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Nathan Liao[1], Susanne Strachota[1], Mathías A. López[1], Bárbara M. Brizuela[1], Maria Blanton[2], Angela Gardiner[2]

[1]Tufts University, USA; nathaniel.liao@tufts.edu

[2]TERC, USA

Abstract

This study consisted of Grades 1 and 2 (ages 6-8) classroom teaching experiments (CTE) with a lesson sequence focused on graphical representations of algebraic relationships. We interviewed students before, during, and after the CTE. Here we report on the progression of one Grade 2 (age 7) student's thinking across the CTE. Prior to the CTE, the student had not previously interacted with representations of algebraic relationships. By the end of the CTE, the student was able to generalize about the functional relationships using graphs. This study illustrates how a learning trajectory modeling students' understandings of function graphs can be used to characterize one children's learning and provides evidence that young students are able to reason with function graphs.

Keywords: learning, learning trajectories, mathematics education, algebra.

Introduction

Prior work has illustrated that young students are capable of algebraic reasoning (e.g., Dougherty, 2008), including using variables (e.g., Blanton et al., 2017; Brizuela, Blanton, Sawrey et al., 2015); reasoning about functional relationships (e.g., Brizuela, Blanton, Gardiner et al., 2015), and representing algebraic ideas using tables (Brizuela et al., 2021). A key takeaway from this prior work is that functional relationships are opportunities for young learner to reason algebraically. We build on emerging work (e.g., Strachota et al., 2024) on students' understandings of Cartesian graphs to reason algebraically.

Our study builds on this previous work and assumes that if students have consistent experience with algebraic reasoning earlier in their K-12 schooling, then algebra would be less of a gatekeeper to higher education and subsequent job opportunities.

In this study, we will address the following research questions:

1. How does one student's understanding of function graphs develop over the course of a classroom teaching experiment (CTE)?
2. What aspects of the CTE lessons in which the student participated seem to be associated with shifts in that student's understandings?

Conceptual framework

We rely on the levels of students' understanding of graphs developed in our prior work (viz., Strachota et al., 2024). To develop these levels, Strachota et al. (2024) identified relevant literature on trajectories and progressions in related contexts (e.g., Brizuela et al., 2021; Battista et al., 1998; Gabucio et al., 2010; García-Milá et al., 2014; Martí, 2009; Martí et al., 2010, 2011; Piaget, 1970; Sfard, 1991) and described the ways of thinking about graphs that might occur in the context of functional relationships. Then, using grounded theory (e.g., Corbin & Strauss, 1990) we iteratively revised the levels based on data that was collected through interviews and CTEs with Grades 1 and 2 students. The process resulted in a description of an eight-level trajectory, modelling students' understandings of function graphs (see Table 1); we use these descriptions to characterize students' understandings as well as help explain the shifts we observed in their understandings over time.

Table 1: Descriptions of the levels in the graphing trajectory

Pre-structural	The student does not understand the representation is a graph. They can engage with the graph but not in a way that is unique to this type of representation or that indicates they understand graphs as different from another type of representation.
Interiorization of component(s)	The student understands what the graph is about, what it is referring to, and/or what information is being recorded. They have interiorized (Sfard, 1991) components, such as axes, variables, what quantities are represented by points, etc., but not the graph as a whole. Thus, they do not yet accurately interpret or plot points.
Emergent coordination of quantities	The student interiorizes the coordination of quantities. This might involve gesturing between two coordinated quantities or connecting two coordinated quantities with a line, but not plotting a point or putting their pencil on the point and indicating the precise location. This might include plotting a point at an imprecise, incorrect location.
Condensed coordination of quantities	The student interiorizes the Cartesian coordinate plane (Sfard, 1991). That is, they have or are developing a conceptualization of the graph as two coordinated axes (i.e., uniformed, structured rows and columns according to Battista et al., 1998) and have interiorized the coordination of quantities. Thus, they have condensed a point. They do not however do anything to provide evidence that they understand how two or more points are related.
Emergent recursive inferences	The student can relate two or more points on the graph by describing a recursive pattern. At the emergent recursive inferences level, students describe the change between points in terms of either the x-axis or the y-axis, but not in terms of both. That is, they describe either the “rise” and the “run” of the slope, but not both.
Condensed recursive inferences	The student can relate two or more points on the graph by describing a recursive pattern. They may interpret consecutive points as always going up by the same amount. They do not however describe the general functional relationship represented in the graph. At the condensed recursive inferences level, students describe the change between points in terms of both the x-axis and the y-axis. That is, they describe both the “rise” and the “run” of the slope.
Functional inferences	The student can relate two or more points on the graph and generalize about a functional relationship. When students generalize about a functional relationship, they describe a relationship that extends beyond consecutive points. The distinction between this understanding and making recursive inferences, is that students’ recursive inferences rely on consecutive points, whereas students’ functional inferences are general and connect across points.
Function graph as object	The student understands the function graph as something that extends beyond what is at hand and exists in relation to other function graphs. They can describe how the graph can be transformed or manipulated without having to recreate it or calculate specific values.

Method

We conducted CTEs in Grade 1 and 2 classes at an elementary school in the Northeastern United States. For this paper, we focus on one Grade 2 student’s reasoning with algebraic relationships and representations in the CTE. The data collection consisted of videorecording 14 CTE lessons in each of the Grades 1 and 2 classrooms (ages 5-8). The CTEs lessons were about 40 minutes and taught by a former classroom teacher. We detail the development and content of the CTE lessons below. To track students’ progress in the instructional sequence, we conducted interviews (Steffe & Thompson, 2000). The interviews took place before, during and after the lesson sequence and were conducted by at least two team members. Lessons and interviews were videorecorded.

Classroom Teaching Experiment

Over the course of the 14 CTE lessons, Grade 2 students reasoned with three different functional relationships within multiple problem contexts: 1) $y = x$; 2) $y = 2x$; and 3) $y = x + 1$. The instructional sequence was designed in the form of cycles of revisiting tabular and graphical representations. For instance, students reasoned with tables and graphs in both the “dog eyes” context ($y = 2x$) and the “cutting string” context ($y = x + 1$). The advantage to this cyclical approach is that it provides opportunities to revise our curriculum plans fluidly to support students in advancing their reasoning (Sandoval, 2014).

The lessons were chosen and adapted from prior research that had included CTEs and instructional sequences (Blanton et al. 2015, 2017; Brizuela, Blanton, Gardiner et al., 2015; Brizuela, Blanton, Sawrey et al., 2015). We intentionally chose lessons that focused on representing, including tabular and graphical, or lessons that could be modified to include these representations to create our instructional sequence. Lessons were piloted with other Grade 1 students prior to the CTE.

Individual Interviews

In addition to the lessons, interviews were conducted before, halfway through and after the CTE to document and understand the students’ progress with 1) reasoning with algebraic representations and 2) ability to think recursively and functionally. In the interviews, students were asked to both reason with and create graphical representations to understand the function $y = 2x$ in the context of the relationship between number of birds and bird wings.

Here we report on three interviews from one Grade 2 student, Levi. We selected Levi’s interviews for analysis because they supported us in answering our research questions. Levi provided rich verbal responses and a willingness to engage with graphs. For each of Levi’s interviews, we noted the highest level within the graphing trajectory that he exhibited.

Data analysis

All of Levi’s interviews were transcribed and coded line by line based on Levi’s utterances and gestures. using MAXQDA software. Codes were applied irrespective of mathematical accuracy. The codes we used are outlined in Table 1. To ensure reliability, we used a primary/secondary coding approach and addressed differences using third-party resolution (Syed & Nelson, 2015).

Findings

In the following we report the most sophisticated level that Levi exhibited in each of his three interviews and provide examples that illustrate the levels.

First interview

Since students were not expected to have seen the graph before, the graph that Levi received in his first interview was pre-constructed with points representing one through seven birds and the corresponding wings. Figure 1 shows the representation that was given to Levi and his annotations.

Levi interpreted some components of the graph, namely that the x-axis represented the number of birds and that the number of bird wings is represented with the points. For instance, the following exchange occurred when Levi was asked to solve for the number of bird wings if there were three birds.

Levi:	Ok, so we started by three (draws line from [3,0] to [1,2] to [2,4] to [3,6]).
Interviewer:	Can you tell me what you’re doing?
Levi:	I’m doing the three and then doing all the way up, with the counting by twos, which number we’re starting with.
Interviewer:	Mhm, why did you stop at this point?

Levi:	Because it (the point) has a six.
Interviewer:	Does it have a three too, or just the six?
Levi:	It has a six, you start with the three and then it goes all the way up to six.

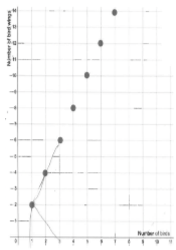

Figure 1: Levi's written work on the graph in his first interview

To solve for the number of bird wings if there were three birds, Levi first marked (3,0) with his pencil, indicating that it is represented three birds on the *x*-axis. Next, Levi drew lines from (3,0) to (1,2) to (2,4) in order to connect (3,0) to (3,6), indicating that (3,6) represented 3 birds and 6 wings. Notably, despite using the *x*-axis as a starting point for his lines, Levi never stated he understood the number of birds is represented by (3,6). When asked if the point "has a three too, or just the six," Levi never stated that three is represented by that point as well. Therefore, we infer that Levi has interiorized *part* of the point, specifically the number of bird wings, evidence of *emergent coordination of quantities* level. He also indicated *emergent recursive inferences* by describing the difference between points by counting up two from the *x*-axis.

At the *emergent coordination of quantities* level, students begin to interiorize how to coordinate quantities in the graph context, such as by drawing lines connecting two covarying quantities. Differing from *condensed coordination of quantities*, students that exhibit this level have not fully interiorized what the points represent.

Second interview

Once again working on the pre-constructed graph, Levi drew lines that connected values on the axes to points on the graph. For instance, when solving for the number of wings if there were two birds, Levi drew a line connecting the *x*-axis (2, 0) to the point (2, 4). This action suggested that he understood two birds were represented within the point (2, 4). When asked where four wings was on the graph, Levi gestured from (0, 4) to (2, 4) and stated that the four is "on this dot," suggesting that he also understood four wings were represented in the point simultaneously. Later in the interview, he drew a line connecting four wings to the point, providing evidence that he was thinking at the *condensed coordination of quantities* level. Levi then accurately coordinated quantities for one, three, and four birds by again drawing guidelines to connect to the points.

Third interview

Similar to his second interview, Levi drew guidelines from the axes to plot points (see Figure 2). The teacher-researcher asked Levi to plot (8, 16) without the use of guidelines. After he accurately plotted the point, the following exchange occurred.

Interviewer:	How are the points helping you figure out where the next point is?
Levi:	Cause they're going like in a pattern, like 1, 2, 3, 4, 5, 6, and they're going diagonal.
Interviewer:	They're going diagonally?
Levi:	Yeah, that's how I know.

Levi's statement reflected the *emergent recursive inferences* level, as he is thinking about the points in relation to each other. By stating "1, 2, 3, 4, 5, 6," Levi described a recursive pattern, indicating he has condensed the points and is now acting on the points, reasoning about they are related.

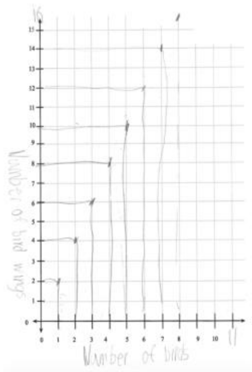

Figure 2: Levi's written work on the graph in his third interview

Summary of findings

To summarize our observations of Levi's progress, we present Table 2, which shows Levi's level of understanding of graphs in each of his three interviews. The table indicates that Levi demonstrated increasingly more sophisticated thinking over the course of the CTE.

Table 2: Levi's levels of understandings of graphs in his first, second, and third interviews

	Interviews		
	1	2	3
Function graph as object			
Functional inferences			
Condensed recursive inferences			
Emergent recursive inferences			
Condensed coordination of quantities			
Emergent coordination of quantities			
Interiorization of components			
Pre-interiorization			

Discussion

The goal of this paper was to describe how one student's understandings of function graphs shifted over the course of a CTE, and to highlight aspects of the lessons of the CTE that seemed associated with those shifts. In the results we presented Levi's understandings of function graphs. Here, we discuss characteristics of the lessons to illustrate how the lessons may have contributed to the shifts in his thinking. For the sake of brevity, we report on two lessons, 4 and 5, that we argue involved content that seemed related to the ways in which Levi reasoned about function graphs.

In lesson 4, students reasoned with a large, human-sized number line taped across the floor that represented the *x*-axis to represent the relationship between number of dogs and dogs' eyes. Students were familiar with a number line but had not yet used the Cartesian coordinate plane, so they practiced using a single number line representing the *x*-axis, with the intention of building on that representation. They reasoned about the relationship, plotting points and stating what quantity represented. When students plotted points on the human-size number line, they walked them out and stood on the point. The goal was to help students *interiorize components* of the graph, namely the *x*-axis and points.

The lesson was also designed to create a need for the *y*-axis. The idea was that students would notice the challenges of representing two different quantities using a single number line and start experimenting with other ways of representing two quantities simultaneously.

In lesson 5, students continued reasoning about the relationship between the number of dogs and eyes. Students continued with the human-sized representation, but this time they used both the *x*- and *y*-axis. Each student had a paper graph as well to record their representation while the class interacted with the human-sized graph.

Because the axes were human-size, students walked out on the axes and Cartesian coordinate plane to represent quantities. We note this activity because we theorize that it may have contributed to developing a visual link between the axes and the quantities, which manifested as drawing guidelines to plot points on a paper graph.

In lesson 5, Levi plotted points on his paper by drawing guidelines from the axes and walked around on the human graph to represent points. Initially, we did not observe an indication that Levi was relating points or noticing a relationship between points. However, a group of students devised a new strategy of plotting new points based on the placement of previously plotted points. This strategy was shared with the whole class on the human-size Cartesian coordinate plane. We theorize that these moments may have helped students *interiorize the y-axis* and begin to think about the points in relation to each other, helping them reach the *emergent recursive inferences* level.

Later in this lesson we also observed that when Levi plotted the point (4, 8), he used the (3, 6) as a reference to "measure." Levi presented this method to the class, and the teacher-researcher asked the students to work individually and try Levi's method of "measuring." This moment affirmed that Levi's strategy was productive. This strategy also indicated that Levi was beginning to relate points.

We share these summaries of lessons 4 and 5 to highlight the ways in which these experiences contributed to developing Levi's reasoning about function graphs. We presume that students' understandings are intimately connected with the curricular progression in which these understandings were developed (Barrett & Battista, 2014; Clements & Sarama, 2014). That is, the trajectory that models Levi's reasoning is dependent on his engagement in the instructional sequence. Moreover, we highlight that these are just two of the 14 lessons in the instructional sequence while reminding readers that lessons were structured similarly throughout the CTE, using repeating representations and functions, but varying the context of the functional relationships.

Conclusion

Levi's thinking in each of his three interviews grew more sophisticated over the course of the CTE. We are impressed with the levels of sophistication of Levi's thinking throughout the CTE, as he only experienced instruction on graphs for a short period of time. We argue that if algebraic representations were woven more consistently throughout mathematics curriculum, we envision that students would more frequently demonstrate these sophisticated ways of algebraic thinking. Aligning with others, particularly those who have conducted longitudinal studies (e.g., Blanton et al., 2015; Dougherty, 2008), we argue that children would benefit from sustained opportunities to reason algebraically to develop "algebraic habits," such as mathematizing unknown quantities and relationships (Blanton et al., 2015, p. 545).

We also view these results as evidence of the power of representations as a tool for enabling students to see things they may not see otherwise. When incorporated properly into instruction, representations are tools that can help students do things more easily or do things they could not do alone (Hiebert et al., 1997). In fact, Levi's reasoning confirms that representations “enable some thoughts that would hardly be possible without them” (Hiebert et al., 1997, p. 53). Consider his observations about relationships between points.

We also highlight the power of representations for cognitive scientists. We presume the representations mediated Levi’s mathematical thinking and supported us to see Levi’s “unseen” ideas (McCormick et al., 1987) about functional relationships. That is, in our study, the graphs were products and interpretations of ideas about how Levi advanced his understandings of functional relationships. With this lens, we can better understand students’ developing understandings of the concept being represented through the representation itself.

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References

Barrett, J. E., & Battista, M. T. (2014). *Two approaches to describing the development of students’ reasoning about length* (Vol. 97). Information Age Publishing. https://doi.org/10.5951/jresematheduc.46.5.0511

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education, 29*(5), 503–532. https://doi.org/10.2307/749731

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics, 95*, 181–202. https://doi.org/10.1007/s10649-016-9745-0

Blanton, M., Brizuela, B., Gardiner, A., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds’ thinking about generalizing functional relationships. *Journal for Research in Mathematics Education, 46*(5), 511–558. https://doi.org/10.5951/jresematheduc.46.5.0511

Brizuela, B. M., Blanton, M., Gardiner, A. M., Newman-Owens, A., & Sawrey, K. (2015). Una alumna de primer grado explora las variables y su notación. [A first grade student’s exploration of variable and variable notation]. *Estudios de Psicología: Studies in Psychology, 36*(1), 138–165. https://doi.org/10.1080/02109395.2014.1000027

Brizuela, B. M., Blanton, M., Sawrey, K., Newman-Owens, A., & Gardiner, A. (2015). Children’s use of variables and variable notation to represent their algebraic ideas. *Mathematical Thinking and Learning, 17*, 1–30. https://doi.org/10.1080/10986065.2015.981939

Brizuela, B. M., Blanton, M., & Kim, Y. (2021). A kindergarten student’s use and understanding of tables while working with function problems. In *Mathematical reasoning of children and adults: Teaching and learning from an interdisciplinary perspective* (pp. 171–190). Springer International Publishing. https://doi.org/10.1007/978-3-030-69657-3_8

Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2017). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). Routledge. https://doi.org/10.4324/9781315097435

Clements, D. H., & Sarama, J. (2014). Learning trajectories: Foundations for effective, research-based education. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 1–30). Charlotte, NC: Information Age Publishing.

Corbin, J. M., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. *Qualitative Sociology, 13*(1), 3–21. https://doi.org/10.1007/BF00988593

Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). Lawrence Erlbaum Associates. https://doi.org/10.4324/9781315097435

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., … Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Heinemann.

McCormick B. H., DeFanti T. A., & Brown M. D. (1987). Visualization in scientific computing: Definition, domain, and recommendations. *Computer Graphics 21*, 3–13.

Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York, NY: W. W. Norton.

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the Learning Sciences, 23*(1), 18–36. https://doi.org/10.1080/10508406.2013.778204

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics, 22*(1), 1–36. https://doi.org/10.1007/BF00302715

Steffe, L. P., & Thompson, P. W. (with von Glasersfeld, E.). (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Lawrence Erlbaum. https://doi.org/10.4324/9781410602725

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS videotape classroom study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States (NCES 99–074).* U.S. Department of Education, National Center for Educational Statistics. Government Printing Office.

Strachota, S., López, M. A., Brizuela, B. M., del Carmen Pérez-Martos, M., Gardiner, A. M., & Blanton, M. (2024). Young Students' Understandings of Function Graphs. In *Lindgren, R., Asino, T. I., Kyza, E. A., Looi, C. K., Keifert, D. T., & Suárez, E. (Eds.), Proceedings of the 18th International Conference of the Learning Sciences - ICLS 2024* (pp. 330–337). International Society of the Learning Sciences. https://doi.org/10.22318/icls2024.380628

Syed, M., & Nelson, S. C. (2015). Guidelines for establishing reliability when coding narrative data. *Emerging Adulthood, 3*(6), 375–387. https://doi.org/10.1177/2167696815587648