Algorithmic aspects of left-orderings of solvable Baumslag–Solitar groups via its dynamical realization*

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Abstract. We answer a question of Calderoni and Clay [4] by showing that the conjugation equivalence relation of left orderings of the Baumslag-Solitar groups $\mathrm{BS}(1,n)$ is hyperfinite for any n. Our proof relies on a classification of $\mathrm{BS}(1,n)$'s left-orderings via its one-dimensional dynamical realizations. We furthermore use the effectiveness of the dynamical realizations of $\mathrm{BS}(1,n)$ to study algorithmic properties of the left-orderings on $\mathrm{BS}(1,n)$.

1 Introduction

A group G is *left-orderable* if there is a linear ordering $\prec$ on the elements of G such that for all $f, g, h \in G$, $g \prec h$ implies $fg \prec fh$. We refer to such a linear ordering as a *left-ordering* of G. The study of (left-)orderable groups has a long tradition in mathematics starting with the work of Dedekind and Hölder in the late 19th and early 20th century. Dedekind famously characterized the real numbers as a complete bi-orderable Abelian group and Hölder showed that any Archimedean ordered group is isomorphic to an additive subgroup of the reals with their standard ordering. These fundamental results led to an influx of interest in orderable groups and established their theory as a cornerstone of group theory; see [18] for a treatment of the classical theory.

While most studies of orderable groups employed algebraic methods, there is a strong connection with one-dimensional dynamics. Indeed, a group is left-orderable if and only if it acts faithfully on the real line by orientation preserving homeomorphisms [14] (see also Section 2.1). Motivated by this observation, Navas [21] systematically applied dynamical ideas to study orderable groups and give new proofs of results previously obtained by algebraic methods, as well as new results. He gave a new dynamical proof of the fact that

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$\mathrm{LO}(F_n)$, the space of left-orderings of the non-abelian free group of rank n, is homeomorphic to the Cantor set if $n \geq 2$ [21, Theorem A], and of the result of Linnell which states that the number of left-orderings of a group is either finite or uncountable [21, Theorem C]. Since then one-dimensional dynamics has become an important tool in orderable group theory with many applications. Generalizing Navas's approach, Rivas proved that for all left-orderable groups G and H, the free-product $G * H$ has no isolated left-orderings [24, Theorem A]. In [19, Theorem 1.1] dynamics was used to find new examples of groups with isolated left-orderings, and in [23,25] characterizations of left-orderings of various solvable groups were obtained.

These developments have led to various natural questions about the space of left-orderings of groups, $\mathrm{LO}(G)$. Of particular interest to this paper is a question by Deroin, Navas, and Rivas [10, Question 2.2.11] that asks if the conjugation equivalence relation of G on $\mathrm{LO}(G)$ is standard. This question has attracted the interest of Calderoni and Clay [3] who initiated the study of the conjugation equivalence of orderings on a fixed group G in the setting of descriptive set theory.

Among other groups, Calderoni and Clay studied the space of linear orders of solvable Baumslag-Solitar groups [4]. Baumslag-Solitar groups are introduced in [1] as an example of non-Hopfian groups, and have served as important examples and counterexamples in group theory [2, Chapter 5]. In particular, the solvable Baumslag-Solitar groups admit nice structural properties and thus provide useful test cases for theories and techniques.

The main contribution of our paper is in this context: We show that the conjugation equivalence relation of the solvable Baumslag-Solitar group $BS(1, n)$ is hyperfinite for every n. This answers a question posed by Calderoni and Clay [4].

The algorithmic aspects of left-orderable groups have also seen attention in the past, mainly focusing on the complexity of orderings of computable groups [12,16,8,9] and their reverse mathematics [27,28]. In Section 4, we explore how a group's dynamics can be used to study algorithmic properties of its orderings, using $\mathrm{BS}(1, n)$ as an example. Our main result shows that the index sets of orderings that are conjugates of a given ordering with irrational base point is Σ^0_3-complete. Our proof relies on the dynamical realizations of $\mathrm{BS}(1, n)$ and the machinery developed in prior sections.

Before we prove the main results of this paper we review the main tools used in their proofs and give the necessary definitions to formally state our results.

2 Left-orderable groups, their dynamical realizations, and E_{lo}

A left ordering $\prec$ on a group G induces a partition of G into disjoint subsets

$$P^+ = \{g \in G \mid g \succ id\}, P^- = (P^+)^{-1} = \{g \in G \mid id \succ g\} \text{ and } \{id\}$$

where P^+ is called the *positive cone* of the left-ordering $\prec$. Notice that the reverse order $\prec^*$, defined as $g \prec^* h$ if and only if $h \prec g$, is also a left-ordering of G with associated positive cone P^-. It is not hard to see that the positive cones on G are precisely the subsets $P \subseteq G$ satisfying

$$P \cap P^{-1} = \varnothing,\ PP \subseteq P \text{ and } P \cup P^{-1} \cup \{id\} = G.$$

Moreover, every positive cone gives rise to an associated left-ordering $\prec_P$ via

$$g \prec_P h \iff g^{-1}h \in P$$

and thus we get a bijection between positive cones and left-orderings on G.

The collection of all positive cones P of G forms a closed subspace, $\mathrm{LO}(G)$, via the subspace topology of 2^G and is thus a Polish space [26]. Given any positive cone $P \in \mathrm{LO}(G)$ and any element $g \in G$, the set $P^g = \{p^g = g^{-1}pg : p \in P\}$ defines a positive cone on G. Consequently, the group G acts on $\mathrm{LO}(G)$ via conjugation simply by defining $g(P) = P^{g^{-1}}$ for all $g \in G$ and $P \in \mathrm{LO}(G)$. It is not hard to see that the action of G on $\mathrm{LO}(G)$ is continuous and, in fact, computable uniformly in G.

Remark 1. A countable group G is *computable* if its domain and group operation are computable. We can assume that the domain of G is all of ω and thus view positive cones $P \subset G$ as subsets of the natural numbers and 2^G simply as 2^ω. Then the above comment that the action of G on $\mathrm{LO}(G)$ is *computable uniformly in* G means that there is a Turing operator Φ such that

$$\Phi(G, P; g) = g(P) \text{ for all left-orderable groups } G, \text{ positive cones } P, \text{ and } g \in G.$$

2.1 Dynamical realizations

Although left-orderability is an algebraic concept, it has a deep connection to one-dimensional dynamics. In particular, the left-orderable countable group can be characterized in dynamical terms.

Theorem 1 ([14, Theorem 6.8]). *Let G be a countable group. Then the following are equivalent:*

1. *G is left-orderable.*
2. *G acts faithfully on the real line by orientation preserving homeomorphisms, i.e., there is a faithful representation $\gamma : G \to \mathrm{Homeo}_+(\mathbb{R})$.*

Let us elaborate on Theorem 1. Given an embedding D of G into $\mathrm{Homeo}_+(\mathbb{R})$ and a dense sequence $(x_1, \dots)$ in $\mathbb{R}$, we can obtain a positive cone $P_D = P_D(x_1, \dots)$ as follows: we define $g \in P_D$ if, for the least i such that $D(g)(x_i)$ is not a fixed point, $D(g)(x_i) > x_i$. The proof of the reverse implication that G is left-orderable implies that G embeds into $\mathrm{Homeo}_+(\mathbb{R})$ is effective. In particular, given a left-ordering on G, there is an associated group action of G on the real line, called a *dynamical realization* of G, constructed as follows. Fix a left-ordering $\prec$ on G and, since G is countable, fix an enumeration of the elements of $G = (g_0, g_1, \dots)$. We define a map $t : G \to \mathbb{R}$ that preserves $\prec$, namely,

$$t(g) < t(h) \iff g \prec h,$$

by defining $t : G \to \mathbb{R}$ inductively starting with $t(g_0) = 0$ and

$$t(g_i) = \begin{cases} \max\{t(g_0), \dots t(g_{i-1})\} + 1 & \text{if } (\forall j < i) g_j \prec g_i \\ \min\{t(g_0), \dots t(g_{i-1})\} - 1 & \text{if } (\forall j < i) g_i \prec g_j \\ \frac{t(g_m) + t(g_n)}{2} & \text{if } g_i \in (g_m, g_n),\ m, n < i \text{ and } (\forall j < i) g_j \notin (g_m, g_n) \end{cases}$$

Then we can define an action of G on $t(G)$ via $g(t(g_i)) = t(gg_i)$ and extend this action continuously to the closure $\overline{t(G)}$. Finally, we extend the action to $\mathbb{R} \setminus \overline{t(G)}$ by affine maps to obtain a faithful orientation-preserving action of G on $\mathbb{R}$ and a faithful representation $D : G \to \mathrm{Homeo}^+(\mathbb{R})$. By construction, this action

– has no global fixed point unless G is the trivial group, and
– the orbit of 0 is free; i.e., the stabilizer of 0 under this action is trivial.

These two properties characterize the dynamical realization up to topological semiconjugacy. In particular, we have

Lemma 1 ([21, Lemma 2.8]). *Let $\prec_1$ be a left-ordering on a non-trivial countable group G, and let D_1 be a dynamical realization of $\prec_1$. Let D_2 be an action of G on $\mathbb{R}$ by orientation-preserving homeomorphisms such that*

– *D_2 has no global fixed point, and*
– *the orbit of 0 is free.*

If $\prec_2$ is a left-ordering on G defined by

$$g \prec_2 h \Leftrightarrow D_2(g)(0) < D_2(h)(0),$$

then the two left-orderings $\prec_1$ and $\prec_2$ coincide if and only if D_2 is topologically semiconjugate *to D_1 relative to 0. That is, there exists a continuous non-decreasing surjective map $\varphi : \mathbb{R} \to \mathbb{R}$ such that $D_1(g) \circ \varphi = \varphi \circ D_2(g)$ for all $g \in G$ and that $\varphi(0) = 0$* [4].

Remark 2. Topological semiconjugacy is not an equivalence relation as φ could collapse some intervals to points. The lemma above says that among all actions satisfying the two conditions, the dynamical realization is "minimal".

It follows that a different choice of enumeration of G yields a different action that is topologically semiconjugate to D. Therefore, we may speak of the dynamical realization of a left-ordering on G without referencing the enumeration of G. From the dynamical realization, we can also recover the original positive cone P on G and its conjugate.

Proposition 1. *Suppose that D is a dynamical realization of a left-ordering P of G. Let $t : G \to \mathbb{R}$ be the order-preserving map used in the construction of D. Then for any $h \in G$, we have*

$$P^h = \{g \in G : D(g)(t(h^{-1})) > t(h^{-1})\}.$$

Proof. When $h = id$, we have

$$P = \{g \in G : D(g)(t(id)) > t(id)\}$$

since t is an order-preserving map. In general, we have

$$\begin{aligned} P^h &= \{h^{-1}gh : g \in P\} \\ &= \{h^{-1}gh : D(g)(t(id)) > t(id)\} \\ &= \{f \in G : D(hfh^{-1})(t(id)) > t(id)\} \\ &= \{f \in G : D(f)(t(h^{-1})) > t(h^{-1})\}. \end{aligned}$$

□

An application of the dynamical realization that is useful for this paper is an effective classification of the left-orderings on the solvable Baumslag–Solitar group $\mathrm{BS}(1, n)$ [23, Theorem 4.2]. We will review this classification in Section 3.

[4] The condition that φ fixes the origin was not stated in [21], but was used in the proof as pointed out in [20, Lemma 3.7]

2.2 Descriptive set theory and E_{lo}

For a fixed group G, the conjugation action of G on $\mathrm{LO}(G)$ defines an orbit equivalence relation, denoted E_{lo}^G, where

$$P\, E_{lo}^G\, Q \Leftrightarrow \exists g \in G, g(P) = Q.$$

When G is countable, every equivalence class of E_{lo}^G (or just E_{lo} if G is clear from context) is countable.

Equivalence relations where each equivalence class is countable are called *countable equivalence relations* and are a major topic in descriptive set theory, where the complexity of equivalence relations is measured using Borel reducibility $\leq_B$. The structure of the quasi-order of countable Borel equivalence relations under Borel reducibility is complicated and its investigation is an active research area, see [17] for an overview of developments. Let us mention three benchmark equivalence relations:

1. The *identity relation* on 2^ω, id^{2^ω}, is the least complicated equivalence relation among countable equivalence relations on uncountable spaces. The equivalence relations reducible to id^{2^ω} are called *smooth*.
2. The equivalence relation of eventual equality on 2^ω,

$$x\ E_0\ y \iff \exists m(\forall n > m) x(n) = y(n)$$

   is the archetypical non-smooth *hyperfinite* equivalence relation (i.e., an increasing union of equivalence relations having only finite classes). By [15] every hyperfinite equivalence relation is either bi-reducible with E_0 or smooth.
3. The orbit equivalence relation S of the shift action of F_2 on 2^{F_2} is *universal* for countable Borel equivalence relations, i.e., every other countable Borel equivalence relation is Borel reducible to it.

While the interval (id^{2^ω}, E_0) is trivial, the interval between E_0 and S is known to be extremely complicated.

A fundamental result due to Feldman and Moore [13] shows that the countable equivalence relations are precisely the orbit equivalence relations of Borel actions of countable groups. One major conjecture in the area, known as Weiss's conjecture, aims to shed light on which groups cannot have complicated orbit equivalence relations. It states that every Borel action of an amenable group has a hyperfinite orbit equivalence relation. So far this conjecture has not been fully confirmed and only partial results are known with the latest advance made in [7].

Deroin, Navas, and Rivas [10] asked if $\mathrm{LO}(G)$ modulo the action of G is standard, which is equivalent to asking if E_{lo}^G is smooth. Calderoni and Clay generalized this to studying the complexity of equivalence relations E_{lo}^G under Borel reductions [3,4,5]. They showed that E_{lo} is universal for free products of countable left-orderable groups and not smooth for many groups, including the Baumslag–Solitar group $\mathrm{BS}(1,n)$ and the Thompson's group F. They also showed that $E_{lo}^{\mathrm{BS}(1,2)}$ is hyperfinite. It is still open if there is a group G with E_{lo}^G being *intermediate*, namely, strictly between E_0 and S. There are also other closely related orbit equivalence relations, including the action of $\mathrm{Aut}(G)$ on the Archimedean ordering of $\mathbb{Z}^n$ or $\mathbb{Q}^n$. In [6, Theorem 1.1], it was shown that the orbit equivalence relation of $\mathrm{Aut}(\mathbb{Q}^2)$ on the space of Archimedean orderings of $\mathbb{Q}^2$ is not smooth. Extending this result, Poulin showed that the action of $\mathrm{Aut}(\mathbb{Q}^n)$ on $\mathrm{LO}(\mathbb{Q}^n)$ is not hyperfinite when $n \geq 3$ [22, Corollary 1.3].

In the next section, we use the affine action of $\mathrm{BS}(1,n)$ on $\mathbb{R}$ to show that the $E_{lo}^{\mathrm{BS}(1,n)}$ is hyperfinite for every n (Theorem 3), answering a question of Calderoni and Clay [5, Question 4.2]. While this result does follow from Weiss's conjecture, the conjecture has not been confirmed for $\mathrm{BS}(1,n)$, $n>2$. In any case, we believe that our proof is of interest as it is quite elementary compared to known proofs of parts of Weiss's conjecture. Furthermore, the result determines exactly the complexity of the conjugation action of $\mathrm{BS}(1,n)$ on $\mathrm{LO}(\mathrm{BS}(1,n))$ which is of interest from the orderable group theory perspective.

3 $E_{lo}^{\mathrm{BS}(1,n)}$ is hyperfinite

The solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$, given by the presentation

$$\langle a, b : b^{-1}ab = a^n \rangle,$$

is an important example in group theory. The normal closure $\langle\langle a \rangle\rangle$ of a is abstractly isomorphic to $\mathbb{Z}[1/n]$, the subgroup of $\mathbb{Q}$ generated by $1/n, 1/n^2, 1/n^3, \dots$, via an isomorphism sending a to 1 and $b^k a b^{-k}$ to $1/n^k$ for every $k \in \mathbb{Z}$. Abusing notation, we will write elements of $\langle\langle a \rangle\rangle$ as a^r where $r \in \mathbb{Z}[1/n]$. The quotient of $\mathrm{BS}(1,n)$ by the normal closure of a is the infinite cyclic group generated by the image of b. Therefore, $\mathrm{BS}(1,n)$ fits into a (split) short exact sequence

$$0 \to \mathbb{Z}[1/n] \to \mathrm{BS}(1,n) \to \mathbb{Z} \to 0,$$

and admits the semidirect product structure $\mathrm{BS}(1,n) = \langle\langle a \rangle\rangle \rtimes \langle b \rangle$. The elements of G have the normal forms $a^r b^s$ where $r \in \mathbb{Z}[1/n]$ and $s \in \mathbb{Z}$. As a warm-up, we first recall a well-known construction of left-orderings using a short exact sequence.

Proposition 2. *Let K and H be left-orderable groups equipped with positive cones $P_K \subset K$ and $P_H \subset H$. Consider the following short exact sequence of groups:*

$$1 \to K \to G \xrightarrow{\pi} H \to 1$$

The set $P_G := \{g \in G \mid \pi(g) \in P_H\} \cup P_K$ defines a positive cone of G.

To order $\mathrm{BS}(1,n)$, we observe that there are exactly two orderings on $\mathbb{Z}[1/n]$ and $\mathbb{Z}$: the ordering coming from the standard ordering on $\mathbb{R}$ and its reversal. Applying Proposition 2, we get four left-orderings on $\mathrm{BS}(1,n)$. The positive cones of these four left-orderings are:

- $P_\infty^{++} = \{a^r b^s : s > 0 \vee (s = 0 \wedge r > 0)\}$,
- $P_\infty^{+-} = \{a^r b^s : s > 0 \vee (s = 0 \wedge r < 0)\}$,
- $P_\infty^{-+} = \{a^r b^s : s < 0 \vee (s = 0 \wedge r > 0)\}$,
- $P_\infty^{--} = \{a^r b^s : s < 0 \vee (s = 0 \wedge r < 0)\}$,

We note that all four left-orderings above are conjugation invariant. In other words, they are all bi-orderings on $\mathrm{BS}(1,n)$ and the action of $\mathrm{BS}(1,n)$ on $\mathrm{LO}(\mathrm{BS}(1,n))$ fixes these four bi-orderings. However, $\mathrm{BS}(1,n)$ admits other left-orderings not coming from Proposition 2. To study $E_{lo}^{\mathrm{BS}(1,n)}$, we will review the classification of all left-orderings on $\mathrm{BS}(1,n)$ for any integer $n \geq 2$ due to Rivas [23, Theorem 4.2]. Note that although Rivas stated the classification of left-orderings only for $\mathrm{BS}(1,2)$, his proof works without modification for any positive integer n. The proof for arbitrary positive integers n is also presented in [10].

A different source of left-orderings on $\mathrm{BS}(1,n)$ comes from the affine action of $\mathrm{BS}(1,n)$ on $\mathbb{R}$. Consider the action of $\mathrm{BS}(1,n)$ on $\mathbb{R}$ given by $\rho : \mathrm{BS}(1,n) \to \mathrm{Aff}^+(\mathbb{R})$ where

$$\rho(a)(x) = x + 1 \quad \text{and} \quad \rho(b)(x) = x/n.$$

It is a straightforward computation that this action is faithful.

Lemma 2. *[23, page 10] Let $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, then the stabilizer $\mathrm{Stab}_\rho(x) \cong \mathbb{Z}$. If x is not in $\mathbb{Q}$, then the stabilizer $\mathrm{Stab}_\rho(x)$ is trivial.*

Proof. First, we observe that the stabilizer of any point must be either trivial or isomorphic to $\mathbb{Z}$. Under ρ, the normal closure of a acts by translation and has no fixed points. Therefore, if the stabilizer of some point on $\mathbb{R}$ is nontrivial, it must be mapped injectively into the quotient $\mathrm{BS}(1,n)/\langle\langle a \rangle\rangle \cong \mathbb{Z}$. Since ρ is a faithful representation, the nontrivial stabilizer must be isomorphic to $\mathbb{Z}$.

For any $r \in \mathbb{Z}[1/n]$ and $s \in \mathbb{Z}$, we have

$$\rho(a^r b^s)(x) = n^{-s}x + r.$$

If $s \neq 0$, then the affine map above has a fixed point which must be rational. Now suppose that $x = p/q \in \mathbb{Q}$. We want to find $r \in \mathbb{Z}[1/n]$ and $s \in \mathbb{Z}$ such that

$$n^s \frac{p}{q} + r = \frac{p}{q}$$

Let $q = dq'$ and $p' = np/d$ where $d = \gcd(q,n)$. The previous equation is equivalent to

$$n^s \frac{np}{dq'} + nr = \frac{np}{dq'} \quad \text{or} \quad (n^s - 1)\frac{p'}{q'} = -nr$$

Since n and q' are relatively prime, q' is in the (multiplicative) group of units of $\mathbb{Z}/n\mathbb{Z}$, so we can find $s \in \mathbb{N} \setminus \{0\}$ such that q' divides $n^s - 1$. Now we set

$$r = -\frac{n^s - 1}{q'}\frac{p'}{n} \in \mathbb{Z}[1/n]$$

since $p' \in \mathbb{Z}$. Therefore $\rho(a^r b^{-s})$ fixes p/q. $\square$

It follows that if $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$, then each of the following subsets defines a positive cone on $\mathrm{BS}(1,n)$:

- $P_\varepsilon^+ = \{g : \rho(g)(\varepsilon) > \varepsilon\}$,
- $P_\varepsilon^- = \{g : \rho(g)(\varepsilon) < \varepsilon\}$

When $\varepsilon \in \mathbb{Q}$, we define the following four positive cones.

- $Q_\varepsilon^{++} = \{g : (\rho(g)(\varepsilon) > \varepsilon) \vee (\rho(g)(\varepsilon) = \varepsilon \wedge \rho(g)(\varepsilon+1) > \varepsilon+1)\}$
- $Q_\varepsilon^{+-} = \{g : (\rho(g)(\varepsilon) > \varepsilon) \vee (\rho(g)(\varepsilon) = \varepsilon \wedge \rho(g)(\varepsilon+1) < \varepsilon+1)\}$
- $Q_\varepsilon^{-+} = \{g : (\rho(g)(\varepsilon) < \varepsilon) \vee (\rho(g)(\varepsilon) = \varepsilon \wedge \rho(g)(\varepsilon+1) > \varepsilon+1)\}$
- $Q_\varepsilon^{--} = \{g : (\rho(g)(\varepsilon) < \varepsilon) \vee (\rho(g)(\varepsilon) = \varepsilon \wedge \rho(g)(\varepsilon+1) < \varepsilon+1)\}$

Theorem 2 ([23, Theorem 4.2], [10]). *P_ε^+ and P_ε^- for $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$, Q_ε^{++}, Q_ε^{+-}, Q_ε^{-+}, and Q_ε^{--} for $\varepsilon \in \mathbb{Q}$, and the 4 positive cones P_∞^{++}, P_∞^{+-}, P_∞^{-+}, and P_∞^{--} corresponding to bi-orderings are all distinct and contain all the left-orderings on* $\mathrm{BS}(1,n)$.

Recall that for a left-orderable group G, a faithful action $D : G \to \mathrm{Homeo}_+(\mathbb{R})$ and a dense sequence $x_1, \dots \in \mathbb{R}$ one can recover an ordering $P_D(x_1, \dots)$ as mentioned after Theorem 1. Theorem 2 tells us that by using the action ρ of $\mathrm{BS}(1,n)$ on $\mathbb{R}$, one can classify all the bi-orderings by considering the first elements $\varepsilon = x_1$ of dense sequences in $\mathbb{R}$. Thus, given a positive cone P of the form P^+_ε, P^-_ε, P°_∞, or Q°_ε where $\circ \in \{++,--,+-,-+\}$ we refer to ε as the *base point* of P and P^+, P^-, P°, and Q° as its type.

We observe that given some $g \in \mathrm{BS}(1,n)$, $\varepsilon \in \mathbb{R}$, and $\circ \in \{+,-,++,+-,-+,--\}$, we have $T^\circ_\varepsilon = (T^\circ_{g(\varepsilon)})^{g^{-1}}$, where $T \in \{P,Q\}$. In particular, ε is rational if and only if $g(\varepsilon)$ is rational and $\mathbb{Q}$ is a countable $\mathrm{BS}(1,n)$-invariant subset. Thus, the conjugation equivalence of the positive cones is Borel equivalent to the orbit equivalence relation of $\mathrm{BS}(1,n) \curvearrowright \mathbb{R}$.

Theorem 3. *The orbit equivalence relation E generated by the affine action* $\mathrm{BS}(1,n) = \langle a, b \mid b^{-1}ab = a^n \rangle \curvearrowright \mathbb{R}$ *via $a(x) = x+1$ and $b(x) = nx$ is hyperfinite.*

Proof. We will reduce E to E_t, the tail equivalence relation, defined on n^ω by AE_tB if $\exists p, q \forall k\ A(p+k) = B(q+k)$. This suffices as the tail equivalence relation is hyperfinite by [11, Section 8].

The reduction $f : \mathbb{R} \to n^\omega$ is given by sending $x \in \mathbb{R}$ to the base n expansion of the fractional part $\{x\}$ of x. To show this is a reduction, let $x, y \in \mathbb{R}$. Assume first that $x\ E^{\mathrm{BS}(1,n)}_{lo}\ y$, so there is some $g \in \mathrm{BS}(1,n)$ such that $g(x) = y$. Assuming $g = a^r b^s$ such that $r \in \mathbb{Z}[1/n]$ and $s \in \mathbb{Z}$, we have $y = g(x) = a^r b^s(x) = n^{-s}x + r$. As $r \in \mathbb{Z}[1/n]$, we can multiply the equation by a power of n to get $n^p y = n^q x + t$, where $p, q \in \mathbb{N}$ and $t \in \mathbb{Z}$. Since t is an integer, we have $\{n^p y\} = \{n^q x\}$. However, in base n, $\{n^q x\}$ can be obtained from $\{x\}$ by truncating the first q digits and shifting the decimal point by q places, and similarly for $\{y\}$ and $\{n^p y\}$. Thus, $\{n^p y\} = \{n^q x\}$ implies that $\{x\}\ E_t\ \{y\}$.

Conversely, assume $\{x\}\ E_t\ \{y\}$ in base n. Then there are some $q, p \in \mathbb{N}$ such that for every k, the $(q+k)$-th decimal place of $\{x\}$ is the same as the $(p+k)$-th decimal place of $\{y\}$. Thus, we have $\{n^q x\} = \{n^p y\}$, namely, there is some $t \in \mathbb{Z}$ with $n^p y = n^q x + t$, or equivalently $y = n^{q-p}x + t/n^p$. This means $y = g(x)$ with $g = a^{t/n^p} b^{p-q}$, so $x\ E\ y$. This shows that f is indeed a reduction. ◻

It follows that $E^{\mathrm{BS}(1,n)}_{lo}$ is hyperfinite, and [5] showed that $E^{\mathrm{BS}(1,n)}_{lo}$ is not smooth.

Corollary 1. $E^{\mathrm{BS}(1,n)}_{lo} \sim_B E_0$.

4 Computability of dynamical realizations

Given a left-ordering of G, it is straightforward to see that the left-ordering (considered as a relation on G) and the corresponding positive cone (considered as a subset of G) are Turing equivalent. It is thus natural to ask if this equivalence extends to the dynamical realization of the left-ordering. We will soon see that this is the case for $\mathrm{BS}(1,n)$.

Towards this fix an enumeration of the dyadic rational numbers $\mathbb{Q}_2$ and recall that a real number $r \in \mathbb{R}$ is *left-c.e.* if its left cut is c.e., i.e., the set $\{q < r : q \in \mathbb{Q}_2\}$ is computably enumerable. If both its left cut and right cut are c.e., then we say that it is *computable*.

Proposition 3. *Let P be a left-ordering of* $\mathrm{BS}(1,n)$*, then P is Turing equivalent to its base point. Furthermore, the reductions are uniform in the type.*

Proof. We non-uniformly fix the type of P. We will assume that the type is P^+ with the construction easily adaptable to work for other types. We enumerate a right cut of its base point ε. For every $q \in \mathbb{Q}_2$ we enumerate P and whenever we see $g = a^r b^s \in P$ such that $\rho(g)(q) = n^{-s}q + r > q$ we enumerate q into C. Say $n^{-s}x + r > x$, then $x > \frac{-r}{n^{-s}-1}$, and so $\rho(g)(\varepsilon) > \varepsilon$ if and only if for every $q > \varepsilon$, $\rho(g)(q) > q$. Thus, C_q is a right cut of ε. Similarly, we can enumerate a left cut.

Similarly, say we can compute a left cut L and right cut R of ε and are given a type T. We will give a proof assuming that $T = P^+$. The proof can be easily adapted to work for other types. For $q \in Q_2$, we can compute whether $g \in T_q$. By the affineness of ρ we have that

$$g \in P^+_\varepsilon \iff (\exists q \in R)\rho(g)(q) > q \iff (\forall q \in R)\rho(g)(q) \geq q.$$

Hence, we can compute P^+_ε from its right cut. □

For G a left-orderable computable group and P a computable positive cone of G the following canonical index sets appear.

$$I(G) = \{e : W_e \text{ is a positive cone}\}$$
$$I(P, G) = \{e : W_e \ E_{lo} \ P\}$$

By definition the set $I(G) \in \Pi^0_2$ as membership in a c.e. set is Σ^0_1. Similarly, it can be seen that $I(P, G) \in \Sigma^0_3$.

Proposition 4. *Let G be an infinite computable group with a computable left-ordering. Then $I(G)$ is Π^0_2-complete.*

Proof. We reduce Inf to $I(G)$. Fix a computable positive cone P and an index e so that $W_e = P$. Given n, we build a total computable function f so that $W_{f(n)} = P$ if and only if $n \in \text{Inf}$. To do this, we enumerate W_n in stages $W_{n,s}$, and, whenever $W_{n,s+1} \neq W_{n,s}$ with $k = |W_{f(n),s+1}|$, we define $W_{f(n),s+1} = W_{e,k}$. The resulting set $W_{f(n)}$ is clearly c.e. and $n \in \text{Inf}$ if and only if $W_{f(n)} = W_e = P$. □

Theorem 4. *1. $I(P^\circ_\infty, \mathrm{BS}(1, n))$ is Π^0_2-complete for every $\circ \in \{++, --, +-, -+\}$.*
2. $I(P^\circ_\varepsilon, \mathrm{BS}(1, n))$ is Σ^0_3-complete for every computable $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$ and $\circ \in \{+, -\}$.

Proof. The proof for Item 1 is analogous to the proof of Proposition 4.

We prove the Σ^0_3-hardness of $I(P^\circ_\varepsilon, \mathrm{BS}(1, n))$ for a fixed computable $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$. That $I(P^\circ_\varepsilon, \mathrm{BS}(1, n)) \in \Sigma^0_3$ follows easily from the definition. Given a Σ^0_3 set S we may assume that there is a computable function $g : \omega^2 \to \omega$ such that

$$n \in S \iff \exists y W_{g(n,y)} \in \text{Inf}.$$

We may also assume that for every $k \in \omega$ there are infinitely many y with $|W_{g(n,y)}| > k$.

Given n, we will define a left cut $C_{f(n)}$ in stages as follows. We let $C^0_{f(n)} = \emptyset$ and at every stage s we choose some $y < s$ and extend $C^s_{f(n)}$ with the goal to make $C_{f(n)}$ a left cut of a real number δ_y. Furthermore, every y will define natural numbers k_y and l_y when it acts at a stage s. Every δ_y will differ from ε at only finitely many digits in their base n expansions and $\delta_0 = \varepsilon$ with $k_0 = l_0 = 0$. The construction is a classical finite injury construction where higher priority "requirements" (i.e., smaller y) will initialize the work of larger y. At the start of the construction, all y are initialized.

Assume we are at stage $s+1$ and that y_0 is least with $W_{g(x,y_0),s+1} \neq W_{g(x,y_0),s}$. Let y_1 be the y that acted at stage s. If y_0 has been initialized since it last acted or has never acted before, do the following:

Set $k_{y_0} = k_{y_1} + 1 = \langle p, q \rangle$. By induction we can assume that $\{n^r \delta_{y_1}\} = \{n^r \varepsilon\}$ for some $r \in \omega$. If $p \neq q$, then since ε is irrational $n^p \delta_{y_1} \neq n^q \varepsilon$. Hence, there must be some least $i > r$ such that the $(p+i)$-th bit of $\{\delta_{y_1}\}$ is not equal to the $(q+i)$th bit of ε. We declare $\delta_{y_0} = \delta_{y_1}$ and $l_{y_0} = \max\{l_{y_1}, p+i\}$.

Now, if $p = q$, then we consider the least $r > \max\{p, l_{y_1}\}$ such that $\delta_{y_1} - n^{-r} \in (\max C^s_{f(n)}, \delta_{y_1})$ and the r-th decimal place of δ_{y_1} is nonzero, let $\delta_{y_0} = \delta_{y_1} - n^{-r}$ and set $l_{y_0} = r$. Note that the first $r-1$ decimal digits of δ_{y_0} and δ_{y_1} are the same.

At last, no matter if y_0 was initialized at the start of this stage or not, let

$$C^{s+1}_{f(n)} = C^s_{f(n)} \cup \{q_i\}$$

where $q_i \in \mathbb{Q}_2$ is least in an enumeration of $\mathbb{Q}_2$ with $q_i < \delta_{y_0}$ and initialize all lower priority requirements. This finishes the construction.

Let $C_{f(n)} = \lim_s C^s_{f(n)}$. We claim that this is a left cut. Indeed, if $q \in C_{f(n)}$ at some stage s_0, then we have $\delta_y[s] > \max(C^s_{f_n}) \geq q$ for $y \in \omega$ and $s > s_0$.[5] As infinitely often some y will act, any $q' < q$ will eventually be enumerated into $C_{f(n)}$, making it a left cut.

If $n \in S$, let y_0 be the least such that $W_{g(n,y_0)} \in \mathrm{Inf}$. Then there is a stage s_0 such that y_0 is never initialized again. As every $y > y_0$ defines $\delta_y[s] \leq \delta_{y_0}$ at every $s > s_0$ and y_0 acts infinitely often $\delta = \delta_{y_0}$ has left cut $C_{f(n)}$. Furthermore, as mentioned in the construction, $\{n^r \delta_{y_0}\} = \{n^r \varepsilon\}$ for some $r \in \omega$. From Proposition 3 we get a positive cone P°_δ and as can be seen in the proof of Theorem 3, $P^\circ_\delta \; E^{\mathrm{BS}(1,n)}_{lo} \; P^\circ_\varepsilon$.

On the other hand, suppose $n \notin S$, so that all $W_{g(n,y)}$ are finite. Given y, let s_y be the stage after which y is never initialized again. Then, by the construction $C_{f(n)}$ is the left cut for $\delta = \lim_y \delta_y[s_y]$ and $\{k_y[s_y] : y \in \omega\} = \omega$. Thus, there cannot be p, q such that $n^p \delta = n^q \varepsilon$. This is the case because if $k_y[s_y] = \langle p, q \rangle$, then it is ensured at stage s_y that $n^p \delta \neq n^q \varepsilon$ since $\{\delta\} \restriction l_y[s_y] = \{\delta_y[s_y]\} \restriction l_y[s_y]$ and already this finite part witnessed that $n^p \delta \neq n^q \varepsilon$. Hence, $\{\delta\} \not\!E_t \; \{\varepsilon\}$ and for the positive cone P°_δ we get from Proposition 3, $P^\circ_\delta \; \not\!E^{\mathrm{BS}(1,n)}_{lo} \; P^\circ_\varepsilon$. $\square$

Question 1. What is the complexity of $I(Q^\circ_\varepsilon, \mathrm{BS}(1,n))$ for $\varepsilon \in \mathbb{Q}$ and $\circ \in \{++, --, +-, -+\}$?

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[5] Following Lachlan $\delta_y[s]$ is the value of δ_y at stage s.

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