

HITTING TIMES FOR SHAMIR'S PROBLEM

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ABSTRACT. For fixed $r \geq 3$ and n divisible by r , let $\mathcal{H} = \mathcal{H}_{n,M}^r$ be the random M -edge r -graph on $V = \{1, \dots, n\}$; that is, $\mathcal{H}$ is chosen uniformly from the M -subsets of $\mathcal{K} := \binom{V}{r}$ ($:= \{r\text{-subsets of } V\}$). *Shamir's Problem* (circa 1980) asks, roughly,

for what $M = M(n)$ is $\mathcal{H}$ likely to contain a perfect matching

(that is, n/r disjoint r -sets)?

In 2008 Johansson, Vu and the author showed that this is true for $M > C_r n \log n$. More recently the author proved the asymptotically correct version of that result: for fixed $C > 1/r$ and $M > Cn \log n$,

$$\mathbb{P}(\mathcal{H} \text{ contains a perfect matching}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

The present work completes a proof, begun in that recent paper, of the definitive “hitting time” statement:

Theorem. *If $A_1, \dots$ is a uniform permutation of $\mathcal{K}$, $\mathcal{H}_t = \{A_1, \dots, A_t\}$, and*

$$T = \min\{t : A_1 \cup \dots \cup A_t = V\},$$

then $\mathbb{P}(\mathcal{H}_T \text{ contains a perfect matching}) \rightarrow 1$ as $n \rightarrow \infty$.

1. INTRODUCTION

A (simple) r -graph (or r -uniform hypergraph) is a set $\mathcal{H}$ of r -subsets (edges) of a vertex set $V = V(\mathcal{H})$; a matching of such an $\mathcal{H}$ is a set of disjoint edges; and a perfect matching (p.m.) is a matching of size $|V|/r$. Write $\mathcal{H}_{n,M}^r$ for the random M -edge r -graph on $[n] := \{1, \dots, n\}$; that is, $\mathcal{H}_{n,M}^r$ is chosen uniformly from the M -subsets of $\mathcal{K} := \binom{[n]}{r}$. (Usage notes are collected at the end of this section.)

This paper completes a proof, begun in [23], of the definitive answer to *Shamir's Problem*, which asks, roughly: for fixed r and n ranging over (large) multiples r ,

for what M is $\mathcal{H}_{n,M}^r$ likely to contain a perfect matching?

In what follows we work with a fixed r and omit it from our notation—so $\mathcal{H}_{n,M}^r$ becomes $\mathcal{H}_{n,M}$ —and restrict to n divisible by r .

The story of Shamir's Problem has been told at some length in [23] and we will be briefer here. The problem first appeared in print in [9], where Erdős says he heard it from Eli Shamir in 1979, and, following initial results of Schmidt and Shamir in [30], became one of the most intensively studied questions in probabilistic combinatorics; for example, [21, Section 4.3] calls Shamir's Problem and its graph factor analogue (see below) “two of the most challenging, unsolved problems in the theory of random structures.”

For precise statements define the threshold for containing a perfect matching, denoted $M_c = M_c(n)$, to be the least the least M for which

$$\mathbb{P}(\mathcal{H}_{n,M} \text{ contains a perfect matching}) \geq 1/2.$$

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(This is also a threshold in the original sense of Erdős and Rényi [11]; see [7] or [21, Theorem 1.24].)

A natural guess—though not recognized as such in [9, 30]—is that *in the random setting* the main obstacle to existence of a perfect matching is isolated vertices (vertices not in any edges), which typically disappear when $M \approx (n/r) \log n$. There are three progressively stronger versions of this intuition that one might hope to establish. The first, called *Erdős-Rényi threshold*, involves the order of magnitude of M_c :

Theorem 1.1. *For each r there is C_r such that if $M > C_r n \log n$ then $\mathcal{H}_{n,M}$ contains a perfect matching w.h.p.¹*

(Equivalently, $M_c = \Theta(n \log n)$, where the implied constant depends on r .) This was shown in [22], with best earlier progress in [15] and [25]. (See also [16, Sec. 13.2] for an exposition.)

The more precise second and third versions—*asymptotics of the threshold* and *hitting time*—are:

Theorem 1.2. *For fixed $C > 1/r$ and $M > Cn \log n$, $\mathcal{H}_{n,M}$ contains a perfect matching w.h.p.*

(Equivalently, $M_c(n) \sim (n/r) \log n$.)

Theorem 1.3. *If $A_1, \dots$ is a uniform permutation of $\mathcal{K}$, $\mathcal{H}_t = \{A_1, \dots, A_t\}$, and*

$$T = \min\{t : A_1 \cup \dots \cup A_t = V\},$$

then $\mathcal{H}_T$ contains a perfect matching w.h.p.

(Here T is the aforementioned *hitting time*. It is easy to see that Theorem 1.3 implies Theorem 1.2.) For $r = 2$, Theorems 1.2 and 1.3 were shown by Erdős and Rényi [12] and Bollobás and Thomason [6] respectively.

Theorem 1.2 was first formally conjectured (in a stronger form corresponding to what's shown in [12]) in [8] and Theorem 1.3 was proposed in [22], though each was probably considered plausible by the time it was recorded. (That Theorem 1.1 was apparently not on the radar in [9, 30]—Erdős specifically says he has no idea what to expect for Shamir's Problem—seems odd in view of [12], but perhaps suggests that the above results were initially thought too much to expect.)

The predecessor, [23], of the present work proved Theorem 1.2 and began a proof of Theorem 1.3 whose completion is our main objective here. The proof proceeds by way of a reduction—given in [23]—to a conditional version of Theorem 1.2 (Theorem 1.5 below). The proof of the conditional statement is similar to the proof of Theorem 1.2; but the conditioning—on a low probability event—makes even formerly routine points tricky to deal with, and the point of the separate proof of Theorem 1.2 was to show the structure of the argument unencumbered by these extra difficulties.

Graph factors (briefly; see [22, 23] for a little more). Recall that, for graphs H and G , an H -factor of G is a collection of copies of H in G whose vertex sets partition $V(G)$. The graph factor counterpart of Shamir's Problem asks (roughly): for a fixed H , when is the random graph $G_{n,M}$ likely to contain an H -factor? This was first suggested (for $H = K_3$) by Ruciński [29].

The factor analogue of Theorem 1.1 was shown in [22] for *strictly balanced* H (more or less those H 's for which one expects it to hold; see [22, Conjecture 1.1] for what should be true in general). For certain nice H 's—e.g. cliques—beautiful coupling arguments of Riordan and Heckel [28, 20] derive the factor versions of Theorems 1.1 and 1.2 from their Shamir versions, a connection that seems unlikely to extend to Theorem 1.3. As suggested in [23], I expect that the work there and here extends to factors, though, at least for what we do here, this looks fairly excruciating absent some simplification of the material below.

¹“with high probability,” meaning with probability tending to 1 as $n \rightarrow \infty$

In recent work, Frankston, Narayanan, Park and the author [14] used a new approach inspired by [2] to prove a general threshold result (a relaxation, suggested by Talagrand [31], of a still open conjecture of Kalai and the author [24]) that easily implies Theorem 1.1 and much more. It is, however, hard to imagine Theorem 1.2 (*a fortiori* Theorem 1.3) being proved along similar lines. It would be very interesting to decide whether the approach of [14] can recover the factor version of Theorem 1.1 proved in [22].

As in [22, 23], the proofs of Theorems 1.2 and 1.3 depend crucially on establishing stronger counting versions; thus for Theorem 1.3, with $\Phi(\mathcal{H})$ denoting the number of perfect matchings of $\mathcal{H}$, we show:

Theorem 1.4. *For $\mathcal{H}_t$ and T as in Theorem 1.3, w.h.p.*

$$(1) \quad \Phi(\mathcal{H}_T) > \left[e^{-(r-1)} \log n \right]^{n/r} e^{-o(n)}.$$

(Up to a subexponential factor, the right-hand side of (1) is the expected value of its left-hand side.)

Our assignment here is to prove the following conditional statement, which, as shown in [23, Section 10], implies Theorem 1.4. (The same reduction gets Theorem 1.3 itself from the weaker version of Theorem 1.5 corresponding to Theorem 1.2, but, again, we don't know how to prove the weaker version without proving the stronger.)

Theorem 1.5. *Fix a small positive ε and suppose $\delta_x \sim \varepsilon \log n$ for each $x \in V := [n]$. Let $M = M(n) \sim (n/r) \log n$ and let $\mathcal{H}$ be distributed as $\mathcal{H}_{n,M}$ conditioned on*

$$\{d_{\mathcal{H}}(x) \geq \delta_x \forall x \in V\}.$$

Then w.h.p.

$$(2) \quad \Phi(\mathcal{H}) > \left[e^{-(r-1)} \log n \right]^{n/r} e^{-o(n)}.$$

In other words: for $\varsigma \ll 1$ there is $\varrho \ll 1$ such that if $M = (1 \pm \varsigma)(n/r) \log n$ and $\delta_x = (1 \pm \varsigma)\varepsilon \log n$ for each x , then

$$\mathbb{P} \left(\Phi(\mathcal{H}) \leq \left[e^{-(r-1)} \log n \right]^{n/r} e^{-\varrho n} \right) < \varrho.$$

(It should perhaps be stressed that our argument doesn't work if we allow $\varepsilon = o(1)$; see the Outline at the end of Section 2 and the note following (101) in Section 9.)

In Section 2 we derive Theorem 1.5 from several statements whose proofs will be the main work of this paper. Outlining that work will be easier once we have the framework of Section 2, so is postponed until then, at which point we'll also say a bit about how what we do here relates to [23]. We won't assume familiarity with [23]—and will wind up more or less repeating parts of it—but, as said above, it shows the present argument in simpler form, and a reader of the present work might find it a useful companion.

Usage

Throughout the paper we fix $r \geq 3$; take $V = [n] := \{1, \dots, n\}$, with n divisible by r ; and use $\mathcal{K}$ for $\binom{V}{r}$. We use v, w, x, y, z for vertices and $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{J}$ for r -graphs (subsets of $\mathcal{K}$), or, often, bold versions of these when the r -graphs in question are random. As above, we abbreviate $\mathcal{H}_{n,M}^r = \mathcal{H}_{n,M}$.

We use $d_{\mathcal{H}}(\cdot)$ and $d_{\mathcal{H}}(\cdot, \cdot)$ for degree and codegree in $\mathcal{H}$ (thus $d_{\mathcal{H}}(x) = |\{A \in \mathcal{H} : x \in A\}|$ and $d_{\mathcal{H}}(x, y) = |\{A \in \mathcal{H} : x, y \in A\}|$), and $\Delta_{\mathcal{H}}, \delta_{\mathcal{H}}$ and $D_{\mathcal{H}}$ for maximum, minimum and average degrees in $\mathcal{H}$. We use $\mathcal{H}_x = \{A \in \mathcal{H} : x \in A\}$ and, for $X \subseteq V$, $\mathcal{H}[X] = \{A \in \mathcal{H} : A \subseteq X\}$ and $\mathcal{H} - X = \mathcal{H}[V \setminus X]$. We will tend to abusively write $Y \cup a$ and $Y \setminus a$ for $Y \cup \{a\}$ and $Y \setminus \{a\}$ (in particular $\mathcal{H} \setminus A$ for $\mathcal{H} \setminus \{A\}$).

For a set X and $p \in [0, 1]$, we use X_p for the random subset in which elements of X appear independently, each with probability p . In all our uses of this X will be some $\mathcal{H}$; so $\mathcal{H}_p$ is formally in conflict with $\mathcal{H}_x$, but there will never be any question as to which is meant. (We will also, beginning with (5), see $\mathcal{H}_t$ —this always with a bold $\mathcal{H}$, though there would be no confusion in any case.)

As above, we will sometimes use bold for random objects: consistently for r -graphs except when we use $\mathcal{H}_p$, but otherwise only when we need to distinguish between a random object and its possible values.

We use mathfrak characters ($\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \dots$) for properties and events. A *property* $\mathfrak{A}$ will usually be a property of r -graphs (thus $\mathfrak{A} \subseteq 2^K$), and we will say, as convenient, “ $\mathcal{H}$ has property $\mathfrak{A}$,” “ $\mathcal{H}$ satisfies $\mathfrak{A}$,” “ $\mathcal{H} \in \mathfrak{A}$ ” or “ $\mathcal{H} \models \mathfrak{A}$.” An *event* is then $\{\mathcal{H} \models \mathfrak{A}\}$ for some property $\mathfrak{A}$ and random $\mathcal{H}$, and may be denoted simply $\mathfrak{A}$ if we have specified $\mathcal{H}$. According to what feels natural (or typographically preferable), we use any of the synonymous $\mathfrak{A} \wedge \mathfrak{B}$, $\mathfrak{A} \cap \mathfrak{B}$, $\mathfrak{A}\mathfrak{B}$. As usual, a property of r -graphs on V is *increasing* if it cannot be destroyed by addition of edges.

We assume (as in Theorem 1.5) that ε is fairly small. We will always assume n is large enough to support our assertions and, following a common abuse, pretend large numbers are integers.

Asymptotic notation is interpreted as $n \rightarrow \infty$. We use $a \ll b$ and $a = o(b)$ interchangeably and, similarly, $a \lesssim b$ is the same as $a < (1 + o(1))b$. We use both “a.e.” and “a.a.” to mean “for all but a $o(1)$ -fraction.” We use $\log$ for natural logarithm and $a \pm b$ for a quantity within b of a .

Where not otherwise stated, implied constants in $\Omega(\cdot)$ and $O(\cdot)$ are allowed to depend on ε . (Usually they won’t, but we will only worry about this when it matters.) A typographical *convention*: in exponents only, we will use c as a substitute for $\Omega(1)$; thus different c ’s in a single statement need not (and will not) be equal. (For consistency we allow dependence on ε , but in our uses of c this will never make any difference.)

Finally, we set $T = |\mathcal{K}| - M$ (M as in Theorem 1.5) and throughout the paper take

$$(3) \quad m_t = |\mathcal{K}| - t \quad (= \binom{n}{r} - t) \text{ and } \mathcal{K}^t = \binom{\mathcal{K}}{m_t}.$$

(So $m_T = M$, but we will usually use m_T . We will always have $t \in [T]$.) We will often use m for m_t (we think of this as a default, but won’t use it without notice). We use D_m for the common value of $D_{\mathcal{H}}$ for $\mathcal{H}$ ’s of size m (so when $m = m_t$ and $\mathcal{H} \in \mathcal{K}^t$, $D_{\mathcal{H}}$ is D_m , not the equally plausible D_t). This may all take a little getting used to, but eventually seemed less annoying than various alternatives.

2. SKELETON

Here we derive Theorem 1.5 from several assertions whose proofs will be the main content of the paper. The discussion here is similar to that of [23, Sec. 2].

Recalling that $T = |\mathcal{K}| - M$, we would like to proceed as in [22, 23], starting from $\mathcal{H}_0 := \mathcal{K}$ and randomly deleting edges one at a time to produce the sequence $\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_T$, with $\mathcal{H}_T$ the $\mathcal{H}$ of Theorem 1.5. Here uniform deletions will not do, but we may proceed as follows.

Let

$$(4) \quad \mathfrak{L} = \{\mathcal{J} \subseteq \mathcal{K} : d_{\mathcal{J}}(x) \geq \delta_x \ \forall x \in V\},$$

and to generate $\{\mathcal{H}_t\}$: choose $\mathcal{H}_T$ uniformly from $\mathfrak{L}_T := \mathfrak{L} \wedge \mathcal{K}^T$; let $A_1, \dots, A_T$ be a uniform ordering of $\mathcal{K} \setminus \mathcal{H}_T$; and for $t \in \{0, \dots, T\}$ set

$$(5) \quad \mathcal{H}_t = \mathcal{K} \setminus \{A_1, \dots, A_t\} \quad (= \mathcal{H}_T \cup \{A_{t+1}, \dots, A_T\}).$$

(One peculiarity of the present approach is that we *start* with the object of interest, $\mathcal{H}_T$, but analyze it as the output of the random sequence it has been used to generate.)

The following two rules governing the law of $\{\mathcal{H}_t\}$ are not needed for the present outline but will be the basis for much of what follows; the easy verifications are left to the reader. For $\mathcal{H} \subseteq \mathcal{K}$ of size at least m_T , we use $\mathcal{U}_{\mathcal{H}}$ for a uniform m_T -subset of $\mathcal{H}$ and set

$$(6) \quad \beta(\mathcal{H}) = \mathbb{P}(\mathcal{U}_{\mathcal{H}} \in \mathfrak{L}).$$

Observation 2.1. Among $\mathcal{H}$'s in $\mathcal{K}^t$,

$$\mathbb{P}(\mathcal{H}_t = \mathcal{H}) \propto \beta(\mathcal{H}).$$

Observation 2.2. Among A 's in $\mathcal{H} \in \mathcal{K}^{t-1}$,

$$\mathbb{P}(A_t = A | \mathcal{H}_{t-1} = \mathcal{H}) \propto \beta(\mathcal{H} \setminus A).$$

Set $\Phi(\mathcal{H}_t) = \Phi_t$ and let ξ_t be the fraction of perfect matchings of $\mathcal{H}_{t-1}$ that contain A_t (so $\xi_t = \Phi(\mathcal{H}_{t-1} - A_t)/\Phi_{t-1}$). Then

$$\Phi_t = \Phi_0(1 - \xi_1) \cdots (1 - \xi_t),$$

or, equivalently,

$$(7) \quad \log \Phi_t = \log \Phi_0 + \sum_{i=1}^t \log(1 - \xi_i).$$

The proof of Theorem 1.5 depends on showing that Φ_t is likely to stay reasonably close to its expectation throughout the above evolution. As will appear, this is self-reinforcing, with past good behavior favoring good behavior going forward.

An issue here is that there are possibilities for the $\mathcal{H}_t$'s that don't support our analysis. (The same was true, but in far milder form, in [22, 23].) To deal with this we define (in Section 5) a collection $\mathfrak{R}$ of ("reasonable" or "generic") $\mathcal{H}$'s, write $\mathfrak{R}_t$ for the event $\{\mathcal{H}_t \in \mathfrak{R}\}$,

and show (mainly in Section 13)

$$(8) \quad \mathbb{P}(\cap_{t \leq T} \mathfrak{R}_t) \rightarrow 1.$$

The resulting license to ignore $\mathcal{H}_t$'s not belonging to $\mathfrak{R}$ will underpin much of what happens below.

Let

$$(9) \quad \Lambda = (r-1)n/r$$

and observe that (recall $\log = \ln$)

$$(10) \quad \log \Phi_0 = \log \frac{n!}{(n/r)!(r!)^{n/r}} = \frac{n}{r} \log \binom{n}{r-1} - \Lambda + O(\log n).$$

Now using m for $m_t (= |\mathcal{K}| - t)$, set

$$(11) \quad \gamma_t = n/(r(m+1)).$$

Then γ_t is the reciprocal of the average degree in $\mathcal{H}_{t-1}$, and would be equal to $\mathbb{E}\xi_t$ if A_t were *uniform* from $\mathcal{H}_{t-1}$. That was the situation in [22, 23], but here the $\mathbb{E}\xi_t$'s will require some care; we will show (recall we are using c for a positive constant)

$$(12) \quad \text{if } \mathcal{H} \in \mathcal{K}^{t-1} \cap \mathfrak{R} \cap \mathfrak{L}, \text{ then } \mathbb{E}[\xi_t | \mathcal{H}_{t-1} = \mathcal{H}] < (1 + n^{-c})\gamma_t.$$

This will mean that, as long as we have not wandered out of $\mathfrak{R}$, we may think of $\mathbb{E}\xi_t$ as essentially γ_t . (Of course if $\mathcal{H} \notin \mathfrak{L}$, the conditioning event is vacuous.)

Let $\mathfrak{A}_t$ be the event

$$(13) \quad \left\{ \log \Phi_t > \log \Phi_0 - \sum_{i=1}^t \gamma_i - o(n) \right\}.$$

(We note, probably unnecessarily, that (13) refers to some specific $o(n)$, so that it makes sense to talk about $\mathfrak{A}_t$ for a particular n . Related points will be common below, and, somewhat departing from common practice, we will elaborate in a couple places where doing so seems possibly helpful.)

Noting that

$$(14) \quad \sum_{i=1}^t \gamma_i = \frac{n}{r} \log \left[\binom{n}{r} / m \right] + o(1),$$

provided $m \gg n$, and recalling the expression for Φ_0 in (10), we find that $\mathfrak{A}_T$ says

$$(15) \quad \log \Phi_T > (n/r) \log(rm_T/n) - \Lambda - o(n),$$

which is the same as (2); so Theorem 1.5 is

$$(16) \quad \mathbb{P}(\overline{\mathfrak{A}}_T) = o(1).$$

(We will in fact show $\mathbb{P}(\cup_{t \leq T} \overline{\mathfrak{A}}_t) = o(1)$; see (22).)

For (16) we use the method of martingales with bounded differences. Here it is natural—though we will need a variant of this—to consider the martingale

$$\{X_t = \sum_{i=1}^t (\xi_i - \mathbb{E}[\xi_i | A_1, \dots, A_{i-1}])\}$$

with associated difference sequence

$$\{Z_i = \xi_i - \mathbb{E}[\xi_i | A_1, \dots, A_{i-1}]\}.$$

In general, proving concentration for such X_t 's depends on maintaining some control over the $|Z_i|$'s, to which end we track, in addition to the $\mathfrak{R}_t$'s, a second sequence of events $\mathfrak{B}_t$. These will be defined in Section 7; roughly $\mathfrak{B}_t$ says that no edge of $\mathcal{H}_t$ is in too much more than its natural share of perfect matchings.

For $t \leq T$ it will follow trivially from $\mathfrak{B}_{t-1}$ (see (87)) that

$$(17) \quad \xi_t = O(\gamma_t).$$

This is more than enough for the desired concentration but can occasionally fail, since $\mathfrak{B}_{t-1}$ may fail. To allow for this, as well as possible failures of the $\mathfrak{R}_j$'s, we slightly modify the above X 's and Z 's, setting

$$(18) \quad Z_i = \begin{cases} \xi_i - \mathbb{E}[\xi_i | A_1, \dots, A_{i-1}] & \text{if } \mathfrak{B}_j \mathfrak{R}_j \text{ holds for all } j < i, \\ 0 & \text{otherwise} \end{cases}$$

(and $X_t = \sum_{i=1}^t Z_i$). As shown in Section 3, a martingale analysis along the lines of Azuma's Inequality then gives

$$(19) \quad \mathbb{P}(X_t > \lambda) < n^{-\omega(1)} \text{ for } \lambda \gg \sqrt{n}.$$

We next observe that if $\mathfrak{B}_i \mathfrak{R}_i$ holds for $i < t \leq T$ —so

$$X_t = \sum_{i=1}^t (\xi_i - \mathbb{E}[\xi_i | A_1, \dots, A_{i-1}])$$

—and $|X_t| < \sqrt{n} \log n$ (say; there is plenty of room here), then we have $\mathfrak{A}_t$. For with these assumptions we have (using (12) and $\sum \gamma_i = O(n \log n)$, the latter from (14), and with sums over $i \in [t]$)

$$\begin{aligned} \sum \xi_i &= X_t + \sum \mathbb{E}[\xi_i | A_1, \dots, A_{i-1}] \\ (20) \quad &\leq X_t + (1 + n^{-c}) \sum \gamma_i = \sum \gamma_i + n^{1-c} \end{aligned}$$

(recall the c 's needn't agree); while $\xi_i = O(\gamma_i)$ for $i \leq t$ (see (17) and recall we have $\mathfrak{B}_{i-1}$) gives (using (11))

$$\begin{aligned} \sum_{i=1}^t \xi_i^2 &= O(\sum_{i=1}^t \gamma_i^2) \\ (21) \quad &< O((n/r)^2 \sum \{j^{-2} : j > m_T\}) = O(n/\log n). \end{aligned}$$

Thus, using (7) (and $\xi_i = o(1)$, as follows from (11) and (17)), we have

$$\log \Phi_t > \log \Phi_0 - \sum (\xi_i + \xi_i^2) > \log \Phi_0 - \sum \gamma_i - O(n/\log n)$$

(where the $O(n/\log n)$ absorbs the smaller error in (20)).

Thus the first failure, if any, of an $\mathfrak{A}_t$ must occur either because X_t is too large or because $\mathfrak{B}_i \mathfrak{R}_i$ fails for some $i < t$; formally, we have

$$(22) \quad \mathbb{P}(\cup_{i \leq t} \overline{\mathfrak{A}}_i < \mathbb{P}(\cup_{i < t} \overline{\mathfrak{R}}_i) + \sum_{i < t} \mathbb{P}(\mathfrak{A}_i \mathfrak{R}_i \overline{\mathfrak{B}}_i) + \sum_{i \leq t} \mathbb{P}((\cap_{j < i} \mathfrak{B}_j \mathfrak{R}_j) \cap \overline{\mathfrak{A}}_i).$$

Here we have already promised in (8) that the first term is $o(1)$; the last is $n^{-\omega(1)}$ by (19) and the discussion following it; and we will show—this is the main point—

$$(23) \quad \text{for } i < T, \quad \mathbb{P}(\mathfrak{A}_i \mathfrak{R}_i \overline{\mathfrak{B}}_i) = n^{-\omega(1)}.$$

Thus the l.h.s. of (22) is $o(1)$, which in particular gives (16) and, as already discussed, Theorem 1.5. $\square$

Outline. The structure of our central argument is described in the easy Section 8, and an early look at that, with the (even easier) Section 7, might be helpful. Here we briefly list contents of the sections and then say a little about the comparison with [23].

After recalling a few large deviation facts, Section 3 records what we need in the way of martingale concentration, in a form convenient for a second application in Section 10, and gives the calculation for (19). Section 4 develops some reasonably simple machinery for dealing with $\beta(\mathcal{H})$'s, the main point being the comparisons of Lemma 4.5. Section 5 introduces the rather long list of requirements for the property $\mathfrak{R}$, with support for (8) mostly postponed to Section 13. Section 6 proves (12), a first application of Lemma 4.5. Section 7 finally defines the central property $\mathfrak{B}$, slightly reformulates (23) (as (86)), and disposes of the trivial (17). The next five sections are then devoted to the proof of (86) (which, as noted following (23), completes the proof of Theorem 1.5), as follows.

Section 8 introduces a few auxiliary properties, with assertions concerning them—Lemmas 8.1–8.4—that together easily imply (86). Lemmas 8.1 and 8.4 are from [23] and are just quoted here; the latter is easy, but the entropy-based Lemma 8.1 was a key ingredient in the earlier paper (and is again here), being a first improvement on [22] that opens the door to the rest. Lemma 8.3 is proved in Section 9. (It is here that the $\Omega(\log n)$ lower bound on degrees provided by $\mathfrak{L}$ becomes crucial; see following (101).) Lemma 8.2, which may be considered the core of the whole business, is proved in Sections 10–12, with Section 10 mostly setting out what needs to be done and Sections 11–12 doing it.

Finally, as mentioned above, Section 13 is concerned with justifying (8), with Sections 13.1–13.2 mainly developing machinery and Section 13.3 applying it. (Sections 13.1–13.2 are largely self-contained and maybe amusing in themselves.)

All that in idea seemed simple became in practice immediately complex; as the waves shape themselves symmetrically from the cliff top, but to the swimmer among them are divided by steep gulfs, and foaming crests.

Virginia Woolf, *To the Lighthouse*

The basic difference between this paper and [23] is that $\mathcal{H}_t$ is now chosen according to Observation 2.1, rather than uniformly from $\mathcal{K}^t$; in a sense all we are doing here is dealing with difficulties occasioned by that change. This has to date proved harder than one might wish (and maybe harder than it needs to be, given that there is quite a lot of room in some of the arguments). We briefly summarize similarities and differences.

As already mentioned, the above sketch is similar to the one in [23, Sec. 2]. The easy Section 3 is nearly the same as [23, Sec. 3] (and gets to skip a couple proofs given there). Sections 4 and 6 have no counterparts in [23]; and Sections 5 and 13 have only a faint, routine echo in the parts of [23] (Section 5 and the appendix) that deal with the present $\mathfrak{R}^0$ (there called $\mathfrak{R}$).

The (central) parts of the argument in the remaining sections have rough parallels in [23], Sections 7-9 (here) corresponding to Sections 5-7 (there) and Sections 10-12 to Section 9. (Sections 4 and 8 of [23] prove the present Lemma 8.1 and correspond to nothing here.) The biggest changes are in Sections 9-11; it is here that we see most clearly the difference between handling the present $\mathcal{H}_t$'s and those of [23], which are ordinary $\mathcal{H}_{n,m}$'s. An additional complication is that we must also deal with $(\mathcal{H} - Z)$'s (with $Z \in \mathcal{K}$), which in our setting—unlike in [23] where they are again $\mathcal{H}_{n,m}$'s—are different from, and trickier than, the already fairly nasty $\mathcal{H}_t$'s. See e.g. the proof of Lemma 8.2, in particular the parallel setup at the beginning of Section 10, and then the arguments of Section 11, which think mainly of $\mathcal{G} = \mathcal{H} - Z$ and simplify considerably when $\mathcal{G} = \mathcal{H}$. (The present Section 12 is just far enough from the corresponding portion of [23, Sec. 9] that it seems necessary to repeat.)

3. CONCENTRATION

Recall that a r.v. ξ is *hypergeometric* if, for some s, a and k , it is distributed as $|X \cap A|$, where A is a fixed a -subset of the s -set S and X is uniform from $\binom{S}{k}$. For the standard bounds in Theorem 3.1, see e.g. [21, Theorems 2.1 and 2.10].

Theorem 3.1. *If ξ is binomial or hypergeometric with $\mathbb{E}\xi = \mu$, then for $t \geq 0$,*

$$(24) \quad \Pr(\xi \geq \mu + t) \leq \exp[-\mu\varphi(t/\mu)] \leq \exp[-t^2/(2(\mu + t/3))],$$

$$(25) \quad \Pr(\xi \leq \mu - t) \leq \exp[-\mu\varphi(-t/\mu)] \leq \exp[-t^2/(2\mu)],$$

where $\varphi(x) = (1+x)\log(1+x) - x$ for $x > -1$ and $\varphi(-1) = 1$.

For larger deviations the following consequence of the finer bound in (24) is helpful.

Theorem 3.2. *For ξ and μ as in Theorem 3.1 and any K ,*

$$\Pr(\xi > K\mu) < \exp[-K\mu \log(K/e)].$$

The next result, proved in [17], will save us some trouble at one point (see Lemma 13.8). Say the $\{0, 1\}$ -valued r.v.'s $\zeta_1, \dots, \zeta_t$ are a *read- k family* if there are independent r.v.'s $\psi_1, \dots, \psi_s$ such that each ζ_j is a function of $(\psi_i : i \in S_j)$ (for some $S_j \subseteq [s]$) and $|\{j : i \in S_j\}| \leq k$ for each $i \in [s]$. (That is, each ψ_i affects at most k of the ζ_j 's.)

Theorem 3.3. For $\zeta_1, \dots, \zeta_t$ a read- k family with $\sum \mathbb{E}\zeta_j \leq t\rho$, and any $\lambda \in [\rho, 1]$,

$$\mathbb{P}(\zeta_1 + \dots + \zeta_t > \lambda t) < \exp[-D(\lambda|\rho)t/k],$$

where $D(\lambda|\rho) = \lambda \log(\lambda/\rho) + (1 - \lambda) \log[(1 - \lambda)/(1 - \rho)]$.

(The statement in [17] assumes $\sum \mathbb{E}\zeta_j = t\rho$, but implies the present version since $D(\lambda|\rho)$ is decreasing in $\rho \leq \lambda$. We will not need the similar lower tail bound.)

We now turn to martingales and (19). Lemma 3.4 and Proposition 3.5, which we use here and again in Section 12, are Lemma 3.3 and Proposition 3.4 of [23] and will not be reproved here. The argument for (19) is really the same as that for (17) in [23], but is superficially different enough that it seems best to repeat it.

Lemma 3.4. If $Z_1, \dots, Z_t$ is a martingale difference sequence with respect to the random sequence $Y_1, \dots, Y_t$ (that is, Z_i is a function of $Y_1, \dots, Y_i$ and $\mathbb{E}[Z_i|Y_1, \dots, Y_{i-1}] = 0$), then for $Z = \sum Z_i$ and any $\vartheta > 0$,

$$(26) \quad \mathbb{E}e^{\vartheta Z} \leq \prod_{i=1}^t \max \mathbb{E}[e^{\vartheta Z_i} | y_1, \dots, y_{i-1}]$$

and, consequently, for any $\lambda > 0$,

$$(27) \quad \mathbb{P}(Z > \lambda) < e^{-\vartheta \lambda} \prod_{i=1}^t \max \mathbb{E}[e^{\vartheta Z_i} | y_1, \dots, y_{i-1}]$$

(where y_i ranges over possibilities for Y_i and the conditioning in (26) and (27) has the natural meaning).

Both here and in Section 12, bounds on the factors in (26) are given by the next observation.

Proposition 3.5. For a r.v. $W \in [0, b]$ with $\mathbb{E}W \leq a$ and $\vartheta \in (0, (2b)^{-1}]$,

$$(28) \quad \max\{\mathbb{E}e^{\vartheta(W - \mathbb{E}W)}, \mathbb{E}e^{-\vartheta(W - \mathbb{E}W)}\} \leq e^{\vartheta^2 ab}.$$

Proof of (19). Let $\varsigma_i = O(\gamma_i)$ be the bound on ξ_i in (17). We will apply Lemma 3.4 with $Y_i = A_i$ and Z_i as in (18) (so $Z = X_t$), using Proposition 3.5 with $b = \varsigma_i$ and $a = a_i$ the bound in (12) (with $t = i$; thus $a = (1 + n^{-c})\gamma_i$) to bound the factors in (27) (or (26)). (For relevance of the proposition notice that, conditioned on any particular values $A_1, \dots, A_{i-1}$, Z_i is either identically zero (as happens if $\mathfrak{B}_j \mathfrak{R}_j$ has failed for some $j < i$) or $Z_i = \xi_i - \mathbb{E}[\xi_i | A_1, \dots, A_{i-1}]$, where $\xi_i \in [0, \varsigma_i]$ and (12) bounds the conditional expectation by a_i .) This combination (i.e. of Lemma 3.4 and Proposition 3.5) gives

$$\mathbb{P}(X_t > \lambda) < \exp[\vartheta^2 \sum_{i=1}^t \varsigma_i a_i - \vartheta \lambda]$$

for any $\lambda > 0$, provided, say, $\vartheta \leq 1$ ($\leq (2 \max \varsigma_i)^{-1}$). So with

$$J = \sum_{i=1}^t \varsigma_i a_i = O(\sum \gamma_i^2) = O(n/\log n)$$

(see (21)) and $\vartheta = \min\{1, \lambda/(2J)\}$, we have

$$\Pr(X_t > \lambda) < \begin{cases} \exp[-\lambda^2/(4J)] & \text{if } \lambda \leq 2J, \\ \exp[-\lambda/2] & \text{otherwise,} \end{cases}$$

and (19) follows (using $\lambda \gg \sqrt{n}$ and, in the first case, $J = O(n/\log n)$). $\square$

4. COMPARING β 'S

A basic difficulty in the present work is that we don't know how to estimate the probabilities $\beta(\mathcal{H})$. As a substitute, this section develops some simple machinery for *comparing* $\beta(\mathcal{H})$'s that will allow us to say, roughly, that under reasonable restrictions, small changes in $\mathcal{H}$ don't cause unmanageable changes in $\beta(\mathcal{H})$.

In general our lives are simpler if we can work with "binomial" surrogates for $\mathcal{U}_{\mathcal{H}}$ (recall from (6) that this is a uniform m_T -subset of $\mathcal{H}$). Set

$$(29) \quad \delta = \varepsilon \log(3/\varepsilon).$$

For perspective note that for $\mathcal{U} = \mathcal{U}_{\mathcal{K}}$ (and its binomial relatives below), any $x \in V$ (and large enough n),

$$(30) \quad \mathbb{P}(d_{\mathcal{U}}(x) < \delta_x) < n^{-1+\delta},$$

since the first, more precise part of (25) bounds the l.h.s. of (30) by $\exp[-(1 - \varepsilon \log(e/\varepsilon) + o(1)) \log n]$. (We note that δ is not meant to suggest δ_x .)

Let

$$(31) \quad \varrho = n^{-(1-\lambda)/2}, \text{ with } \lambda > \delta \text{ small and fixed}$$

(for concreteness we may take $\lambda = 2\delta$, but the actual value barely matters), and for $\mathcal{H} \subseteq \mathcal{K}$ with $|\mathcal{H}| = m \geq (1 + \varrho)m_T$, define

$$(32) \quad \mathcal{X}_{\mathcal{H}} = \mathcal{H}_p, \quad \mathcal{Z}_{\mathcal{H}} = \mathcal{H}_q,$$

where $p = (1 + \varrho)m_T/m$ and $q = (1 - \varrho)m_T/m$. (Recall that $\mathcal{H}_p$ includes edges of $\mathcal{H}$ independently, with probability p . The restriction on m is needed to keep p below 1; of course $\mathcal{Z}_{\mathcal{H}}$ could be defined more generally.) We will compare $\mathcal{U}_{\mathcal{H}}$, $\mathcal{X}_{\mathcal{H}}$ and $\mathcal{Z}_{\mathcal{H}}$, with the basic observations in Lemma 4.2 and the main point of the section Lemma 4.5.

Set

$$(33) \quad \xi = \exp[-\varrho^2 m_T/3] (= n^{-\Omega(n^\lambda)}).$$

This value, which is far smaller than we will really need it to be, is chosen so that (by Theorem 3.1)

$$(34) \quad \max\{\mathbb{P}(|\mathcal{Z}| > m_T), \mathbb{P}(|\mathcal{X}| < m_T)\} < \xi.$$

Before turning to the main business of this section we record one crude observation that will sometimes allow us to more or less ignore very small values of m_t (recall $\mathfrak{L}$ was defined in (4)):

Lemma 4.1. *If $\mathcal{H} \in \mathfrak{L}$, $|\mathcal{H}| = m < (1 + n^{-(2\delta+c)})m_T$, and*

$$(35) \quad |\{x : d_{\mathcal{H}}(x) < 1.5\varepsilon \log n\}| = O(n^{2\delta}),$$

then $\beta(\mathcal{H}) > 1 - n^{-c}$.

Proof. Set $\mathcal{U} = \mathcal{U}_{\mathcal{H}}$ and let X be the set in (35). If $\mathcal{U} \notin \mathfrak{L}$ then $\mathcal{H} \setminus \mathcal{U}$ must contain either

- (i) an edge on some $x \in X$, or
- (ii) for some $x \in V \setminus X$, at least $1.5\varepsilon \log n - \delta_x > 0.4\varepsilon \log n$ of any given $1.5\varepsilon \log n$ edges of $\mathcal{H}_x$;

and the probability that one of these occurs is at most

$$O(n^{2\delta} \log n \cdot n^{-(2\delta+c)}) + n \binom{1.5\epsilon \log n}{0.4\epsilon \log n} n^{-(2\delta+c)0.4\epsilon \log n} < n^{-c}.$$

□

Call $\mathcal{J} \subseteq \mathcal{K}$ *anemic* if

$$(36) \quad \mathcal{J} \text{ has at least } 2rn^{2\delta} \text{ vertices of degree less than } 2\epsilon \log n.$$

In Lemma 4.2 and Corollary 4.3, $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Z}$ are $\mathcal{U}_{\mathcal{G}}$, $\mathcal{X}_{\mathcal{G}}$ and $\mathcal{Z}_{\mathcal{G}}$ for some $\mathcal{G} \subseteq \mathcal{K}$ of size at least $(1+\varrho)m_T$, and $\mathfrak{S}$ is a property of r -graphs. As in much of this work, there is room in the bounds—also, e.g., in the definition of anemic—and we aim for (relative) simplicity rather than anything like optimality.

Lemma 4.2. (a) *If $\mathfrak{S}$ is increasing, then*

$$(37) \quad \mathbb{P}(\mathcal{Z} \in \mathfrak{S}) - \xi \leq \mathbb{P}(\mathcal{U} \in \mathfrak{S}) \leq (1 - \xi)^{-1} \mathbb{P}(\mathcal{X} \in \mathfrak{S}).$$

(b) *Suppose $\mathfrak{S}$ is increasing and membership of $\mathcal{J}$ in $\mathfrak{S}$ is determined by $\{x : d_{\mathcal{J}}(x) \geq \rho_x\}$, with $\rho := \max \rho_x \leq 1.5\epsilon \log n$. If*

$$(38) \quad \mathbb{P}(\mathcal{X} \text{ anemic}) < \eta \mathbb{P}(\mathcal{X} \in \mathfrak{S}),$$

then $\mathbb{P}(\mathcal{Z} \in \mathfrak{S}) > (1 - \eta - n^{-c}) \mathbb{P}(\mathcal{X} \in \mathfrak{S})$.

(Note (38), though we will need to check it whenever we use (b), is not much of a requirement.)

Proof. (a) Since $\mathfrak{S}$ is increasing, we have

$$\mathbb{P}(\mathcal{Z} \in \mathfrak{S}) \leq \mathbb{P}(|\mathcal{Z}| > m_T) + \mathbb{P}(\mathcal{U} \in \mathfrak{S})$$

(yielding the first inequality in (37)) and

$$\mathbb{P}(\mathcal{X} \in \mathfrak{S}) \geq \mathbb{P}(|\mathcal{X}| \geq m_T) \mathbb{P}(\mathcal{U} \in \mathfrak{S})$$

(yielding the second).

(b) We couple $\mathcal{X}$ and $\mathcal{Z}$ in the usual way: $\mathcal{Z} = \mathcal{X}_{1-\varsigma}$, with $\varsigma = 1 - (1 - \varrho)/(1 + \varrho) < 2\varrho$. The desired inequality is then

$$(39) \quad \mathbb{P}(\mathcal{Z} \in \mathfrak{S} | \mathcal{X} \in \mathfrak{S}) > 1 - \eta - n^{-c}.$$

From (38) we have

$$\mathbb{P}(\mathcal{X} \text{ anemic} | \mathcal{X} \in \mathfrak{S}) \leq \mathbb{P}(\mathcal{X} \text{ anemic}) / \mathbb{P}(\mathcal{X} \in \mathfrak{S}) < \eta.$$

On the other hand, $\{\mathcal{X} \in \mathfrak{S}, \mathcal{Z} \notin \mathfrak{S}\}$ implies existence of an x for which

$$(40) \quad d_{\mathcal{Z}}(x) < \rho_x \leq d_{\mathcal{X}}(x),$$

meaning that the passage from $\mathcal{X}$ to $\mathcal{Z}$ deletes at least $d_{\mathcal{X}}(x) - \rho_x + 1$ members of $\mathcal{X}_x$. But the probability of deleting at least k edges at x is at most $\binom{d_{\mathcal{X}}(x)}{k} \varsigma^k$; so whenever $\mathcal{X}$ is non-anemic, the probability that (40) occurs for some x is less than

$$2rn^{2\delta} \rho \varsigma + n \binom{2\epsilon \log n}{0.5\epsilon \log n} \varsigma^{0.5\epsilon \log n}.$$

Combining these observations gives (39):

$$\begin{aligned} \mathbb{P}(\mathcal{Z} \in \mathfrak{S} | \mathcal{X} \in \mathfrak{S}) &\geq \mathbb{P}(\mathcal{X} \text{ non-anemic} | \mathcal{X} \in \mathfrak{S}) \mathbb{P}(\mathcal{Z} \in \mathfrak{S} | \mathcal{X} \in \mathfrak{S}, \mathcal{X} \text{ non-anemic}) \\ &> 1 - \eta - n^{-c}. \end{aligned}$$

□

In Corollary 4.3 and Lemma 4.5 we use $a \succ b$ for $a > (1 - n^{-c})b$.

Corollary 4.3. *If $\mathfrak{S}$ is as in Lemma 4.2(b) and*

$$(41) \quad \mathbb{P}(\mathcal{X} \in \mathfrak{S}) > n^c \max\{\xi, \mathbb{P}(\mathcal{X} \text{ anemic})\},$$

then

$$(42) \quad \mathbb{P}(\mathcal{Z} \in \mathfrak{S}) \succ \mathbb{P}(\mathcal{U} \in \mathfrak{S}) \succ \mathbb{P}(\mathcal{X} \in \mathfrak{S}) \ (\geq \mathbb{P}(\mathcal{Z} \in \mathfrak{S})).$$

(Like (38), (41) should be considered a minor annoyance.)

Proof. This is three applications of Lemma 4.2: the second part of (a) of the lemma gives $\mathbb{P}(\mathcal{X} \in \mathfrak{S}) \succ \mathbb{P}(\mathcal{U} \in \mathfrak{S})$; (b), with the second bound in (41), gives $\mathbb{P}(\mathcal{Z} \in \mathfrak{S}) \succ \mathbb{P}(\mathcal{X} \in \mathfrak{S})$; and combining this with the first bound in (41), and using the first part of (a), gives $\mathbb{P}(\mathcal{U} \in \mathfrak{S}) \succ \mathbb{P}(\mathcal{Z} \in \mathfrak{S})$. $\square$

Corollary 4.4. $\beta(\mathcal{K}) \gtrsim \exp[-n^\delta]$.

(In fact $\beta(\mathcal{K}) \sim \exp[-n^\delta]$, but we don't need this.)

Proof. With $\mathcal{Z} = \mathcal{Z}_{\mathcal{K}}$, Theorem 3.1 gives

$$\mathbb{P}(d_{\mathcal{Z}}(x) < \delta_x) < n^{-1+\delta}$$

(the calculation, which is essentially the same as that for (30), is valid provided $\varrho = o(1)$), and combining this with Harris' Inequality [19] yields

$$\mathbb{P}(\mathcal{Z} \in \mathfrak{L}) > (1 - n^{-1+\delta})^n \sim \exp[-n^\delta].$$

The corollary then follows from Lemma 4.2(a) (with $\rho_x = \delta_x$ and $\mathfrak{S} = \mathfrak{L}$, the subtracted ξ in (37) being obviously irrelevant here). $\square$

Say x is *dangerous* for $\mathcal{J} (\subseteq \mathcal{K})$ if

$$(43) \quad d_{\mathcal{J}}(x) < 1.5\varepsilon D_{\mathcal{J}}$$

(recall $D_{\mathcal{J}}$ is average degree in $\mathcal{J}$). For the next lemma we assume $\mathcal{J}, \mathcal{J}' \subseteq \mathcal{K}, \mathcal{E} = \mathcal{J} \cap \mathcal{J}'$,

$$(44) \quad |\mathcal{J}| = |\mathcal{J}'| = m \geq (1 + \varrho)m_T,$$

$$\mathcal{J} \setminus \mathcal{E} = \{A_1, \dots, A_\sigma\} \text{ and } \mathcal{J}' \setminus \mathcal{E} = \{B_1, \dots, B_\sigma\},$$

with

$$(45) \quad \{A_1, \dots, A_\sigma\} \text{ a matching}$$

and

$$(46) \quad (1 \leq) \sigma = n^{o(1)};$$

let κ be the number of A_i 's containing vertices that are dangerous for $\mathcal{J}$; and set $W = A_1 \cup \dots \cup A_\sigma$, noting that

$$(47) \quad |W| = \sigma r \text{ and } |\{x \in W : x \text{ dangerous for } \mathcal{J}\}| \leq \kappa r.$$

Lemma 4.5. *With the above setup, assume: $\mathcal{J}' \in \mathfrak{L}$;*

$$(48) \quad \text{if } m > 2rm_T \text{ then } d_{\mathcal{E}}(x) \geq 1.5\varepsilon D_{\mathcal{J}} \quad \forall x \in W$$

(so $\kappa = 0$); and

$$(49) \quad \beta(\mathcal{J}) > n^{\Omega(\sigma)} \max\{\xi, \mathbb{P}(\mathcal{X}_{\mathcal{J}} \text{ anemic})\},$$

where the implied constant doesn't depend on ε . Then

$$(50) \quad \beta(\mathcal{J}') \succ n^{-O(\kappa\varepsilon)} \beta(\mathcal{J})$$

(where, of course, the implied constant is again universal).

Notes. The assumptions in (46) and (49), while supporting the lemma, are much weaker than what we will have when we come to use it; on the other hand, the ε in (50) will be critical at one point (see (137)). Of course the " $\succ$ " (vs. " $>$ ") in (50) is unnecessary if $\kappa \neq 0$. (It is only in our first application of Lemma 4.5—in the proof of (12) in Section 6—that the precision of " $\succ$ " in (50) is needed.)

Proof. Recalling that $\beta(\mathcal{H}) = \mathbb{P}(\mathcal{U}_{\mathcal{H}} \in \mathfrak{L})$, we first observe that it suffices to show (50) with $\mathcal{Z}$ in place of $\mathcal{U}$; that is,

$$(51) \quad \mathbb{P}(\mathcal{Z}_{\mathcal{J}'} \in \mathfrak{L}) \succ n^{-O(\kappa\varepsilon)} \mathbb{P}(\mathcal{Z}_{\mathcal{J}} \in \mathfrak{L}).$$

To see that this is enough, notice that Corollary 4.3, with $\mathfrak{S} = \mathfrak{L}$ (and $\rho_x = \delta_x$) and (41) given by (49) and Lemma 4.2(a) (the latter to say $\mathbb{P}(\mathcal{X}_{\mathcal{J}} \in \mathfrak{S}) \gtrsim \beta(\mathcal{J})$), gives

$$(52) \quad \mathbb{P}(\mathcal{Z}_{\mathcal{J}} \in \mathfrak{L}) \succ \beta(\mathcal{J});$$

that this with (51) and (49) gives

$$\mathbb{P}(\mathcal{Z}_{\mathcal{J}'} \in \mathfrak{L}) > n^{-O(\kappa\varepsilon) + \Omega(\sigma)} \xi \quad (> n^c \xi),$$

which by Lemma 4.2(a) gives

$$(53) \quad \beta(\mathcal{J}') \succ \mathbb{P}(\mathcal{Z}_{\mathcal{J}'} \in \mathfrak{L});$$

and, finally, that the combination of (53), (51) and (52) gives (50).

For the proof of (51) set $\mathcal{Z}_{\mathcal{J}} = \mathcal{Z}$ and $\mathcal{Z}_{\mathcal{J}'} = \mathcal{Z}'$. We may couple these by taking $\mathcal{Z}' \cap \mathcal{E} = \mathcal{Z} \cap \mathcal{E}$, with the remaining decisions (those involving the A_i 's and B_i 's) made independently, and show a mild strengthening of (51):

$$(54) \quad \mathbb{P}(\mathcal{Z}' \in \mathfrak{L} | \mathcal{Z} \in \mathfrak{L}) \succ n^{-O(\kappa\varepsilon)}.$$

Here Harris' Inequality gives

$$(55) \quad \begin{aligned} \mathbb{P}(\mathcal{Z}' \in \mathfrak{L} | \mathcal{Z} \in \mathfrak{L}) &= \mathbb{P}(d_{\mathcal{Z}'}(x) \geq \delta_x \quad \forall x \in W | \mathcal{Z} \in \mathfrak{L}) \\ &\geq \prod_{x \in W} \mathbb{P}(d_{\mathcal{Z}'}(x) \geq \delta_x). \end{aligned}$$

On the other hand, now using $\mathcal{J}' \in \mathfrak{L}$ and (48) (the latter just to say that if the first case in (56) is not vacuous then $q = (1 - \varrho)m_T/m = \Omega(1)$), we have, for $x \in W$,

$$(56) \quad \mathbb{P}(d_{\mathcal{Z}'}(x) \geq \delta_x) \geq \begin{cases} q^{\delta_x} = n^{-O(\varepsilon)} & \text{if } d_{\mathcal{E}}(x) < 1.5\varepsilon D_{\mathcal{J}} - 1, \\ 1 - n^{-\Omega(\varepsilon)} & \text{otherwise.} \end{cases}$$

Here the second bound is given by Theorem 3.1 since $\mathbb{E}d_{\mathcal{Z}'}(x) \gtrsim 1.5\varepsilon \log n$, as follows from $d_{\mathcal{J}'}(x) \geq d_{\mathcal{E}}(x) \geq 1.5\varepsilon D_{\mathcal{J}} - 1$. Finally, inserting the bounds from (56) in (55), noting that (45) implies that all x 's in the first part of (56) are dangerous for $\mathcal{J}$, and using (46) and (47), gives (54). $\square$

5. GENERICS

Here we define the property $\mathfrak{R}$, with most of the discussion supporting (8) postponed to Section 13. As we will do elsewhere (see Sections 7, 8 and 10), we give the definition for a general r -graph $\mathcal{H}$, with $n := |V(\mathcal{H})|$ and $m := |\mathcal{H}| \gg n$, so average degree

$$D_{\mathcal{H}} = mr/n.$$

The event $\mathfrak{R}_t$ of Section 2 is then $\{\mathcal{H}_t \in \mathfrak{R}\}$. (In line with our “default,” we may think of $t = \binom{n}{r} - m$, but, apart from a pseudo-exception at (61), the present definitions don’t involve t .)

We number the several requirements for $\mathfrak{R}$ —thus $\mathfrak{R} = \mathfrak{R}^0 \cap \dots \cap \mathfrak{R}^4$ (with, again, $\mathfrak{R}_t^i = \{\mathcal{H}_t \in \mathfrak{R}^i\}$)—to allow later pointers to what exactly is being used. The first of these, $\mathfrak{R}^0$ consists of standardish genericity conditions for degrees; *viz.*

$$(57) \quad \text{a.a. degrees in } \mathcal{H} \text{ are asymptotic to } D_{\mathcal{H}}$$

(formally: there is $\varsigma = o(1)$ such that $d_{\mathcal{H}}(x) = (1 \pm \varsigma)D_{\mathcal{H}}$ for all but $(1 - \varsigma)n$ vertices x);

$$(58) \quad \Delta_{\mathcal{H}} = O(D_{\mathcal{H}}), \delta_{\mathcal{H}} = \Omega(\varepsilon D_{\mathcal{H}});$$

and

$$(59) \quad \max d_{\mathcal{H}}(x, y) = o(D_{\mathcal{H}})$$

(the max over distinct $x, y \in V$).

Note that $\mathfrak{R}^0$ is robust in that, for any fixed C ,

$$(60) \quad \text{if } \mathcal{H} \text{ satisfies } \mathfrak{R}^0 \text{ then so does } \mathcal{H} - Z \text{ for each } Z \subseteq V \text{ with } |Z| \leq C.$$

(Note—though it doesn’t matter—this refers to a slightly changed n and m .) We omit the easy justification, just noting that (58) implies $D_{\mathcal{H}-Z} \sim D_{\mathcal{H}}$ and that (57) and (the lower bound in) (58) for $\mathcal{H} - Z$ depend on having (59) for $\mathcal{H}$. We will only use (60) with Z a member of $\mathcal{K}$ or the union of two disjoint members.

For the next item, m_T is the $M = M(n)$ of Theorem 1.5 (so the definition makes sense for a general $\mathcal{H}$, though we will use it only with $\mathcal{H} = \mathcal{H}_t$). Say $\mathcal{H}$ is in $\mathfrak{R}^1$ provided that

$$(61) \quad \text{if } |\mathcal{H}| > 2rm_T \text{ then } \delta_{\mathcal{H}} > 2\varepsilon D_{\mathcal{H}}.$$

(So—irrelevantly—smaller $\mathcal{H}$ are automatically in $\mathfrak{R}^1$. This item is minor but will sometimes allow us to focus on $|\mathcal{H}|$ closer to m_T , which is where most of the interest lies. Of course for $\mathcal{H} = \mathcal{H}_t$ (61) implies the second part of (58), since $\mathcal{H}_t \in \mathfrak{L}$.)

That

$$(62) \quad \mathbb{P}(\cap_{t \leq T} (\mathfrak{R}_t^0 \cap \mathfrak{R}_t^1)) \rightarrow 1$$

is shown in Section 13.3.

For $Z \subseteq V$ (and $\mathcal{H} \subseteq \mathcal{K}$), set

$$(63) \quad \mathfrak{J}_Z(\mathcal{H}) = \{\mathcal{H}' \subseteq \mathcal{K} : |\mathcal{H}'| = |\mathcal{H}|, \mathcal{H}' - Z = \mathcal{H} - Z\},$$

$$(64) \quad \varphi_Z(\mathcal{H}) = \sum \{\beta(\mathcal{H}') : \mathcal{H}' \in \mathfrak{J}_Z(\mathcal{H})\}$$

and

$$(65) \quad \bar{\varphi}_Z(\mathcal{H}) = |\mathfrak{J}_Z(\mathcal{H})|^{-1} \varphi_Z(\mathcal{H})$$

(the average of $\beta(\mathcal{H}')$ over $\mathcal{H}'$ of size $|\mathcal{H}|$ agreeing with $\mathcal{H}$ off Z ; note each of the items in (63)-(65) is determined by $|\mathcal{H}|$ and $\mathcal{H} - Z$). Let

$$(66) \quad \mathfrak{R}^2 = \{\mathcal{H} : \bar{\varphi}_Z(\mathcal{H}) > n^{-(2r+1)} \beta(\mathcal{K}) \ \forall Z \in \{\emptyset\} \cup \mathcal{K}\},$$

noting in particular (taking $Z = \emptyset$ and recalling Corollary 4.4) that

$$(67) \quad \mathcal{H} \in \mathfrak{R}^2 \Rightarrow \beta(\mathcal{H}) > n^{-(2r+1)} \beta(\mathcal{K}) > \exp[-(1+o(1))n^\delta].$$

In Section 13.3 we will show that for any $Z \subseteq V$ (and any $t \in [T]$ and $\eta > 0$),

$$(68) \quad \mathbb{P}(\bar{\varphi}_Z(\mathcal{H}_t) < \eta \beta(\mathcal{K})) < \eta,$$

whence

$$(69) \quad \mathbb{P}(\exists t \ \mathcal{H}_t \notin \mathfrak{R}^2) < n^{-1} (= o(1)).$$

With $\alpha = m_T/m$ and, as in (36), $\mathcal{J}$ anemic if $d_{\mathcal{J}}(v) < 2\varepsilon \log n$ for at least $2rn^{2\delta}$ vertices v , let

$$\mathfrak{R}^3 = \{\mathcal{H} : \mathbb{P}(\mathcal{H}_\alpha \text{ anemic}) < \exp[-(1-o(1))n^{2\delta}]\}.$$

Again in Section 13.3, we will show

$$(70) \quad \mathbb{P}(\mathcal{H}_t \notin \mathfrak{R}^3) < \exp[-(1-o(1))n^{2\delta}].$$

We will usually use membership in $\mathfrak{R}^3$ in combination with the next little point.

Observation 5.1. For fixed $c > 0$, $\mathcal{H}$ of size asymptotic to m and $\zeta \sim m_T/m$, if

$$(71) \quad \mathbb{P}(\mathcal{H}_\zeta \text{ anemic}) < \exp[-\Omega(n^{2\delta})],$$

then

$$(72) \quad |\{x : d_{\mathcal{H}}(x) < (2-c)\varepsilon D_{\mathcal{H}}\}| < 2rn^{2\delta}.$$

In particular, (72) holds when $\mathcal{H} \in \mathfrak{R}^3$.

Proof. If W is a $(2rn^{2\delta})$ -subset of the set in (72), then Harris' Inequality (with Theorem 3.1) gives

$$\mathbb{P}(\mathcal{H}_\zeta \text{ anemic}) \geq \mathbb{P}(d_{\mathcal{H}_\zeta}(x) < 2\varepsilon \log n \ \forall x \in W) > (1-o(1))^{|W|} = \exp[-o(n^{2\delta})],$$

contradicting (71). □

For $\mathfrak{R}^4$ we use a parameter ω that tends to infinity *slowly*; precisely, we want

$$(73) \quad 1 \ll \omega \ll \nu,$$

where conditions on ν are included in the parameter requirements of (143)-(147). Those requirements could instead have been given here, but really belong in Section 12, where they become relevant. As explained there, they are functions of some $\gamma = o(1)$ that in turn depends on the “quality” of $\mathfrak{A}$ and $\mathfrak{R}^0$ (e.g. the speed of the $o(1)$ in $\mathfrak{A}$), meaning the most we can ask of ν is that it tend to infinity; this leaves room for (73), but no more. (We could also make ω a suitably large constant, but this feels less natural.)

For $Z \subseteq V$, let

$$(74) \quad \mathfrak{D}_Z = \{\mathcal{H} \in \mathcal{K}^T : \sum_{y \in V \setminus Z} (\delta_y - d_{\mathcal{H}-Z}(y))^+ > \omega\},$$

and let $\mathfrak{D} = \cup_{Z \in \mathcal{K}} \mathfrak{D}_Z$ (and $\mathfrak{D}_x = \mathfrak{D}_{\{x\}}$). It is shown in Lemma 13.9 (see (188)) that

$$(75) \quad \mathbb{P}(\mathcal{H}_T \in \mathfrak{D}) < n^{-\eta\omega}$$

for some fixed $\eta > 0$ (not depending on ε , though we don't need this).

For $t \in [T]$ and $Z \in \mathcal{K}$, say $\mathcal{H} \in \mathcal{K}^t$ is in $\mathfrak{R}^4(Z)$ if

$$(76) \quad \mathbb{P}(\mathcal{H}_T \in \mathfrak{D}_Z | \mathcal{H}_t - Z = \mathcal{H} - Z) < n^{-\eta\omega/2},$$

and set $\mathfrak{R}^4 = \cap_{Z \in \mathcal{K}} \mathfrak{R}^4(Z)$. Then for any $Z \in \mathcal{K}$,

$$(77) \quad \begin{aligned} n^{-\eta\omega} &> \mathbb{P}(\mathcal{H}_T \in \mathfrak{D}) \\ &\geq \mathbb{P}(\mathcal{H}_t \notin \mathfrak{R}^4(Z)) \mathbb{P}(\mathcal{H}_T \in \mathfrak{D}_Z | \mathcal{H}_t \notin \mathfrak{R}^4(Z)) \\ &\geq \mathbb{P}(\mathcal{H}_t \notin \mathfrak{R}^4(Z)) n^{-\eta\omega/2}, \end{aligned}$$

implying $\mathbb{P}(\mathcal{H}_t \notin \mathfrak{R}^4(Z)) < n^{-\eta\omega/2}$ and

$$(78) \quad \mathbb{P}(\exists t \mathcal{H}_t \notin \mathfrak{R}^4) < n^{2r} n^{-\omega/2} = n^{-\Omega(\omega)} (= o(1)).$$

(To make sense of (77) notice that membership of $\mathcal{H}_t$ in $\mathfrak{R}^4(Z)$ is decided by $\mathcal{H}_t - Z$, and the conditioning says precisely that (76) does not hold.)

To recap: (8) will follow from (62), (68) (which implies (69)), (70) and (75) (which implies (78)); as noted above, these are all shown in Section 13.

6. EXPECTATIONS

In this section we take $m = m_t$ and prove (12); recall this said (with $\gamma_t = n/(r(m+1))$); see (11))

$$(79) \quad \mathcal{H} \in \mathcal{K}^{t-1} \cap \mathfrak{R} \cap \mathfrak{L} \Rightarrow \mathbb{E}[\xi_t | \mathcal{H}_{t-1} = \mathcal{H}] < (1 + n^{-c})\gamma_t.$$

Given $\mathcal{H}$ as in (79), set, for $A \in \mathcal{H}$,

$$p_A = p_{\mathcal{H}}(A) = \mathbb{P}(A_t = A | \mathcal{H}_{t-1} = \mathcal{H}) \ (\propto \beta(\mathcal{H} \setminus A); \text{ see Observation 2.2})$$

and

$$\xi_A = \xi_{\mathcal{H}}(A) = \Phi(\mathcal{H} - A) / \Phi(\mathcal{H}).$$

Then ξ_A is the fraction of perfect matchings of $\mathcal{H}$ that contain A , and

$$(80) \quad \mathbb{E}[\xi_t | \mathcal{H}_{t-1} = \mathcal{H}] = \sum p_A \xi_A.$$

We will show that no p_A is much more than the average; concretely,

$$(81) \quad p_A < (1 + n^{-c}) / (m + 1) \quad \forall A \in \mathcal{H}.$$

Since $(m+1)^{-1} \sum_{A \in \mathcal{H}} \xi_A = \gamma_t$ (equivalently, $\sum \Phi(\mathcal{H} - A) = (n/r)\Phi(\mathcal{H})$), this bounds the r.h.s. of (80) by $(1 + n^{-c})\gamma_t$, as desired.

Proof of (81). Set

$$\mathcal{H}^0 = \{B \in \mathcal{H} : d_{\mathcal{H}}(x) \geq 1.5\varepsilon D_{\mathcal{H}} + 1 \ \forall x \in B\}$$

and notice that, by Observation 5.1 and (61) (so we use $\mathcal{H} \in \mathfrak{R}^3 \cap \mathfrak{R}^1$),

$$|\mathcal{H}^0| \begin{cases} \geq m + 1 - 3\varepsilon r n^{2\delta} D_{\mathcal{H}} > (1 - n^{-c})(m + 1) & \text{if } m \leq 2rm_T, \\ = m + 1 & \text{otherwise.} \end{cases}$$

This implies that for (81) it is enough to show

$$(82) \quad \beta(\mathcal{H} \setminus A) < (1 + n^{-c})\beta(\mathcal{H} \setminus B) \quad \forall B \in \mathcal{H}^0,$$

since then

$$p_A = \frac{\beta(\mathcal{H} \setminus A)}{\sum_{B \in \mathcal{H}} \beta(\mathcal{H} \setminus B)} \leq \frac{\beta(\mathcal{H} \setminus A)}{\sum_{B \in \mathcal{H}^0} \beta(\mathcal{H} \setminus B)} < (1 + n^{-c})/(m + 1).$$

For $m < (1 + n^{-3\delta})m_T$ (say), (82) is give by Lemma 4.1, according to which $\beta(\mathcal{H} \setminus B) > 1 - n^{-c}$ for any $B \in \mathcal{H}^0$ (the lemma's hypotheses, $\mathcal{H} \setminus B \in \mathfrak{L}$ and (35), following from $B \in \mathcal{H}^0$ (with $\mathcal{H} \in \mathfrak{L}$) and the combination of $\mathcal{H} \in \mathfrak{R}^3$ and Observation 5.1 respectively).

For larger m we show (82) assuming (81) fails (which suffices for our purposes). We have

$$\frac{1}{m+1} \sum_{B \in \mathcal{H}} \beta(\mathcal{H} \setminus B) = \beta(\mathcal{H}) > \exp[-(1 + o(1))n^\delta].$$

[The inequality holds since $\mathcal{H} \in \mathfrak{R}^2$ (see (67)), and for the equality we have, with sums over $B \in \mathcal{H}$ (and $\mathfrak{L}_T = \mathfrak{L} \cap \mathcal{K}^T$),

$$\begin{aligned} \sum \beta(\mathcal{H} \setminus B) &= \binom{m}{m_T}^{-1} \sum |\{\mathcal{U} \in \mathfrak{L}_T : \mathcal{U} \subseteq \mathcal{H} \setminus B\}| \\ &= \binom{m}{m_T}^{-1} (m + 1 - m_T) |\{\mathcal{U} \in \mathfrak{L}_T : \mathcal{U} \subseteq \mathcal{H}\}| = (m + 1)\beta(\mathcal{H}). \end{aligned}$$

We thus have

$$(83) \quad p_A = \frac{\beta(\mathcal{H} \setminus A)}{\sum_{B \in \mathcal{H}} \beta(\mathcal{H} \setminus B)} < \frac{1}{m+1} \frac{\beta(\mathcal{H} \setminus A)}{\exp[-(1 + o(1))n^\delta]}$$

and may assume

$$(84) \quad \beta(\mathcal{H} \setminus A) > \exp[-(1 + o(1))n^\delta],$$

since otherwise (83) implies (81).

Then for (82) we apply Lemma 4.5 with $\mathcal{J} = \mathcal{H} \setminus A$ and $\mathcal{J}' = \mathcal{H} \setminus B$ (so $\mathcal{E} = \mathcal{H} \setminus \{A, B\}$, $\sigma = 1$ and, unfortunately, $A_1 = B$ and $B_1 = A$). Here $B \in \mathcal{H}^0$ implies $\kappa = 0$, so the lemma, if applicable, does give (82); but its first hypothesis, (48), holds because $B \in \mathcal{H}^0$, and its second, (49), is a weak consequence of (84) and $\mathcal{H} \in \mathfrak{R}^3$, which implies $\mathbb{P}(\mathcal{X}_{\mathcal{H}} \text{ anemic}) < \exp[-(1 - o(1))n^{2\delta}]$. (Note Lemma 4.5 also assumes the lower bound in (44), but here we have the stronger $m \geq (1 + n^{-3\delta})m_T$.) $\square$

7. PROPERTIES $\mathfrak{A}$ AND $\mathfrak{B}$

As in Section 5, properties in this section, as well as Sections 8 and 10, are defined for a general r -graph $\mathcal{H}$, and $\mathfrak{S}_t$ is the event $\{\mathcal{H}_t \in \mathfrak{S}\}$. Here and in Section 8—but not quite in Section 10—we again use n and m for the numbers of vertices and edges of $\mathcal{H}$, and $\mathcal{K}$ for $\binom{V(\mathcal{H})}{r}$.

For the remainder of the paper we will tend to use A for edges and Z or U for general r -sets. We assume throughout that we have fixed some positive ε (it will be essentially the one in Theorem 1.5), upon which the implied constants in “ $O(\cdot)$ ” and “ $\Omega(\cdot)$ ” may depend.

We say $\mathcal{H}$ has the property $\mathfrak{A}$ if

$$(85) \quad \log \Phi(\mathcal{H}) > \log \Phi_0 - \frac{n}{r} \log \left[\binom{n}{r} / m \right] - o(n).$$

(Recall from (14) that for $m = m_t$, the main subtracted term here is essentially $\sum_{i=1}^t \gamma_i$; so, as promised, $\{\mathcal{H}_t \in \mathfrak{A}\}$ is the $\mathfrak{A}_t$ of Section 2.)

For $\mathfrak{B}$ a little notation will be helpful. For a finite set S and $w : S \rightarrow \mathbb{R}^+ (:= [0, \infty))$, set

$$\bar{w}(S) = |S|^{-1} \sum_{a \in S} w(a),$$

$$\max w(S) = \max_{a \in S} w(a),$$

and

$$\max_{\mathcal{R}} w(S) = \bar{w}(S)^{-1} \max w(S).$$

For $\mathcal{H} \subseteq \mathcal{K}$ define $w_{\mathcal{H}} : \mathcal{K} \rightarrow \mathbb{R}^+$ by

$$w_{\mathcal{H}}(Z) = \Phi(\mathcal{H} - Z),$$

and say $\mathcal{H}$ has the property $\mathfrak{B}$ if

$$\max_{\mathcal{R}} w_{\mathcal{H}}(\mathcal{H}) = O(1).$$

(So $\mathfrak{B}$ says the number of p.m.s containing any particular $A \in \mathcal{H}$ is not too large compared to the average. Note that the implied constant here *does* depend on ε , its natural value being roughly $1/\varepsilon$: on average over $A \ni x$, the fraction of p.m.s of $\mathcal{H}$ containing A is $1/d_{\mathcal{H}}(x)$; and the $d_{\mathcal{H}}(x)$'s, while typically around $D_{\mathcal{H}}$, can be as small as (about) $\varepsilon D_{\mathcal{H}}$.)

Then $\mathfrak{B}_t (= \{\mathcal{H}_t \models \mathfrak{B}\})$ is as in Section 2 and (23) is

$$(86) \quad \text{for } t < T, \quad \mathbb{P}(\mathcal{H}_t \models \mathfrak{A}\mathfrak{R}\mathfrak{B}) = n^{-\omega(1)}.$$

(More formally: there is a fixed C , depending on the $o(\cdot)$'s and implied constants in $\mathfrak{A}$ and $\mathfrak{R}$, such that $\mathbb{P}(\{\mathcal{H}_t \models \mathfrak{A}\mathfrak{R}\} \wedge \{\max_{\mathcal{R}} w_{\mathcal{H}_t}(\mathcal{H}_t) > C\}) = n^{-\omega(1)}.$)

As mentioned at the end of Section 2, (86) is shown in Sections 8-12, and with (8) (likelihood of the $\mathfrak{R}_t$'s, established in Section 13) will complete the proof of Theorem 1.5.

We conclude this section with the promised

$$(87) \quad \mathfrak{B}_{t-1} \text{ implies (17).}$$

(Recall (17) says $\xi_t = O(\gamma_t)$, where, as in (11), $\gamma_t = n/(r(m+1)).$)

Proof. If $\mathcal{H}_{t-1} = \mathcal{H}$, then $\xi_t \leq \max_{A \in \mathcal{H}} w_{\mathcal{H}}(A)/\Phi(\mathcal{H})$, while γ_t is the average of these ratios, since

$$\sum_{A \in \mathcal{H}} w_{\mathcal{H}}(A) = \Phi(\mathcal{H})n/r$$

(and $|\mathcal{H}| = m+1$). This gives (87). □

8. MORE PROPERTIES

We will get at $\mathfrak{B}$ *via* several auxiliary properties. We introduce the first three of these here (there will be a couple more in Section 10), together with assertions concerning them that, as shown below, easily imply (86). As mentioned earlier, two of these assertions are from [23] and the others (which also have counterparts in [23]) are proved in the next four sections.

With n and $\mathcal{K}$ again the size and collection of r -subsets of $V(\mathcal{H})$, the properties of interest here are:

$\mathfrak{C}$: if $Z \in \mathcal{K}$ satisfies

$$(88) \quad w_{\mathcal{H}}(Z) > \Phi(\mathcal{H})e^{-o(n)},$$

then for any $x \in Z$,

$$(89) \quad w_{\mathcal{H}}((Z \setminus x) \cup y) \gtrsim w_{\mathcal{H}}(Z)d(x)/D_{\mathcal{H}} \text{ for a.e. } y \in V \setminus Z;$$

$\mathfrak{E}$: $w_{\mathcal{H}}(A) \sim \Phi(\mathcal{H})/D_{\mathcal{H}}$ for a.e. $A \in \mathcal{H}$;

$\mathfrak{F}$: $w_{\mathcal{H}}(Z) \sim \Phi(\mathcal{H})/D_{\mathcal{H}}$ for a.e. $Z \in \mathcal{K}$.

(More formally, e.g. for $\mathfrak{E}$: there is $\varsigma = \varsigma(n) = o(1)$ such that $|\{A \in \mathcal{H} : w_{\mathcal{H}}(A) \neq (1 \pm \varsigma)\Phi(\mathcal{H})/D_{\mathcal{H}}\}| < \varsigma|\mathcal{H}|$.) For perspective on $\mathfrak{E}$ and $\mathfrak{F}$, notice that

$$(\overline{w}_{\mathcal{H}}(\mathcal{H}) =) |\mathcal{H}|^{-1} \sum_{A \in \mathcal{H}} w_{\mathcal{H}}(A) = |\mathcal{H}|^{-1} \Phi(\mathcal{H})n/r = \Phi(\mathcal{H})/D_{\mathcal{H}}.$$

For the next little bit we use

$$\mathfrak{X} \xRightarrow{*} \mathfrak{Z}$$

to mean $\mathbb{P}(\mathfrak{X}\overline{\mathfrak{Z}}) = n^{-\omega(1)}$; e.g. the probability bound of (86) is

$$(90) \quad \{\mathcal{H}_t \models \mathfrak{A}\mathfrak{R}\} \xRightarrow{*} \{\mathcal{H}_t \models \mathfrak{B}\}.$$

The aforementioned assertions are as follows. (In Lemmas 8.2 and 8.3 we assume $t \in [T]$.)

Lemma 8.1. *If $\mathcal{H}$ satisfies $\mathfrak{A}\mathfrak{R}^0$ then it satisfies $\mathfrak{E}$.*

Lemma 8.2. *With $\wedge_Z$ ranging over Z as in (88) (with $\mathcal{H} = \mathcal{H}_t$),*

$$\{\mathcal{H}_t \models \mathfrak{A}\mathfrak{R}\} \xRightarrow{*} \{\mathcal{H}_t \models \mathfrak{F}\} \wedge \wedge_Z \{\mathcal{H}_t - Z \models \mathfrak{F}\}.$$

Lemma 8.3. *For $x \in Z \in \mathcal{K}$,*

$$(91) \quad \{\mathcal{H}_t \models \mathfrak{R}\} \wedge \{\mathcal{H}_t - Z \models \mathfrak{F}\} \xRightarrow{*} \{(\mathcal{H}_t, Z, x) \models (89)\}.$$

Lemma 8.4. *If $\mathcal{H}$ satisfies $\mathfrak{R}^0\mathfrak{F}\mathfrak{E}$ then it satisfies $\mathfrak{B}$.*

Lemmas 8.2-8.4 immediately imply (86) (in the form (90)): the first two give

$$\{\mathcal{H}_t \models \mathfrak{A}\mathfrak{R}\} \xRightarrow{*} \{\mathcal{H}_t \models \mathfrak{F}\mathfrak{E}\}$$

and the story is then completed by Lemma 8.4.

Remarks. The crucial contribution of Lemma 8.1 is that it allows us to replace $\mathfrak{A}\mathfrak{R}$ by $\mathfrak{A}\mathfrak{R}\mathfrak{E}$ in Lemma 8.2; more precisely: recall from (60) that $\mathcal{H} \models \mathfrak{R}^0$ implies $\mathcal{H} - Z \models \mathfrak{R}^0$ for every $Z \in \mathcal{K}$, and note that $\mathcal{H} \models \mathfrak{A}$ easily implies $\mathcal{H} - Z \models \mathfrak{A}$ for any Z as in (88); these observations, with Lemma 8.1, say that in proving any of the assertions $\{\mathcal{H}_t \models \mathfrak{A}\mathfrak{R}\} \xRightarrow{*} \{\mathcal{G} \models \mathfrak{F}\}$ of Lemma 8.2, we may replace the l.h.s. by

$$(92) \quad \{\mathcal{H}_t \models \mathfrak{A}\mathfrak{R}\} \wedge \{\mathcal{G} \models \mathfrak{E}\}.$$

Lemma 8.2 then embodies the idea that $\mathfrak{E}\mathfrak{F}$ is unlikely for a random $\mathcal{G}$ (here either $\mathcal{H}_t$ or $\mathcal{H}_t - Z$) because the distribution of the $w_{\mathcal{G}}(A)$'s ($A \in \mathcal{G}$) should reflect that of the $w_{\mathcal{G}}(Z)$'s ($Z \in \mathcal{K}$). As said in [23], we regard this natural point as the heart of our argument.

The more important part of Lemma 8.2 is that involving $\mathcal{H}_t - Z$, which provides input for Lemma 8.3. Its other use, allowing us to assume $\mathfrak{F}$ in Lemma 8.4, is convenient but less critical: with more effort one can show directly that $\{\mathcal{H}_t \models \mathfrak{R}\mathfrak{E}\mathfrak{B}\}$ is unlikely.

The nonprobabilistic Lemmas 8.1 and 8.4 are Lemmas 6.1 and 6.4 of [23] (the $\mathfrak{R}$ there being the present $\mathfrak{R}^0$), and their proofs will not be repeated here. (As mentioned in Section 2, Lemma 8.4 is easy, but Lemma 8.1 was one of the main points of [23].) Lemmas 8.3 and 8.2 are proved (in this reverse order) in Sections 9 and 10-12 respectively.

9. PROOF OF LEMMA 8.3

Fix $x \in Z \in \mathcal{K}$ and let $\mathcal{H} = \mathcal{H}_t$, $Y = Z \setminus x$ and $W = V \setminus Z$. We now use d_x for $d_{\mathcal{H}}(x)$.

Recall (see (5)) that $\mathcal{H} = \mathcal{H}_T \cup \mathcal{J}$ with $\mathcal{H}_T$ uniform from $\mathcal{L}_T$ and $\mathcal{J}$ uniform from the $(T-t)$ -subsets of $\mathcal{K} \setminus \mathcal{H}_T$. This gives a joint distribution on $(\mathcal{G}, \mathcal{G}_T, \mathcal{N}_T, \mathcal{H}_x \setminus \mathcal{N}_T)$, where $\mathcal{G} = \mathcal{H} - x$, $\mathcal{G}_T = \mathcal{G} \cap \mathcal{H}_T$ and $\mathcal{N}_T = \mathcal{H}_T \setminus \mathcal{G}_T (= (\mathcal{H}_T)_x)$. Given $\mathcal{G} = \mathcal{G}$ we choose $\mathcal{G}_T, \mathcal{N}_T, \mathcal{H}_x \setminus \mathcal{N}_T$ (in this order), and set $g = |\mathcal{G}|$ and $g_T = |\mathcal{G}_T|$.

The law of $\mathcal{G}_T$ will not concern us, but the next two observations will be helpful. First, the law of $\mathcal{N}_T$ depends only on $\mathcal{G}_T$: it is uniform measure on

$$(93) \quad \{\mathcal{N} \subseteq \mathcal{K}_x : |\mathcal{N}| = m_T - g_T, \mathcal{N} \cup \mathcal{G}_T \in \mathcal{L}_T\}.$$

Second, $\mathcal{H}_x \setminus \mathcal{N}_T$ is chosen uniformly from the $(m - g - |\mathcal{N}_T|)$ -subsets of $\mathcal{K}_x \setminus \mathcal{N}_T$; in particular its law depends only on $\mathcal{N}_T$ and g .

We are really interested in the choice of $\mathcal{H}_x$, for which we may assume whatever consequences of $\mathcal{H} \in \mathfrak{R}$ can be read off from $(\mathcal{G}, \mathcal{G}_T)$; of these we will use just $D_{\mathcal{H}-Z} \sim D_{\mathcal{H}}$, $d_x = \Omega(\varepsilon \log n)$ (both given by (58)) and $\mathcal{H} \in \mathfrak{R}^4(Z)$. We are then also entitled to assume

$$(94) \quad \mathcal{H}_T \notin \mathfrak{D}_x$$

(see (74)), since $\mathcal{H} \in \mathfrak{R}^4(Z)$ says that the probability that $\mathcal{H}_T \in \mathfrak{D}_Z (\supseteq \mathfrak{D}_x)$ —an event decided by $\mathcal{G}_T$ —is $n^{-\Omega(\omega)}$ (recall ω was introduced at (73)). Armed with these assumptions, we continue.

Set $\Phi' = \Phi(\mathcal{H} - Z)/D_{\mathcal{H}}$ (a function of $\mathcal{G}$) and recall that $\mathcal{H} - Z \in \mathfrak{F}$ says (using $D_{\mathcal{H}-Z} \sim D_{\mathcal{H}}$)

$$(95) \quad w_{\mathcal{H}-Z}(U) \sim \Phi' \text{ for a.e. } U \in \mathcal{K}[W].$$

On the other hand, for any $y \in W$,

$$(96) \quad w_{\mathcal{H}}(Y \cup y) = \sum \{w_{\mathcal{H}-Z}(S \cup y) : S \in \binom{W \setminus y}{r-1}, S \cup x \in \mathcal{H}\},$$

and (95) implies

$$(97) \quad \text{for a.e. } y \in W, w_{\mathcal{H}-Z}(S \cup y) \sim \Phi' \text{ for a.e. } S \in \binom{W \setminus y}{r-1}.$$

It is thus enough to show that (under the assumptions in the preceding paragraph) the inequality in (89) is unlikely to fail for any y as in (97).

We are now choosing $\mathcal{H}_x = \mathcal{N}_T \cup (\mathcal{H}_x \setminus \mathcal{N}_T)$. Given y as in (97), let²

$$\mathfrak{J} = \{S \in \binom{W \setminus y}{r-1} : w_{\mathcal{H}-Z}(S \cup y) \not\sim \Phi'\} \cup \{S \in \binom{V \setminus x}{r-1} : S \cap (Y \cup y) \neq \emptyset\}$$

and $\tilde{\mathfrak{J}} = \{S \cup x : S \in \mathfrak{J}\}$, and notice that

$$(98) \quad |\tilde{\mathfrak{J}}| = |\mathfrak{J}| = o(n^{r-1}).$$

(The first set in the definition of $\mathfrak{J}$ consists of the exceptional S 's in (97), so is of size $o(n^{r-1})$; the second has size $\Theta(n^{r-2})$ and should be ignored.)

²A more formal version: (97) says there is $\varsigma = o(1)$ so that for all but ςn y 's,

$$|\{S \in \binom{W \setminus y}{r-1} : w_{\mathcal{H}-Z}(S \cup y) \neq (1 \pm \varsigma)\Phi'\}| < \varsigma n^{r-1}.$$

Then “ y as in (97)” is one of these, and the condition in $\mathfrak{J}$ is $w_{\mathcal{H}-Z}(S \cup y) \neq (1 \pm \varsigma)\Phi'$.

Noting that $d_x (= d_{\mathcal{H}}(x))$ is determined by $\mathcal{G}$, we observe that the sum in (96) is at least $(1 - o(1))d_x\Phi'$ —so the inequality in (89) holds—*provided*

$$|\mathcal{H}_x \cap \tilde{\mathcal{J}}| = o(d_x).$$

So it is enough to show, for some $\zeta = o(1)$,

$$(99) \quad \max\{\mathbb{P}(|\mathcal{N}_T \cap \tilde{\mathcal{J}}| > \zeta d_x), \mathbb{P}(|(\mathcal{H}_x \setminus \mathcal{N}_T) \cap \tilde{\mathcal{J}}| > \zeta d_x)\} = n^{-\omega(1)};$$

as we will see, this is true whenever

$$(100) \quad \zeta \gg \max\{(\log(n^{r-1}/|\tilde{\mathcal{J}}|))^{-1}, \omega/\log n\}.$$

(Recall we assume $\omega \rightarrow \infty$ *slowly*, so the second bound is small.)

Note that for $\mathcal{H}_x$ uniform from $\binom{\mathcal{K}_x}{d_x}$ and ζ satisfying (just) the first bound in (100), $\mathbb{P}(|\mathcal{H}_x \cap \tilde{\mathcal{J}}| > \zeta d_x) = n^{-\omega(1)}$ is given by Theorem 3.2, using (98) and $d_x = \Omega(\varepsilon \log n)$; namely, since $\mathbb{E}|\mathcal{H}_x \cap \tilde{\mathcal{J}}| \asymp n^{-(r-1)}|\tilde{\mathcal{J}}|d_x$, Theorem 3.2 bounds the probability in question by

$$(101) \quad \exp[-\zeta \log(\zeta n^{r-1}/(e|\tilde{\mathcal{J}}|))d_x] = \exp[-\omega(\log n)].$$

(It is this point—more precisely, its analogues below—that collapses if Theorem 1.5 allows $\delta_x = o(\log n)$.) So we are trying to show that the present distribution (see (93)) doesn't behave too differently.

For any choice of $\mathcal{N}_T$, $\mathcal{H}_x \setminus \mathcal{N}_T$ is a uniform subset of some size less than d_x from a universe of size $\Theta(n^{r-1})$, so, as above,

$$\mathbb{P}(|(\mathcal{H}_x \setminus \mathcal{N}_T) \cap \tilde{\mathcal{J}}| > \zeta d_x) = n^{-\omega(1)}.$$

While something similar is clearly true for $\mathbb{P}(|\mathcal{N}_T \cap \tilde{\mathcal{J}}| > \zeta d_x)$, I don't see how to say it's just trivial. A nice fly-with-a-sledgehammer argument runs as follows.

Given $\mathcal{G}_T$, let $u_z = \delta_z - d_{\mathcal{G}_T}(z)$, $J = \{z \in V \setminus x : u_z > 0\}$ and

$$(102) \quad u = \sum_{z \in J} u_z \leq \omega$$

(with the inequality from (94); again see (74)). Then $\mathcal{N}_T$ is distributed as a uniform $(m_T - g_T)$ -subset, $\mathcal{N}$, of $\mathcal{K}_x$ conditioned on

$$\mathfrak{S} := \{d_{\mathcal{N}}(z) \geq u_z \forall z \in J\}.$$

Set $\mathfrak{N} = \{|\mathcal{N} \cap \tilde{\mathcal{J}}| > \zeta d_x\}$ and $\mathfrak{N}' = \{|\mathcal{N} \cap \tilde{\mathcal{J}}| > \zeta d_x - u\}$, and notice that if $\mathfrak{S}\mathfrak{N}$ holds (at $\mathcal{N}$), then $\mathfrak{S}$ and $\mathfrak{N}'$ occur *disjointly* at $\mathcal{N}$; that is, there are disjoint $\mathcal{A}, \mathcal{B} \subseteq \mathcal{N}$ such that $\mathcal{N} \supseteq \mathcal{A}$ implies $\mathfrak{S}$ (for $\mathcal{N}$) and $\mathcal{N} \supseteq \mathcal{B}$ implies $\mathfrak{N}'$ (see [4] or e.g. [18]). But a beautiful result of van den Berg and Jonasson [3] bounds the probability of this disjoint occurrence by $\mathbb{P}(\mathfrak{S})\mathbb{P}(\mathfrak{N}')$, yielding

$$(\mathbb{P}(|\mathcal{N}_T \cap \tilde{\mathcal{J}}| > \zeta d_x) =) \mathbb{P}(\mathfrak{N}|\mathfrak{S}) \leq \mathbb{P}(\mathfrak{N}') = n^{-\omega(1)},$$

with the $n^{-\omega(1)}$ again given by Theorem 3.2 (now using $u \leq \omega$, the second bound in (100) and, again, $d_x = \Omega(\varepsilon \log n)$ to say the subtracted u in $\mathfrak{N}'$ is irrelevant).

10. PROOF OF LEMMA 8.2: SETTING UP

In this and the next two sections, $\mathcal{H}$ is $\mathcal{H}_t$, we use m for m_t and D for D_m , and $\mathcal{G}$ is either $\mathcal{H}$ or $\mathcal{H} - Z$ (with Z as in (88)), with law in either case denoted φ ; thus

$$(103) \quad \varphi(\mathcal{G}) := \varphi(\mathcal{G} = \mathcal{G}) \propto \begin{cases} \beta(\mathcal{G}) & \text{if } \mathcal{G} = \mathcal{H}, \\ \sum_{\mathcal{H}-Z=\mathcal{G}} \beta(\mathcal{H}) & \text{if } \mathcal{G} = \mathcal{H} - Z, \end{cases}$$

where $\mathcal{G}$ ranges over possibilities for $\mathcal{G}$ and $\mathcal{H}$ over $\mathcal{K}^t$ (or just over possibilities for $\mathcal{H}$, since the rest don't contribute to (103)). Note that the sum in the second part of (103) is the common value of $\varphi_Z(\mathcal{H})$ for $\mathcal{H} \in \mathcal{K}^t$ with $\mathcal{H} - Z = \mathcal{G}$; see (64).

We will refer to the two parts of (103) as Case 1 and Case 2 (respectively). Note that Case 2 includes Case 1 if we allow $Z = \emptyset$; we have included the first part of (103) to emphasize the distinction, but from this point (until the end of Section 12) Z is either empty or as in (88), and we set $V' = V \setminus Z$ and $\mathcal{K}' = \mathcal{K} - Z$. In what follows we will usually be thinking of the more demanding Case 2; but the arguments also make sense in Case 1, where they often simplify, sometimes drastically. A few comments on Case 1 appear in square brackets.

As observed at (92), Lemma 8.1 says that Lemma 8.2 is equivalent to the assertion that for each $\mathcal{G}$ in (103),

$$(104) \quad \mathbb{P}(\mathcal{H} \models \mathfrak{A}\mathfrak{R}, \mathcal{G} \models \mathfrak{E}\mathfrak{F}) = n^{-\omega(1)}.$$

Note we may assume here that

$$(105) \quad |\mathcal{K}| - m = \Omega(|\mathcal{K}|);$$

for $\mathcal{H} \models \mathfrak{R}^0$ implies that $|\mathcal{G}| \sim m$ for each $\mathcal{G}$ in (103), so if (105) fails then for each such $\mathcal{G}$, $\mathfrak{E}$ and $\mathfrak{F}$ are equivalent and (104) is vacuous. (This rather silly point will be needed for (140).)

It will be convenient to further reformulate as follows. For any $\mathcal{G}$ set

$$\eta(\mathcal{G}) = \inf\{\eta : |\{U \in \mathcal{K} : w_{\mathcal{G}}(U) \neq (1 \pm \eta)\Phi(\mathcal{G})/D_{\mathcal{G}}\}| < \eta|\mathcal{K}|\}.$$

Then $\{\mathcal{G} \models \mathfrak{F}\} = \{\eta(\mathcal{G}) = o(1)\}$ and (104) is equivalent to³

$$(106) \quad \text{for any fixed } \theta > 0, \mathbb{P}(\mathcal{H} \models \mathfrak{A}\mathfrak{R}, \mathcal{G} \models \mathfrak{E}, \eta(\mathcal{G}) > 2\theta) = n^{-\omega(1)}.$$

(The 2θ will be convenient below.) So for the rest of this section we fix $\theta > 0$ and aim for (106).

Set

$$(107) \quad \Phi' = \Phi(\mathcal{G})/D_{\mathcal{G}}.$$

Notice that $\{\mathcal{G} \models \mathfrak{E}\} \wedge \{\eta(\mathcal{G}) > 2\theta\}$ implies

$$\Omega: w_{\mathcal{G}}(A) \sim \Phi' \text{ for a.e. } A \in \mathcal{G}, \text{ but } w_{\mathcal{G}}(U) \neq (1 \pm 2\theta)\Phi' \text{ for at least a } (2\theta)\text{-fraction of the } U\text{'s in } \mathcal{K}' \setminus \mathcal{G}.$$

So it is enough to show

$$(108) \quad \mathbb{P}(\mathcal{H} \models \mathfrak{A}\mathfrak{R}, \mathcal{G} \models \Omega) = n^{-\omega(1)}.$$

For the proof of this we work with an auxiliary random set $\mathcal{T}$ chosen uniformly from $\binom{\mathcal{G}}{\tau}$, where τ , which will be specified later (see the paragraph containing (143)-(147)), will at least satisfy

$$(109) \quad \omega \log n \ll \tau \ll \log^2 n.$$

(As explained following (73), ω is really chosen *after* the parameters of (143)-(147).) We set $\mathcal{F} = \mathcal{G} \setminus \mathcal{T}$ and

$$\zeta = e^{-\tau/D},$$

³With $\mathfrak{G} = \{\mathcal{H} \models \mathfrak{A}\mathfrak{R}, \mathcal{G} \models \mathfrak{E}\}$ and $\mathfrak{H}(\nu) = \{\eta(\mathcal{G}) > \nu\}$, (104) says

$$\text{there is } \varsigma = o(1) \text{ such that } \mathbb{P}(\mathfrak{G} \wedge \mathfrak{H}(\varsigma)) = n^{-\omega(1)},$$

while (106) implies

$$\forall k, \mathbb{P}(\mathfrak{G} \wedge \mathfrak{H}(1/k)) < n^{-k} \text{ for } n \geq n_k;$$

and we get the former from the latter by taking $\varsigma(n) = (\max\{k : n_k \leq n\})^{-1}$.

and will be interested in a property of the pair $(\mathcal{G}, \mathcal{T})$ (or $(\mathcal{F}, \mathcal{T})$), viz.

$\mathfrak{V}$: $w_{\mathcal{F}}(A) \sim \zeta \Phi'$ for a.e. $A \in \mathcal{T}$, but $w_{\mathcal{F}}(U) \neq (1 \pm \theta)\zeta \Phi'$ for at least a θ -fraction of the U 's in $\mathcal{K}' \setminus \mathcal{G}$,

Note $\zeta w_{\mathcal{G}}(U)$ is a natural approximate value for $w_{\mathcal{F}}(U)$, since each p.m. of $\mathcal{G} - U$ survives in $\mathcal{F}$ with probability roughly (actually, asymptotically) $(1 - \tau/m)^{n/r} \sim \zeta$; cf. (162).

We will exploit the familiar leverage derived from the interplay of two natural ways of generating $(\mathcal{G}, \mathcal{T})$:

- (A) choose $\mathcal{G}$ and then $\mathcal{T}$ (as above);
- (B) choose $\mathcal{F}$ and then $\mathcal{T}$ (determining $\mathcal{G} = \mathcal{F} \cup \mathcal{T}$).

(In Case 2, analysis for (A) will involve choosing $\mathcal{H}$ rather than just $\mathcal{G}$.)

Notice that, given $\mathcal{F} = \mathcal{F}$, the law of $\mathcal{T}$ is given by

$$(110) \quad \mathbb{P}(\mathcal{T} = \mathcal{T}) = \mathbb{P}(\mathcal{F} = \mathcal{F}, \mathcal{T} = \mathcal{T}) / \mathbb{P}(\mathcal{F} = \mathcal{F}) = \mathbb{P}(\mathcal{G} = \mathcal{F} \cup \mathcal{T}) \binom{|\mathcal{F}| + \tau}{\tau}^{-1} / \mathbb{P}(\mathcal{F} = \mathcal{F}) \propto \varphi(\mathcal{F} \cup \mathcal{T}),$$

where $\mathcal{T}$ ranges over τ -subsets of $\mathcal{K}' \setminus \mathcal{F}$ (and φ is as in (103)).

We will not need to know much about the law of $\mathcal{F}$, but will want to restrict attention to reasonably well-behaved possibilities. Thus we will define a property $\mathfrak{N}$ specifying a few desirable features of $\mathcal{F}$, and, now writing $\mathfrak{A}\mathfrak{N}$ for $\{\mathcal{H} \models \mathfrak{A}\mathfrak{N}\}$, $\mathfrak{Q}$ for $\{\mathcal{G} \models \mathfrak{Q}\}$, $\mathfrak{N}$ for $\{\mathcal{F} \models \mathfrak{N}\}$, and $\mathfrak{V}$ for $\{(\mathcal{G}, \mathcal{T}) \models \mathfrak{V}\}$, show

$$(111) \quad \mathbb{P}(\mathfrak{V}\mathfrak{N}|\mathfrak{A}\mathfrak{N}\mathfrak{Q}) = 1 - o(1)$$

and

$$(112) \quad \mathbb{P}(\mathfrak{V}\mathfrak{N}) = n^{-\omega(1)}.$$

These give (108), since

$$\mathbb{P}(\mathfrak{A}\mathfrak{N}\mathfrak{Q}) = \mathbb{P}(\mathfrak{A}\mathfrak{N}\mathfrak{Q}\mathfrak{V}\mathfrak{N}) / \mathbb{P}(\mathfrak{V}\mathfrak{N}|\mathfrak{A}\mathfrak{N}\mathfrak{Q}) \leq \mathbb{P}(\mathfrak{V}\mathfrak{N}) / \mathbb{P}(\mathfrak{V}\mathfrak{N}|\mathfrak{A}\mathfrak{N}\mathfrak{Q}).$$

(So (111) is more than is really needed here.)

Noting that we have specified m and Z , we take $\mathfrak{N}$ to be the property comprising (118)-(121) below. The first of these takes a little preparation. Set, for $g \leq m$ (in what follows, g will be $|\mathcal{G}|$):

$$(113) \quad \mathfrak{I}(g) = \binom{\mathcal{K}_Z}{m - g},$$

with $\mathcal{K}_Z := \cup_{x \in Z} \mathcal{K}_x$, and

$$(114) \quad \mathfrak{I}^*(g) = \{\mathcal{I} \in \mathfrak{I}(g) : d_{\mathcal{I}}(x) \geq \delta_x \ \forall x \in Z\};$$

for $\mathcal{G}' \in \binom{\mathcal{K}'}{g}$,

$$\bar{\varphi}(\mathcal{G}') = |\mathfrak{I}(g)|^{-1} \varphi(\mathcal{G}')$$

(with φ as in (103)); and for $\mathcal{F} \subseteq \mathcal{K}'$ with $|\mathcal{F}| = g - \tau$,

$$(115) \quad \mathfrak{H}(\mathcal{F}) = \{\mathcal{G}' \in \binom{\mathcal{K}'}{g} : \mathcal{G}' \supseteq \mathcal{F}\}$$

and

$$(116) \quad \mathfrak{H}^*(\mathcal{F}) = \{\mathcal{G}' \in \mathfrak{H}(\mathcal{F}) : d_{\mathcal{G}'}(x) \geq \delta_x \ \forall x \in V'\}.$$

Like φ itself, $\bar{\varphi}$ should recall $\mathfrak{N}^2$: $\bar{\varphi}(\mathcal{G}')$ is $\bar{\varphi}_Z(\mathcal{H})$ for any $\mathcal{H} \in \mathcal{K}^t$ with $\mathcal{H} - Z = \mathcal{G}'$. [In Case 1, $g = m$, $\mathfrak{I}^*(g) = \mathfrak{I}(g) = \{\emptyset\}$, and $\bar{\varphi}(\mathcal{G}) = \varphi(\mathcal{G}) = \beta(\mathcal{G})$.] Note that $\mathfrak{H}(\mathcal{F})$ includes possibilities for $\mathcal{G}$ given $\mathcal{F} = \mathcal{F}$, but typically also some (irrelevant) impossibilities; e.g. in Case 1 anything in $\mathfrak{H}(\mathcal{F}) \setminus \mathfrak{H}^*(\mathcal{F})$.

In two places below it will be convenient to first dispose of the easy case of very small m , allowing us to restrict attention to (say)

$$(117) \quad m > (1 + n^{-3\delta})m_T.$$

With $g = |\mathcal{F}| + \tau$, the first requirement for $\mathcal{F} \models \mathfrak{N}$ is

$$(118) \quad \bar{\varphi}(\mathcal{G}') \gtrsim \begin{cases} |\mathfrak{I}^*(g)|/|\mathfrak{I}(g)| \forall \mathcal{G}' \in \mathfrak{H}^*(\mathcal{F}) & \text{if } m \text{ violates (117),} \\ n^{-(2r+1)\beta(\mathcal{K})} \forall \mathcal{G}' \in \mathfrak{H}(\mathcal{F}) & \text{if } m \text{ satisfies (117).} \end{cases}$$

A little perspective: For (112) we will use viewpoint (B), bounding $\mathbb{P}(\mathfrak{V}|\mathcal{F} = \mathcal{F})$ for $\mathcal{F} \in \mathfrak{N}$, and will want to say that the law of $\mathcal{G}$ under this conditioning is not too awful, meaning that the ratios between different $\mathbb{P}(\mathcal{G}')$'s are not *too* large; but as these probabilities are governed by φ (again, see (103)), the lower bounds of (118) will be useful in bounding their ratios. (See e.g. (139) for the role of (118) in the (more important) case where (117) holds.)

The other, more easily stated requirements (for $\mathcal{F} \models \mathfrak{N}$) are

$$(119) \quad \text{if } m > 2rm_T \text{ then } d_{\mathcal{F}}(x) > 1.5\varepsilon D \forall x \in V';$$

$$(120) \quad \text{with } \alpha = m_T/m, \mathbb{P}(\mathcal{F}_\alpha \text{ anemic}) < \exp[-(1 - o(1))n^{2\delta}]$$

(see (36) for “anemic”); and (with ω as in (73) and (109))

$$(121) \quad \sum_{y \in V'} (\delta_y - d_{\mathcal{F}}(y))^+ \leq \omega.$$

Of course for (111) it is enough to show (as above using $\mathfrak{N}$ for $\mathcal{F} \models \mathfrak{N}$ and so on)

$$(122) \quad \text{for any } \mathcal{H} \in \mathcal{K}^t \cap \mathfrak{R}, \mathbb{P}(\mathfrak{N}|\mathcal{H} = \mathcal{H}) = 1 - o(1)$$

and (our main point)

$$(123) \quad \mathbb{P}(\mathfrak{V}|\mathfrak{N}\mathfrak{N}) > 1 - o(1).$$

We prove (122) and (112) in Section 11 and (123) in Section 12, organizing in this way because the proofs of (122) and (112) are slightly similar (mainly in their use of Lemma 4.5) and unrelated to the proof of (123).

11. PROOFS OF (122) AND (112)

Proof of (122). We will show that (120) follows (deterministically) from $\mathcal{H} \in \mathfrak{R}$, while the other parts of $\mathfrak{N}$ hold (for $\mathcal{H} \in \mathfrak{R}$) whenever

$$(124) \quad \mathcal{T} \text{ is a matching}$$

and

$$(125) \quad \mathcal{T} \text{ covers no } x \text{ for which } d_{\mathcal{G}}(x) \leq 1.5\varepsilon D.$$

(Recall we are using $\mathcal{H}$, m and D for $\mathcal{H}_t$, m_t and D_m .) This gives (122) since (under $\mathcal{H} \in \mathfrak{R}$) the upper bound on τ in (109) implies that each of (124), (125) holds with probability $1 - o(1)$: for (124) this is a weak consequence of $\Delta_{\mathcal{H}} = O(D_{\mathcal{H}})$ (see (58)); and for (125) it holds because $\mathcal{H} \in \mathfrak{R}^3$ (with Observation 5.1) and the codegree condition (59) bound the number of x 's in (125) by $2rn^{2\delta}$.

Turning to the deterministic assertions preceding (124), we first note that (119) is immediate from $\mathcal{H} \in \mathfrak{R}$ (specifically, (61) and (59)) and (124), while (121) follows easily from $\mathcal{H} \in \mathfrak{R}^4$, (124) and (125) (the first trivially implies (121) with $\mathcal{G}$ in place of $\mathcal{F}$, and the others say the passage to $\mathcal{F}$ doesn't affect this).

To get (120) from $\mathcal{H} \in \mathfrak{R}$, notice that

$$\exp[-(1 - o(1))n^{2\delta}] > \mathbb{P}(\mathcal{H}_\alpha \text{ anemic}) \geq \mathbb{P}(\mathcal{F}_\alpha \text{ anemic}) \cdot (1 - \alpha)^{|\mathcal{H} \setminus \mathcal{F}|}$$

(the first inequality is $\mathcal{H} \in \mathfrak{R}^3$ and the second is trivial). Thus for (120) it is enough to show

$$(126) \quad (1 - \alpha)^{|\mathcal{H} \setminus \mathcal{F}|} = \exp[-o(n^{2\delta})]$$

—which isn't close: we have (using (58))

$$(127) \quad |\mathcal{H} \setminus \mathcal{F}| = |\mathcal{H} \setminus \mathcal{G}| + \tau = O(D) + \tau = O(m/n) + \tau;$$

so the l.h.s. of (126) is (crudely)

$$(1 - \alpha)^{O(m/n) + \tau} = \begin{cases} \exp[-O(\frac{m_T}{m} \{ \frac{m}{n} + \tau \})] & \text{if } \alpha < 1/2 \text{ (say),} \\ \exp[-O(\frac{m}{n} + \tau) \log n] & \text{otherwise} \end{cases}$$

(the latter since $1 - \alpha > 1/m$), and (126) follows easily, using the upper bound on τ in (109) and $m_T < n \log n$ (which when $\alpha \geq 1/2$ also implies $m = O(n \log n)$).

Finally, we turn to the two cases of (118). For m violating (117), we use Lemma 4.1. Here the assumption (35) holds even for $\mathcal{F}$ —as follows from $\mathcal{H} \in \mathfrak{R}$ (specifically, (72) and (59)), (124) and (125)—so also for any $\mathcal{G}' \in \mathfrak{H}(\mathcal{F})$. Thus the lemma gives $\beta(\mathcal{G}' \cup \mathcal{I}) = 1 - o(1)$ whenever $\mathcal{G}' \cup \mathcal{I} \in \mathfrak{L}$ (with $\mathcal{G}' \in \mathfrak{H}(\mathcal{F})$ and $\mathcal{I} \in \mathfrak{I}(g)$), which in particular is true whenever $\mathcal{G}' \in \mathfrak{H}^*(\mathcal{F})$ and $\mathcal{I} \in \mathfrak{I}^*(g)$; so we have (118) in this case.

For m satisfying (117), recall from (66) [or, in Case 1, its specialization (67)] that $\mathcal{H} \in \mathfrak{R}^2$ implies the inequality in (118) for $\mathcal{G}' = \mathcal{G}$ (since $\overline{\varphi}(\mathcal{G})$ is the same as $\overline{\varphi}_Z(\mathcal{H})$); so it is enough to show

$$(128) \quad \varphi(\mathcal{G}') \gtrsim \varphi(\mathcal{G}) \quad \forall \mathcal{G}' \in \mathfrak{H}(\mathcal{F}).$$

It's also easy to see that $\overline{\varphi}(\mathcal{G}) \gtrsim n^{-(2r+1)}\beta(\mathcal{K})$ (as in (118)) implies (say)

$$(129) \quad \varphi(\mathcal{G}) \sim \sum \{ \beta(\mathcal{G} \cup \mathcal{I}) : \mathcal{I} \in \mathfrak{I}(g), \beta(\mathcal{G} \cup \mathcal{I}) > n^{-(2r+2)}\beta(\mathcal{K}) \}.$$

But we claim that for any $\mathcal{I}$ as in (129) and $\mathcal{G}' \in \mathfrak{H}(\mathcal{F})$,

$$(130) \quad \beta(\mathcal{G}' \cup \mathcal{I}) \gtrsim \beta(\mathcal{G} \cup \mathcal{I}),$$

which (in view of (129)) gives (128).

For (130) we apply Lemma 4.5 with $\mathcal{J} = \mathcal{G} \cup \mathcal{I}$ and $\mathcal{J}' = \mathcal{G}' \cup \mathcal{I}$ (so $\mathcal{E} \supseteq \mathcal{F} \cup \mathcal{I}$, $\sigma \leq \tau$ and $W \subseteq V'$). Note that here the κ of the lemma is zero, since all A_i 's lie in $\mathcal{T}$, so by (125) are not dangerous for $\mathcal{J}$. So the lemma's conclusion is (130), and we just need to check its hypotheses (assuming $\mathcal{J} \in \mathfrak{L}$, without which the r.h.s. of (130) is zero):

First, (44) holds because we assume (117), and (45) is given by (124). Second, $\mathcal{J}' \in \mathfrak{L}$ follows from $\mathcal{J} \in \mathfrak{L}$, using $\mathcal{J} \setminus \mathcal{J}' \subseteq \mathcal{T}$ with (124) and (125). Third, (124) implies $d_{\mathcal{E}}(x) \geq d_{\mathcal{G}}(x) - 1$ for each $x \in V' (\supseteq W)$, which gives (48) since (using (61) and (59), and noting $|\mathcal{H}| = |\mathcal{J}|$)

$$m > 2rm_T \Rightarrow [d_{\mathcal{H}}(x) > 2\varepsilon D_{\mathcal{J}} \quad \forall x \in V] \Rightarrow [d_{\mathcal{G}}(x) > (2\varepsilon - o(1))D_{\mathcal{J}} \quad \forall x \in V'].$$

Last, (49) holds because $\beta(\mathcal{J}) > \exp[-(1 - o(1))n^\delta]$ (by Corollary 4.4, since $\mathcal{I}$ is as in (129)) and, by (120), $\mathbb{P}(\mathcal{X}_{\mathcal{J}} \text{ anemic}) (\leq \mathbb{P}(\mathcal{F}_\alpha \text{ anemic})) < \exp[-(1 - o(1))n^{2\delta}]$. $\square$

Proof of (112). Implied constants in this argument do not depend on ε or θ . We actually show

$$(131) \quad \text{for any } \mathcal{F} \in \mathfrak{N}, \mathbb{P}(\mathfrak{V}|\mathcal{F} = \mathcal{F}) = e^{-\Omega(\theta\tau)}$$

(which is $n^{-\omega(1)}$ by (109)). Here we use viewpoint (B). The (natural) idea is roughly: $\mathcal{F}$ determines the weights $w_{\mathcal{F}}(U)$ (for all $U \in \mathcal{K}'$, though here we are only interested in $U \in \mathcal{K}' \setminus \mathcal{F}$), and $\mathfrak{V}$ then requires that $\mathcal{T}$ be (pathologically) drawn almost entirely from U 's with weights close to $\zeta\Phi'$, though this group excludes a constant fraction of $\mathcal{K}' \setminus \mathcal{F}$.

In fact (112) would be more or less routine if we were choosing $\mathcal{T}$ *uniformly* rather than as in (110). The crude comparison in Lemma 11.1 below will allow us to move between these two regimes.

Fix $\mathcal{F} \in \mathfrak{N}$ and set $g = |\mathcal{F}| + \tau$, $\mathfrak{J} = \mathfrak{J}(g)$, $\mathfrak{J}^* = \mathfrak{J}^*(g)$, $\mathfrak{H} = \mathfrak{H}(\mathcal{F})$ and $\mathfrak{H}^* = \mathfrak{H}^*(\mathcal{F})$ (see (113)-(116)). We now regard the $\mathbb{P}$ of (110) as a probability measure on $\mathfrak{H}$ —thus

$$(132) \quad \mathbb{P}(\mathcal{G}') \propto \varphi(\mathcal{G}') \text{ for } \mathcal{G}' \in \mathfrak{H}$$

—and use μ for uniform measure on $\mathfrak{H}$. (It's perhaps worth noting—though this won't matter—that, unlike $\mathbb{P}$, μ can assign positive probability to $\mathcal{T}$'s for which $\mathcal{F} \cup \mathcal{T}$ is not a possible value of $\mathcal{G}$.)

Lemma 11.1. *With notation as above, if $\mathfrak{X} \subseteq \mathfrak{H}$ and $\mu(\mathfrak{X}) = e^{-\Omega(\theta\tau)}$, then*

$$\mathbb{P}(\mathfrak{X}) = e^{-\Omega(\theta\tau)}.$$

Proof. Notice first that (121) and $\tau \geq \omega$ (see (109)) imply

$$(133) \quad \mu(\mathfrak{H}^*) > n^{-\omega}.$$

For m violating (117) we now finish easily: combining (118) and the trivial $\varphi(\mathcal{G}') \leq |\mathfrak{J}^*|$ ($\forall \mathcal{G}' \in \mathfrak{H}$) with (133) (and (132)) gives $\mathbb{P}(\mathfrak{X}) \lesssim n^{\omega} \mu(\mathfrak{X})$ for any $\mathfrak{X} \subseteq \mathfrak{H}$; and this gives the lemma since $\tau \gg \omega \log n$ (again see (109)). So we assume from now on that m satisfies (117).

For $\mathcal{G}' \in \mathfrak{H}$, let $\lambda(\mathcal{G}')$ be the number of edges of $\mathcal{G}' \setminus \mathcal{F}$ containing vertices x with

$$(134) \quad d_{\mathcal{F}}(x) < 1.5\varepsilon D.$$

By (120) and Observation 5.1, the number of such vertices is less than $2n^{2\delta}$, implying that, for any b ,

$$(135) \quad \mu(\mathcal{G}' : \lambda(\mathcal{G}') \geq b) = n^{-\Omega(b)}$$

(In more detail: if there *are* x 's as in (134), then a very crude application of (119)—all that's needed here is $|\mathcal{K}' \setminus \mathcal{F}| = \Omega(n^r)$ —bounds the fraction of members of $\mathcal{K}' \setminus \mathcal{F}$ containing such x 's by $O(n^{-1+2\delta})$, and the upper bound in (109)—again, very crudely; $\tau = n^{o(1)}$ would suffice—then gives (135).)

Fix $\mathcal{G}^0 \in \mathfrak{H}^*$ with $\varphi(\mathcal{G}^0)$ minimum. We will show that for any $\mathcal{G}' \in \mathfrak{H}$,

$$(136) \quad \varphi(\mathcal{G}')/\varphi(\mathcal{G}^0) \lesssim n^{O(\varepsilon\lambda(\mathcal{G}'))}.$$

(Of course we can replace " $\lesssim$ " by " $<$ " if $\lambda(\mathcal{G}') \neq 0$.)

Before proving (136) we show it gives Lemma 11.1 (for m satisfying (117)). Since (by (133))

$$\sum \{\varphi(\mathcal{G}') : \mathcal{G}' \in \mathfrak{H}\} > n^{-\omega} |\mathfrak{H}| \varphi(\mathcal{G}^0),$$

we have, for any $\mathcal{G}' \in \mathfrak{H}$ (using (136), and with $\mathcal{G}''$ running over $\mathfrak{H}$),

$$\mathbb{P}(\mathcal{G}') = \frac{\varphi(\mathcal{G}')}{\sum \varphi(\mathcal{G}'')} < \frac{n^\omega \varphi(\mathcal{G}')}{|\mathfrak{H}| \varphi(\mathcal{G}^0)} \lesssim \frac{n^\omega n^{O(\varepsilon\lambda(\mathcal{G}'))}}{|\mathfrak{H}|} = n^\omega \mu(\mathcal{G}') n^{O(\varepsilon\lambda(\mathcal{G}'))}.$$

Combining this with (135), $\tau \gg \omega \log n$ and $\mu(\mathfrak{X}) = e^{-\Omega(\theta\tau)}$ gives the desired bound: with $\lambda_0 = \theta\tau / \log n$,

$$(137) \quad \mathbb{P}(\mathfrak{X}) \lesssim n^\omega [\mu(\mathfrak{X}) n^{O(\varepsilon\lambda_0)} + \sum_{b > \lambda_0} \mu(\mathcal{G}' : \lambda(\mathcal{G}') = b) n^{O(\varepsilon b)}] = e^{-\Omega(\theta\tau)}. \quad \square$$

Proof of (136). Set $\mathfrak{I}^0 = \{\mathcal{I} \in \mathfrak{I} : \beta(\mathcal{G}' \cup \mathcal{I}) > n^{-(2r+2)} \beta(\mathcal{K})\} (\subseteq \mathfrak{I}^*)$ and notice that it is enough to show

$$(138) \quad \beta(\mathcal{G}' \cup \mathcal{I}) \lesssim n^{O(\varepsilon\lambda(\mathcal{G}'))} \beta(\mathcal{G}^0 \cup \mathcal{I}) \quad \forall \mathcal{I} \in \mathfrak{I}^0;$$

for if this is true then, recalling that $\mathcal{F} \in \mathfrak{N}$ implies $\bar{\varphi}(\mathcal{G}^0) \gtrsim n^{-(2r+1)} \beta(\mathcal{K})$ (see (118)), we have

$$(139) \quad \begin{aligned} \bar{\varphi}(\mathcal{G}') &\leq |\mathfrak{I}|^{-1} \sum \{\beta(\mathcal{G}' \cup \mathcal{I}) : \mathcal{I} \in \mathfrak{I}^0\} + (1 + o(1)) n^{-1} \bar{\varphi}(\mathcal{G}^0) \\ &\lesssim n^{O(\varepsilon\lambda(\mathcal{G}'))} \varphi(\mathcal{G}^0) / |\mathfrak{I}| + n^{-1} \bar{\varphi}(\mathcal{G}^0) \lesssim n^{O(\varepsilon\lambda(\mathcal{G}'))} \bar{\varphi}(\mathcal{G}^0). \end{aligned}$$

[In Case 1, (118) says $\mathfrak{I}^0 = \{\emptyset\} (= \mathfrak{I})$, and (138) is (136).]

For (138) we will again use Lemma 4.5, now with $\mathcal{J} = \mathcal{G}' \cup \mathcal{I}$ and $\mathcal{J}' = \mathcal{G}^0 \cup \mathcal{I}$ (and $\mathcal{E} = (\mathcal{G}' \cap \mathcal{G}^0) \cup \mathcal{I} \supseteq \mathcal{F}$), so should check hypotheses: first, (44) holds since we assume (117), and (45) is given by (124) (since $\mathcal{J} \setminus \mathcal{E} \subseteq \mathcal{T}$); second, assuming (as we may) that $\mathcal{J} \in \mathfrak{L}$, we have $\mathcal{J}' \in \mathfrak{L}$, since $\mathcal{G}^0 \in \mathfrak{H}^*$ and $\mathcal{J}, \mathcal{J}'$ agree on edges meeting Z ; third, (48) holds since (for $m > 2rm_\tau$) (119) gives $d_{\mathcal{E}}(x) \geq d_{\mathcal{F}}(x) > 1.5\varepsilon D (= 1.5\varepsilon D_{\mathcal{J}})$ for $x \in V' (\supseteq W)$; last, (49) follows from $\mathcal{I} \in \mathfrak{I}^0$ (with Corollary 4.4) and (120) (with $\mathbb{P}(\mathcal{X}_{\mathcal{J}} \text{ anemic}) \leq \mathbb{P}(\mathcal{F}_{\alpha} \text{ anemic})$).

So the lemma applies and we just need to check that κ (the number of edges of $\mathcal{J} \setminus \mathcal{J}'$ containing vertices dangerous for $\mathcal{J}$) is at most $\lambda(\mathcal{G}')$; but this is true because $\mathcal{J} \setminus \mathcal{J}' \subseteq \mathcal{G}' \setminus \mathcal{F}$ and any x that is dangerous for $\mathcal{J}$ satisfies (134) (since $d_{\mathcal{F}}(x) \leq d_{\mathcal{J}}(x)$). $\square$

This completes the proof of Lemma 11.1.

We return to (131), which by Lemma 11.1 will follow from its “ μ -version,” viz.

$$(140) \quad \text{for any } \mathcal{F} \in \mathfrak{N}, \quad \mu(\mathfrak{V} | \mathcal{F} = \mathcal{F}) = e^{-\Omega(\theta\tau)}.$$

A small complication here is that $\mathcal{F}$ doesn't determine the “target” $\zeta\Phi'$ appearing in $\mathfrak{V}$. Among several ways of dealing with this, the following seems nicest.

Given $\mathcal{F}$, let $U_1, \dots$ be an ordering of $\mathcal{K}' \setminus \mathcal{F}$ with $w_{\mathcal{F}}(U_1) \leq w_{\mathcal{F}}(U_2) \leq \dots$, and let $\mathcal{Y}$ and $\mathcal{Z}$ be (resp.) the first and last $\theta|\mathcal{K}' \setminus \mathcal{F}|/3$ of the U_i 's. Then, *whatever* Φ' turns out to be, the second part of $\mathfrak{V}$ requires that at least one of $\mathcal{Y}, \mathcal{Z}$ be contained in

$$\mathcal{W} := \{U \in \mathcal{K}' \setminus \mathcal{F} : w_{\mathcal{F}}(U) \neq (1 \pm \theta)\zeta\Phi'\}$$

(or $|\mathcal{W}| < |\mathcal{Y}| + |\mathcal{Z}| < 2(\theta|\mathcal{K}' \setminus \mathcal{G}| + \tau)/3 < \theta|\mathcal{K}' \setminus \mathcal{G}|$, the last inequality since (105) and (109) imply $\tau \ll |\mathcal{K}' \setminus \mathcal{G}|$). But then, since (now regarding μ as the law of $\mathcal{T}$)

$$\mathbb{E}_{\mu} |\mathcal{T} \cap \mathcal{Y}| = \mathbb{E}_{\mu} |\mathcal{T} \cap \mathcal{Z}| = \theta\tau/3$$

and θ is fixed, Theorem 3.1 bounds the probability that the first part of $\mathfrak{V}$ holds by (say)

$$\mu(\max\{|\mathcal{T} \cap \mathcal{Y}|, |\mathcal{T} \cap \mathcal{Z}|\} < \theta\tau/4) = e^{-\Omega(\theta\tau)}.$$

$\square$

12. PROOF OF (123)

(We continue to use $\mathcal{H} = \mathcal{H}_t$, $m = m_t$ and $D = D_m$.) We now need to pay some attention to parameters. We first observe that if $\mathcal{H} \models \mathfrak{A}\mathfrak{R}^0$ and $\mathcal{G} = \mathcal{H} - Z$ (with Z either empty or as in (88)), then there is $\gamma = o(1)$ (depending on the $o(n)$ in $\mathfrak{A}$ and, in $\mathfrak{R}^0$, the (explicit or implicit) $o(\cdot)$'s in (57) and (59), and the implied constants in (58)), such that for each $U \in \mathcal{K}'$ with (say)

$$(141) \quad (\mathbf{w}_{\mathcal{G}}(U) =) \Phi(\mathcal{G} - U) > \Phi(\mathcal{G})n^{-r},$$

$\mathcal{G}^* := \mathcal{G} - U$ and $\Phi^* := \Phi(\mathcal{G}^*)$ satisfy

$$(142) \quad \sum \{\mathbf{w}_{\mathcal{G}^*}(A) : A \in \mathcal{G}^*, \mathbf{w}_{\mathcal{G}^*}(A) \neq (1 \pm \gamma)\Phi^*/D\} < \gamma n \Phi^*.$$

To see this, recall from the remarks following Lemma 8.4 that each relevant $\mathcal{G}^*$ satisfies $\mathfrak{A}\mathfrak{R}^0$, so also $\mathfrak{E}$ by Lemma 8.1. But then $\mathcal{G}^*$ contains $(1 - o(1))|\mathcal{G}^*| \sim nD/r$ edges A with $\mathbf{w}_{\mathcal{G}^*}(A) \sim \Phi^*/D_{\mathcal{G}^*} \sim \Phi^*/D$, with (both) asymptotics following easily from $\mathcal{G} \models \mathfrak{R}^0$ (see (59)); so such edges account for all but a $o(1)$ -fraction of the total weight $\Phi^*(n - r)/r \sim \Phi^*n/r$. This gives (142) for a suitable $\gamma = o(1)$.

We now choose $\tau = \nu \min\{\log n, D\}$ ($\sim \nu \log n$)—noting that then

$$(143) \quad \zeta (= e^{-\tau/D}) \geq e^{-\nu}$$

—together with M and η , satisfying

$$(144) \quad \log n \gg \nu \gg \omega$$

(which is (109));

$$(145) \quad e^{-\nu} \gg \gamma;$$

$$(146) \quad \tau \gg M \begin{cases} \gg \gamma\tau, \\ > 1 + \gamma; \end{cases}$$

and

$$(147) \quad e^{-\nu} \gg \eta \gg \sqrt{\tau M} / \log n.$$

Note this is possible: we may choose $\nu \rightarrow \infty$ as slowly as we like (which in particular gives (144) and (145)); we then want to choose M as in (146) satisfying (to leave room for η) $e^{-\nu} \gg \sqrt{\tau M} / \log n$; and this is possible if $e^{-\nu} \gg \max\{\nu\sqrt{\gamma}, \sqrt{\nu/\log n}\}$, which is true for a slow enough ν .

For the proof of (123) we use viewpoint (A) (choose $\mathcal{H}$ —so also $\mathcal{G}$ —and then $\mathcal{T}$). We assume we have chosen $\mathcal{H} = \mathcal{H}$, with $\mathcal{H} \models \mathfrak{A}\mathfrak{R}$ and $\mathcal{G} := \mathcal{H} - Z \models \mathfrak{Q}$; so $\mathbb{P}$ now refers just to the choice of $\mathcal{T}$, and (123) will follow from

$$(148) \quad \mathbb{P}((\mathcal{G}, \mathcal{T}) \models \mathfrak{V}) = 1 - o(1).$$

It will be enough to show that for $U \in \mathcal{K}'$ as in (141) (i.e. $\mathbf{w}_{\mathcal{G}}(U) > \Phi(\mathcal{G})n^{-r}$),

$$(149) \quad \mathbb{P}(\mathbf{w}_{\mathcal{F}}(U) \sim \zeta \mathbf{w}_{\mathcal{G}}(U)) = 1 - o(1) \quad \text{if } U \in \mathcal{K}' \setminus \mathcal{G},$$

$$(150) \quad \mathbb{P}(\mathbf{w}_{\mathcal{F}}(U) \sim \zeta \mathbf{w}_{\mathcal{G}}(U) | U \in \mathcal{T}) = 1 - o(1) \quad \text{if } U \in \mathcal{G}.$$

Before proving this we show that it does give (148). If $\mathcal{G}$ satisfies $\mathfrak{Q}$ then for a suitable $\varsigma = o(1)$,

$$(151) \quad |\{A \in \mathcal{G} : \mathbf{w}_{\mathcal{G}}(A) \neq (1 \pm \varsigma)\Phi^*\}| \ll |\mathcal{G}|$$

(where $\Phi' = \Phi(\mathcal{G})/D_{\mathcal{G}}$; see (107)). Thus, with $\mathcal{G}^0$ the set in (151), we have $\mathbb{E}|\mathcal{T} \cap \mathcal{G}^0| = \tau|\mathcal{G}^0|/|\mathcal{G}| \ll \tau$, so

$$|\mathcal{T} \cap \mathcal{G}^0| \ll \tau \text{ w.h.p.}$$

(by Theorem 3.1 or just Markov's Inequality). But for the first part of $\mathfrak{V}$ to fail we must have either $|\mathcal{T} \cap \mathcal{G}^0| = \Omega(\tau)$, which we have just said occurs with probability $o(1)$, or

$$|\{A \in \mathcal{T} \setminus \mathcal{G}^0 : w_{\mathcal{F}}(A) \not\sim \zeta w_{\mathcal{G}}(A)\}| = \Omega(\tau),$$

which has probability $o(1)$ by (150) (and Markov).

Similarly, failure of the second part of $\mathfrak{V}$ implies

$$(152) \quad w_{\mathcal{F}}(U) = (1 \pm \theta)\zeta w_{\mathcal{G}}(U) \not\sim \zeta w_{\mathcal{G}}(U)$$

for at least $\theta|\mathcal{K}' \setminus \mathcal{G}|$ of those U 's in the second part of $\mathfrak{Q}$ that satisfy

$$(153) \quad w_{\mathcal{G}}(U) > (1 - \theta)\zeta\Phi' > n^{-o(1)}\Phi(\mathcal{G})/D$$

(since those failing (153) cannot satisfy (152); for the second bound in (153) see (143) and (144)). But since the bound in (153) is larger than the one in (141), (149) implies that the probability that (152) holds for such a set of U 's is $o(1)$.

Finally, we prove (149); the proof of (150) is almost literally the same and is omitted. (Note the probability in (150) is just $\mathbb{P}(w_{\mathcal{G} \setminus \mathcal{T}_0}(U) \sim \zeta w_{\mathcal{G}}(U))$, with $\mathcal{T}_0$ uniform from $\binom{\mathcal{G} \setminus \{U\}}{\tau-1}$.)

Proof of (149). We now fix U as in (141) (and recall $\mathcal{G}^* = \mathcal{G} - U$ and $\Phi^* = \Phi(\mathcal{G}^*)$). (We will, pedantically, keep track of the microscopic numerical differences between Cases 1 and 2—in Case 2 the number of vertices is $n - r$ and $|\mathcal{G}|$ is not exactly m —but stress they are wholly irrelevant.)

Say $A \in \mathcal{G}$ is *heavy* if $A \in \mathcal{G}^*$ and $w_{\mathcal{G}^*}(A) > M\Phi^*/D$, and note that by (142) (and $M > 1 + \gamma$; see (146)),

$$(154) \quad \text{the number of heavy edges in } \mathcal{G} \text{ is less than } \gamma n D / M = \gamma m r / M,$$

implying

$$(155) \quad \mathbb{P}(\mathcal{T} \text{ contains a heavy edge}) \lesssim \gamma \tau r / M = o(1)$$

(see (146); we need " $\lesssim$ " because we only have $|\mathcal{G}| \sim m$). So it is enough to show (149) conditioned on

$$(156) \quad \{\mathcal{T} \text{ contains no heavy edges}\}.$$

We will instead show a slight variant, replacing $\mathcal{T}$ by $\mathcal{T}' = \{A_1, \dots, A_\tau\}$, with the A_i 's chosen uniformly and *independently* from the non-heavy edges of $\mathcal{G}$; thus:

$$(157) \quad \mathbb{P}(w_{\mathcal{G} \setminus \mathcal{T}'}(U) \sim \zeta w_{\mathcal{G}}(U)) = 1 - o(1).$$

Of course this suffices: we may couple $\mathcal{T}$ (conditioned on (156)) and $\mathcal{T}'$ so they agree whenever the edges of $\mathcal{T}'$ are distinct, which occurs w.h.p. (more precisely, with probability greater than $1 - \tau^2/m$), and the probability in (149) is then at least the probability in (157) minus $\mathbb{P}(\mathcal{T}' \neq \mathcal{T})$.

For the proof of (157), let

$$X = X(A_1, \dots, A_\tau) = \Phi(\mathcal{G}^* \setminus \{A_1, \dots, A_\tau\}) = w_{\mathcal{G} \setminus \mathcal{T}'}(U).$$

Since $\eta \ll \zeta$ (see (143) and (147)), (157) will follow from (recall $w_{\mathcal{G}}(U) = \Phi^*$)

$$(158) \quad \mathbb{E}X \sim \zeta \Phi^*$$

and

$$(159) \quad \mathbb{P}(|X - \mathbb{E}X| > \eta\Phi^*) = o(1).$$

Proof of (158). Let M_i run through the p.m.s of $\mathcal{G}^*$ and let x_i be the number of heavy edges in M_i . Then with m' the number of non-heavy edges in $\mathcal{G}$, we have (with, irrelevantly, $\iota = j$ in Case j for $j = 1, 2$),

$$\mathbb{E}X = \sum_i (1 - (n/r - \iota - x_i)/m')^\tau$$

and, by (142),

$$(160) \quad \sum x_i = \sum \{w_{\mathcal{G}^*}(A) : A \in \mathcal{G}^*, A \text{ heavy}\} < \gamma n \Phi^*.$$

These imply, with $\varrho = (n/r - \iota)/m'$,

$$(161) \quad \begin{aligned} (1 - \varrho)^\tau \Phi^* &\leq \mathbb{E}X < \sum_i e^{-(\varrho - x_i/m')\tau} \\ &< [e^{-\varrho\tau} + \gamma r n / (n - \iota r)] \Phi^*. \end{aligned}$$

Here the last inequality follows from (160) and convexity of the exponential function, which imply that the sum in (161) is at most what it would be with $\frac{\gamma n \Phi^*}{n/r - \iota} = \frac{\gamma r n \Phi^*}{n - \iota r}$ of the x_i 's equal to $n/r - \iota$ and the rest (the number of which we just bound by Φ^*) equal to zero.

In view of (161), (158) will follow from

$$(162) \quad (1 - \varrho)^\tau \sim e^{-\varrho\tau} \sim e^{-\tau/D} (= \zeta)$$

(and $\gamma \ll \zeta$, which is given by (143) and (145)). For the two parts of (162) we need (resp.) $\varrho^2 \ll 1/\tau$ and $|\varrho - 1/D| \ll 1/\tau$. The first of these follows from (154) (which gives $m' \sim m$, though here $m' = \Omega(m)$ would suffice) and (109). For the second, now using (154) more precisely (and recalling $D = mr/n$), we have

$$\left| \frac{n/r - \iota}{m'} - \frac{n/r}{m} \right| \leq \frac{\iota}{m'} + \frac{n}{r} \frac{m - m'}{mm'} < \frac{\iota}{m'} + \frac{n}{r} \frac{\gamma r}{Mm'} \ll \frac{1}{\tau},$$

with the last inequality a (weak) consequence of (146). □

Proof of (159). We consider the (Doob) martingale

$$(163) \quad X_i = X_i(A_1, \dots, A_i) = \mathbb{E}[X | A_1, \dots, A_i] \quad (i = 0, \dots, \tau),$$

with difference sequence $Z_i = X_i - X_{i-1}$ ($i \in [\tau]$) and $Z = \sum Z_i (= X - \mathbb{E}X)$. For the next little bit we use $\mathbb{E}_S$ for expectation with respect to $(A_i : i \in S)$.

Given $A_1, \dots, A_{i-1}$ we may express

$$(164) \quad Z_i = \mathbb{E}W - W,$$

where $\mathbb{E}$ refers to A chosen uniformly from the non-heavy edges of $\mathcal{G}$ and

$$(165) \quad \begin{aligned} W(A) &= \mathbb{E}_{[i+1, \tau]} \Phi(\mathcal{G}^* \setminus \{A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_\tau\}) \\ &\quad - \mathbb{E}_{[i+1, \tau]} \Phi(\mathcal{G}^* \setminus \{A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_\tau\}). \end{aligned}$$

For (164) just notice that

$$X_i(A_1, \dots, A_{i-1}, A) = \mathbb{E}_{[i+1, \tau]} \Phi(\mathcal{G}^* \setminus \{A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_\tau\}),$$

while

$$X_{i-1}(A_1, \dots, A_{i-1}) = \mathbb{E}_{[i, \tau]} \Phi(\mathcal{G}^* \setminus \{A_1, \dots, A_{i-1}, A_i, A_{i+1}, \dots, A_\tau\}).$$

(The first term on the r.h.s. of (165) is chosen to give (166) and, not depending on A , doesn't affect (164).)

We also have

$$(166) \quad 0 \leq W(A) \leq w_{\mathcal{G}^*}(A),$$

since these bounds hold even if we remove the $\mathbb{E}$'s in (165). Thus W satisfies the conditions in Proposition 3.5 with $b = M\Phi^*/D$ and $a \sim \Phi^*/D$ (the latter since $|\mathcal{G}|^{-1} \sum_{A \in \mathcal{G}} w_{\mathcal{G}^*}(A) = |\mathcal{G}|^{-1} \Phi^*(n/r - \iota) \lesssim \Phi^*/D$ —note $w_{\mathcal{G}^*}(A) := 0$ if $A \in \mathcal{G} \setminus \mathcal{G}^*$ —and averaging instead only over non-heavy edges can only decrease this). So for any

$$(167) \quad \vartheta \in [0, (2b)^{-1}],$$

we may apply Lemma 3.4 to each of $Z, -Z$, using Proposition 3.5 (with (164)) to bound the factors in (26), yielding

$$\max\{\mathbb{E}e^{\vartheta Z}, \mathbb{E}e^{-\vartheta Z}\} \leq e^{\tau\vartheta^2 ab} = \exp[(1 + o(1))\tau\vartheta^2 M(\Phi^*/D)^2]$$

and, for any $\lambda > 0$,

$$(168) \quad \max\{\mathbb{P}(Z > \lambda), \mathbb{P}(Z < -\lambda)\} < \exp[(1 + o(1))\tau\vartheta^2 M(\Phi^*/D)^2 - \vartheta\lambda].$$

For (159) we use (168) with $\lambda = \eta\Phi^*$ and

$$(169) \quad \vartheta = \min \left\{ \frac{\eta\Phi^*}{2\tau M(\Phi^*/D)^2}, \frac{D}{2M\Phi^*} \right\} = \frac{D}{2M\Phi^*} \min \left\{ \frac{\eta D}{\tau}, 1 \right\}$$

(the first value in “min” essentially minimizes the r.h.s. of (168) and the second enforces (167)), and should show that the exponent in (168) is then $-\omega(1)$.

Suppose first that $\eta D \leq \tau$, so ϑ takes the first value(s) in (169). Then the negative of the exponent in (168) is asymptotically (using (147) and $D \gtrsim \log n$)

$$\frac{(\eta\Phi^*)^2}{4\tau M(\Phi^*/D)^2} = \frac{\eta^2 D^2}{4\tau M} = \omega(1).$$

If instead $\eta D > \tau$, then $\vartheta = D/(2M\Phi^*)$ and the exponent in (168) is

$$(1 + o(1)) \frac{D^2}{(2M\Phi^*)^2} \tau M \left(\frac{\Phi^*}{D} \right)^2 - \frac{D}{2M\Phi^*} \eta\Phi^* = (1 + o(1)) \frac{\tau}{4M} - \frac{\eta D}{2M} = -\omega(1),$$

where we used $\eta D > \tau$ and, from (146), $\tau \gg M$. □

This completes the proof of (149). □

13. FOUNDATION

Finally, we return to the assertions listed at the end of Section 5 that will complete the proof of (8). As suggested earlier, this is currently a much longer story than it seems ought to be necessary, but we do the best we can, as usual aiming for “simplicity” rather than strongest statements.

Most of this involves behavior at m_T . This “foundation” is covered in Sections 13.1–13.2, with the final points needed for (8) mostly in Section 13.3.

13.1. Configurations and simplicity. With $V = [n]$, the *degree sequence* of $\mathcal{H} \subseteq \mathcal{K}$ is $\underline{d}(\mathcal{H}) = (d_{\mathcal{H}}(1), \dots, d_{\mathcal{H}}(n))$. In what follows $\underline{d}$ is always in

$$(170) \quad \mathcal{S} = \{(d_1, \dots, d_n) : \sum d_i = mr\}$$

(m for now unspecified), and we set

$$(171) \quad \mathcal{K}(\underline{d}) = \{\mathcal{H} \subseteq \mathcal{K} : \underline{d}(\mathcal{H}) = \underline{d}\}.$$

We will work with the hypergraph version of the “configuration model” of Bollobás [5] (see [32] for a good discussion of the model and antecedents). Let T be a set of size mr and $T_1 \cup \dots \cup T_n$ a partition of T (into *pre-verts*) with $|T_j| = d_j \forall j$ (so $mr = \sum d_i$). A *configuration* is an (*unordered*) partition of T into *pre-edges* of size r ; it is *simple* if

$$(172) \quad \text{no pre-edge meets any pre-vert more than once}$$

and

$$\text{no two pre-edges meet exactly the same pre-verts.}$$

The projection $\pi : T \rightarrow V$ given by $\pi(T_j) = \{j\} \forall j$ maps each *simple* configuration to some $\mathcal{H} \in \mathcal{K}(\underline{d})$, and for any such $\mathcal{H}$ we have

$$|\pi^{-1}(\mathcal{H})| = \prod d_j!.$$

Thus

$$(173) \quad |\mathcal{K}(\underline{d})| = \Psi \gamma(\underline{d}) (\prod d_j!)^{-1},$$

where $\Psi = \Psi(m, r)$ is the number of configurations (which of course depends only on m and r) and $\gamma(\underline{d})$ is the probability that a uniformly chosen configuration is simple. The (easily calculated) Ψ is irrelevant here, since we are only interested in ratios, but we *will* need some crude information on the $\gamma(\underline{d})$'s. (Much better estimates can be gotten by adapting the switching methods of McKay and Wormald; see [26, 27] or, again, [32].)

Since it costs nothing to do so, and perhaps helps clarify what's relevant, we state our basic result here in some generality, assuming the setup in the paragraph containing (172), with $r \geq 3$ fixed and $D = \max d_i$.

Lemma 13.1. *If*

$$(174) \quad m^{2r-3} > n^{r-1} D^{2r-1},$$

then $\gamma(\underline{d}) = e^{-O(D)}$.

(We will use this with $D = n^{o(1)}$ —or, really, with D growing at most a little faster than $\log n$ —so (174) won't be an issue.)

Proof of Lemma 13.1. Here we think of configurations in terms of maps, as follows. Let $E = E_1 \cup \dots \cup E_m$, with the E_i 's disjoint r -sets. A bijection $\sigma : E \rightarrow T$ gives the configuration $\{\sigma(E_1), \dots, \sigma(E_m)\}$, and we say σ is simple if the configuration is. Thus for a uniform σ , $\gamma(\underline{d}) = \mathbb{P}(\sigma \text{ is simple})$ and Lemma 13.1 becomes

$$(175) \quad \text{under the assumptions of Lemma 13.1, } \mathbb{P}(\sigma \text{ is simple}) = e^{-O(D)}.$$

The proof of this uses the Lovász Local Lemma [10] in the following form (see [1], Lemma 5.1.1 and the remark beginning near the bottom of p. 71).

Lemma 13.2. *Let $A_1, \dots, A_s$ be events in a probability space, Γ a graph on $[s]$ (thought of as a set of edges), and $x_1, \dots, x_s \in [0, 1]$. Suppose that for any $i \in [s]$ and $S \subseteq [s] \setminus (\{i\} \cup \{j : ij \in \Gamma\})$,*

$$(176) \quad \mathbb{P}(A_i | \bigwedge_{j \in S} \bar{A}_j) \leq x_i \prod_{ij \in \Gamma} (1 - x_j).$$

Then

$$\mathbb{P}(\bigwedge_i \bar{A}_i) \geq \prod (1 - x_i).$$

Our use of this, which is reminiscent of [13] (or see [1, Sec. 5.6]), depends on the following observation.

We consider bijections $\sigma : [N] \rightarrow [N]$, each for now regarded as a set of N cells $(i, \sigma(i))$ of an $N \times N$ array M . We use *pattern* to mean a set of cells (in M), no two on a line (i.e. row or column), and define patterns X, Y to be adjacent ($X \sim Y$) if some line meets both. Let σ be a uniform bijection and, for a pattern X , let A_X be the event $\{\sigma \supseteq X\}$, noting that $(a)_t = a(a-1) \cdots (a-t+1)$

$$(177) \quad \mathbb{P}(A_X) = 1/(N)_{|X|}.$$

Proposition 13.3. *If $X, X_1, \dots, X_t$ are patterns with $X \not\sim X_i \forall i$, then*

$$\mathbb{P}(A_X | \bigwedge_{i=1}^t \bar{A}_{X_i}) \leq \mathbb{P}(A_X).$$

Proof. We may assume $X = \{(i, i) : i \in [k]\}$, so

$$(178) \quad \bigcup X_j \subseteq \{k+1, \dots, N\}^2.$$

Set $\mathcal{R} = \{\sigma : \sigma(i) = i \forall i \in [k]\}$ (so $A_X = \{\sigma \in \mathcal{R}\}$). With $B = \bigwedge_{i=1}^t \bar{A}_{X_i}$, it is enough to exhibit, for any distinct $j_1, \dots, j_k \in [N]$, an injection $\psi : \mathcal{R} \rightarrow \mathcal{T} := \{\sigma : \sigma(i) = j_i \forall i \in [k]\}$ satisfying

$$(179) \quad \sigma \in B \Rightarrow \psi(\sigma) \in B.$$

Here it's convenient to interpret a bijection σ as a perfect matching of $K_{N,N}$ (whose vertex set we regard as two copies of $[N]$). For $\sigma \in \mathcal{R}$ and $\tau := \{(i, j_i) : i \in [k]\}$ (also thought of as a matching of $K_{N,N}$), the components of $\sigma \cup \tau$ are paths and cycles, each alternating with respect to (σ, τ) (with the obvious meaning; in particular an edge of $\sigma \cap \tau$ is considered an alternating 2-cycle), and with the ends of the paths the vertices not covered by τ . (Some—many—of these paths may be single edges of σ .)

We then take $\psi(\sigma)$ to consist of τ together with all edges that complete path components of $\sigma \cup \tau$ to cycles. It is straightforward to check that ψ has the desired properties; that is, it maps $\mathcal{R}$ injectively to $\mathcal{T}$ and satisfies (179). (Both of these follow from the observation that the edges of $\psi(\sigma)$ not in σ are precisely those not of the form (i, i) that meet at least one of the two copies of $[k]$.) $\square$

We return to (175). We will use Proposition 13.3 with $N = mr$ and E and T our two copies of $[N]$ (so M is an $E \times T$ array). Define a *block* to be a subarray indexed by some $E_i \times T_j$ (denoted B_{ij}). We consider two types of patterns (“loops” and “repeats”):

- (L) two cells in the same block;
- (R) for some $i \neq j$ and distinct $l_1, \dots, l_r, 2r$ cells, one in each of the blocks indexed by $\{i, j\} \times \{l_1, \dots, l_r\}$.

Then a bijection $\sigma : E \rightarrow T$ is simple iff it contains none of these patterns, and

$$(180) \quad \text{each cell lies in } O(D) \text{ patterns of type L and } O(mnr^{r-1}D^{2r-1}) \text{ of type R.}$$

Now let $X_1, \dots, X_s$ run over patterns of types L and R, write A_i for A_{X_i} , and let Γ be the graph on $[s]$ with adjacency corresponding to adjacency of patterns as in Proposition 13.3 (so $i \sim j$ iff $X_i \sim X_j$). Since

lines have size $mr = O(m)$ (and patterns have size $O(1)$), each pattern is adjacent to $O(mD)$ patterns of type L and $O(m^2 n^{r-1} D^{2r-1})$ of type R. So if we take (say)

$$x_i = \begin{cases} x := 2N^{-2} & \text{if } X_i \text{ is of type L,} \\ y := 2N^{-2r} & \text{if } X_i \text{ is of type R,} \end{cases}$$

then each of the products $\prod_{i \in \Gamma} (1 - x_j)$ in (176) is

$$(1 - x)^{O(mD)} (1 - y)^{O(m^2 n^{r-1} D^{2r-1})} \sim 1$$

(the asymptotic following from (174)), which with (177) implies (176).

Thus, since (180) bounds the numbers of type L and R patterns by $O(m^2 D)$ and $O(m^3 n^{r-1} D^{2r-1})$ respectively, Lemma 13.2 gives (again using (174))

$$(181) \quad \mathbb{P}(\sigma \text{ is simple}) \geq (1 - x)^{O(m^2 D)} (1 - y)^{O(m^3 n^{r-1} D^{2r-1})} = e^{-O(D)},$$

which is (175) and completes the proof of Lemma 13.1. (Note (181) fails for $r = 2$ —as it should, since Lemma 13.1 is not true in this case.) $\square$

13.2. Degrees. In this section *only* we take $m = m_T$ and $\mathcal{H} = \mathcal{H}_T$. Notice that we may choose $\mathcal{H}$ by first choosing $\underline{d} := \underline{d}(\mathcal{H}) \in \mathcal{S} (= \{(d_1, \dots, d_n) : \sum d_i = mr\})$ as in (170) and then $\mathcal{H}$ itself uniformly from $\mathcal{K}(\underline{d})$ (see (171)).

Now thinking of the law of $\underline{d}(\mathcal{H})$, we set

$$\mathcal{L} = \{\underline{d} \in \mathcal{S} : d_i \geq \delta_i \forall i\}.$$

(The δ_i 's are our usual δ_x 's, so are asymptotic to $\varepsilon \log n$.) Then with $\mathbb{P}_h$ the probability measure on $\mathcal{S}$ given by

$$\mathbb{P}_h(\underline{d}) \propto |\mathcal{K}(\underline{d})|,$$

we have

$$\mathbb{P}_h(\mathcal{L}) = \beta(\mathcal{K})$$

and

$$(182) \quad \mathbb{P}(\underline{d}(\mathcal{H}) = \underline{d}) = \mathbb{P}_h(\underline{d} | \mathcal{L})$$

($= \mathbb{P}_h(\underline{d}) / \mathbb{P}_h(\mathcal{L})$ if $\underline{d} \in \mathcal{L}$). We compare $\mathbb{P}_h$ to the probability measure $\mathbb{P}_u$ on $\mathcal{S}$ given by

$$\mathbb{P}_u(\underline{d}) \propto (\prod d_i!)^{-1}.$$

Thus $\mathbb{P}_u(\underline{d})$ is the probability that $mr (= m_T r)$ balls placed uniformly and independently in urns $U_1, \dots, U_n$ produce the occupation statistics $\underline{d}$, and (by (173))

$$\mathbb{P}_h(\underline{d}) \propto \gamma(\underline{d}) \mathbb{P}_u(\underline{d}).$$

For better understanding the law of $\underline{d}(\mathcal{H})$ (as in (182)) we will use Lemma 13.1 and the following easy observations, whose verifications we omit.

Observation 13.4. Under $\mathbb{P}_u$, for any $L \subseteq [n]$, with $|L| = l$,

$$\sum_{i \in L} d_i \sim \text{Bin}(mr, l/n).$$

Observation 13.5. If $\gamma(\underline{d}) > \xi$ for all $\underline{d} \in \mathcal{J} \subseteq \mathcal{S}$, then for any $\mathcal{E} \subseteq \mathcal{S}$,

$$\mathbb{P}_h(\mathcal{E}) / \mathbb{P}_h(\mathcal{J}) < \xi^{-1} \mathbb{P}_u(\mathcal{E}) / \mathbb{P}_u(\mathcal{J}).$$

Observation 13.6. For any $i \in [n]$, $\mathbb{P}_u(\mathcal{L}|d_i = k)$ is decreasing on $\{k \geq \delta_i\}$, implying that, for any $J \geq \delta_i$,

$$\begin{aligned} \mathbb{P}_u(d_i \geq J|\mathcal{L}) &= \frac{\sum_{k \geq J} \mathbb{P}_u(d_i = k) \mathbb{P}_u(\mathcal{L}|d_i = k)}{\sum_{k \geq \delta_i} \mathbb{P}_u(d_i = k) \mathbb{P}_u(\mathcal{L}|d_i = k)} \\ &\leq \frac{\mathbb{P}_u(d_i \geq J)}{\mathbb{P}_u(d_i \geq \delta_i)} \sim \mathbb{P}_u(d_i \geq J). \end{aligned}$$

(The initial assertion is a trivial coupling argument and, since $m \sim (n/r) \log n$, the “ $\sim$ ” is a tiny consequence of Observation 13.4 and Theorem 3.1.)

We next note that Theorem 3.1 and Observation 13.4 give (for any i , using $\mathbb{E}d_i = mr/n \sim \log n$)

$$\mathbb{P}_u(d_i > 3 \log n) < \exp \left[-(1 - o(1)) \frac{4 \log n}{2(1+2/3)} \right] = n^{-6/5+o(1)}.$$

Thus, now using Observation 13.6,

$$(183) \quad \mathbb{P}_u(\max d_i > 3 \log n | \mathcal{L}) < n^{-1/5+o(1)} = o(1).$$

Lemma 13.7. For large enough κ ,

$$(184) \quad \mathbb{P}(\max d_i(\mathcal{H}) > \kappa \log n) < n^{-\kappa}.$$

Proof. Set

$$\mathcal{J} = \{\max d_i \leq 3 \log n\} \wedge \mathcal{L} \text{ and } \mathcal{E} = \{\max d_i > \kappa \log n\} \wedge \mathcal{L}.$$

Lemma 13.1 gives

$$(185) \quad \text{if } \max d_i \leq 3 \log n \text{ then } \gamma(\underline{d}) > n^{-K}$$

for some fixed K , while Observation 13.4 and Theorem 3.2 imply

$$(186) \quad \mathbb{P}_u(d_i > \kappa \log n) < n^{-2\kappa}$$

for large enough κ (the actual bound being essentially $n^{-\kappa \log(\kappa/e)}$). The l.h.s. of (184) is then, again for large enough κ ,

$$\begin{aligned} \mathbb{P}_h(\mathcal{E})/\mathbb{P}_h(\mathcal{L}) &\leq \mathbb{P}_h(\mathcal{E})/\mathbb{P}_h(\mathcal{J}) < n^K \mathbb{P}_u(\mathcal{E})/\mathbb{P}_u(\mathcal{J}) \\ &\sim n^K \mathbb{P}_u(\max d_i > \kappa \log n | \mathcal{L}) < (1 + o(1)) n^{-2\kappa+K+1} < n^{-\kappa}, \end{aligned}$$

with the second inequality given by (185) and Observation 13.5; the “ $\sim$ ” by (183); and the third inequality by (186) and Observation 13.6. $\square$

Lemma 13.8. With $\lambda = m/|\mathcal{K}|$, $\mathbb{P}(\mathcal{K}_\lambda \text{ anemic}) < \exp[-2n^{2\delta}]$.

(We remind once more that $m = m_T$ and recall that “anemic” was defined in (36).)

Proof. With ζ_x the indicator of $\{d_{\mathcal{K}_\lambda}(x) < 2\varepsilon \log n\}$, an easy calculation gives

$$\mathbb{E}\zeta_x < n^{-1+2\delta} =: \rho.$$

(Like (30), this uses the first bound in (25) of Theorem 3.1; of course ζ_x is binomial while its counterpart in (30) was hypergeometric, but the bound applies to both.)

On the other hand the ζ_x ’s form a read- r family (with corresponding ψ_i ’s the indicators $\mathbf{1}_{\{A \in \mathcal{K}_\lambda\}}$), so Theorem 3.3 gives

$$\begin{aligned} \mathbb{P}(\mathcal{K}_\lambda \text{ anemic}) &= \mathbb{P}(\sum \xi_x \geq 2rn^{2\delta}) \\ &< \exp[-D(2r\rho\| \rho)n/r] < \exp[-2n^{2\delta}] \end{aligned}$$

(using $D(K\rho\|\rho) = (K \log(K/e) + 1)\rho + O(\rho^2)$ for fixed K and small ρ).

□

Lemma 13.9. *For sufficiently large $\omega \ll \log n$,*

$$(187) \quad \mathbb{P}(\max d_{\mathcal{H}}(x, y) > \omega) < n^{-\Omega(\omega)}$$

(the maximum over distinct vertices x, y) and

$$(188) \quad \mathbb{P}(\max_{Z \in \mathcal{K}} \sum_{y \notin Z} (\delta_y - d_{\mathcal{H}-Z}(y))^+ > \omega) < n^{-\Omega(\omega)}.$$

(Recall that (188) was promised at (75) and used there to show that $\cap_t \mathfrak{R}_t^4$ is likely (see (78).)

Proof. Let

$$\mathcal{D} = \{\underline{d} : \max d_i < \kappa \log n\},$$

with $\kappa = \Theta(\omega)$ chosen so

$$(189) \quad \underline{d} \in \mathcal{D} \Rightarrow \gamma(\underline{d}) > n^{-\omega/(2r)}$$

(see Lemma 13.1), and

$$\mathcal{N} = \{\underline{d} : |\{i : d_i < 2\varepsilon \log n\}| < 2rn^{2\delta}\}$$

(so $\{\underline{d}(\mathcal{H}) \notin \mathcal{N}\} = \{\mathcal{H} \text{ anemic}\}$). Then Lemma 13.7, its use justified by our assumption that ω is somewhat large, says

$$(190) \quad \mathbb{P}(\underline{d}(\mathcal{H}) \notin \mathcal{D}) < n^{-\kappa} \quad (= n^{-\Omega(\omega)}),$$

while Lemma 13.8 implies

$$(191) \quad \mathbb{P}(\underline{d}(\mathcal{H}) \notin \mathcal{N}) \quad (= \mathbb{P}(\mathcal{H} \text{ anemic})) < \exp[-(2 - o(1))n^{2\delta}];$$

this follows from

$$\exp[-2n^{2\delta}] > \mathbb{P}(\mathcal{K}_\lambda \text{ anemic}) > \mathbb{P}(|\mathcal{K}_\lambda| = m) \mathbb{P}(\mathcal{K}_\lambda \in \mathfrak{L} | |\mathcal{K}_\lambda| = m) \mathbb{P}(\mathcal{K}_\lambda \text{ anemic} | \mathcal{K}_\lambda \in \mathfrak{L}_T),$$

since (i) $\mathbb{P}(|\mathcal{K}_\lambda| = m) \asymp m^{-1/2}$ (this is standard and easy); (ii) $\mathbb{P}(\mathcal{K}_\lambda \in \mathfrak{L} | |\mathcal{K}_\lambda| = m) = \beta(\mathcal{K}) \gtrsim \exp[-n^\delta]$ (see Corollary 4.4); and (iii) on $\{\mathcal{K}_\lambda \in \mathfrak{L}_T\}$, $\mathcal{K}_\lambda$ is distributed as $\mathcal{H}$.

By (190) and (191), Lemma 13.9 will follow if we show that, for each $\underline{d} \in \mathcal{D} \cap \mathcal{N} \cap \mathcal{L}$, (187) and (188) hold with $\mathcal{H}$ replaced by $\mathcal{G}$ chosen uniformly from $\mathcal{K}(\underline{d})$; so we fix such a $\underline{d}$ and choose $\mathcal{G}$ in this way.

We will again get at this using the configuration model; thus we fix the partition $T = \cup T_i$ with $|T_i| = d_i$, let $\pi : T \rightarrow V (= [n])$ be the corresponding projection, and for a configuration F use “ $F \in \mathfrak{G}$ ” (with $\mathfrak{G}$ TBA) to mean F is simple and $\pi(F) \in \mathfrak{G}$. Then for a uniformly chosen configuration $\mathbf{F}$ we have

$$(192) \quad \mathbb{P}(\mathcal{G} \in \mathfrak{G}) = \mathbb{P}(\mathbf{F} \in \mathfrak{G} | \mathbf{F} \text{ simple}) < \mathbb{P}(\mathbf{F} \in \mathfrak{G}) n^{\omega/(2r)}$$

(with the inequality given by (189)).

As earlier, we think of random maps; say $\mathbf{F} = \{\sigma(E_1), \dots, \sigma(E_m)\}$, with $\sigma : E \rightarrow T$ a uniform bijection (and $E = \cup E_i$ as in the proof of Lemma 13.1).

For (187), we have, with $\mathfrak{G} = \{\mathcal{G} : \max d_{\mathcal{G}}(x, y) > \omega\}$,

$$(193) \quad \mathbb{P}(\mathbf{F} \in \mathfrak{G}) < \binom{n}{2} \binom{m}{\omega} (r(r-1))^\omega \left(\frac{\kappa \log n}{mr} \right)^{2\omega} = O(n^{-\omega+o(\omega)+2}).$$

The first three terms of the first bound correspond to choosing (i) $x, y \in [n]$, (ii) ω of the E_i 's to map to preimages of edges containing x, y , and (iii) elements of these E_i 's to map to T_x and T_y ; and the last term bounds the probability that these choices behave as desired, using (177) and $\max d_i < \kappa \log n$. For the final

bound recall $\kappa = \Theta(\omega) \ll \log n$ ($\omega = n^{o(1)}$ is enough) and $mr \sim n \log n$. The combination of (192) and (193) then gives (187) (with $\mathcal{G}$ in place of $\mathcal{H}$). The $\omega/(2r)$ in (189) and (192) is overkill here, but is needed for (188).

For (188), fix Z and let $I = \{i \in V \setminus Z : d_i < 2\varepsilon \log n\}$ —so $|I| < 2rn^{2\delta}$ since $\underline{d} \in \mathcal{N}$ —and let

$$(194) \quad \mathfrak{G} = \{\mathcal{G} : |\{A \in \mathcal{G} : A \cap Z \neq \emptyset \neq A \cap I\}| > \omega/r\}.$$

Then for $\Sigma(Z) := \sum_{y \notin Z} (\delta_y - d_{\mathcal{G}-Z}(y))^+ > \omega$, we must have either $d_{\mathcal{G}}(x, y) > (2\varepsilon \log n - \delta_y) (> \omega)$ for some $x \in Z$ and $y \in V \setminus (Z \cup I)$ —which we have just shown happens with probability $n^{-\Omega(\omega)}$ —or $\mathcal{G} \in \mathfrak{G}$ (since, absent such a large codegree, the only y 's that can contribute to $\Sigma(Z)$ are those in I and $\Sigma(Z)$ is at most $r - 1$ times the cardinality in (194)). We then have (with justification similar to that for (193))

$$\begin{aligned} \mathbb{P}(\mathbf{F} \in \mathfrak{G}) &< \binom{m}{\omega/r} (r(r-1))^{\omega/r} \left(\frac{r\kappa \log n}{mr} \right)^{\omega/r} \left(\frac{|I| \cdot 2\varepsilon \log n}{mr} \right)^{\omega/r} \\ &< n^{-(1+2\delta-o(1))\omega/r}, \end{aligned}$$

and combining with (192) and multiplying by n^r for the choice of Z (and recalling ω is “sufficiently large”) gives (188). $\square$

13.3. Back to $\mathfrak{R}$. Here, finally, we fill in the remaining promises from Section 5, namely (62) (which means dealing with (57)-(59) and (61)), (68) and (70).

We now use $d_t(\cdot)$ for degree in $\mathcal{H}_t$ and revert to the “default” $m = m_t$. As in Section 9, we work with the generation of $\mathcal{H}_t$ in (5):

$$(195) \quad \mathcal{H}_t = \mathcal{H}_T \cup \mathcal{G},$$

with $\mathcal{H}_T$ uniform from $\mathfrak{L}_T$ and $\mathcal{G}$ uniform from $\binom{\mathcal{K} \setminus \mathcal{H}_T}{m-m_T}$. This supports the following little device, which will be useful in the arguments for (57), (68) and (70).

Given t , let $(\mathcal{H}, \mathcal{U})$ be the random pair gotten by choosing

$$(196) \quad \mathcal{H} \text{ uniformly from } \binom{\mathcal{K}}{m} (= \mathcal{K}^t) \text{ and then } \mathcal{U} \text{ uniformly from } \binom{\mathcal{H}}{m_T}.$$

In this section $\mathcal{H}$ will always be as in (196) (not $\mathcal{H}_t$ as it was in Section 13.2). In each application we will have some property $\mathfrak{G}$ for which we would like to show $\mathbb{P}(\mathcal{H}_t \in \mathfrak{G})$ is small, and (slightly echoing Section 9) will exploit information gotten by reversing the order in (196); that is, by choosing

$$\mathcal{U} \text{ uniformly from } \binom{\mathcal{K}}{m_T} \text{ and } \mathcal{H} \text{ uniformly from } \{\mathcal{H} \in \mathcal{K}^t : \mathcal{H} \supseteq \mathcal{U}\}.$$

Thinking of the process in this way and setting

$$(197) \quad \Theta = \mathbb{P}(\mathcal{H} \in \mathfrak{G}, \mathcal{U} \in \mathfrak{L}),$$

we have

$$(198) \quad \Theta = \mathbb{P}(\mathcal{U} \in \mathfrak{L}) \mathbb{P}(\mathcal{H} \in \mathfrak{G} | \mathcal{U} \in \mathfrak{L}) = \beta(\mathcal{K}) \mathbb{P}(\mathcal{H}_t \in \mathfrak{G})$$

(since on $\{\mathcal{U} \in \mathfrak{L}\}$, $\mathcal{H}$ is distributed as $\mathcal{H}_t$), which we will combine with upper bounds on Θ based on the viewpoint in (196).

Proof of (62). Recall this says that w.h.p. $\mathcal{H}_t$ satisfies (57)-(59) and (61) for all $t \leq T$.

For (57) we may appeal to [23]: as shown there—see the paragraph containing (131)—the probability that $\mathcal{H}$ as in (196) violates (57) is $e^{-\Omega(n)}$. (Precisely, with $\theta = (\log n)^{-1/3}$, it is shown that $e^{-\Omega(n)}$ bounds the probability that $d_{\mathcal{H}}(x) \neq (1 \pm \theta)D_{\mathcal{H}}$ for at least θn vertices x .)

Then with $\mathfrak{G} = \{\mathcal{H} \subseteq \mathcal{K} : \mathcal{H} \text{ violates (57)}\}$ (and Θ as in (197)), we have $\Theta < \mathbb{P}(\mathcal{H} \in \mathfrak{G}) = \exp[-\Omega(n)]$ and, using (198) and Corollary 4.4,

$$(\mathbb{P}(\mathcal{H}_t \text{ violates (57)}) =) \mathbb{P}(\mathcal{H}_t \in \mathfrak{G}) = \Theta/\beta(\mathcal{K}) = \exp[-\Omega(n)].$$

For (58) and (61) (with the lower bound in the former contained in the latter, as observed following (61)), we use (195), noting that, given $\mathcal{H}_T$, each $d_{\mathcal{G}}(x)$ is hypergeometric with

$$(199) \quad \mu_x := \mathbb{E}d_{\mathcal{G}}(x) = \frac{D_{\mathcal{K}} - d_T(x)}{|\mathcal{K}| - m_T}(m - m_T) \sim (1 - m_T/m)D_m$$

(since $D_m = D_{\mathcal{K}}m/|\mathcal{K}|$).

For the upper bound in (58) we first note that $\Delta_{\mathcal{H}_T} = O(\log n)$ w.h.p. by Lemma 13.7 (in which, recall, $\mathcal{H}$ was $\mathcal{H}_T$). Then for $\mathcal{G}$, Theorem 3.2 gives (very wastefully but we don't care)

$$\mathbb{P}(\exists x, t \ d_t(x) > 3rD_m) < n^{r+1} \exp[-3rD_m \log(3r/e)] = o(1).$$

For (61) it will be enough to consider $\mathcal{G}$. Here (199) and $D_m \sim (m/m_T) \log n$ imply $\mu_x \gtrsim (2r-1) \log n$, and then Theorem 3.1 (see the first bound in (25)) gives (for any x)

$$(200) \quad \mathbb{P}(d_{\mathcal{G}}(x) < 2\varepsilon D_m) < \exp[-\mu_x \varphi(-1 + 2\varepsilon D_m/\mu_x)] < n^{-(2r-1)+O(\varepsilon \log(1/\varepsilon))}.$$

For (59), we again use (195), noting to begin that Lemma 13.9 says that w.h.p. $\mathcal{H}_T$ has maximum codegree $O(1)$. For $\mathcal{G}$ we again use Theorem 3.2 (with plenty of room): each $d_{\mathcal{G}}(x, y)$ is hypergeometric with mean

$$\frac{d_{\mathcal{K}}(x, y) - d_T(x, y)}{|\mathcal{K}| - m_T}(m - m_T) < rD_m/n,$$

whence $\mathbb{P}(\max d_t(x, y) > CD_m/n) < n^{-(r+1)}$ for a suitable fixed C . □

Proof of (68). Given $Z \in \mathcal{K}$, set

$$\mathfrak{G} = \{\mathcal{H} \subseteq \mathcal{K} : \bar{\varphi}_Z(\mathcal{H}) < \eta\beta(\mathcal{K})\},$$

so (68) is

$$(201) \quad \mathbb{P}(\mathcal{H}_t \in \mathfrak{G}) < \eta.$$

Let $(\mathcal{H}, \mathcal{U})$ and Θ again be as in (196) and (197), and set $\mathcal{G} = \mathcal{H} - Z$. Note that, as $\bar{\varphi}_Z(\mathcal{H})$ depends only on $|\mathcal{H}|$ and $\mathcal{H} - Z$, membership of $\mathcal{H}$ in $\mathfrak{G}$ is decided by $\mathcal{G}$. We now think of choosing first $\mathcal{G}$ and then $\mathcal{H} \setminus \mathcal{G}$ and $\mathcal{U}$. The law of $\mathcal{G}$ plays no role here; what matters is that $\mathcal{H} \setminus \mathcal{G}$ is uniform from $\mathfrak{J}_Z(\mathcal{H})$ ($= \{\mathcal{H}' \in \mathcal{K}^t : \mathcal{H}' - Z = \mathcal{G}\}$; see (63)), so that (essentially by definition; see (64), (65))

$$\mathbb{P}(\mathcal{U} \in \mathfrak{L}|\mathcal{G}) = \bar{\varphi}_Z(\mathcal{H}).$$

Thus

$$\Theta = \mathbb{P}(\mathcal{H} \in \mathfrak{G})\mathbb{P}(\mathcal{U} \in \mathfrak{L}|\mathcal{H} \in \mathfrak{G}) < \mathbb{P}(\mathcal{H} \in \mathfrak{G}) \cdot \eta\beta(\mathcal{K}).$$

We then sacrifice the first factor on the r.h.s. and combine with (198) to get (201). (The sacrifice in this case is substantial, but we aren't asking much and can afford it.) □

Proof of (70). (Recall this said $\mathbb{P}(\mathcal{H}_t \notin \mathfrak{R}^3) < \exp[-(1 - o(1))n^{2\delta}] \forall t \leq T$.) Set $q = m/|\mathcal{K}|$ and (as in $\mathfrak{R}^3$ and Lemma 13.8 resp.) $\alpha = m_T/m$ and $\lambda = m_T/|\mathcal{K}|$; so $\mathcal{K}_\lambda = (\mathcal{K}_q)_\alpha$. From Lemma 13.8 and the definition of $\mathfrak{R}^3$ (and the fact that on $\{|\mathcal{K}_q| = m\}$, $\mathcal{K}_q$ is distributed as $\mathcal{H}$), we have

$$\begin{aligned} \exp[-2n^{2\delta}] &> \mathbb{P}(\mathcal{K}_\lambda \text{ anemic}) \\ &> \mathbb{P}(|\mathcal{K}_q| = m) \mathbb{P}(\mathcal{H} \notin \mathfrak{R}^3) \exp[-(1 - o(1))n^{2\delta}], \end{aligned}$$

which, since $\mathbb{P}(|\mathcal{K}_q| = m) \asymp m^{-1/2}$, implies $\mathbb{P}(\mathcal{H} \notin \mathfrak{R}^3) < \exp[-(1 - o(1))n^{2\delta}]$. We then set $\mathfrak{G} = \{\mathcal{H} \subseteq \mathcal{K} : \mathcal{H} \notin \mathfrak{R}^3\}$ to obtain, as in the above treatment of (57),

$$\begin{aligned} \mathbb{P}(\mathcal{H}_t \notin \mathfrak{R}^3) &= \mathbb{P}(\mathcal{H}_t \in \mathfrak{G}) = \Theta/\beta(\mathcal{K}) \\ &\leq \mathbb{P}(\mathcal{H} \notin \mathfrak{R}^3)/\beta(\mathcal{K}) < \exp[-(1 - o(1))n^{2\delta}]. \end{aligned}$$

□

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