Symbolic forms analysis of expressions for probability in Dirac and wave function notations for spins-first students

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The ability to relate physical concepts and phenomena to multiple mathematical representations—and to move fluidly between these representations—is a critical outcome expected of physics instruction. In upper-division quantum mechanics, students must work with multiple symbolic notations, including some that they have not previously encountered. Thus, developing the ability to generate and translate expressions in these notations is of great importance, and the extent to which students can relate these expressions to physical quantities and phenomena is crucial to understand. To investigate student understanding of the expressions used in these notations and the ways they relate, clinical think-aloud interviews were conducted with students enrolled in an upper-division quantum mechanics course. Analysis of these interviews used the symbolic forms framework to determine the ways that participants interpret and reason about these expressions. Multiple symbolic forms—internalized connections between symbolic templates and their conceptual interpretations—were identified in both Dirac and wave function notations, suggesting that students develop an understanding of expressions for probability both in terms of their constituent pieces and as larger composite expressions.

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I. INTRODUCTION

Physics students enrolled in upper-division quantum mechanics (QM) courses are expected to learn and use Dirac formalism to represent quantum systems and compute relevant values such as probabilities and expectation values for specific measurements and physical observables, respectively. Students in these courses are also expected to learn and use other, more familiar mathematical notations, such as matrix-vector and wave function notations, that they may have used previously in linear algebra or modern physics contexts. Perhaps most crucially, they are expected to learn how these different notational styles interrelate, as calculating properties related to different physical observables will often require a student to work partially in one notation before needing to finish a calculation using another. If this translation is not required, it is nonetheless often preferred, as certain calculations are less computationally demanding with a given notation for a given context [1,2].

The symbolic forms framework was proposed by Sherin in an attempt to capture the ways in which students reason about formal mathematical expressions in physics [3]. In particular, it functions on the premise that students learn to interpret expressions via a vocabulary of smaller elements arranged via some syntactical rules. One goal of instruction is to assist students in developing and refining an understanding of these elements so that the students are eventually able to both make sense of new expressions they encounter and generate mathematical expressions to describe physical phenomena.

Viewing upper-division quantum mechanics courses through a symbolic forms lens thus provides a means of studying the mathematical and physical interpretations of quantum mechanical quantities that students develop in these courses. That students in this context are typically learning an entirely new mathematical representation (in the form of Dirac formalism) makes the application of this lens even more interesting, as it allows for an investigation into the mathematical and physical interpretations that students develop for expressions that are entirely new to them, and that will be of great relevance should they continue on to graduate study in physics. This study investigates the ways that students reason about expressions commonly used in upper-division quantum mechanics courses, particularly those used to represent probabilities in Dirac and wave function notations. To this end, we sought to determine the various symbolic forms students develop relating to probability concepts throughout a spins-first, upper-division quantum mechanics course. A spins-first QM course is one that traditionally begins by studying the Stern-Gerlach experiments and spin-$1/2$ systems [4,5]. Dirac notation is thus introduced at the very beginning of the course and is used extensively at the beginning of the course before wave function notation and the Schrödinger equation is introduced later on. This is in contrast with wave functions-first courses, wherein the Schrödinger equation and wave functions are used in the beginning, and Dirac formalism is often introduced much later in the course [6].

II. BACKGROUND

The symbolic forms framework [3] was developed by Sherin as an extension of the knowledge-in-pieces framework [7] in order to describe the building blocks of meaning encoded in symbolic mathematical expressions. Symbolic forms thus represent a marriage of form (e.g., the shapes and squiggles on paper and their orientation relative to each other) and meaning (i.e., the relationships ascribed to the given arrangements of shapes and squiggles). These are referred to within the framework as symbol templates and conceptual schemata, respectively, and their combinations are dubbed symbolic forms. These are intended to be viewed as the simplest possible relationships; a given equation in physics may be constructed of multiple symbolic forms. An example symbolic form is "opposition," represented by the symbol template "$\square - \square$" and the conceptual schema "two influences working against each other." An example application of the "opposition" symbolic form would be in the sum of vertical forces acting on a block resting on a surface: $N - mg$. This framework allows for theory building where new building blocks of symbolic reasoning are discovered and has been used as such both for studying mathematical sensemaking in electrostatics contexts with vector calculus [8,9] and linear algebra concepts in QM [10,11]. As these symbolic representations and their implied meaning are often not taught explicitly (unlike much of the conceptual knowledge built from conceptual resources, which is largely the curricular focus), a greater understanding of these forms and more explicit curricular goals related to symbolic understanding is largely the goal of research within this framework.

Originally developed in chemistry education, representational competence is a theoretical framework targeting students' understanding of and ability to work with multiple representations—including symbolic and graphical representations [12]. They found that students possess an impressive ability to generate, refine, and judge the quality of representations of physical phenomena. This particular aspect of representational competence was dubbed metarepresentational competence (MRC) [13], as it was not only their competence in generating and refining representations that proved valuable, but their ability to reason about the representations as well—critiquing and refining them as deemed necessary *by the students*. They found that even young children possess a "deep, rich, and generative (if intuitive and sometimes limited) understanding of representations" [14] and that this inherent MRC may be key to deepening student understanding of the power and limitations of representations both in physics and more generally [15]. This framework has been used by both mathematics and physics education researchers to study student understanding of linear algebra representations in quantum mechanics [16].

Related work within PER also describes two skills that are needed to benefit from using multiple representations in physics: representational fluency and flexibility [17]. Representational fluency refers to "the ability to construct or interpret certain representations like equations, diagrams, or graphs, but also to what extent someone can switch between different representations on demand," and representational flexibility involves "making appropriate representational choices when solving problems" [18]. The idea of representational fluency has been used to investigate the challenges students face when working with symbolic and graphical representations of vector fields [18].

Research in physics education has shown that the ways in which conceptual meaning is tied to mathematical representations are multifaceted, and authors often use different combinations of the various frameworks discussed above to analyze their student data and to inform their claims. Redish and Kuo used some aspects of cognitive semantics to compare how meaning is made in language and to "show how those same mechanisms can be used to understand how meaning is made with mathematical expressions in both science and math" [19]. They discussed how embodied cognition can be extended to mathematical reasoning, using the symbolic forms framework [3] as an example. They argued that the conceptual schema is obtained through embodied experience, citing the *parts-of-a-whole* symbolic form, where the concept of pieces of a larger whole is inherently connected to physical experiences with real-life objects that are made up of smaller objects [19].

III. SYMBOLIC FORMS' SUITABILITY

Before discussing the experimental design and digging into our analyses, it will be helpful to first discuss the normative expressions for these probability concepts in these notations and to discuss the suitability of the symbolic forms framework for this analysis.

Given that the focus of this work is on the interpretation of expressions for probability that commonly occur within upper-division quantum mechanics courses, there are two normative expressions within Dirac notation that are of explicit interest to this study. The first of these is inner products, which represent probability *amplitudes*: $\langle a_n|\psi\rangle$. These are also expressed as c_n, the coefficient of an eigenstate in that operator's basis expansion. The associated probabilities are found by taking the complex square of these inner products: $\mathcal{P}_{a_n} = |\langle a_a|\psi\rangle|^2 = |c_n|^2$. Thus, the principal (normative) expression in Dirac notation that is expected to show up within students' responses as representing probability is the complex square of an inner product, $|\langle a_n|\psi\rangle|^2$. It would also be expected for students to have developed conceptual knowledge about the constituent parts of this expression, such as the inner product, the ket, and/or the bra, separately. It should be noted that bras and kets are formally distinct mathematical objects: a bra is the covector of its associated ket and the set of bras forms a dual vector space to the vector space within which the set of ket vectors reside. These nuances of dual spaces are not discussed in depth in most upper-division quantum mechanics courses and thus students are expected to view these distinctions as relatively "fuzzy."

The principal (normative) wave function expressions expected of students as representative of probabilities in quantum mechanics would be twofold. For situations where a probability for a single value of a discrete observable is being represented in wave function notation, a complex square of an integral with two different functions, similar to $\mathcal{P}_n = |\int_{-\infty}^{\infty} \varphi_n^*(x)\Psi(x)dx|^2$ would be expected (in this example, as in our data collection, the wave functions are written as functions of position). For scenarios where the students discuss probabilities for a system to have a measured value of a continuous observable within a given region, an integral over the range of values of the complex square of a single function should be expected, either expressed as exactly that (e.g., $\int_a^b |\psi(x)|^2 dx$) or as the product of the wave function and its complex conjugate (e.g., $\int_a^b \psi^*(x)\psi(x)dx$). As was the case with the expressions in Dirac notation, it is reasonable to suspect that students would also learn to view the components of these expressions—the wave functions $\psi(x)$ and $\varphi_n(x)$ as well as their complex conjugates—as individual objects with their own interpretations as well.

Because these expressions for probability concepts in principle follow a simple formula as inner product expressions and complex squares of inner product expressions, the symbolic forms framework is particularly apt for this analysis. The symbolic forms framework describes consistent symbolic templates to which students learn to ascribe specific meaning, and the types of expressions discussed earlier in this section very much fit the description as fitting certain symbolic templates. In particular, Dirac brackets and inner product integrals, as well as their complex squares and their constituent pieces (the bras, kets, and functions), are seen and used extensively enough within these courses that it is reasonable to expect that students would learn to recognize them quickly and treat them as representative of physical and mathematical objects and/or processes. In short, we would expect students to develop symbolic forms for many of the expression types discussed above by connecting recognizable symbol templates with distinct conceptual schemata.

IV. RESEARCH DESIGN AND METHODOLOGY

To determine the ways that students interpret expressions in these two different notations, think-aloud interviews were conducted over several years with students enrolled in the upper-division quantum mechanics course at a public research university in the U.S. This course is offered every fall semester and follows the spins-first curricular structure. All students enrolled in these classes were offered financial compensation for agreeing to participate in the interviews, and all students who expressed interest were interviewed.

The first set of interviews was planned before the COVID-19 pandemic, which greatly affected both the distribution of and participation rates for the interviews conducted. Thus, our first set of interviews was virtual, while in subsequent semesters, they were able to be conducted in person. The differences inherent in virtual and in-person interview settings required the structure of these interviews to differ significantly, and participation rates were lower than expected for all three instances of interview data collection. In total, two individual virtual interviews, a single in-person interview with a pair of students, and two individual in-person interviews were conducted.

Pseudonyms have been selected for all six students, with Aaliyah and Bilbo as the two virtual interviewees, Castor and Delilah as the participants in the pair interview, and Enoch and Frodo as the participants in the individual interviews. The perceived gender of participants' pseudonyms does not necessarily correspond to the participants' own gender identities.

A. Virtual interviews

Because the virtual interview participants could not be expected to have equipment at hand such as tablets and styluses or other means of writing that would be convenient and visible in a virtual environment, the interview tasks were necessarily designed to be conducted solely with a computer mouse. Accordingly, two tasks were administered to the participants: a card-sorting task and an expression-construction task.

1. Card-sorting interview task

The card-sorting task made use of the card-sorting functionality in Desmos and effectively tasked participants with categorizing and recategorizing a number of expressions by whatever means they deemed appropriate. An example of the expressions and a potential sorting by a participant are shown in Fig. 1. Expressions were selected for this task for a number of reasons. First, they were all

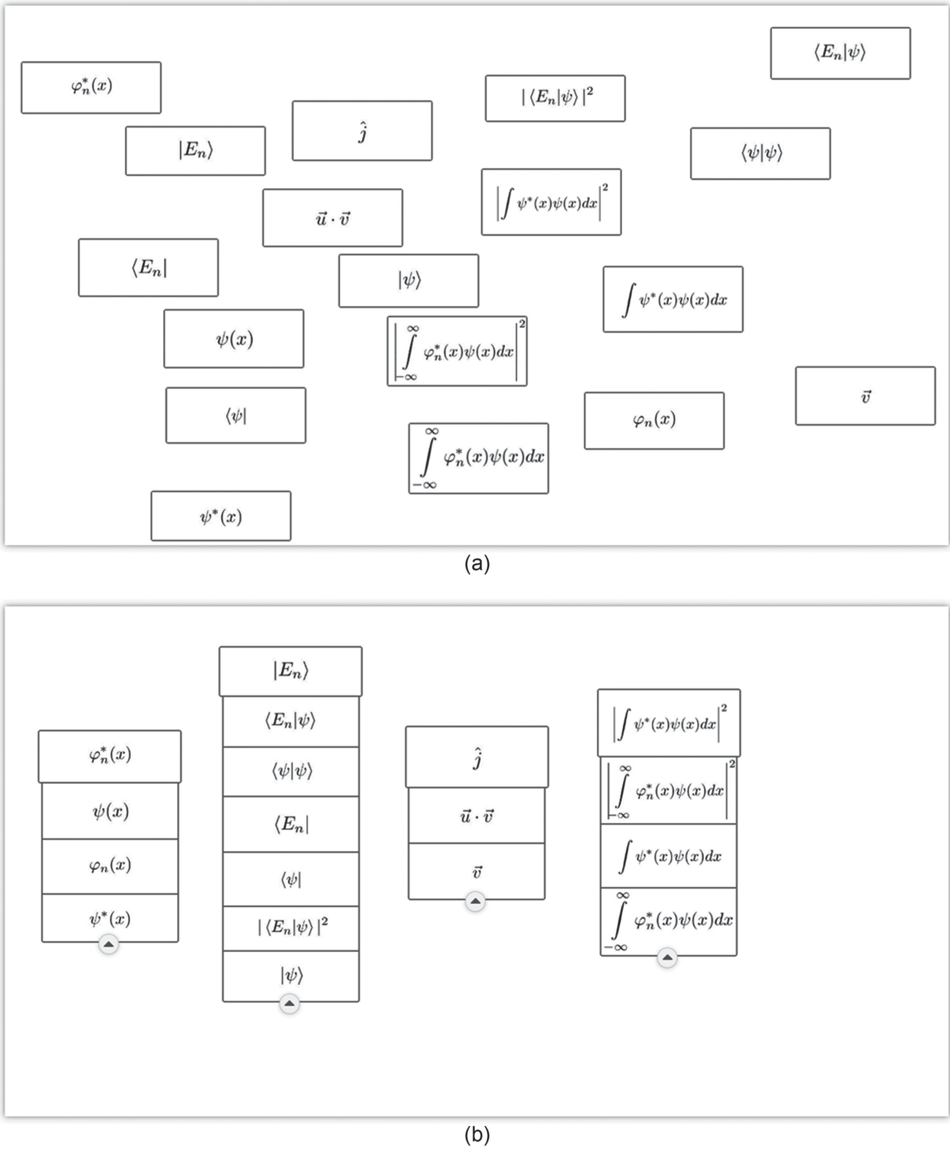

FIG. 1. (a) The original view of the expression cards in the card-sorting task as seen by the participants. (b) One way that a participant chose to sort the given expression cards.

expressions with which the students should have been familiar, either from the quantum mechanics course they were enrolled in or from earlier physics courses (e.g., $\vec{v}$, $\vec{u}\cdot\vec{v}$, and $\hat{j}$). Second, they largely covered every type of expression for probability and their constituent parts within Dirac and wave function notation, such that groups could be constructed containing analogous or equivalent expressions from both notations. It is worth noting that one expression, $|\int\psi^*(x)\psi(x)dx|^2$, is not a normatively correct expression in a QM context—it is either a complex square of a normalization integral (if integrated over all space) or a complex square of a probability over a region in space (if integrated over that region). It was included, despite its non-normative nature, due to its similarity in form to the probability of measuring an eigenfunction's associated eigenvalue (e.g., $|\int\varphi_n^*(x)\psi(x)dx|^2$, where $\varphi_n(x)$ is an eigenfunction of an operator), in order to determine whether participants would focus on this surface-level similarity or recognize its distinction from a probability expression. Participants were expected to determine categories that "made sense" for these expressions and to group them accordingly. They were asked to reason aloud about their thought processes and were encouraged to sort the expressions multiple times to capture as many different categorizations that they thought made sense.

The goals of this interview task were to allow for insights into (a) the ways that students think of expressions conceptually, by seeing the categories into which they would sort the expressions and (b) the ways that various expressions interrelate for students, both within and between the given representations. This task generally took the first 20 min of the interview; upon the student being satisfied with their categorizations, the second task was initiated.

2. Expression-construction interview task

The second task was an expression-construction task, where students were provided with an assortment of expressions and parts of expressions that were commonly used in their quantum mechanics course. They were then tasked with constructing as many expressions as possible that they deemed as representative of a quantum mechanical probability. This was conducted via a shared Google Slides file, with the student clicking and dragging the parts of expressions around to form the desired probability expressions. The expressions used, as well as an example of some expressions formed by participants, are shown in Fig. 2. Similar to the card-sorting task, participants were asked to think aloud and explain their reasoning for each expression they constructed. Due to the nature of this task, follow-up questions posed to participants depended greatly on the particular expressions they constructed and the expression elements they indicated, selected, and manipulated.

The goals of this interview task were to elicit student thinking on (a) what the individual components of expressions mean and (b) how these expression components interact to form larger expressions with their own meaning. Due to the structure of the questions, the meaning ascribed to these expressions would be expected to relate to probability concepts. By observing the language students used to describe both the components and larger expressions as they were in the process of constructing them, these goals could be addressed. The remaining 40 min of interview time were generally dedicated to this task.

B. In-person interviews

The questions and tasks posed to the participants of the in-person interviews differed from the virtual interviews but were identical between the 2 years these in-person interviews were conducted, with the exception of one additional question asked during the two individual in-person interviews. These interviews took place in a room with a whiteboard and markers, with both the participants and their writing captured by a video camera. The initial prompts given to the students are shown in Table I and generally required students to either generate or translate expressions in Dirac and/or wave function notations.

The primary goal of these interviews was to ascertain students' functional understanding of the expressions used to represent probability concepts in quantum mechanics, by which we mean that understanding these students exhibit in an authentic setting such as in the classroom, on homework problem sets, or on an exam [20]. Prompts 3 and 4 were thus designed to be similar to homework problems the participants would have seen throughout the course. Prompts 1 and 2 were chiefly focused on how students generate an expression with only verbal prompting, as well

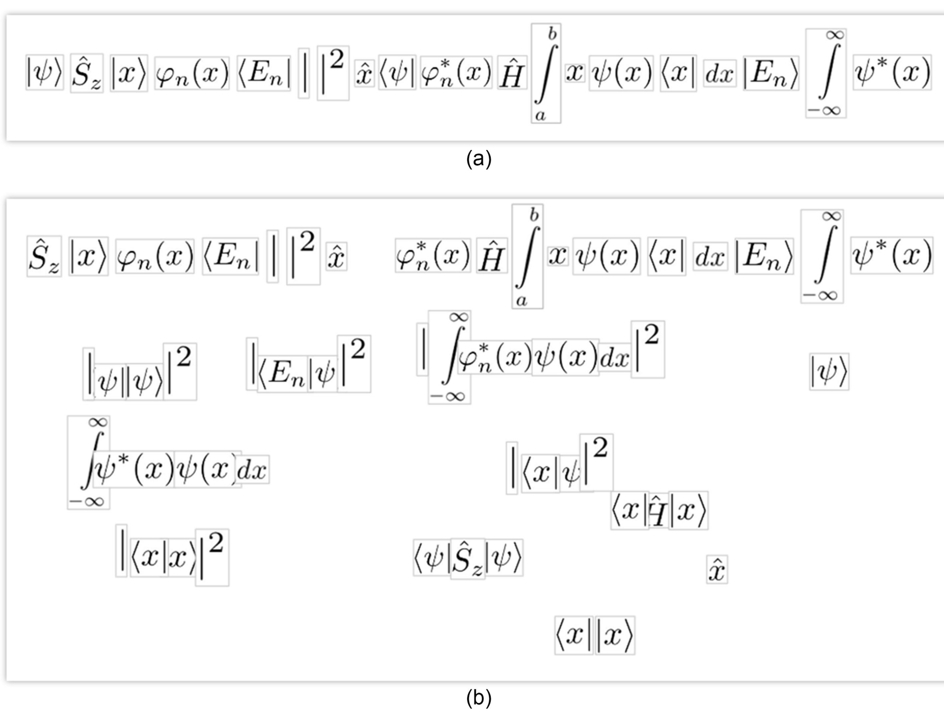

FIG. 2. (a) The list of expression components provided to the students for the expression-construction task. (b) An example of a participant's constructed expressions. Some expression components are free floating from the process of constructing other expressions.

TABLE I. The structured prompts given in the in-person interviews. Note: *Not asked during pair interview but added for individual interviews.

Prompt 1	How would you express the probability for an electron within a potential well to be measured as having the ground state energy of that well?
Prompt 2*	Let's say we have an electron in a potential well—perhaps an infinite square well. If we know that it has an even 33% chance of having any of the three lowest possible measurable energy values for that well, how could you express its current quantum state mathematically?
Prompt 3a	Let's say we have a particle in an infinite square well ("particle-in-a-box") potential. It is currently in the superposition state described by $\lvert\psi\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}\lvert E_1\rangle + \lvert E_2\rangle + 2\lvert E_3\rangle)$
	How would you go about finding the probability of measuring that particle to be in the left half of the square well?
Prompt 3b	How would you go about finding the probability of measuring that particle to be in the lowest energy state?
Prompt 4a	Let's say we have a particle in an infinite square well ("particle-in-a-box") potential. The particle is described by the following wave function: $\psi(x) = \frac{4}{\sqrt{5L}}\sin^3(\frac{4\pi x}{L})$
	How would you go about finding the probability of measuring that particle to be in the lowest energy state?
Prompt 4b	How would you go about finding the probability of measuring that particle to be on the left half of the square well?

as seeing what participants' first choice for notational style would be. These prompts and their respective follow-up questions were thus intended to gain an understanding of the ways students reason about these expressions, including what they mean as a whole, what their constituent parts mean, and how they are related to their analogous expressions in other notational styles. These responses were analyzed through a symbolic forms lens.

V. SYMBOLIC FORMS ANALYSIS RESULTS AND DISCUSSION

Analysis of students' responses in both virtual and in-person interviews was conducted to determine any conceptual interpretations that were consistently applied to expressions and components of expressions when working with or discussing them. We note a convention when presenting student excerpts. When a symbolic expression is spoken, and the interpretation is clear, our convention is to write out the symbolic expression rather than the verbal equivalent, e.g., E_n is written when a student says "E-sub-n", for both ease of reading mathematical expressions and efficiency.

The results of this analysis are broken down primarily by notational representation and secondarily by the types of expressions identified as separable by students. There is then a discussion about the instances of students identifying conceptual overlap between the two notations, as often occurred in situations where they translated between the different representations.

The symbolic forms we have identified are presented in Table II, along with their associated symbol templates. It is notable that multiple forms seen in this table share identical symbol templates; this is not unprecedented, as Sherin noted cases where templates can appear similar or identical, depending on the context. This includes the "base $\pm$ change" symbolic form having the symbol template $[\square \pm \Delta]$, which can appear identical to the templates used for "parts-of-a-whole" or "whole − part" symbolic forms depending on context ($[\square + \square + \square + \cdots]$ and $[\square - \square]$, respectively). The thing that distinguishes two symbolic forms with identical symbol templates is the interpretations that students apply to them—that is, their conceptual schemata. The conceptual schemata applied to each symbolic form are summarized somewhat by their titles and are made explicit within the following sections where they are discussed in detail. To assist the reader, the end of each section includes a smaller table containing the specific symbolic forms and symbol templates identified in that section.

A. Symbolic forms identified within Dirac notation expressions

Many symbolic forms were identified within Dirac notation expressions for probability concepts. We begin by discussing the symbolic forms identified for the smallest constituent pieces of these expressions: Dirac bras and kets.

1. Dirac bras and kets as quantum states

One consistent observation is that students appeared to consider bras and kets as representative of quantum states. Aaliyah was seen doing this in the virtual card-sorting task when discussing the elements within a category containing several Dirac bras and kets and a Dirac inner product:

> This [$\lvert E_n\rangle$] represents a ket energy eigenstate, and this [$\langle E_n\rvert$] represents a bra energy eigenstate. So these [$\lvert\psi\rangle$ and $\langle\psi\rvert$] are general ones, these [$\lvert E_n\rangle$] and $\langle E_n\rvert$] are specific energy eigenstates, and this thing [$\langle E_n\vert\psi\rangle$]- this inner product represents the amplitude of the energy eigenstate E_n if I […] map out all the […] energy eigenstates that make up the ψ.

Here Aaliyah called out the E_n bras and kets as being different from the ψ bras and kets, as she drew a distinction

TABLE II. The symbolic forms identified through our analysis, as well as their associated symbol templates. Symbolic forms are divided into two categories, based on whether their associated conceptual schemata imply more physical or mathematical interpretations of their associated symbol templates. Note that "probability amplitude" and "probability" refer to the probability (amplitude) of a physical measurement, hence their placement in the "more physical" column.

(More) mathematical symbolic forms	Symbol templates	(More) physical symbolic forms
Ket as vector	$\vert\ \rangle$	Ket as quantum state
Bra as vector	$\langle\ \vert$	Bra as quantum state
Bracket as projection	$\langle\vert\rangle$	Bracket as probability amplitude
	$\vert\langle\vert\rangle\vert^2$	Square bracket as probability
Function as vector	$f(\square)$	Function as superposition state
Conjugate function as vector	$f^*(\square)$	Conjugate function as superposition state
	$f_n(\square)$	Function as specific state
	$f_n^*(\square)$	Conjugate function as specific state
Inner product integral of two identical functions as projection	$\int f^*(\square)f(\square)d\square$	Inner product integral of two identical functions as probability amplitude
		Inner product integral of two identical functions as probability
Inner product integral of two different functions as projection	$\int f^*(\square)g(\square)d\square$	Inner product integral of two different functions as probability amplitude
		Inner product integral of two different functions as probability
	$\vert\int f^*(\square)f(\square)d\square\vert^2$	Complex square of inner product integral of two identical functions as probability
	$\vert\int f^*(\square)g(\square)d\square\vert^2$	Complex square of inner product integral of two different functions as probability
Coefficient as component	c_n	
	$\vert c_n\vert^2$	Squared coefficient as probability

between "specific" eigenstates and "general" states. This distinction shows up again later in the interview when Aaliyah discusses an expression for the probability that she constructed ($|\langle E_n|\psi\rangle|^2$): "The $E_n\varphi_n$ [gestures at $|\langle E_n|\psi\rangle|^2$] represents, like the probability of finding ψ, which is a general state, in a particular energy eigenstate E_n." This suggests that, for Aaliyah at least, there may be a distinction between the symbolic forms for a "general state" and for a "specific eigenstate." The exact nature of the meaning of a general state may be hinted at by Aaliyah's discussion of ψ being "[made] up" of the energy eigenstates. This may be indicative of an interpretation of ψ and general states as relating to superposition states made of a combination of specific eigenstates.

Bilbo, in his interview, discussed the ket $|x\rangle$ in the following terms: "you could make x an eigenstate, you could make it a spin state […] put anything in there […] I just need it to be a ket." This implies that Bilbo was very much treating the ket symbol as a marker for a quantum state. Interestingly, he appeared to treat x here as a mathematical variable, with it being a possibility to swap it with some other symbol to signify a given quantum state; so long as it is a ket, it represents some kind of quantum state, with the marker inside the ket determining the exact type of state.

Castor and Delilah quite frequently discussed E_n bras and kets as representing quantum states, such as early in their interview when explaining an expression they wrote to represent an electron in a potential well. They had written

$$|\psi(x)\rangle_E = c_1|E_1\rangle_E + c_2|E_2\rangle_E + \cdots,$$

and Delilah described it with: "if psi is written in terms of the energy states, in Dirac notation like this [points to kets in the expression] then $\mathcal{P}$ [probability] is, you know, as [Castor] said, is just [writes $\mathcal{P}_{E_0} = {c_0}^2$]." Much later in the interview, Delilah reflected on their earlier response to prompt 3a, where they were given the expression

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$$

as representing a particle in an infinite square well potential. She discussed her interpretation of the expression as a whole and why it was written that way:

> I think it's just, by design. The point of this is to give information. And so the coefficients are designed to give us the probability. And well, […] these [points to the different terms in $|\psi\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle)$] are all our possible values. […] And so we just represent it as […] [points to $\sqrt{3}$] the square root of the

> probability times the first state [points to $|E_1\rangle$] plus the square root of probability [points in front of $|E_2\rangle$] times the second state [points to $|E_2\rangle$] plus the square root of the probability [points to the 2] times the third state [points to $|E_3\rangle$] and then do that. We can do that infinitely. […] So yeah, […] we essentially just have square root of probability times each state.

As can be seen from this excerpt, Delilah directly referred to the ket symbols as states, with the E_n's representing "all our possible values" (presumably the possible measurable energy values for each state). She also notably described the coefficients in front of each eigenstate as "the square root of the probability" for measuring that state's energy, which is a normatively correct interpretation, albeit without explicitly referring to the possibility of complex coefficients and the necessity of a complex square to attain the probabilities.

Delilah also wrote an expression equating $\langle E_n|$ and $\hat{x}$,$\hat{y}$, and $\hat{z}$ with a large and exaggerated "≈" sign (see Fig. 3). When asked to explain her expression, she replied with "I'm just trying to say that's how I reconciled the complete orthonormal basis of the E_n energy states," calling out a conceptual similarity to them between a bra representing an "energy state" and Cartesian unit vectors. This vector interpretation of bras and kets will be discussed in more depth in Sec. V A 2.

Although Aaliyah, Bilbo, Castor, and Delilah all claimed bras and kets represented quantum states, there was some ambiguity as to what they meant by "state." Did they have a clear conceptual interpretation for that phrase or was that simply a learned name from the lecture? Evidence that they did in fact have a clear conceptual understanding of what a quantum state "is" or describes can be found in the interviews with Enoch and Frodo. When Enoch interpreted the expression given in prompt 3a ($|\psi\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle)$), he identified the kets as energy states: "What this equation is saying is that […] the particle could be in any of the three energy states [gestures at the three E_n kets]." Later, Enoch discussed his error in writing $\psi(x)$ on the LHS of the expression he initially wrote for prompt 2 ($\psi(x) = \frac{1}{\sqrt{3}}|E_1\rangle + \frac{1}{\sqrt{3}}|E_2\rangle + \frac{1}{\sqrt{3}}|E_3\rangle$): "[$\psi(x)$] is a function of x, and then I wrote it as a sum of vectors of the distinct energy levels." Frodo likewise discussed kets as representing explicit energy levels, discussing the terms in his expression for prompt 2 (written as $\psi = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$) with "so each of these [gestures to the three kets]… these are the three lowest energy levels," later describing the state of the particle as "a mixture of the three different energy states." Frodo did explain that he was being imprecise with calling it a "mixture" and was conscientious of the difference between superposition and mixed states. In these examples, Enoch and Frodo were clear about what a (quantum) state means to them—that it was related to a property associated with a particle, be it describing a particle's (possible) energy value or a superposition of possible energy values.

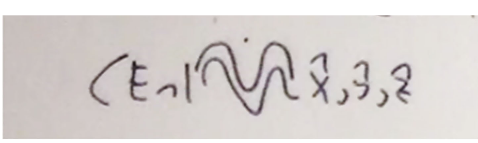

FIG. 3. Delilah's expression relating an energy eigenstate bra to Cartesian unit vectors.

TABLE III. Symbolic forms identified for Dirac bras and kets as describing quantum states and their associated symbol templates.

Symbolic form	Symbol template
Ket as quantum state	$\vert\ \rangle$
Bra as quantum state	$\langle\ \vert$

Based on the responses discussed in this section, two symbol templates are present here—$|\ \rangle$ and $\langle\ |$—both of which appear to share a conceptual schema of representing a particle or system in a specific quantum state (either a general state or an eigenstate of an observable), with their associated physical properties or eigenvalues. These symbol template-conceptual schema pairs then define the "*ket as quantum state*" and "*bra as quantum state*" symbolic forms (Table III). Looking at the responses from Aaliyah, it is also possible that there are distinctions to be made between a "general" state—typically with a ψ inside the ket or bra—and a "specific" eigenstate—with some other label within the bras and kets. This would suggest that perhaps there should be a differentiation between two types of kets and bras as symbol templates based on the symbol within them and that there are distinct conceptual schemata for the two "flavors" of kets and bras. While this is representative of a normative understanding as there is indeed a distinction between a general ket that can represent any state (typically expressed as $|\psi\rangle$) and kets such as $|+_z\rangle$ or $|E_2\rangle$ that describe specific eigenstates of physical observables, without more evidence of this from the other five interviewees there does not appear to be enough evidence to entirely support that claim. Thus, we will only lay out the two symbolic forms discussed above (ket and bra as quantum state).

2. Dirac bras and kets as vectors, and Dirac brackets as dot products

Another cluster of symbolic forms that appeared in student responses relates to treating bras and kets as vectors, and Dirac brackets as vector dot products. As students often discussed brackets in terms of bras and kets—particularly in the context of bras and kets as vectors—and brackets as dot products between bra and ket vectors, these three symbol templates and their associated vectorlike conceptual schemata will all be discussed together.

In the virtual card-sorting task, Aaliyah discussed both the $\langle E_n|\psi\rangle$ and $\langle\psi|\psi\rangle$ brackets in terms relating to a geometric interpretation of dot products, where she calculated a projection of one vector onto another. When discussing the $\langle E_n|\psi\rangle$ bracket, she said, "I will get the amplitude from this inner product […] like in vector form, like how one vector along the other, like projection of one thing along the other, so that is like the projection of ψ function along the E_n-basis." When discussing the $\langle\psi|\psi\rangle$ bracket, she said "why would I do ψ of ψ? Because physically like I'm thinking in terms of vectors, it represents ψ along ψ." Upon being asked what she meant by "____ along ____," she explicitly connected their interpretation of the Dirac bracket to that of a dot product's projection idea: "it's a traditional way to think about vectors, like because our dot product represents—like $\vec{a}\cdot\vec{b}$ represents, basically, the projection of $\vec{a}$ along $\vec{b}$ or projection of $\vec{b}$ along $\vec{a}$." Later, Aaliyah was thinking aloud about the meaning of $|\langle x|x\rangle|^2$ and asked herself "what does a vector dot-product-ed with itself represent? The magnitude? Squared…" She eventually settled on the convention that a dot product of a vector with itself would in fact produce the magnitude of that vector squared (e.g., $\vec{v}\cdot\vec{v}=|v|^2$), and thus the complex square on the outside of the bracket was redundant. In all of these cases, the connection between Dirac brackets and the projection ideas she associated with dot products is clear. She also appeared to relate the Dirac bracket as a combination of a bra and a ket and associated them as her $\vec{a}$ and $\vec{b}$ vectors (e.g., treating the bracket as an analog: $\langle\,|\,\rangle\leftrightarrow\vec{a}\cdot\vec{b}$), where the bracket is explicitly the projection of a ket vector onto a bra vector with a geometric spatial interpretation to both the vectors and the inner product.

During the virtual card-sorting task, Bilbo provided explicit categories for "vector" and "vector inner product," wherein he placed the bras/kets and Dirac brackets, respectively. He initially grouped the $\vec{v}$ and $|\psi\rangle$ together, saying "I'm going to be grouping these things as vectors," then added $\hat{j}$ and $|E_n\rangle$ sequentially, explicitly calling out that they go into that group because they are all vectors. Bilbo briefly grouped $\langle E_n|$ and $\langle\psi|$ together, saying "bras are effectively vectors as well, they're just conjugate vectors," before then combining the two groups together, stating "I could combine the bras with the kets, surely, because to me, those are just—they're all—they're vectors." He crystallized this point for himself by declaring "If you're gonna take an inner product between a bra and a ket, you can only have an inner product of two vectors." This connected in with his earlier grouping of Dirac brackets and dot product expressions, which began by grouping $\langle\psi|\psi\rangle$ together with $\vec{u}\cdot\vec{v}$, with him stating "yeah so, I mean, got some dot products here" and continued with adding $\langle E_n|\psi\rangle$, saying, "this is also going to be a dot product." Bilbo applied other rules of dot products and vectors to this category when he stated "$\langle E_n|\psi\rangle$, $\langle\psi|\psi\rangle$, $\vec{u}\cdot\vec{v}$] should be scalars because they're inner products"—upon being asked why that meant they were necessarily scalars, he expounded with "a dot product produces a scalar. Due to mathematics […] that's the way it goes—there are two types of vector multiplication. You're doing an inner product that is the […] scalar multiplication." He later clarified that "[he'd] been taught to think of [a dot/inner product] as like a projection, so then you know how much does one vector project onto the other." Bilbo also treated kets and bras as geometric vectors in contexts beyond inner products. For example, during the expression construction task, he discussed what operating an $\hat{S}_z$ operator on $|\psi\rangle$ or $\langle\psi|$ would do.

> Certainly changes the state […] well actually […] does it have to change the state? I mean […] if the state is purely in z, I believe it'll still change it, but I think by only lengthwise stretching […] rather than rotating.
> [Asked why it would only stretch and not rotate] Because it would be an eigenstate of that matrix […] we just know if you have a vector […] that is an eigenvector of the operator, then when you operate you just get you know 'λ your eigenvalue times your vector,' which is thus the same vector and not rotated at all, but its magnitude may have changed.

Here it is clear that Bilbo was treating these bras and kets geometrically, with "stretching" and "rotating" as viable operations that could occur to them. It is also of note that Bilbo discussed this action as occurring not only to a vector, but to a state as well. This is further evidence of the *ket as quantum state* symbolic form from Sec. VA 1, as well as evidence of Bilbo thinking fluidly about these symbolic forms.

Castor and Delilah likewise treated brackets as dot products, and even had a discussion on the distinctions between an inner product and a dot product, starting when the interviewer asked them about their calling $\langle E_1|\psi(x)\rangle$, an expression they wrote prior, a dot product.

Interviewer: Okay, so you called the–this thing [$\langle E_1|\psi\rangle(x)$] a dot product.
Delilah: Uh, inner product, yes.
Castor: They're basically the same.
Delilah: I think a dot product is a form—one of them is a form of the other.

Castor: Like, one is more broad than the other, but they're basically the same thing.
Delilah: Which one is which though […] I think dot product is a form of an inner product.

Here they can be seen determining (correctly) that an inner product is a generalization of a dot product, but they nonetheless referred to Dirac brackets as dot products. Later in their interview, they made the assertion that for $n \neq m$, $\langle E_n|E_m\rangle = 0$. When asked to explain why that was the case, Castor stated "because of like orthonormality, the eigenstates are perpendicular in a space," ascribing spatial geometric properties such as orthogonality and perpendicularity to a Dirac bracket. As was discussed in Sec. V A 1, Delilah wrote an exaggerated version of $\langle E_n| \approx \hat{x}, \hat{y}, \hat{z}$ on the board. Later in the interview, the two discussed it in the following way:

Delilah: But like that [$\langle E_n| \approx \hat{x},\hat{y},\hat{z}$], that helps me realize, why it's orthonormal- why it's orthogonal. And complete.
Castor: And like the dot product, or inner product is like the, if you do it with just with vectors, it's like, how much is a projection onto the other… thing.
Delilah: Yeah, how much of them are in the same direction.
Castor: So if they're 90 degrees from each other, then their components are just in their directions. They're not, like, a superposition or like a vector that has multiple pieces. [sketches an arrow lying between two perpendicular dotted arrows (presumably two axes)]

Here Castor and Delilah were discussing the similarities between the bra $\langle E_n|$ and the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. They explicitly connected $\langle E_n|$ to ideas of dot products, projection, and directionality. Later on, when working through some calculations for prompt 3b, they determined that $\langle E_2|E_2\rangle = 1$. When asked to explain that step, Castor responded with "because, like 100% of E_2 [points to the $\langle E_2|$ in $\langle E_2|E_2\rangle$] is in the direction of E_2 [points to the $|E_2\rangle$ in $\langle E_2|E_2\rangle$]."

Enoch and Frodo also described Dirac bras/kets and brackets in terms of a vector- and dot productlike interpretation, respectively. While explaining his answer to prompt 3b, Enoch described $|\langle E_1|\psi\rangle|^2$ as "giving the component of this [$|\psi\rangle$] in a particular direction or in this case of the particular energy, and then norm squaring it." Enoch was referring to the Dirac bracket within the complex square as a process of determining a component along a direction, a clearly geometric dot product interpretation. Enoch also discussed the ket $|\psi\rangle$ alone, referring to it as "a vector sum of each of the probabilities of the different energy states or observables that you can do," where he explicitly connected a ket for a superposition state to a vector sum. Very early in Enoch's interview, he wrote $\psi(x) = \frac{1}{\sqrt{3}}|E_1\rangle + \frac{1}{\sqrt{3}}|E_2\rangle + \frac{1}{\sqrt{3}}|E_3\rangle$ for prompt 2. Upon reflection near the end of the interview, he corrected himself: "Yeah, that's wrong […] [$\psi(x)$] is a function of x, and then I wrote it as a sum of vectors of the distinct energy levels." Here, Enoch again referred to a sum of kets as a vector sum. It is notable that there are two ways that Enoch could have corrected this expression: he could have changed $\psi(x)$ into $|\psi\rangle$, thus matching it to the "vector" ontology he identifies the E_n kets as sharing; alternatively, he could have changed the E_n kets into their corresponding eigenfunctions written as functions of position to match the $\psi(x)$'s "function of x" identity. It is also worth noting the somewhat "sloppy" language Enoch uses, referring earlier to a vector sum of probabilities (which are scalar quantities) and "vectors of […] energy levels." Based on his other responses, we believe this sloppiness is not indicative of a low level of understanding and view his first statement as referring to the probability amplitudes being the coefficients in front of the basis vectors and his reference to "energy levels" referring to energy eigenstates (which describe states at certain energy levels). Frodo, meanwhile, used perpendicularity to explain the Dirac bracket $\langle E_1|E_2\rangle$: "so, these two [gestures at $\langle E_1|E_2\rangle$], when they're not the same state, they're perpendicular to each other. Like these [gestures at $\langle E_1|$ and $|E_2\rangle$ in $\langle E_1|E_2\rangle$] are each orthogonal to each other." Similarly to Castor and Delilah, Frodo conceptually connected inner products and dot products together, saying "we were calling these inner products in class. […] But I mean, it's the same as a dot product."

As can be seen from these interview excerpts, these students all made very strong conceptual connections between bras and kets and vector ideas, and between projection/dot product ideas and Dirac brackets. In this case, there are three symbol templates: $|\ \rangle$, $\langle\ |$, and $\langle\ |\ \rangle$. The conceptual schemata that these students appear to have connected to the $|\ \rangle$ and $\langle\ |$ symbol templates include ideas related to vectors in a geometric sense, such as length/magnitude and directionality. They appear to have identical conceptual schemata tied to both: although Bilbo does potentially draw a distinction between $|\ \rangle$ as a "vector" and $\langle\ |$ as a "conjugate vector," he quickly sorted the two together into one overarching vector category so a strong conceptual distinction does not seem likely. We name the symbolic forms formed from these symbol template-conceptual schema pairs "*ket as vector*" and "*bra as vector*" (Table IV). The $\langle\ |\ \rangle$ symbol template, meanwhile,

TABLE IV. Symbolic forms identified for Dirac bras, kets, and brackets in the context of vectorlike conceptualizations.

Symbolic form	Symbol template
Ket as vector	$\|\ \rangle$
Bra as vector	$\langle\ \|$
Bracket as projection	$\langle\ \|\ \rangle$

appears to elicit a very strong conceptual response as representing a dot product, complete with a geometric projection interpretation. We call the combination of this symbol template with these ideas of two vectors being projected together via a dot product the "*bracket as projection*" symbolic form (Table IV).

3. Dirac brackets—and squared brackets—as probability concepts

Another common interpretation of Dirac brackets was that of probabilities or probability amplitudes (meaning a quantity that will represent a probability upon being multiplied with its complex conjugate). Recall from Sec. II A that the complex square of a Dirac bracket between a state vector (often represented as $|\psi\rangle$) and an eigenstate of an operator is generally representative of a probability (e.g., $|\langle a_n|\psi\rangle|^2$ is the probability of measuring the eigenvalue associated with $|a_n\rangle$, the nth eigenstate of the operator $\hat{A}$). The bracket alone generally represents the probability amplitude. The only deviation from this rule is when the bracket includes two identical vectors (e.g., $\langle\psi|\psi\rangle$ or $\langle a_n|a_n\rangle$). These expressions have two interpretations: as a step in the process of normalizing the vectors within the inner product or as a sum of probabilities of all states included within a superposition expansion of the vector within the bracket in a basis (which, not coincidentally, need to sum to one—hence the normalization condition).

In the card-sorting task, Aaliyah discussed the bracket $\langle E_n|\psi\rangle$ as representing a probability amplitude in this way, saying "this inner product represents the amplitude of the energy eigenstate E_n if I, you know, map out all the […] energy eigenstates that make up the ψ […] and this will give me- squaring it will give me the probability." Aaliyah also discussed a similar interpretation of the complex square of the bracket $\langle\psi|\psi\rangle$ in the expression-construction task, constructing $|\langle\psi|\psi\rangle|^2$ and saying "I'm trying to represent probability in bra-ket notation […] and this will be just one, if psi is normalized […] that should represent probability of one." In these excerpts, Aaliyah was connecting the square of a Dirac bracket with the means of calculating a probability. Similarly, Aaliyah later described $|\langle E_n|\psi\rangle|^2$ as "the probability of finding ψ, which is a general state, in a particular energy eigenstate E_n." Aaliyah also explicitly connected $|\langle\psi|\psi\rangle|^2$ and $|\langle E_n|\psi\rangle|^2$ later in the interview during the expression-construction task:

> for me, this [$|\langle\psi|\psi\rangle|^2$] also represents probability in the sense like if I go all–to like all space […] This [$|\langle\psi|\psi\rangle|^2$] is like a summation of all the possible variations of this [$|\langle E_n|\psi\rangle|^2$], okay. If I add all of these guys [$|\langle E_n|\psi\rangle|^2$] together I'll get–end up getting a one […] since every ψ is made up of all the possible energy eigenstates […] so if I find the individual probabilities of all of these E_n's [$|\langle E_n|\psi\rangle|^2$] and add them together what do I get? I get 1. Because that's how we represent probability. […] One means hundred percent of the time so—so that's what like this [$|\langle\psi|\psi\rangle|^2$] is like more broader [sic] representation of the right thing [$|\langle E_n|\psi\rangle|^2$], but again, essentially […] they would both represent probabilities to me.

Aaliyah was clearly treating both $|\langle\psi|\psi\rangle|^2$ and $|\langle E_n|\psi\rangle|^2$ as representations of probability; she also noted (incorrectly) that $|\langle\psi|\psi\rangle|^2$ is the sum of all possible $|\langle E_n|\psi\rangle|^2$ probabilities (in fact, $\langle\psi|\psi\rangle$ is the sum of all possible $|E_n|\psi|^2$'s). Earlier in her interview, however, Aaliyah also explained a Dirac bracket she constructed—$\langle x|\psi\rangle$—as representing a probability, despite the fact that it lacked the complex square: "this will also represent the probability of finding x—sorry, the probability of finding the general state ψ in the eigenstate x." It is unclear if this is a mere slip of the tongue for Aaliyah, as she was consistent in requiring the complex square in all other cases, or if the position representation for this expression had a different meaning for her.

Bilbo also treated $|\langle E_n|\psi\rangle|^2$ as a probability in the card-sorting task, saying "I'm looking at [$|\langle E_n|\psi\rangle|^2$] and I'm thinking like, probability. […] In this case, probability of being in that eigenstate. Of this wave function [indicates ψ] being in that [indicates E_n] eigenstate." Later in the card-sorting task, Bilbo grouped $|\langle E_n|\psi\rangle|^2$, $|\int\psi^*(x)\psi(x)dx|^2$, and $|\int\varphi_n^*(x)\psi(x)dx|^2$ together and said "okay, here we got our inner products—our probabilities, excuse me, because they're […] magnitude squared of inner products." Here he explicitly stated the necessity of taking the complex square of an inner product to effectively represent a probability. Later on, Bilbo expressed some confusion as to whether he should square $\langle\psi|\psi\rangle$, asking "do we want to square [$\langle\psi|\psi\rangle$] here? Do we need to square this again? I'm not so sure here, because it's already a magnitude. It is already a scalar. That is just an inner product, though I had been saying the inner product squared is a probability and that this [$\langle\psi|\psi\rangle$] is just a density." Here Bilbo exhibited a behavior that was observed quite often: mixing up terminology, particularly probability amplitude and probability density. He also appeared to be confused due to the repeated label within both the bra and ket of $\langle\psi|\psi\rangle$, asking if it was necessary to "square [it] again." Taking the statement about it being a magnitude, this is presumably a result of his conceptualizing the result of taking a dot product of a vector with itself as the magnitude of the vector squared, hence the question of whether he was squaring it "again." Regardless, it is clear that Bilbo thought both an inner product and a square of some kind was required (be it implicit in the repeated ψ or explicit in the complex square). Enoch showcased similar reasoning to Aaliyah and Bilbo, even describing his own pseudosymbol template when asked how to write a probability in prompt 1: "The probability, which I will call squiggly P, is going to be something like the norm squared of some business with some kets and… [writes $\mathcal{P} = |\langle\ |\ \rangle|^2$] something like that."

Frodo likewise discussed the complex squares of Dirac brackets representing probabilities, discussing his answer to prompt 2 ($\psi = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$) by declaring the criteria for its correctness as "when you do the square—when you do this… [writes $|\langle\psi|\psi\rangle|^2 =$] you need it to spit out the one third for the probability." While the expression he gave for the desired probability was incorrect (we believe he meant to write $|\langle 1|\psi\rangle|^2$), he nonetheless wrote a complex square of a Dirac bracket. Similarly to Enoch, he then went on to (correctly) write for their solution to prompt 3b that the probability could be written $\mathcal{P}_{E_1} = |\langle E_1|\psi\rangle|^2$. When asked why the square was necessary, he explained "because we're looking for the probability, and without it, we just have the probability amplitude."

These students have developed some symbol templates—$\langle\ |\ \rangle$ and $|\langle\ |\ \rangle|^2$—as well as some consistent conceptual schemata tied to them. The latter template has developed a strong conceptual association as a representation of a probability, while the former, commonly referred to as a probability amplitude but occasionally as a probability density, appears to have a strong association as a quantity that is squared to become a probability. In some cases, as with Aaliyah with $\langle x|\psi\rangle$, the Dirac bracket is declared a probability despite lacking the square. In another case, Bilbo gets confused about whether $|\langle\psi|\psi\rangle|^2$ is squaring a quantity one too many times, leading him to wonder if $\langle\psi|\psi\rangle$ is in fact a probability and not a probability amplitude. While these fuzzy distinctions do appear to exist, for the majority of cases, these students appeared to have robust conceptual schemata ascribed to these symbol templates. We name these symbolic forms the "square bracket as probability" symbolic form, and the "bracket as probability amplitude" symbolic form, as seen in Table V. While some students referred to the probability amplitude as effectively the square root of the probability, rather than using the technically rigorous term, we have elected to use the normatively correct term for this symbolic form, but note that most students appear to interpret it in terms of an object that needs to be squared.

B. Symbolic forms identified within wave function notation expressions

Just as the focus of our work on expressions for probability determined the particular expressions of interest within Dirac notation to primarily be Dirac brackets and their constituent parts, this same focus is true for the wave function notation. As in Sec. V A, we begin by looking at symbolic forms identified for the smallest constituent components of these expressions: the wave functions themselves.

TABLE V. Symbolic forms identified for Dirac brackets (and complex squares of Dirac brackets) in the context of probability concepts.

Symbolic form	Symbol template
Square bracket as probability	$\|\langle\ \|\ \rangle\|^2$
Bracket as probability amplitude	$\langle\ \|\ \rangle$

1. Functions as quantum state

A common interpretation of wave function expressions was that—similar to Dirac bras and kets—they represented quantum states. During the card-sorting task, Aaliyah sorted $\psi(x)$, $\varphi_n(x)$, $\psi^*(x)$, and $\varphi_n^*(x)$ all into the same category and said they "would represent a general eigenstate $\psi(x)$ or its conjugate [indicates $\psi^*(x)$], or like a specific energy eigenstate phi of- … Those represent states […] some of them represent general states [$\psi(x)$ and $\psi^*(x)$], some of them represent specific energy states [$\varphi_n(x)$ and $\varphi_n^*(x)$], but they represent states." Here Aaliyah drew a parallel to the distinction she drew between "general" states and "specific" states in Dirac notation, as discussed in Sec. V A 1. If taken in conjunction with her discussion in that section of the general states $\langle\psi|$ and $|\psi\rangle$ as being "made up of" the specific states $\langle E_n|$ and $|E_n\rangle$, it is reasonable that this interpretation of general vs specific applies to the ψ and φ_n wave functions here.

Bilbo also discussed wave functions as representing quantum states, particularly while discussing them in the context of inner product integrals. While discussing the expression $\int \varphi_n^*(x)\psi(x)dx$, he said "I'm thinking okay, you have this state [indicates $\psi(x)$] … and you want to ask the question of, you know, 'what about that state [indicates the $\psi(x)$] being in this [indicates the $\varphi_n^*(x)$] eigenstate.'" Later in his interview, he discussed the expression $|\int \psi^*(x)\psi(x)dx|^2$ on two different occasions. First, he seemed somewhat puzzled by what it could mean but suggested that "it's like the probability of a state being… in… its own state? I- yeah I'm not quite sure honestly." He later came back to it and declared that "[$|\int \psi^*(x)\psi(x)dx|^2$] I believe should be one […] because it's a state with itself- what's the probability of a state being in itself? What's the probability of a heads-up coin being heads? It's one." Here he drew a parallel between the wave function $\psi(x)$ and a heads-up coin. Both objects have a certain quality that describes them—whatever quantum state a system is in and that the heads side is facing up, respectively. The analogy is not a perfect one, as heads-up would be more analogous to an eigenstate of a coin flip and thus the quantum state described by $\psi(x)$ would need to be in a postmeasurement eigenstate to be perfectly analogous to a coin with a predetermined coin flip result. Regardless, Bilbo did appear to treat the functions inside of the integral as representing a quantum state, and moreover, the complex conjugate of the wave function ($\psi^*(x)$) as representing the same state as $\psi(x)$. He also treats the integral as determining the probability of one function "being in" the other.

While responding to prompt 1 in his interview, Frodo described his thinking after writing $\psi(x) = c_1\varphi_1(x)$ as "$\psi(x)$ is […] c_1 times $\varphi_1(x)$ […] but I think these [gestures at the $\varphi_1(x)$] are the […] energy eigenstates written in the position basis." Later, during his response to prompt 2, Frodo wrote an expression in Dirac notation for a superposition state, $\psi = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$. When asked whether he could translate it into another notation, he wrote a φ_1 directly above the $|1\rangle$ and discussed the difference between the ψ and φ_1, saying "so this [gestures at ψ] would be this- like the whole state, and then these [gestures at the φ_1] are the individual energy functions, I believe." Frodo can be seen here describing $\psi(x)$ and $\varphi_1(x)$ as representative of "whole" states and "energy" eigenstates, respectively. This is reminiscent of the distinction between "general" and "specific" states drawn by Aaliyah earlier.

As was seen with the Dirac bras and kets, these wave function expressions appear to be treated as representative of quantum states. There appear to be two symbol templates present: that of a function $f(\square)$ and of a function with a subscript $f_n(\square)$, as well as distinct symbol templates for their complex conjugates (i.e., $f^*(\square)$ and $f_n^*(\square)$). We use generic function notation (the f) for these symbol templates because we do not wish to make a claim about what specific letters the students cue these templates from (if there exist specific letters at all). The conceptual schemata that students have applied to these functions seem to consistently differ, with $f(\square)$ connoting a whole or general state. This suggests they regard these functions as representative of superposition states, which is distinct from their conception of $f_n(\square)$ as representing a specific state associated with a given quantity, i.e., an eigenstate; this manifested as a specific energy in the situations prompted by the interview setting. This is similar to the *quantum state* conceptual schemata observed for Dirac bras and kets in Sec. V A 1, though there appears to be a more obvious symbolic distinction between the general states and the specific states in the context of wave function expressions. We note that while the symbol templates differ between the functions and their respective complex conjugates ($f(\square)/f^*(\square)$ and $f_n(\square)/f_n^*(\square)$), there were no real differences between the conceptual schemata applied to the complex conjugate functions when compared to their respective nonconjugated function—i.e., these students did not appear to draw a physical or mathematical distinction between the wave functions or their dual functions. We name the symbolic forms associated with the $f(\square)$, $f^*(\square)$, $f_n(\square)$, and $f_n^*(\square)$ symbol templates with their associated conceptual schemata the "function as superposition state," "conjugate function as superposition state," "function as eigenstate," and "conjugate function as eigenstate," respectively. These symbolic forms and their symbol templates are summed up in Table VI.

TABLE VI. Symbolic forms identified for functions in the context of describing quantum states.

Symbolic form	Symbol template
Function as superposition state	$f(\square)$
Conjugate function as superposition state	$f^*(\square)$
Function as eigenstate	$f_n(\square)$
Conjugate function as eigenstate	$f_n^*(\square)$

2. Functions as vectors, and integrals as dot products

One potentially surprising conceptualization of wave function expressions is the combination of wave functions representing vectors and integrals representing dot products. This was most often exhibited within the context of inner product integrals, such as Aaliyah describing $|\int \psi^*(x)\psi(x)dx|^2$ with "it basically represents an inner product of ψ with itself. So in the Cartesian world, it will look like the projection of ψ along itself." Aliyah drew an explicit analogy between the integral and projection in a Cartesian space. During the card-sorting task, after discussing how he interpreted the expression, Bilbo added $\int \varphi_n^*(x)\psi(x)dx$ to the group containing $\psi|\psi$ and $E_n|\psi$: "So this integral without being squared [referring to $\int \varphi_n^*(x)\psi(x)dx$], also a dot product, I believe, effectively- or an inner product of those functions over space." He similarly remarked upon subsequently adding $\int \psi^*(x)\psi(x)dx$ to the same group, saying that it is also "just a dot product." Later, he discussed his interpretation of $\int \varphi_n^*(x)\psi(x)dx$:

> in this case here […] I'm thinking okay, you have this state [indicates the $\psi(x)$] […] and you want to ask the question of, you know, 'what about that state [indicates the $\psi(x)$] being in this [indicates the $\varphi_n(x)$] eigenstate.' […] to me I'm looking at this thing I'm thinking, 'what is the projection of this eigenstate onto this wave function,' or maybe vice versa, but I don't think it should matter—dot products are […] commutative.

Bilbo seemed to be categorizing these integrals into a group of what he considered to be dot products, which bear the conceptualization of a geometric projection. He also sorted $\psi(x)$ and $\varphi_n(x)$ together with $|E_n$, $\vec{v}$, $|\psi$, and $\hat{j}$, saying "these [$\psi(x)$, $\varphi_n(x)$] could also, you know, be kets in functional form, so we could think of all of these [$|E_n$, $\vec{v}$, $|\psi$, $\hat{j}$, $\psi(x)$, and $\varphi_n(x)$] as just vectors, kets I guess," explicitly referring to the wave functions as representing vectors. Bilbo also discussed wave functions in terms of representing vectors to him in the context of operating an operator on a function. During the expression construction task, Bilbo constructed $|\int \varphi_n^*(x)x\psi(x)dx|^2$, which is something of a conflation of an expression for a probability of measuring an energy value and that of an expectation value for position. He discussed the effect of placing the x in this expression in the following way: "if I am saying [the x in $|\int \varphi_n^*(x)x\psi(x)dx|^2$] is an operator, an operator on a vector is just […] a vector, so it's the same thing. It's just another probability […] it's a different state after being operated on." In this context, Bilbo was expressing that the expression still represented a probability because he treated the x as an operator, which acts on one of the wave functions like it would on a vector, which will just generate another vector (albeit one that has been scaled and/or rotated from the original). This is further evidence that these wave functions represented vector objects to these students.

As has been shown, some students appear to have developed an understanding by relating wave functions to vector properties and inner product integrals to dot products. The symbol templates at play here include $f(\square)$, $f^*(\square)$, $\int f^*(\square)f(\square)d\square$, and $\int f^*(\square)g(\square)d\square$. While there does not appear to be any conceptual distinction between wave functions with and without subscripts when students are interpreting these functions in this way, students did discuss integrals in this manner when the two functions within the integrand were both the same (e.g., $\int \psi^*(x)\psi(x)dx$) and different (e.g., $\int \varphi_n^*(x)\psi(x)dx$). These students did not appear to focus on the bounds of integration (i.e., they did not distinguish between an indefinite integral, a definite integral over all space, and a definite integral over a finite region), and so these are excised from the symbol template that they have developed. The conceptual schema tied to both of the first two symbol templates ($f(\square)$ and $f^*(\square)$) appears to be that of a vector with an associated directionality, which is the same as was connected to the bras and kets in Sec. V A 2. Thus, we call the symbolic forms formed from these symbol template-conceptual schema pairs "*function as vector*" and "*conjugate function as vector*." The latter symbol templates ($\int f^*(\square)f(\square)d\square$ and $\int f^*(\square)g(\square)d\square$) appear to also both share a conceptual schema that appeared in Sec. V A 2 when discussing the Dirac brackets: that of a dot product's projection along a direction. In this context, the "direction" is that of one of the two functions' vector interpretation. We name these symbolic forms "*inner product integral of two identical/different functions as projection*." These symbolic forms and their symbol templates are again shown in Table VII.

TABLE VII. Symbolic forms identified for describing functions and integrals in vectorlike terms.

Symbolic form	Symbol template
Function as vector	$f(\square)$
Conjugate function as vector	$f^*(\square)$
Inner product integral of two identical functions as projection	$\int f^*(\square)f(\square)d\square$
Inner product integral of two different functions as projection	$\int f^*(\square)g(\square)d\square$

3. Integrals as probabilities

Finally, students also viewed complex squares of inner product integral expressions as representative of probabilities. Because the expressions for probability, probability density, and probability amplitude for continuous variables are quite similar, a brief refresher of the distinctions is warranted. As in Dirac notation, a *probability amplitude* for measuring a given observable's eigenvalue is the inner product of the initial state and the corresponding eigenstate (e.g., $\int_{-\infty}^{\infty} \varphi_n^*(x)\psi(x)dx$). Thus, the *probability* of that eigenvalue will be the complex square of this expression: $|\int_{-\infty}^{\infty} \varphi_n^*(x)\psi(x)dx|^2$. Unlike in Dirac notation, however, there is another potential expression for a probability: that of measuring a continuous observable (e.g., position). Due to it being a probability for a continuous variable, we instead define a *probability density* for that variable as the complex square of the wave function as a function of the continuous variable (e.g., $|\psi(x)|^2$), thus the probability for a region is found by integrating the probability density over the region of interest: $\int_a^b |\psi(x)|^2 dx$ (alternatively, $\int_a^b \psi^*(x)\psi(x)dx$). While the first few expressions have an obvious probability analog in Dirac notation, the latter are more related to a subset of the normalization condition $\langle\psi|\psi\rangle$. These different integrals and their interpretations can appear quite similar both visually and conceptually, and it is thus reasonable that they could prove challenging to students in the development of firm symbolic forms.

During the expression-construction task, Aaliyah compared two pairs of expressions, with each pair containing a Dirac expression and a wave function expression. She discussed the distinction between one pair ($|\langle\psi|\psi\rangle|^2$ and $\int \psi^*(x)\psi(x)dx$) and the other ($|\langle E_n|\psi\rangle|^2$ and $\int \varphi_n^*(x)\psi(x)dx$) as follows:

> So this [$|\langle\psi|\psi\rangle|^2$] and this [$\int \psi^*(x)\psi(x)dx$] will represent the same thing, [$|\langle E_n|\psi\rangle|^2$ and $\int \varphi_n^*(x)\psi(x)dx$] will represent the same thing, which is probability- […] [$|\langle\psi|\psi\rangle|^2$ and $\int \psi^*(x)\psi(x)dx$] will be just [makes finger quotations] "one" if ψ is normalized […] and then [$|\langle E_n|\psi\rangle|^2$ and $\int \varphi_n^*(x)\psi(x)dx$] will be like some number, less than one, unless it's an eigenstate itself.

Here, Aaliyah was drawing parallels between the Dirac and wave function notation expressions she viewed as representative of probabilities. Interestingly, the squares that are present in the Dirac expressions are missing from their associated integrals. It is not entirely clear why she paired these together despite their visual (and mathematical/physical) dissimilarity, though it is perhaps evidence of some confusion as to whether and/or how a complex square

TABLE VIII. Symbolic forms identified for inner product integrals in the context of describing probability concepts.

Symbolic form	Symbol template
Inner product integral of two identical functions as probability amplitude	$\int f^*(\square)f(\square)d\square$
Inner product integral of two identical functions as probability	$\int f^*(\square)f(\square)d\square$
Inner product integral of two different functions as probability amplitude	$\int f^*(\square)g(\square)d\square$
Inner product integral of two different functions as probability	$\int f^*(\square)g(\square)d\square$
Complex square of inner product integral of two identical functions as probability	$\lvert\int f^*(\square)f(\square)d\square\rvert^2$
Complex square of inner product integral of two different functions as probability	$\lvert\int f^*(\square)g(\square)d\square\rvert^2$

translates between Dirac and wave function notation. She also drew a distinction again between the expressions that contained an eigenstate (and thus have a probability less than 1) and expressions that contained two identical states (which have a probability of 1).

Bilbo sorted $|\langle E_n|\psi\rangle|^2$, $|\int \psi^*(x)\psi(x)dx|^2$, and $|\int \varphi_n^*(x)\psi(x)dx|^2$ all together and explained the grouping as "okay, here we got our inner products—our probabilities, excuse me, because they're […] magnitude squared of inner products." The square on the outside of the integrals appeared to be crucial to him for delineating which expressions represented probabilities.

When Enoch was asked the question in prompt 3a about finding the probability for measuring a particle (provided as in a superposition state in Dirac notation) within the left half of an infinite square well, his immediate response was "I remember it being an integral of sorts […] it was something like the probability equals integral of ψ of x complex conjugate $\psi x dx$ [writes $\mathcal{P} = \int \psi^*(x)\psi(x)dx$]." Later in his interview, while working with the expression given in prompt 3 ($|\psi\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle)$), Enoch explained the difference between the probabilities calculated by $|\langle E_1|\psi\rangle|^2$ and $\int \psi^*(x)\psi(x)dx$:

> The particle could be in any of the three energy states [gestures at $\frac{1}{2\sqrt{2}}(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle)$] by a probability given by whatever you calculate this to be [gestures at $|\langle E_1|\psi\rangle|^2$]. Whereas this [gestures at $\int \psi^*(x)\psi(x)dx$] is saying […] what is the probability of the particle to be at a certain position, regardless of which energy state you're looking at.

Enoch later successfully translated the expression given in prompt 3 into generic wave function notation:

$$\psi(x) = \frac{1}{2\sqrt{2}}\left[\sqrt{3}\varphi_1(x) + \varphi_2(x) + 2\varphi_3(x)\right].$$

He was then asked how he would find the probability for measuring the particle to be in the left half of the well and said that he would "take this whole conflagration [draws brackets around the expression he translated, and squares it all] […] just do that from 0 to L/2." It seems that Enoch had developed a fairly consistent symbol template of an integral of a product of two functions that he drew from when expressing probabilities in wave function notation.

Again, a few symbol templates appear to capture the work of these students when generating or selecting function-based expressions for probability. $\int f^*(\square)f(\square)d\square$ shows up here as it did in Sec. V B 2, though this time it is called upon to represent either a probability for a measurement or a quantity that must be complex-squared to get a probability (i.e., a probability amplitude). Unlike in Sec. V B 2, however, it appears that some of the students treated integrals differently depending on whether they contained a product of the same function (albeit with one being the function's complex conjugate) or a product of two different functions. Because this appears to be a meaningful distinction within this context, we will propose another symbol template here, $\int f^*(\square)g(\square)d\square$, with the same (or at least fundamentally similar) interpretations as a (square root of) probability as was applied to the $\int f^*(\square)f(\square)d\square$ symbol template. Relatedly, $|\int f^*(\square)f(\square)d\square|^2$ appears to represent probabilities of a measurement as well and thus shares a conceptual schema. We name these symbolic forms the "inner product integral of two identical/different functions as probability amplitude," "inner product integral of two identical/different functions as probability," and "complex square of inner product integral of two identical/different functions as probability" symbolic forms, as shown in Table VIII. Similar to what was noted in Sec. V A 3, we are naming some of these forms "probability amplitudes" to better reflect convention, though students often appeared to reason about them as a thing that must be squared. Given the discussion of normative interpretations of inner product integrals at the beginning of this section, it is worth pointing out which of these symbolic forms are generally normatively correct. These would be the "inner product integral of two identical functions as probability," "inner product integral of two different functions as probability amplitude," and "complex square of inner product integral of two different functions as probability" symbolic forms.

C. Castor and Delilah's focus on coefficients as an intermediate step between inner products and probability expressions

While Aaliyah, Bilbo, Enoch, and Frodo all shared very similar interpretations of the ways that inner products expressed in both notations related to probability concepts, Castor and Delilah appeared to connect the expressions and concepts differently. They instead seemed to prefer to make an additional symbolic step between inner products and the probabilities they represent: that of first converting the inner product to a coefficient associated with a term in the initial state's representation as a superposition of orthonormal eigenstates. For example, after they had finished with prompt 3a, where they were given $|\psi\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle)$ and asked to find the probability for the left half of the well, they were asked to calculate the probability for measuring the lowest energy value for the given particle. The following exchange then occurred:

Castor: I mean, it's written in the energy basis.
Delilah: Yeah. I mean, I'd go back to the energy basis
Castor: And just square the coefficient.
[…]
[Castor writes $|\frac{\sqrt{3}}{2\sqrt{2}}|^2$]
Interviewer: Okay. Why is that the probability of the first energy state?
Castor: Because it's the coefficient for the first energy state.

In this exchange, Castor has squared the coefficient in front of the lowest energy state's ket and claimed that as the probability of a measurement of E_1. Earlier in their interview, they defined their own initial state for prompt 1 as $|\psi(x)\rangle_E = c_1|E_1\rangle_E + c_2|E_2\rangle_E + \cdots$, and, when explaining how they would find the probability for the lowest energy, Delilah said, "if ψ is written in terms of the energy states, in Dirac notation like this [gestures at their expression for $|\psi(x)\rangle_E$], then $\mathcal{P}$ is […] just [writes $\mathcal{P}_{E_1} = c_1{}^2$]," defining the probability as an eigenstate's associated component (squared) when expressed as a superposition state. Later, they expanded that expression for the probability and added complex square bars to make it $\mathcal{P}_{E_1} = |c_1|^2 = |\langle E_1|\psi\rangle|^2$; Castor justified their addition of the complex square by saying, "because this [$\langle E_1|\psi(x)\rangle$] gives you the coefficient. […] Because when you do, like, the inner product of that they end up like, the dot product of something times itself [referring to the $\langle E_1|$ multiplication being distributed through the superposition state, specifically the $|E_1\rangle$ component to form $\langle E_1|E_1\rangle$] it all works out to give you the coefficient." Again, they were very focused on the coefficient as the expression that represents the probability amplitude and treated the inner product as a means of obtaining the coefficient rather than the probability itself. However, they then got confused as to whether they needed to square the integrand of a wave function version of an inner product or square the result of the integral (i.e., whether the complex square happens inside or outside the integral). In the end, they made sense of this problem through the following conversation:

Castor: Isn't it inside the integral?
Delilah: Well, I'm not sure. I have to think…
Castor: I'm pretty sure- I don't think we've ever seen-
Delilah: -No, but I think, I think this equation is [writes $c_1 = \int_1^2$[squiggle]]
Castor: Oh, yeah.
[…]
Castor: Because that's where you get your coefficient and then you square it. I was like, thinking I'm like, we've never really seen that on the outside.
Delilah: Yeah, no, we've never done that. But I think that's just because we write it as this [gestures to $c_1 = \int_1^2$[squiggle]]
Castor: Because typically we write it as the coefficient and square it to be able to get the probability

Here Castor and Delilah are calling back to a recognizable expression in the $c_1 = \int_1^2$[squiggle]. Their statements, such as "I don't think we've ever seen ____," "we write it as ____," and Castor's "Oh, yeah" upon seeing Delilah's expression template, fit very well with the symbolic forms framework.

While Castor and Delilah do occasionally make direct connections between inner products in both notations and probabilities (displaying use of the symbolic forms discussed in Secs. V A 3 and V B 3), they appear to have two symbol templates that show up with much more importance in their thinking than did in the other students' interviews: c_n and $|c_n|^2$. Given the role these symbol templates appear to play in their reasoning and interpretation of these expressions, the symbolic forms framework suggests there are different conceptual schemata tied to each of them. The first is that of a coefficient in a linear combination of terms, as seen in their superposition state $|\psi(x)\rangle_E = c_1|E_1\rangle_E + c_2|E_2\rangle_E + \cdots$, with the interpretation of the relative importance or size of its associated component in the sum. The inner products' symbolic forms relating to projection along axes thus interface with this conceptual schema with the interpretation of the inner product "picking out" the coefficient. This manifests in their equality, $c_1 = \langle E_1|\psi\rangle$. This conceptual schema pairs with the c_n symbol template in what we call the "coefficient as component" symbolic form. The conceptual schema connected to the $|c_n|^2$ symbol template is that of the squared coefficient being a representation of the probability for the coefficient's associated component. This pair forms the symbolic form "squared coefficient as probability." These two symbolic forms are ultimately manifested in their

TABLE IX. Symbolic forms identified for coefficients and complex squares of coefficients as describing probability concepts.

Symbolic form	Symbol template
Coefficient as component	c_n
Squared coefficient as probability	$\lvert c_n \rvert^2$

equality $\mathcal{P}_{E_1} = |c_1|^2 = |\langle E_1|\psi\rangle|^2$. Given this interpretation, this equality can be read out as "the probability of measuring E_1 is the complex square of the coefficient for the $|E_1\rangle$ term in the expansion of $|\psi\rangle$, which can be obtained by taking the inner product of $E_1|\psi$," and understood as relating $|\langle E_1|\psi\rangle|^2$ to by means of picking out the E_1 component of $|\psi\rangle$. These two symbolic forms and their symbol templates are shown in Table IX.

VI. CONCLUSIONS AND FUTURE WORK

One conclusion that can be drawn from these excerpts can be seen by comparing the lengths of Secs. V A and V B: these students used and referred to wave function expressions less often than they did those expressed in Dirac notation. This is perhaps explainable by the curricular focus of the course, as in a spins-first course, students may be expected to be more comfortable with Dirac notation than with wave function notation. Viewed through a symbolic forms lens, it could be that the increased time spent working with and thinking about expressions in Dirac notation increased the strength of the connections between symbol templates and conceptual schemata for expressions in that notation. This is potentially significant as, due to the timing of these interviews during or following the course, students had likely used more wave functions than Dirac expressions in the weeks before the interview. This relative recency nonetheless does not appear to override the comfort working with Dirac expressions that has been engendered by working within this notation since day one of the course.

It is also apparent from Sec. V and Table II that students in this course developed numerous symbolic forms to aid them in interpreting, generating, and translating expressions in one or both notations explored in this study. Of note is that the vast majority of the symbolic forms explored in this work are normative; most non-normative symbolic forms are explainable by a lack of a complex square. This is potentially a problem when it leads to students failing to take complex squares when calculating probabilities in quantum mechanical contexts. This confusion did appear to manifest more often within wave function contexts (with students not being sure where to put the square or if a square is necessary at all), though often students would mix up the terms "probability," "probability amplitude," and "probability density" in both contexts—they would simply write the correct expressions more often in Dirac notation, regardless of what they called them. Confusion regarding the different expressions for probability in QM has been noted in prior work [21], though not this particular difficulty in correctly applying the complex square. Other work has also identified challenges in correctly translating between Dirac and wave function inner product expressions [22]. While one would like to believe that these data suggest that students mostly develop normative symbolic forms in these courses, it is possible that the tasks given within these interviews were not conducive to capturing any other non-normative symbolic forms. It is also possible that there is a selection bias to the participants, as the interviews were entirely opt-in, and thus perhaps only those students with well-developed normative symbolic forms (and thus likely well-performing students in the class) were willing to volunteer to answer questions about quantum mechanics. Regardless, these symbolic forms provide a useful means for understanding the interpretations students hold for common expressions for probabilities in quantum mechanics, both of the expressions as a whole and their constituent parts.

Indeed, the symmetry between the symbol templates for the Dirac and wave function expressions—due largely to very similar conceptual schemata cropping up within both notations—is perhaps evidence of the symbolic forms framework's usefulness in explaining how students translate between expressions that they deem "equivalent." Within the symbolic forms framework, shared conceptual schemata—such as that between the "bracket as dot product" and "integral as dot product"—may be the means by which students coordinate the expressions they choose as a direct translation from one to the other. Misapplying these shared schemata is a possible explanation for the difficulties in translating between inner products reported by Wan *et al.* [22].

This work is limited by a number of factors, which provide obvious avenues for future research. First, this work is entirely concerned with expressions for probability. There are many other types of expressions that are commonly used in upper-division quantum mechanics, such as eigenequations and expectation values, that would benefit from further investigations into the symbolic forms students learn to apply when reasoning about these expressions. Second, the subject pool for this study is limited to students from a single institution using a spins-first curriculum. It is very likely that gathering data from other spins-first institutions and/or from wave functions-first institutions would expand the pool of symbolic forms students develop within their curriculum at their institutions and may even show distinctions due to instructional approach. Third, the COVID-19 pandemic impacted every class studied within this work and very likely affected individual students' learning (and thus the number and/or type of symbolic forms that they were able to develop within the course). Fourth, COVID-19 also noticeably lowered interview participation rates, and thus, this study is only representative of the symbolic forms developed by six students. It is likely that with a larger pool of participants, there would have been a larger pool of symbolic forms observed as well.

VII. IMPLICATIONS FOR INSTRUCTION

The findings presented above allow for some potential insight into instruction. The number of normative symbolic forms expressed by the participants is encouraging and perhaps serves as an endorsement of the spins-first curriculum with which the students were engaged. Given the novel nature of Dirac notation to the students taking this course, the clear level at which they were able to engage with and understand the nuances of this notation is an encouraging sign that this curricular structure was generally successful in this aspect. It is likely that the explicit connections between Dirac and vector-matrix notation both in the text and in instruction supported the understanding of the geometric aspects of Dirac notation. It is worth noting that, even for this potentially biased sample of high-achieving students, there were consistent difficulties in correctly applying complex squares when calculating probabilities. Given our symbolic forms theoretical framework, this difficulty is explicable by students developing the "complex square of inner product integral of two identical functions as probability" symbolic form (symbol template: $|\int f^*(\square)f(\square)d\square|^2$). It is likely that this unproductive symbolic form is developed due to its visual similarity to the "complex square of inner product integral of two different functions as probability" symbolic form (symbol template: $|\int f^*(\square)g(\square)d\square|^2$). This suggests that perhaps time should be explicitly spent, either in or out of class, drawing attention to the times when a complex square is needed and, in the case of wave function notation, whether it belongs inside or outside the integral. For example, Castor and Delilah's focus on the expansion coefficients as an in-between step for finding expressions for probability may prove to be a useful connection to focus on during instruction, as this learned connection between expressions ($c_1 = \int_1^2$[squiggle])—i.e., a symbolic form connecting an integral to an expansion coefficient—ultimately helped them to recognize where the complex squaring should take place. Focusing on this connection may therefore help students avoid this common pitfall.

One specific potential instructional intervention to help students learn to correctly identify expressions for probability and correct uses of complex squares would be a simple clicker question-type sequence or other in-class activity, paced throughout the course, asking students to select correct statements for probability from a list of expressions in the notations they have learned up to that point. For example, in a spins-first course, they may first be given expressions in Dirac notation of the form: $\langle n|\psi\rangle$, $|\langle n|\psi\rangle|^2$, $\langle\psi|\psi\rangle$, and $|\langle\psi|\psi\rangle|^2$ and be asked to select those representing probabilities. Then, once they have learned wave function notation, they can be asked to choose from a larger list of the form: $\int_a^b \psi^*(x)\psi(x)dx$, $\int_a^b |\psi(x)|^2 dx$, $\int_a^b \varphi_n^*(x)\psi(x)dx$, $\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx$, $\int_{-\infty}^{\infty} \varphi_n^*(x)\psi(x)dx$, $|\int_a^b \psi^*(x)\psi(x)dx|^2$, $|\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx|^2$, and $|\int_{-\infty}^{\infty} \varphi_n^*(x)\psi(x)dx|^2$, as this covers most permutations of integration bounds (either over a finite or infinite extent), complex square inside vs outside of the inner product, and inner products between matching wave functions or wave functions and eigenfunctions. Alternatively, these could be divided up along any of these variables (integration bounds, complex square, and/or functions) into multiple different questions in sequence. In any case, an in-class discussion following these questions would allow for an opportunity to explicitly draw attention to the relevant details and to tease apart the important distinctions between what makes for a valid or invalid expression for probability—and whether they should consider a normalization condition as a statement of 100% probability. It is our intention that interventions such as these may help students to better be able to tackle the challenge in distinguishing these expressions as observed in our student accounts.

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