

NON-ORIENTABLE LINK COBORDISMS AND TORSION ORDER IN FLOER HOMOLOGIES

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ABSTRACT. We use unoriented versions of instanton and knot Floer homology to prove inequalities involving the Euler characteristic and the number of local maxima appearing in unorientable cobordisms, which mirror results of a recent paper by Juhasz, Miller, and Zemke concerning orientable cobordisms. Most of the subtlety in our argument lies in the fact that maps for non-orientable cobordisms require more complicated decorations than their orientable counterparts. We introduce unoriented versions of the band unknotting number and the refined cobordism distance and apply our results to give bounds on these based on the torsion orders of the Floer homologies. Finally, we show that the difference between the unoriented refined cobordism distance of a knot K from the unknot and the non-orientable slice genus of K can be arbitrarily large.

1. INTRODUCTION

A classical problem in low-dimensional topology is the study of embedded orientable surfaces in 4-manifolds. The special case of surfaces with boundary has been a particularly popular topic for a very long time, and it includes for example questions pertaining to the slice genus of a knot or the complexity of a knot or link cobordism.

On the other hand, the study of *non-orientable* surfaces and knot cobordisms in $I \times S^3$ has received increasing attentions in the last decade [Bat14, OSSz17a, GM18, Fan19], and there are now several bounds to the non-orientable slice genus of a knot. However, if a knot bounds a non-orientable surface of a given ‘genus’, it is not clear how complicated the embedding must be. In this paper we tackle this question by proving a non-orientable analogue of a recent result of Juhász–Miller–Zemke. In a recent paper [JMZ20], they proved an inequality involving the number of local maxima and the genus appearing in an oriented knot cobordism using a version of knot Floer homology. In this paper, we prove similar inequalities for non-orientable knot cobordisms using the torsion orders of unoriented versions of knot Floer homology and instanton Floer homology.

As for knot Floer homology, we use Ozsváth–Stipsicz–Szabó’s *unoriented knot Floer homology* HFK' [OSSz17b], which is a module over $\mathbb{F}[U]$. For a

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knot $K \subset S^3$ we define its unoriented knot Floer torsion order as

$$\text{Ord}_U(K) = \min \{n \geq 0 \mid U^n \cdot \text{Tors} = 0\},$$

where $\text{Tors} \subset \text{HFK}'(K)$ denotes the $\mathbb{F}[U]$ -torsion subgroup.

In the instanton setting, we use Kronheimer and Mrowka's *instanton Floer homology with local coefficients*, denoted by $I^\sharp(K)$, which is a module over a Noetherian domain $\mathcal{S}$ which has a special element P [KM21a]. We will restrict our attention to certain domains $\mathcal{S}$, for which $I^\sharp(K)$ is functorial for nonorientable knot cobordisms with singular bundles represented by surfaces ω with $\partial\omega$ on the cobordism. In this case, it can be shown that for a knot K and for the torsion part Tors of $I^\sharp(K)$, there is a positive integer n such that $P^n \cdot \text{Tors} = 0$. Thus, we define

$$\text{Ord}_I(K) = \min \{n \geq 0 \mid P^n \cdot \text{Tors} = 0\}.$$

For an unorientable surface Σ with n boundary components, recall that its *non-orientable genus* is

$$\gamma(\Sigma) = 2 - \chi(\Sigma) - n.$$

For example, $\mathbb{RP}^2$ (with an arbitrary number of punctures) has non-orientable genus 1. Note that for unorientable knot cobordisms $\gamma(\Sigma) = -\chi(\Sigma)$. With this notation in mind, we state our main theorem.

Theorem 1.1. *Let K_1 and K_2 be knots in S^3 . Suppose that there is an unorientable knot cobordism Σ in $I \times S^3$ from K_1 to K_2 with M local maxima. Then*

$$(1) \quad \text{Ord}_I(K_1) \leq \max \{\text{Ord}_I(K_2), M\} + \gamma(\Sigma)$$

and

$$(2) \quad \text{Ord}_U(K_1) \leq \max \{\text{Ord}_U(K_2), M\} + \gamma(\Sigma).$$

From a formal viewpoint, Theorem 1.1 is analogous to [JMZ20, Theorem 1.1]. The proof of Theorem 1.1 uses the functorial properties of HFK' and $I^\sharp$ (see [Fan19, KM21a]), in a similar way as [JMZ20, Theorem 1.1] relies on knot Floer homology cobordism maps [Juh16, Zem19c]. Despite being inspired by [JMZ20, Theorem 1.1], the proof of Theorem 1.1 must necessarily deviate from it. Recall that in order to define a cobordism map in knot Floer homology one needs to choose a properly embedded 1-manifold on the surface, often called the set of *decorations*. In [JMZ20] the chosen decorations were a pair of parallel arcs, which make the computations of the cobordism maps more tractable. This choice does not work for non-orientable cobordisms in HFK' , so we are forced to choose different decorations, which make the cobordism map more complicated. To circumvent this problem we related the resulting unorientable cobordism to an orientable one, then use a stabilization lemma proved by Ian Zemke (cf. Lemma 5.4). In the case of $I^\sharp$, for a cobordism Σ to define a map, one needs a surface ω with boundary $\partial\omega \subset \Sigma$. The natural choice for orientable cobordisms would be $\omega = \emptyset$, in

which case [JMZ20] applies verbatim to the case of $I^\sharp$. The map can be defined for the non-orientable surfaces we are interested in, but it will usually vanish. To overcome this problem, we choose a particular ω that allows us to control the induced map.

Remark 1.2. Note that while Theorem 1.1 is stated for unorientable cobordisms, both inequalities also hold for *orientable* cobordisms too. The proof follows verbatim from [JMZ20], by replacing knot Floer homology with the desired Floer theory.

We prove several applications of Theorem 1.1, which mirror those of [JMZ20, Theorem 1.1].

1.1. Unorientable ribbon cobordism. A knot cobordism in $I \times S^3$ is called *ribbon* if it has no local maxima. For example, a ribbon concordance (i.e., a cobordism of genus 0) from the unknot to a knot K is equivalent to a ribbon disc for K . Theorem 1.1 has a straightforward application to non-orientable ribbon cobordisms.

Corollary 1.3. *Let K_1 and K_2 be knots in S^3 . Suppose that there is an unorientable ribbon cobordism Σ in $I \times S^3$ from K_1 to K_2 . Then*

$$\text{Ord}_I(K_1) \leq \text{Ord}_I(K_2) + \gamma(\Sigma)$$

and

$$\text{Ord}_U(K_1) \leq \text{Ord}_U(K_2) + \gamma(\Sigma).$$

1.2. The unorientable refined cobordism distance. The standard cobordism distance between two knots K_1 and K_2 is given by $d_o(K_1, K_2) = 2g_4(K_1 \# \overline{K_2})$, where g_4 denotes the standard slice genus. This is not a distance on the set of knots, because concordant knots have distance 0, but it descends to a well defined distance on the concordance group [Baa12]. In [JMZ20], Juhász, Miller, and Zemke define a refined cobordism distance on the set of knots, and give lower bounds to it in terms of the torsion order in knot Floer homology.

There are analogous unorientable notions too. For an (orientable or unorientable) cobordism Σ in $I \times S^3$ from K_1 to K_2 with m local minima and M local maxima, let

$$|\Sigma| = \max\{m, M\} - \chi(\Sigma).$$

Definition 1.4. Given knots $K_1, K_2 \subset S^3$, we define the *standard unorientable cobordism distance* d_u and the *refined unorientable cobordism distance* d_u^r between them as

$$d_u(K_1, K_2) = \min\{-\chi(\Sigma)\} \quad \text{and} \quad d_u^r(K_1, K_2) = \min\{|\Sigma|\},$$

where in both cases Σ varies over all unorientable connected cobordisms and all genus-0 orientable cobordisms (i.e., concordances).

Remark 1.5. The orientable counterparts, the *standard orientable cobordism distance* d_o from [Baa12] and the *refined orientable cobordism distance* d_o^r from [JMZ20] are defined in the same way as in Definition 1.4, but the surface Σ now varies over all *orientable* connected cobordisms. One can also define analogous notions d_a and d_a^r , which we can call *all-surface cobordism distances*, where Σ varies over all (orientable or unorientable) connected cobordisms.

It is immediate to see that d_o , d_u , and d_a are distances on the concordance group and d_o^r , d_u^r , and d_a^r are distances on the set of knots.

Theorem 1.1 implies the following lower bounds.

Corollary 1.6. *If K_1 and K_2 are knots in S^3 , then*

$$|\text{Ord}_I(K_1) - \text{Ord}_I(K_2)| \leq d_u^r(K_1, K_2)$$

and

$$|\text{Ord}_U(K_1) - \text{Ord}_U(K_2)| \leq d_u^r(K_1, K_2).$$

In view of Remark 1.2, one can in fact replace d_u^r with d_a^r . However, for orientable cobordisms one can also use the standard versions of instanton and knot Floer homology, which should give better bounds.

We use Corollary 1.6 to show that the difference between $d_u^r(K_1, K_2)$ and $d_u(K_1, K_2)$ can be arbitrarily large.

Corollary 1.7. *For all $\gamma \geq 1$ and $m \geq 1$, there exists a knot $K_{\gamma,m}$ with $d_u(K_{\gamma,m}, U_1) = \gamma_4(K_{\gamma,m}) = \gamma$, and such that $d_u^r(K_{\gamma,m}, U_1) \geq \gamma + m$.*

Thus, each unorientable surface $\Sigma \subset B^4$ with $\partial\Sigma = K_{\gamma,m}$ and $\gamma(\Sigma) = \gamma$ has at least m local minima (with respect to the radial function).

The knots $K_{\gamma,m}$ that we consider in the proof of Corollary 1.7 are a subfamily of torus knots, for which Ord_U can be computed explicitly.

1.3. The unoriented band-unlinking number. For a knot K in S^3 , the *oriented band-unlinking number* $u_b(K)$ is defined as the minimum number of oriented band surgeries that turn K into the unknot. This was called *SH(2)-unknotting number* in [HNT90]. Its unoriented counterpart $u_b^u(K)$, called *H(2)-unknotting number* in [HNT90], seems to pre-date $u_b(K)$ in the literature, since Lickorish proved that there exist knots with $u_b^u(K) > 1$ in [Lic86]. Note that in the definition of $u_b^u(K)$ we allow both orientable and unorientable band surgeries.

Juhász–Miller–Zemke [JMZ20] introduced a variation, called the *oriented band-unlinking number* $ul_b(K)$, is defined as the minimum number of oriented band surgeries that turn K into an unlink. Of course $ul_b(K) \leq u_b(K)$, and they proved that $\text{Ord}_v(K) \leq ul_b(K)$ for all knots K in S^3 . Using Theorem 1.1 we can derive a similar result for the corresponding unoriented notion.

Definition 1.8. The *unoriented band-unlinking number* $ul_b^u(K)$ of a knot K in S^3 is defined as the minimum number of (orientable or unorientable) band surgeries that turn K into an unlink.

Clearly, we have

$$\begin{array}{ccc} ul_b^u(K) & \leq & u_b^u(K) \\ | \wedge & & | \wedge \\ ul_b(K) & \leq & u_b(K) \end{array}$$

Corollary 1.9. For a knot K in S^3 ,

$$\text{Ord}_I(K) \leq ul_b^u(K) \quad \text{and} \quad \text{Ord}_U(K) \leq ul_b^u(K).$$

Remark 1.10. In private communication [Won], Wong has informed us of a proof, using methods analogous to [AE20], that if there is a cobordism $\Sigma \subset I \times S^3$ from K_1 and K_2 (no matter whether orientable or unorientable) with m minima, b saddles, and M maxima, then

$$|\text{Ord}_U(K_1) - \text{Ord}_U(K_2)| \leq m + b + M.$$

Since the unlink has vanishing torsion order, this would recover the inequality of Corollary 1.9 involving Ord_U .

Our paper is one of several recent papers related to ribbon cobordisms. In [Zem19a], Zemke showed that knot Floer homology obstructs ribbon concordance, a result that prompted a flurry of interesting results in this area, including [LZ19], [MZ21], [DLVW19], [Kan19], and [CGL⁺20]. Other papers in the area are Sarkar's paper on the ribbon distance [Sar20] and the already cited paper of Juhász, Miller, and Zemke [JMZ20], which is the closest paper to ours.

Organisation. The first two sections of the paper are on instanton Floer homology: we review the necessary background in Section 2, and we prove the main instanton technical result (Proposition 3.3) in Section 3. In the following two sections we do the same for knot Floer homology: after a review in Section 4, we prove the main knot Floer technical result (Proposition 5.5) in Section 5. In Section 6 we prove Theorem 1.1 and the applications discussed in the introduction (Corollaries 1.3, 1.6, and 1.9). Finally, in Section 7 we compute the torsion order Ord_U for a subfamily of torus knots and prove Corollary 1.7.

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2. BACKGROUND ON INSTANTON HOMOLOGY WITH LOCAL SYSTEMS

2.1. Instanton homology groups. Kronheimer and Mrowka introduced singular instanton homology with local systems in [KM11b], and introduced several more involved variants of it in [KM21a]. In this paper we will be working with a variant from the latter. Let us now review the relevant definitions and properties, following [KM21a] and [KM21b].

Let Y be a closed, oriented 3-manifold, let L be a link in Y and let y_0 be a basepoint in Y , and let B_{y_0} be a ball around y_0 that does not intersect L . Let $\theta_0 \subset Y$ be a standard θ -web in B_{y_0} . Let ω be a 1-dimensional submanifold of Y , which consists of components that are circles disjoint from L and B_{y_0} and arcs which have endpoints on L and are otherwise disjoint from L .

Then there is an associated space, $\mathcal{B}^\sharp(Y, L)_\omega$ of $SO(3)$ connections on Y that are singular at $L \cup \theta_0$, which lift to $SU(2)$ away from the $L \cup \omega \cup \theta_0$, such that the $SU(2)$ holonomy around ω is -1 and the $SU(2)$ holonomies around components of L and arcs of θ_0 are conjugate to $I \in SU(2)$, when we regard $SU(2)$ as unit quaternions and $1, I, J, K$ are the fundamental quaternion units.

The local system Γ is defined using three maps $h_i : \mathcal{B}^\sharp(Y, L)_\omega \rightarrow \mathbb{R}/\mathbb{Z}$, for $i = 1, 2, 3$, given by taking holonomy along the three arcs of the θ web, which gives three maps to $SU(2)$ and then composing with a character $SU(2) \rightarrow U(1) = \mathbb{R}/\mathbb{Z}$ to get maps to $\mathbb{R}/\mathbb{Z}$. Let $\mathcal{R} = \mathbb{F}_2[\mathbb{Z}^3]$ be the group ring, which we can also write as the ring of Laurent polynomials in three variables,

$$\mathcal{R} = \mathbb{F}_2[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}].$$

Then Γ is defined as the pull back via (h_1, h_2, h_3) of a particular local system over $(\mathbb{R}/\mathbb{Z})^3$ with fibre the free rank 1 module over $\mathcal{R}$. For a commutative ring $\mathcal{S}$ and a homomorphism $\sigma : \mathcal{R} \rightarrow \mathcal{S}$, let Γ_σ denote the induced local system of $\mathcal{S}$ modules.

The instanton homology group $I^\sharp(Y, L; \Gamma_\sigma)_\omega$ is defined as the Floer homology of $\mathcal{B}^\sharp(Y, L)_\omega$ with a perturbed Chern Simons functional and with the local system Γ_σ . (In [KM21a], there is an additional map $h_0 : \mathcal{B}^\sharp(Y, L)_\omega \rightarrow \mathbb{R}/\mathbb{Z}$ coming from taking holonomy along the link itself, and $\mathcal{R}$ is defined to be $\mathbb{F}_2[\mathbb{Z}^4]$, but for our purposes, we will only be using the local system coming from h_1, h_2 , and h_3 .)

2.2. Maps induced by cobordisms with dots. We now review the functoriality of $I^\sharp(Y, L; \Gamma_\sigma)_\omega$. Keeping previous notation, let $\sigma : \mathcal{R} \rightarrow \mathcal{S}$ be a map of commutative rings.

For $i = 1, 2$, let Y_i denote a closed, oriented 3-manifolds, with a link L_i and a 1-manifold ω_i embedded in Y_i with boundary on L_i and otherwise not intersecting L_i .

For a cobordism of pairs (X, S) from (Y_1, L_1) to (Y_2, L_2) , and ω a 2 manifold with corners, whose boundary pieces are ω_1 and ω_2 in Y_1 and Y_2 respectively, together with arcs and circles in S , then there is an induced map

$$I^\sharp(X, S; \Gamma_\sigma)_\omega : I^\sharp(Y_1, L_1; \Gamma_\sigma)_{\omega_1} \rightarrow I^\sharp(Y_2, L_2; \Gamma_\sigma)_{\omega_2}$$

of $\mathcal{S}$ modules.

This functoriality can be extended to morphisms given by cobordisms of pairs with dots on the surfaces. That is, for a cobordism of pairs (X, S) , define a *dot* on S to be an interior point $p \in S$ along with an orientation of $T_p S$. Then for dots $p_1, p_2, \dots, p_d$ on S , there is an induced map of $\mathcal{S}$ modules

$$I^\sharp(X, S, p_1, p_2, \dots, p_d; \Gamma_\sigma)_\omega : I^\sharp(Y_1, L_1; \Gamma_\sigma)_{\omega_1} \rightarrow I^\sharp(Y_2, L_2; \Gamma_\sigma)_{\omega_2}.$$

In our computations, we will always have $Y_1, Y_2 = S^3$ and $X = S^3 \times [0, 1]$. Moreover we will be using the same Γ_σ . Thus, we will denote our cobordisms by

$$I^\sharp(S, p_1, p_2, \dots, p_d)_\omega = I^\sharp(X, S, p_1, p_2, \dots, p_d; \Gamma_\sigma)_\omega$$

2.3. Properties of the induced maps. Before going over some of the properties of the maps of $\mathcal{S}$ modules induced by cobordisms, let us recall two particular elements of the rings $\mathcal{R}$ and $\mathcal{S}$. Writing $\mathcal{R} = \mathbb{F}_2[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}]$, the elements P and Q are given by

$$P = T_1 T_2 T_3 + T_1 T_2^{-1} T_3^{-1} + T_1^{-1} T_2 T_3^{-1} + T_1^{-1} T_2^{-1} T_3$$

and

$$Q = T_1^2 + T_1^{-2} + T_2^2 + T_2^{-2} + T_3^2 + T_3^{-2}.$$

For $\sigma : \mathcal{R} \rightarrow \mathcal{S}$, the elements $\sigma(P), \sigma(Q) \in \mathcal{S}$ will also be denoted P, Q respectively.

- (1) (Lemma 3.2 of [KM21b]) Let S be an oriented cobordism. Suppose S' is obtained from S by adding an internal 1-handle connecting points $p, q \in S$, where p, q both have the same orientation as S . Then

$$I^\sharp(S') = I^\sharp(S, p) + I^\sharp(S, q) + P I^\sharp(S).$$

Here and throughout we assume that $\omega = \emptyset$ when it is not denoted.

- (2) (Lemma 4.2 of [KM21a]) Let (S, ω) be a cobordism between (L_1, ω_1) and (L_2, ω_2) . Let R_+ and R_- be the two standard embedded copies of $\mathbb{R}P^2$ in S^4 with self intersection $+2$ and -2 respectively. Let π be a disk whose boundary is the generator of $H_1(\mathbb{R}P^2)$. Then

$$I^\sharp(S \# R_+)_{\omega + \pi} = I^\sharp(S)_\omega$$

and

$$I^\sharp(S \# R_-)_{\omega + \pi} = P I^\sharp(S)_\omega.$$

- (3) (Section 5.5 of [KM11a], Section 2.2 of [KM19], Section 5.3 of [KM21a], “Kunneth formula for split links”) Let L be a split link, so that $L = L_0 \amalg L_1$, and L_0 and L_1 are contained in disjoint balls in S^3 . Then

$$I^\sharp(L) \simeq I^\sharp(L_0) \otimes I^\sharp(L_1),$$

and this is natural with respect to cobordisms with dots.

This is shown in Section 5.5 of [KM11a] using a version of excision without local coefficients, Hopf link instead of a θ web, and without dots. There is an argument in Section 2.2 of [KM19] for why it does not matter whether one uses a θ web or a Hopf link, and it is explained in Section 5.3 of [KM21a] why it still works with local coefficients. The proof of functoriality in [KM11a] carries over with no problems to the situation of cobordisms with dots.

- (4) (Section 5.2 of [KM21a]) Let U_ℓ be the ℓ component unlink. Then $I^\sharp(U_0)$ is a free module of rank 1 over $\mathcal{S}$, which we write as $I^\sharp(U_0) = \mathcal{S}u_0$, and $I^\sharp(U_1)$ is the free module over $\mathcal{S}$ of rank 2, which we write as $I^\sharp(U_1) = \mathcal{S}u_+ \oplus \mathcal{S}u_-$. For D the standard disk viewed as a cobordism from the empty link to the unknot, and q a point with orientation compatible with the choice of orientation of the knot,

$$I^\sharp(D)(u_0) = u_+ \text{ and } I^\sharp(D, q)(u_0) = u_-.$$

Moreover, if D_o is the standard disk viewed as a cobordism from the unknot to the empty link, and q a point with orientation compatible with the choice of orientation of the knot,

$$I^\sharp(D_o)(u_-) = 1, I^\sharp(D_o)(u_+) = 0, I^\sharp(D_o, q)(u_+) = 1, I^\sharp(D_o, q)(u_-) = P.$$

For U_ℓ , by the previous point, we have

$$I^\sharp(U_\ell) = (\mathcal{S}u_+ \oplus \mathcal{S}u_-)^{\otimes \ell}.$$

- (5) (Section 5.4 of [KM21a]) Let m and Δ denote the standard “pair of pants” cobordisms between the two component unlink U_2 and the unknot U_1 , the merge,

$$m : U_2 \rightarrow U_1$$

and the split

$$\Delta : U_1 \rightarrow U_2.$$

The map on $I^\sharp$ induced by m (with no dots) is given by

$$\begin{aligned} (3) \quad & u_+ \otimes u_+ \mapsto u_+ \\ & u_\pm \otimes u_\mp \mapsto u_- \\ & u_- \otimes u_- \mapsto Pu_- + Qu_+, \end{aligned}$$

and the map induced by Δ (with no dots) is given by

$$\begin{aligned} (4) \quad & u_+ \mapsto u_+ \otimes u_- + u_- \otimes u_+ + Pu_+ \otimes u_+ \\ & u_- \mapsto u_- \otimes u_- + Qu_+ \otimes u_+. \end{aligned}$$

3. A TECHNICAL RESULT FOR INSTANTONS

In this section we prove the main technical result for instanton Floer homology $I^\sharp$ which we will use to prove Theorem 1.1. To do so, we will use a classical result in Morse theory, Lemma 3.2 below. We state it in the most convenient form for us, and give a quick sketch its proof.

Definition 3.1. Given a knot K in S^3 and a band B for K , i.e. an embedded rectangle B in S^3 which intersects K in two opposite sides, we say that B is *orientable with respect to K* if the knot K and the result of band surgery on K along B can be given coherent orientations (equivalently, if surgering K along B gives a 2-component link).

Lemma 3.2. *Let $\Sigma \subset I \times S^3$ be a non-orientable cobordism between knots K and K' with m local minima, b saddles, and M local maxima. Then, after an isotopy rel boundary, we can break it into a sequence of cobordisms as follows.*

- (1) m births (from $K_1 = K$ to L_1);
- (2) m band surgeries that join the various components of the link (from L_1 to K'_1);
- (3) $b - (m + M + 1)$ band surgeries orientable with respect to K'_1 (this cobordism ends with a knot or a 2-component link L');
- (4) 1 band surgery unorientable with respect to K'_1 (this cobordism goes from L' to a knot K'_2);
- (5) M band surgeries that split the knot K'_2 into $M + 1$ components;
- (6) M deaths.

Moreover, in this decomposition, the attaching arcs of the b bands on K'_1 can be assumed to be all disjoint, and we can assume that both attaching arcs of the unorientable band are already contained in K_1 .

Sketch of the proof. We can arrange all births to appear first and all deaths to appear last (steps (1) and (6)). We can also find bands that connect the various components (steps (2) and (5)). Thus, we can restrict to the part of the cobordism between K'_1 and K'_2 , which consists of saddles (i.e., band surgeries). Note that both K'_1 and K'_2 are knots.

If all bands are orientable with respect to K'_1 , then all Σ would be orientable, so there is at least one band unorientable with respect to K'_1 .

Arrange for all bands from K'_1 to K'_2 to appear all at the same time.

If there is more than one band unorientable with respect to K'_1 , pick one of them (call it B) and slide it following the surgery of K'_1 along all the other bands. When B slides over an orientable band, it stays unorientable. When B slides over an unorientable band, it becomes orientable. Note that eventually it must slide over an unorientable band because K'_2 is connected, so $K'_2 \setminus B$ consists of just two arcs.

Repeat until you have only one unorientable band left.

If B is the unique unorientable band, then you can slide its endpoints along L' so that they are disjoint from all the other (oriented) bands, so we can think of it as in K_1 . $\square$

The main technical result of this section, needed to prove Theorem 1.1, is the following Proposition.

Proposition 3.3. *Let S be a cobordism from K to K' with m local minima, b saddles, and M local maxima. Then there is a surface ω that meets S cleanly and only at $\partial\omega \subset S$, whose boundary is a circle in S such that for $\bar{\omega}$ its mirror, we have*

$$(5) \quad P^M I^\#(\bar{S} \circ S)_{\omega \cup \bar{\omega}} = P^{b-m} \text{Id}.$$

Towards this goal, let us start by doing some computations of maps induced by cobordisms with ω .

First let us understand the dependence of $I^\#(\Sigma)_\omega$ on ω , when ω is a surface with boundary on Σ , which intersects Σ cleanly and only at $\partial\omega \subset S$. Note that for a link L in S^3 , up to isomorphism, $I^\#(S^3, L)_\omega$ depends only on the homology of $[\partial\omega] \in H_0(L; \mathbb{Z}/2)$, because it counts flat connections and instantons on spaces determined by the homology class.

Similarly, $I^\#(\Sigma)_\omega$ depends only on the homology class $[\partial\omega] \in H_1(\Sigma, \mathbb{Z}/2)$. This is because the map counts instantons on a moduli space built from Σ , $[\partial\omega] \in H^1(\Sigma, \mathbb{Z}/2)$ and $[\omega, \partial\omega] \in H_2(X, \Sigma, \mathbb{Z}/2)$, and $H_1(\Sigma) \simeq H_2(X, \Sigma)$, for $X = S^3 \times \mathbb{R}$.

From here, we can see that for a cylinder Σ and ω given by either a small disk or a small tube with boundary on Σ , as in Figure 1, $I^\#(\Sigma)_\omega$ induces the identity: here $\partial\omega$ is trivial in $H_1(\Sigma)$, and $[\omega, \partial\omega]$ is also trivial in $H_2(X, \Sigma, \mathbb{Z}/2)$.

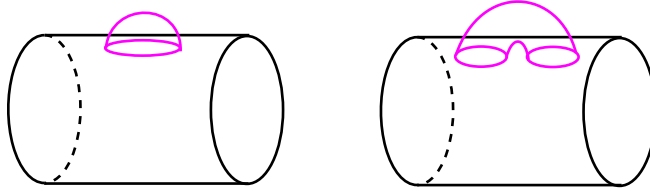


FIGURE 1. Cylinders with the magenta surfaces depicting ω .

When the cobordism in Figure 2 is composed with its inverse, the map induced is the identity. Moreover, up to isomorphism $I^\#(U, \omega)$ depends only on $[\partial\omega] \in H_0(U; \mathbb{Z}/2)$, so the two ends of the cobordism have the same instanton Floer homology. Thus, the cobordism in Figure 2 induces an isomorphism.

We will call the two generators of the instanton Floer homology of the unknot with an arc ω on the right, which is depicted in Figure 3, x_+ and x_- , so that in the $u_\pm$ and $x_\pm$ bases, the cobordism depicted in Figure 2 is the identity matrix.

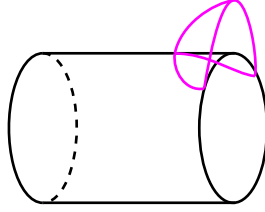


FIGURE 2

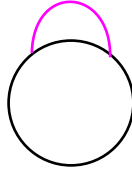


FIGURE 3

The cobordism from the two component unlink to itself induced by two standard cylinders with ω as a tube between them, as depicted in Figure 4, induces the identity map, because in this situation, $(\omega, \partial\omega)$ is trivial in homology in $(S^3 \times I, \Sigma)$.

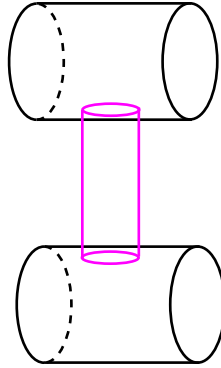


FIGURE 4

The same is true for the map depicted in Figure 5 precomposed with its mirror. Thus, the map induced by the cobordism depicted in Figure 5 is an isomorphism whose inverse is its mirror image. Here, we are identifying the link with ω on the right end of Figure 5 with the unlink with empty ω via the isomorphism induced by Figure 2, and the link with ω on the left has isomorphic instanton Floer homology.

For the link on the left in Figure 5, its homology is then a free module of rank 4 over $\mathcal{S}$. Let $x_{++}, x_{+-}, x_{-+}, x_{--}$ be a basis of this homology so that if we choose the $x_+ \otimes x_+, x_+ \otimes x_-, x_- \otimes x_+, x_- \otimes x_-$ basis for the two component unlink on the right, the matrix the cobordism induces is the identity. (Recall that $x_{\pm}$ are the basis elements of the instanton homology

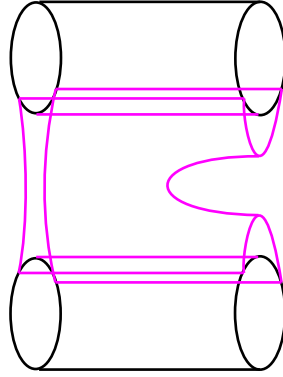


FIGURE 5

of the unknot with an arc, so that the cobordism of Figure 2 induces the identity matrix.

A central step in our proof will be dealing with a cobordism that flips an unknot, but does not change ω . To describe this, consider a link L with decoration ω , which has an unknot component U that is split from the rest of L ; we may isotope U so that it is a geometric circle. Suppose that ω has two endpoints on U , p and q , which we may isotope to be the endpoints of a diameter of U . Then the flip cobordism is a cobordism in $I \times S^3$ that is traced by the isotopy obtained by rotating U by π about the diameter pq . So this is an isotopy that does not change ω and reverses the orientation of one of the two components.

Claim 1. The map on the instanton homology of U_2 with ω consisting of two arcs, each going between the two components that results from flipping one of the unknots (as described above) in a way that does not change ω is the identity map.

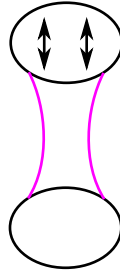


FIGURE 6

Proof. By composing with the isomorphisms induced by the cobordism depicted in Figure 5, if Φ is the matrix associated to the flip in the $x_{++}, x_{+-}, x_{-+}, x_{--}$

basis, then

$$\Phi = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

where $\Phi_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the flip on the unknot with an arc in the x_+, x_- basis, depicted in Figure 7. This is because Φ is the matrix for $\Phi_1 \otimes \text{Id}$, for Id the identity map, in the $x_+ \otimes x_+, x_+ \otimes x_-, x_- \otimes x_+, x_- \otimes x_-$ basis, and we are using the basis of the instanton homology of Figure 6 corresponding to this basis.

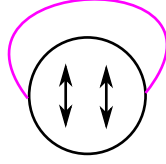


FIGURE 7

Now let us compute some of the entries of Φ_1 . Note that if we pre-compose or post-compose Φ_1 with caps like those in Figure 8, we get back the cap itself. These caps induce the maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[0, 1]$, so from these compositions, we can see that $a = d = 1$ and $c = 0$.

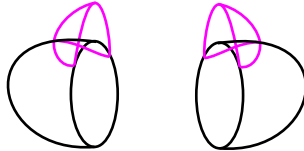


FIGURE 8

Note that if we did not have ω , then we could do the same argument with a cap with a dot, and using the fact that doing a flip and then a cap with a dot is the same as doing a negative dot, we would be able to get the remaining entry, b , and recover Proposition 5.8 of [KM21a], in which the flip map does *not* induce the identity. However, because we have ω here, this does not work: the flip changes which side of $\partial\omega$ the dot is on.

Going back to our computation, we now have that

$$\Phi = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We would now like to show that b is 0. Consider the pair of pants cobordism with ω as two half-disks from the unlink with two arcs going between components to the unknot, as depicted in Figure 9. Because we can pre-compose with isomorphisms to make a regular merge with a null-homotopic disk on top, as in Figure 10, we see that Figure 9 induces the same as the merge map, if we use the $x_{++}, x_{+-}, x_{-+}, x_{--}$ basis. Here we are using that the reverse of the map in Figure 5 is also the identity matrix with our choice of basis.

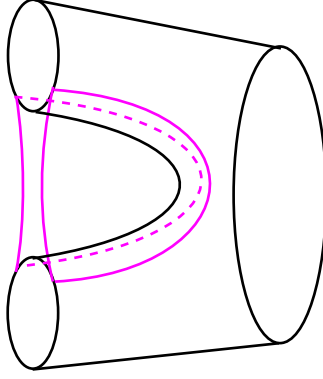


FIGURE 9

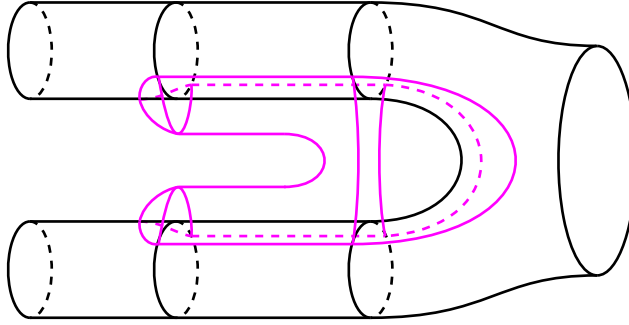


FIGURE 10

Thus, in this basis, it induces the map

$$m = \begin{bmatrix} 1 & 0 & 0 & Q \\ 0 & 1 & 1 & P \end{bmatrix}.$$

Similarly, the reverse of this cobordism induces the same map as Δ , so it induces

$$\Delta = \begin{bmatrix} P & Q \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, composing $m \circ \Phi \circ \Delta$, we get the map

$$m \circ \Phi \circ \Delta = \begin{bmatrix} b+P & 0 \\ 0 & b+P \end{bmatrix}.$$

However, if we compose these cobordisms, we get a Klein bottle, which is a connected sum of $\mathbb{R}P_+^2$ and $\mathbb{R}P_-^2$, with ω given by two disks, one on each $\mathbb{R}P^2$, such that the boundary circle of the disk is the generator of $H_1(\mathbb{R}P^2)$. It is shown in [KM21a] that this Klein bottle with these ω induces the map $P \cdot \text{Id}$, so $b = 0$, as desired. $\square$

Claim 2. Let $S \subset S^3 \times [0, 2]$ be a cobordism from K_1 to K_2 such that in $S^3 \times [0, 1]$ it is the cylinder on K_1 and in $S^3 \times [1, 2]$ it consists of adding a single non-orientable band. More precisely, we may consider a band $B \subset S^3$ with vertices $A_1 A_2 A_3 A_4 \subset S^3$ with $A_1 A_2$ and $A_3 A_4$ on K_1 , as in Figure 11. In $S^3 \times [1, 2]$, S then looks like $(K_1 \setminus (A_1 A_2 \cup A_3 A_4)) \times [1, 2]$ away from the band $B \times [1, 2]$, and within the band it goes from $A_1 A_2 \cup A_3 A_4$ at time 1 to $A_2 A_3 \cup A_4 A_1$ at time 2.

The cobordism is depicted in frames in Figure 12.

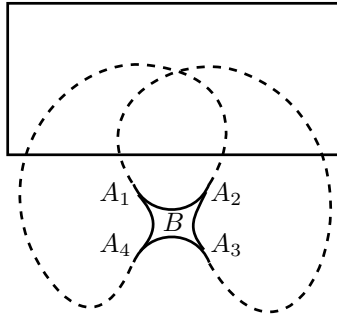


FIGURE 11. The cobordism S is cylindrical on the dotted part.

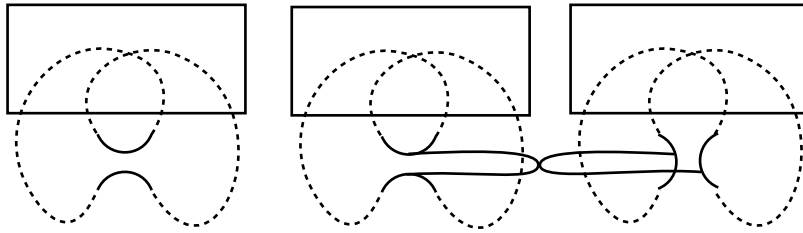


FIGURE 12

Then there is a surface ω with boundary in the interior of S , such that ω meets S only at the boundary, where they meet cleanly, such that for $\bar{S}$ the reverse of S , with corresponding $\bar{\omega}$,

$$I^\sharp(\bar{S})_{\bar{\omega}} \circ I^\sharp(S)_\omega = P \cdot \text{Id} : I^\sharp(K_1) \rightarrow I^\sharp(K_1).$$

Proof. Observe that A_1 and A_3 split K_1 into two parts, which we call a and b (these are coloured magenta and blue respectively in Figure 13. Let c be the diagonal on the band that goes from A_1 to A_3 .

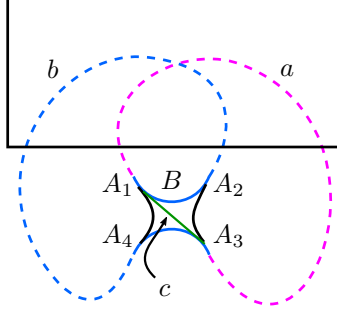


FIGURE 13. The cobordism S is cylindrical on the dotted part.

Consider $a \cup c$ as a knot in S^3 and let F_0 be a Seifert surface of it. Then, F_0 is a surface with corners, with boundary $a \cup c$, and which meets b at the ends, A_1 and A_3 . We may isotope a, b, c so that F_0 meets b cleanly at the ends and transversely in the interior, as in Figure 14

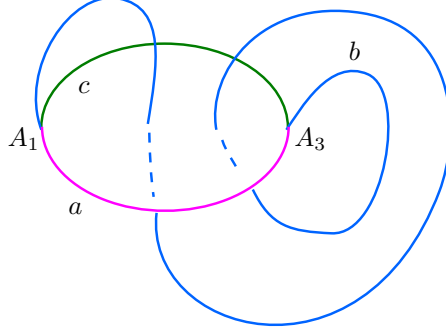


FIGURE 14. Here F_0 is depicted as a disk though it could have higher genus.

If we choose an orientation of F_0 and b , then the intersection points may have positive or negative sign. We can increase the number of positive or negative intersection points without changing the isotopy type of the embedding of $K_1 \cup c$ into S^3 by twisting b around A_1 or A_3 , as in Figure 15. Let us do this, adding either positive or negative intersections as needed until there are the same number of positive as negative interior intersection points between b and F_0 .

Now say that the intersection points are $A_1, p_1, p_2, \dots, p_{2k}, A_3$, in order along b . Then if p_i and p_{i+1} are intersection points with opposite sign, we may remove a small disk around each of p_i and p_{i+1} and replace it with a small tube around the part of b that goes from p_i to p_{i+1} , thus reducing the number of intersection points. We may continue in this manner, removing adjacent, opposite-sign intersection points until none remain.

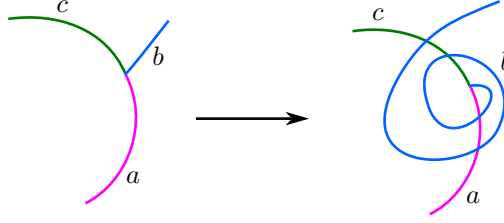


FIGURE 15

We now have a surface, which we call F_1 with boundary $a \cup c$, which intersects b only at A_1 and A_3 , where the intersection is clean.

We now consider a surface $F_2 \subset S^3 \times [0, 2]$ with boundary on S which is given by the union of $F_1 \subset S^3 \times \{1\}$ with a disk sitting between $c \times 1 \subset S^3 \times [1, 2]$ and S , as in Figure 16.

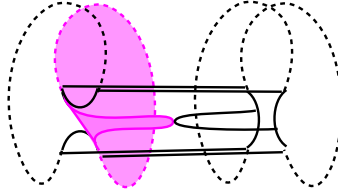


FIGURE 16

Then this F_2 can have its corners smoothed out to a surface with boundary ω .

Let us now show that for this ω , we have

$$I^\sharp(\overline{S})_{\overline{\omega}} \circ I^\sharp(S)_{\omega} = P \cdot \text{Id} : I^\sharp(K_1) \rightarrow I^\sharp(K_1).$$

Let Σ denote the composition of S with $\overline{S}$, and let $\omega_\Sigma = \omega \cup \overline{\omega}$ be the decoration on this cobordism. See Figure 17 for an illustration.

Let γ denote the circle composed of the co-core of the band and its mirror, depicted in blue in Figure 17. A regular neighbourhood of γ in Σ is a tube, represented in Figure 18. If we cut the surface along γ , we get the twice punctured cylinder as a cobordism from K_1 to itself.

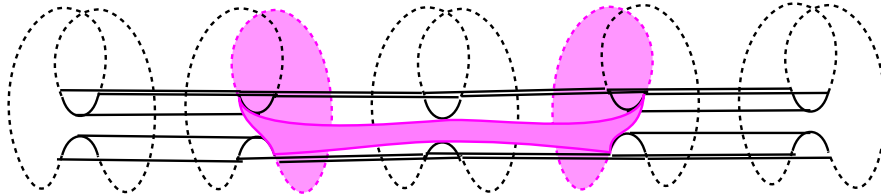
FIGURE 17. This is S with ω composed with the reverses $\overline{S}$ with $\overline{\omega}$.

Figure 18 shows $\partial\omega_\Sigma$ as well. The mod 2 homology class $[\partial\omega_\Sigma]$ on the surface Σ is the same as $[\gamma]$. One way to see it is to perform surgery on

$\partial\omega_\Sigma$ along the green arc in Figure 18: this operation does not change the homology class and it yields a curve which is easily checked to be isotopic to γ in Σ .

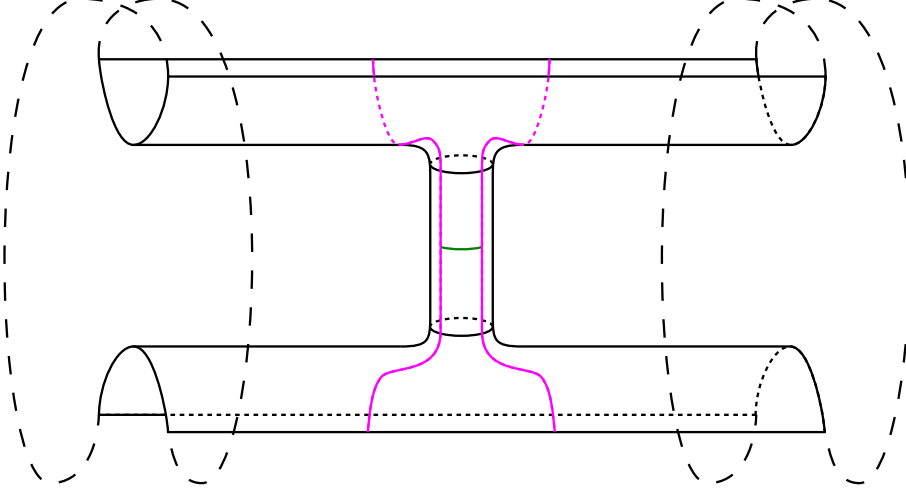


FIGURE 18

Let Σ' be the cobordism obtained from Σ by inserting a flip in the tube in the centre of Figure 18, with the same decoration ω_Σ . Using Claim 1, we will see below that $I^\#(\Sigma)_{\omega_\Sigma} = I^\#(\Sigma')_{\omega_\Sigma}$. However, the curve $\partial\omega_\Sigma$ is homologically trivial in Σ' . One can check it again by doing surgery on the green arc, but this time the extra flip ensured that the obtained curve is not γ , but a homotopically trivial one. Thus, $I^\#(\Sigma')_{\omega_\Sigma} = I^\#(\Sigma')_\emptyset$, since the map depends only on $[\partial\omega_\Sigma]$. If $\omega = \emptyset$ one can apply the neck cutting relation (Property (1) in Section 2.3) to obtain that

$$I^\#(\Sigma') = P \cdot I^\#(I \times K_1) = P \cdot \text{Id}_{I^\#(K_1)}.$$

We still have to show that $I^\#(\Sigma)_{\omega_\Sigma} = I^\#(\Sigma')_{\omega_\Sigma}$. To see this, isotope the tube in the middle as shown in Figure 19.

Let's restrict our attention to the piece contained in the cylinder in green, which is the identity cobordism on a 2-component unlink. By Claim 1 the map induced by this cobordism is the same that we get if we introduce a flip on one of the two components. Since instanton Floer maps respect composition of cobordisms and disjoint unions, the map induced by the whole cobordism is not affected by the insertion of the flip, i.e. $I^\#(\Sigma)_{\omega_\Sigma} = I^\#(\Sigma')_{\omega_\Sigma}$. $\square$

Claim 3. Let S be a cobordism from a knot K_1 to a knot K_2 such that S consists of only b bands. That is, there are no births nor deaths. Then there is a surface ω with boundary on S such that

$$I^\#(\overline{S})_{\overline{\omega}} \circ I^\#(S)_\omega = P^b \cdot \text{Id}.$$

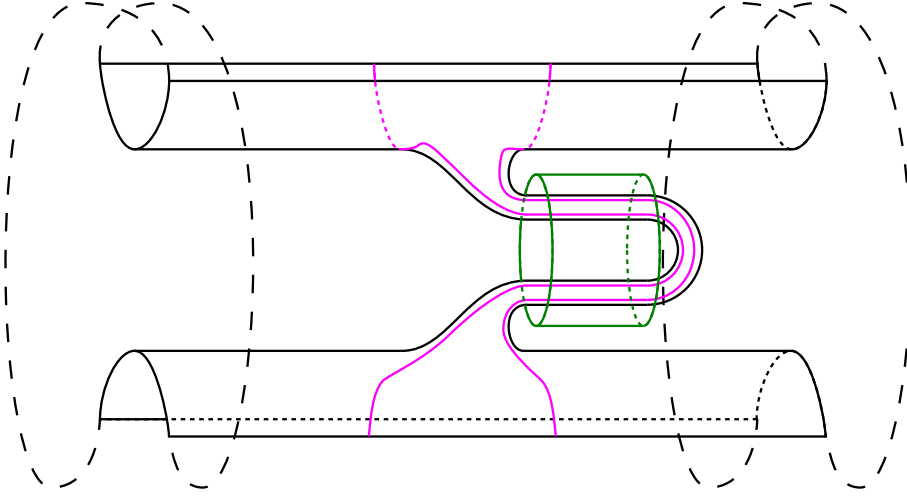


FIGURE 19

Proof. We proceed by induction on b . The base case $b = 0$, is obvious.

For the inductive step, we divide into two cases. If S is orientable, then the statement holds for ω empty, because the cobordism $\overline{S} \circ S$ is the same as the cylinder on K_1 with b orientable tubes, and the result follows from the tube cutting formula.

In the case that S is not orientable, at least one of the bands of S must be non-orientable with respect to K . In this case, let us write $S = S_r \circ S_u$, where S_u is a cobordism consisting of the non-orientable band, and S_r is the rest of the cobordism, which may or may not be orientable.

Then by the induction hypothesis there is some ω_r such that

$$I^\sharp(\overline{S_r})_{\overline{\omega_r}} \circ I^\sharp(S_r)_{\omega_r} = P^{b-1} \cdot \text{Id}$$

Applying Claim 2, there is a surface ω_u with boundary on S_u such that

$$I^\sharp(\overline{S_u})_{\overline{\omega_u}} \circ I^\sharp(S_u)_{\omega_u} = P \cdot \text{Id}.$$

The statement

$$I^\sharp(\overline{S})_{\overline{\omega}} \circ I^\sharp(S)_{\omega} = P^b \cdot \text{Id}$$

now follows. □

Now, we can proceed with the proof of Proposition 3.3.

Proof of Proposition 3.3. Applying Lemma 3.2, we may break S into pieces of the form.

- (1) m births (from K_1 to L_1);

- (2) m band surgeries that joint the various components of the link (from L_1 to K'_1);
- (3) $b - (m + M)$ band surgeries which may or may not be orientable, ending in a knot K'_2 .
- (4) M band surgeries that split the knot K'_2 into $M + 1$ components;
- (5) M deaths.

Let us call the cobordisms corresponding to the five steps $S_1, S_2, \dots, S_5$. We may isotope the cobordism in $S^3 \times \mathbb{R}$ so that S_i is in the $S^3 \times [i, i + 1]$.

We will choose ω to be in $S^3 \times [2, 3]$, so that its boundary is in S_3 as in Claim 3 so that

$$I^\#(\overline{S_3})_{\overline{\omega}} \circ I^\#(S_3)_\omega = P^{b-m-M} \cdot \text{Id}.$$

The proof now proceeds the same way as the proof of Proposition 4.1 of [JMZ20]. The main argument is by considering a cobordism Σ that comes from adding M tubes connecting points on the death-caps to their mirrors.

By Lemma 3.2 of [KM21b], for a connected, oriented cobordism Σ if Σ' is obtained from Σ by adding a tube between points p and q , then

$$(6) \quad I^\#(\Sigma') = I^\#(\Sigma, p) + I^\#(\Sigma, q) + PI^\#(\Sigma) = PI^\#(\Sigma),$$

where the second equality is because Σ is connected, so $I^\#(\Sigma, p)$ and $I^\#(\Sigma, q)$ induce the same map, and since we are working over characteristic two, they cancel.

Let Σ_1 denote the cobordism coming from taking $\overline{S} \circ S$ and adding M tubes, one for each death connecting a point in the death to its reverse, so that $\Sigma_1 = S_1 S_2 S_3 S_4 \overline{S_4} \overline{S_3} \overline{S_2} \overline{S_1}$. Applying Equation (6) for each death, to the part of the cobordism from K_3 to itself coming from doing S_4, S_5 and their reverses, we see that

$$I^\#(\Sigma_1) = P^M I^\#(\overline{S} \circ S).$$

Here, we are allowed to use the above result because $S_4 S_5 \overline{S_5} \overline{S_4}$ and $S_4 \overline{S_4}$ are both orientable and connected.

In $\Sigma_1 = (S_1 S_2 S_3 S_4) \overline{(S_1 S_2 S_3 S_4)}$, M splitting bands of S_4 and their reverses, cap off the ends, and call the resulting cobordism Σ_2 . Then for the same reason as the above, we have

$$I^\#(\Sigma_1) = P^M I^\#(\Sigma_2),$$

because again, $S_3 S_4 \overline{S_4} \overline{S_3}$ and $S_3 \overline{S_3}$ are both orientable and connected.

Now we have $\Sigma_2 = (S_1 S_2 S_3) \overline{(S_1 S_2 S_3)}$.

Because of our construction of ω , $S_3 \overline{S_3}$ with $\omega \cup \overline{\omega}$ falls under the setting of Claim 3, so the map it induces is $P^{b-m-M} \cdot \text{Id}_{I^\#(K_2)}$. Thus, if we let $\Sigma_3 = (S_1 S_2) \overline{(S_1 S_2)}$, then

$$I^\#(\Sigma_2) = P^{b-m-M} I^\#(\Sigma_3).$$

Now Σ_3 is given by a cylinder on K_1 and m S^2 s, with m tubes, with the tubes connecting the S^2 s and the cylinder in a tree-like fashion. Applying the tube-cutting formula, Lemma 3.2 of [KM21b], and observing that a sphere

without any dots induces the 0 map and the sphere with one dot induces the identity, we see that $I^\sharp(\Sigma_3)$ induces the same map as the cylinder, which is to say, the identity.

Putting all of this together, we get

$$P^M I^\sharp(\bar{S} \circ S) = I^\sharp(\Sigma_1) = P^{b-m} \cdot \text{Id},$$

as desired. $\square$

4. BACKGROUND ON UNORIENTED KNOT FLOER HOMOLOGY

Unoriented knot Floer homology was introduced by Ozsváth–Stipsicz–Szabó [OSSz17a, OSSz17b]. Fan [Fan19] showed that an unorientable cobordism (with some extra data) induces maps on the unoriented knot Floer homology. We now review the relevant definitions, following mostly [Zem19c] and [Fan19].

4.1. Zemke’s oriented TQFT. Cobordism maps in link Floer homology were first defined by [Juh16]. In this paper we use Zemke’s setup [Zem19c], specified to unoriented link Floer homology in the case $Y = S^3$.

Definition 4.1. An *oriented multi-based link* in S^3 is a triple $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ consisting of an oriented, embedded link $L \subset S^3$, with two disjoint collections of basepoints $\mathbf{w}$ and $\mathbf{z}$ on L , such that each component of L has at least two basepoints and the basepoints alternate between those in $\mathbf{w}$ and those in $\mathbf{z}$ as one traverses a component of L .

To an oriented multi-based link $\mathbb{L}$, Zemke’s most general construction gives a curved $\mathbb{F}[U_{\mathbf{w}}, V_{\mathbf{z}}]$ -complex $\mathcal{CFL}^-(\mathbb{L})$ up to $\mathbb{F}[U_{\mathbf{w}}, V_{\mathbf{z}}]$ -equivariant chain homotopy. Here $\mathbb{F}[U_{\mathbf{w}}, V_{\mathbf{z}}]$ denotes the polynomial ring generated by a U variable for each $\mathbf{w}$ basepoint and a V variable for each $\mathbf{z}$ basepoint. The curved complex is also endowed with gradings and a filtration.

In our case, we only need a simpler version of Zemke’s complex, namely unoriented link Floer homology. This is defined as

$$\text{CFL}'(\mathbb{L}) := \mathcal{CFL}^-(\mathbb{L}) \otimes_{\mathbb{F}[U_{\mathbf{w}}, V_{\mathbf{z}}]} \mathbb{F}[U],$$

where all variables act on $\mathbb{F}[U]$ as the multiplication by U . For the reader familiar with Heegaard Floer homology, this is the free $\mathbb{F}[U]$ -module generated by the intersection points $\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ in the symmetric product, with differential given by

$$(7) \quad \partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \# \overline{\mathcal{M}}(\phi) \cdot U^{n_{\mathbf{o}}(\phi)} \cdot \mathbf{y},$$

where $n_{\mathbf{o}}(\phi) = \sum_{w \in \mathbf{w}} n_w(\phi) + \sum_{z \in \mathbf{z}} n_z(\phi)$.

Definition 4.2. For a doubly-based knot $\mathbb{K} = (K, w, z)$ we also use the notations $\text{CFK}'(\mathbb{K})$ and $\text{HFK}'(\mathbb{K})$ for $\text{CFL}'(\mathbb{K})$ and $\text{HFL}'(\mathbb{K})$, respectively.

If $\mathbb{K}_1 = (K, w_1, z_1)$ and $\mathbb{K}_2 = (K, w_2, z_2)$ are two doubly-based knot with the same underlying knot K , then $\text{HFK}'(\mathbb{K}_1)$ and $\text{HFK}(\mathbb{K}_2)$ are non-canonically isomorphic as $\mathbb{F}[U]$ -modules. Thus, the following number is well-defined.

Definition 4.3. If K is a knot, we define its *unoriented torsion order* as

$$\text{Ord}_U(K) = \min \{n \geq 0 \mid U^n \cdot \text{Tors} = 0\},$$

where Tors is the torsion submodule of $\text{HFK}'(\mathbb{K})$, considered as a module over $\mathbb{F}[U]$. Here $\mathbb{K}$ is any doubly-based knot with underlying knot K .

Remark 4.4. CFL' enjoys the following properties:

- (1) $\text{CFL}'(\mathbb{L})$ is a genuine chain complex (i.e., its curvature vanishes), so one can compute its homology $\text{HFL}'(\mathbb{L})$, known as the *unoriented link Floer homology* of $\mathbb{L}$. This is still an $\mathbb{F}[U]$ -module.
- (2) For a doubly-based knot $\mathbb{K} = (K, w, z)$, $\text{HFK}'(\mathbb{K}) \cong \mathbb{F}[U] \oplus \text{Tors}$, where Tors is the torsion as an $\mathbb{F}[U]$ -module.
- (3) For a doubly-based unknot $\mathbb{U}_1 = (U_1, w, z)$, $\text{HFK}'(\mathbb{U}_1) \cong \mathbb{F}[U]$.
- (4) Given doubly based knots $\mathbb{K}_1$ and $\mathbb{K}_2$,

$$\text{CFK}'(\mathbb{K}_1 \# \mathbb{K}_2) = \text{CFK}'(\mathbb{K}_1) \otimes_{\mathbb{F}[U]} \text{CFK}'(\mathbb{K}_2).$$

As a consequence, for knots K_1 and K_2 in S^3 ,

$$\text{Ord}_U(K_1 \# K_2) = \max \{\text{Ord}_U(K_1), \text{Ord}_U(K_2)\}.$$

- (5) If $\overline{\mathbb{L}}$ is the mirror of $\mathbb{L}$ (with same basepoints), then by [OSSz17b, Proposition 2.17]

$$\text{CFL}'(\overline{\mathbb{L}}) = \text{hom}_{\mathbb{F}[U]}(\text{CFL}'(\mathbb{L}), \mathbb{F}[U]).$$

As a consequence, for a knot K in S^3 ,

$$\text{Ord}_U(\overline{K}) = \text{Ord}_U(K).$$

Definition 4.5. If $\mathbb{L}_1 = (L_1, \mathbf{w}_1, \mathbf{z}_1)$ and $\mathbb{L}_2 = (L_2, \mathbf{w}_2, \mathbf{z}_2)$ are two oriented multi-based links, an (*oriented*) *decorated link cobordism* from $\mathbb{L}_1$ to $\mathbb{L}_2$ is a pair $\mathbb{S} = (\Sigma, \mathcal{A})$ such that

- (1) $\Sigma \subset I \times S^3$ is a properly embedded, compact, oriented surface with $\Sigma \cap \{0\} \times S^3 = \{0\} \times (-L_1)$ and $\Sigma \cap \{1\} \times S^3 = \{1\} \times L_2$;
- (2) $\mathcal{A} \subset \Sigma$ is a properly embedded 1-manifold, which we refer to as the *decorations*;
- (3) the components of $\Sigma \setminus \mathcal{A}$ are partitioned into two sub-surfaces, $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$, which meet along $\mathcal{A}$;
- (4) each component of $L_i \setminus \mathcal{A}$ contains exactly one basepoint of $\mathbf{w}_i \sqcup \mathbf{z}_i$;
- (5) $\mathbf{w}_1 \sqcup \mathbf{w}_2 \subset \Sigma_{\mathbf{w}}$ and $\mathbf{z}_1 \sqcup \mathbf{z}_2 \subset \Sigma_{\mathbf{z}}$.

Definition 4.6. The identity (decorated link) cobordism $\text{id}_{\mathbb{L}}$ from $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ to itself is given by the surface $\Sigma = I \times L$ with decorations $\mathcal{A} = I \times Q$, where $Q \subset L \setminus (\mathbf{w} \cup \mathbf{z})$ is a finite set such that the inclusion induces an isomorphism in π_0 .

By the work of Zemke [Zem19c], an oriented decorated link cobordism $\mathbb{S}$ from $\mathbb{L}_1$ to $\mathbb{L}_2$ induces an $\mathbb{F}[U]$ -equivariant map

$$F_{\mathbb{S}}^Z : \text{HFL}'(\mathbb{L}_1) \rightarrow \text{HFL}'(\mathbb{L}_2).$$

Remark 4.7. The map F^Z enjoys the following properties:

- (1) $F_{\mathbb{S}}^Z$ is invariant under isotopy of Σ in $I \times S^3$ while fixing the boundary, and under isotopy of $\mathcal{A}$ in Σ while keeping

$$\partial \mathcal{A} \subset (L_1 \setminus (\mathbf{w}_1 \cup \mathbf{z}_1)) \cup (L_2 \setminus (\mathbf{w}_2 \cup \mathbf{z}_2)).$$

- (2) If $\text{id}_{\mathbb{L}}$ is the identity cobordism from $\mathbb{L}$ to itself, then

$$F_{\text{id}_{\mathbb{L}}}^Z = \text{id}_{\text{HFL}'(\mathbb{L})}.$$

- (3) If $\mathbb{S}_1$ and $\mathbb{S}_2$ are oriented decorated link cobordisms from $\mathbb{L}_1$ to $\mathbb{L}_2$ and from $\mathbb{L}_2$ to $\mathbb{L}_3$ respectively, then one can stack $\mathbb{S}_2$ on top of $\mathbb{S}_1$ (after isotoping the decorations so that they match on the $\mathbb{L}_2$ level), and obtain a new oriented decorated link cobordism $\mathbb{S}_2 \circ \mathbb{S}_1$ from $\mathbb{L}_1$ to $\mathbb{L}_3$. In such a case,

$$F_{\mathbb{S}_2 \circ \mathbb{S}_1}^Z = F_{\mathbb{S}_2}^Z \circ F_{\mathbb{S}_1}^Z.$$

- (4) If $\mathbb{S}' = (\Sigma', \mathcal{A}')$ is obtained from $\mathbb{S} = (\Sigma, \mathcal{A})$ by attaching a tube with both feet in $\Sigma_{\mathbf{z}}$ (or both feet in $\Sigma_{\mathbf{w}}$), then $F_{\mathbb{S}'}^Z = U \cdot F_{\mathbb{S}}^Z$.

4.2. Fan's unoriented TQFT. By the work of Fan [Fan19], the link Floer TQFT can be extended to the non-orientable case. We review the relevant definitions.

Definition 4.8. A *disoriented link* in S^3 is a tuple $\mathcal{L} = (L, \mathbf{p}, \mathbf{q})$ consisting of an unoriented, embedded link $L \subset S^3$, with two disjoint collections of points $\mathbf{p}$ and $\mathbf{q}$ on L , called the *dividing set*, such that each component of L has at least two points in the dividing set and the points in the dividing set alternate between those in $\mathbf{p}$ and those in $\mathbf{q}$ as one traverses a component of L .

Each component of $L \setminus (\mathbf{p} \cup \mathbf{q})$ is given a canonical orientation from $\mathbf{q}$ to $\mathbf{p}$. We denote the oriented manifold $L \setminus (\mathbf{p} \cup \mathbf{q})$ by $\mathbf{l}$. Note that these orientations do not glue to an orientation of L .

As it is customary, we consider isotopic disoriented knots as different disoriented knots. It is well known that isotopies can induce non-trivial maps in knot Floer homology, such as the moving basepoint maps [Sar15, Zem17].

Definition 4.8 looks exactly the same as Definition 4.1, except that the link is now unoriented. However, we emphasise that the basepoints $\mathbf{w} \cup \mathbf{z}$ from 4.1 are ontologically different from the dividing set from 4.8. From a Morse theoretical viewpoint, the former arise as the intersection between L and the middle level surface of a Morse function, whereas the latter are the index-0 and index-3 critical points of the function.

However, we can define a notion of compatibility between oriented decorated links and disoriented links.

Definition 4.9. We say that an oriented decorated link $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ and a disoriented link $\mathcal{L} = (L, \mathbf{p}, \mathbf{q})$ are *compatible* if:

- the underlying unoriented link L is the same (but note that in $\mathbb{L}$ it also comes with an orientation);
- $\mathbf{p} \cup \mathbf{q}$ is disjoint from $\mathbf{w} \cup \mathbf{z}$;
- each component of $L \setminus (\mathbf{p} \cup \mathbf{q})$ contains exactly one basepoint in $\mathbf{w} \cup \mathbf{z}$;
- the components of $L \setminus (\mathbf{w} \cup \mathbf{z})$ containing the $\mathbf{p}$ point are oriented from z to w (with orientation induced from $\mathbb{L}$).

Remark 4.10. Every disoriented link admits a (non-canonical) compatible oriented decorated link. Likewise, every oriented decorated link admits a (non-canonical) compatible disoriented link.

If two oriented decorated links $\mathbb{L}_1$ and $\mathbb{L}_2$ are compatible with the disoriented link $\mathcal{L}$, then $\text{HFL}'(\mathbb{L}_1)$ and $\text{HFL}'(\mathbb{L}_2)$ are canonically isomorphic. Thus, we can define $\text{HFL}'(\mathcal{L})$ as $\text{HFL}'(\mathbb{L})$, for any $\mathbb{L}$ compatible with $\mathcal{L}$. (More precisely, $\text{HFL}'(\mathcal{L})$ is the transitive system over all compatible oriented decorated links.)

Note that $\text{HFL}'(\mathcal{L})$ does not depend on the orientation chosen on L . If $(L, \mathbf{w}, \mathbf{z})$ is a compatible oriented decorated link, then the orientation reversal L^r is also part of a compatible oriented decorated link, namely $(L^r, \mathbf{z}, \mathbf{w})$. The swap of the $\mathbf{w}$ and $\mathbf{z}$ basepoints does not affect the homology, since the differential was defined to be symmetric in $\mathbf{w}$ and $\mathbf{z}$ (cf. Equation (7)). This justifies the name *unoriented knot Floer homology* used in [OSSz17b].

Remark 4.11. Fan [Fan19] defines other categories of unoriented links, which he calls *bipartite links* and *bipartite disoriented links*. These are essential to define a TQFT framework for disoriented links, but we do not recall them here.

We now revise the cobordism maps defined by Fan [Fan19].

Definition 4.12. A *disoriented link cobordism* from $\mathcal{L}_1 = (L_1, \mathbf{p}_1, \mathbf{q}_1)$ to $\mathcal{L}_2 = (L_2, \mathbf{p}_2, \mathbf{q}_2)$ is a pair $\mathcal{S} = (\Sigma, \mathcal{M})$ such that

- (1) $\Sigma \subset I \times S^3$ is a properly embedded, compact surface with $\Sigma \cap \{0\} \times S^3 = \{0\} \times (-L_1)$ and $\Sigma \cap \{1\} \times S^3 = \{1\} \times L_2$;
- (2) $\mathcal{M} \subset \Sigma$ is a properly embedded, compact, oriented 1-manifold, which we refer to as the *motion* of the critical points;
- (3) the components of $\Sigma \setminus \mathcal{M}$ are compact, oriented surfaces with orientation induced by $\mathcal{M}$;
- (4) $\partial \mathcal{M} = \mathbf{q}_1 - \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{q}_2$.

Note that, with the orientation given in point (3), $\partial(\Sigma \setminus \mathcal{M}) = \mathbf{l}_2 - \mathbf{l}_1 + 2\mathcal{M}$. The surface Σ does not need to be oriented.

There is a natural notion of identity cobordism, in the same spirit as Definition 4.6. We do not write such a definition explicitly.

By the work of Fan [Fan19], an disoriented link cobordism $\mathcal{S}$ from $\mathcal{L}_1$ to $\mathcal{L}_2$ induces an $\mathbb{F}[U]$ -equivariant map

$$F_{\mathcal{S}}^F : \text{HFL}'(\mathcal{L}_1) \rightarrow \text{HFL}'(\mathcal{L}_2).$$

Remark 4.13. The map F^F enjoys the following properties:

- (1) $F_{\mathcal{S}}^F$ is invariant under isotopy of Σ in $I \times S^3$ while fixing the boundary, and under isotopy of $\mathcal{M}$ in Σ while fixing the boundary.
- (2) If $\text{id}_{\mathcal{L}}$ is the identity cobordism from $\mathcal{L}$ to itself, then

$$F_{\text{id}_{\mathcal{L}}}^F = \text{id}_{\text{HFL}'(\mathcal{L})}.$$

- (3) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are oriented disoriented link cobordisms from $\mathcal{L}_1$ to $\mathcal{L}_2$ and from $\mathcal{L}_2$ to $\mathcal{L}_3$ respectively, then one can stack $\mathcal{S}_2$ on top of $\mathcal{S}_1$, and obtain a new oriented disoriented link cobordism $\mathcal{S}_2 \circ \mathcal{S}_1$ from $\mathcal{L}_1$ to $\mathcal{L}_3$. In such a case,

$$F_{\mathcal{S}_2 \circ \mathcal{S}_1}^F = F_{\mathcal{S}_2}^F \circ F_{\mathcal{S}_1}^F.$$

4.3. Relation between Zemke's TQFT and Fan's TQFT.

Definition 4.14. For $i = 1, 2$, suppose that $\mathbb{L}_i = (L_i, \mathbf{w}_i, \mathbf{z}_i)$ and $\mathcal{L}_i$ are compatible. We say that a decorated link cobordism $\mathbb{S} = (\Sigma, \mathcal{A})$ from $\mathbb{L}_1$ to $\mathbb{L}_2$ and a disoriented link cobordism $\mathcal{S} = (\Sigma, \mathcal{M})$ from $\mathcal{L}_1$ to $\mathcal{L}_2$ are *compatible* if

- the underlying unoriented surface Σ is the same (but note that in $\mathbb{S}$ it also comes with an orientation);
- after isotoping $\mathcal{A}$ without crossing $\mathbf{w}_1 \sqcup \mathbf{z}_1 \sqcup \mathbf{w}_2 \sqcup \mathbf{z}_2$, $\mathcal{A} = \mathcal{M}$.

Remark 4.15. For $i = 1, 2$, suppose that $\mathbb{L}_i$ and $\mathcal{L}_i$ are compatible. Moreover, suppose that $\mathbb{S}$ is a decorated link cobordism from $\mathbb{L}_1$ to $\mathbb{L}_2$ and $\mathcal{S}$ is a compatible disoriented link cobordism from $\mathcal{L}_1$ to $\mathcal{L}_2$. Then

$$F_{\mathbb{S}}^Z = F_{\mathcal{S}}^F.$$

5. A TECHNICAL RESULT FOR HFK'

5.1. The flip cobordism in HFK' .

Definition 5.1. The standard disoriented unknot is $\mathcal{U}_1 = (U_1, p, q)$, where $U_1 = \{x^2 + y^2 = 1\} \cap \{z = 0\}$, $p = (1, 0, 0)$ and $q = (-1, 0, 0)$.

Definition 5.2. The *flip cobordism* $\mathcal{F} = (\Sigma_{\mathcal{F}}, \mathcal{M}_{\mathcal{F}})$ from the standard disoriented unknot $\mathcal{U}_1 = (U_1, p, q)$ to itself is the disoriented cobordism traced by the isotopy obtained by rotating U_1 by π along the x -axis. The points p and q stay fixed throughout the isotopy, so we can set $\mathcal{M}_{\mathcal{F}} = I \times \{p, q\}$.

Note that the surface underlying a flip cobordism is orientable, although no orientation of the surface restricts to the same orientation on the two standard disoriented unknots on the boundary.

Lemma 5.3. *The map $F_{\mathcal{F}}^F$ induced by the flip cobordism is the identity map on $\text{HFK}'(\mathcal{U}_1) \cong \mathbb{F}[U]$.*

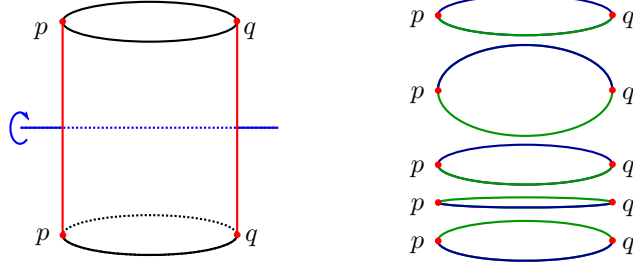


FIGURE 20. The figure on the left is our notation for the flip cobordism. On the right we sketched a few sections of the cobordism. We used different colours for the two components of $U_1 \setminus \{p, q\}$ to help the visualisation.

Proof. The fourth iteration $\mathcal{F}^4$ is the disoriented cobordism traced by a 4π rotation about the x -axis. Since $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, the rotation by 4π is isotopic to the identity. Thus, $\mathcal{F}^4$ is isotopic to the identity cobordism, and

$$(8) \quad (F_{\mathcal{F}}^F)^4 = F_{\mathcal{F}^4}^F = \text{id}_{\mathbb{F}[U]}.$$

The map

$$F_{\mathcal{F}}^F : \mathbb{F}[U] \rightarrow \mathbb{F}[U]$$

is U -equivariant, so it is completely determined by the image of 1. If we set $p(U) := F_{\mathcal{F}}^F(1) \in \mathbb{F}[U]$, Equation (8) implies that $(p(U))^4 = 1$. Since every invertible element of $\mathbb{F}[U]$ must be in $\mathbb{F}$, we deduce $p(U) = 1$. $\square$

5.2. A stabilisation lemma. In this subsection only, we will need to work in a more general setting than the one outlined in Section 4.

First, we will consider decorated links $\mathbb{L}$ in a 3-manifold Y , and decorated link cobordisms $(\Sigma, \mathcal{A})$ in a 4-manifold W . In Section 4, we have stated the definitions of decorated link and decorated link cobordism only when $Y = S^3$ and $W = I \times S^3$. The general definitions are only needed in this subsection, and they can be found in [Zem19c, Definitions 2.1 and 2.4]. Again in this subsection only, we will consider the full chain complex $\mathcal{CFL}^-(Y, \mathbb{K})$ associated to a decorated knot, which is a complex over $\mathbb{F}[U, V]$, up to chain homotopy equivalence.

We will also use the homological action on link Floer homology. (See [OS04, Section 4.2.5] for the original definition of Heegaard Floer homology, [OS06, Theorem 3.1] for the cobordism action, and [Zem19b, Section 12.2] for its extension to link Floer homology.) Given a decorated link $\mathbb{L}$ in a 3-manifold Y there is a homological action

$$A : \Lambda^*(H_1(Y; \mathbb{Z})/\text{Tors}) \otimes \mathcal{CFL}^-(Y, \mathbb{L}) \rightarrow \mathcal{CFL}^-(Y, \mathbb{L}),$$

and given a decorated link cobordism $(\Sigma, \mathcal{A})$ in a 4-manifold W there is a version of the cobordism map incorporating the homological action

$$F_{W, \Sigma, \mathcal{A}}^H : \Lambda^*(H_1(W; \mathbb{Z})/\text{Tors}) \otimes \mathcal{CFL}^-(Y_1, \mathbb{L}_1) \rightarrow \mathcal{CFL}^-(Y_2, \mathbb{L}_2).$$

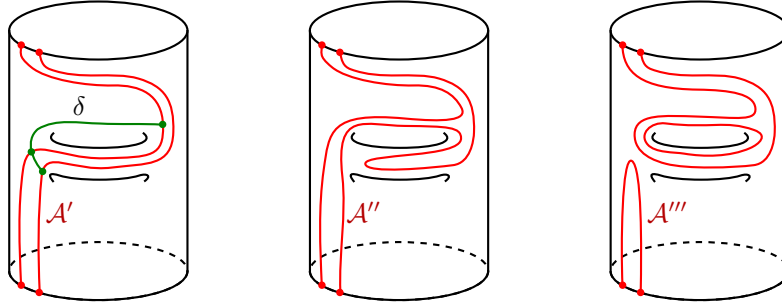


FIGURE 21. The three figures show the three decorations $\mathcal{A}'$, $\mathcal{A}''$, and $\mathcal{A}'''$ on the surface Σ' appearing in the bypass relation arising from the arc δ .

(We use the notations F^H to distinguish it from the cobordism map F^Z which does not incorporate the homological action.) If the 4-manifold W is obtained by adding 1-handles to B^4 , then the map F^H can be recovered from F^Z by post-composing with the homological action on Y_1 . The following lemma, which is needed to establish Proposition 5.5, was proved by Ian Zemke. A related argument appeared in [JZ20, Section 5] (see in particular [JZ20, Lemma 5.3]).

Lemma 5.4. *Let $\Sigma = I \times K \subset I \times S^3$ be the identity cobordism from the knot K to itself, and let Σ' denote a surface obtained by adding a compressible 1-handle to Σ . If $\gamma \subset \Sigma$ denotes an embedded arc joining the feet of the 1-handle, define decorations $\mathcal{A}'$ on Σ' as two parallel embedded arcs from $\{0\} \times K$ to $\{1\} \times K$ such that:*

- $\mathcal{A}'$ does not intersect γ ;
- each arc of $\mathcal{A}'$ crosses the co-core of the 1-handle exactly once;
- the arcs of $\mathcal{A}'$ join the points $(0, p)$ and $(0, q)$ to the points $(1, p)$ and $(1, q)$ in $I \times K$ respectively;
- the decorations $\mathcal{A}$, obtained by restricting $\mathcal{A}'$ to Σ and by reconnecting each pair of arcs with an arc parallel to γ , is isotopic rel boundary to a product decoration $I \times \{p, q\}$.

Then, if $\mathbb{K} = (K, w, z)$ for some points w, z alternated to p, q , the cobordism map

$$F_{\Sigma', \mathcal{A}'}^Z : \text{HFK}'(\mathbb{K}) \rightarrow \text{HFK}'(\mathbb{K})$$

coincides with the map $U \cdot \text{id}_{\text{HFK}'(\mathbb{K})}$.

Proof. If a_1 and a_2 denote the two components of $\mathcal{A}'$, let δ be an arc on Σ' which starts from a_1 near a foot of the 1-handle, then traverses a_2 , follows γ to the other foot of the 1-handle, and ends on a_2 . See Figure 21 on the left for an illustration.

We apply Zemke's bypass relation on a disc $\Delta \subset \Sigma'$ obtained as a regular neighbourhood of the arc δ in Σ' . If $\mathcal{A}''$ and $\mathcal{A}'''$ denote the other decorations

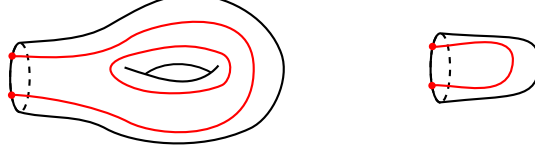


FIGURE 22. The figure on the left shows the decorated surface $(\widetilde{\Sigma}_1, \widetilde{\mathcal{A}}_1)$. The punctured torus $\widetilde{\Sigma}_1$ sits in $S^1 \times D^3$ in such a way that its longitude generates $H_1(S^1 \times D^3)$, while its meridian is null-homotopic. The figure on the right shows the decorated surface $(\overline{\Sigma}_1, \overline{\mathcal{A}}_1)$.

appearing in the bypass relation as in Figure 21, we have that

$$F_{\Sigma', \mathcal{A}'}^Z = F_{\Sigma', \mathcal{A}''}^Z + F_{\Sigma', \mathcal{A}'''}^Z.$$

The decorations $\mathcal{A}''$ can be isotoped away from the co-core of the 1-handle. After compressing the 1-handle the surface becomes isotopic to Σ and the decorations become isotopic to the product decorations $\mathcal{A}$. Thus, by Remark 4.7.(4),

$$F_{\Sigma', \mathcal{A}''}^Z = U \cdot F_{\Sigma, \mathcal{A}}^Z = U \cdot \text{id}_{\text{HFK}'(\mathbb{K})}.$$

Thus, we only need to show that $F_{\Sigma', \mathcal{A}'''}^Z = 0$. From this point until the end of the proof we will work on the chain level $\mathcal{CFL}^-(\mathbb{K})$, considered as an $\mathbb{F}[U, V]$ -complex, up to chain homotopy equivalence. (The variable U is associated to the basepoint w .)

We split the cobordism $(I \times S^3, \Sigma', \mathcal{A}''')$ as the composition of two cobordisms. The first one, which we call $\mathcal{W}_1 = (W_1, \Sigma_1, \mathcal{A}_1)$, is obtained by taking as W_1 the (disjoint) union of a regular neighbourhood of $\{0\} \times S^3$ and a neighbourhood of $\gamma \cup c$ (where c denotes the core of the 1-handle) containing the 1-handle entirely. Note that the latter component of W_1 is diffeomorphic to $S^1 \times D^3$. The decorated surface $(\Sigma_1, \mathcal{A}_1)$ is obtained by intersecting W_1 with $(\Sigma', \mathcal{A}''')$. The second cobordism $\mathcal{W}_2$ is obtained by taking the closure of the complement of W_1 in $I \times S^3$. Thus, we have

$$F_{\Sigma, \mathcal{A}'''}^Z = F_{(I \times S^3, \Sigma', \mathcal{A}''')}^H = F_{\mathcal{W}_2}^H \circ F_{\mathcal{W}_1}^H.$$

(In the first equality we used the fact that $H_1(I \times S^3) = 0$.)

We focus on the map $F_{\mathcal{W}_1}^H$. Since W_1 has two connected components (one of which is an identity cobordism over $\mathbb{K}$), the map splits as a tensor product

$$(9) \quad F_{\mathcal{W}_1}^H = \text{id}_{\mathcal{CFL}^-(\mathbb{K})} \otimes F_{\widetilde{\mathcal{W}}_1}^H,$$

where $\widetilde{\mathcal{W}}_1 = (S^1 \times D^3, \widetilde{\Sigma}_1, \widetilde{\mathcal{A}}_1)$ is a cobordism from the empty link in the empty 3-manifold to a doubly pointed unknot $\mathbb{U}$ in $S^1 \times S^2$, illustrated in Figure 22 on the left.

The knot Floer complex $\mathcal{CFL}^-(S^1 \times S^2, \mathbb{U})$ is generated over $\mathbb{F}[U, V]$ by two homogeneous elements $\mathbf{x}_+$ and $\mathbf{x}_-$. Their $\mathbf{gr}_w$ and $\mathbf{gr}_z$ gradings (as

defined in [Zem19b]) are given by the formulas

$$(\mathbf{gr}_w, \mathbf{gr}_z)(\mathbf{x}_\pm) = \left(\pm \frac{1}{2}, \pm \frac{1}{2} \right).$$

By grading reasons [Zem19b], we have that

$$(10) \quad F_{\mathcal{W}_1}^H(1) = k \cdot \mathbf{x}_-$$

for some $k \in \mathbb{Z}/2\mathbb{Z}$. An explicit computation of the action of $H_1(S^1 \times S^2) = \mathbb{Z}\langle \zeta \rangle$ shows that $A(\zeta \otimes \mathbf{x}_+) = \mathbf{x}_-$. From this fact, a direct computation shows that

$$(11) \quad F_{\mathcal{W}_1}^H(\zeta \otimes 1) = \mathbf{x}_-,$$

where $\overline{\mathcal{W}_1} = (S^1 \times S^2, \overline{\Sigma_1}, \overline{\mathcal{A}_1})$ is the cobordism shown in Figure 22 on the right.

Recall that the cobordism $\mathcal{W}_1$ is the disjoint union of an identity cobordism over $\mathbb{K}$ and the cobordism $\widehat{\mathcal{W}_1}$. If $\widehat{\mathcal{W}_1}$ denotes the cobordism obtained by replacing the $\widehat{\mathcal{W}_1}$ component with $\overline{\mathcal{W}_1}$, then by combining Equations (9), (10), and (11) we have

$$(12) \quad F_{\mathcal{W}_1}^H(\mathbf{x}) = \mathbf{x} \otimes F_{\mathcal{W}_1}^H(1) = k \cdot \mathbf{x} \otimes F_{\mathcal{W}_1}^H(\zeta \otimes 1) = k \cdot F_{\mathcal{W}_1}^H(\zeta \otimes \mathbf{x}).$$

Finally, let $\widehat{\mathcal{W}}$ denote the composition of $\widehat{\mathcal{W}_1}$ and $\mathcal{W}_2$. Note that the 4-manifold underlying $\widehat{\mathcal{W}}$ is still $I \times S^3$ (same as $\mathcal{W}$), since the replacement of $\mathcal{W}_1$ with $\widehat{\mathcal{W}_1}$ did not affect the underlying 4-manifold. Then, by Equation (12),

$$\begin{aligned} F_{\Sigma, \mathcal{A}'''}^Z(\mathbf{x}) &= F_{\mathcal{W}_2}^H \circ F_{\mathcal{W}_1}^H(\mathbf{x}) \\ &= k \cdot F_{\mathcal{W}_2}^H \circ F_{\mathcal{W}_1}^H(\zeta \otimes \mathbf{x}) \\ &= k \cdot F_{\widehat{\mathcal{W}}}^H(\iota_*(\zeta) \otimes \mathbf{x}) = 0. \end{aligned}$$

The last term vanishes because the map $\iota_*: H_1(S^1 \times D^3) \rightarrow H_1(I \times S^3) = 0$ induced by the inclusion of $\widehat{\mathcal{W}_1}$ into $\widehat{\mathcal{W}}$ must map ζ to 0. $\square$

5.3. The main theorem in HFK'

Proposition 5.5. *Suppose that Σ is a connected unorientable knot cobordism from K_1 to K_2 in $I \times S^3$ with m local minima, b saddles, and M local maxima, and let $\overline{\Sigma}$ denote the mirrored upside down cobordism from K_2 to K_1 .*

Then there are choices of motions of the critical points such that the disoriented knot cobordisms $\mathcal{S} = (\Sigma, \mathcal{M}_1)$ and $\overline{\mathcal{S}} = (\overline{\Sigma}, \mathcal{M}_2)$ can be composed to $\overline{\mathcal{S}} \circ \mathcal{S}$, and

$$(13) \quad U^M \cdot F_{\overline{\mathcal{S}}}^F \circ F_{\mathcal{S}}^F = U^{b-m} \cdot \text{id}_{\text{HFK}'(K_1)}.$$

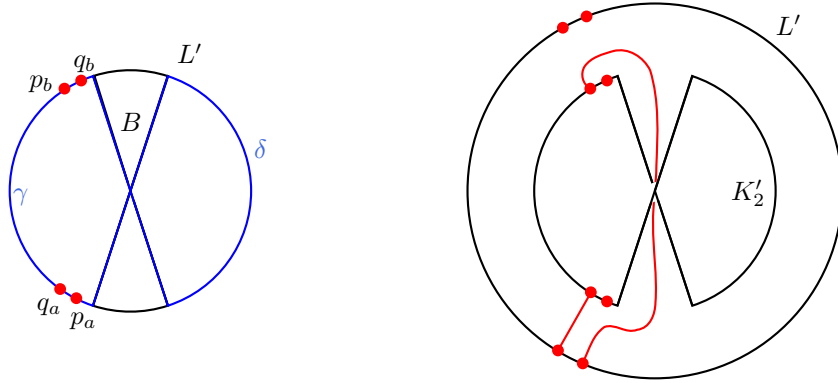


FIGURE 23. The circle on the left hand side represents L' after step (3), in case it is a knot. The figure on the right hand side shows step (4) of the cobordism Σ , from L' to K'_2 , together with the motion chosen to define $\mathcal{S}$.

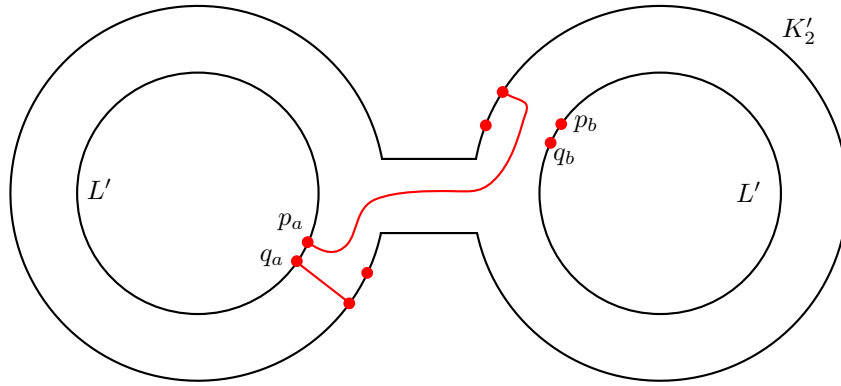


FIGURE 24. The figure shows step (4) of $\mathcal{S}$ in case L' is a 2-component link, represented above by the two inner circles.

Proof. Using Lemma 3.2, we can break the cobordism Σ into the composition of cobordisms labelled (1)-(6). Let K'_1 and K'_2 be the knots after steps (2) and (4) respectively, as in the statement of Lemma 3.2, and let L' be the link after step (3). Note that L' differs from K'_2 by a single band surgery.

By removing the two attaching arcs of the unorientable band B from L' , we are left with two arcs γ and δ . If L' is a knot, let p_a, q_a, p_b, q_b be points on γ , appearing in this order, such that p_a and q_a are close to one end of γ and p_b and q_b are close to the other end of γ , so that all the intersections of $L' \cap \gamma$ with the oriented bands are between p_b and q_a . See the left hand side of Figure 23. If instead L' is a 2-component link, let p_a and q_a be on γ and p_b and q_b be on δ , such that they are near opposite corners of the band, and p_a (resp. q_b) is closer to the band than q_a (resp. p_b). See Figure 24.

Let $\mathcal{S}$ be the disoriented cobordism from (K_1, p_a, q_a) to (K_2, p_b, q_a) obtained by endowing Σ with the following motion of basepoints:

- on steps (1)-(3), the motion consists of straight arcs $I \times \{p_a, q_a\}$;
- on step (4), the motion consists of a straight arc $I \times \{q_a\}$ and of an arc that starts from p_a , goes through the unorientable saddle, and ends at p_b (see Figure 23 for an illustration);
- on steps (5)-(6), the motion consists of straight arcs $I \times \{p_b, q_a\}$.

A crucial condition in Definition 4.12 is that each component of $\Sigma \setminus \mathcal{M}_1$ must be orientable. In fact, we show that $\Sigma \setminus \mathcal{M}_1$ consists of a single and orientable component. If L' is a knot, one can check from Figure 23 that $\Sigma \setminus \mathcal{M}_1$ restricted to step (4) has a single component; in steps (1)-(3) the surface Σ is orientable and the motion is given by two parallel arcs, so there are two components of $\Sigma \setminus \mathcal{M}_1$, which are then glued to the unique component in step (4); steps (5)-(6) define a concordance of disoriented knots, which does not change the abstract topology of the disoriented cobordism. The compatibility of the orientation of $\Sigma \setminus \mathcal{M}_1$ with the orientation of $\mathcal{M}_1$ is dealt with in an analogous way.

If instead L' is a 2-component link, then one should consider Figure 24 instead. Let ζ be the closed component of L' containing p_b and q_b (which appears on the right in Figure 24), and let ε be the component containing p_a and q_a , minus the short arc connecting p_a and q_a (which appears on the left in Figure 24). From Figure 24 it is immediate to check that $\Sigma \setminus \mathcal{M}_1$ has 2 components in step (4), which deformation retract on ζ and on ε . Since K_1 is a knot, $\Sigma \setminus \mathcal{M}_1$ in steps (1)-(3) also has two components: a “small” rectangular one (S), spanned by the short arc connecting p_a and q_a , and a large one (L) the complement of it, which contains all the genus. When you glue steps (1)-(3) to step (4), the rectangular component S is glued to the component containing ζ , without affecting the topology, whereas the component L glues to both component of step (4). Thus, we see that there is only one component of $\Sigma \setminus \mathcal{M}_1$. Its orientability and the compatibility with the orientation of $\mathcal{M}_1$ is left to the reader (it basically follows from the fact that cutting along $\mathcal{M}_1$ effectively cuts the unorientable saddle, leaving an orientable cobordism). As before, we do not worry about steps (5)-(6), since they define a concordance, which does not change the abstract topology.

We next introduce a disoriented cobordism $\bar{\mathcal{S}}$ from (K_2, p_b, q_a) to (K_1, p_a, q_a) with underlying surface $\bar{\Sigma}$. To define it, we play the steps of the cobordism $\mathcal{S}$ in reverse order, but we use a different motion of basepoints:

- on the reversed steps (6)-(5), the motion consists of straight arcs $I \times \{p_b, q_a\}$;
- on the reversed step (4), the motion consists of a straight arc $I \times \{p_b\}$ and of an arc that starts from q_a , goes through the (dual) unorientable saddle, and ends at q_b ;
- on the reversed steps (3)-(1), the motion consists of straight arcs $I \times \{p_b, q_b\}$;
- finally, in a collar of the K_1 boundary component, the motion of the basepoints brings p_b and q_b back to p_a and q_a .

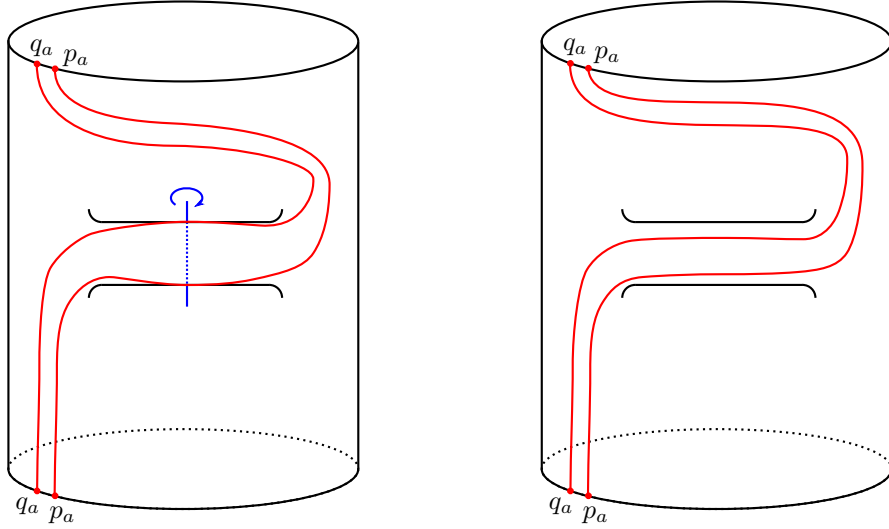


FIGURE 25. The figure on the left represents the disoriented cobordism $\mathcal{T}^u$. The figure on the right represents the disoriented cobordism $\mathcal{T}^o$.

Note that $\bar{\mathcal{S}}$ is not $\mathcal{S}$ turned upside down as disoriented cobordisms (even if the disoriented knots at the boundary are not the same).

We also define a disoriented cobordism $\mathcal{T}^u$ from (K_1, p_a, q_a) to (K_1, p_a, q_a) , obtained in three steps:

- the first step is the same disoriented cobordism as in Figure 23, except that the knot L' is replaced with K_1 ; more explicitly, the surface Σ in the first step consists of the cylinder $I \times K_1$, with the unorientable band B attached on the upper end (recall that by Lemma 3.2 all bands have disjoint attaching arcs, so we can attach B to K_1), and the motion of the basepoints consists of $I \times \{q_a\}$ and of an arc that starts from p_a , goes through the band, and ends at p_b ;
- the surface in the second step is simply the surface from the first step turned upside down, and the motion consists of a straight arc $I \times \{p_b\}$ and of an arc that starts from q_a , goes through the (dual) band, and ends at q_b ;
- finally, in a collar of the end boundary component, the motion of the basepoints brings p_b and q_b back to p_a and q_a .

Note that the surface Σ of the disoriented cobordism $\mathcal{T}^u$ is a genus-1 surface, consisting of a cylinder $I \times K_1$ with a flipped tube attached to it. The flipped tube is made up of the two unorientable bands. See the left hand side of Figure 25.

Lastly, we define a variant of $\mathcal{T}^u$: the disoriented cobordism $\mathcal{T}^o$ from (K_1, p_a, q_a) to (K_1, p_a, q_a) is obtained by replacing the flipped tube in $\mathcal{T}^u$ with an orientable tube, so that the underlying surface Σ is orientable (in other words, the unorientable bands are replaced with orientable bands);

the motion of the basepoints divides Σ into a disc and a punctured torus; see the right hand side of Figure 25. Note that Lemma 5.3 implies that $F_{\mathcal{T}^u}^F = F_{\mathcal{T}^o}^F$, since it is possible to isolate a flip cobordism.

In order to prove Proposition 5.5 we argue in a similar way as in [JMZ20, Proposition 4.1]: we define a cobordism $\mathcal{G}^u$, and we compute the map $F_{\mathcal{G}^u}^F$ in two different ways, which will be the two sides of equation (13).

The disoriented cobordism $\mathcal{G}^u$, from (K_1, p_a, q_a) to itself, is obtained by playing all the steps of $\mathcal{S}$ except (6) followed by all the reversed steps of $\overline{\mathcal{S}}$ except (6). (In $\overline{\mathcal{S}}$ we also play the basepoint moving step in a collar of K_1 .)

Since the attaching arc of the unoriented band can be isotoped in K_1 , we can move step (4) of $\mathcal{S}$ and the corresponding reversed step of $\overline{\mathcal{S}}$ up past all the other steps of $\mathcal{S}$ and $\overline{\mathcal{S}}$. These two steps together make up the cobordism $\mathcal{T}^u$, which we can replace with the orientable cobordism $\mathcal{T}^o$. The replacement yields a new disoriented cobordism $\mathcal{G}^o$ from $\mathcal{G}^u$, with $F_{\mathcal{G}^o}^F = F_{\mathcal{G}^u}^F$. The advantage of $\mathcal{G}^o$ over $\mathcal{G}^u$ is that the underlying surface of the former is orientable, so $F_{\mathcal{G}^o}^F = F_{\mathbb{G}}^Z$ for a compatible decorated link cobordism $\mathbb{G}$, and we can use the properties of Zemke's TQFT on $F_{\mathbb{G}}^Z$, in particular the one about compressing discs.

Note that in the definition of $\mathcal{G}^u$ (or $\mathcal{G}^o$) we do not play the M deaths of $\mathcal{S}$ (step (6)) and the M births of $\overline{\mathcal{S}}$, obtained by mirroring the deaths of $\mathcal{S}$. Thus, $\overline{\mathcal{S}} \circ \mathcal{S}$ is obtained from $\mathcal{G}^u$ by compressing M discs with boundary in the complement of the motion of the basepoints. By transiting through their orientable replacements, and by Remark 4.7.(4), we get

$$(14) \quad F_{\mathcal{G}^u}^F = U^M \cdot \circ F_{\overline{\mathcal{S}}}^F \circ F_{\mathcal{S}}^F.$$

On the other hand, we saw earlier that the cobordism $\mathcal{G}^o$ can be rearranged so that we have $\mathcal{T}^o$ at the top. The first part consists of a cylindrical cobordism from (K_1, p_a, q_a) to itself with $b - 1 - m$ tubes added, as in [JMZ20]. (The -1 here comes from the fact that we have moved the unorientable band to the top of the cobordism.) Thus, we can compress the cobordism $\mathcal{G}^o$ $b - 1 - m$ times to get $\mathcal{T}^o$, so

$$F_{\mathcal{G}^o}^F = U^{b-1-m} \cdot F_{\mathcal{T}^o}^F.$$

But the cobordism $\mathcal{T}^o$ is of the form studied in Lemma 5.4, so the map that it induces is the multiplication by U . Thus:

$$(15) \quad F_{\mathcal{G}^u}^F = F_{\mathcal{G}^o}^F = U^{b-m} \cdot \text{id}_{\text{HFL}'(K_1)}.$$

By combining Equations (14) and (15) we finish the proof. $\square$

Remark 5.6. The careful reader will note that the motion of the basepoints play an important role in the proof of Proposition 5.5. This is by contrast with [JMZ20, Proposition 4.1], where the decorations of the cobordism were the simplest possible, i.e. two parallel arcs from the bottom to the top. In the unoriented setting it is impossible to choose two parallel arcs as the motion of basepoints, otherwise the cobordism would not fall in the correct category.

6. APPLICATIONS

In this section we prove Theorem 1.1, which we restate below, together with its corollaries.

Theorem 1.1. *Let K_1 and K_2 be knots in S^3 . Suppose that there is an unorientable knot cobordism Σ in $I \times S^3$ from K_1 to K_2 with M local maxima. Then*

$$(1) \quad \text{Ord}_I(K_1) \leq \max \{ \text{Ord}_I(K_2), M \} + \gamma(\Sigma)$$

and

$$(2) \quad \text{Ord}_U(K_1) \leq \max \{ \text{Ord}_U(K_2), M \} + \gamma(\Sigma).$$

Proof. The proof closely follows that of [JMZ20, Theorem 1.1].

Add decorations on Σ and $\bar{\Sigma}$ (in the instanton or unoriented knot Floer sense) to obtain cobordisms with decorations $\mathcal{S}$ and $\bar{\mathcal{S}}$ that satisfy the relation in Proposition 3.3 or 5.5

$$(16) \quad v^M \cdot F_{\bar{\mathcal{S}}} \circ F_{\mathcal{S}} = v^{b-m} \cdot \text{id}_{H(K_1)}.$$

Here m is the number of local minima and b is the number of saddles on Σ , H denotes either $I^\sharp$ or HFK' , F denotes the corresponding map induced by an unoriented cobordisms with decorations, and v denotes the relevant variable P or U .

Suppose that $x \in H(K_1)$ is a torsion element. Since $F_{\mathcal{S}}(x)$ must be torsion in $H(K_2)$, $v^\ell \cdot F_{\bar{\mathcal{S}}} \circ F_{\mathcal{S}}(x) = F_{\bar{\mathcal{S}}}(v^\ell \cdot F_{\mathcal{S}}(x)) = 0$ whenever $\ell \geq \text{Ord}(K_2)$. Thus, in view of Equation (16), $v^{\ell+b-m-M} \cdot x = 0$ whenever $\ell \geq \max \{ \text{Ord}(K_2), M \}$. Since this holds for every torsion element $x \in H(K_1)$, we obtain

$$\text{Ord}(K_1) \leq \max \{ \text{Ord}(K_2), M \} + b - m - M,$$

and we conclude by noticing that $\gamma(\Sigma) = -\chi(\Sigma) = b - m - M$. $\square$

We now focus on the proofs of the Corollaries from the introduction. Corollary 1.3 follows immediately from Theorem 1.1 by setting $M = 0$, so we move directly to the following corollary, about the refined unoriented cobordism distance.

Recall that for a cobordism Σ in $I \times S^3$ from K_1 to K_2 , we define

$$|\Sigma| = \max \{ m, M \} - \chi(\Sigma),$$

and that the *refined unorientable cobordism distance* between two knots K_1 and K_2 is given by

$$d_u^r(K_1, K_2) = \min \{ |\Sigma| \},$$

where Σ varies over all connected unorientable cobordisms and oriented concordances from K_1 to K_2 .

Corollary 1.6. *If K_1 and K_2 are knots in S^3 , then*

$$|\text{Ord}_I(K_1) - \text{Ord}_I(K_2)| \leq d_u^r(K_1, K_2)$$

and

$$|\text{Ord}_U(K_1) - \text{Ord}_U(K_2)| \leq d_u^r(K_1, K_2).$$

Proof. The proof follows closely that of [JMZ20, Corollary 1.5]. Given a cobordism Σ from K_1 to K_2 with M maxima and m minima of the kind considered in the definition of d_u^r , by Theorem 1.1 (if Σ is unorientable) and Remark 1.2 (if Σ is an orientable concordance) we have

$$\text{Ord}(K_1) \leq \max\{\text{Ord}(K_2), M\} - \chi(\Sigma) \leq \text{Ord}(K_2) + M - \chi(\Sigma),$$

where Ord is either Ord_I or Ord_U . From here we get

$$\text{Ord}(K_1) - \text{Ord}(K_2) \leq M - \chi(\Sigma) \leq \max\{m, M\} - \chi(\Sigma),$$

and we conclude by exchanging the roles of K_1 and K_2 , and taking the minimum on the right hand side. $\square$

Recall that the *unoriented band-unlinking number* $ul_b^u(K)$ of a knot K in S^3 is defined as the minimum number of (orientable or unorientable) band surgeries that turn K into an unlink.

Corollary 1.9. *For a knot K in S^3 ,*

$$\text{Ord}_I(K) \leq ul_b^u(K) \quad \text{and} \quad \text{Ord}_U(K) \leq ul_b^u(K).$$

Proof. The proof is similar to the one of [JMZ20, Corollary 1.6]. If $b = ul_b^u(K)$, one can build a cobordism Σ from K to the unknot U with b saddles and M local maxima, by attaching b bands to K to get an $(M+1)$ -component unlink and then capping off M components of the latter. By applying Theorem 1.1, and Remark 1.2 if necessary (i.e., if Σ is orientable), we get (for $I^\#$ or HFK')

$$\text{Ord}(K) \leq \max\{\text{Ord}(U), M\} - \chi(\Sigma) = M - \chi(\Sigma),$$

since the unknot has vanishing torsion order in both $I^\#$ and HFK' . We conclude by noticing that $\chi(\Sigma) = M - b$. $\square$

7. EXAMPLES

Lemma 7.1. *For the torus knot $T_{n,n+1}$, $\text{Ord}_U(T_{n,n+1}) = \lfloor n/2 \rfloor$.*

Proof. Any torus knot is an L -space knot, so its Alexander polynomial determines the full knot Floer complex CFK^∞ up to chain homotopy equivalence, and in turn the unoriented knot Floer homology. See [OS05, OSSz17a, JMZ20].

If K is an L -space knot, its Alexander polynomial takes the form

$$(17) \quad \Delta_K(t) = \sum_{k=0}^{2l} (-1)^k t^{\alpha_k}$$

for a decreasing sequence of integers $\alpha_0, \dots, \alpha_{2l}$. Let $d_1, \dots, d_{2l}$ denote the gaps, i.e., $d_k = \alpha_k - \alpha_{k-1}$. Then the full knot Floer complex is (up to chain homotopy equivalence) a staircase $\mathbb{F}[U, U^{-1}]$ -module, generated by $\mathbf{x}_0, \dots, \mathbf{x}_{2l}$, with

$$\partial \mathbf{x}_{2k} = 0, \quad \partial \mathbf{x}_{2k+1} = \mathbf{x}_{2k} + \mathbf{x}_{2k+2}.$$

Moreover, the filtration over $\mathbb{Z} \oplus \mathbb{Z}$ is determined up to an overall shift by the following properties:

- the element $\mathbf{x}_{2k+1}$ has the same j -filtration as $\mathbf{x}_{2k}$, but the i -filtration differs by d_{2k+1} ;
- the element $\mathbf{x}_{2k+1}$ has the same i -filtration as $\mathbf{x}_{2k+2}$, but the j -filtration differs by d_{2k+2} .

Then, the unoriented knot Floer complex $\text{CFK}'(K)$ (up to chain homotopy equivalence) is generated over $\mathbb{F}[U]$ by $\mathbf{y}_0, \dots, \mathbf{y}_{2l}$, with differential

$$\partial \mathbf{y}_{2k} = 0, \quad \partial \mathbf{y}_{2k+1} = U^{d_{2k+1}} \cdot \mathbf{y}_{2k} + U^{d_{2k+2}} \cdot \mathbf{y}_{2k+2}.$$

In the language of [DM19], this is a *standard complex associated to a graded root*. Graded roots were introduced by Némethi [Ném05] to study HF^+ of plumbed 3-manifolds. We instead consider the ‘upside-down’ graded roots as in [DM19], which are used to describe HF^- . Note that the generators that we call $\mathbf{y}_0, \dots, \mathbf{y}_{2l+1}$ were called $v_1, \alpha_1, v_2, \alpha_2, \dots, \alpha_{n-1}, v_n$ in [DM19]. The numbers d_i determine the graded root up to an overall shift: the (relative) grading is given by

$$\chi(\mathbf{y}_{2k}) - \chi(\mathbf{y}_{2k+1}) = d_{2k+1}, \quad \chi(\mathbf{y}_{2k+2}) - \chi(\mathbf{y}_{2k+1}) = d_{2k+2}.$$

We now determine the numbers d_i in the case of the torus knot $T_{n,n+1}$. Recall that the Alexander polynomial of $T_{p,q}$ is

$$\Delta_{p,q}(t) = \frac{(t^{pq} - 1) \cdot (t - 1)}{(t^p - 1) \cdot (t^q - 1)}.$$

The coefficients of $\Delta_{p,q}(t)$ have been computed in the general case (see for example [Len78, (1.6) and (2.16)] or [LL96]). In our case $p = n$ and $q = n+1$, and $\Delta_{p,q}$ is simple enough to be computed explicitly. After simplifying

$$\Delta_{n,n+1} = \frac{(x^n)^n + (x^n)^{n-1} + \dots + x^n + 1}{x^n + x^{n-1} + \dots + x + 1},$$

one can carry out the long division explicitly and find that $\Delta_{n,n+1}$ is in the form of Equation (17), with

$$(d_1, d_2, d_3, d_4, \dots, d_{2l-1}, d_{2l}) = (1, n-1, 2, n-2, \dots, n-1, 1).$$

From this one can check that the graded root has a picture with n branches, of lengths $1, 2, \dots, 2, 1$. See Figure 26. The longest branch is in the middle, of length $\lceil n/2 \rceil$. This is also the top graded branch, so it generates the infinite tower. Thus, the order of HFK' is given by the next longest branch, which has length $\lfloor n/2 \rfloor$. Thus, $\text{Ord}_U(T_{n,n+1}) = \lfloor n/2 \rfloor$. $\square$

We now restrict the attention to the torus knots of the form $T_{2r-1,2r}$.

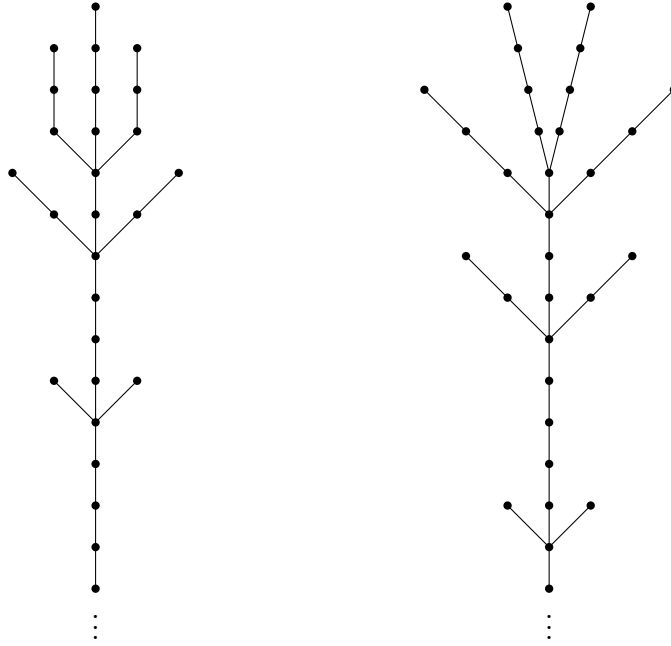


FIGURE 26. The figure shows the graded roots homotopy equivalent to $\text{CFK}'(T_{7,8})$ and $\text{CFK}'(T_{8,9})$ respectively. Each dot in the figure denotes a generator of the complex over $\mathbb{F}$, and the edges encode the U -action: for a dot $\mathbf{x}$, $U \cdot \mathbf{x}$ is the dot where you get by following the edge exiting from the bottom of the dot $\mathbf{x}$. The height of the dot denotes its (relative) Maslov grading, and the U action decreases the Maslov grading by 2. Note that when n is odd (e.g. $T_{7,8}$) there is one branch of length $\lceil n/2 \rceil$ and two branches of length $\lfloor n/2 \rfloor$, whereas when n is even (e.g. $T_{8,9}$) there are two branches of length $n/2$.

Remark 7.2. Batson [Bat14] first proved that $\gamma_4(T_{2r-1,2r}) = r - 1$. This can be proved with any of the bounds from [Bat14, OSSz17b, GM18] (for $T_{2r-1,2r}$ or $\overline{T_{2r-1,2r}}$), which all give the same sharp obstruction. We choose to use v from [OSSz17b] because it is an additive quantity, like the knot signature. In [OSSz17b, Theorem 1.2], Ozsváth–Stipsicz–Szabó proved that for a knot K in S^3

$$(18) \quad \gamma_4(K) \geq v(K) - \frac{\sigma(K)}{2}.$$

Batson [Bat14] computed that $\sigma(T_{2r-1,2r}) = -2r^2 + 2$, and by [OSSz17b, Theorem 1.3] one can compute $v(T_{2r-1,2r}) = -r^2 + r$. Thus,

$$(19) \quad v(T_{2r-1,2r}) - \frac{\sigma(T_{2r-1,2r})}{2} = r - 1.$$

We now restate and prove Corollary 1.7 from the introduction.

Corollary 1.7. *For all $\gamma \geq 1$ and $m \geq 1$, there exists a knot $K_{\gamma,m}$ with $d_u(K_{\gamma,m}, U_1) = \gamma_4(K_{\gamma,m}) = \gamma$, and such that $d_u^r(K_{\gamma,m}, U_1) \geq \gamma + m$.*

Thus, each unorientable surface $\Sigma \subset B^4$ with $\partial\Sigma = K_{\gamma,m}$ and $\gamma(\Sigma) = \gamma$ has at least m local minima (with respect to the radial function).

Proof. Let $K_{\gamma,m} = T_{2r-1,2r} \# \overline{T_{2s-1,2s}}$, for $r = \gamma + m$ and $s = m$. By Equation (18), the additivity of σ and v , and Equation (19), we have that

$$\begin{aligned} \gamma_4(K_{\gamma,m}) &\geq \left(v(T_{2r-1,2r}) - \frac{\sigma(T_{2r-1,2r})}{2} \right) - \left(v(T_{2s-1,2s}) - \frac{\sigma(T_{2s-1,2s})}{2} \right) \\ &= (r-1) - (s-1) = r-s = \gamma. \end{aligned}$$

On the other hand, Batson showed that there is a sequence $r-s$ unoriented band surgeries from $T_{2r-1,2r}$ to $T_{2s-1,2s}$ [Bat14, Figure 7]. Thus, we get a sequence of $r-s$ unoriented band surgeries from $K_{\gamma,m}$ to $T_{2s-1,2s} \# \overline{T_{2s-1,2s}}$, which is slice, so $\gamma_4(K_{\gamma,m}) = r-s$.

Now let Σ be a (possibly non-orientable) surface $\Sigma \subset B^4$ with $\partial\Sigma = K_{\gamma,m}$ and $b_1(\Sigma) = \gamma$. Theorem 1.1 gives a lower bound on the number of local minima. More precisely, if Σ has n local minima, by removing a small ball from B^4 we get a cobordism from $K_{\gamma,m}$ to the unknot U_1 with $M = n-1$ maxima (note that the cobordism is upside down, so the minima are turned into maxima, and one of them disappears when we remove the ball). Thus, Theorem 1.1 implies that

$$\text{Ord}_U(K_{\gamma,m}) \leq (n-1) + (r-s) = n-s+r-1.$$

We also know that

$$\text{Ord}_U(K_{\gamma,m}) = \max \{ \text{Ord}_U(T_{2r-1,2r}), \text{Ord}_U(T_{2s-1,2s}) \} = r-1$$

by Remark 4.4 (points (4) and (5)) and Lemma 7.1, so we get

$$n \geq s = m.$$

The statement about d_u and d_u^r follows from the computation of $\gamma_4(K_{\gamma,m})$ above and from Corollary 1.6. $\square$

Remark 7.3. We do not know if the bound on $d_u^r(K_{\gamma,m}, U_1)$ and on the number of minima of Σ in Corollary 1.7 is sharp on the knots used in the proof of the corollary. Recall that we set

$$K_{\gamma,m} := T_{2r-1,2r} \# \overline{T_{2s-1,2s}}$$

for $r = \gamma + m$ and $s = m$. Batson showed that with γ bands we can get to $K_{0,m} = T_{2s-1,2s} \# \overline{T_{2s-1,2s}}$, and Juhász–Miller–Zemke showed that $K_{0,m}$ bounds a ribbon disc with $2m-1$ local minima. Thus,

$$d_u^r(K_{\gamma,m}, U_1) \leq \gamma + 2m - 2.$$

We conjecture that this inequality is actually an equality.

REFERENCES

- [AE20] Akram Alishahi and Eaman Eftekhary. Knot Floer homology and the unknotting number. *Geometry & Topology*, 24(5):2435–2469, 2020.
- [Baa12] Sebastian Baader. Scissor equivalence for torus links. *Bulletin of the London Mathematical Society*, 44(5), 2012.
- [Bat14] Joshua Batson. Nonorientable slice genus can be arbitrarily large. *Mathematical Research Letters*, 21(3):423–436, 2014.
- [CGL⁺20] Carmen Caprau, Nicolle González, Christine Ruey Shan Lee, Adam M Lowrance, Radmila Sazdanović, and Melissa Zhang. On Khovanov homology and related invariants. *arXiv preprint arXiv:2002.05247*, 2020.
- [DLVW19] Aliakbar Daemi, Tye Lidman, David Shea Vela-Vick, and C.-M. Michael Wong. Ribbon homology cobordisms. *arXiv preprint arXiv:1904.09721*, 2019.
- [DM19] Irving Dai and Ciprian Manolescu. Involutive Heegaard Floer homology and plumbed three-manifolds. *Journal of the Institute of Mathematics of Jussieu*, 18(6):1115–1155, 2019.
- [Fan19] Haofei Fan. *Unoriented cobordism maps on link Floer homology*. University of California, Los Angeles, 2019.
- [GM18] Marco Golla and Marco Marengon. Correction terms and the nonorientable slice genus. *The Michigan Mathematical Journal*, 67(1):59–82, 2018.
- [HNT90] Jim Hoste, Yasutaka Nakanishi, and Kouki Taniyama. Unknotting operations involving trivial tangles. *Osaka Journal of Mathematics*, 27(3):555–566, 1990.
- [JMZ20] András Juhász, Maggie Miller, and Ian Zemke. Knot cobordisms, bridge index, and torsion in floer homology. *Journal of Topology*, 13(4):1701–1724, 2020.
- [Juh16] András Juhász. Cobordisms of sutured manifolds and the functoriality of link Floer homology. *Advances in Mathematics*, 299:940–1038, 2016.
- [JZ20] András Juhász and Ian Zemke. New heegaard floer slice genus and clasp number bounds. *arXiv preprint arXiv:2007.07106*, 2020.
- [Kan19] Sungkyung Kang. Link homology theories and ribbon concordances. *arXiv preprint arXiv:1909.06969*, 2019.
- [KM11a] Peter B. Kronheimer and Tomasz S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.
- [KM11b] Peter B. Kronheimer and Tomasz S. Mrowka. Knot homology groups from instantons. *J. Topol.*, 4(4):835–918, 2011.
- [KM19] Peter B. Kronheimer and Tomasz S. Mrowka. A deformation of instanton homology for webs. *Geom. Topol.*, 23(3):1491–1547, 2019.
- [KM21a] Peter B. Kronheimer and Tomasz S. Mrowka. Instantons and Bar-Natan homology. *Compositio Mathematica*, 157(3):484–528, 2021.
- [KM21b] Peter B. Kronheimer and Tomasz S. Mrowka. Instantons and some concordance invariants of knots. *Journal of the London Mathematical Society*, 2021.
- [Len78] Hendrik W. Lenstra. Vanishing sums of roots of unity. *Proc. Bicentennial Congress Wiskundig Genootschap*, 2:249–268, 1978.
- [Lic86] William B. R. Lickorish. Unknotting by adding a twisted band. *Bulletin of the London Mathematical Society*, 18(6):613–615, 1986.
- [LL96] Tsit-Yeun Lam and Ka Hin Leung. On the cyclotomic polynomial $\phi_{p,q}(x)$. *The American Mathematical Monthly*, 103(7):562–564, 1996.
- [LZ19] Adam Simon Levine and Ian Zemke. Khovanov homology and ribbon concordances. *Bull. Lond. Math. Soc.*, 51(6):1099–1103, 2019.
- [MZ21] Maggie Miller and Ian Zemke. Knot Floer homology and strongly homotopy-ribbon concordances. *Mathematical Research Letters*, 28(3):849–861, 2021.

- [Ném05] András Némethi. On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds. *Geometry & Topology*, 9(2):991–1042, 2005.
- [OS04] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Annals of Mathematics*, pages 1027–1158, 2004.
- [OS05] Peter S. Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44(6):1281–1300, 2005.
- [OS06] Peter Ozsváth and Zoltán Szabó. Holomorphic triangles and invariants for smooth four-manifolds. *Advances in Mathematics*, 202(2):326–400, 2006.
- [OSSz17a] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó. Concordance homomorphisms from knot Floer homology. *Advances in Mathematics*, 315:366–426, 2017.
- [OSSz17b] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó. Unoriented knot Floer homology and the unoriented four-ball genus. *International Mathematics Research Notices*, 2017(17):5137–5181, 2017.
- [Sar15] Sucharit Sarkar. Moving basepoints and the induced automorphisms of link Floer homology. *Algebraic & Geometric Topology*, 15(5):2479–2515, 2015.
- [Sar20] Sucharit Sarkar. Ribbon distance and Khovanov homology. *Algebraic & Geometric Topology*, 20(2):1041–1058, 2020.
- [Won] C.-M. Michael Wong. Personal communication.
- [Zem17] Ian Zemke. Quasistabilization and basepoint moving maps in link Floer homology. *Algebraic & geometric topology*, 17(6):3461–3518, 2017.
- [Zem19a] Ian Zemke. Knot Floer homology obstructs ribbon concordance. *Ann. of Math. (2)*, 190(3):931–947, 2019.
- [Zem19b] Ian Zemke. Link cobordisms and absolute gradings on link Floer homology. *Quantum Topology*, 10(2):207–323, 2019.
- [Zem19c] Ian Zemke. Link cobordisms and functoriality in link Floer homology. *Journal of Topology*, 12(1):94–220, 2019.

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