, with $\alpha = 1 \ldots N$. These fields transform under a "Higgs-flavor" symmetry, $G_{\rm HF}$, so that the Yukawa couplings x^α_{ij} and $\tilde{x}^\alpha_{ij}$ are sparse with many zero entries. The Higgs fields all transform differently under $G_{\rm HF}$ so that the Higgs potential can be written in the form

$$V = M^2_\alpha \phi^\dagger_\alpha \phi_\alpha + \mu^2_{\alpha\beta} \phi^\dagger_\alpha \phi_\beta + \lambda_{\alpha\beta} \phi^\dagger_\alpha \phi_\alpha \phi^\dagger_\beta \phi_\beta + \lambda_{\alpha\beta\gamma\delta} \phi^\dagger_\alpha \phi_\beta \phi^\dagger_\gamma \phi_\delta + \text{H.c.} \quad (2)$$

While the quartics $\lambda_{\alpha\beta}$ are present in all models, the quartics $\lambda_{\alpha\beta\gamma\delta}$ are model-dependent, occurring only when there happens to be a $G_{\rm HF}$ invariant with at least three of $(\alpha, \beta, \gamma, \delta)$ different. CP and $G_{\rm HF}$ are conserved in all interactions and masses, except for the soft mass terms described by $\mu^2_{\alpha\beta}$, which are complex and have $\alpha \neq \beta$. While $|\mu^2_{\alpha\beta}|$ may take a variety of values, they are typified by a scale μ^2. $N-1$ of the Higgs doublets have mass of order the scale M and the other is fine-tuned to be much lighter. Hence, the relevant mass scales and EFTs for this framework are illustrated in Fig. 1. The unitary matrix U rotates the Higgs fields from flavor to mass eigenstates, $\phi_\alpha = U_{\alpha\beta} h_\beta$, and only h_1, which denotes the SM Higgs, acquires a vacuum expectation value, $\langle h_1 \rangle = v \simeq 174$ GeV. Below the scale M, we obtain the SM EFT with SM Yukawa coupling matrices

$$y^u_{ij} = x^\alpha_{ij} U_{\alpha 1}, \qquad y^d_{ij} = \tilde{x}^\alpha_{ij} U^*_{\alpha 1}. \quad (3)$$

The phases appearing in y^u and y^d arise from matrix elements of U and U^*, and hence are correlated. We study theories

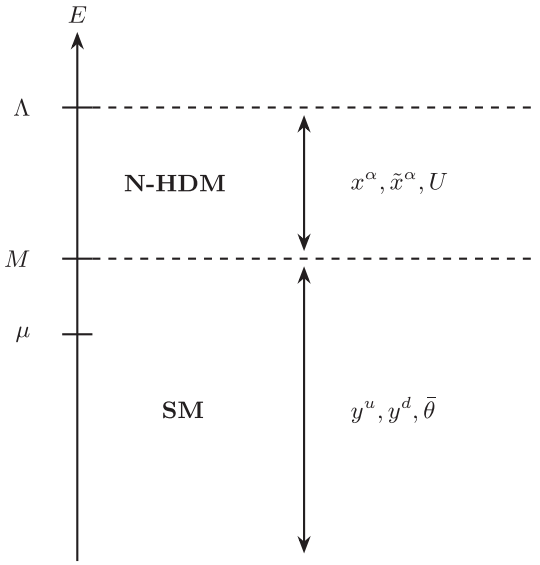

FIG. 1. The framework: energy scales, EFTs, and their flavor parameters. Λ is the UV cutoff of theories with N Higgs doublets, all with the same gauge quantum numbers. All but one of these Higgs doublets acquires a mass of order M. Below M, the EFT is the minimal SM.

where the Yukawa interactions have flavor determinants that involve a *single* product of scalar fields: $\det(x^\alpha \phi_\alpha) \propto \Pi(\phi_\alpha)$ and $\det(\tilde{x}^\alpha \phi^*_\alpha) \propto \Pi(\phi^*_\alpha)$. The determinants of y^u and y^d then involve the same single product of $U_{\alpha 1}$ matrix elements,

$$\det(y^u) \propto \Pi(U_{\alpha 1}), \qquad \det(y^d) \propto \Pi(U^*_{\alpha 1}), \tag{4}$$

so that $\bar{\theta}$ vanishes at tree level. For example, in the theory with three Higgses studied below, the Yukawa matrices are such that $\det(y^u) \propto U_{11} U_{31} U_{31}$ and $\det(y^d) \propto U^*_{11} U^*_{31} U^*_{31}$, even though the textures of the two matrices are quite different. However, the experimental limit from the nEDM is so severe that $\bar{\theta} = 0$ at tree level may not be sufficient to ensure that these theories solve the strong *CP* problem. Given the softness of *CP* violations, loop contributions to $\bar{\theta}$ arise as threshold corrections at the mass scale M of the heavy Higgs doublets and take the form $\delta\bar{\theta} = f(x^\alpha_{ij}, \tilde{x}^\alpha_{ij}, U_{\alpha\beta})$. In the SM, there are two distinct small number problems associated with the quark mass matrices $y^{u,d}$: the strong *CP* problem and the flavor hierarchy problem. In the latter, small Yukawa couplings are needed to account for the small quark masses and mixings. From (3), we see that small values for some matrix elements of $y^{u,d}$ may be partially accounted for by small values of $|U_{\alpha 1}|$ for $\alpha \neq 1$, that is, by $\mu \ll M$. At the same time, $\delta\bar{\theta}$ is proportional to $U_{\alpha\beta}$, so that it is advantageous for them to be small. Hence, our framework offers the prospect of making progress on small flavor hierarchies as well as solving the strong *CP* problem.

III. RADIATIVE CORRECTIONS

Radiative corrections to Yukawa interactions modify the quark mass matrices and can lead to a nonvanishing value for $\bar{\theta} = \arg\det(y^u y^d)$. Below the scale M, the SM provides the effective description of the model, and it was shown by Ellis and Gaillard [11] that the first finite contribution to $\bar{\theta}$ appears at three-loop order (giving $\bar{\theta} \sim 10^{-16}$) and there is no divergent contribution before seven loops. Between Λ and M there may be additional contributions through loop diagrams with a scalar exchange[1], as illustrated in Fig. 2.

The corrections to the SM Yukawa coupling matrix y^u at renormalization point μ read

$$\delta y^u = \frac{1}{16\pi^2} \tilde{x}^\alpha \tilde{x}^{\dagger\beta} x^\gamma U_{\beta 1} \times \sum_\delta U^*_{\alpha\delta} U_{\gamma\delta} \left(\frac{1}{\epsilon} + 1 + \log\left(\frac{\mu^2}{M^2_\delta} \right) \right), \tag{5}$$

and a similar expression for δy^d. Here, dimensional regularization has been used, and sums on α, β, and γ are understood. The index δ runs over the N mass eigenstate scalars, h^δ. As already commented above, the softness of *CP* violations ensures that loop contributions to $\bar{\theta}$ arise as threshold corrections at the mass scale M. In fact, the unitarity of the U matrix implies that terms involving $(1/\epsilon + 1 + \log\mu^2)$ are proportional to $\tilde{x}^\alpha \tilde{x}^{\dagger\beta} x^\alpha U_{\beta 1}$ and $x^\alpha x^{\dagger\beta} \tilde{x}^\alpha U^*_{\beta 1}$ for δy^u and δy^d, respectively. One can easily prove that $\tilde{x}^\alpha \tilde{x}^{\dagger\beta} x^\alpha (x^\alpha x^{\dagger\beta} \tilde{x}^\alpha)$ has the same matrix structure of $x^\beta (\tilde{x}^\beta)$ and, since at tree level $y^u = x^\beta U_{\beta 1}$ and $y^d = \tilde{x}^\beta U^*_{\beta 1}$, this does not lead to corrections to $\bar{\theta}$ in the multi-Higgs theory. Keeping only the $\log(M^2_\delta)$ terms relevant for $\bar{\theta}$,

$$\delta y^u = \frac{1}{16\pi^2} \tilde{x}^\alpha \tilde{x}^{\dagger\beta} x^\gamma U_{\beta 1} \left[U^*_{\alpha 1} U_{\gamma 1} \log\left(\frac{M^2_2}{m^2_h} \right) + \sum_{\delta=3}^{N} U^*_{\alpha\delta} U_{\gamma\delta} \log\left(\frac{M^2_2}{M^2_\delta} \right) \right], \tag{6}$$

and a similar result follows for δy^d. The first term above is the one-loop correction due to the SM Higgs in the SM EFT below M_2, which does not contribute to $\bar{\theta}$ [11]. The only contribution to $\bar{\theta}$ comes from the second term in Eq. (6) and depends logarithmically on the mass ratios of the heavy scalars.

An important result follows from Eq. (6): there are no corrections to $\bar{\theta}$ from one-loop scalar exchange in theories with two Higgs doublets. A similar statement can be made for higher-order corrections to the quark Yukawa matrices. Below M, we recover the SM EFT and, therefore, there is a negligible finite contribution to $\bar{\theta}$ at three loops and no divergent contributions up to six loops [11]. Above M, where the two-Higgs doublet model is the correct EFT, the finite corrections to the Yukawa matrices can be constant terms, terms proportional to powers of m_h/M_2, and terms proportional to $\log(M_2/m_h)$ (due to the softness of *CP* violation). At two loops, we check diagram by diagram that the constant pieces do not contribute to $\bar{\theta}$. This is easy to prove using the unitarity of the U matrix and noting that each propagator for the scalar H_β comes with the insertion of a $U_{\alpha\beta}$ and a $U^*_{\gamma\beta}$ matrix. Terms proportional to powers of

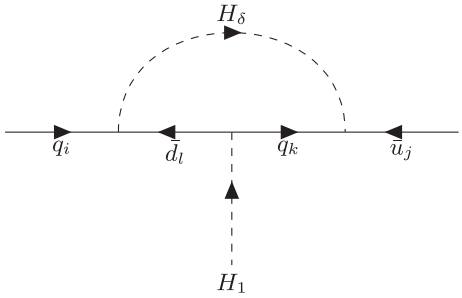

FIG. 2. Feynman diagrams for the one-loop *CP*-violating contributions to the quark Yukawa couplings x^α_{ij}. Similar diagrams contribute to $\tilde{x}^\alpha_{ij}$.

[1]There are no contributions from wave function renormalization as they have the structure $\delta y^u = y^u B$, where B is a hermitian matrix.

m_h/M_2 correspond to higher-dimensional operators in the SM EFT and are negligible in the limit where $M_2 \gg m_h$. Finally, terms proportional to $\log(M_2/m_h)$ can be understood as coming from renormalization group (RG) scaling in the SM EFT below the heavy Higgs mass, where we know there are no corrections to $\bar{\theta}$ at two loops. Similar arguments apply to any loop order; in theories with two Higgs doublets, we conclude there are no radiative corrections to $\bar{\theta}$ beyond those of the SM. On the other hand, for theories with $N > 2$, there are corrections to $\bar{\theta}$ already at one loop. These are proportional to a combination of the $x, \tilde{x}$, and U matrices and are, therefore, model-dependent. In general, one needs to compute $\bar{\theta} = \arg\det[(y^u + \delta y^u)(y^d + \delta y^d)]$ to assess whether a theory solves the strong CP problem for a reasonable choice of parameters.

IV. MODELS

A complete study and categorization of theories within the framework described above is beyond the scope of this work. In the following, we discuss a representative example that captures the main features of this class of models.

We consider as an example theories with three Higgs doublets and $G_{\rm HF} = \mathrm{U}(1)$. Restricting the charges of scalar and quark fields to be smaller than two, and requiring the theory to generate quark masses, Cabibbo-Kobayashi-Maskawa (CKM) mixing angles at first order in the off-diagonal Yukawa entries, and a nonzero Jarkslog invariant, we found 2160 possible charge assignments. In each of these theories, the scalar sector is the same, with three real diagonal and three complex off-diagonal mass parameters. For simplicity and with the same conventions as in Eq. (2), we assume $\epsilon_{\alpha\beta} = \mu^2_{\alpha\beta}/M^2 \ll 1$, such that the off-diagonal elements of the rotation matrix are $U_{ij} = -U^*_{ij} \simeq \epsilon_{ij} e^{i\beta_{ij}}$, while $U_{ii} \simeq 1$. The number of parameters in the quark Yukawas matrices depends on the number of nonzero entries, as dictated by the flavor symmetry. A representative $\mathrm{U}(1)_{\rm HF}$ charge assignment, which has the virtue of reducing the number of free parameters, is shown in Table I, leading to the Yukawa interactions

$$\mathcal{L}_{\rm Y} = (q_1 \;\; q_2 \;\; q_3) \begin{pmatrix} 0 & 0 & x_{13}\phi_3 \\ 0 & x_{22}\phi_1 & 0 \\ x_{31}\phi_3 & x_{32}\phi_2 & x_{33}\phi_1 \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} + (q_1 \;\; q_2 \;\; q_3) \begin{pmatrix} 0 & \tilde{x}_{12}\phi_3^\dagger & 0 \\ \tilde{x}_{21}\phi_3^\dagger & \tilde{x}_{22}\phi_1^\dagger & \tilde{x}_{23}\phi_2^\dagger \\ 0 & 0 & \tilde{x}_{33}\phi_1^\dagger \end{pmatrix} \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \end{pmatrix}. \quad (7)$$

TABLE I. $\mathrm{U}(1)_{\rm HF}$ charges for the representative three-Higgs doublet model discussed in this letter.

	ϕ_1	ϕ_2	ϕ_3	q_1	q_2	q_3	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$
$\mathrm{U}(1)_{\rm HF}$	0	2	−1	0	1	−1	2	−1	1	−2	−1	1

The SM Yukawa coupling matrices are obtained by inserting $\phi_\alpha = U_{\alpha 1} h_1$. At tree level, $\bar{\theta} = 0$ as $\det(y^u) = x_{13}x_{22}x_{31}U_{11}U_{31}U_{31}$ and $\det(y^d) = \tilde{x}_{12}\tilde{x}_{21}\tilde{x}_{33}U^*_{11}U^*_{31}U^*_{31}$. This theory has 13 real parameters and three phases. Quark masses and mixings fix nine real parameters and the CKM fixes one phase, according to

$$\begin{aligned} &y_u \simeq x^u_{31}|U^*_{31}V_{ub}|, \quad y_c \simeq x^u_{22}|U_{11}|, \quad y_t \simeq x^u_{33}|U_{11}|, \\ &y_d \simeq x^d_{21}|U^*_{31}V_{us}|, \quad y_s \simeq x^d_{22}|U^*_{11}|, \quad y_b \simeq x^d_{33}|U^*_{11}|, \\ &|V_{us}|y_s \simeq x^d_{12}|U^*_{31}|, \quad |V_{ub}|y_t \simeq x^u_{13}|U_{31}|, \\ &|V_{cb}|y_b \simeq x^d_{23}|U^*_{21}|, \quad \delta_{\rm CKM} \simeq \beta_{12} + 2\beta_{13}, \end{aligned} \quad (8)$$

to leading order for each of V_{ub} (from y^u), V_{us}, and V_{cb} (from y^d). This leaves four free real parameters, $|U_{21}|, |U_{31}|, |U_{23}|$, and x_{32}. Requiring x_{13} and $\tilde{x}_{23}$ to be perturbative gives the lower bounds $\epsilon_{13} \gtrsim 0.004$ and $\epsilon_{12} \gtrsim 0.001$. Furthermore, we have the requirement of sufficiently small one-loop corrections to $\bar{\theta}$. Using Eq. (6), setting ϵ_{23} to its minimal natural value, $\epsilon_{23} \simeq \epsilon_{12}\epsilon_{13}$, and taking phases of order unity, the dominant terms are

$$\delta\bar{\theta} \simeq 10^{-5} x_{32}\epsilon_{12} - 3 \times 10^{-9}\epsilon^2_{13}. \quad (9)$$

For example, taking $\epsilon_{12} \simeq 10^{-3}, \epsilon_{13} \simeq 10^{-2}, \epsilon_{23} \simeq 10^{-5}$, and $x_{32} \simeq 10^{-2}$ gives $\delta\bar{\theta} \simeq 10^{-10}$, with numerical values of the Yukawa matrices,

$$x \simeq \begin{pmatrix} 0 & 0 & 0.4 \\ 0 & 0.005 & 0 \\ 0.1 & 0.01 & 1 \end{pmatrix}, \quad \tilde{x} \simeq \begin{pmatrix} 0 & 0.006 & 0 \\ 0.005 & 0.0003 & 0.8 \\ 0 & 0 & 0.02 \end{pmatrix}. \quad (10)$$

This theory illustrates how $G_{\rm HF}$ leads to a close relation between the strong CP and flavor problems. In the SM, all we know about the quark Yukawa couplings $y^{u,d}$ is that they are general complex matrices that have certain eigenvalues, ranging over 5 orders of magnitude, and are diagonalized by certain mixing angles. In this theory, solving the strong CP problem leads to a very different set of real flavor matrices, as shown in (10). Many entries vanish, and the hierarchies in x are much less than the hierarchies in the up-type quark masses. Of course, we do not provide an understanding for these flavor matrices, but stress that the flavor problem has been transformed by our attempt to solve the strong CP problem. In Appendixes, we discuss theories with $N = 2$ and a different theory with $N = 3$, highlighting the vast and diverse space of models within our framework.

V. CONCLUSION

Building on early work [8–10], we introduced a new framework for studying flavor and the strong *CP* problem. The scalar sector of the SM is augmented from 1 to N Higgs doublets, and there was no additions to the fermion or gauge sectors. All dimension 4 interactions are symmetric under *CP* and a Higgs-flavor symmetry $G_{\rm HF}$ carried by both quarks and Higgs doublets. The SM Yukawa matrices are replaced by real couplings, due to *CP*, and have many zero entries, due to $G_{\rm HF}$. *CP* and $G_{\rm HF}$ are softly broken via the off-diagonal entries of the Higgs mass matrix. We restricted our attention to theories that have $N-1$ heavy Higgs doublets with mass of order M and one much lighter Higgs h. Thus, the SM appears as the EFT below M, with Yukawa couplings and a strong *CP* parameter arising from integrating out the heavy Higgs doublets. In this framework, the flavor and strong *CP* problems are inextricably linked.

We have computed the one-loop threshold corrections to $\bar{\theta}$ resulting from integrating out the heavy Higgs doublets at the mass scale M. In this class of theories, all nonvanishing contributions are proportional to $\log(M_\alpha/M_\beta)$, where α and β label the heavy Higgs doublet mass eigenstates. Thus, $N=2$ theories, with a single heavy Higgs, don't have any contribution to $\bar{\theta}$ at one loop. We studied radiative contributions to y^u and y^d at two-loop order with $N=2$ and find only contributions from RG scaling in the SM EFT below M, which are known to not induce $\bar{\theta}$. Hence, models with $N=2$ were particularly powerful at solving the strong *CP* problem. Contributions from higher-dimensional operators at the scale Λ in general depended on the UV completion of these multi-Higgs theories, but they are proportional to $(M^2/\Lambda^2)\ln(\Lambda^2/M^2)$. Order of magnitude estimates allowed us to conclude that they are negligible if the scale M is a few orders of magnitude below Λ. Furthermore, contributions to the neutron EDM through higher-dimensional operators at the scale M are negligible if the latter is taken to be well above the electroweak scale.

In this framework, the requirement of a "Higgs-flavor" symmetry that gives $\bar{\theta}=0$ at tree level may be a useful guide in constructing predictive theories of flavor. A small value for μ/M may suppress radiative corrections to $\bar{\theta}$ and contribute to the smallness of quark masses and mixing angles.

This framework is easily extended to the lepton sector to give a fully realistic description of nature, whether neutrinos are Majorana or Dirac. This leads to relations between the phases appearing in the quark and lepton mass matrices, with important implications for models where the cosmological baryon asymmetry arises from leptogenesis. Moreover, $G_{\rm HF}$ charges can be found for quarks and leptons that are consistent with either SU(5) or SO(10) unification. Finally, our framework offers diverse and model-dependent experimental signatures, including flavor observables like K, B, and D mixing, as well as opportunities for direct searches at current and future collider experiments. Given the richness of the phenomenology in this class of multi-Higgs theories, we have intentionally remained agnostic on the UV completion and we defer its discussion to future works.

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APPENDIX A: SMALL PARAMETERS AND $U(1)^9$

In the class of theories discussed in this letter, although many entries in the Yukawa matrices $x^\alpha_{ij}, \tilde{x}^\alpha_{ij}$ vanish, the nonzero entries may not all be determined from data on quark masses and mixings. Furthermore, radiative corrections to $\bar{\theta}$ depend on these unknown parameters. While $\bar{\theta} < 10^{-10}$ results if these parameters are sufficiently small, how can we determine if such small values constitute a solution to the strong *CP* problem? One possibility is that the sizes of the x and $\tilde{x}$ matrix elements are controlled by conventional flavor symmetries. A convenient and simple description may be possible in terms of symmetry breaking parameters, $\epsilon_{q_i}, \epsilon_{u_i}$, and ϵ_{d_i}, of the flavor symmetry $\mathrm{U}(1)^3_q \times \mathrm{U}(1)^3_u \times \mathrm{U}(1)^3_d$, such that, when nonzero, $x^\alpha_{ij} \sim \epsilon_{q_i}\epsilon_{u_i}$ and $\tilde{x}^\alpha_{ij} \sim \epsilon_{q_i}\epsilon_{d_i}$. Where possible, in computing the value of radiative corrections to $\bar{\theta}$, we estimate any $x^\alpha_{ij}, \tilde{x}^\alpha_{ij}$ that cannot be determined from data using this approximate flavor symmetry. The nine real symmetry breaking parameters $\epsilon_{q_i}, \epsilon_{u_i}$, and ϵ_{d_i} are all $\lesssim 1$ and can be determined from the six quark masses and three quark mixing angles, so that $\bar{\theta}$ can be estimated in terms of $U_{\alpha\beta}$. In fact, since the top Yukawa is close to unity, it determines two of these parameters, $\epsilon_{q_3} \sim \epsilon_{u_3} \sim 1$, implying that fitting to quark masses and mixings leads to a relation between the various $U_{\alpha 1}$. Using this procedure, the radiative correction to $\bar{\theta}$ can then be evaluated in terms of the remaining independent values of $U_{\alpha\beta}$, allowing a determination of whether the strong *CP* problem is naturally solved. The size of the required $\epsilon_{q_i}, \epsilon_{u_i}$, and ϵ_{d_i} parameters will also allow an assessment of any progress on the flavor problem. On the other hand, for some models, a fit to the quark masses and mixings using natural values for $U_{\alpha\beta}$ yields values of x^α_{ij} and $\tilde{x}^\alpha_{ij}$ that cannot be consistently understood from an approximate $\mathrm{U}(1)^9$ flavor symmetry (see the theory discussed in the main text). In this case, one can derive upper bounds on those $x, \tilde{x}$ matrix elements not determined from data in order that $\bar{\theta} < 10^{-10}$. For these models, an assessment of whether the strong *CP* problem is solved can

TABLE II. The 24 unique Yukawa textures that appear in viable $N = 2$ theories with $G_{\rm HF} = \mathbb{Z}_3$.

1: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}$	2: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & \epsilon & 0 \\ \epsilon & 1 & \epsilon \end{pmatrix}$	3: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \end{pmatrix}$	4: $\begin{pmatrix} 1 & \epsilon & 1 \\ 0 & 1 & 0 \\ \epsilon & 0 & \epsilon \end{pmatrix}$	5: $\begin{pmatrix} 0 & 1 & 1 \\ \epsilon & 0 & 0 \\ 1 & \epsilon & \epsilon \end{pmatrix}$	6: $\begin{pmatrix} 0 & 1 & 0 \\ \epsilon & 0 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}$
7: $\begin{pmatrix} 0 & 0 & 1 \\ \epsilon & \epsilon & 0 \\ 1 & 1 & \epsilon \end{pmatrix}$	8: $\begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & 1 & \epsilon \\ 1 & 0 & 1 \end{pmatrix}$	9: $\begin{pmatrix} \epsilon & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \epsilon & \epsilon \end{pmatrix}$	10: $\begin{pmatrix} \epsilon & 0 & 0 \\ 1 & \epsilon & \epsilon \\ 0 & 1 & 1 \end{pmatrix}$	11: $\begin{pmatrix} \epsilon & \epsilon & 1 \\ 1 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$	12: $\begin{pmatrix} \epsilon & \epsilon & 0 \\ 1 & 1 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$
13: $\begin{pmatrix} 1 & 1 & 0 \\ \epsilon & \epsilon & 1 \\ 0 & 0 & \epsilon \end{pmatrix}$	14: $\begin{pmatrix} 1 & 1 & \epsilon \\ \epsilon & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$	15: $\begin{pmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 1 \\ 0 & \epsilon & \epsilon \end{pmatrix}$	16: $\begin{pmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	17: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & \epsilon & 1 \\ \epsilon & 0 & \epsilon \end{pmatrix}$	18: $\begin{pmatrix} 0 & 0 & \epsilon \\ 1 & 1 & 0 \\ \epsilon & \epsilon & 1 \end{pmatrix}$
19: $\begin{pmatrix} 0 & \epsilon & 0 \\ 1 & 0 & 1 \\ \epsilon & 1 & \epsilon \end{pmatrix}$	20: $\begin{pmatrix} 0 & \epsilon & \epsilon \\ 1 & 0 & 0 \\ \epsilon & 1 & 1 \end{pmatrix}$	21: $\begin{pmatrix} \epsilon & 1 & \epsilon \\ 0 & \epsilon & 0 \\ 1 & 0 & 1 \end{pmatrix}$	22: $\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 1 \\ 1 & \epsilon & \epsilon \end{pmatrix}$	23: $\begin{pmatrix} \epsilon & 0 & \epsilon \\ 0 & 1 & 0 \\ 1 & \epsilon & 1 \end{pmatrix}$	24: $\begin{pmatrix} \epsilon & \epsilon & 0 \\ 0 & 0 & 1 \\ 1 & 1 & \epsilon \end{pmatrix}$

be made according to how small these parameters must be made.

APPENDIX B: THEORIES WITH $N = 2$ AND $G_{\rm HF} = \mathbb{Z}_3$

We found 864 models with two Higgs bosons and $G_{\rm HF} = \mathbb{Z}_3$ that could accommodate quark masses, CKM angles (from diagonalization of Yukawa matrices at first order), a nonzero Jarlskog invariant, and $\bar{\theta} = 0$ at tree level. Notably, all of these theories have precisely three zeros in each Yukawa coupling matrix at tree level. Among these theories there occur 24 unique textures of Yukawa matrices that appear in 48 unique $\{y^u, y^d\}$ pairings. Here, by "texture" we mean the full form of the tree-level Yukawas, keeping track of where the U_{21} and U_{11} factors appear, as well as zeros. Each of the 24 unique textures of y^u correspond to one of the 24 textures of y^d by complex conjugation, as one would expect at the level of charges. As such, we list the 24 textures in the following format: 0 where a coupling is absent at tree level, 1 where the coupling to ϕ_1 (or ϕ_1^*) is allowed and thus a factor of U_{11} is present, and ϵ where the coupling to ϕ_2 (or ϕ_2^*) is allowed and thus a factor of U_{21} (or U_{21}^*) is present. For example, the texture corresponding to the theory discussed in the next appendix would be written as

$$y^u \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \end{pmatrix}, \qquad y^d \sim \begin{pmatrix} 1 & 1 & \epsilon \\ \epsilon & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B1}$$

With this convention, the 24 textures are listed in Table II. They are arbitrarily numbered for reference, e.g., the above theory would be said to have texture $\{3, 14\}$. Note that for each of these matrices, there is another one obtained exchanging $\epsilon \leftrightarrow 1$. The 24 distinct pairings of textures are listed in Table III.

TABLE III. All 24 pairs of Yukawa textures in viable $G_{HF} = \mathbb{Z}_3$ models. Textures are labeled according to Table II.

y^u	y^d
1	16, 21
2	13
3	14
4	15, 17, 24
5	13
6	14
7	17
8	16, 18, 21
9	15, 17, 24
10	16, 21
11	13, 19, 22
12	14, 20, 23

APPENDIX C: A THEORY WITH $N = 2$

Theories with two Higgs doublets are particularly simple and interesting. CP and Higgs-flavor symmetry is broken by a single soft scalar mass $\mu_{12}^2 \equiv \mu^2 e^{i\beta}$ (with $\mu^2, \beta \in \mathbb{R}$). The unitary matrix transforming from Higgs-flavor to mass eigenstates is

$$U = \begin{pmatrix} \cos\xi & -\sin\xi e^{i\beta} \\ \sin\xi e^{-i\beta} & \cos\xi \end{pmatrix}, \tag{C1}$$

with $(\cos 2\xi, \sin 2\xi) = \frac{(M_1^2 - M_2^2, 2\mu^2)}{\sqrt{(M_1^2 - M_2^2)^2 + 4\mu^4}}$. Here, we study the case of a $\mathbb{Z}_3$ Higgs-flavor symmetry, yielding $\det(y^u y^d) \in \mathbb{R}$. A discussion of these theories and realistic charge assignments can be found in Appendixes. As discussed in the main text, there are no radiative corrections to $\bar{\theta}$ in addition to those already present in the SM and, therefore, the strong CP problem is solved in these theories. A two-Higgs doublet theory at the TeV scale with a $\mathbb{Z}_3$ symmetry was discussed in Ref. [12]. After electroweak symmetry breaking, they found that CP violation appeared only in the $W^\pm$ and $H^\pm$ couplings, and that the one-loop contribution to $\bar{\theta}$ from a charged Higgs loop vanished, in agreement with our more general discussion.

TABLE IV. The Higgs-flavor charges in a two-Higgs doublet model with $G_{\rm HF}$ taken to be a discrete $\mathbb{Z}_3$ global symmetry. Here, α is the generator of $\mathbb{Z}_3$, i.e., $\alpha^3 = 1$.

	ϕ_1	ϕ_2	q_1	q_2	q_3	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$
$\mathbb{Z}_3$	1	α^2	α^2	α	1	α	1	1	α	α	1

For concreteness, we discuss the model with charges given in Table IV, leading to Yukawa interactions

$$\mathcal{L}_{\rm Y} = (q_1\ \ q_2\ \ q_3)\begin{pmatrix} x_{11}\phi_1 & 0 & 0 \\ 0 & x_{22}\phi_2 & x_{23}\phi_2 \\ x_{31}\phi_2 & x_{32}\phi_1 & x_{33}\phi_1 \end{pmatrix}\begin{pmatrix}\bar{u}_1\\ \bar{u}_2\\ \bar{u}_3\end{pmatrix} + (q_1\ \ q_2\ \ q_3)\begin{pmatrix} \tilde{x}_{11}\phi_1^\dagger & \tilde{x}_{12}\phi_1^\dagger & \tilde{x}_{13}\phi_2^\dagger \\ \tilde{x}_{21}\phi_2^\dagger & \tilde{x}_{22}\phi_2^\dagger & 0 \\ 0 & 0 & \tilde{x}_{33}\phi_1^\dagger \end{pmatrix}\begin{pmatrix}\bar{d}_1\\ \bar{d}_2\\ \bar{d}_3\end{pmatrix}, \quad \text{(C2)}$$

where we suppress the scalar index on x and $\tilde{x}$, as each entry of these matrices receives a contribution from only one scalar. Thus, in this model there are 13 real parameters (12 couplings x_{ij}, $\tilde{x}_{ij}$, and ξ) and one phase (β). The measured quark Yukawa eigenvalues, y_i, and CKM parameters constrain 10 of them. Moreover, it is straightforward to check that $\det(y^u y^d) = 0$ and, therefore, $\bar{\theta} = 0$ at tree level. For the case that $\xi \ll 1$, the x and $\tilde{x}$ matrices take the form

$$x = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c/\xi & V_{cb}y_t/\xi \\ x_{31} & x_{32} & y_t \end{pmatrix}, \qquad \tilde{x} = \begin{pmatrix} y_d & V_{us}y_s & V_{ub}y_b/\xi \\ \tilde{x}_{21} & y_s/\xi & 0 \\ 0 & 0 & y_b \end{pmatrix}. \quad \text{(C3)}$$

The 11 and 33 entries of the $x, \tilde{x}$ matrices are the same as in the y^u, y^d matrices, so there is no progress here on the flavor problem. However, the 22 entries, and the entries generating CKM mixing, can be much larger than in the SM Yukawas. For example, all entries in the heavy 2×2 space of x could be large and order unity. Note also that the requirement of zero $\bar{\theta}$ at tree level has forced six texture zeros.

TABLE V. $\mathrm{U}(1)_{\rm F}$ charges for the theory with $N = 3$ and $G_{\rm HF} = \mathrm{U}(1)$ discussed in Appendix D.

	ϕ_1	ϕ_2	ϕ_3	q_1	$\bar{u}_1$	$\bar{d}_1$	q_2	$\bar{u}_2$	$\bar{d}_2$	q_3	$\bar{u}_3$	$\bar{d}_3$
$\mathrm{U}(1)_{\rm HF}$	0	1	2	1	-1	-1	0	0	0	-1	1	1

APPENDIX D: A DIFFERENT $N = 3$ MODEL

We study here another theory with $N = 3$ and $G_{\rm HF} = \mathrm{U}(1)$ to show how a different charge assignment under the Higgs-flavor symmetry leads to a different result for flavor and the strong *CP* problem. The $\mathrm{U}(1)_{\rm HF}$ charges are as presented in Ref. [10] and reported in Table V. The allowed Yukawa interactions are as follows:

$$\mathcal{L}_{\rm Y} = (q_1\ \ q_2\ \ q_3)\begin{pmatrix} x_{11}\phi_1 & 0 & 0 \\ x_{21}\phi_2 & x_{22}\phi_1 & 0 \\ x_{31}\phi_3 & x_{32}\phi_2 & x_{33}\phi_1 \end{pmatrix}\begin{pmatrix}\bar{u}_1\\ \bar{u}_2\\ \bar{u}_3\end{pmatrix} + (q_1\ \ q_2\ \ q_3)\begin{pmatrix} \tilde{x}_{11}\phi_1^\dagger & \tilde{x}_{12}\phi_2^\dagger & \tilde{x}_{13}\phi_3^\dagger \\ 0 & \tilde{x}_{22}\phi_1^\dagger & \tilde{x}_{23}\phi_2^\dagger \\ 0 & 0 & \tilde{x}_{33}\phi_1^\dagger \end{pmatrix}\begin{pmatrix}\bar{d}_1\\ \bar{d}_2\\ \bar{d}_3\end{pmatrix}. \quad \text{(D1)}$$

This theory has 15 real parameters and three phases, and the pattern of quark masses and CKM parameters is easily reproduced. We obtain

$$\begin{aligned} &y_u \simeq x_{11}|U_{11}|, \quad y_c \simeq x_{22}|U_{11}|, \quad y_t \simeq x_{33}|U_{11}|, \\ &y_d \simeq \tilde{x}_{11}|U_{11}^*|, \quad y_s \simeq \tilde{x}_{22}|U_{11}^*|, \quad y_b \simeq \tilde{x}_{33}|U_{11}^*|, \\ &|V_{us}|y_s \simeq \tilde{x}_{12}|U_{21}^*|, \quad |V_{ub}|y_b \simeq \tilde{x}_{13}|U_{31}^*|, \\ &|V_{cb}|y_b \simeq \tilde{x}_{23}|U_{21}^*|, \quad \delta_{\rm CKM} \simeq 2\beta_{12} - \beta_{13}. \end{aligned} \quad \text{(D2)}$$

Together with the requirement of perturbative Yukawa couplings, these results give the lower bounds $|U_{21}| \gtrsim 10^{-3}$ and $|U_{31}| \gtrsim 10^{-4}$. These relations and the unitarity of the U matrix leave only six real free parameters in this theory: $|U_{12}|, |U_{13}|, |U_{23}|, x^u_{21}, x^u_{31}$, and x^u_{32}, and two phases. Using the result of Eq. (6), we compute the one-loop correction to $\bar{\theta}$, which in the small mixing approximation reads

$$\begin{aligned} \delta\bar{\theta} \simeq{}& \left(5\times 10^{-5}\frac{\epsilon_{23}x^u_{21}}{\epsilon_{31}} + 6\times 10^{-10}\frac{\epsilon_{23}x^u_{31}}{\epsilon_{21}} - 2\times 10^{-4}\epsilon_{23}x^u_{21}x^u_{31} - 0.2\frac{\epsilon_{21}\epsilon_{23}x^u_{21}x^u_{32}}{\epsilon_{31}} \right.\\ &\left. + 4\times 10^{-4}\epsilon_{21}\epsilon_{23}\epsilon_{31}x^u_{21}x^u_{32}\right)\sin(\beta_{21}+\beta_{23}) \\ &+ (5\times 10^{-5}\epsilon_{21}x^u_{21} + 6\times 10^{-10}\epsilon_{31}x^u_{31} - 3\times 10^{-4}\epsilon_{21}\epsilon_{31}x^u_{21}x^u_{31})\sin(2\beta_{21}-\beta_{31}) \\ &- \left(6\times 10^{-9}\frac{\epsilon_{23}\epsilon_{31}}{\epsilon_{21}} - 3\times 10^{-5}\epsilon_{23}\epsilon_{31}x^u_{21} - 2\times 10^{-5}\epsilon_{23}\epsilon_{31}x^u_{32}\right)\sin(\beta_{21}-\beta_{23}-\beta_{31}). \end{aligned} \quad \text{(D3)}$$

To evaluate whether this radiative contribution is sufficiently small for the model to solve the strong CP problem, we are faced with the problem that x_{21}, x_{31}, and x_{32} are not determined by data. Here, we use an approximate $U(1)^9$ flavor symmetry, described in Appendix A, and estimate

$$x_{21} \simeq 2 \times 10^{-5} U_{21}, \qquad x_{31} \simeq 5 \times 10^{-4} U_{21}^2,$$
$$x_{32} \simeq 10^{-1} U_{21}, \qquad U_{31} \simeq 0.5 U_{21}^2. \tag{D4}$$

Furthermore, for the symmetry breaking parameter ϵ_{q_1} to be less than unity requires $U_{21} \gtrsim 0.1$. While U_{32} is not determined by this approximate symmetry, it has the same Higgs-flavor charge as U_{21}, and hence we expect $U_{32} \simeq U_{21}$. Using these results, and taking U_{21} near its lower bound, the first term of (D3) is of order 10^{-9}, as estimated in [10], which within uncertainties is consistent with data. However, the fourth term in (D3), not considered in [10], is larger and is similarly estimated to be 10^{-8}. Hence, we conclude that while this model makes significant progress on the strong CP problem, it does not convincingly solve it. Perhaps an alternative flavor symmetry scheme can be found that further suppresses these one-loop corrections.

APPENDIX E: HIGHER-DIMENSIONAL OPERATORS

Higher-dimensional operators, involving fields $q_i, \bar{u}_i, \bar{d}_i$, and ϕ_α, that can potentially spoil the solution to the strong CP problem, may be generated at the scale Λ where the multi-Higgs theory is UV completed. The leading such operators occur at dimension 6,

$$\mathcal{L}_6 \supset \frac{C_{ij}^{\alpha\beta\gamma}}{\Lambda^2} q_i \bar{u}_j \phi_\alpha \phi_\beta \phi_\gamma^\dagger + \frac{\tilde{C}_{ij}^{\alpha\beta\gamma}}{\Lambda^2} q_i \bar{d}_j \phi_\alpha^\dagger \phi_\beta \phi_\gamma^\dagger, \tag{E1}$$

where the C_{ij} and $\tilde{C}_{ij}$ tensors respect Higgs-flavor and CP symmetries. These operators generate corrections to x and $\tilde{x}$ Yukawa interactions at one loop by connecting two scalar legs, as shown in Fig. 3. For the diagram on the left, the loop integral is quadratically divergent, corresponding to a threshold correction at scale Λ. However, this contribution preserves the Higgs-flavor symmetry, and only corrects the size of nonzero entries of x and $\tilde{x}$, which we take to be free parameters. Introducing the flavor and CP-violating scalar mass terms $\mu_{\beta\gamma}^2$, represented by a cross in the diagram on the right in Fig. 3, leads to complex contributions to elements of x and $\tilde{x}$ (including those that vanish at tree level), and hence can give a contribution to $\bar{\theta}$. With the $\mu_{\beta\gamma}^2$ insertion, the loop integral is logarithmically divergent, so that these flavor-violating contributions are generated by RG scaling from the scale Λ to M and are proportional to $\ln(\Lambda/M)$,

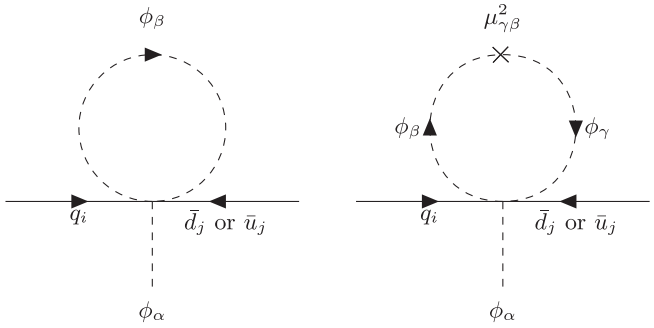

FIG. 3. One-loop radiative corrections to the x_{ij}^α and $\tilde{x}_{ij}^\alpha$ matrices from dimension 6 operators and soft scalar masses in the interaction basis.

$$\delta x_{ij}^\alpha = \frac{1}{8\pi^2\Lambda^2} \ln\left(\frac{\Lambda}{M}\right) \sum_{\beta\gamma} C_{ij}^{\alpha\beta\gamma} \mu_{\beta\gamma}^2,$$
$$\delta \tilde{x}_{ij}^\alpha = \frac{1}{8\pi^2\Lambda^2} \ln\left(\frac{\Lambda}{M}\right) \sum_{\beta\gamma} \tilde{C}_{ij}^{\alpha\beta\gamma} \mu_{\beta\gamma}^2. \tag{E2}$$

We ignore the threshold corrections at scale M, which have constant terms and terms proportional to logs of heavy Higgs mass ratios.

The resulting radiative contribution to $\bar{\theta}$ is obtained by inserting these corrections into y^u and y^d. A precise numerical evaluation is not possible, because $C_{ij}^{\alpha\beta\gamma}$ and $\tilde{C}_{ij}^{\alpha\beta\gamma}$ are unknown. However, using approximate $U(1)^9$ flavor symmetries, the nonzero entries are estimated to be $C_{ij} \sim \epsilon_{q_i} \epsilon_{u_j}$ and $\tilde{C}_{ij} \sim \epsilon_{q_i} \epsilon_{d_j}$. Therefore, the magnitudes of all contributions to the determinant of y^u and y^d, whether at tree or loop level, are proportional to $\epsilon_{q_1}\epsilon_{q_2}\epsilon_{q_3}\epsilon_{u_1}\epsilon_{u_2}\epsilon_{u_3}$ and $\epsilon_{q_1}\epsilon_{q_2}\epsilon_{q_3}\epsilon_{d_1}\epsilon_{d_2}\epsilon_{d_3}$, respectively, which drops out of the result for $\bar{\theta}$, giving the rough estimate

$$\delta\bar{\theta}_\Lambda \simeq \frac{1}{8\pi^2} \frac{M^2}{\Lambda^2} \ln\left(\frac{\Lambda}{M}\right) \sum_{\beta\gamma} \frac{\mu_{\gamma\beta}^2}{M^2} f_{\gamma\beta}(U). \tag{E3}$$

Here, $f_{\gamma\beta}(U)$ are model-dependent cubic polynomials of matrix elements of U. The off-diagonal matrix elements of U are small if the scale of the soft parameters $\mu_{\alpha\beta}^2$ are less than the mass scale M of the heavy Higgs. While the suppression from this is limited, since realistic quark masses and mixings must result from perturbative values of the x and $\tilde{x}$ matrices, the general expectation is that Λ need not be many orders of magnitude above M. We conclude by discussing higher-dimensional operators in the two models presented in the main letter.

1. $N = 2$

For the model with two Higgs doublets, the contributions to $\bar{\theta}$ at $\mathcal{O}(\frac{1}{\Lambda^2})$, from one-loop diagrams involving dimension 6 operators, read

$$\begin{aligned}\delta\bar{\theta}_\Lambda \simeq [&-2\tilde{C}_{31}^{212}\sin\xi + 3408\tilde{C}_{32}^{212}\sin^2\xi\tilde{x}_{21} + 10\tilde{C}_{31}^{212}\cos\xi\sin\xi x_{32}\\ &- 2\times 10^4\tilde{C}_{32}^{212}\cos\xi\sin^2\xi\tilde{x}_{21}x_{32} - (7\times 10^3 C_{12}^{212} - 10^3 C_{13}^{212})\sin^2\xi x_{31}\\ &+ (5\times 10^7 C_{12}^{212} - 8\times 10^6 C_{13}^{212})\sin^3\xi\tilde{x}_{21}x_{31}]\frac{\mu^2}{\Lambda^2}\log\left(\frac{\Lambda^2}{M^2}\right)\sin 3\beta. \end{aligned} \tag{E4}$$

Using a $U(1)^9$ flavor symmetry, we find $\sin\xi \sim 0.4$ and $M/\Lambda \lesssim 8\times 10^{-5}$ to not spoil the solution to the strong *CP* problem.

2. $N = 3$

For the model with three Higgs doublets, the contributions to $\bar{\theta}$ at $\mathcal{O}(\frac{1}{\Lambda^2})$, from one-loop diagrams involving dimension 6 operators, read

$$\begin{aligned}\delta\bar{\theta}_\Lambda \simeq \frac{M^2}{\Lambda^2}\log\left(\frac{\Lambda^2}{M^2}\right)[&(4\times 10^2\tilde{C}_{11}^{121} - 10^3 C_{11}^{112})\epsilon_{12}\sin(\beta_{12}+2\beta_{13})\\ &+ 4\times 10^2\tilde{C}_{31}^{132}\epsilon_{12}\epsilon_{13}\sin(2\beta_{12}) - 3\times 10^2 C_{21}^{123}\epsilon_{12}^2\epsilon_{13}x_{32}\sin(2\beta_{12}-2\beta_{13})\\ &+ (3\times 10^2 C_{21}\epsilon_{12}\epsilon_{12}x_{32} - \tilde{C}_{31}^{123}\epsilon_{23})\sin(\beta_{12}-\beta_{13}-\beta_{23})]. \end{aligned} \tag{E5}$$

In this model we cannot use $U(1)^9$ approximate flavor symmetries to estimate the sizes of nonzero entries in the $x, \tilde{x}, C$, and $\tilde{C}$ matrices given the large size of some parameters. Instead, we use our illustrative example of $\epsilon_{12} \simeq 10^{-3}, \epsilon_{13} \simeq 10^{-2}, \epsilon_{23} \simeq 10^{-5}$, and $x_{32} \simeq 10^{-2}$, which gives $\delta\bar{\theta} \simeq 10^{-10}$. We find $\delta\bar{\theta}_\Lambda \lesssim 10^{-10}$ for $M/\Lambda \lesssim 10^{-3}$ when we take $C_{11}^{112}, \tilde{C}_{11}^{121} \simeq 10^{-4}$, $\tilde{C}_{31}^{132} \simeq 10^{-2}$, and other C and $\tilde{C}$ elements of order unity.

APPENDIX F: SPONTANEOUS *CP* BREAKING

In most of this paper we treat $\mu_{\alpha\beta}^2$ as soft symmetry-breaking spurions for $G_{\rm HF}$ and *CP*. However, they may arise from spontaneous $G_{\rm HF}$ and *CP* symmetry breaking via a set of SM singlet vacuum expectation values (vevs), ξ. The dimension of the operator linking the Higgs bilinear $\phi_\alpha^\dagger\phi_\beta$ to the ξ fields depends on the $G_{\rm HF}$ charges of the bilinear and of ξ. Thus $|\mu_{\alpha\beta}^2|$, which we take less than or of order M^2, could be of order $M\xi, \xi^2, \xi^3/\Lambda_\phi, \xi^4/\Lambda_\phi^2, \ldots$, where Λ_ϕ is the UV scale at which relevant interactions appear that connect ξ and ϕ_α. The various $\phi_\alpha^\dagger\phi_\beta$ operators can have different charges, and the corresponding $\mu_{\alpha\beta}^2$ can differ by orders of magnitude. In the case that two such operators have the same charge, we expect the corresponding soft masses to be comparable. There could be higher-dimensional interactions, such as $g_1(\xi/\Lambda_q)q_i\bar{u}_j\phi^\alpha$ or $[g_2(\xi/\Lambda_G) - g_2^\dagger(\xi/\Lambda_G)]G\tilde{G}$, leading to $G_{\rm HF}$ and *CP* breaking effects suppressed by powers of $\xi/\Lambda_{q,G}$. The polynomials $g_{1,2}$ are model-dependent, and g_2 is at least cubic order. We assume these can be neglected compared to breaking terms involving powers of ξ/M that arise in the Higgs doublet mass matrix. In this paper, the breaking of *CP* and $G_{\rm HF}$ is communicated to the quark sector through the Higgs Portal. We conclude the discussion showing how the soft breaking of *CP* and $G_{\rm HF}$ could be obtained spontaneously in the model with three Higgs doublets discussed in the main text. We introduce the scalar fields $\xi_{1,2}$ of $G_{\rm HF}$ charges (1, 2) that spontaneously break $G_{\rm HF}$ and *CP* via the vevs,

$$\begin{aligned}\xi_2^* &= \frac{\mu_{12}^2}{M_2}e^{i\beta_{12}} = \epsilon_{12}M_2 e^{i\beta_{12}} \simeq 10^{-3}M_2 e^{i\beta_{12}},\\ \xi_1 &= \frac{\mu_{13}^2}{M_3}e^{i\beta_{13}} = \epsilon_{13}M_3 e^{i\beta_{13}} \simeq 10^{-2}M_3 e^{i\beta_{13}}. \end{aligned} \tag{F1}$$

The mass matrix for the scalars (ϕ_1, ϕ_2, ϕ_3) of $G_{\rm HF}$ charges $(0, 2, -1)$ can take the form

$$\begin{aligned}M_\phi^2 &\simeq \begin{pmatrix} M_1^2 & M_2\xi_2^* & M_3\xi_1\\ M_2\xi_2 & M_2^2 & \xi_1\xi_2^*\\ M_3\xi_1^* & \xi_1^*\xi_2 & M_3^2\end{pmatrix}\\ &\simeq M_2^2\begin{pmatrix} r^2 10^{-4} & 10^{-3}e^{i\beta_{12}} & 10^{-2}r^2 e^{i\beta_{13}}\\ 10^{-3}e^{-i\beta_{12}} & 1 & 10^{-5}re^{i(\beta_{12}+\beta_{13})}\\ 10^{-2}r^2e^{-i\beta_{13}} & 10^{-5}re^{-i(\beta_{12}+\beta_{13})} & r^2\end{pmatrix}, \end{aligned} \tag{F2}$$

where $r = M_3/M_2$. Note that the 11 entry is small by the fine-tuning to obtain the weak scale much below the masses of the heavy Higgs doublets.

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