Electronic Journal of Probability

Electron. J. Probab. **30** (2025), article no. 25, 1–27.
ISSN: 1083-6489 https://doi.org/10.1214/25-EJP1284

The small-mass limit for some constrained wave equations with nonlinear conservative noise*

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Abstract

We study the small-mass limit, also known as the Smoluchowski-Kramers diffusion approximation (see [16] and [22]), for a system of stochastic damped wave equations, whose solution is constrained to live in the unitary sphere of the space of square integrable functions on the interval $(0, L)$. The stochastic perturbation is given by a nonlinear multiplicative Gaussian noise, where the stochastic differential is understood in Stratonovich sense. Due to its particular structure, such noise not only conserves $\mathbb{P}$-a.s. the constraint, but also preserves a suitable energy functional. In the limit we derive a deterministic system, that remains confined to the unit sphere of L^2, but includes additional terms. These terms depend on the reproducing kernel of the noise and account for the interaction between the constraint and the particular conservative noise we choose.

Keywords: stochastic nonlinear damped wave equations; Smoluchowski-Kramers approximation; SPDEs with constraints.
MSC2020 subject classifications: 60H15; 60F10; 35R60; 35L15.
Submitted to EJP on September 19, 2024, final version accepted on January 24, 2025.

1 Introduction

In recent years, there has been a considerable research activity on the Smoluchowski-Kramers diffusion approximation for infinite-dimensional systems. This is related to the study of the limiting behavior of the solution of a stochastic wave equation with damping, when the mass vanishes. The first results in this direction dealt with the case of constant damping term, with smooth noise and regular coefficients (see [9], [10], [20], and [17]). More recently, the case of constant friction has been studied in [13] and [24] for equations perturbed by space-time white noise in dimension $d = 2$, and in [15] for equations with Hölder continuous coefficients in dimension $d = 1$. In all these papers, the fact that the damping coefficient is constant leads to a perturbative result, in

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the sense that, in the small-mass limit, the solution u_μ of the stochastic damped wave equation converges to the solution of the stochastic parabolic problem formally obtained by taking $\mu = 0$. The case of SPDEs with state-dependent damping was considered first in [11] for a single equation and later, in [8], in the case of systems of equations (see also [12]). Notably, this scenario differs drastically from the previous one, as the non-constant friction leads to an additional noise-induced term in the small-mass limit.

An analogous phenomenon has been identified in [1], where the case of SPDEs constrained to live on a manifold in the functional space of square-integrable functions L^2 was considered. The study of deterministic and stochastic constrained PDEs is not a new field of study. In this context, we would like to mention the paper [19] by Rybka and the paper [6] by Caffarelli and Lin, where, in order to find a gradient flow approach to a specific minimization problem, deterministic heat flows in Hilbert manifolds were explored. A constrained version of the deterministic 2-D Navier-Stokes equation was studied in [4] by Brzezniak, Dhariwal and Mariani, as well as in [7] by Caglioti, Pulvirenti, and Rousset, and later its stochastic version was investigated in [3] and [2] by Brzezniak and Dhariwal.

In [1] we have introduced for the first time a class of damped stochastic wave equations constrained to evolve within the unitary sphere of L^2 and we have shown that the Smoluchowski-Kramers approximation leads to a stochastic parabolic problem, whose solution is still confined to the unitary L^2-sphere and where, as in [11] and [8], an additional drift term appears. Somewhat surprisingly, such extra drift does not account for the Stratonovich-to-Itô correction term.

In the present paper we continue the work started in [1] and we introduce the following system of stochastic wave equations on the interval $(0, L)$

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) + \mu|\partial_t u_\mu(t)|^2_{L^2(0,L)} u_\mu(t,x) = \partial_x^2 u_\mu(t,x) + |\partial_x u_\mu(t)|^2_{L^2(0,L)} u_\mu(t,x) \\ \qquad\qquad -\gamma\partial_t u_\mu(t,x) + \sqrt{\mu}\left(u_\mu(t) \times \partial_t u_\mu(t)\right) \circ \partial_t w(t,x), \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \quad u_\mu(t,0) = u_\mu(t,L) = 0, \end{cases} \tag{1.1}$$

depending on a parameter $0 < \mu << 1$. Here $u_\mu(t,x) \in \mathbb{R}^3$, for every $(t,x) \in [0,+\infty) \times (0,L)$, the friction coefficient γ is strictly positive, and $w(t)$ is a cylindrical Wiener process, white in time and colored in space, defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, with $\circ$ denoting the Stratonovich stochastic differential. The solution $u_\mu(t)$ is subject to the finite-codimension constraint of living on $M = S_{L^2(0,L)}(0,1)$, the unitary sphere of $L^2(0,L)$, with the initial data (u_0, v_0) in $\mathcal{M}$, the tangent bundle of M.

The key and fundamental distinction between the present paper and [1] lies in the nature of the random perturbation considered. Actually, unlike any previous work related to the Smoluchowski-Kramers diffusion-approximation, both in finite and infinite dimensions, here we consider a diffusion coefficient σ_μ which is nonlinear and includes both the position $u_\mu(t)$ and the velocity $\partial_t u_\mu(t)$, through the vector product $\sqrt{\mu}\, u_\mu(t) \times \partial_t u_\mu(t)$. The reason why in all previous works the diffusion does not depend on the velocity is that, while one expects a limit for u_μ, there is no limit for $\partial_t u_\mu$, and it is not clear how to make sense of the limit in the equation, especially when it comes to the martingale term. However, as shown in previous work, the term $\sqrt{\mu}\,\partial_t u_\mu$ can have non trivial limiting behavior, and with this new work we are trying to understand what happens when $\sigma_\mu(u) = \sqrt{\mu}\, u \times v$.

Since in [1] the diffusion coefficient did not include the velocity $\partial_t u_\mu(t)$, the Itô and Stratonovich interpretations of the stochastic differential yielded the same equation. In the current setting, however, the Itô and Stratonovich differentials lead to different equations, and our choice to interpret the stochastic differential in the Stratonovich sense has significant implications. Because of the special structure of the diffusion coefficient, both the Stratonovich and Itô integrals ensure that $(u_\mu(t), \partial_t u_\mu(t))$ remains within the tangent bundle $\mathcal{M}$, for every $t \geq 0$. However, the noise in the Stratonovich sense exhibits a more substantial conservative behavior by preserving also the energy

$$\mathcal{E}_\mu(t) := |u_\mu(t)|^2_{H^1_0(0,L)} + \mu\,|\partial_t u_\mu(t)|^2_{L^2(0,L)} + \int_0^t |\partial_t u_\mu(s)|^2_{L^2(0,L)}\,ds, \tag{1.2}$$

almost surely with respect to $\mathbb{P}$. This phenomenon, well-understood in other contexts, particularly in the parabolic setting, plays a critical role in the scenario considered here, as it serves as a key tool in proving the necessary bounds for $u_\mu(t)$ and $\sqrt{\mu}\,\partial_t u_\mu(t)$ in the appropriate functional spaces, uniformly with respect to $\mu \in (0,1)$. And those bounds are fundamental in the proof of the tightness and in the identification of the limit.

After showing that for every fixed $\mu \in (0,1)$ and $p \geq 1$, and any initial condition $(u_0, v_0) \in [H^1_0(0,L) \times L^2(0,L)] \cap \mathcal{M}$ and $p \geq 1$ there exists a unique mild solution

$$z_\mu = (u_\mu, \partial_t u_\mu) \in L^p(\Omega; C([0,+\infty); [H^1_0(0,L) \times L^2(0,L)] \cap \mathcal{M})),$$

we study the limiting behavior of u_μ, as $\mu \downarrow 0$. Our main result consists in proving that if $(u_0, v_0) \in [H^2(0,L) \times H^1(0,L)] \cap \mathcal{M}$, then, for every $T > 0$ and $\delta < 2$ and for every $\eta > 0$ we have

$$\lim_{\mu \to 0} \mathbb{P}\Big(|u_\mu - u|_{C([0,T];H^\delta)} > \eta\Big) = 0. \tag{1.3}$$

Here u is the unique solution of the deterministic problem

$$\left\{\begin{array}{l} \gamma \partial_t u(t,x) + \dfrac{1}{2}\varphi\,\partial_t(|u(t,x)|^2 u(t,x)) = \partial_x^2 u(t,x) + |\partial_x u(t)|^2_H u(t,x) \\[2ex] \qquad\qquad\qquad + \dfrac{3\varphi(x)}{2\gamma}\left(\left[\partial_x^2 u(t,x) + |\partial_x u(t)|^2_H u(t,x)\right] \cdot u(t,x)\right) u(t,x), \\[2ex] u(0,x) = u_0(x), \quad u(t,0) = u(t,L) = 0, \end{array}\right. \tag{1.4}$$

where

$$\varphi(x) = \sum_{i=1}^\infty \xi_i^2(x), \qquad x \in [0,L],$$

and $\{\xi_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for the reproducing kernel K of the noise $w(t)$. In particular, this means that u_μ converges to a deterministic limit u, which solves a deterministic problem, where the constraint to stay on the unitary sphere of $L^2(0,L)$ is preserved. Remarkably, as in the previously mentioned cases - where however only stochastic limits are obtained - in the small-mass limit several noise-induced terms appear in the limiting equation, and such terms depend on the noise present in the second-order problem through the function φ.

It is important to remark that this non trivial behavior emerges only in the $\sqrt{\mu}$ scaling for the diffusion coefficient, as in the case of μ^α, with $\alpha > 1/2$, the limiting equation (1.4) has to be replaced by the constrained parabolic problem

$$\gamma \partial_t u(t,x) = \partial_x^2 u(t,x) + |\partial_x u(t)|^2_H u(t,x),$$

with the same initial and boundary conditions, where there are no noise-induced terms. As for the limiting behavior of u_μ in the scaling μ^α, with $\alpha \in [0, 1/2)$, at this stage it is not clear what we should expect. We believe that if any limiting point exists, it should satisfy the deterministic equation

$$|u(t)|^2 u(t) = |u_0|^2 u_0 + \frac{3}{\gamma}\int_0^t \big(\partial_x^2 u(s) \cdot u(s)\big) u(s) ds + \frac{3}{\gamma}\int_0^t |\partial_x u(s)|^2_H |u(s)|^2 u(s) ds. \tag{1.5}$$

However, in order to prove the convergence of any limiting point of the family $\{u_\mu\}_{\mu\in(0,1)}$ to a solution of (1.5), tightness in at least $L^2(0,T;H^1_0(0,L))$ would be necessary, and, because of the nature of the diffusion coefficient $\sigma(u,v)=u\times v$, the uniform bounds required for its proof seem to be out of reach.

Under the $\sqrt{\mu}$ scaling assumption, in addition to the energy identity (1.2), we can prove suitable bounds for u_μ and $\partial_t u_\mu$ in spaces of higher regularity than the space $C([0,T];H^1_0(0,L))\cap L^2(0,T;H^2(0,L))$ for u_μ and the space $L^2(0,T;L^2(0,L))$ for $\partial_t u_\mu$, which are uniform with respect to $\mu\in(0,1)$. Those bounds allow to show that the family $\{\mathcal{L}(u_\mu)\}_{\mu\in(0,1)}$ is tight in $C([0,T];H^\delta(0,L))$, for every $\delta<2$, and any weak limit point for u_μ is a solution of (1.4). Equation (1.4) is highly non trivial and the existence of solutions is obtained only as a consequence of the small-mass limit. However, any limiting point for $\{u_\mu\}_{\mu\in(0,1)}$ turns out to belong to the space of functions in $C([0,T];H^1_0(0,L))\cap L^2(0,T;H^2(0,L))$, which admit a weak derivative in time in $L^2(0,T;H)$. What is relevant here is that despite its complex form, we can prove the uniqueness of the solution to equation (1.4) in these functional spaces. Consequently, we can identify any limit point for $\{u_\mu\}_{\mu\in(0,1)}$ with the unique solution of equation (1.4) and limit (1.3) follows.

Before concluding, let us outline the structure of this paper. In Section 2, we introduce all the assumptions and notations that will be used throughout the paper. Section 3 presents the main results. In Section 4, we study the well-posedness of equation (1.1), and in Section 5, we establish bounds for the solution u_μ and $\sqrt{\mu}\,\partial_t u_\mu$ which hold uniformly with respect to $\mu\in(0,1)$. Section 6 focuses on the limiting equation (1.4); we introduce an equivalent formulation and prove the uniqueness of the solution in a suitable functional space. Finally, in Section 7, we demonstrate the validity of limit (1.3). This is achieved by first integrating (1.4) with respect to time and rearranging all terms in a proper way, and then by proving tightness and identifying any weak limit as the unique solution of (1.4).

2 Notations and assumptions

Let H denote the Hilbert space $L^2(0,L;\mathbb{R}^3)$, for some fixed $L>0$, endowed with the inner product

$$\langle u,v\rangle_H=\sum_{i=1}^3\langle u_i,v_i\rangle_{L^2(0,L)}=\int_0^L(u(x)\cdot v(x))\,dx,$$

and the corresponding norm $|\cdot|_H$. Notice that here and in what follows, we shall denote the scalar product of two vectors $h,k\in\mathbb{R}^3$ by $(h\cdot k)$. Moreover we shall denote the norm of a vector h in $\mathbb{R}^3$ by $|h|_{\mathbb{R}^3}$, or just by $|h|$, when there is no risk of confusion.

Next, for every $k\in\mathbb{N}$ we shall denote by H^k the closure of $C_0^\infty([0,L])$ in $W^{k,2}(0,L)$, where $W^{k,2}(0,L)$ is the space of all functions $u\in H$ such that $D^h u$ exists in the weak sense, for every $h\le k$, and $D^h u\in H$. Due to the Poincaré inequality, we can endow H^k with the norm

$$|u|_{H^k}:=|D^k u|_H.$$

Moreover, we shall set $\mathcal{H}_k:=H^{k+1}\times H^k$. When $k=0$, we will simply denote $\mathcal{H}_0$ by $\mathcal{H}$. Finally, for every function $u:(0,L)\to\mathbb{R}^3$ we shall denote

$$|u|_\infty=\sup_{x\in[0,L]}|u(x)|_{\mathbb{R}^3}.$$

Notice that since $[0,L]\subset\mathbb{R}$, we have $H^1\hookrightarrow L^\infty(0,L)$.

If E and F are Banach spaces, the class of all bounded linear operators from E to F will be denoted by $\mathcal{L}(E,F)$. We will use a shortcut notation $\mathcal{L}(E)$ for $\mathcal{L}(E,E)$. It is known that $\mathcal{L}(E,F)$ is also a Banach space. By $\mathcal{L}_2(E,E;F)$ we will denote the Banach space of all bounded bilinear operators from $E\times E=: E^2$ to F. If K is another Hilbert space, by $\mathcal{T}_2(K,F)$, we will denote the Hilbert space of all Hilbert-Schmidt operators from K to F, endowed with the natural inner product and norm. It is known that $\mathcal{T}_2(K,F)\hookrightarrow\mathcal{L}(K,F)$ continuously. If $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of a separable Hilbert space K which is continuously embedded into a Banach space E and

$$\sum_{j=1}^\infty |e_j|_E^2<\infty,$$

then for every $\Lambda\in\mathcal{L}_2(E\times E;F)$ we put

$$\mathrm{tr}_K(\Lambda)=\sum_{i=1}^\infty \Lambda(e_j,e_j). \tag{2.1}$$

In what follows, we denote by M the unit sphere in H

$$M=\big\{u\in H: |u|_H=1\big\},$$

and by $\mathcal{M}$ the corresponding tangent bundle

$$\mathcal{M}=\big\{(u,v)\in M\times H: \big\langle u,v\big\rangle_H=0\big\}.$$

Now, we rewrite equation (1.1) as the following system

$$\begin{cases} du_\mu(t)=v_\mu(t)dt,\\[2mm] dv_\mu(t)=\dfrac{1}{\mu}\left[\partial_x^2 u_\mu(t)+|\partial_x u_\mu(t)|_H^2 u_\mu(t)-\mu|v_\mu(t)|_H^2u_\mu(t)-\gamma v_\mu(t)\right]dt\\[2mm] \qquad\qquad\qquad +\dfrac{1}{\sqrt{\mu}}\big(u_\mu(t)\times v_\mu(t)\big)\circ dw(t),\\[2mm] u_\mu(0)=u_0,\quad v_\mu(0)=v_0,\qquad\qquad u_\mu(t,0)=u_\mu(t,L)=0.\end{cases} \tag{2.2}$$

Here $w(t)$ is a cylindrical Wiener process in H. Thus, if we denote by K its reproducing kernel Hilbert space, we have

$$w(t,x)=\sum_{i=1}^\infty \xi_i(x)\beta_i(t),\qquad\qquad (t,x)\in\ [0,+\infty)\times(0,L),$$

where $\{\beta_i(t)\}_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions, all defined on the stochastic basis $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, and $\{\xi_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of K. In what follows we shall denote by E a Banach space containing K, such that the embedding of K in E is Hilbert-Schmidt. In particular

$$\sum_{i=1}^\infty |\xi_i|_E^2<\infty.$$

Moreover, we shall assume the following conditions are satisfied.

Hypothesis 1. All functions ξ_i belong to $C^1([0,L])$. Moreover, if we denote

$$\varphi(x):=\sum_{i=1}^\infty|\xi_i(x)|^2,\qquad \varphi_1(x):=\sum_{i=1}^\infty|\xi_i'(x)|^2,\qquad x\in(0,L),$$

we have that φ and φ_1 belong to $L^\infty(0,L)$.

Remark 2.1. *Since for every $x \in (0,L)$, we have*

$$\Big|\sum_{i=1}^\infty \xi_i(x)\xi_i'(x)\Big| \le \sqrt{\varphi(x)\varphi_1(x)},$$

thanks to Hypothesis 1 we have that φ is weakly differentiable and

$$\varphi'(x) = 2\sum_{i=1}^\infty \xi_i(x)\xi_i'(x), \qquad x \in\ (0,L).$$

In particular, this allows to conclude that $\varphi \in\ W^{1,\infty}(0,L)$ with

$$|\varphi'|_\infty \le c\,(|\varphi|_\infty + |\varphi_1|_\infty).$$ □

3 Main results

In what follows, we denote by A the realization in H of the second derivative operator, endowed with Dirichlet boundary conditions, and for all $\mu > 0$ define

$$\mathcal{A}_\mu z := \big(v, \mu^{-1}Au\big), \quad z=(u,v) \in D(\mathcal{A}_\mu) = \mathcal{H}_1.$$

Moreover, for every $\mu > 0$ we define

$$F_\mu(z) := -\mu\,|v|_H^2 u + |\partial_x u|_H^2 u, \qquad z=(u,v)\in\mathcal{H}.$$

With these notations, system (2.2) can be rewritten as

$$dz_\mu(t) = \mathcal{A}_\mu z_\mu(t)dt + \frac{1}{\mu}\ (0, F_\mu(z_\mu(t)) - \gamma\, v_\mu(t))\,dt + \frac{1}{\sqrt{\mu}}\big(0, u_\mu(t)\times v_\mu(t)\big)\circ dw(t). \tag{3.1}$$

with $z_\mu(0) = z_0 = (u_0, v_0)$.

The first result we will prove in this paper is the following well-posedness result for system (2.2) in $\mathcal{H}$.

Theorem 3.1. *For every $\mu > 0$ and $z_0 = (u_0,v_0) \in \mathcal{H}\cap\mathcal{M}$, there exists a unique mild solution to the stochastic constrained wave equation* (2.2)*. Namely, there exists a unique $\mathcal{H}\cap\mathcal{M}$-valued continuous and adapted process $z_\mu(t) = (u_\mu(t), v_\mu(t))$, $t\ge 0$, such that the following hold.*

1. *The process $u_\mu(t)$ has M-valued trajectories of class C^1 and*
$$v_\mu(t) = \partial_t u_\mu(t), \quad t \ge 0.$$
2. *The process $z_\mu(t)$ satisfies the equation*
$$z_\mu(t) = \mathcal{S}_\mu(t)z_0 + \frac{1}{\mu}\int_0^t \mathcal{S}_\mu(t-s)\big(0, F_\mu(z_\mu(s)) - \gamma v_\mu(s)\big)ds$$
$$+\frac{1}{\sqrt{\mu}}\int_0^t \mathcal{S}_\mu(t-s)\big(0, (u_\mu(s)\times v_\mu(s))\circ\, dw(s)\big),$$
for every $t\ge 0$, $\mathbb{P}$-almost surely.
3. *The following identity holds for every $t \ge 0$*
$$|u_\mu(t)|_{H^1}^2 + \mu\,|v_\mu(t)|_H^2 + 2\,\gamma\int_0^t |v_\mu(s)|_H^2\,ds = |u_0|_{H^1}^2 + \mu\,|v_0|_H^2, \qquad \mathbb{P}-\text{a.s.} \tag{3.2}$$

The second result, which represents the main goal of this paper, concerns the limiting behavior of the process $u_\mu(t)$, as $\mu \downarrow 0$.

Theorem 3.2. *Fix* $(u_0, v_0) \in \mathcal{H}_1 \cap \mathcal{M}$. *Then, for every* $T > 0$ *and* $\delta < 2$ *and for every* $\eta > 0$ *we have*

$$\lim_{\mu \to 0} \mathbb{P}\Big(|u_\mu - u|_{C([0,T];H^\delta)} > \eta\Big) = 0,$$

where u *is the unique solution of the following deterministic problem*

$$\begin{cases} \partial_t \Big[\Big(\gamma + \dfrac{1}{2}\varphi|u(t)|^2\Big)u(t)\Big] = \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{3\varphi}{2\gamma}\left(\left[\partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t)\right] \cdot u(t)\right) u(t), \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,L) = 0. \end{cases}$$

4 The well-posedness of system (2.2)

As known (see e.g. [5, Definition 3.1]), for an arbitrary function $G : F \to \mathcal{L}(E,F)$ of class C^1

$$\int_0^t G(z(s)) \circ dW(s) := \int_0^t G(z(s))\, dW(s) + \frac{1}{2}\int_0^t \mathrm{tr}_K\left[G'(z(s))G(z(s))\right] ds, \tag{4.1}$$

with tr_K defined as in (2.1). Note that

$$G'(z)\, G(z) \in \mathcal{L}(E; \mathcal{L}(E,F)) \equiv \mathcal{L}_2(E \times E, F), \qquad z \in F,$$

and

$$G'(z)G(z)(e_1, e_2) = [G'(z)\big(G(z)e_1\big)]e_2, \qquad (e_1, e_2) \in E \times E, \quad z \in F.$$

This means that $\mathrm{tr}_K\left[G'(z)G(z)\right]$ is a well defined element of F and satisfies

$$\mathrm{tr}_K[G'(z)G(z)] = \sum_{i=1}^\infty [G'(z)\big(G(z)e_j\big)]e_j,$$

where $\{e_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of K. In particular, if we take

$$G(u,v)k := (0, \sigma(u,v)k), \qquad (u,v) \in \mathcal{H}, \quad k \in E,$$

for some $\sigma : \mathcal{H} \to \mathcal{L}(E, H^1)$, we have

$$[G'(u,v)G(u,v)](k,h) = (0, \partial_v \sigma(u,v)[\sigma(u,v)(k)](h)),$$

so that

$$\mathrm{tr}_K[G'(z)G(z)] = (0, \mathrm{tr}_K[\partial_v \sigma(u,v)\sigma(u,v)]). \tag{4.2}$$

In what follows, we will take

$$\sigma(u,v)k = (u \times v)k.$$

The following result holds.

Lemma 4.1. *The map* $\sigma : \mathcal{H}_k \to \mathcal{T}_2(K, H^k)$ *is Lipschitz-continuous on balls and has polynomial growth, for* $k = 0, 1$. *Moreover, its Fréchet derivative along any direction* $v \in H^k$ *is given by*

$$\partial_v \sigma(z) y = u \times y, \quad z = (u,v) \in \mathcal{H}_k, \quad y \in H^k,$$

and the function

$$\mathcal{H}_k \ni z = (u, v) \mapsto \mathrm{tr}_K [\partial_v \sigma(z)\sigma(z)] = \mathrm{tr}_K [u \times (u \times v)] = \varphi [u \times (u \times v)] \in H^k, \tag{4.3}$$

is Lipschitz-continuous on balls with polynomial growth.

Proof. Case $k = 0$. For every $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ in $\mathcal{H}$ we have

$$\begin{aligned}
\|\sigma(z_1) - \sigma(z_2)\|^2_{\mathcal{T}_2(K,H)} &= \sum_{i=1}^{\infty} |(\sigma(z_1) - \sigma(z_2))\xi_i|^2_H \\
&= \int_0^L |u_1(x) \times v_1(x) - u_2(x) \times v_2(x)|^2 \sum_{i=1}^{\infty} |\xi_i(x)|^2 \, dx \leq c \, |\varphi|_\infty |u_1 \times v_1 - u_2 \times v_2|^2_H \\
&\leq c \, |\varphi|_\infty \Big(|u_1 - u_2|^2_{H^1} |v_1|^2_H + |u_2|^2_{H^1} |v_1 - v_2|^2_H \Big) \leq c \, |\varphi|_\infty \big(|z_1|^2_{\mathcal{H}} + |z_2|^2_{\mathcal{H}} \big) |z_1 - z_2|^2_{\mathcal{H}}.
\end{aligned}$$

Since $\sigma(0) = 0$, this implies that $\sigma : \mathcal{H} \to \mathcal{T}_2(K, H)$ is well defined and locally Lipschitz continuous, with quadratic growth.

For every $u \in H^1$ fixed, the mapping $\sigma(u, \cdot) : H \to \mathcal{L}(E, H)$ is linear and its derivative $\partial_v \sigma(u, \cdot)$ is given by

$$\partial_v \sigma(z) \, y = u \times y \in \mathcal{L}(E, H), \qquad z = (u, v) \in \mathcal{H}, \quad y \in H.$$

In particular $\partial_v \sigma(z) \, \sigma(z) \in \mathcal{L}(E \times E, H)$ and

$$\mathrm{tr}_K [\partial_v \sigma(z)\sigma(z)] = \mathrm{tr}_K [u \times (u \times v)] = \sum_{i=1}^{\infty} (u \times (u \times v)\xi_i) \, \xi_i = \varphi \, (u \times (u \times v)) \, .$$

Now, for every $h, k \in \mathbb{R}^3$ we have

$$h \times (h \times k) = -|h|^2 k + (h \cdot k) h, \tag{4.4}$$

so that we can write

$$\mathrm{tr}_K [\partial_v \sigma(z)\sigma(z)] = \varphi(-|u|^2 v + (u \cdot v) u),$$

and

$$|\mathrm{tr}_K [u \times (u \times v)]|^2_H = \int_0^L \varphi(x)^2 \left| -|u(x)|^2 v(x) + (u(x) \cdot v(x)) u(x) \right|^2 dx$$

$$= \int_0^L \varphi(x)^2 \big(|u(x)|^4 |v(x)|^2 - |u(x)|^2 (u(x) \cdot v(x))^2 \big) dx \leq c \, |\varphi|^2_\infty |u|^4_{H^1} |v|^2_H \leq c |\varphi|^2_\infty |z|^6_{\mathcal{H}}.$$

Moreover,

$$\begin{aligned}
&\big| \mathrm{tr}_K (u_1 \times (u_1 \times v_1)) - \mathrm{tr}_K (u_2 \times (u_2 \times v_2)) \big|^2_H \\
&\leq c \int_0^L \varphi(x)^2 \, \Big(\big(|u_1(x)|^2 + |u_2(x)|^2 \big) \big(|v_1(x)|^2 + |v_2(x)|^2 \big) |u_1(x) - u_2(x)|^2 \\
&\qquad + \big(|u_1(x)|^4 + |u_2(x)|^4 \big) |v_1(x) - v_2(x)|^2 \Big) \, dx \\
&\leq c |\varphi|^2_\infty \Big(\big(|u_1|^2_{H^1} + |u_2|^2_{H^1} \big) \big(|v_1|^2_H + |v_2|^2_H \big) |u_1 - u_2|^2_{H^1} + \big(|u_1|^4_{H^1} + |u_2|^4_{H^1} \big) |v_1 - v_2|^2_H \Big) \\
&\leq c |\varphi|^2_\infty \big(|z_1|^4_{\mathcal{H}} + |z_2|^4_{\mathcal{H}} \big) |z_1 - z_2|^2_{\mathcal{H}}.
\end{aligned}$$

This implies that the mapping $z \in \mathcal{H} \mapsto \text{tr}_K\,[\partial_v\sigma(z)\sigma(z)] \in H$ is well-defined and locally Lischitz-continuous, with cubic growth.

Case $k = 1$. For any $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ in $\mathcal{H}_1$, we have

$$\|\sigma(z_1) - \sigma(z_2)\|^2_{\mathcal{T}_2(K,H^1)} = \sum_{i=1}^{\infty} |(\sigma(z_1) - \sigma(z_2))\xi_i|^2_{H^1}$$

$$\leq 2\sum_{i=1}^{\infty} \left|\xi_i\,(u_1 \times v_1 - u_2 \times v_2)'\right|^2_H + 2\sum_{i=1}^{\infty} \left|\xi_i'\,(u_1 \times v_1 - u_2 \times v_2)\right|^2_H$$

$$\leq c|\varphi|_\infty\Big(|(u_1 - u_2)' \times v_1|^2_H + |(u_1 - u_2) \times v_1'|^2_H + |u_2' \times (v_1 - v_2)|^2_H + |u_2 \times (v_1 - v_2)'|^2_H\Big)$$

$$+c|\varphi_1|_\infty\Big(|(u_1 - u_2) \times v_1|^2_H + |u_2 \times (v_1 - v_2)|^2_H\Big)$$

$$\leq c\big(|\varphi|_\infty + |\varphi_1|_\infty\big)\big(|u_1 - u_2|^2_{H^1}|v_1|^2_{H^1} + |u_2|^2_{H^1}|v_1 - v_2|^2_{H^1}\big)$$

$$\leq c\big(|\varphi|_\infty + |\varphi_1|_\infty\big)\big(|z_1|^2_{\mathcal{H}_1} + |z_2|^2_{\mathcal{H}_1}\big)|z_1 - z_2|^2_{\mathcal{H}_1}.$$

This implies the local Lipschitz-continuity and polynomial growth of the mapping $\sigma : \mathcal{H}_1 \to \mathcal{T}_2(K, H^1)$. Finally, for every $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ in $\mathcal{H}_1$

$$\left|\text{tr}_K(u_1 \times (u_1 \times v_1)) - \text{tr}_K(u_2 \times (u_2 \times v_2))\right|^2_{H^1}$$

$$\leq c\,|\varphi|^2_\infty\Big(|(u_1 - u_2)' \times (u_1 \times v_1)|^2_H + |(u_1 - u_2) \times (u_1 \times v_1)'|^2_H\Big)$$

$$+c\,|\varphi|^2_\infty\Big(|u_2' \times (u_1 \times v_1 - u_2 \times v_2)|^2_H + |u_2 \times (u_1 \times v_1 - u_2 \times v_2)'|^2_H\Big)$$

$$+c\,|\varphi'|^2_\infty|u_1 \times (u_1 \times v_1) - u_2 \times (u_2 \times v_2)|^2_H$$

$$\leq c\left(|\varphi|^2_\infty + |\varphi'|^2_\infty\right)\left(|z_1|^2_{\mathcal{H}_1} + |z_2|^2_{\mathcal{H}_1}\right)^2|z_1 - z_2|^2_{\mathcal{H}_1},$$

and this implies that the mapping

$$\mathcal{H}_1 \ni\mapsto \text{tr}_K\,[u \times (u \times v)] \in H^1,$$

is locally Lipschitz continuous and has cubic growth. □

As a consequence of (4.1), (4.2) and (4.3), we can rewrite system (2.2) as

$$\begin{cases} du_\mu(t) = v_\mu(t)dt, \\ dv_\mu(t) = \dfrac{1}{\mu}\big[\partial_x^2 u_\mu(t) + |\partial_x u_\mu(t)|^2_H u_\mu(t) - \mu|v_\mu(t)|^2_H u_\mu(t) - \gamma v_\mu(t) \\ \qquad\qquad +\dfrac{1}{2\mu}\,\text{tr}_K(u_\mu(t) \times (u_\mu(t) \times v_\mu(t)))\big]\,dt + \dfrac{1}{\sqrt{\mu}}\big(u_\mu(t) \times v_\mu(t)\big)dw(t), \\ u_\mu(0) = u_0, \quad v_\mu(0) = v_0, \qquad u_\mu(t,0) = u_\mu(t,L) = 0. \end{cases} \tag{4.5}$$

In particular, equation (3.1) can be rewritten as

$$dz_\mu(t) = \mathcal{A}_\mu z_\mu(t)dt + \frac{1}{\mu}\left(0, F_\mu(z_\mu(t)) - \gamma v_\mu(t)\right)dt$$

$$+\frac{1}{2\mu}\left(0, \mathrm{tr}_K\left(u_\mu \times (u_\mu(t) \times v_\mu(t))\right)\right)dt + \frac{1}{\sqrt{\mu}}\left(0, u_\mu(t) \times v_\mu(t)\right)dw(t),$$

with $z_\mu(0) = z_0 = (u_0, v_0)$. This allows to say that if a process $u_\mu(t)$ has M-valued trajectories of class C^1 and

$$v_\mu(t) = \partial_t u_\mu(t), \quad t \geq 0,$$

and $z_\mu(t) = (u_\mu(t), v_\mu(t))$, then the process z_μ is a mild solution of equation (2.2), with initial condition z_0, if for every $t \geq 0$, $\mathbb{P}$-almost surely,

$$z_\mu(t) = \mathcal{S}_\mu(t)z_0 + \frac{1}{\mu}\int_0^t \mathcal{S}_\mu(t-s)\big(0, F_\mu(z_\mu(s)) - \gamma v_\mu(s)\big)ds$$

$$+\frac{1}{2\mu}\int_0^t \mathcal{S}_\mu(t-s)\big(0, \mathrm{tr}_K\left(u_\mu(s) \times (u_\mu(s) \times v_\mu(s))\right)\big)ds$$

$$+\frac{1}{\sqrt{\mu}}\int_0^t \mathcal{S}_\mu(t-s)\big(0, (u_\mu(s) \times v_\mu(s))\,dw(s)\big).$$

4.1 Proof of Theorem 3.1

It is immediate to check that $F_\mu : \mathcal{H}_k \to H^k$ is Lipschitz continuous when restricted to balls, for all $k \geq 0$, and has cubic growth. Thus, in view of Lemma 4.1, the proof follows from a modification of the arguments introduced in [1, proof of Theorems 2.9 and 2.10].

Due to the local Lipschitz continuity of all coefficients in $\mathcal{H}$, equation (3.1) admits a unique maximal local mild solution $z_\mu \in C([0, \tau_\mu); \mathcal{H})$, defined up to a certain stopping time τ_μ. Our purpose is showing that $z_\mu(t) \in \mathcal{M}$, for all $t \in [0, \tau_\mu)$, and

$$\mathbb{P}(\tau_\mu = \infty) = 1. \tag{4.6}$$

In this way, we get the existence and uniqueness of a global mild solution z_μ in the space $C([0, +\infty); \mathcal{H} \cap \mathcal{M})$.

In order to prove the invariance of the tangent bundle $\mathcal{M}$, we introduce the following processes

$$\vartheta_\mu(t) := \frac{1}{2}\left(|u_\mu(t)|_H^2 - 1\right), \qquad \eta_\mu(t) := \langle u_\mu(t), v_\mu(t)\rangle_H, \quad t \in [0, \tau_\mu).$$

If we show that they satisfy the linear system

$$\begin{cases} d\vartheta_\mu(t) = \eta_\mu(t)\,dt \\ d\eta_\mu(t) + \dfrac{\gamma}{\mu}\,d\vartheta_\mu(t) = \left(\dfrac{1}{\mu}|u_\mu(t)|_{H^1}^2 - |v_\mu(t)|_H^2\right)\vartheta_\mu(t), \qquad t \in [0, \tau_\mu), \end{cases} \tag{4.7}$$

since $\vartheta_\mu(0) = \eta_\mu(0) = 0$, we obtain that $\vartheta_\mu(t) = \eta_\mu(t) = 0$, for every $t \in [0, \tau_\mu)$, $\mathbb{P}$-a.s., and this implies that $z_\mu(t) \in \mathcal{M}$, for every $t \in [0, \tau_\mu)$, $\mathbb{P}$-a.s.

As in [1], it can be shown the following fact.

Lemma 4.1. Assume that a local process $z_\mu(t) = (u_\mu(t), v_\mu(t))$, $t \in [0, \sigma_\mu)$ is a solution to

$$z_\mu(t) = \mathcal{S}_\mu(t)z_0 + \int_0^t \mathcal{S}_\mu(t-s)\big(0, f(s)\big)\,ds + \int_0^t \mathcal{S}_\mu(t-s)\big(0, g(s)\big)\,dw(s), \qquad t \in [0, \sigma_\mu). \tag{4.8}$$

where all processes are progressively measurable, f is H-valued and g is $\mathcal{T}_2(K,H)$-valued. Then, for every $t \in [0,\sigma_\mu)$, $\mathbb{P}$-almost surely,

$$\begin{aligned}\langle u_\mu(t), v_\mu(t)\rangle_H = \langle u_\mu(0), v_\mu(0)\rangle_H - \frac{1}{\mu}\int_0^t |\partial_x u_\mu(s)|_H^2\,ds + \int_0^t \langle u_\mu(s), f(s)\rangle_H\,ds \\ + \int_0^t |v_\mu(s)|_H^2\,ds + \int_0^t \langle u_\mu(s), g(s)\,dw(s)\rangle_H,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}|v_\mu(t)|_H^2 + \frac{2}{\mu}|\partial_x u_\mu(t)|_H^2 = |v_\mu(0)|_H^2 + \frac{2}{\mu}|\partial_x u_\mu(0)|_H^2 + 2\int_0^t \langle v_\mu(s), f(s)\rangle_H\,ds \\ +2\int_0^t \langle v_\mu(s), g(s)dw(s)\rangle_H + \int_0^t \|g(s)\|_{\mathcal{T}_2(K,H)}^2\,ds.\end{aligned} \tag{4.10}$$

□

Now, the local solution $z_\mu(t) = (u_\mu(t), v_\mu(t))$ of equation (3.1) satisfies equation (4.8), with

$$f(s) := -|v_\mu(s)|_H^2 u_\mu(s) + \frac{1}{\mu}|u_\mu(s)|_{H^1}^2 u_\mu(s) - \frac{\gamma}{\mu}v_\mu(s) + \frac{1}{2\mu}\mathrm{tr}_K\left[u_\mu(s)\times(u_\mu(s)\times v_\mu(s))\right],$$

and

$$g(s) := \frac{1}{\sqrt{\mu}}\sigma(z_\mu(s)) = \frac{1}{\sqrt{\mu}}\,u_\mu(s)\times v_\mu(s).$$

Notice that

$$\langle u_\mu(s), f(s)\rangle_H = -|v_\mu(s)|_H^2|u_\mu(s)|_H^2 + \frac{1}{\mu}|u_\mu(s)|_{H^1}^2|u_\mu(s)|_H^2 - \frac{\gamma}{\mu}\eta_\mu(s),$$

and for every $\xi \in K$

$$\langle u_\mu(s), g(s)\xi\rangle_H = 0.$$

Thus, thanks to identity (4.9) in Lemma 4.1, we have

$$\begin{aligned}\eta_\mu(t) - \eta_\mu(0) &= \frac{1}{\mu}\int_0^t |u_\mu(s)|_{H^1}^2\left(|u_\mu(s)|_H^2 - 1\right)\,ds \\ &\quad - \int_0^t |v_\mu(s)|_H^2\left(|u_\mu(s)|_H^2 - 1\right)\,ds - \frac{\gamma}{\mu}\int_0^t \langle u_\mu(s), v_\mu(s)\rangle_H\,ds \\ &= \frac{1}{\mu}\int_0^t |u_\mu(s)|_{H^1}^2\,\vartheta_\mu(s)\,ds - \int_0^t |v_\mu(s)|_H^2\,\vartheta_\mu(s)\,ds - \frac{\gamma}{\mu}\int_0^t \eta_\mu(s)\,ds.\end{aligned}$$

In particular, the processes $\vartheta_\mu(t)$ and $\eta_\mu(t)$ satisfy equation (4.7), and, as explained above, this implies that $z_\mu(t) \in \mathcal{M}$, for every $t \in [0,\tau_\mu)$, $\mathbb{P}$-a.s.

Next, let us prove (4.6). Since $(u_\mu(t), v_\mu(t)) \in \mathcal{M}$, we have

$$\langle v_\mu(t), f(t)\rangle_H = -\frac{\gamma}{\mu}\,|v_\mu(t)|_H^2 + \frac{1}{2\mu}\langle v_\mu(t), \mathrm{tr}_K\left[u_\mu(s)\times(u_\mu(s)\times v_\mu(s))\right]\rangle_H.$$

As

$$\langle v_\mu(t), g(t)\xi\rangle_H = 0, \qquad \xi \in K,$$

for every $t < \tau_\mu$ this gives

$$d|v_\mu(t)|_H^2 = \frac{2}{\mu}\big\langle v_\mu(t), \partial_x^2 u_\mu(t)\big\rangle_H dt - \frac{2\gamma}{\mu}|v_\mu(t)|_H^2 dt + \frac{1}{\mu}\ \|u(t)\times v(t)\|^2_{\mathcal{T}_2(K,H)}\,dt$$

$$+\frac{1}{\mu}\langle v_\mu(t), \mathrm{tr}_K\,[u_\mu(s)\times(u_\mu(s)\times v_\mu(s))]\rangle_H\,dt.$$

In view of (4.4), for every $h,k\in\mathbb{R}^3$ we have

$$|h\times k|^2 + (k\cdot[h\times(h\times k)]) = |h\times k|^2 + |(h\cdot k)|^2 - |h|^2|k|^2 = 0.$$

Therefore, we obtain

$$d|v_\mu(t)|_H^2 = -\frac{1}{\mu}\,d|u_\mu(t)|^2_{H^1} - \frac{2\gamma}{\mu}|v_\mu(t)|_H^2 dt,$$

and this implies that for every $t < \tau_\mu$

$$|u_\mu(t)|^2_{H^1} + \mu\,|v_\mu(t)|_H^2 + 2\gamma\int_0^t |v_\mu(s)|_H^2 ds = |u_0|^2_{H^1} + \mu\,|v_0|_H^2, \quad \mathbb{P}\text{-a.s.} \tag{4.11}$$

In particular, we can conclude that for every $\mu > 0$ there exists some deterministic constant $\kappa_\mu > 0$ such that

$$\sup_{t\in[0,\tau_\mu)} |z_\mu(t)|_{\mathcal{H}} \leq \kappa_\mu, \qquad \mathbb{P}-\text{a.s.},$$

and (4.6) follows. Finally, by combining together (4.6) and (4.11), we obtain (3.2).

5 Energy estimates

In Theorem 3.1 we have seen that for every $(u_0,v_0)\in\mathcal{H}\cap\mathcal{M}$ and $\mu>0$, equation (2.2) (and, equivalently, equation (4.5)) admits a unique mild solution $z_\mu = (u_\mu, v_\mu) \in C([0,+\infty);\mathcal{H})$. Moreover, for every $\mu > 0$ and $t > 0$ the following identity holds

$$|u_\mu(t)|^2_{H^1} + \mu|v_\mu(t)|_H^2 + 2\gamma\int_0^t |v_\mu(s)|_H^2 ds = |u_0|^2_{H^1} + \mu|v_0|_H^2, \quad \mathbb{P}\text{-a.s.} \tag{5.1}$$

In this section, our purpose is proving that for every $(u_0,v_0)\in\mathcal{H}_1\cap\mathcal{M}$, there exists a constant $c>0$ such that for every $\mu\in(0,1)$ and $t\geq 0$

$$\mathbb{E}\sup_{r\in[0,t]}\Big(|u_\mu(r)|^2_{H^2} + \mu|v_\mu(r)|^2_{H^1}\Big) + \mathbb{E}\int_0^t |v_\mu(s)|^2_{H^1} ds \leq c. \tag{5.2}$$

The inequality above is a consequence of the following Lemma.

Lemma 5.1. For every $(u_0,v_0)\in\mathcal{H}_1\cap\mathcal{M}$, there exist two constants $c_1,c_2>0$ depending only on $|(u_0,\sqrt{\mu}\,v_0)|_{\mathcal{H}_1}$, φ and φ_1, such that for every $\mu\in(0,1)$ and $t\geq 0$

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\Big(|u_\mu(r)|^2_{H^2} + \mu|v_\mu(r)|^2_{H^1} + \mu|u_\mu(r)|^2_{H^1}|v_\mu(r)|^2_H\Big)\\
&+\mathbb{E}\int_0^t\Big(|v_\mu(s)|^2_{H^1} + |u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H + \mu|v_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H + \mu|u_\mu(s)|^2_{H^1}|v_\mu(s)|^4_H\Big)ds\\
&\leq c_1\Big(|u_0|^2_{H^2} + \mu|v_0|^2_{H^1} + \mu|u_0|^2_{H^1}|v_0|^2_H\Big)\exp\Big(c_2\big(|u_0|^2_{H^1} + \mu|v_0|^2_H\big)\Big).
\end{aligned} \tag{5.3}$$

Proof. By applying Itô's formula to $|v_\mu(t)|^2_{H^1}$, we get

$$\begin{aligned}
d|v_\mu(t)|^2_{H^1} &= \frac{2}{\mu}\Big(\big\langle v_\mu(t), \partial_x^2 u_\mu(t)\big\rangle_{H^1} + |u_\mu(t)|^2_{H^1}\big\langle v_\mu(t), u_\mu(t)\big\rangle_{H^1} \\
&-\mu|v_\mu(t)|^2_H\big\langle v_\mu(t), u_\mu(t)\big\rangle_{H^1} - \gamma|v_\mu(t)|^2_{H^1} + \frac{1}{2}\big\langle v_\mu(t), \mathrm{tr}_K\big(u_\mu(t)\times(u_\mu(t)\times v_\mu(t))\big)\big\rangle_{H^1} \\
&+\frac{1}{2}\,\|u_\mu(t)\times v_\mu(t)\|^2_{\mathcal{T}_2(K,H^1)}\Big)\,dt + \frac{2}{\sqrt{\mu}}\big\langle v_\mu(t), (u_\mu(t)\times v_\mu(t))dw(t)\big\rangle_{H^1} \\
&= -\frac{1}{\mu}d|u_\mu(t)|^2_{H^2} + |u_\mu(t)|^2_{H^1}\,d\left(\frac{1}{\mu}|u_\mu(t)|^2_{H^1} + |v_\mu(t)|^2_H\right) - d\Big(|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H\Big) \\
&+\frac{1}{\mu}\Big(-2\gamma|v_\mu(t)|^2_{H^1} + \big\langle v_\mu(t), \mathrm{tr}_K\big(u_\mu(t)\times(u_\mu(t)\times v_\mu(t))\big)\big\rangle_{H^1} \\
&+\|u_\mu(t)\times v_\mu(t)\|^2_{\mathcal{T}_2(K,H^1)}\Big)\,dt + \frac{2}{\sqrt{\mu}}\big\langle v_\mu(t), (u_\mu(t)\times v_\mu(t))dw(t)\big\rangle_{H^1}.
\end{aligned} \tag{5.4}$$

According to (5.1), this implies that

$$\begin{aligned}
&d\Big(|u_\mu(t)|^2_{H^2} + \mu|v_\mu(t)|^2_{H^1} + \mu|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H\Big) \\
&= -2\gamma|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H dt - 2\gamma|v_\mu(t)|^2_{H^1}dt + \big\langle v_\mu(t), \mathrm{tr}_K\big(u_\mu(t)\times(u_\mu(t)\times v_\mu(t))\big)\big\rangle_{H^1}dt \\
&\quad+\|u_\mu(t)\times v_\mu(t)\|^2_{\mathcal{T}_2(K,H^1)}\,dt + 2\sqrt{\mu}\big\langle v_\mu(t), (u_\mu(t)\times v_\mu(t))dw(t)\big\rangle_{H^1}.
\end{aligned}$$

Now, let us define

$$Y_{\mu,a}(t) := \exp\left(-a\int_0^t |v_\mu(s)|^2_H ds\right), \quad t \geq 0, \quad \mu > 0, \tag{5.5}$$

for some constant $a > 0$ to be determined later. We have

$$\begin{aligned}
&d\Big(Y_{\mu,a}(t)\Big(|u_\mu(t)|^2_{H^2} + \mu|v_\mu(t)|^2_{H^1} + \mu|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H\Big)\Big) \\
&= Y_{\mu,a}(t)\Big(-a|u_\mu(t)|^2_{H^2}|v_\mu(t)|^2_H - a\mu|v_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H - a\mu|u_\mu(t)|^2_{H^1}|v_\mu(t)|^4_H \\
&\quad-2\gamma|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H - 2\gamma|v_\mu(t)|^2_{H^1} + \big\langle v_\mu(t), \mathrm{tr}_K\big(u_\mu(t)\times(u_\mu(t)\times v_\mu(t))\big)\big\rangle_{H^1} \\
&\quad+\|u_\mu(t)\times v_\mu(t)\|^2_{\mathcal{T}_2(K,H^1)}\Big)dt + 2\sqrt{\mu}\,Y_{\mu,a}(t)\big\langle v_\mu(t), (u_\mu(t)\times v_\mu(t))dw(t)\big\rangle_{H^1}.
\end{aligned}$$

Thus, if we integrate with respect to $t \geq 0$, we get

$$\begin{aligned}
&Y_{\mu,a}(t)\Big(|u_\mu(t)|^2_{H^2} + \mu|v_\mu(t)|^2_{H^1} + \mu|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H\Big) + \int_0^t Y_{\mu,a}(s)\Big(a|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H \\
&\quad+a\mu|v_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H + a\mu|u_\mu(s)|^2_{H^1}|v_\mu(s)|^4_H + 2\gamma|u_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H + 2\gamma|v_\mu(s)|^2_{H^1}\Big)ds \\
&= |u_0|^2_{H^2} + \mu|v_0|^2_{H^1} + \mu|u_0|^2_{H^1}|v_0|^2_H \\
&\quad+\int_0^t Y_{\mu,a}(s)\,J(u_\mu(s), v_\mu(s))ds + 2\sqrt{\mu}\int_0^t Y_{\mu,a}(s)\,G(u_\mu(s), v_\mu(s))dw(s),
\end{aligned} \tag{5.6}$$

where

$$J(u,v) := \big\langle v, \mathrm{tr}_K(u\times(u\times v))\big\rangle_{H^1} + \|u\times v\|^2_{\mathcal{T}_2(K,H^1)}\,, \quad (u,v)\in\mathcal{H}_1,$$

and

$$G(u,v)\xi := \big\langle v, (u\times v)\xi\big\rangle_{H^1}, \quad (u,v)\in\mathcal{H}_1, \quad \xi\in K.$$

Note that for every $(u,v)\in\mathcal{H}_1$

$$D\big(\mathrm{tr}_K\big(u\times(u\times v)\big)\big) = \big(-|u|^2 v + (u\cdot v)u\big)\,\varphi'$$

$$+\Big(-2(Du\cdot u)v - |u|^2 Dv + (Du\cdot v)u + (u\cdot Dv)u + (u\cdot v)Du\Big)\varphi,$$

so that we have

$$\begin{aligned}\big\langle v, \mathrm{tr}_K\big(u\times(u\times v)\big)\big\rangle_{H^1} &= \int_0^L \Big(-|u|^2(v\cdot Dv) + (u\cdot v)(u\cdot Dv)\Big)\varphi' dx\\ &+\int_0^L \Big(-2(Du\cdot u)(Dv\cdot v) - |u|^2|Dv|^2 + (Du\cdot v)(u\cdot Dv)\Big)\varphi dx\\ &+\int_0^L \Big(-2(Du\cdot u)(Dv\cdot(u\cdot Dv)^2 + (u\cdot v)(Du\cdot Dv)\Big)\varphi dx.\end{aligned} \tag{5.7}$$

Moreover,

$$\begin{aligned}&\|u\times v\|^2_{\mathcal{T}_2(K,H^1)}\\ &= \sum_{i=1}^\infty \int_0^L \big|(u\times v)\xi_i' + D(u\times v)\xi_i\big|^2 dx = \int_0^L \sum_{i=1}^\infty \big|(u\times v)\xi_i' + D(u\times v)\xi_i\big|^2 dx\\ &= \int_0^L |u\times v|^2\varphi_1 dx + \int_0^L \big|D(u\times v)\big|^2\varphi dx + 2\int_0^L (u\times v\cdot D(u\times v))\sum_i \xi_i\xi_i' dx\\ &= \int_0^L |u\times v|^2\varphi_1 dx + \int_0^L \big|Du\times v + u\times Dv\big|^2\varphi dx\\ &\quad +2\int_0^L (u\times v)\cdot\big(Du\times v + u\times Dv\big)\sum_i \xi_i\xi_i' dx.\end{aligned} \tag{5.8}$$

Due to the well-known identity

$$((a\times b)\cdot(c\times d)) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c), \qquad a,b,c,d\in\mathbb{R}^3,$$

from (5.7) and (5.8) we get

$$\begin{aligned}J(u,v) &= \int_0^L (u\times v)^2\varphi_1 dx + 2\int_0^L \big((u\times v)\cdot(Du\times v)\big)\sum_i \xi_i\xi_i'\, dx\\ &+\int_0^L \big[(Du\times v)^2 - \big((Du\times u)\cdot(Dv\times v)\big)\big]\varphi\, dx.\end{aligned} \tag{5.9}$$

In particular, this implies that for any $\epsilon>0$

$$\begin{aligned}|J(u,v)| &\le c|\varphi_1|_\infty |u|^2_{H^1}|v|^2_H + c\sqrt{|\varphi|_\infty|\varphi_1|_\infty}|u|_{H^1}|u|_{H^2}|v|^2_H\\ &+c|\varphi|_\infty |u|_{H^2}|u|_{H^1}|v|_{H^1}|v|_H \le \epsilon\,|u|^2_{H^1}|v|^2_{H^1} + c(\epsilon)|u|^2_{H^2}|v|^2_H,\end{aligned}$$

for some constant $c(\epsilon) = c(\epsilon, |\varphi|_\infty, |\varphi_1|_\infty) > 0$. Hence, according to (5.1), we obtain

$$\mathbb{E} \sup_{r\in[0,t]} \left|\int_0^r Y_{\mu,a}(s)\, J(u_\mu(s), v_\mu(s))ds\right|$$

$$\leq \epsilon \left(|u_0|^2_{H^1} + \mu|v_0|^2_H\right) \mathbb{E} \int_0^t Y_{\mu,a}(s)|v_\mu(s)|^2_{H^1}\, ds + c(\epsilon)\, \mathbb{E} \int_0^t Y_{\mu,a}(s)|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H\, ds. \tag{5.10}$$

Next, for every $(u, v) \in \mathcal{H}_1$ and $k \in K$ we have

$$G(u,v)k = \int_0^L \left(Dv \cdot (u\times v)k' + [Du \times v + u \times Dv]\, k\right)\, dx$$

$$= \int_0^L \left(Dv \cdot (u\times v)k' + (Du \times v)k\right)\, dx,$$

so that

$$\|G(u,v)\|^2_{\mathcal{T}_2(K,\mathbb{R})} = \sum_{i=1}^\infty \Big| \int_0^L \left[(Dv\cdot(u\times v))\,\xi_i' + (Dv\cdot(Du\times v))\,\xi_i\right]\, dx\Big|^2$$

$$\leq |v|^2_{H^1} \sum_{i=1}^\infty \left(|(u\times v)\xi_i|^2_H + |(Du\times v)\xi_i|^2_H\right) \leq c\big(|\varphi|_\infty + |\varphi_1|_\infty\big)|u|^2_{H^2}|v|^2_{H^1}|v|^2_H. \tag{5.11}$$

This implies that for every $\epsilon > 0$ we can fix some $c(\epsilon) = c(\epsilon, |\varphi|_\infty, |\varphi_1|_\infty) > 0$ such that,

$$\sqrt{\mu}\, \mathbb{E} \sup_{r\in[0,t]} \Big| \int_0^r Y_{\mu,a}(s) G(u_\mu(s), v_\mu(s))dw(s)\Big|$$

$$\leq c\, \mathbb{E} \left(\int_0^t \mu Y^2_{\mu,a}(s)\, \|G(u_\mu(s), v_\mu(s))\|^2_{\mathcal{T}_2(K,\mathbb{R})}\, ds\right)^{\frac{1}{2}}$$

$$\leq c\left(|\varphi|_\infty + |\varphi_1|_\infty\right)^{\frac{1}{2}} \mathbb{E} \left(\int_0^t \mu Y^2_{\mu,a}(s)|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H\, ds\right)^{\frac{1}{2}}$$

$$\leq \epsilon\mu\, \mathbb{E} \sup_{r\in[0,t]} Y_{\mu,a}(r)|v_\mu(r)|^2_{H^1} + c(\epsilon)\mathbb{E} \int_0^t Y_{\mu,a}(s)|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H ds. \tag{5.12}$$

Therefore, if we pick

$$\bar{\epsilon} := \frac{1}{2} \wedge \gamma\, \left(|u_0|^2_{H^1} + \mu|v_0|^2_H\right)^{-1},$$

and $\bar{a} = a(\bar{\epsilon}) > 0$ large enough in (5.10) and (5.12), from (5.6) we get

$$\mathbb{E} \sup_{r\in[0,t]} \left(Y_{\mu,\bar{a}}(r)\Big(|u_\mu(r)|^2_{H^2} + \mu|v_\mu(r)|^2_{H^1} + \mu|u_\mu(r)|^2_{H^1}|v_\mu(r)|^2_H\Big)\right)$$

$$+ c\, \mathbb{E} \int_0^t Y_{\mu,\bar{a}}(s)\Big(|v_\mu(s)|^2_{H^1} + |u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H\Big)ds$$

$$+ c\, \mathbb{E} \int_0^t Y_{\mu,\bar{a}}(s)\Big(\mu|v_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H + \mu|u_\mu(s)|^2_{H^1}|v_\mu(s)|^4_H\Big)ds$$

$$\leq |u_0|^2_{H^2} + \mu|v_0|^2_{H^1} + \mu|u_0|^2_{H^1}|v_0|^4_H.$$

Finally, since (5.1) gives for every $t \geq 0$

$$\int_0^t |v_\mu(s)|_H^2 ds \leq \frac{1}{2\gamma}\big(|u_0|_{H^1}^2 + \mu|v_0|_H^2\big), \quad \mathbb{P}-\text{a.s.},$$

we conclude

$$\mathbb{E}\sup_{r\in[0,t]}\Big(|u_\mu(r)|_{H^2}^2 + \mu|v_\mu(r)|_{H^1}^2 + \mu|u_\mu(r)|_{H^1}^2|v_\mu(r)|_H^2\Big)$$

$$+c\,\mathbb{E}\int_0^t \Big(|v_\mu(s)|_{H^1}^2 + |u_\mu(s)|_{H^2}^2|v_\mu(s)|_H^2 + \mu|v_\mu(s)|_{H^1}^2|v_\mu(s)|_H^2 + \mu|u_\mu(s)|_{H^1}^2|v_\mu(s)|_H^4\Big)ds$$

$$\leq \Big(|u_0|_{H^2}^2 + \mu|v_0|_{H^1}^2 + \mu|u_0|_{H^1}^2|v_0|_H^2\Big)\exp\Big(\frac{\overline{a}}{2\gamma}\big(|u_0|_{H^1}^2 + \mu|v_0|_H^2\big)\Big),$$

and this implies (5.3). □

6 The limiting equation

We consider the following deterministic equation

$$\begin{cases} \partial_t\Big[\Big(\gamma + \frac{1}{2}\varphi|u(t,x)|^2\Big)u(t,x)\Big] = \partial_x^2 u(t,x) + |\partial_x u(t)|_H^2 u(t,x) \\ \qquad\qquad + \frac{3\varphi}{2\gamma}\left(\left[\partial_x^2 u(t,x) + |\partial_x u(t)|_H^2 u(t,x)\right]\cdot u(t,x)\right)u(t,x), \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,L) = 0, \end{cases} \tag{6.1}$$

where, we recall, $\varphi(x) := \sum_{i=1}^\infty |\xi_i(x)|^2$, for $x \in (0,L)$.

Definition 6.1. Let $u_0 \in H^1 \cap M$. We say that u is a solution to equation (6.1) in $[0,T]$ if

$$u \in C([0,T];H^1) \cap L^2(0,T;H^2), \qquad \partial_t u \in L^2(0,T;H),$$

and the identity

$$\Big(\gamma + \frac{1}{2}\varphi|u(t)|^2\Big)u(t) = \Big(\gamma + \frac{1}{2}\varphi|u_0|^2\Big)u_0$$

$$+\int_0^t \Big(\partial_x^2 u(s) + |\partial_x u(s)|_H^2 u(s) + \frac{3\varphi}{2\gamma}\left(\left[\partial_x^2 u(s) + |\partial_x u(s)|_H^2 u(s)\right]\cdot u(s)\right)u(s)\Big)\,ds,$$

holds in H, for a.e. $t \in [0,T]$.

In the following lemma we show that equation (6.1) has an equivalent formulation.

Lemma 6.1. *Let $u_0 \in H^1 \cap M$. Then any function $u \in C([0,T];H^1) \cap L^2(0,T;H^2)$, with $\partial_t u \in L^2(0,T;H)$, satisfies equation (6.1) if and only if satisfies the following equation*

$$\begin{cases} \gamma\partial_t u(t,x) = \partial_x^2 u(t,x) + |\partial_x u(t)|_H^2 u(t,x) + \frac{1}{2}\,\mathrm{tr}_K\big(u(t,x)\times(u(t,x)\times\partial_t u(t,x))\big), \\ u(0,x) = u_0(x), \qquad u(t,0) = u(t,L) = 0. \end{cases} \tag{6.2}$$

Proof. If u satisfies equation (6.2), then we have

$$\gamma\partial_t u(t) = \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u + \frac{1}{2}\varphi\big(-|u(t)|^2\partial_t u(t) + (u(t)\cdot\partial_t u(t))u(t)\big)$$

$$= \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) + \frac{1}{2}\varphi\Big(-\partial_t(|u(t)|^2 u(t)) + 3(u(t)\cdot\partial_t u(t))u(t)\Big),$$

the identity holding in $L^2(0,T;H)$. Then, since

$$\gamma u(t)\cdot\partial_t u(t)=\partial_x^2 u(t)\cdot u(t)+|\partial_x u(t)|_H^2|u(t)|^2,$$

we have

$$\gamma(u(t)\cdot\partial_t u(t))u(t)=(\partial_x^2 u(t)\cdot u(t))u(t)+|\partial_x u(t)|_H^2|u(t)|^2u(t).$$

This implies that

$$\begin{aligned}&\gamma\partial_t u(t)+\frac{1}{2}\varphi\partial_t(|u(t)|^2u(t))\\&\quad=\partial_x^2 u(t)+|\partial_x u(t)|_H^2u(t)+\frac{3\varphi}{2\gamma}(\partial_x^2 u(t)\cdot u(t))u(t)+\frac{3\varphi}{2\gamma}|\partial_x u(t)|_H^2|u(t)|^2u(t).\end{aligned}$$

On the other hand, if u is a solution of equation (6.1), in order to prove that it is also a solution to (6.2) it suffices to show that

$$\gamma u(t)\cdot\partial_t u(t)=\partial_x^2 u(t)\cdot u(t)+|\partial_x u(t)|_H^2|u(t)|^2.$$

Indeed, note that

$$\begin{aligned}&\gamma\partial_t u(t)+\frac{1}{2}\varphi\Big(2(u(t)\cdot\partial_t u(t))u(t)+|u(t)|^2\partial_t u(t)\Big)\\&\quad=\partial_x^2 u(t)+|\partial_x u(t)|_H^2u(t)+\frac{3\varphi}{2\gamma}(\partial_x^2 u(t)\cdot u(t))u(t)+\frac{3\varphi}{2\gamma}|\partial_x u(t)|_H^2|u(t)|^2u(t),\end{aligned}$$

so that if we take the scalar product by u of both sides, we get

$$\Big(1+\frac{3}{2\gamma}\varphi|u(t)|^2\Big)\Big((\gamma u(t)\cdot\partial_t u(t))-(\partial_x^2 u(t)\cdot u(t))-|\partial_x u(t)|_H^2|u(t)|^2\Big)=0,$$

which completes the proof. □

Remark 6.2. By using the (6.2) formulation of equation (6.1), it is immediate to check that if $u_0\in M$ then $u(t)\in M$, for every $t\in[0,T]$. Actually, for any $t>0$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(|u(t)|_H^2-1\Big)&=\big\langle\partial_x^2 u(t),u(t)\big\rangle_H+|\partial_x u(t)|_H^2|u(t)|_H^2\\&\quad+\frac{1}{2}\big\langle\varphi(u(t)\times(u(t)\times\partial_t u(t))),u(t)\big\rangle_H=|\partial_x u(t)|_H^2\Big(|u(t)|_H^2-1\Big).\end{aligned}$$

Combined with the fact that $|u_0|_H-1=0$ and $\partial_x u\in C([0,T];H)$, this implies that $|u(t)|_H=1$, for any $t\geq 0$.

Lemma 6.3. *Let u be a solution to equation* (6.1)*, with $u_0\in H^1\cap M$. Then for every $t\geq 0$ we have*

$$|u(t)|_{H^1}^2+2\gamma\int_0^t|\partial_t u(s)|_H^2\,ds\leq|u_0|_{H^1}^2. \tag{6.3}$$

Proof. If we use the (6.2) formulation of equation (6.1), we get

$$\begin{aligned}\gamma|\partial_t u(t)|_H^2&=\big\langle\partial_x^2 u(t),\partial_t u(t)\big\rangle_H+\big\langle|\partial_x u(t)|_H^2u(t),\partial_t u(t)\big\rangle_H\\&\quad-\frac{1}{2}\big\langle\varphi|u(t)|^2\partial_t u(t),\partial_t u(t)\big\rangle_H+\frac{1}{2}\big\langle\varphi(u(t)\cdot\partial_t u(t))u(t),\partial_t u(t)\big\rangle_H.\end{aligned}$$

Recalling that $|u(t)|_H = 1$, this gives

$$\gamma\,|\partial_t u(t)|_H^2 = -\frac{1}{2}\frac{d}{dt}|u(t)|_{H^1}^2 - \frac{1}{2}\big\langle \varphi |u(t)|^2 \partial_t u(t), \partial_t u(t)\big\rangle_H + \frac{1}{2}\big\langle \varphi(u(t)\cdot \partial_t u(t))u(t), \partial_t u(t)\big\rangle_H \leq -\frac{1}{2}\frac{d}{dt}|u(t)|_{H^1}^2,$$

and (6.3) follows once we integrate both sides in time. □

Proposition 6.2. Let u_1 and u_2 be any two solutions of equation (6.1), with initial conditions $u_{1,0}, u_{2,0} \in H^1 \cap M$, respectively. Then there exist some constants $c_1, c_2 > 0$, depending only on $u_{1,0}$, $u_{2,0}$ and φ, such that for every $t \geq 0$

$$|u_1(t) - u_2(t)|_{H^1}^2 + \int_0^t |\partial_t u_1(s) - \partial_t u_2(s)|_H^2 ds \leq c_1 |u_{1,0} - u_{2,0}|_{H^1}^2 e^{c_2 t}. \tag{6.4}$$

In particular, there is at most one solution to equation (6.1) in $C([0,T];H^1)\cap L^2(0,T;H^2)$, with $\partial_t u \in\ L^2(0,T;H)$.

Proof. Let us fix $u_{1,0}, u_{2,0} \in H^1 \cap M$ and let u_1, u_2 be solutions of equation (6.1) with initial conditions $u_{1,0}, u_{2,0}$, respectively. If we denote $v_1 = \partial_t u_1$ and $v_2 = \partial_t u_2$, then, recalling (5.5)

$$\begin{aligned}\gamma|(v_1 - v_2)(t)|_H^2 &= \big\langle \partial_x^2 (u_1 - u_2)(t), (v_1 - v_2)(t)\big\rangle_H \\ &\quad + \big\langle |\partial_x u_1(t)|_H^2 u_1(t) - |\partial_x u_2(t)|_H^2 u_2(t), (v_1 - v_2)(t)\big\rangle_H \\ &\quad - \frac{1}{2}\big\langle \varphi\big(|u_1(t)|^2 v_1(t) - |u_2(t)|^2 v_2(t)\big), (v_1 - v_2)(t)\big\rangle_H \\ &\quad + \frac{1}{2}\big\langle \varphi\big((u_1(t)\cdot v_1(t))u_1(t) - (u_2(t)\cdot v_2(t))u_2(t)\big), (v_1 - v_2)(t)\big\rangle_H.\end{aligned}$$

Hence, if a is an arbitrary positive constant and we denote

$$Y_a(t) := \exp\left(-a\int_0^t \big(1 + |v_2(s)|_H^2\big) ds\right), \quad t \geq 0, \tag{6.5}$$

we get

$$\begin{aligned}\frac{d}{dt}\Big(Y_a(t)|(u_1 - u_2)(t)|_{H^1}^2\Big) &= Y_a(t)\Big(\frac{d}{dt}|(u_1 - u_2)(t)|_{H^1}^2 - a\big(1 + |v_2(t)|_H^2\big)|(u_1 - u_2)(t)|_{H^1}^2\Big) \\ &= Y_a(t)\Big(- a\big(1 + |v_2(t)|_H^2\big)|(u_1 - u_2)(t)|_{H^1}^2 \\ &\quad + 2\big\langle |u_1(t)|_{H^1}^2 u_1(t) - |u_2(t)|_{H^1}^2 u_2(t), (v_1 - v_2)(t)\big\rangle_H \\ &\quad - 2\gamma|(v_1 - v_2)(t)|_H^2 - \big\langle \varphi\big(|u_1(t)|^2 v_1(t) - |u_2(t)|^2 v_2(t)\big), (v_1 - v_2)(t)\big\rangle_H \\ &\quad + \big\langle \varphi\big((u_1(t)\cdot v_1(t))u_1(t) - (u_2(t)\cdot v_2(t))u_2(t)\big), (v_1 - v_2)(t)\big\rangle_H\Big).\end{aligned} \tag{6.6}$$

Now, for any $\epsilon > 0$ we can find $c(\epsilon) > 0$ such that

$$\begin{aligned}
&\big\langle |u_1(t)|^2_{H^1} u_1(t) - |u_2(t)|^2_{H^1} u_2(t), (v_1 - v_2)(t)\big\rangle_H \\
&= |u_1(t)|^2_{H^1} \big\langle (u_1 - u_2)(t), (v_1 - v_2)(t)\big\rangle_H + \big\langle \big(|u_1(t)|^2_{H^1} - |u_2(t)|^2_{H^1}\big) u_2(t), (v_1 - v_2)(t)\big\rangle_H \\
&\quad \leq |u_1(t)|^2_{H^1} \Big(\epsilon |(v_1 - v_2)(t)|^2_H + c(\epsilon) |(u_1 - u_2)(t)|^2_H\Big) \\
&\qquad\quad + c|u_2(t)|_H \big(|u_1(t)|_{H^1} + |u_2(t)|_{H^1}\big) \Big(\epsilon |(v_1 - v_2)(t)|^2_H + c(\epsilon) |(u_1 - u_2)(t)|^2_{H^1}\Big).
\end{aligned} \tag{6.7}$$

Moreover

$$\begin{aligned}
&-\big\langle \varphi\big(|u_1(t)|^2 v_1(t) - |u_2(t)|^2 v_2(t)\big), (v_1 - v_2)(t)\big\rangle_H \\
&= -\big\langle \varphi |u_1(t)|^2 (v_1 - v_2)(t), (v_1 - v_2)(t)\big\rangle_H - \big\langle \varphi\big(|u_1(t)|^2 - |u_2(t)|^2\big) v_2(t), (v_1 - v_2)(t)\big\rangle_H \\
&\quad \leq -\big\langle \varphi |u_1(t)|^2 (v_1 - v_2)(t), (v_1 - v_2)(t)\big\rangle_H \\
&\qquad\quad + |\varphi|_\infty \big(|u_1(t)|_{H^1} + |u_2(t)|_{H^1}\big) |(u_1 - u_2)(t)|_{H^1} |v_2(t)|_H |(v_1 - v_2)(t)|_H \\
&\quad \leq -\big\langle \varphi |u_1(t)|^2 (v_1 - v_2)(t), (v_1 - v_2)(t)\big\rangle_H \\
&\quad + c|\varphi|_\infty \big(|u_1(t)|_{H^1} + |u_2(t)|_{H^1}\big) \Big(\epsilon |(v_1 - v_2)(t)|^2_H + c(\epsilon) |v_2(t)|^2_H |(u_1 - u_2)(t)|^2_{H^1}\Big),
\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}
&\big\langle \varphi\big((u_1(t) \cdot v_1(t)) u_1(t) - (u_2(t) \cdot v_2(t)) u_2(t)\big), (v_1 - v_2)(t)\big\rangle_H \\
&\quad = \big\langle \varphi\big(u_1(t) \cdot (v_1 - v_2)(t)\big) u_1(t), (v_1 - v_2)(t)\big\rangle_H \\
&\qquad\quad + \big\langle \varphi\big((u_1 - u_2)(t) \cdot v_2(t)\big) u_1(t), (v_1 - v_2)(t)\big\rangle_H \\
&\qquad\qquad + \big\langle \varphi(u_2(t) \cdot v_2(t))(u_1 - u_2)(t), (v_1 - v_2)(t)\big\rangle_H \\
&\quad \leq \big\langle \varphi\big(u_1(t) \cdot (v_1 - v_2)(t)\big) u_1(t), (v_1 - v_2)(t)\big\rangle_H \\
&\qquad + c|\varphi|_\infty \big(|u_1(t)|_{H^1} + |u_2(t)|_{H^1}\big) \Big(\epsilon |(v_1 - v_2)(t)|^2_H + c(\epsilon) |v_2(t)|^2_H |(u_1 - u_2)(t)|^2_{H^1}\Big).
\end{aligned} \tag{6.9}$$

Since $\varphi \geq 0$, we have

$$-\big\langle \varphi |u_1|^2 (v_1 - v_2), v_1 - v_2\big\rangle_H + \big\langle \varphi\big(u_1 \cdot (v_1 - v_2)\big) u_1, v_1 - v_2\big\rangle_H \leq 0.$$

Hence, thanks to (6.3) we can take $\bar{\epsilon} > 0$ sufficiently small and $\bar{a} = \bar{a}(\bar{\epsilon}) > 0$ sufficiently large so that if we replace (6.7), (6.8) and (6.9) into (6.6), we get

$$\begin{aligned}
&Y_{\bar{a}}(t) |u_1(t) - u_2(t)|^2_{H^1} + c \int_0^t Y_{\bar{a}}(s) \big(1 + |v_2(s)|^2_H\big) |u_1(s) - u_2(s)|^2_{H^1} ds \\
&\qquad\qquad + c \int_0^t Y_{\bar{a}}(s) |v_1(s) - v_2(s)|^2_H ds \leq |u_{1,0} - u_{2,0}|^2_{H^1},
\end{aligned} \tag{6.10}$$

for some constant $c = c(u_{1,0}, u_{2,0}, \varphi) > 0$. Finally, since from (6.3)

$$Y_{\bar{a}}(t) \geq \exp\Big(-\bar{a}\Big(t + |u_{2,0}|^2_{H^1}/2\gamma\Big)\Big), \quad t > 0,$$

we can complete the proof of (6.4). □

7 Proof of the validity of the small-mass limit

In this final section, we conclude the proof of Theorem 3.2. We first prove some identities, then we investigate tightness and finally we proceed with the proof of the theorem.

7.1 An identity for the solution of system (2.2)

Lemma 7.1. For every $\mu > 0$ and $(u_0, v_0) \in \mathcal{H}_1 \cap \mathcal{M}$ the solution $(u_\mu(t), v_\mu(t))$ of system (2.2) (or, equivalently system (4.5)) satisfies the following identity, for every $t \geq 0$

$$\begin{aligned}
&\gamma u_\mu(t) + \frac{1}{2}\varphi|u_\mu(t)|^2 u_\mu(t) + \mu v_\mu(t) \\
&\quad= \gamma u_0 + \frac{1}{2}\varphi|u_0|^2 u_0 + \mu v_0 + \int_0^t \partial_x^2 u_\mu(s)ds + \int_0^t |\partial_x u_\mu(s)|_H^2 u_\mu(s)ds \\
&\qquad+\frac{3}{2\gamma}\varphi\int_0^t \big(\partial_x^2 u_\mu(s)\cdot u_\mu(s)\big)u_\mu(s)ds + \frac{3}{2\gamma}\varphi\int_0^t |\partial_x u_\mu(s)|_H^2 |u_\mu(s)|^2 u_\mu(s)ds + R_\mu(t),
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
R_\mu(t) &= \frac{3\mu}{2\gamma}\varphi(u_0\cdot v_0)u_0 - \frac{3\mu}{2\gamma}\varphi(u_\mu(t)\cdot v_\mu(t))u_\mu(t) - \mu\int_0^t |v_\mu(s)|_H^2 u_\mu(s)ds \\
&\quad+\frac{3\mu}{2\gamma}\varphi\int_0^t (u_\mu(s)\cdot v_\mu(t))v_\mu(s)ds + \frac{3\mu}{2\gamma}\varphi\int_0^t |v_\mu(s)|^2 u_\mu(s)ds \\
&\quad-\frac{3\mu}{2\gamma}\varphi\int_0^t |v_\mu(s)|_H^2 |u_\mu(s)|^2 u_\mu(s)ds + \sqrt{\mu}\int_0^t (u_\mu(s)\times v_\mu(s))dw(s) \\
&=: \frac{3\mu}{2\gamma}\varphi(u_0\cdot v_0)u_0 + \sum_{i=1}^6 J_{\mu,i}(t).
\end{aligned} \tag{7.2}$$

Proof. In view of (4.5), we have

$$\begin{aligned}
&d(u_\mu(t)\cdot v_\mu(t)) = |v_\mu(t)|^2 dt + \frac{1}{\mu}\big(\partial_x^2 u_\mu(t)\cdot u_\mu(t)\big)dt + \frac{1}{\mu}|\partial_x u_\mu(t)|_H^2 |u_\mu(t)|^2 dt \\
&-|v_\mu(t)|_H^2 |u_\mu(t)|^2 dt - \frac{\gamma}{\mu}(u_\mu(t)\cdot v_\mu(t))dt \\
&\qquad+\frac{1}{2\mu}\varphi\Big(-|u_\mu(t)|^2 v_\mu(t) + (u_\mu(t)\cdot v_\mu(t))u_\mu(t)\Big)\cdot u_\mu(t)dt \\
&\qquad+\frac{1}{\sqrt{\mu}}u_\mu(t)\cdot(u_\mu(t)\times v_\mu(t))dw(t) = |v_\mu(t)|^2 dt + \frac{1}{\mu}\big(\partial_x^2 u_\mu(t)\cdot u_\mu(t)\big)dt \\
&\qquad+\frac{1}{\mu}|\partial_x u_\mu(t)|_H^2 |u_\mu(t)|^2 dt - |v_\mu(t)|_H^2 |u_\mu(t)|^2 dt - \frac{\gamma}{\mu}(u_\mu(t)\cdot v_\mu(t))dt.
\end{aligned}$$

This implies that

$$d\Big((u_\mu(t)\cdot v_\mu(t))u_\mu(t)\Big)$$

$$= (u_\mu(t)\cdot v_\mu(t))v_\mu(t)dt + |v_\mu(t)|^2 u_\mu(t)dt + \frac{1}{\mu}\big(\partial_x^2 u_\mu(t)\cdot u_\mu(t)\big)u_\mu(t)dt$$

$$+\frac{1}{\mu}|\partial_x u_\mu(t)|_H^2|u_\mu(t)|^2 u_\mu(t)dt - |v_\mu(t)|_H^2|u_\mu(t)|^2 u_\mu(t)dt - \frac{\gamma}{\mu}(u_\mu(t)\cdot v_\mu(t))u_\mu(t)dt.$$

Hence, for every $t\geq 0$, we obtain

$$\begin{aligned}
&\gamma\int_0^t (u_\mu(s)\cdot v_\mu(s))u_\mu(s)ds\\
&= \int_0^t \big(\partial_x^2 u_\mu(s)\cdot u_\mu(s)\big)u_\mu(s)ds + \int_0^t |\partial_x u_\mu(s)|_H^2|u_\mu(s)|^2 u_\mu(s)ds\\
&\quad -\mu(u_\mu(t)\cdot v_\mu(t))u_\mu(t) + \mu(u_0\cdot v_0)u_0 + \mu\int_0^t (u_\mu(s)\cdot v_\mu(s))v_\mu(s)ds\\
&\quad +\mu\int_0^t |v_\mu(s)|^2 u_\mu(s)ds - \mu\int_0^t |v_\mu(s)|_H^2|u_\mu(s)|^2 u_\mu(s)ds.
\end{aligned} \tag{7.3}$$

By using the fact that

$$d(|u_\mu(t)|^2 u_\mu(t)) = 2(u_\mu(t)\cdot v_\mu(t))u_\mu(t) + |u_\mu(t)|^2 v_\mu(t),$$

this implies

$$\begin{aligned}
&\int_0^t \Big(-|u_\mu(s)|^2 v_\mu(s) + (u_\mu(s)\cdot v_\mu(s))u_\mu(s)\Big)ds\\
&= -|u_\mu(t)|^2 u_\mu(t) + |u_0|^2 u_0 + 3\int_0^t (u_\mu(s)\cdot v_\mu(s))u_\mu(s)ds = -|u_\mu(t)|^2 u_\mu(t) + |u_0|^2 u_0\\
&\quad +\frac{3}{\gamma}\int_0^t \big(\partial_x^2 u_\mu(s)\cdot u_\mu(s)\big)u_\mu(s)ds + \frac{3}{\gamma}\int_0^t |\partial_x u_\mu(s)|_H^2|u_\mu(s)|^2 u_\mu(s)ds\\
&\quad -\frac{3\mu}{\gamma}(u_\mu(t)\cdot v_\mu(t))u_\mu(t) + \frac{3\mu}{\gamma}(u_0\cdot v_0)u_0 + \frac{3\mu}{\gamma}\int_0^t (u_\mu(s)\cdot v_\mu(s))v_\mu(s)ds\\
&\quad +\frac{3\mu}{\gamma}\int_0^t |v_\mu(s)|^2 u_\mu(s)ds - \frac{3\mu}{\gamma}\int_0^t |v_\mu(s)|_H^2|u_\mu(s)|^2 u_\mu(s)ds.
\end{aligned} \tag{7.4}$$

Finally, we rewrite system (4.5) in the following form

$$\begin{aligned}
&\gamma u_\mu(t) + \mu v_\mu(t) = \gamma u_0 + \mu v_0\\
&\quad + \int_0^t \partial_x^2 u_\mu(s)ds + \int_0^t |\partial_x u_\mu(s)|_H^2 u_\mu(s)ds - \mu\int_0^t |v_\mu(s)|_H^2 u_\mu(s)ds\\
&\quad +\frac{1}{2}\,\varphi\int_0^t \Big(-|u_\mu(s)|^2 v_\mu(s) + (u_\mu(s)\cdot v_\mu(s))u_\mu(s)\Big)ds + \sqrt{\mu}\int_0^t (u_\mu(s)\times v_\mu(s))dw(s).
\end{aligned} \tag{7.5}$$

Therefore, if we replace (7.4) into (7.5), we get (7.1), with $R_\mu(t)$ defined by (7.2). □

Lemma 7.2. For every $(u_0, v_0) \in \mathcal{H}_1 \cap \mathcal{M}$ and $t > 0$ we have

$$\lim_{\mu\to 0} \mathbb{E} \sup_{r\in[0,t]} |R_\mu(r)|_H^2 = 0. \tag{7.6}$$

Proof. First, note that, since

$$|J_{\mu,1}(t)|_H \leq c\,|\varphi|_\infty \mu |u_\mu(t)|_{H^1}^2 |v_\mu(t)|_H,$$

due to (5.1) we have

$$\lim_{\mu\to 0} \mathbb{E} \sup_{r\in[0,t]} |J_{\mu,1}(r)|_H^2 = 0, \tag{7.7}$$

and since

$$|J_{\mu,2}(r)|_H^2 \leq c\,t\mu^2 \int_0^t |v_\mu(s)|_H^4 ds \leq c\,t\mu^2 \int_0^t |v_\mu(s)|_H^4 |u_\mu(s)|_{H^1}^2\, ds, \qquad r \in [0,t],$$

due to (5.3) we have

$$\lim_{\mu\to 0} \mathbb{E} \sup_{r\in[0,t]} |J_{\mu,2}(r)|_H^2 = 0. \tag{7.8}$$

Moreover, thanks to (5.1), (5.3) and

$$|J_{\mu,3}(r)|_H^2 + |J_{\mu,4}(r)|_H^2 \leq c\,t|\varphi|_\infty^2 \mu^2 \int_0^t |u_\mu(s)|_{H^1}^2 |v_\mu(s)|_{H^1}^2 |v_\mu(s)|_H^2 ds, \qquad r \in [0,t],$$

we have

$$\lim_{\mu\to 0} \mathbb{E} \sup_{r\in[0,t]} \left(|J_{\mu,3}(r)|_H^2 + |J_{\mu,4}(r)|_H^2 \right) = 0. \tag{7.9}$$

From (5.1), (5.3), and

$$|J_{\mu,5}|_H^2 \leq c\,t|\varphi|_\infty^2 \mu^2 \mathbb{E} \int_0^t |u_\mu(s)|_{H^1}^6 |v_\mu(s)|_H^4 ds, \qquad r \in [0,t],$$

it follows that

$$\lim_{\mu\to 0} \mathbb{E} \sup_{r\in[0,t]} |J_{\mu,5}(r)|_H^2 = 0. \tag{7.10}$$

Furthermore, we have

$$\mathbb{E} \sup_{r\in[0,t]} |J_{\mu,6}(r)|_H^2 \leq c\mu \mathbb{E} \int_0^t \|u_\mu(s) \times v_\mu(s)\|_{\mathcal{T}_2(K,H)}^2\, ds$$

$$\leq c|\varphi|_\infty \mu \mathbb{E} \int_0^t |u_\mu(s)|_{H^1}^2 |v_\mu(s)|_H^2 ds,$$

so that, thanks again to (5.1),

$$\lim_{\mu\to 0} \mathbb{E} \sup_{r\in[0,t]} |J_{\mu,6}(r)|_H^2 = 0. \tag{7.11}$$

Finally, combining (7.7)-(7.11), we complete our proof. □

7.2 Tightness

Proposition 7.1. *For every $(u_0, v_0) \in \mathcal{H}_1 \cap \mathcal{M}$ and $T > 0$, the family of probability measures $\left\{\mathcal{L}(u_\mu)\right\}_{\mu\in(0,1)}$ is tight in $C([0,T]; H^\delta)$, for every $\delta < 2$.*

Proof. According to (5.2), we have that

$$\sup_{\mu\in(0,1)} \mathbb{E}\left(\sup_{t\in[0,T]}|u_\mu(t)|^2_{H^2}+\int_0^T|\partial_t u_\mu(s)|^2_{H^1}\,ds\right)<+\infty. \tag{7.12}$$

This, in particular, implies that for every $\epsilon>0$ there exists $L_\epsilon>0$ such that if we denote by K_ϵ the ball of radius L_ϵ in $C([0,T];H^2)\cap W^{1,2}(0,T;H^1)$, then

$$\inf_{\mu\in(0,1)} \mathbb{P}(u_\mu\in K_\epsilon)\geq 1-\epsilon.$$

Due to the Aubin-Lions lemma, we know that the set K_ϵ is compact in $C([0,T];H^\delta)$, for every $\delta<2$. □

7.3 Proof of Theorem 3.2

Proof. Thanks to Proposition 7.1 and (5.2), we have the family $\left\{\mathcal{L}(u_\mu,\mu\,\partial_t u_\mu)\right\}_{\mu\in(0,1)}$ is tight in $C([0,T];H^\delta)\times C([0,T];H^1)$, for any $\delta<2$. If for every $T>0$ and $\delta<2$ we define

$$\Gamma_{T,\delta}:=C([0,T];H^\delta)\times C([0,T];H^1)\times C([0,T];E),$$

where E is any Banach space such that the embedding $K\subset E$ is Hilbert-Schmidt, then as a consequence of Skorokhod's theorem, for any sequence $\{\mu_k\}_{k\in\mathbb{N}}$, converging to zero, there exists a subsequence, still denoted by $\{\mu_k\}_{k\in\mathbb{N}}$, and some $\Gamma_{T,\delta}$-valued random variables

$$\mathcal{Y}_k:=\big(\varrho_k,\mu_k\vartheta_k,\hat{w}_k\big),\quad \mathcal{Y}:=\big(\varrho,\vartheta,\hat{w}\big),\quad k\in\mathbb{N},$$

all defined on some probability space $\big(\hat{\Omega},\hat{\mathcal{F}},\{\hat{\mathcal{F}}_t\}_{t\in[0,T]},\hat{\mathbb{P}}\big)$, such that

$$\mathcal{L}(\mathcal{Y}_k)=\mathcal{L}\big(u_{\mu_k},\mu_k\partial_t u_{\mu_k},w\big),\quad k\in\mathbb{N}, \tag{7.13}$$

and

$$\lim_{k\to\infty}\Big(|\varrho_k-\varrho|_{C([0,T];H^\delta)}+\mu_k|\vartheta_k|_{C([0,T];H^1)}+|\hat{w}_k-\hat{w}|_{C([0,T];E)}\Big)=0,\qquad \hat{\mathbb{P}}-\text{a.s.} \tag{7.14}$$

In particular, due to (5.1), (5.2), (7.12) and (7.13), we have

$$\sup_{k\in\mathbb{N}}\left(\hat{\mathbb{E}}\sup_{t\in[0,T]}\Big(|\varrho_k(t)|^2_{H^2}+\mu_k|\vartheta_k(t)|^2_{H^1}\Big)+\hat{\mathbb{E}}\int_0^T|\vartheta_k(s)|^2_{H^1}ds\right)<+\infty, \tag{7.15}$$

and there exists a deterministic $c>0$ such that

$$\sup_{k\in\mathbb{N}}\Big(\sup_{t\in[0,T]}|\varrho_k(t)|_{H^1}+\int_0^T|\partial_t\varrho_k(s)|^2_H\,ds\Big)\leq c,\quad \hat{\mathbb{P}}\text{-a.s.} \tag{7.16}$$

Thanks to (7.14) and (7.16), it follows that $\varrho\in L^2(\hat{\Omega};L^\infty(0,T;H^\delta))$, for every $\delta<2$, and

$$\sup_{t\in[0,T]}|\varrho(t)|_{H^1}\leq c,\quad \hat{\mathbb{P}}\text{-a.s.} \tag{7.17}$$

Moreover, from (7.15) and (7.16) we get that ϱ is weakly differentiable in time, with $\partial_t\varrho\in L^2(0,T;H)$ and

$$\int_0^T|\partial_t\varrho(s)|^2_H\,ds\leq c,\qquad \hat{\mathbb{P}}\text{-a.s.} \tag{7.18}$$

Finally, as a consequence of (7.14) and (7.15), we have

$$\int_0^T\hat{\mathbb{E}}|\varrho(s)|^2_{H^2}\,ds<\infty. \tag{7.19}$$

Now, if we can show that ϱ solves equation (6.1), then by the uniqueness of solutions for equation (6.1), due to the classical argument by Gyongy and Krylov (see [14]), we can conclude that u_{μ_k} converges to u in $C([0,T];H^\delta)$, for every sequence $\mu_k \downarrow 0$, and the convergence is in probability.

Due to (7.13) and identities (7.1) and (7.2), we have that for every $\psi \in C_0^\infty([0,L])$

$$\begin{aligned}
&\langle \gamma\varrho_k(t) + \frac{1}{2}\varphi|\varrho_k(t)|^2\varrho_k(t) + \mu_k\vartheta_k(t), \psi\rangle_H \\
&\quad = \langle \gamma u_0 + \frac{1}{2}\varphi|u_0|^2 u_0 + \mu_k v_0, \psi\rangle_H + \int_0^t \langle \partial_x^2\varrho_k(s), \psi\rangle_H ds \\
&\quad + \int_0^t \langle |\varrho_k(s)|_{H^1}^2\varrho_k(s), \psi\rangle_H ds + \frac{3}{2\gamma}\int_0^t \langle \varphi\big(\partial_x^2\varrho_k(s)\cdot\varrho_k(s)\big)\varrho_k(s), \psi\rangle_H ds \\
&\quad + \frac{3}{2\gamma}\int_0^t \langle \varphi|\varrho_k(s)|_{H^1}^2|\varrho_k(s)|^2\varrho_k(s), \psi\rangle_H ds + \langle \hat{R}_k(t), \psi\rangle_H,
\end{aligned}$$

where

$$\begin{aligned}
\widehat{R}_k(t) &= \frac{3\mu_k}{2\gamma}\varphi(u_0\cdot v_0)u_0 - \frac{3\mu_k\varphi}{2\gamma}(\varrho_k(t)\cdot\vartheta_k(t))\varrho_k(t) - \mu_k\int_0^t |\vartheta_k(s)|_H^2\varrho_k(s)ds \\
&\quad + \frac{3\mu_k\varphi}{2\gamma}\int_0^t (\varrho_k(s)\cdot\vartheta_k(t))\vartheta_k(s)ds + \frac{3\mu_k\varphi}{2\gamma}\int_0^t |\vartheta_k(s)|^2\varrho_k(s)ds \\
&\quad - \frac{3\mu_k}{2\gamma}\int_0^t |\vartheta_k(s)|_H^2|\varrho_k(s)|^2\varrho_k(s)ds + \sqrt{\mu_k}\int_0^t (\varrho_k(s)\times\vartheta_k(s))d\hat{w}_k(s).
\end{aligned}$$

We have

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big|\varphi\Big(|\varrho_k(t)|^2\varrho_k(t) - |\varrho(t)|^2\varrho(t)\Big)\Big|_H \\
&\quad \leq c|\varphi|_\infty \sup_{t\in[0,T]} \Big(\big|\big(|\varrho_k(t)|^2 - |\varrho(t)|^2\big)\varrho_k(t)\big|_H + \big||\rho(t)|^2(\varrho_k(t)-\varrho(t))\big|_H\Big) \\
&\quad \leq c|\varphi|_\infty \sup_{t\in[0,T]} \Big(\big(|\varrho_k(t)|_{H^1}^2 + |\varrho(t)|_{H^1}^2\big)|\varrho_k(t)-\varrho(t)|_H\Big).
\end{aligned}$$

Then, in view of (7.14), (7.16) and (7.17) we conclude

$$\lim_{k\to\infty}\sup_{t\in\,[0,T]}\left|\langle \gamma\varrho_k(t) + \frac{1}{2}\varphi|\varrho_k(t)|^2\varrho_k(t) + \mu_k\vartheta_k(t), \psi\rangle_H - \langle \gamma\varrho(t) + \frac{1}{2}\varphi|\varrho(t)|^2\varrho(t), \psi\rangle_H\right| = 0, \tag{7.20}$$

$\hat{\mathbb{P}}$-a.s. Since

$$\sup_{t\in[0,T]}\left|\int_0^t \langle \partial_x^2(\varrho_k - \varrho)(s), \psi\rangle_H ds\right| \leq |\psi|_{H^1}\int_0^T |\varrho_k(s) - \varrho(s)|_{H^1} ds,$$

due to (7.14) we have

$$\lim_{k\to\infty}\sup_{t\in\,[0,T]}\left|\int_0^t \langle \partial_x^2\varrho_k(s), \psi\rangle_H ds - \int_0^t \langle \partial_x^2\varrho(s), \psi\rangle_H\, ds\right| = 0, \qquad \widehat{\mathbb{P}}\text{-a.s.}, \tag{7.21}$$

and since

$$\sup_{t\in[0,T]}\Big|\int_0^t\Big(|\varrho_k(s)|^2_{H^1}\varrho_k(s)-|\varrho(s)|^2_{H^1}\varrho(s)\Big)ds\Big|_H$$

$$\leq\int_0^T\big||\varrho_k(s)|^2_{H^1}-|\varrho(s)|^2_{H^1}\big|ds+\int_0^T|\varrho(s)|^2_{H^1}|\varrho_k(s)-\varrho(s)|_H ds,$$

we get

$$\lim_{k\to\infty}\sup_{t\in\,[0,T]}\left|\int_0^t\langle|\varrho_k(s)|^2_{H^1}\varrho_k(s),\psi\rangle_H ds-\int_0^t\langle|\varrho(s)|^2_{H^1}\varrho(s),\psi\rangle_H\,ds\right|=0,\qquad\hat{\mathbb{P}}\text{-a.s.}\quad(7.22)$$

Moreover, for every $\eta\in\ C([0,T];H^1)\cap L^2(0,T;H^2)$ and $s\in\ [0,T]$ we have

$$\langle\big(\partial_x^2\eta(s)\cdot\eta(s)\big)\eta(s)\varphi,\psi\rangle_H=-\langle\big(\partial_x\eta(s),\eta(s)\big)\eta(s)\varphi,\psi'\rangle_H$$

$$-\langle\big(\partial_x\eta(s),\eta(s)\big)\big(\partial_x\eta(s)\varphi+\eta(s)\varphi'\big)+|\partial_x\eta(s)|^2\eta(s)\varphi,\psi\rangle_H.$$

This implies that

$$\int_0^t\langle\big(\partial_x^2\varrho_k(s)\cdot\varrho_k(s)\big)\varrho_k(s)\varphi-\big(\partial_x^2\varrho(s)\cdot\varrho(s)\big)\varrho(s)\varphi,\psi\rangle_H\,ds$$

$$=-\int_0^t\langle\big[\big(\partial_x\varrho_k(s),\varrho_k(s)\big)\varrho_k(s)-\big(\partial_x\varrho(s),\varrho(s)\big)\varrho(s)\big]\varphi,\psi'\rangle_H\,ds$$

$$-\int_0^t\langle\big[\big(\partial_x\varrho_k(s),\varrho_k(s)\big)\big(\partial_x\varrho_k(s)\varphi+\varrho_k(s)\varphi'\big)-\big(\partial_x\varrho(s),\varrho(s)\big)\big(\partial_x\varrho(s)\varphi+\varrho(s)\varphi'\big)\big],\psi\rangle_H\,ds$$

$$-\int_0^t\langle\big[|\partial_x\varrho_k(s)|^2\varrho_k(s)-|\partial_x\varrho(s)|^2\varrho(s)\big]\varphi,\psi\rangle_H\,ds=:\sum_{i=1}^3J_{i,k}(t).$$

We have

$$|J_{1,k}(t)|\leq c\,|\varphi|_\infty|\psi'|_\infty\int_0^T|\varrho_k(s)-\varrho(s)|_{H^1}|\varrho_k(s)|^2_{H^1}\,ds$$

$$+c\,|\varphi|_\infty|\psi'|_\infty\int_0^T|\varrho_k(s)-\varrho(s)|_{H^1}|\varrho(s)|_{H^1}\left(|\varrho_k(s)|_{H^1}+|\varrho(s)|_{H^1}\right)\,ds,$$

and, thanks to (7.14), (7.16) and (7.17), we conclude that

$$\lim_{k\to\infty}\sup_{t\in\,[0,T]}|J_{1,k}(t)|=0,\qquad\hat{\mathbb{P}}-\text{a.s.}\tag{7.23}$$

Next, we have

$$|J_{2,k}(t)|$$

$$\leq c\,|\psi|_\infty\int_0^T|\varrho_k(s)-\varrho(s)|_{H^1}\left(|\varrho_k(s)|_{H^1}+|\varrho(s)|_{H^1}\right)\left(|\varrho_k(s)|_{H^2}|\varphi|_\infty+|\varrho_k(s)|_{H^1}|\varphi'|_\infty\right)\,ds$$

$$+c\,|\psi|_\infty\left(|\varphi|_\infty+|\varphi'|_\infty\right)\int_0^T|\varrho_k(s)-\varrho(s)|_{H^1}|\varrho(s)|_{H^2}|\varrho(s)|_{H^1}\,ds.$$

Thus, as a consequence of (7.14) and bounds (7.15), (7.16), (7.17) and (7.19), we can conclude that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in [0,T]} |J_{2,k}(t)| = 0. \tag{7.24}$$

Finally, we have

$$|J_{3,k}(t)| \leq c\,|\varphi|_\infty |\psi|_\infty \int_0^T |\varrho_k(s)-\varrho(s)|_{H^1} \left[|\varrho_k(s)|_{H^1}\left(|\varrho_k(s)|_{H^2}+|\varrho(s)|_{H^2}\right)+|\varrho(s)|^2_{H^1}\right]\,ds,$$

and, due again to (7.14), (7.15) and(7.19), we conclude

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in [0,T]} |J_{3,k}(t)| = 0. \tag{7.25}$$

Therefore, as a consequence of (7.23), (7.24) and (7.25), we conclude that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in [0,T]} \left|\int_0^t \langle \left(\partial_x^2\varrho_k(s)\cdot\varrho_k(s)\right)\varrho_k(s)\varphi - \left(\partial_x^2\varrho(s)\cdot\varrho(s)\right)\varrho(s)\varphi,\psi\rangle_H\,ds\right| = 0. \tag{7.26}$$

Next, for every $t\in\ [0,T]$ we have

$$\begin{aligned}
&\left|\varphi\int_0^t \Big(|\varrho_k(s)|^2_{H^1}|\varrho_k(s)|^2\varrho_k(s)-|\varrho(s)|^2_{H^1}|\varrho(s)|^2\varrho(s)\Big)ds\right|_H\\
&\leq c|\varphi|_\infty\Bigg(\int_0^T \left||\varrho_k(s)|^2_{H^1}-|\varrho(s)|^2_{H^1}\right|\cdot\left||\varrho_k(s)|^2\varrho_k(s)\right|_H ds\\
&\quad+\int_0^T|\varrho(s)|^2_{H^1}\left|(|\varrho_k(s)|^2-|\varrho(s)|^2)\varrho_k(s)\right|_H ds+\int_0^T|\varrho(s)|^2_{H^1}\left||\varrho(s)|^2(\varrho_k(s)-\varrho(s))\right|_H ds\Bigg)\\
&\leq c|\varphi|_\infty\Big(\int_0^T|\varrho_k(s)|^2_{H^1}\left(|\varrho_k(s)|_{H^1}+|\varrho(s)|_{H^1}\right)|\varrho_k(s)-\varrho(s)|_{H^1}ds\\
&\qquad\qquad+\int_0^T\left(|\varrho_k(s)|^4_{H^1}+|\varrho(s)|^4_{H^1}\right)|\varrho_k(s)-\varrho(s)|_H ds\Big).
\end{aligned}$$

Thus, thanks again to (7.16) and (7.17), we get

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left|\int_0^t\langle\varphi|\varrho_k(s)|^2_{H^1}|\varrho_k(s)|^2\varrho_k(s),\psi\rangle_H\,ds-\int_0^t\langle\varphi|\varrho(s)|^2_{H^1}|\varrho(s)|^2\varrho(s),\psi\rangle_H\,ds\right|=0, \tag{7.27}$$

$\hat{\mathbb{P}}$-a.s. By using the same arguments as in the proof of Lemma 7.2, we conclude that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in [0,T]} \left|\langle \widehat{R}_k(t),\psi\rangle_H\right|^2 = 0. \tag{7.28}$$

Finally, combining (7.20), (7.21), (7.22), (7.26) and (7.27) together with (7.28), we can conclude that ϱ satisfies equation (6.1). As we have seen above, this allows to conclude the proof of Theorem 3.2. □

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