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Collaborating with mathematicians to use active learning in university mathematics courses: the importance of attending to mathematicians' obligations

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Abstract
In this paper, we discuss our experience in collaborating with mathematicians to increase their use of active learning pedagogy in a proof-based linear algebra course. The mathematicians we worked with valued using active learning pedagogy to increase student engagement but were reluctant to use active learning pedagogy due to time constraints. Our mathematicians perceived obligations in their teaching that increased the time it would take to implement some of the active learning pedagogy that we suggested, leading them to view this pedagogy as inviable. By attending to mathematicians' obligations, we were able to design active learning strategies that met the interests and needs of the mathematics educators and mathematicians collaborating on this project.

Keywords Active learning · Changing pedagogical practice · Linear algebra · University mathematics instruction

1 Introduction

When Michèle Artigue (2016) reflected on her career as a researcher in undergraduate mathematics education, she lamented that one of the field's biggest weaknesses was its lack of impact on how undergraduate mathematics courses were taught. Specifically, Artigue was alarmed by the

> insufficient dissemination of research results towards the relevant communities or practitioners, and the very limited influence of our research on university teaching

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practices. Reading recent publications… I find nearly the same description of standard university practices at undergraduate level as decades ago. (p. 12)

Indeed, it is still the case that many university mathematics courses are largely taught by lecture (Fukawa-Connelly et al., 2016; Johnson et al., 2018), even though effective, research-based alternatives exist (see Freeman et al., 2014).

To improve the impact of undergraduate mathematics education research, Artigue (2016) recommended that mathematics educators form "collaborative projects, building and negotiating, jointly with mathematicians and other university teachers, problématiques that make sense for all those involved, and meet their respective interests and needs" (p. 12). In this paper, we follow Artigue's suggestion of utilizing collaborative teams between mathematics education researchers and mathematicians to offer insight into instructional change. We describe a collaborative project between mathematics education researchers and mathematicians in which we together attempted to integrate active learning pedagogy into mathematicians' lectures in a proof-based linear algebra course.

Our thesis is that mathematicians have *obligations* (in the sense of Chazan et al., 2016). These obligations increase the workload of the mathematician implementing instructional tasks and increase the time that it takes to implement such tasks. Consequently, active learning activities that seem easy to integrate into a lecture from the perspective of mathematics education researchers may be time-intensive and infeasible from the perspective of the mathematician. By accounting for the obligations of the mathematicians teaching the course, mathematics education researchers were able to propose active learning strategies that mathematicians valued and were willing to use.

2 Literature review and theoretical perspective

2.1 Lecturing in advanced mathematics courses

Lecturing is the most common way in which advanced mathematics courses are taught. In a survey with 131 mathematician respondents who had recently taught a course in abstract algebra in the USA, 85% of the respondents said they taught by lecture (Johnson et al., 2018). This finding is consistent with observational studies (e.g., Fukawa-Connelly et al., 2017).[1] Lectures in advanced university mathematics classes are not strictly formal affairs in which mathematicians recite definitions, theorems, and proofs; mathematicians often describe the concepts intuitively, exemplify concepts, provide heuristics for solving problems and writing proofs, and describe what it means to do mathematics (Fukawa-Connelly et al., 2017; see Melhuish et al., 2022 for a review of the related literature). During lectures, instructors frequently ask students questions about the course content, although these questions typically have very short wait times and are often answered by the instructors themselves (Paoletti et al., 2018).

Many mathematics educators and some mathematicians believe lecturing is an ineffective mode of instruction in university mathematics (e.g., Bressoud, 2011; Larsen, 2017). Research suggests that students have difficulty grasping the main points of the lectures that

[1] It is important to note that several surveys of US mathematicians have found that most mathematicians claim they do not spend all of their class time lecturing and spend at least some class time on active learning strategies (e.g., Apkarian et al., 2020; Johnson, 2019).

they attend (e.g., Lew et al., 2016) and that most students emerge from university proof-oriented courses with a limited understanding of the course content (e.g., Dubinsky et al., 1994) and a limited ability to write proofs (e.g., Mejía-Ramos & Weber, 2019). In a large meta-analysis of the research literature, Freeman et al. (2014) found that students from STEM courses that employed active learning pedagogy performed significantly better than students from STEM courses that were taught solely by lecture. For these reasons, most researchers in mathematics education (e.g., Larsen, 2017) and many mathematicians (e.g., Braun et al., 2017; Bressoud, 2011) encourage mathematicians to lecture less and incorporate more active learning pedagogy.

An important question in undergraduate mathematics education research is why mathematicians continue to lecture in their proof-oriented mathematics courses, even though research suggests that teaching by lecturing alone may be contributing to undesirable learning outcomes. Additionally, Fukawa-Connelly et al. (2016) found that among lecturers "56 percent agreed (and 26 percent more slightly agreed) with the statement I think students learn better when they do mathematical work (in addition to taking notes and attending to the lecture) in class" (p. 279). Yet when mathematics educators have observed class meetings in university mathematics courses, active learning strategies were rarely used (e.g., Fukawa-Connelly et al., 2017). This juxtaposition is curious and raises many possible explanations. It may be the case that some mathematicians feel that the questions that they ask during class time are sufficient to engage students and have them actively think about the material, or it may be the case that some mathematicians believe the limited active learning that teacher questioning provides is all these teachers can offer if they have the obligation to cover the course material (Woods & Weber, 2020). Alternatively, some mathematicians might be unaware of plausible alternatives to lecturing, and still others might feel they lack the time to learn a new way of teaching due to other professional obligations (Johnson et al., 2018). Or perhaps, mathematics educators do not fully understand what it means to mathematicians for students to "do mathematical work" in class.

2.2 Research-based active learning pedagogy in advanced mathematics

Braun et al. (2017) observed that although recommendations to introduce active learning into university mathematics classrooms are commonplace, there is not a shared understanding of what active learning pedagogy is. In this paper, we follow the Conference Board of the Mathematical Sciences in defining *active learning pedagogy* broadly, where active learning pedagogy refers "to classroom practices that engage students in activities, such as reading, writing, discussion, or problem-solving that promote higher-order thinking" (as cited in Braun et al., 2017, p. 124). In other words, active learning occurs when students are engaged in some activity involving the mathematical content that does not involve passively listening to a lecture or practicing procedures.

In terms of scope, we distinguish between two types of active learning pedagogy in advanced mathematics: pedagogy to *replace* lecturing and pedagogy to *augment* lecturing. There has been substantial research in undergraduate mathematics education on the former type of pedagogy. In particular, many scholars have focused on inquiry-based and/or inquiry-oriented instruction (c.f., Laursen & Rasmussen, 2019), developing entire curricula where students spend most of their class time engaged in doing authentic mathematics (e.g., Larsen, 2013). However, as Artigue (2016) noted, and Johnson et al. (2018) documented, there remains a challenge in disseminating these curricula to mathematicians to change university instructional practice at scale.

An alternative approach is to use shorter active learning activities, such as think-pair-share questions or clicker questions, that can augment lectures. While these activities have been used in primary and secondary mathematics classrooms, as well as university physics classrooms (e.g., Crouch & Mazur, 2001), there has been comparatively limited research in undergraduate proof-based mathematics courses (Melhuish et al., 2022, but see Alcock, 2018, and Iannone & Miller, 2019, for exceptions).[2]

Thus, we agree with Braun et al. (2017) that there is a wide range of pedagogy that might be referred to as active learning. Some active learning pedagogy may consist of occasionally augmenting lectures with brief activities that require students to think about a conceptual question or interact with one another. For instance, Alcock (2018) described how she augmented her real analysis lectures by starting the class with true–false questions to prime what she expects the students to know, asking students to decide if a theorem is true or false before proving the theorem, and having students explain the meaning of a mathematical statement to their neighbor. Other active learning pedagogy, such as inquiry-based learning, represents a radical break from lecturing and dramatically shifts how mathematics class time is spent.

2.3 The role of teaching obligations in changing mathematical practice

Herbst and Chazan (2003) observed that when mathematics teachers do not adapt reform-oriented recommendations for changing instruction, some mathematics educators attribute this to the teachers suffering from a "lack of knowledge or a paucity of vision" (p. 3). Herbst and Chazan argue that this stance is counterproductive. Teachers are not free to teach in any manner that they choose. Rather, mathematics teachers, and especially university mathematics teachers, are members of a "disciplinary culture [sharing] a common set of intellectual values, a common territory" (Becher, 1994, p. 153). As members of a disciplinary practice, mathematicians may "feel bound by norms, rules, or practices" (Berthiaume, 2023, p. 17). As a result, in analyzing what active learning strategies mathematicians may adopt, revise, or reject, we need to understand what mathematicians' perceptions of the norms of teaching are, and what they consider to be "legitimate [or] illegitimate in the exercise of his or her profession" (Bridoux et al., 2023, p. 77).

Chazan et al. (2016) argued that mathematics teachers have *obligations* to their discipline, to their students, and to their institutions. In some cases, these obligations are formal rules that mathematics teachers are required to uphold. For instance, mathematicians teaching different sections of the same course may be required to give the same final exams to their students. In other cases, the obligations may be informal and may relate to commitments that a mathematician feels they should satisfy to be a respectable member of their disciplinary culture. For instance, a mathematician may feel obligated to lecture on topics that would be on the final exam, even if there is no formal edict requiring them to do so. As Chazan et al. (2016) noted, mathematics teachers often cite obligations to justify norms and explain why pedagogical actions may be legitimate or illegitimate.

Herbst and Chazan (2003) presented an interesting case on whether it was appropriate for a secondary geometry teacher to encourage students to work off provisional

[2] This observation might be accounted for by the fact that those who encourage inquiry-oriented instruction have as a goal to engage students in authentic mathematical practice, and they want students to generate core disciplinary ideas, which cannot be accomplished with think-pair-share activities.

assumptions when writing a geometry proof, only to justify those provisional assumptions at a later time. Some geometry teachers appreciated that students were behaving as real mathematicians do, but nonetheless had reservations about this practice. As Herbst and Chazan put matters,

> seeing students engage in activity that exhibits valuable, authentic mathematical characteristics might delight a teacher. But, at the same time, a teacher might also realize that such an action could jeopardize students' ability to know when a result has been proven. (p. 11)

One point to be drawn is that if a teacher refuses to allow the use of provisional assumptions when orchestrating a classroom discussion around writing a proof, we should not infer that the teacher does not value engaging students in authentic mathematical practice. It may be the case that this pedagogy is in conflict with other teacher commitments. A central aim of this paper is to illustrate how mathematicians' perceived obligations may prevent them from using some types of active learning pedagogy.

3 Methods

3.1 Research context

Rasmussen and Nardi (2020) observed that in the mathematics education literature, there is a growing body of research involving discussions between mathematics educators and mathematicians on issues of pedagogy, so that mathematics educators may have a better perspective on mathematicians' beliefs about teaching (e.g., Alcock, 2010; Nardi, 2008; Woods & Weber, 2020). Our study builds upon this literature, where mathematics educators and mathematicians collaborate to develop active learning strategies that are used in a proof-based linear algebra course. This study took place at Rutgers University, a large state university in the northeast United States. The proof-based linear algebra course (taught using the textbook by Friedberg et al., 2019) was typically taken by third- or fourth-year mathematics, physics, and engineering students. The course presented an axiomatic treatment of abstract vector spaces and linear transformations on those spaces and covered topics such as eigenvectors and eigenvalues, the Jordan canonical form and its associated theorems, and inner product spaces. It was a successor to a second-year calculation-based linear algebra course that focused on procedures on real-valued matrices, covering topics such as how to invert a matrix or compute its determinant. The decision to conduct the study in the context of the proof-based linear algebra course was pragmatic; the course had five sections, which enabled the research team to recruit multiple participants from the same proof-based mathematics course.

In the mathematics department, there were initiatives to bring active learning pedagogy to the first- and second-year courses (as well as some third-year courses), where mathematics instructors were encouraged, although not required, to devote 30 to 50% of their class time to have students working collaboratively to solve problems with learning assistants facilitating students' work. Participation was uneven: some instructors declined to work with learning assistants and many preferred to continue using their lecture-based instruction. Thus, the institutional context supported, but did not require, instructors to adopt active learning practices in the classroom, although many individual instructors resisted using them.

3.2 Participants

The mathematics education team was comprised of the authors of this paper. One co-author (Carbone) is a research-active mathematician in the mathematics department who has coordinated efforts to bring active learning pedagogy into some of the third-year mathematics courses (though not previously the linear algebra course that is the focus of this study). A second co-author (Mahmoudian) was a Ph.D student in mathematics who served as a research assistant on this project. The other members of the research team specialize in research in undergraduate mathematics education. Four mathematicians agreed to participate in this study. All were research-active, tenured, male mathematicians, whose research interests and experience teaching linear algebra are summarized in Table 1. Mathematician D could only attend the first half of the research meetings and contributed less to the conversation in the meetings even when he was present. Our analysis focuses primarily on the other three mathematicians.

3.3 Meeting structure

Going into the study, the research team had a hypothetical trajectory where the active learning strategies we proposed required little class time and effort for the mathematician, at least from our perspective. As the semester progressed, we planned for more class time to be spent on active learning strategies. Our initial theoretical expectation was that each active learning strategy would provide the mathematician with observations and insights into student thinking and that mathematicians would appreciate these insights. They would then be willing to devote more class time to active learning, giving them even greater access to student thinking.

In general, our study was our attempt to meet Artigue's (2016) call to build a genuine collaboration where mathematics educators and mathematicians worked jointly to design instruction that met both groups' interests and needs. We also followed principles that Alcock (2010) proposed for mathematics educators who work with mathematicians: to foster a respectful relationship, to appreciate and capitalize on mathematicians' extensive experience teaching undergraduate mathematics, and to understand and appreciate mathematicians' needs. Finally, our instruction is consistent with Henderson et al.'s (2011) principles for effective instructor change in STEM education: we developed a lengthy and iterative intervention where we proceeded in an incremental fashion, simultaneously addressing mathematicians' practices and beliefs.

During each meeting, the mathematics education research team proposed an active learning pedagogical strategy that mathematicians were asked to incorporate into their linear algebra lessons. Mathematicians could decline to use any strategy but were asked to provide a reason if they did so. If the mathematicians declined to use a strategy, the

Table 1 Summary of mathematician participants

Mathematician	Research area(s)
A	Mathematical Physics
B	Gauge Theory, Mathematical Physics
C	Infinite Dimensional Lie Operators, Vector Operator Algebras
D	Complex Analysis, Cauchy-Riemann Geometry

research team and mathematicians suggested modifications to the active learning strategy until it was one that mathematicians were willing to try in their classes. After an active learning strategy was agreed upon, the implementation of the strategy was discussed. The implementation involved choosing specific questions (e.g., if a true–false question was to be used, what would the specific prompt be?) and the practical issues of how the strategy would be employed in the classroom context (e.g., if a true–false question was to be used, how would the prompt be given to students and how would the instructor collect data on students' responses?). In general, the mathematics education research team would provide mathematicians with an array of questions they might use to implement the strategy. The mathematicians could then choose to use or adapt the questions that they liked.

In the next meeting between the mathematics education research team and the mathematicians, the groups would debrief on how the active learning strategy went, what benefits and shortcomings this might have had from the mathematicians' perspectives, and what changes might be needed for this to be a viable active learning strategy in the future. The first meeting occurred roughly halfway through the 15-week semester. Meetings took place weekly until near the end of the semester.[3] There were six meetings in total.

3.4 Data collection

All collaboration meetings were recorded over Zoom and transcribed; field notes for further analysis on these meetings were taken and shared within the research team. Before the study commenced, each mathematician met individually with the two members of the Rutgers mathematics education research team (Carbone and Weber) for an in-person pre-interview. The topics discussed during the pre-interview were their experience teaching mathematics courses, their teaching practices, their beliefs about teaching advanced mathematics in general and teaching linear algebra in particular, and their beliefs about the benefits, shortcomings, and viability of active learning. These interviews were audio-recorded and transcribed.

One semester after the study concluded, a member of the mathematics education research team (Weber) invited each participant for a final interview. Mathematicians A, B, and C agreed to participate. Mathematician D was traveling and did not respond to several e-mail invitations. The first part of the final interview was brief and consisted of questions about whether the mathematicians used any of the instructional techniques that we discussed in their subsequent semester of teaching.

The second part consisted of a "member-checking" interview — i.e., our research team sought confirmation that we accurately captured the thoughts and intentions that we were assigning to the mathematicians. After the collaboration meetings had concluded, the mathematics education research team generated a list of hypotheses about what obligations the mathematicians had, what forms of active learning the mathematicians believed were viable (and why), what forms of active learning the mathematicians did not believe were viable (and why not), what the mathematicians believed were the most important learning goals of a proof-based linear algebra course, and their general beliefs about instruction. Each mathematician was asked if they agreed with our interpretations of what they expressed during our interviews and was given the opportunity to clarify or elaborate on

[3] The delay was due to funding for the project being received in October, rather than September, when the semester began.

any issues. The member-checking part of the interview took up the bulk of the interview. Finally, each mathematician was asked to offer suggestions for how the collaboration could be enhanced. These interviews took place over Zoom and lasted between 30 min and 1 h. All interviews were video-recorded and transcribed.

3.5 Data analysis

After our collaborative meetings concluded, our initial data analysis focused on which active learning strategies mathematicians thought were viable and why others were not. As the mathematicians primarily lectured, we viewed our active learning strategies as a breach of their traditional practice. Following Herbst and Chazan (2003), we analyzed how they attended to the breach: did the participants acknowledge that we were asking them to do something unusual? If they thought the active learning strategy was not suitable for their classroom, what rationale did they provide for this and how (if at all) did they try to repair the breach so the practice in question would still satisfy their values and obligations? If they thought the active learning strategy was viable, what rationale did they provide? Often, when a mathematician judged an active learning strategy to not be viable in the classroom, they either provided a specific claim or we could infer an obligation that the mathematician perceived for teaching linear algebra with which the active learning strategy conflicted.

The result of our analysis was the reasons that mathematicians would or would not use active learning strategies, framed in terms of how they aligned with mathematicians' perceived obligations (Chazan et al., 2016). These reasons were presented to the mathematicians in the final member-checking interview (described above). All mathematicians interviewed affirmed that our analysis reflected their intentions, strengthening the validity and trustworthiness of our findings.

4 Results

4.1 Benefits of active learning pedagogy

During the pre-interviews, each mathematician indicated that they believed active learning pedagogy was potentially beneficial. They claimed that they wished students were more engaged in their classroom and welcomed hearing about active learning strategies that could increase students' engagement. For instance, when asked about the suitability of active learning pedagogy, Math A said, "I think the students learn better. I think that's been established. I have no doubt about it".[4] Math A was also dissatisfied with his own teaching that relied heavily on lecture:

> I would like to think that there should be some way of increasing the level of engagement even more than what I'm doing. Right now, engagement is limited to them talking to me. But what about them talking to each other? Is that possible? […] I don't really know how to do it. I haven't figured it out yet.

[4] Transcripts were lightly edited to increase their readability. We removed stutters, repeated words or phrases, and short fragments of text that did not carry meaning. At no point did we add words to the transcript and we do not believe we changed the meaning.

During the collaboration meetings, mathematicians frequently cited two benefits of using active learning pedagogy. First, students can become passive during lectures to the point that they drift off and are no longer paying attention. Second, the mathematicians benefit from receiving data about what students are understanding and what they are not. The mathematicians confirmed that active learning pedagogy had these benefits in the final interviews.

The mathematicians indicated that they tried to incorporate some active learning pedagogy into their lectures. For instance, Math C said that when he teaches, "you try and do as much active learning as possible." As an example, Math C described an activity where his students needed to think about $sin(x+1)$ as a linear combination of $sin(x)$ and $cos(x)$. While he had intended for this to be a quick "do now" question to start class, his students ended up working on this example for nearly half of the class period. As such, Math C felt that he could only use such a task a few times during the semester.

4.2 Obligations

Covering content The mathematicians all expressed an obligation to cover the course content as it was described in the syllabus and course catalog. Content coverage was reflected as an obligation to their institution (e.g., the department chair would be unhappy if different sections of the same course covered different topics), their students (e.g., students would be unprepared for graduate school if relevant content was not covered), and their discipline (e.g., a course that did not cover the Jordan Canonical Form theorems would not be a legitimate linear algebra course). Of course, mathematicians' perceived obligation to cover course content has been reported extensively elsewhere (e.g., Johnson et al., 2018; Woods & Weber, 2020), so we will not belabor the point here.

What we wish to emphasize is how the obligation to cover content limited the active learning strategies that the mathematicians were willing to use. A primary consideration that was continuously raised was how mathematicians felt constrained by time due to the challenge of covering course content:

> Math A: Time is a really important factor […] My biggest adversary in the classroom when I'm teaching is the clock on the wall. Sometimes I just want to reach up and pull the handle back.

In our initial interviews, all mathematicians thought there were benefits to active learning, but all said active learning pedagogy could only be used sparingly if all the content was to be covered. In particular, the mathematicians all expressed doubt whether there were active learning strategies that did not take up most of the class period:

> Math C: The learning goals are something like reasoning with proofs in linear algebra. It is possible. Several people described these learning activities [we had suggested using think-pair-share questions during lecture] as quick. Keith mentioned something that took two to five minutes. I'm not sure there's anything like that.
> Math B: We want people to think. Thinking will not take a very short time to actually finish. That is the main problem.

Abstraction The mathematicians claimed that a key disciplinary goal for teaching the proof-based linear algebra course was that students should recognize that linear algebra did not only apply to the familiar vector spaces R^n but also to arbitrary vector spaces, including function spaces and vector spaces over the complex field. They cited this as an obligation to the students, where the applications of the content that they are covering will usually not be in R^n:

> Math A: Physics students in their quantum mechanics classes, they're going to see Hilbert spaces. They're going to come try to grapple with this notion of a state of a quantum system being a vector over a Hilbert space. And that's not a vector in R^n. That's very far removed from a vector in R^n.

Indeed, the mathematicians avoided giving examples involving R^n as this defeated the goal of recognizing that vector spaces are more general and abstract than this:

> Math B: I always try to give examples, using polynomial space and a matrix space instead of R^n. Otherwise, it doesn't make sense to keep repeating the material of 250 [the calculation-based linear algebra course].
> Math A: I think too much emphasis on R^n is going to make it hard for students to see the connections with other disciplines where linear algebra is used heavily [...] I tried very much to de-emphasize R^n and come up with other examples of vector spaces, and in particular, a lot of function spaces.

This obligation led to the need to use more abstract or esoteric examples than the familiar R^n in our question prompts. Because a variety of vector spaces were used, the different examples and questions required carefully stated and quantified structure for precision and clarity; however, at times, this implied students might need more time reading the questions, interpreting notation, and recalling definitions to make sense of what was being asked. Thus, this obligation for abstraction could at times be at odds with the obligation for content coverage and its implied issue of time.

Providing rigorous justifications to answer all questions Whenever mathematicians used a conceptual question that we suggested proposing to their class, out of an obligation to their discipline, they felt required to give a rigorous answer[5] to the question:

> Math C: In mathematics, we're constantly trying to keep a very clear boundary of things that we understand and that we don't understand. And I think if you start giving lots of activities where the students weren't really sure if they understood the answer or not, there would be no clear boundary and it just gets very disorganized.
> Keith: Okay. So it's one of the distinctive features of the discipline that you have the accepted statements and then you have the open questions.
> Math C: Or the questions you haven't looked at or explored. Everything sort of builds on previous stuff. So you want to have very clear line of things that have been established and understood.
> Keith: And the discipline has clear criteria when something becomes established.

[5] By "rigorous answer," we mean an acceptable proof by mathematicians' standards. The mathematicians claimed that the level of detail that they provided in a proof could vary, depending on the importance of the theorem and time constraints.

Math C: Right. That's right.

Similarly, Math B remarked that if questions were posed, but their answers were not proven, this could potentially confuse students, saying proofs "definitely should be provided after these discussions, because otherwise it will mislead students." This is especially important as "whether a proof is correct or not, this is something that students might not be able to judge."

4.3 Exit tickets

In our first meeting, we proposed that mathematicians use "exit tickets." That is, we asked that at the end of class, mathematicians ask students to submit a response to a question about a key topic covered in the class.[6] We use this topic to illustrate how obligations can prevent mathematicians from using an active learning strategy and to explain why our initial trajectory for how our collaborative meetings could progress was based on faulty assumptions. Our rationale for beginning with exit prompts was based on the following three hypotheses:

(i) Mathematicians would find it useful to know that many students were not understanding the main points in their lecture, which might motivate the need for alternative forms of pedagogy.
(ii) By reading students' responses to the exit ticket, mathematicians would come to appreciate the value of attending to student reasoning.
(iii) An exit ticket would take minimum class time.

Each of these three hypotheses was wrong.

The mathematicians were willing to try using the exit ticket, but they each indicated that they did not expect students to fully understand the lecture immediately after attending it. Math C said, "I think most [students] would say they did not understand yet," and Math B immediately added, "After I proved, the students are still puzzled" and suggested students would understand better after reflection and working on homework. All the mathematicians thought the exit ticket prompts would yield more useful data if they were posed at the beginning of the next class after students had a chance to digest the material. Our second hypothesis was incorrect for similar reasons. The exit tickets did not lead mathematicians to see the value in attending to students' reasoning. They already saw the value in doing so.

Regarding our third hypothesis, mathematicians' implementation of the exit ticket strategy took much longer than we anticipated. One issue was that Math A[7] and Math C chose to use exit tickets at the start of the next lesson as an in-class activity rather than at the end of the previous lesson (for reasons we discussed above). A larger issue is that once exit

[6] Exit tickets are a popular pedagogical technique in the USA and are used often in primary and secondary mathematics classrooms.

[7] To illustrate what exit tickets look like, Math A used his exit ticket after a lecture introducing eigenvalues and eigenvectors. The prompt he gave was "Let $T{:}P^3(R)\rightarrow P^3(R)$ be defined by $T(f(x))=xf'(x)$. True or false: x^2 is an eigenvector of T. If true, what is the associated eigenvalue." Here $P^3(R)$ is the set of cubic polynomials with real-valued coefficients and T is a linear transformation. This was one of a collection of prompts that our research team generated. The mathematicians were free to choose one of our suggested prompts or create their own.

ticket data was collected, the mathematicians would give a rigorous answer to the question on the exit ticket (even if they had not yet looked at the student responses). As Math A said, "of course, after I collected it, I went over this." Math C remarked that providing answers to the questions took over 20 min of class time. This reflected the mathematicians' obligation to rigorously answer each question that was posed. Math A and Math C both saw the benefit in assigning the exit ticket questions as launch activities. They valued identifying what students knew and what they were confused about. Further, Math A found the activity "useful" since he started the class by having students think about the material:

> Math A: They had at least five minutes to think about something. It's very rare that we have this kind of luxury to make them think about something for five minutes, and then talk about it [...] Putting something in front of them in the first five minutes has that effect, they get to put their linear algebra hat on.

Unfortunately, due to the time it took to implement the activity, all the mathematicians thought this was a pedagogical strategy that they could not employ often, at least not in the way we had suggested they implement it.

Mathematicians' reactions to exit tickets were revealing to our research team. Mathematicians were weighing the benefit of using an exit ticket against the cost of the time it would take to implement the exit ticket. By using exit tickets, we were trying to increase mathematicians' awareness of the benefits of collecting data on student thinking, but mathematicians were already aware of these benefits. (To avoid misinterpretation, we are not claiming that mathematicians were aware of the benefits of *all* active learning pedagogical strategies, like orchestrating whole class discussions.) What we underestimated were what mathematicians' concerns about the cost in terms of class time for the activities. Further, the cost was higher than we realized, since mathematicians had the obligation to offer rigorous answers to each question that was posed. For mathematicians to value our activity, it was not enough for the activities to be beneficial. We also needed to mitigate the cost of the activities.

4.4 A solution that was viable

Through negotiation between the mathematics education research team and the mathematicians, we designed a format for asking students questions that were akin to clicker questions that had the following format:

(i) The questions would be a true–false question or a multiple choice question. (e.g., True or false: Any vector that is orthogonal to itself must have an inner product of zero).
(ii) The questions were chosen to align with the flow of the lecture, focusing on vector spaces and theorems that were either just covered or were going to be covered.
(iii) The questions that were asked would be short and easy to comprehend.
(iv) The participant would be given a short time to answer the question, but they had to submit an answer[8]; an educated guess was okay.

[8] How the question was answered varied by the mathematician. Math A had students raise colored panels to express their choice, Math B relied on a show of hands, and Math C asked students to write their answers on a sheet of paper that he checked as he circled the room. Each had detailed reasons for their choice, but we do not discuss these for the sake of brevity.

Each of these conditions was put into place to manage the cost of time. Condition (ii) did so subtly. Prior to presenting a theorem, mathematicians might ask if a theorem, or an application of the theorem, was true or false. Before presenting a proof of a theorem, the mathematician may ask if a particular inference within the proof always held. For instance, in a proof of the Jordan Canonical Form theorem, there is the statement "Clearly, the null space of $(T - \lambda I)^m$ is contained in K_λ." (where T is a linear transformation on a vector space and the generalized eigenvector space K_λ is the subspace of vectors such that v is in K_λ if $(T - \lambda I)^n v = 0$ for some natural number n). Participants can be asked if a member of the null space of $(T - \lambda I)^m$ is a member of the generalized eigenvector space. The virtue in (ii) in terms of time is that while mathematicians had the obligation to rigorously answer the question that was posed, this would not cost them extra time since they were going to answer that question anyway. Condition (iv) was also important to mathematicians as there was a concern that not all students would participate in an activity that was posed. By requiring all students to commit to something, the hope was that all students would think about the question that was posed.

Mathematicians expressed two benefits to using these questions. First, students were spending some class time thinking about the mathematics, which increased their understanding of the content that was being discussed. For instance, Math A described a benefit of these questions as follows:

> Overall, I think it is helpful for students to understand these concepts. [When I pose these questions] it looks like they were still thinking, and their CPUs [central processing units] were overheating. Let's put it that way. Eventually, I think most of them ended up on the right side [answered the question correctly] and the ones who know what to do really got it right.

Another benefit that was discussed was found in the mathematicians recognizing what students did not understand. For instance, Math A observed that students had trouble with conceptual questions requiring them to relate eigenvalues and eigenvectors to null spaces, in part because these topics were presented in different parts of the course. Math A concluded, "I think these connections are probably something that we could think more about and be more strategic about." Both benefits were confirmed by the three mathematicians who completed the final interview.

In the final interviews, each mathematician was asked if they used ideas from our collaborative meeting in their subsequent teaching. Although Math A and Math C were not teaching linear algebra, they confirmed they had used the ideas that we had discussed. For instance, Math A described his experience teaching general relativity:

> Math A: I tried to incorporate some of the discussion techniques that we talked about in the linear algebra course. The setting, of course, was different, because this was more of a special topics course where students don't really know very much about what they were talking about.
> Keith: Could you give me an example of something specific that you did?
> Math A: I was teaching, say, the principles of relativity. So I would describe a typical situation in relativistic dynamics. I would ask a student to tell me what they think the solution is, or what's going to happen next. So this guy goes and comes back, is he going to be younger or older than his twin and things like that? That sort of allows students to discuss and then come up with an answer […]
> Keith: Right. And how did it go?

> Math A: It was very positive. I think the students seem to be getting more out of it in this way.

Math B said he did not use the ideas in the course since he was teaching a fast-paced lower-level computational course that did not have theoretical discussions. However, he said he planned to use the techniques we discussed the next time he taught the proof-oriented linear algebra course. He was also quite complimentary of the structure that we provided and the collaboration sessions and was grateful that the mathematics education research team listened to his concerns and took them seriously.

> Math B: I wanted to see the active learning in the classroom setting, what kind of thing is effective. You bring your perspective on this. Then see, we actually told you what is our concern, then we combined together, we actually get a much better way of understanding… My main worry about active learning is always the time constraint. I just feel that if I use the usual way of suggesting to do active learning, we do not have enough time, because we need to finish the material. When we work with you, you listen to what we are talking about. I think that now, the format works […] In the future, we can do this systematically. I always worry about the time constraint. When we talked to you, you actually listened, and then we work together to actually have a format where we do not need to spend a lot of time on things, but still will be effective to a certain extent. That part is very, very, very helpful. Thank you.
> Math B: One thing I just want to say. I feel this study is much more effective than when the study started. At the beginning, I have to say that I'm a little bit skeptical about whether this activity will be useful. But I see now I think it seems to be much, much, much more effective than I thought.

5 Discussion

Our study was a highly contextual case study on a collaboration between mathematics educators and mathematicians teaching a specific linear algebra course in a specific context. Accordingly, we caution the reader not to generalize our findings to other mathematicians or contexts. In particular, we do not know the extent that the obligations that our mathematicians cited are shared by other mathematicians. Rather, our situated qualitative study makes the following theoretical contributions.

First, we entered the study believing we needed to change mathematicians' beliefs about the *benefits* of having students engage in active learning. Even before our study began, mathematicians already saw benefits to active learning pedagogy and expressed a willingness to employ active learning in their classroom. This finding is consistent with the broader literature (e.g., Johnson et al., 2018; Melhuish et al., 2022; Woods & Weber, 2020). There may be benefit to exposing mathematicians to other benefits of active learning pedagogy (e.g., developing students' mathematical identities and increasing student autonomy). Nonetheless, the lack of a perception of benefits was not what inhibited mathematicians from using the active learning pedagogy that we proposed. Rather, what dissuaded mathematicians was the *cost* in terms of time entailed by their implementation. We suggest that to persuade mathematicians to use active learning strategies, mathematics educators must be mindful of what they are asking mathematicians to sacrifice in terms of time and content coverage.

Second, we found that mathematicians' *obligations increased the cost* of implementing even basic active learning strategies, such as using an exit ticket or a think-pair-share question. In our mathematicians' case, the obligation to use abstract examples considerably lengthened the time it took students to understand and process many of the questions that we asked and the obligation to give rigorous answers to each question that was presented to students extended the time required to complete the activity. We cannot be sure that other mathematicians will have the same obligations as the mathematicians in our collaboration. However, our work does illustrate how mathematicians may reject some active learning pedagogy, even though they perceive the pedagogy as beneficial because they believe the pedagogy fails to meet their obligations (c.f., Herbst & Chazan, 2003). We caution mathematics education researchers to be mindful that mathematicians' obligations, whatever they may be, may increase the time and effort, and limit the feasibility, of even seemingly simple active learning strategies, such as those proposed by Alcock (2018) and Braun et al. (2017).

Third, our work is consistent with Becher (1994) and Bridoux et al. (2023) in that mathematicians' obligations appeared to be *discipline-specific*. As Becher (1994) and Bridoux et al. (2023) noted, mathematicians highly value abstraction, perhaps more than other professional communities. Using rigorous proof to distinguish between mathematical theorems that have been conclusively established and mathematical claims that are merely supported also may be specific to mathematics. Our results extend beyond this and note that our mathematicians' obligations depend on the specific course being taught. The mathematicians in our study were comfortable situating the first linear algebra course in R^n but thought references to R^n should be avoided in the second linear algebra course. The broad theme that we wish to highlight is that when it comes to mathematicians' obligations, these are not fully determined by mathematicians' membership in a professional discipline. Context matters, including the specific class being taught.

Finally, our collaboration illustrates how mathematics education researchers can collaborate with mathematicians to shift their practice by following Artigue's (2016) call for formulating "problématiques that make sense for all those involved" (p. 12) and honoring Alcock's (2010) commitments to interact with mathematicians in a respectful manner and consider their goals and needs. By listening to mathematicians and adapting our pedagogical suggestions to align with their obligations, we were able to design pedagogical strategies that mathematicians implemented and valued and indeed claimed they were willing to use after our collaboration concluded.

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Declarations

Competing interests The authors declare no competing interests.

Human subjects All human subjects data was collected with informed consent under the approval of Rutgers University Human Research Protection Program's Institutional Review Board.

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