

A REMARK ON INVERSE PROBLEMS FOR NONLINEAR MAGNETIC SCHRÖDINGER EQUATIONS ON COMPLEX MANIFOLDS

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ABSTRACT. We show that the knowledge of the Dirichlet-to-Neumann map for a nonlinear magnetic Schrödinger operator on the boundary of a compact complex manifold, equipped with a Kähler metric and admitting sufficiently many global holomorphic functions, determines the nonlinear magnetic and electric potentials uniquely.

1. INTRODUCTION

Let M be an n -dimensional compact complex manifold with C^∞ boundary, equipped with a Kähler metric g . Consider the nonlinear magnetic Schrödinger operator

$$L_{A,V}u = d_{A(\cdot,u)}^* d_{A(\cdot,u)} u + V(\cdot, u),$$

acting on $u \in C^\infty(M)$. Here the nonlinear magnetic $A : M \times \mathbb{C} \rightarrow T^*M \otimes \mathbb{C}$ and electric $V : M \times \mathbb{C} \rightarrow \mathbb{C}$ potentials are assumed to satisfy the following conditions:

- (i) the map $\mathbb{C} \ni w \mapsto A(\cdot, w)$ is holomorphic with values in $C^\infty(M, T^*M \otimes \mathbb{C})$,
- (ii) $A(z, 0) = 0$ for all $z \in M$,
- (iii) the map $\mathbb{C} \ni w \mapsto V(\cdot, w)$ is holomorphic with values in $C^\infty(M)$,
- (iv) $V(z, 0) = \partial_w V(z, 0) = 0$ for all $z \in M$.

Thus, A and V can be expanded into power series

$$A(z, w) = \sum_{k=1}^{\infty} A_k(z) \frac{w^k}{k!}, \quad V(z, w) = \sum_{k=2}^{\infty} V_k(z) \frac{w^k}{k!}, \quad (1.1)$$

converging in $C^\infty(M, T^*M \otimes \mathbb{C})$ and $C^\infty(M)$ topologies, respectively. Here

$$A_k(z) := \partial_w^k A(z, 0) \in C^\infty(M, T^*M \otimes \mathbb{C}), \quad V_k(z) := \partial_w^k V(z, 0) \in C^\infty(M).$$

We write $T^*M \otimes \mathbb{C}$ for the complexified cotangent bundle of M ,

$$d_{A(\cdot,w)} = d + iA(\cdot, w) : C^\infty(M) \rightarrow C^\infty(M, T^*M \otimes \mathbb{C}), \quad w \in \mathbb{C}, \quad (1.2)$$

where $d : C^\infty(M) \rightarrow C^\infty(M, T^*M \otimes \mathbb{C})$ is the de Rham differential, and $d_{A(\cdot, w)}^* : C^\infty(M, T^*M \otimes \mathbb{C}) \rightarrow C^\infty(M)$ is the formal L^2 -adjoint of $d_{A(\cdot, w)}$ taken with respect to the Kähler metric g .

It is established in [28, Appendix B] that under the assumptions (i)-(iv), there exist $\delta > 0$ and $C > 0$ such that for any $f \in B_\delta(\partial M) := \{f \in C^{2,\alpha}(\partial M) : \|f\|_{C^{2,\alpha}(\partial M)} < \delta\}$, $0 < \alpha < 1$, the Dirichlet problem for the nonlinear magnetic Schrödinger operator

$$\begin{cases} L_{A,V}u = 0 & \text{in } M^{\text{int}}, \\ u|_{\partial M} = f, \end{cases} \quad (1.3)$$

has a unique solution $u = u_f \in C^{2,\alpha}(M)$ satisfying $\|u\|_{C^{2,\alpha}(M)} < C\delta$. Here $C^{2,\alpha}(M)$ and $C^{2,\alpha}(\partial M)$ stand for the standard Hölder spaces of functions on M and ∂M , respectively, and $M^{\text{int}} = M \setminus \partial M$ stands for the interior of M . Associated to (1.3), we introduce the Dirichlet-to-Neumann map

$$\Lambda_{A,V}f = \partial_\nu u_f|_{\partial M}, \quad f \in B_\delta(\partial M), \quad (1.4)$$

where ν is the unit outer normal to the boundary of M .

The inverse boundary problem for the nonlinear magnetic Schrödinger operator that we are interested in asks whether the knowledge of the Dirichlet-to-Neumann map $\Lambda_{A,V}$ determines the nonlinear magnetic A and electric V potentials in M . Such inverse problems have been recently studied in [28] in the case of conformally transversally anisotropic manifolds and in [30] and [38] in the case of partial data in the Euclidean space and on Riemann surfaces, respectively.

To state our result, following [20], we assume that the manifold M satisfies the following additional assumptions, which occur in the definition of a Stein manifold, see [16, Section 2.2, p. 49]:

- (a) M is holomorphically separable in the sense that if $x, y \in M$ with $x \neq y$, there is some $f \in \mathcal{O}(M) := \{f \in C^\infty(M) : f \text{ is holomorphic in } M^{\text{int}}\}$ such that $f(x) \neq f(y)$,
- (b) M has local charts given by global holomorphic functions in the sense that for every $p \in M$ there exist $f_1, \dots, f_n \in \mathcal{O}(M)$ which form a complex coordinate system near p .

The definition of a Stein manifold demands also that M should be holomorphically convex, see [16, Section 2.2, p. 49]. However, this property is not needed in this work. As explained in [20], examples of complex manifolds satisfying all of the assumptions above including (a) and (b) are as follows:

- any compact C^∞ subdomain of a Stein manifold, equipped with a Kähler metric,
- any compact C^∞ subdomain of a complex submanifold of $\mathbb{C}^N$, equipped with a Kähler metric,

- any compact C^∞ subdomain of a complex coordinate neighborhood on a Kähler manifold.

The main result of this note is as follows.

Theorem 1.1. *Let M be an n -dimensional compact complex manifold with C^∞ boundary, equipped with a Kähler metric g , satisfying assumptions (a) and (b). Let $A^{(1)}, A^{(2)} : M \times \mathbb{C} \rightarrow T^*M \otimes \mathbb{C}$ and $V^{(1)}, V^{(2)} : M \times \mathbb{C} \rightarrow \mathbb{C}$ be such that the assumptions (i)–(iv) hold. If $\Lambda_{A^{(1)}, V^{(1)}} = \Lambda_{A^{(2)}, V^{(2)}}$ then $A^{(1)} = A^{(2)}$ and $V^{(1)} = V^{(2)}$ in $M \times \mathbb{C}$.*

Remark 1.2. *Theorem 1.1 in the case of a semilinear Schrödinger operator, i.e. when $A = 0$, was obtained in [39].*

Remark 1.3. *The corresponding inverse problems for the linear Schrödinger operator $-\Delta_g + V_0$, $V_0 \in C^\infty(M)$, as well as for the linear magnetic Schrödinger operator $d_{A_0}^* d_{A_0} + V_0$, $A_0 \in C^\infty(M, T^*M \otimes \mathbb{C})$, in the geometric setting of Theorem 1.1 are open. Theorem 1.1 can be viewed as a manifestation of the phenomenon, discovered in [29], that the presence of nonlinearity may help to solve inverse problems. We refer to [20] for the solution of the linearized inverse problem for the linear Schrödinger operator in this geometric setting, and would like to emphasize that our proof of Theorem 1.1 is based crucially on this result. We also refer to [19], [17], [18] for solutions to inverse boundary problems for the linear Schrödinger and magnetic Schrödinger operators on Riemann surfaces.*

Remark 1.4. *The known results for the inverse boundary problem for the linear Schrödinger and magnetic Schrödinger operators on Riemannian manifolds of dimension ≥ 3 with boundary beyond the Euclidean ones, see [47], [42], [23], and real analytic ones, see [33], [34], [35], all require a certain conformal symmetry of the manifold as well as some additional assumptions about the injectivity of geodesic ray transforms, see [12], [13], [11], [24]. The known results for inverse problems for the nonlinear Schrödinger operators $L_{0,V}$ [15], [32], and nonlinear magnetic Schrödinger operators $L_{A,V}$ [28] still require the same conformal symmetry of the manifold, while the injectivity of the geodesic transform is no longer needed.*

Note that the need to require a certain conformal symmetry of the manifold in all of the known results in dimensions $n \geq 3$ is due to the existence of limiting Carleman weights on such manifolds, see [12], which are crucial for the construction of complex geometric optics solutions used for solving inverse problems for elliptic PDE since the fundamental work [47]. However, it is shown in [37], [1] that a generic manifold of dimension $n \geq 3$ does not admit limiting Carleman weights.

Remark 1.5. *As in [20, Theorem 1.1], manifolds considered in Theorem 1.1 need not admit limiting Carleman weights. For example, it was established in [2] that*

$\mathbb{CP}^2$ with the Fubini-Study metric g does not admit a limiting Carleman weight near any point. However, $(\mathbb{CP}^2, g)$ is a Kähler manifold, and as explained in [20], compact C^∞ subdomains of it provide examples of manifolds where Theorem 1.1 applies.

Remark 1.6. *In contrast to the inverse boundary problem for the linear magnetic Schrödinger equation, where one can determine the magnetic potential up to a gauge transformation only, see for example [42], [23], in Theorem 1.1 the unique determination of both nonlinear magnetic and electric potentials is achieved. This is due to our assumptions (ii) and (iv) which lead to the first order linearization of the nonlinear magnetic Schrödinger equation given by $-\Delta_g u = 0$ rather than by the linear magnetic Schrödinger equation, see also [28] for a similar unique determination in the case of conformally transversally anisotropic manifolds.*

Let us finally mention that inverse problems for the semilinear Schrödinger operators and for nonlinear conductivity equations have been investigated intensively recently, see for example [15], [31], [32], [36], [26], [27], and [9], [22], [7], [8], [41], [44], respectively.

Theorem 1.1 is a direct consequence of the main result of [20], combined with some boundary determination results of [39] and of Appendix A, as well as the higher order linearization procedure introduced in [29] in the hyperbolic case, and in [15], [32] in the elliptic case. We refer to [21] where the method of a first order linearization was pioneered in the study of inverse problems for nonlinear PDE, and to [3], [10], [45], and [46] where a second order linearization was successfully exploited. The crucial fact used in the proof of the main result of [20], indispensable for our Theorem 1.1, is that both holomorphic and antiholomorphic functions are harmonic on Kähler manifolds. The assumptions (a) and (b) in Theorem 1.1 are needed as they are used in [20] to construct suitable holomorphic and antiholomorphic functions by extending the two dimensional arguments of [6] and [17] to the case of higher dimensional complex manifolds.

The plan of the note is as follows. The proof of Theorem 1.1 is given in Section 2. Appendix A contains the boundary determination result needed in the proof of Theorem 1.1.

2. PROOF OF THEOREM 1.1

First using that $d_A^* = d^* - i\langle \bar{A}, \cdot \rangle_g$ and (1.2), we write the nonlinear magnetic Schrödinger operator $L_{A,V}$ as follows,

$$\begin{aligned} L_{A,V}u &= d_{A(\cdot,u)}^* d_{A(\cdot,u)} u + V(\cdot, u) \\ &= -\Delta_g u + d^*(iA(\cdot, u)u) - i\langle A(\cdot, u), du \rangle_g + \langle A(\cdot, u), A(\cdot, u) \rangle_g u + V(\cdot, u), \end{aligned}$$

for $u \in C^\infty(M)$. Here $\langle \cdot, \cdot \rangle_g$ is the pointwise scalar product in the space of 1-forms induced by the Riemannian metric g , compatible with the Kähler structure.

Using the m th order linearization of the Dirichlet-to-Neumann map $\Lambda_{A,V}$ and induction on $m = 2, 3, \dots$, we shall show that the coefficients A_{m-1} and V_m in (1.1) can all be recovered from $\Lambda_{A,V}$.

First, let $m = 2$ and let us proceed to carry out a second order linearization of the Dirichlet-to-Neumann map. To that end, let $f_1, f_2 \in C^\infty(\partial M)$ and let $u_j = u_j(x, \varepsilon) \in C^{2,\alpha}(M)$ be the unique small solution of the following Dirichlet problem,

$$\begin{cases} -\Delta_g u_j + id^*(\sum_{k=1}^\infty A_k^{(j)}(x) \frac{u_j^k}{k!} u_j) - i\langle \sum_{k=1}^\infty A_k^{(j)}(x) \frac{u_j^k}{k!}, du_j \rangle_g \\ \quad + \langle \sum_{k=1}^\infty A_k^{(j)}(x) \frac{u_j^k}{k!}, \sum_{k=1}^\infty A_k^{(j)}(x) \frac{u_j^k}{k!} \rangle_g u_j + \sum_{k=2}^\infty V_k^{(j)}(x) \frac{u_j^k}{k!} = 0 \text{ in } M^{\text{int}}, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 \text{ on } \partial M, \end{cases} \quad (2.1)$$

for $j = 1, 2$. It was established in [28, Appendix B] that for all $|\varepsilon|$ sufficiently small, the solution $u_j(\cdot, \varepsilon)$ depends holomorphically on $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \text{neigh}(0, \mathbb{C}^2)$. Applying the operator $\partial_{\varepsilon_l}|_{\varepsilon=0}$, $l = 1, 2$, to (2.1) and using that $u_j(x, 0) = 0$, we get

$$\begin{cases} -\Delta_g v_j^{(l)} = 0 & \text{in } M^{\text{int}}, \\ v_j^{(l)} = f_l & \text{on } \partial M, \end{cases}$$

where $v_j^{(l)} = \partial_{\varepsilon_l} u_j|_{\varepsilon=0}$. By the uniqueness and the elliptic regularity, it follows that $v^{(l)} := v_1^{(l)} = v_2^{(l)} \in C^\infty(M)$, $l = 1, 2$. Applying $\partial_{\varepsilon_1} \partial_{\varepsilon_2}|_{\varepsilon=0}$ to (2.1), we obtain the second order linearization,

$$\begin{cases} -\Delta_g w_j + 2id^*(A_1^{(j)} v^{(1)} v^{(2)}) - i\langle A_1^{(j)}, d(v^{(1)} v^{(2)}) \rangle_g + V_2^{(j)} v^{(1)} v^{(2)} = 0 \text{ in } M^{\text{int}}, \\ w_j = 0 \text{ on } \partial M, \end{cases} \quad (2.2)$$

where $w_j = \partial_{\varepsilon_1} \partial_{\varepsilon_2} u_j|_{\varepsilon=0}$, $j = 1, 2$. Using that

$$d^*(Bv) = (d^*B)v - \langle B, dv \rangle_g, \quad (2.3)$$

for any $B \in C^\infty(M, T^*M \otimes \mathbb{C})$ and $v \in C^\infty(M)$, (2.2) implies that

$$\begin{cases} -\Delta_g w_j - 3i\langle A_1^{(j)}, d(v^{(1)} v^{(2)}) \rangle_g + (2id^*(A_1^{(j)}) + V_2^{(j)}) v^{(1)} v^{(2)} = 0 \text{ in } M^{\text{int}}, \\ w_j = 0 \text{ on } \partial M, \end{cases} \quad (2.4)$$

$j = 1, 2$. The equality $\Lambda_{A^{(1)}, V^{(1)}}(\varepsilon_1 f_1 + \varepsilon_2 f_2) = \Lambda_{A^{(2)}, V^{(2)}}(\varepsilon_1 f_1 + \varepsilon_2 f_2)$ yields that $\partial_\nu u_1|_{\partial M} = \partial_\nu u_2|_{\partial M}$, and hence, $\partial_\nu w_1|_{\partial M} = \partial_\nu w_2|_{\partial M}$. Multiplying the difference of two equations in (2.4) by a harmonic function $v^{(3)} \in C^\infty(M)$, integrating over

M and using Green's formula, we obtain that

$$\int_M (3i\langle A, d(v^{(1)}v^{(2)}) \rangle_g v^{(3)} - (2id^*(A) + V)v^{(1)}v^{(2)}v^{(3)})dV_g = 0, \quad (2.5)$$

valid for all harmonic functions $v^{(l)} \in C^\infty(M)$, $l = 1, 2, 3$. Here $A = A_1^{(1)} - A_1^{(2)}$ and $V = V_2^{(1)} - V_2^{(2)}$. Interchanging $v^{(3)}$ and $v^{(1)}$ in (2.5), we also have

$$\int_M (3i\langle A, d(v^{(3)}v^{(2)}) \rangle_g v^{(1)} - (2id^*(A) + V)v^{(1)}v^{(2)}v^{(3)})dV_g = 0. \quad (2.6)$$

Subtracting (2.6) from (2.5) and letting $v^{(3)} = 1$, we get

$$\int_M \langle A, dv^{(1)} \rangle_g v^{(2)}dV_g = 0, \quad (2.7)$$

for all harmonic functions $v^{(1)}, v^{(2)} \in C^\infty(M)$. Applying Proposition A.1 to (2.7), we conclude that $A|_{\partial M} = 0$. Using this together with Stokes' formula,

$$\int_M \langle dw, \eta \rangle_g dV_g = \int_M wd^*\eta dV_g + \int_{\partial M} \omega(\mathbf{n}\eta), \quad \omega \in C^\infty(M), \quad \eta \in C^\infty(M, T^*M \otimes \mathbb{C}),$$

where the $(2n-1)$ -form $\mathbf{n}\eta$ on the boundary is the normal trace of η , see [43, Proposition 2.1.2], we obtain from (2.5) that

$$\int_M (3id^*(Av^{(3)}) - (2id^*(A) + V)v^{(3)})v^{(1)}v^{(2)}dV_g = 0, \quad (2.8)$$

for all harmonic functions $v^{(l)} \in C^\infty(M)$, $l = 1, 2, 3$. Applying [20, Theorem 1.1] together with the boundary determination result of [39, Proposition 3.1] to (2.8), we get

$$3id^*(Av^{(3)}) - (2id^*(A) + V)v^{(3)} = 0, \quad (2.9)$$

for every harmonic function $v^{(3)} \in C^\infty(M)$. Using (2.3), we obtain from (2.9) that

$$(id^*(A) - V)v^{(3)} - 3i\langle A, dv^{(3)} \rangle_g = 0, \quad (2.10)$$

for every harmonic function $v^{(3)} \in C^\infty(M)$. Letting $v^{(3)} = 1$ in (2.10), we get

$$id^*(A) - V = 0, \quad (2.11)$$

and therefore,

$$\langle A, dv^{(3)} \rangle_g = 0, \quad (2.12)$$

for every harmonic function $v^{(3)} \in C^\infty(M)$. Let $p \in M^{\text{int}}$ and by assumption (b), there exist $f_1, \dots, f_n \in \mathcal{O}(M)$ which form a complex coordinate system near p . Hence, $df_j(p)$, $\overline{df_j(p)}$ is a basis for $T_p^*M \otimes \mathbb{C}$. Since on a Kähler manifold the Laplacian on functions satisfies

$$\Delta_g = d^*d = 2\partial^*\partial = 2\overline{\partial}^*\overline{\partial},$$

see [20, Lemma 2.1], [40, Theorem 8.6, p. 45], we have that all functions $f_1, \dots, f_n$ as well as $\bar{f}_1, \dots, \bar{f}_n$ are harmonic, and therefore, it follows from (2.12) that

$$\langle A, df_j \rangle_g(p) = 0, \quad \langle A, d\bar{f}_j \rangle_g(p) = 0.$$

Hence, $A = 0$, and therefore, $A_1^{(1)} = A_1^{(2)}$ in M . It follows from (2.11) that $V = 0$, and therefore, $V_2^{(1)} = V_2^{(2)}$ in M .

Let $m \geq 3$ and let us assume that

$$A_k^{(1)} = A_k^{(2)}, \quad k = 1, \dots, m-2, \quad V_k^{(1)} = V_k^{(2)}, \quad k = 2, \dots, m-1. \quad (2.13)$$

To prove that $A_{m-1}^{(1)} = A_{m-1}^{(2)}$ and $V_m^{(1)} = V_m^{(2)}$, we shall use the m th order linearization of the Dirichlet-to-Neumann map. Such an m th order linearization with $m \geq 3$ is performed in [28], and combining with (2.13), it leads to the following integral identity,

$$\int_M ((m+1)i\langle A, d(v^{(1)} \dots v^{(m)}) \rangle_g v^{(m+1)} - (mid^*(A) + V)v^{(1)} \dots v^{(m+1)}) dV_g = 0, \quad (2.14)$$

for all harmonic functions $v^{(l)} \in C^\infty(M)$, $l = 1, \dots, m+1$, see [28, Section 5]. Here $A = A_{m-1}^{(1)} - A_{m-1}^{(2)}$ and $V = V_m^{(1)} - V_m^{(2)}$. Letting $v^{(1)} = \dots = v^{(m-2)} = 1$ in (2.14) and arguing as in the case $m = 2$, we complete the proof of Theorem 1.1.

Remark 2.1. *Thanks to the density of products of two harmonic functions in the geometric setting of Theorem 1.1 established in [20], we recover the nonlinear magnetic and electric potentials of the general form (1.1) here. On the other hand, in the case of conformally transversally anisotropic manifolds of real dimension ≥ 3 , only the density of products of four harmonic functions is available, see [15], [32], [28], and therefore, the nonlinear magnetic and electric potentials of the form (1.1) with $k \geq 2$ and $k \geq 3$, respectively, were determined from the knowledge of the Dirichlet-to-Neumann map in [28].*

APPENDIX A. BOUNDARY DETERMINATION OF A 1-FORM ON A RIEMANNIAN MANIFOLD

When proving Theorem 1.1, we need the following essentially known boundary determination result on a general compact Riemannian manifold with boundary, see [5], [25, Appendix A], [28, Appendix C], [38] for similar results. We present a proof for completeness and convenience of the reader.

Proposition A.1. *Let (M, g) be a compact smooth Riemannian manifold of dimension $n \geq 2$ with smooth boundary. If $A \in C(M, T^*M \otimes \mathbb{C})$ satisfies*

$$\int_M \langle A, du \rangle_g \bar{u} dV_g = 0, \quad (A.1)$$

for every harmonic function $u \in C^\infty(M)$, then $A|_{\partial M} = 0$.

Proof. In order to show that $A|_{\partial M} = 0$, we shall construct a suitable harmonic function $u \in C^\infty(M)$ to be used in the integral identity (A.1). When doing so, we shall use an explicit family of functions v_λ , constructed in [4], [5], whose boundary values have a highly oscillatory behavior as $\lambda \rightarrow 0$, while becoming increasingly concentrated near a given point on the boundary of M . We let $x_0 \in \partial M$ and we shall work in the boundary normal coordinates centered at x_0 so that in these coordinates, $x_0 = 0$, the boundary ∂M is given by $\{x_n = 0\}$, and M^{int} is given by $\{x_n > 0\}$. We have $T_{x_0}\partial M = \mathbb{R}^{n-1}$, equipped with the Euclidean metric. The unit tangent vector τ is then given by $\tau = (\tau', 0)$ where $\tau' \in \mathbb{R}^{n-1}$, $|\tau'| = 1$. Associated to the tangent vector τ' is the covector $\sum_{\beta=1}^{n-1} g_{\alpha\beta}(0)\tau'_\beta = \tau'_\alpha \in T_{x_0}^*\partial M$. Letting $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$ and following [5], see also [14, Appendix A], we set

$$v_\lambda(x) = \lambda^{-\frac{\alpha(n-1)}{2} - \frac{1}{2}} \eta\left(\frac{x}{\lambda^\alpha}\right) e^{\frac{i}{\lambda}(\tau' \cdot x' + ix_n)}, \quad 0 < \lambda \ll 1,$$

where $\eta \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ is such that $\text{supp } (\eta)$ is in a small neighborhood of 0, and

$$\int_{\mathbb{R}^{n-1}} \eta(x', 0)^2 dx' = 1.$$

Here τ' is viewed as a covector. Thus, we have $v_\lambda \in C^\infty(M)$ with $\text{supp } (v_\lambda)$ in $\mathcal{O}(\lambda^\alpha)$ neighborhood of $x_0 = 0$. A direct computation shows that

$$\|v_\lambda\|_{L^2(M)} = \mathcal{O}(1), \quad (\text{A.2})$$

as $\lambda \rightarrow 0$, see also [14, Appendix A, (A.8)]. Furthermore, we have

$$\|dv_\lambda\|_{L^2(M)} = \mathcal{O}(\lambda^{-1}), \quad (\text{A.3})$$

as $\lambda \rightarrow 0$, see [28, Appendix C, bound (C.42)].

Following [5], we set

$$u = v_\lambda + r, \quad (\text{A.4})$$

where $r \in H_0^1(M^{\text{int}})$ is the unique solution to the Dirichlet problem,

$$\begin{cases} -\Delta_g r = \Delta_g v_\lambda & \text{in } M^{\text{int}}, \\ r|_{\partial M} = 0. \end{cases} \quad (\text{A.5})$$

Boundary elliptic regularity implies $r \in C^\infty(M)$, and hence, $u \in C^\infty(M)$. Following [14, Appendix A], we fix $\alpha = 1/3$. The following bound, proved in [14, Appendix A, bound (A.15)], will be needed here,

$$\|r\|_{L^2(M)} = \mathcal{O}(\lambda^{1/12}), \quad (\text{A.6})$$

as $\lambda \rightarrow 0$. The proof of (A.6) relies on elliptic estimates for the Dirichlet problem for the Laplacian in Sobolev spaces of low regularity. We shall also need the following rough bound

$$\|r\|_{H^1(M^{\text{int}})} = \mathcal{O}(\lambda^{-1/3}), \quad (\text{A.7})$$

as $\lambda \rightarrow 0$, established in [28, Appendix C, bound (C.41)].

Substituting u into (A.1), and multiplying (A.1) by λ , we get

$$0 = \lambda \int_M \langle A, dv_\lambda + dr \rangle_g (\bar{v}_\lambda + \bar{r}) dV_g = \lambda(I_1 + I_2 + I_3), \quad (\text{A.8})$$

where

$$I_1 = \int_M \langle A, dv_\lambda \rangle_g \bar{v}_\lambda dV_g, \quad I_2 = \int_M \langle A, dr \rangle_g (\bar{v}_\lambda + \bar{r}) dV_g, \quad I_3 = \int_M \langle A, dv_\lambda \rangle_g \bar{r} dV_g.$$

It was computed in [28, Appendix C], see bounds (C.44) and (C.45) there, that

$$\lim_{\lambda \rightarrow 0} \lambda I_1 = \frac{i}{2} \langle A(0), (\tau', i) \rangle. \quad (\text{A.9})$$

It follows from (A.7), (A.2), and (A.6) that

$$\lambda |I_2| \leq \mathcal{O}(\lambda) \|dr\|_{L^2(M)} \|v_\lambda + r\|_{L^2(M)} = \mathcal{O}(\lambda^{2/3}). \quad (\text{A.10})$$

Using (A.3) and (A.6), we get

$$\lambda |I_3| \leq \mathcal{O}(\lambda) \|dv_\lambda\|_{L^2(M)} \|r\|_{L^2(M)} = \mathcal{O}(\lambda^{1/12}). \quad (\text{A.11})$$

Passing to the limit $\lambda \rightarrow 0$ in (A.8) and using (A.9), (A.10), (A.11), we obtain that $\langle A(0), (\tau', i) \rangle = 0$, and arguing as in [28, Appendix C], we get $A|_{\partial M} = 0$. This completes the proof of Proposition A.1. $\square$

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