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CscK metrics near the canonical class

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Abstract. Let X be a Kähler manifold with semiample canonical bundle K_X. It is proved in [*W. Jian*, *Y. Shi* and *J. Song*, A remark on constant scalar curvature Kähler metrics on minimal models, Proc. Amer. Math. Soc. **147** (2019), no. 8, 3507–3513] that, for any Kähler class γ, there exists $\delta > 0$ such that, for all $t \in (0, \delta)$, there exists a unique cscK metric g_t in $K_X + t\gamma$. In this paper, we prove that $\{(X, g_t)\}_{t \in (0,\delta)}$ have uniformly bounded Kähler potentials, volume forms and diameters. As a consequence, these metric spaces are pre-compact in the Gromov–Hausdorff sense.

1. Introduction

The existence of constant scalar curvature Kähler (cscK) metrics and the related moduli problem are fundamental problems in complex differential geometry. The work of Chen–Cheng [2] proves that the cscK metric equation can be solved if the Mabuchi K-energy is proper. Such properness of the Mabuchi K-energy is closely related to the J-equation in the case when the canonical class of the underlying Kähler manifold is semipositive [1, 25]. In fact, if X is a minimal model, i.e., the canonical bundle K_X is nef, it is proved in [12, 16] that there always exists a unique cscK metric in any Kähler class sufficiently close to K_X. Naturally, one would like to establish a compactness result and to gain further understanding of geometric degeneration for such cscK metrics in relation to the moduli problem.

Let X be a compact Kähler manifold of complex dimension n. Suppose the canonical line bundle K_X is *semiample*, i.e., K_X^m is base point free for some $m \geq 1$. For sufficiently large $m \in \mathbb{Z}^+$, the linear system $|mK_X|$ induces a holomorphic map

$$\pi\colon X \to \mathbb{CP}^N \tag{1.1}$$

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for some $N \in \mathbb{N}$. The image of X via π, $\pi(X)$, coincides with the unique algebraic canonical model X_{can} determined by the canonical ring of X. The dimension $\kappa = \dim X_{\mathrm{can}}$ is the Kodaira dimension of X, where X is of general type if $\kappa = n$, and in this case, there exists a geometric (singular) Kähler–Einstein metric ω_{can} on X_{can} (see [17]) satisfying $\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}}$. If $\kappa < n$, then $\pi\colon X \to X_{\mathrm{can}}$ is a holomorphic fibration over X_{can}, whose generic fiber is a Calabi–Yau manifold. There exists a unique canonical (singular) Kähler metric ω_{can} (see [18, 19]) on X_{can} defined by

$$\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}},$$

where ω_{WP} is a positive current induced by the L^2 or Weil–Petersson metric of Calabi–Yau fibration $\pi\colon X \to X_{\mathrm{can}}$. Furthermore, ω_{can} are smooth on X°_{can}, away from the critical values of π.

We now fix a Kähler class γ and consider the perturbation of K_X by

$$\gamma_t = K_X + t\gamma \tag{1.2}$$

for sufficiently small $t > 0$. By [12], there exists $\delta = \delta(\gamma) > 0$ such that, for all $t \in (0, \delta)$, there exists a unique cscK metric $\omega_t \in \gamma_t$. It is natural to ask what is the asymptotic behavior of this family of cscK metrics when $t \to 0$. The following is a natural extension of the conjecture in [12].

Conjecture 1.1. *The above cscK metric spaces (X, ω_t) converge to*

$$\overline{(X^{\circ}_{\mathrm{can}}, \omega_{\mathrm{can}})},$$

the metric completion of $(X^{\circ}_{\mathrm{can}}, \omega_{\mathrm{can}})$, in Gromov–Hausdorff topology as $t \to 0$. Furthermore, $\overline{(X^{\circ}_{\mathrm{can}}, \omega_{\mathrm{can}})}$ is homeomorphic to the algebraic variety X_{can}.

When X is of general type, it is proved in [15] that ω_t converges smoothly to ω_{can} on X°_{can}. Furthermore, (X, g_t) have uniformly bounded diameter [9]. The main goal of this paper is to establish uniform geometric bounds for (X, g_t) for all X with semiample K_X.

When X is not of general type, i.e., $\kappa < n$, it is much more challenging to obtain both analytic and geometric estimates since the total volume approaches 0 as $t \to 0$. In particular, the corresponding cscK metrics must collapse, whereas there is very limited understanding for the behavior and regularity of collapsing canonical Kähler metrics. We would like to point out that, concerning the compactness of non-Einstein cscK metrics, the known results [3, 24] all implicitly require certain non-collapsing conditions and integral control of curvatures, neither of which holds in our study when $\kappa < n$.

In [8,9], geometric estimates such as diameter, lower bound of Green's function, Sobolev constants are established under the assumptions of normalized Nash entropy for the Monge–Ampère measures, where collapsing is allowed to take place. Such estimates also lead to a relative volume non-collapsing

$$\frac{\mathrm{Vol}(B(x,R))}{\mathrm{Vol}(X)} \geq cR^{\alpha} \quad \text{for all } R \in (0, \mathrm{diam}(X, \omega)),$$

for some constants $c > 0$ and $\alpha > 0$, which suffices to conclude the Gromov–Hausdorff compactness in many cases. Our goal is to apply the Sobolev inequality to our study and to establish uniform L^{∞}-estimates for local potentials and volume measure of the cscK metrics ω_t, which we will also write as g_t. Indeed, we prove the following theorem.

Theorem 1.1. *Let X be an n-dimensional compact Kähler manifold with semiample canonical bundle K_X. For any Kähler class γ, there exists $\delta = \delta(\gamma) > 0$ such that there exists a unique cscK metric $\omega_t \in K_X + t\gamma$ for $t \in (0, \delta)$ as in* [12]. *Then there exist $\alpha = \alpha(n) > 0$ and $C = C(n, X, \gamma, \delta)$ such that, for any $t \in (0, \delta)$, we have*

$$\operatorname{diam}(X, \omega_t) \leq C, \quad \frac{\operatorname{Vol}_{\omega_t}(B_{\omega_t}(x, R))}{\operatorname{Vol}_{\omega_t}(X)} \geq C^{-1} R^{\alpha} \tag{1.3}$$

for any $R \in (0, 1)$, where $B_{\omega_t}(x, R)$ denotes the geodesic ball in (X, ω_t) with center $x \in X$ and radius $R > 0$. Consequently, the family of metric spaces $\{(X, \omega_t)\}_{t \in (0,\delta)}$ is pre-compact with respect to the Gromov–Hausdorff topology.

In fact, we obtain uniform estimates for the Sobolev constant and a lower bound of Green's function associated to g_t as in [8, 9] due to the uniform estimates in Theorem 2.1. Consequently, for any sequence $\{t_j\} \to 0$, the metric spaces (X, g_{t_j}) subsequently converge in Gromov–Hausdorff sense to a compact metric space Z. It is interesting to investigate the geometry of the limit space Z. If Conjecture 1.1 holds, then the twisted Kähler–Einstein space $(X_{\text{can}}, g_{\text{can}})$ arises as the unique geometric limit of cscK metrics in the Kähler classes near the canonical K_X. These phenomena should be compared to the normalized Kähler–Ricci flow on X, where the solution converges to g_{can} pointwise on X°_{can} with bounded diameter and scalar curvature [13, 20].

In the next section, we shall prove Theorem 1.1 assuming a uniform L^∞ estimate of the cscK system. Then, in Section 3, we prove a uniform entropy bound based on a uniform L^∞ estimate of J-equations. Finally, in Section 4, we prove the uniform L^∞-estimate (Theorem 2.1) based on the entropy bound in Section 3.

2. Reduction to uniform a priori estimates for cscK system

Let ω_{FS} be the Fubini–Study metric on $\mathbb{CP}^N$ from the pluricanonical map $\pi\colon X \to \mathbb{CP}^N$ in (1.1) induced by $|mK_X|$ and let

$$\eta = \frac{1}{m}\pi^*\omega_{\text{FS}} \in K_X,$$

which is a semipositive $(1,1)$-form. By Yau's theorem [26], there is a unique Kähler metric $\theta \in \gamma$ such that $\operatorname{Ric}(\theta) = -\eta$. Let ω_t be the unique cscK metric $\omega_t \in \gamma_t$ and let

$$\theta_t = \eta + t\theta \in \gamma_t$$

as in (1.2) be the reference metric for each $t \in (0, \delta)$ for fixed $\delta = \delta(\gamma) > 0$. Then there exists a unique φ_t satisfying

$$\omega_t = \theta_t + i\partial\bar{\partial}\varphi_t, \quad \sup_X \varphi_t = 0.$$

Furthermore, φ_t solves the following coupled system:

$$\begin{cases} (\theta_t + i\partial\bar{\partial}\varphi_t)^n = V_t e^{F_t}\theta^n \\ \qquad\qquad \Delta_{\omega_t} F_t = -\overline{R}_t - \operatorname{tr}_{\omega_t}(\eta), \end{cases} \tag{2.1}$$

where

$$V_t = [\gamma_t]^n = \int_X \theta_t^n \quad \text{and} \quad \overline{R}_t = \frac{n[K_X] \cdot [\gamma_t]^{n-1}}{[\gamma_t]^n}.$$

We also have $\int_X e^{F_t} \theta^n = 1$ from the first equation of (2.1).

We shall prove that Theorem 1.1 follows from the following uniform L^∞ estimates.

Theorem 2.1. *There exists a uniform constant* $C = C(n, X, \gamma, \delta, \eta) > 0$ *such that, for all* $t \in (0, \delta)$*, we have*

$$\sup_{t \in (0,\delta)} (\|\varphi_t\|_{L^\infty(X)} + \|F_t\|_{L^\infty(X)}) \leq C.$$

We remark that, by Theorem 2.1, the metrics ω_t (after possibly passing to a subsequence) converge weakly to a positive current with bounded local potentials on X_{can}.

Proof of Theorem 1.1 *assuming Theorem* 2.1. We first recall the following results of [9] for the convenience of readers.

Let (X, ω_X) be a compact Kähler manifold. The p-Nash entropy of another Kähler form ω is

$$\mathcal{N}_{\omega_X,p}(\omega) := \frac{1}{[\omega]^n} \int_X \left|\log\left(\frac{1}{[\omega]^n} \frac{\omega^n}{\omega_X^n}\right)\right|^p \omega^n.$$

Denote by $K(X)$ the set of Kähler metrics on X. Consider the class of Kähler metrics

$$W(\omega_X, A, p, K; \sigma) := \left\{\omega \in K(X) \;\middle|\; [\omega] \cdot [\omega_X]^{n-1} \leq A,\ \mathcal{N}_{\omega_X,p}(\omega) \leq K,\ \frac{1}{[\omega]^n} \frac{\omega^n}{\omega_X^n} \geq \sigma\right\},$$

where $\sigma \geq 0$ is a continuous function. Through Green function's estimates, Guo–Phong–Song–Sturm proved in [9] that if $\dim_H\{\sigma = 0\} < 2n - 1$ and $p > n$, then we can find constants $C = C(\omega_X, n, A, p, K, \sigma) > 0$, $c = c(\omega_X, n, A, p, K, \sigma) > 0$ and $\alpha = \alpha(n, p) > 0$ such that, for any $\omega \in W(\omega_X, A, p, K; \sigma)$, $\mathrm{diam}(X, \omega) \leq C$ and, for any $x \in X$ and $R \in (0, 1]$,

$$\frac{\mathrm{Vol}_\omega(B_\omega(x, R))}{\mathrm{Vol}_\omega(X)} \geq c R^\alpha.$$

In our case, note that, since

$$\mathcal{N}_{\theta,p}(\omega_t) := \frac{1}{V_t} \int_X |F_t|^p \omega_t^n,$$

if we have Theorem 2.1, then $\mathcal{N}_{\theta,p}(\omega_t) \leq C^p$ for any p, and

$$\frac{1}{[\omega_t]^n} \frac{\omega_t^n}{\theta^n} = e^{F_t} \geq e^{-C},$$

where C is the constant in Theorem 2.1. So we can simply take σ to be the constant function e^{-C} and hence the conditions of the main theorem of [9] are all fulfilled. Consequently, we obtain the desired uniform diameter bound and relative non-collapsing estimate (1.3).

The proof of Gromov–Hausdorff pre-compactness from (1.3) is standard: from the relative non-collapsing estimate, for any $\epsilon > 0$ sufficiently small, the maximal packing number using disjoint geodesic balls of radius ϵ is uniformly bounded from above. Then the family is pre-compact by Gromov's pre-compactness theorem [6, Proposition 5.2]. □

3. Uniform entropy bounds

To prove Theorem 2.1, we first need a 1-Nash entropy bound. Let $t \in (0, \delta)$ be fixed. All the relevant constants in this section are independent of t. Let (φ_t, F_t) be the solution to the coupled system (2.1). Note that, since

$$\mathcal{N}_{\theta,1}(\omega_t) = \frac{1}{V_t}\int_X |F_t|\omega_t^n = \int_X |F_t| e^{F_t}\theta^n \le \int_X F_t e^{F_t}\theta^n + \frac{2}{e}[\theta]^n,$$

it suffices to bound $\int_X F_t e^{F_t}\theta^n$ by the following proposition.

Proposition 3.1. *There is a constant $C > 0$ that depends on n, θ, η such that*

$$\frac{1}{V_t}\int_X \log\Big(\frac{\omega_t^n}{V_t\theta^n}\Big)\omega_t^n = \int_X F_t e^{F_t}\theta^n \le C.$$

To prove this upper bound, we start with a family version of the well-known α-invariant argument in [23], based on a local version of [11].

Lemma 3.1. *There is a constant $c_0 = c_0(n, \theta, \eta) > 0$ such that*

$$\frac{1}{V_t}\int_X \log\Big(\frac{\omega_t^n}{V_t\theta^n}\Big)\omega_t^n \ge 2c_0(I_{\theta_t}(\varphi_t) - J_{\theta_t}(\varphi_t)) - C$$

for some uniform constant $C > 0$, where $I_{\theta_t}, J_{\theta_t}$ are Aubin's functionals.

Proof. Note that $\theta_t = \eta + t\theta \le C\theta$ for some uniform constant $C > 0$. So any function in $\mathrm{PSH}(X, \theta_t)$ satisfies a uniform α-invariant estimate (see [11,23])

$$\int_X e^{-\alpha_0\varphi_t}\theta^n \le C$$

for some $\alpha_0 = \alpha_0(n, X, \eta, \theta) > 0$. This implies that

$$\frac{1}{V_t}\int_X e^{-\alpha_0\varphi_t - F_t}\omega_t^n = \int_X e^{-\alpha_0\varphi_t - F_t}e^{F_t}\theta^n \le C.$$

Taking log on both sides and applying Jensen's inequality, we obtain

$$\frac{1}{V_t}\int_X (-\alpha_0\varphi_t - F_t)\omega_t^n \le \log C.$$

Rearranging the terms gives

$$\frac{1}{V_t}\int_X F_t\omega_t^n \ge \frac{\alpha_0}{V_t}\int_X (-\varphi_t)\omega_t^n - \log C. \tag{3.1}$$

The lemma then follows from equivalence of the functionals $I_{\theta_t} - J_{\theta_t}$ and $\frac{1}{V_t}\int_X(-\varphi_t)\omega_t^n$ and choosing $2c_0 = \alpha_0 > 0$. □

The proof of Proposition 3.1 makes use of the fact that the entropy of F_t is a component of the Mabuchi K-energy. We recall the following modified form of Chen–Tian's formula for the K-energy: for any $\varphi \in \mathrm{PSH}(X, \theta_t)$,

$$K_{\theta_t}(\varphi) = \frac{1}{V_t}\int_X \log\Big(\frac{\theta_{t,\varphi}^n}{V_t\theta^n}\Big)\theta_{t,\varphi}^n - \frac{1}{V_t}\int_X \log\Big(\frac{\theta_t^n}{V_t\theta^n}\Big)\theta_t^n + J_{\eta,\theta_t}(\varphi), \tag{3.2}$$

where $\theta_{t,\varphi} = \theta_t + i\partial\bar\partial\varphi$. In fact, the usual Chen–Tian formula gives

$$K_{\theta_t}(\varphi) = \frac{1}{V_t}\int_X \log\Big(\frac{\theta_{t,\varphi}^n}{\theta_t^n}\Big)\theta_{t,\varphi}^n + J_{-\mathrm{Ric}(\theta_t),\theta_t}(\varphi)$$

since

$$\begin{aligned}\frac{1}{V_t}\int_X \log\Big(\frac{\theta_{t,\varphi}^n}{\theta_t^n}\Big)\theta_{t,\varphi}^n &= \frac{1}{V_t}\int_X \log\Big(\frac{\theta_{t,\varphi}^n}{V_t\theta^n}\Big)\theta_{t,\varphi}^n - \frac{1}{V_t}\int_X \log\Big(\frac{\theta_t^n}{V_t\theta^n}\Big)\theta_{t,\varphi}^n \\ &= \frac{1}{V_t}\int_X \log\Big(\frac{\theta_{t,\varphi}^n}{V_t\theta^n}\Big)\theta_{t,\varphi}^n - \frac{1}{V_t}\int_X \log\Big(\frac{\theta_t^n}{V_t\theta^n}\Big)\theta_t^n \\ &\quad - \frac{1}{V_t}\int_X \log\Big(\frac{\theta_t^n}{V_t\theta^n}\Big)(\theta_{t,\varphi}^n - \theta_t^n).\end{aligned}$$

By writing $\theta_{t,\varphi}^n - \theta_t^n$ as $\int_0^1 \frac{d}{ds}\theta_{t,s\varphi}^n\,ds$, it is straightforward to check that the last term equals $J_{\eta+\mathrm{Ric}(\theta_t),\theta_t}(\varphi)$, which in turn implies (3.2).

It is well known that cscK metrics are minimizers of K-energy; hence we have

$$K_{\theta_t}(\varphi_t) \le K_{\theta_t}(0) = 0,$$

which implies that (recall that $\omega_t = \theta_{t,\varphi_t}$)

$$\begin{aligned}\frac{1}{V_t}\int_X \log\Big(\frac{\theta_t^n}{V_t\theta^n}\Big)\theta_t^n &\ge \frac{1}{V_t}\int_X \log\Big(\frac{\omega_t^n}{V_t\theta^n}\Big)\omega_t^n + J_{\eta,\theta_t}(\varphi_t) \\ &\ge \frac{1}{2}\frac{1}{V_t}\int_X \log\Big(\frac{\omega_t^n}{V_t\theta^n}\Big)\omega_t^n + J_{\eta+c_0\theta_t,\theta_t}(\varphi_t) - C,\end{aligned}$$

where we use Lemma 3.1 and the following equation of the J-functionals:

$$c_0\big(I_{\theta_t}(\varphi_t) - J_{\theta_t}(\varphi_t)\big) + J_{\eta,\theta_t}(\varphi_t) = J_{\eta+c_0\theta_t,\theta_t}(\varphi_t).$$

By straightforward calculations, we have

$$\frac{1}{V_t}\int_X \log\Big(\frac{\theta_t^n}{V_t\theta^n}\Big)\theta_t^n \le C$$

for some constant $C = C(n, \eta, \theta) > 0$. Combining these inequalities, we get

$$\frac{1}{V_t}\int_X \log\Big(\frac{\omega_t^n}{V_t\theta^n}\Big)\omega_t^n + 2J_{\eta+c_0\theta_t,\theta_t}(\varphi_t) \le C. \tag{3.3}$$

To get an upper bound of

$$\frac{1}{V_t}\int_X \log\Big(\frac{\omega_t^n}{V_t\theta^n}\Big)\omega_t^n,$$

from (3.3), we see that it suffices to prove a uniform *lower* bound of $J_{\eta+c_0\theta_t,\theta_t}(\varphi_t)$.

In the next step, we will use the existence of a minimizer of the $J_{\eta+c_0\theta_t,\theta_t}$-functional to show the lower bound. For notational convenience, we denote $\chi_t = \eta + c_0\theta_t$, which satisfies

$$c_0\theta_t \le \chi_t \le (1 + c_0)\theta_t.$$

By [12,21], the minimizer $\phi_t \in \mathrm{PSH}(X, \theta_t)$ of J_{χ_t,θ_t} exists and solves the J-equation

$$(\theta_t + i\partial\bar\partial\phi_t)^{n-1} \wedge \chi_t = c_t(\theta_t + i\partial\bar\partial\phi_t)^n, \tag{3.4}$$

where we normalize $\sup_X \phi_t = 0$ and $c_t = c_0 + a_t \geq c_0 > 0$ with

$$a_t := \frac{1}{V_t}\int_X \theta_t^{n-1} \wedge \eta. \tag{3.5}$$

We claim that if we can prove a uniform L^∞ bound for the solutions ϕ_t of (3.4), then we will finish the proof of Proposition 3.1.

In fact, if ϕ_t is uniformly bounded, we obtain a uniform lower bound for $J_{\eta+c_0\theta_t,\theta_t}(\varphi_t)$ as follows:

$$J_{\chi_t,\theta_t}(\varphi_t) \geq J_{\chi_t,\theta_t}(\phi_t) = \frac{1}{V_t}\int_X\int_0^1 n\phi_t(\chi_t - c_t\theta_{t,s\phi_t}) \wedge (\theta_{t,s\phi_t})^{n-1}\,ds \geq -C$$

for some uniform $C = C(\|\phi_t\|_{L^\infty(X)}) > 0$. Consequently, by (3.3), we immediately get a uniform upper bound for $\int_X F_t e^{F_t}\theta^n$.

To prove the uniform L^∞ bound for ϕ_t, we will apply the trick of [22] by Moser's iteration. However, we would need the uniform Sobolev inequality from [8] for the reference metric θ_t.

Lemma 3.2. *There exists $C > 0$ such that, for all $t \in (0, \delta)$, we have*

$$\left|a_t - \frac{\kappa}{n}\right| \leq Ct$$

for a_t in (3.5).

Proof. By direct computation, it follows that

$$a_t = \frac{\sum_{i=0}^{\kappa-1} t^{n-1-i}\binom{n-1}{i}\int_X \eta^{i+1}\wedge\theta^{n-1-i}}{\sum_{i=0}^{\kappa} t^{n-i}\binom{n}{i}\int_X \eta^i \wedge \theta^{n-i}} = \frac{\binom{n-1}{\kappa-1}}{\binom{n}{\kappa}} + O(t) = \frac{\kappa}{n} + O(t). \qquad \square$$

Lemma 3.3 ([12]). *There exists a uniform $\delta_0 > 0$ such that*

$$nc_t(\theta_t)^{n-1} - (n-1)\chi_t \wedge (\theta_t)^{n-2} \geq \delta_0(\theta_t)^{n-1} \tag{3.6}$$

if $0 < t \leq \bar t$ for some $\bar t = \bar t(n, \theta, \eta)$ sufficiently small.

Proof. This lemma follows from straightforward calculations as in [12]. Indeed, we have

$$\begin{aligned}
&nc_t\theta_t^{n-1} - (n-1)\chi_t \wedge \theta_t^{n-2}\\
&\quad = nc_t(\eta + t\theta)^{n-1} - (n-1)\eta \wedge (\eta + t\theta)^{n-2} - (n-1)c_0\theta_t^{n-1}\\
&\quad = (na_t + c_0)\sum_{i=0}^{\kappa}\binom{n-1}{i}\eta^i \wedge (t\theta)^{n-1-i} - (n-1)\sum_{i=0}^{\kappa-1}\binom{n-2}{i}\eta^{i+1} \wedge (t\theta)^{n-2-i}\\
&\quad = (na_t + c_0)(t\theta)^{n-1} + \sum_{i=1}^{\kappa} A_i\eta^i \wedge (t\theta)^{n-1-i},
\end{aligned}$$

where the coefficients (for $i = 1, \dots, \kappa$) are

$$
\begin{aligned}
A_i &= (na_t + c_0)\binom{n-1}{i} - (n-1)\binom{n-2}{i-1} \\
\text{(by Lemma 3.2)} \quad &= (\kappa + c_0 + O(t))\binom{n-1}{i} - (n-1)\binom{n-2}{i-1} \\
&= \binom{n-1}{i}(\kappa + c_0 + O(t) - i) \geq \frac{c_0}{2}
\end{aligned}
$$

if $t \leq \bar{t}$ for some sufficiently small $\bar{t} = \bar{t}(n, \theta, \eta) > 0$. Combining the above inequalities, we finally arrive at

$$
nc_t\theta_t^{n-1} - (n-1)\chi_t \wedge \theta_t^{n-2} \geq \frac{c_0}{2}\sum_{i=0}^{\kappa} \eta^i \wedge (t\theta)^{n-1-i} \geq \delta_0 \theta_t^{n-1},
$$

where we may take

$$
\delta_0 = \frac{c_0}{2\max_{i=0,\dots,\kappa}\{\binom{n-1}{i}\}}. \qquad \square
$$

From now on, we additionally impose that $0 < t \leq \bar{t}$. For any $s \in [0, 1]$, we denote

$$
\theta_{t,s} = \theta_t + i\partial\bar{\partial}(s\phi_t),
$$

where ϕ_t is the solution to the J-equation (3.4).

Lemma 3.4 ([22]). *There exists a uniform constant* $c_1 = c_1(n, \theta, \eta) > 0$ *such that*

$$
nc_t(\theta_{t,s})^{n-1} - (n-1)\chi_t \wedge (\theta_{t,s})^{n-2} \geq c_1(1-s)^{n-1}(\theta_t)^{n-1}.
$$

Proof. The proof of this lemma is the same as that of [22, Lemma 2.3]. The point is that the constant c_1 here is independent of t. For completeness, we include a proof here. We view χ_t as the reference form in the definition of Hessian operators: for any positive $(1, 1)$-form θ,

$$
\sigma_n(\theta) = \frac{\theta^n}{\chi_t^n}, \quad \sigma_{n-1}(\theta) = \frac{n\theta^{n-1} \wedge \chi_t}{\chi_t^n}.
$$

We write

$$
\widehat{\theta}_t = \theta_t + i\partial\bar{\partial}\phi_t = \theta_{t,1}.
$$

It is clear that $\theta_{t,s} = s\widehat{\theta}_t + (1-s)\theta_t$. Since, for each $i = 1, \dots, n$, the function

$$
s \mapsto \frac{\sigma_{n-1;i}(\theta_{t,s})}{\sigma_{n-2;i}(\theta_{t,s})}
$$

is concave, it follows that

$$
\frac{\sigma_{n-1;i}(\theta_{t,s})}{\sigma_{n-2;i}(\theta_{t,s})} \geq s\frac{\sigma_{n-1;i}(\widehat{\theta}_t)}{\sigma_{n-2;i}(\widehat{\theta}_t)} + (1-s)\frac{\sigma_{n-1;i}(\theta_t)}{\sigma_{n-2;i}(\theta_t)}. \tag{3.7}
$$

The first term on the right-hand side of (3.7),

$$
\frac{\sigma_{n-1;i}(\widehat{\theta}_t)}{\sigma_{n-2;i}(\widehat{\theta}_t)},
$$

is bigger than $\frac{1}{nc_t}$ (see [21]). By the cone condition (3.6), the second term on the right-hand side of (3.7),

$$\frac{\sigma_{n-1;i}(\theta_t)}{\sigma_{n-2;i}(\theta_t)},$$

is no less than

$$\frac{1}{(1-\frac{\delta_0}{nc_t})nc_t} \geq \Big(1+\frac{\delta_0}{nc_t}\Big)\frac{1}{nc_t} \geq (1+\bar{\delta}_0)\frac{1}{nc_t},$$

where

$$\bar{\delta}_0 = \frac{\delta_0}{\max_{t\in(0,\bar{t}]} nc_t}$$

is a uniform positive constant. Thus inequality (3.7) yields

$$\frac{\sigma_{n-1;i}(\theta_{t,s})}{\sigma_{n-2;i}(\theta_{t,s})} \geq \frac{s}{nc_t} + \frac{(1-s)(1+\bar{\delta}_0)}{nc_t}. \tag{3.8}$$

In terms of $(n-1, n-1)$-forms, (3.8) is equivalent to

$$nc_t\theta_{t,s}^{n-1} - (n-1)\chi_t \wedge \theta_{t,s}^{n-2} \geq \bar{\delta}_0(1-s)(n-1)\chi_t \wedge \theta_{t,s}^{n-2}. \tag{3.9}$$

On the other hand, since the function $s \mapsto \sigma_{n-1;i}(\theta_{t,s})^{1/(n-1)}$ is concave, we have

$$\begin{aligned}\sigma_{n-1;i}(\theta_{t,s})^{1/(n-1)} &\geq s\sigma_{n-1;i}(\widehat{\theta}_t)^{1/(n-1)} + (1-s)\sigma_{n-1;i}(\theta_t)^{1/(n-1)}\\ &\geq (1-s)\sigma_{n-1;i}(\theta_t)^{1/(n-1)},\end{aligned}$$

which implies that

$$\chi_t \wedge \theta_{t,s}^{n-2} \geq (1-s)^{n-1}\chi_t \wedge \theta_t^{n-2}.$$

Combining this with (3.9) gives that

$$nc_t\theta_{t,s}^{n-1} - (n-1)\chi_t \wedge \theta_{t,s}^{n-2} \geq \bar{\delta}_0(1-s)^n(n-1)\chi_t \wedge \theta_t^{n-2} \geq c_0\bar{\delta}_0(n-1)(1-s)^n\theta_t^{n-1}.$$

The lemma is proved with $c_1 = c_0\bar{\delta}_0(n-1)$. □

We will use the Moser iteration argument to prove the C^0 estimates of ϕ_t. To this end, we need the following uniform Sobolev inequality for the reference metrics θ_t. (Note that, by direct computations, the metric θ_t satisfies the conditions in [8]; see also [8, Example 4.1].)

Lemma 3.5 ([8, Theorem 2.1 and (4.10)]). *There exist a constant $q = q(n, X) > 1$ and a constant $C = C(n, \theta, \eta) > 0$ such that, for any $u \in C^1(X)$,*

$$\Big(\frac{1}{V_t}\int_X |u|^{2q}\theta_t^n\Big)^{1/q} \leq \frac{C}{V_t}\int_X (u^2 + |\nabla u|^2_{\theta_t})\theta_t^n. \tag{3.10}$$

From these, we can now prove the uniform L^∞ estimate for ϕ_t, which finishes the proof of Proposition 3.1.

Proposition 3.2. *There exists a uniform constant $C > 0$ that is independent of $t \in (0, \bar{t}]$ such that* $\sup_X(-\phi_t) \leq C$.

Proof. We follow the arguments in [22] closely. For any $p > 1$, we consider the integral

$$\int_X e^{-p\phi_t}\big(c_t(\theta^n_{t,\phi_t} - \theta^n_t) - \chi_t \wedge (\theta^{n-1}_{t,\phi_t} - \theta^{n-1}_t)\big). \tag{3.11}$$

On one hand, this integral is

$$\int_X e^{-p\phi_t}(-c_t\theta^n_t + \chi_t \wedge \theta^{n-1}_t) \le C \int_X e^{-p\phi_t}\theta^n_t. \tag{3.12}$$

On the other hand, the integral in (3.11) is

$$\begin{aligned}
\int_X & e^{-p\phi_t} i\partial\bar\partial\phi_t \wedge \Big(\int_0^1 nc_t\theta^{n-1}_{t,s} - (n-1)\chi_t \wedge \theta^{n-2}_{t,s}\, ds\Big) \\
&= p\int_X e^{-p\phi_t}\sqrt{-1}\partial\phi_t \wedge \bar\partial\phi_t \wedge \Big(\int_0^1 nc_t\theta^{n-1}_{t,s} - (n-1)\chi_t \wedge \theta^{n-2}_{t,s}\, ds\Big) \\
&\ge p\int_X e^{-p\phi_t}\sqrt{-1}\partial\phi_t \wedge \bar\partial\phi_t \wedge \Big(\int_0^1 c_1(1-s)^n\, ds\, \theta^{n-1}_t\Big) \\
&\ge c_2 p\int_X e^{-p\phi_t}\sqrt{-1}\partial\phi_t \wedge \bar\partial\phi_t \wedge \theta^{n-1}_t
\end{aligned}$$

for some $c_2 > 0$ that depends on n, θ, η but is independent of t and p. This inequality together with (3.12) yield that, for some uniform constant $C' > 0$,

$$\frac{1}{V_t}\int_X |\nabla e^{-\frac{p}{2}\phi_t}|^2_{\theta_t}\theta^n_t \le \frac{C'p}{V_t}\int_X e^{-p\phi_t}\theta^n_t. \tag{3.13}$$

Applying the Sobolev inequality (3.10) to $u := e^{-p\phi_t/2}$ and using (3.13), we obtain

$$\Big(\frac{1}{V_t}\int_X e^{-qp\phi_t}\theta^n_t\Big)^{1/q} \le \frac{Cp}{V_t}\int_X e^{-p\phi_t}\theta^n_t. \tag{3.14}$$

We now apply inequality (3.14) with $p_k = q^k$ for $k = 1, 2, \ldots$, and (3.14) reads

$$\Big(\frac{1}{V_t}\int_X e^{-p_{k+1}\phi_t}\theta^n_t\Big)^{1/p_{k+1}} \le C^{1/q}p_k^{1/q}\Big(\frac{1}{V_t}\int_X e^{-p_k\phi_t}\theta^n_t\Big)^{1/p_k}. \tag{3.15}$$

Iterating (3.15) gives

$$\begin{aligned}
\Big(\frac{1}{V_t}\int_X e^{-p_{k+1}\phi_t}\theta^n_t\Big)^{1/p_{k+1}} &\le C^{\sum_{j=1}^k q^{-j}} q^{\sum_{j=1}^k jq^{-j}}\Big(\frac{1}{V_t}\int_X e^{-q\phi_t}\theta^n_t\Big)^{1/q} \\
&\le C\Big(\frac{1}{V_t}\int_X e^{-q\phi_t}\theta^n_t\Big)^{1/q}.
\end{aligned}$$

Letting $k \to \infty$ yields that

$$\sup_X e^{-\phi_t} \le C\Big(\frac{1}{V_t}\int_X e^{-q\phi_t}\theta^n_t\Big)^{1/q}. \tag{3.16}$$

Finally, note that $\frac{1}{V_t}\theta^n_t \le C\theta^n$ for a uniform constant $C > 0$, so

$$\begin{aligned}
\Big(\frac{1}{V_t}\int_X e^{-q\phi_t}\theta^n_t\Big)^{1/q} &\le C\big(\sup_X e^{-\phi_t}\big)^{\frac{q-\alpha_0}{q}}\Big(\int_X e^{-\alpha_0\phi_t}\theta^n\Big)^{1/q} \\
\text{(by } \alpha\text{-invariant)} \quad &\le C\big(\sup_X e^{-\phi_t}\big)^{\frac{q-\alpha_0}{q}},
\end{aligned}$$

which combined with (3.16) gives the desired estimate $\sup_X e^{-\phi_t} \le C$. □

4. From entropy bound to L^∞ estimates

Given the uniform entropy bound of F_t, the L^∞ estimates of φ_t and F_t have been proved in [7]. For completeness, we provide a sketched proof.

Note that, by Proposition 3.1, we have

$$\int_X |F_t| e^{F_t} \theta^n \le C, \tag{4.1}$$

and from (3.1), we also have

$$\frac{1}{V_t} \int_X (-\varphi_t) \omega_t^n \le C. \tag{4.2}$$

Denote $\beta = 1/10$. We solve the auxiliary complex Monge–Ampère equations as in [7, 10],

$$(\theta_t + i\partial\bar{\partial}\psi_k)^n = \frac{\tau_k(-\varphi_t + \beta F_t)}{A_k} V_t e^{F_t} \theta^n, \quad \sup_X \psi_k = 0,$$

where $\tau_k(x)\colon \mathbb{R} \to \mathbb{R}_+$ is a family of positive smooth function that decreases to $x\chi_{\mathbb{R}_+}(x)$, and A_k is a constant that makes the equation solvable,

$$A_k = \int_X \tau_k(-\varphi_t + \beta F_t) e^{F_t} \theta^n \to \int_\Omega (-\varphi_t + \beta F_t) e^{F_t} \theta^n =: A_\infty,$$

and here $\Omega = \{-\varphi_t + \beta F_t > 0\}$. Equations (4.1), (4.2) imply that A_∞ is uniformly bounded from above. So we can find a uniform constant $C > 0$ such that, for any $t > 0$, we can find a k_0 (possibly depending on t) such that $A_k \le C$ for any $k \ge k_0$. In the following, we always assume that $k \ge k_0$.

Consider the test function

$$\Psi = -\varepsilon(-\psi_k + \Lambda)^{\frac{n}{n+1}} - \varphi_t + \beta F_t,$$

with the constants chosen such that

$$\Lambda^{\frac{1}{n+1}} = \frac{2n}{n+1}\varepsilon, \quad \varepsilon = \frac{[(n+1)(n+\beta\overline{R}_t)]^{n/(n+1)}}{n^{2n/(n+1)}} A_k^{1/(n+1)}.$$

We claim that $\sup_X \Psi \le 0$. If the maximum of Ψ is obtained at some point in $X\backslash\Omega$, we are done. So assume Ψ takes a maximum at $x_{\max} \in \Omega$; then, at $x_{\max}$,

$$\begin{aligned} 0 \ge \Delta_{\omega_t}\Psi \ge{} & \frac{n\varepsilon}{n+1}(-\psi_k + \Lambda)^{-\frac{1}{n+1}} \operatorname{tr}_{\omega_t} \theta_{t,\psi_k} \\ & - \frac{n\varepsilon}{n+1}(-\psi_k + \Lambda)^{-\frac{1}{n+1}} \operatorname{tr}_{\omega_t} \theta_t \\ & - n + \operatorname{tr}_{\omega_t} \theta_t - \beta\overline{R}_t - \beta \operatorname{tr}_{\omega_t} \eta \\ \ge{} & \frac{n^2\varepsilon}{n+1}(-\psi_k + \Lambda)^{-\frac{1}{n+1}} \Big(\frac{\theta_{t,\psi_k}^n}{\omega_t^n}\Big)^{\frac{1}{n}} - n - \beta\overline{R}_t \end{aligned}$$

by the choice of the constants ε, Λ. This implies that, at $x_{\max}$, $\Psi \le 0$. Hence the claim is proved. Since $\varepsilon \le C$ and $\Lambda \le C$, we obtain

$$\beta F_t \le -\varphi_t + \beta F_t \le C(-\psi_k + \Lambda)^{\frac{n}{n+1}}.$$

Then, for any $\epsilon > 0$, we can find a constant $C_\epsilon > 0$ such that $\beta F_t \leq \epsilon(-\psi_k) + C_\epsilon$. Again, using the α-invariant, this shows that, for any $p > 1$,

$$\int_X e^{pF_t}\theta^n \leq C_p. \tag{4.3}$$

By the family version of Kołodziej's uniform estimate [14], developed in [4, 5], we have $\|\varphi_t\|_{L^\infty} \leq C$ (see also [10]).

To show the L^∞ estimates of F_t, we need the following mean-value inequality in [9] for the Laplace operator Δ_{ω_t}.

Lemma 4.1 ([9, Lemma 5.1]). *Under condition* (4.3) *on* F_t, *there is a uniform constant* $C = C(n, p, \theta, \eta) > 0$ *such that, for any* C^2 *function* u *with* $\Delta_{\omega_t} u \geq -a$ *for some* $a > 0$, *the following inequality holds:*

$$\sup_X u \leq C\Big(a + \frac{1}{V_t}\int_X |u|\omega_t^n\Big).$$

We first apply Lemma 4.1 to the function $u := F_t - \varphi_t$, which satisfies

$$\Delta_{\omega_t} u = -\overline{R}_t - \operatorname{tr}_{\omega_t}\eta - n + \operatorname{tr}_{\omega_t}\theta_t \geq -\overline{R}_t - n,$$

and this implies that

$$\sup_X F_t \leq \sup_X u \leq C\Big(\overline{R}_t + n + \int_X (|F_t| + |\varphi_t|)e^{F_t}\theta^n\Big) \leq C.$$

To get the lower bound of F_t, we apply Lemma 4.1 to the function $u := -F_t$, which fulfills the equation

$$\Delta_{\omega_t} u = -\Delta_{\omega_t} F_t = \overline{R}_t + \operatorname{tr}_{\omega_t}\eta \geq -\overline{R}_t,$$

and we obtain

$$\sup_X(-F_t) \leq C\Big(|\overline{R}_t| + \int_X |F_t|e^{F_t}\theta^n\Big) \leq C;$$

thus the lower bound of F_t follows, and we finish the proof of Theorem 2.1.

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