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Diameter estimates in Kähler geometry II: removing the small degeneracy assumption

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Abstract
In this short note, we remove the small degeneracy assumption in our earlier works (Guo et al. in Diameter estimates in Kähler geometry. Commun Pure Appl Math. arXiv:2209.09428; Sobolev inequalities on Kähler spaces. arXiv:2311.00221). This is achieved by a technical improvement of Corollary 5.1 in Guo et al. As a consequence, we establish the same geometric estimates for diameter, Green's functions and Sobolev inequalities under an entropy bound for the Kähler metrics, without any small degeneracy assumption.

1 Introduction

The classical works of Yau and his collaborators have built a wide range of geometric estimates for Riemannian manifolds. A lower bound for the Ricci curvature is usually required to guarantee uniformity and compactness (c.f. [1, 2, 15]). In our developing program to build geometric analysis on complex varieties with singularities [10, 11], we managed to establish uniform diameter bounds, Sobolev inequalities and the spectral theorem for a large family of Kähler metrics on both smooth compact Kähler manifolds and normal Kähler varieties. Furthermore, these uniform estimates do not require any Ricci curvature bound. Instead they only depend on an entropy bound and a small degeneracy assumption (c.f. (1.4)).

In his celebrated work [17] on the Calabi conjecture, Yau initiated the theory of global complex Monge-Ampère equations. The analytic theory was subsequently developed by

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Kolodziej [14] in the framework of pluripotential theory. It was further extended to complex Monge–Ampère equations on singular Kähler varieties [3, 4]. Recently the PDE methods developed by Guo-Phong-Tong in [12] gave an alternative proof of Kolodziej's estimates. This approach has had many important geometric consequences [7, 10, 11].

We begin by reviewing the set-up and results of [10, 11]. Let (X, θ_X) be an n-dimensional compact Kähler manifold equipped with a Kähler metric θ_X. Let $\mathcal{K}(X)$ be the space of Kähler metrics on X and let the p-th Nash–Yau entropy of a Kähler metric $\omega \in \mathcal{K}(X)$ associated to (X, θ_X) be defined by

$$\mathcal{N}_{X,\theta_X,p}(\omega) = \frac{1}{V_\omega}\int_X \left|\log\left((V_\omega)^{-1}\frac{\omega^n}{\theta_X^n}\right)\right|^p \omega^n,\ V_\omega = \int_X \omega^n = [\omega]^n, \tag{1.1}$$

for $p > 0$.

We introduce the following set of admissible metrics for given parameters $A, K > 0$, $p > n$,

$$\mathcal{V}(X, \theta_X, n, A, p, K) = \left\{\omega \in \mathcal{K}(X) : [\omega] \cdot [\theta_X]^{n-1} \le A,\ \mathcal{N}_{X,\theta_X,p}(\omega) \le K\right\}. \tag{1.2}$$

Let γ be a non-negative continuous function. We further define a subset of $\mathcal{V}(X, \theta_X, n, A, p, K)$ by

$$\mathcal{W}(X, \theta_X, n, A, p, K; \gamma) = \left\{\omega \in \mathcal{V}(X, \theta_X, n, A, p, K) :\ (V_\omega)^{-1}\frac{\omega^n}{\theta_X^n} \ge \gamma\right\}. \tag{1.3}$$

In the earlier works [10, 11] of the authors, uniform geometric estimates for the Green's function, Sobolev constant and diameter were established for Kähler metrics in $\mathcal{W}(X, \theta_X, n, A, p, K; \gamma)$ if γ is a non-negative continuous function on X satisfying

$$\dim_{\mathcal{H}}\{\gamma = 0\} < 2n - 1. \tag{1.4}$$

The small degeneracy assumption (1.4) requires the Monge-Ampère measure to be uniformly positive away from a closed subset of X of Hausdorff co-dimension strictly greater than 1. In fact, this assumption naturally arises in most of geometric applications, because degeneration usually occurs along an analytic subvariety of X, which is closed and of complex codimension no less than 1. The small degeneracy assumption was removed in [5] for the uniform diameter estimate if the underlying Kähler class lies in a compact set in the Kähler cone of a smooth Kähler manifold. Very recently, Vu [16] applied the Sobolev inequality of [11] to derive the diameter bounds for $\mathcal{V}(X, \theta_X, n, A, p, K)$. This leads us to try and remove the small degeneracy assumption in general, particularly for the Green's function and the Sobolev inequality in [10, 11].

Indeed in this note, we will remove the small degeneracy assumption (1.4) for all the results of [9–11]. This is achieved by the following simple technical improvement of Corollary 5.1 of [10].

Proposition 1.1 *Suppose $\omega \in \mathcal{V}(X, \theta_X, n, A, p, K)$. If $v \in C^2$ satisfies $|\Delta_\omega v| \le 1$ and $\int_X v\omega^n = 0$, then there is a uniform constant $C = C(X, \theta_X, n, A, p, K) > 0$ such that*

$$\sup_X |v| \le C.$$

The proof of Proposition 1.1 is a straightforward iteration of the original argument in [10]. It utilizes the same auxiliary complex Monge-Ampère equation and is entirely based on the standard maximum principle. With Proposition 1.1, all the results of [9–11] will hold without the small degeneracy assumption (1.4). We will summarize these results in the general setting of normal Kähler spaces.

Definition 1.1 Let X be an n-dimensional compact normal Kähler space. Let $\pi : Y \to X$ be a log resolution of singularities and let θ_Y be a smooth Kähler metric on the nonsingular model Y. We define the set of admissible semi-Kähler currents

$$\mathcal{AK}(X, \theta_Y, n, p, A, K),$$

to be the set of any semi-Kähler current ω on X satisfying the following conditions.

(1) $[\omega]$ is a Kähler class on X and ω has bounded local potentials.
(2) $[\pi^*\omega] \cdot [\theta_Y]^{n-1} \le A$.
(3) The p-th Nash-Yau entropy is bounded for some $p > n$, i.e.

$$\mathcal{N}_p(\omega) = \frac{1}{V_\omega} \int_Y \left| \log \frac{1}{V_\omega} \frac{(\pi^*\omega)^n}{(\theta_Y)^n} \right|^p (\pi^*\omega)^n \le K,$$

where $V_\omega = [\omega]^n$.
(4) The log volume measure ratio

$$\log \left(\frac{(\pi^*\omega)^n}{V_\omega (\theta_Y)^n} \right)$$

has log type analytic singularities (c.f. Definition 7.2 of [11]).

The space $\mathcal{AK}(X, \theta_X, n, p, A, K)$ is larger than the $\mathcal{AK}(X, \theta_X, n, p, A, K, \gamma)$ defined in Definition 3.1 of [11] by removing the small degeneracy assumption as well as the volume non-collapsing condition $[\omega]^n \ge A^{-1}$. If X is a smooth Kähler manifold,

$$\mathcal{AK}(X, \theta_X, n, p, A, K) = \mathcal{V}(X, \theta_X, n, A, p, K)$$

by considering the identity map $\pi = id : X \to X$. Hence $\mathcal{AK}(X, \theta_X, n, p, A, K)$ is the natural generalization of $\mathcal{V}(X, \theta_X, n, A, p, K)$ on normal Kahler spaces. Proposition 1.1 enables us to enlarge the $\mathcal{AK}$-space in [11] to the $\mathcal{AK}$-space in Definition 1.1.

Let X be an n-dimensional normal Kähler variety. For any $p > n$ and any $\omega \in \mathcal{AK}(X, \theta_Y, n, p, A, K)$, we let $(\overline{X}, d, \omega^n)$ the metric measure space associated to (X, ω) as in [11]. The Sobolev space $W^{1,2}(\overline{X}, d, \omega^n)$ and its spectral theory are established in [11] on the metric measure space $(\overline{X}, d, \omega^n)$ associated to ω.

We now state the geometric consequence of Proposition 1.1 for the works in [11].

Theorem 1.1 *Let X be an n-dimensional compact normal Kähler space. For any $\omega \in \mathcal{AK}(X, \theta_Y, n, p, A, K)$, the metric measure space $(\overline{X}, d, \omega^n)$ associated to (X, ω) satisfies the following properties.*

(1) There exists $C = C(X, \theta_Y, n, p, A, K) > 0$ such that

$$\operatorname{diam}(\overline{X}, d) \le C.$$

In particular, $(\overline{X}, d)$ is a compact metric space.
(2) There exist $q > 1$ and $C_S = C_S(X, \theta_Y, n, p, A, K, q) > 0$ such that

$$\left(\frac{1}{V_\omega} \int_{\overline{X}} |u|^{2q} \omega^n \right)^{1/q} \le C_S \left(\frac{I_\omega}{V_\omega} \int_{\overline{X}} |\nabla u|^2 \, \omega^n + \frac{1}{V_\omega} \int_{\overline{X}} u^2 \omega^n \right).$$

for all $u \in W^{1,2}(\overline{X}, d, \omega^n)$.

(3) *There exists* $C = C(X, \theta_Y, n, p, A, K) > 0$ *such that the following trace formula holds for the heat kernel* $H(x, y, t)$ *of* $(\overline{X}, d, \omega^n)$

$$H(x, x, t) \leq \frac{1}{V_\omega} + \frac{C}{V_\omega} I_\omega^{\frac{q}{q-1}} t^{-\frac{q}{q-1}}$$

on $\overline{X} \times (0, \infty)$.

(4) *Let* $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ *be the increasing sequence of eigenvalues of the Laplacian* $-\Delta_\omega$ *on* $(\overline{X}, d, \omega^n)$*. Then there exists* $c = c(X, \theta_Y, n, p, A, K) > 0$ *such that*

$$\lambda_k \geq c I_\omega^{-1} k^{\frac{q-1}{q}}.$$

We make the final remark that the estimates in Theorem 1.1 of [10] also hold for the Green's functions and local volume non-collapsing for Kähler currents in $\mathcal{AK}(X, \theta_Y, n, p, A, K)$.

2 Proof of Proposition 1.1

We will prove Proposition 1.1 in this section. The following lemma is proved in [10] (Corollary 5.1).

Lemma 2.1 *Suppose* $\omega \in \mathcal{V}(X, \theta_X, n, A, p, K)$*. If* $v \in C^2(X)$ *satisfies*

$$|\Delta_\omega v| \leq 1 \text{ and } \int_X v\omega^n = 0, \tag{2.5}$$

then there is $C = C(X, \theta_X, n, A, p, K) > 0$ *such that*

$$\sup_X |v| \leq C\left(1 + \frac{1}{[\omega]^n}\int_X |v|\omega^n\right).$$

Proposition 2.1 *Suppose* $\omega \in \mathcal{V}(X, \theta_X, n, A, p, K)$*. If* $v \in C^2(X)$ *satisfies* $|\Delta_\omega v| \leq 1$ *and* $\int_X v\omega^n = 0$*, then there is* $C = C(X, \theta_X, n, A, p, K) > 0$ *such that*

$$\sup_X |v| \leq C.$$

Proof The proof is to repeat the argument of Lemma 5.1 of [10] by the maximum principle. For convenience, we denote by $C > 0$ uniform constants that only depend on X, θ_X, n, A, p, K throughout the argument. Without loss of generality we may assume that

$$\frac{1}{V_\omega}\int_{\{v>0\}} \omega^n \leq \frac{1}{2}, \tag{2.6}$$

otherwise we consider $-v$.

Given $v_+ = \max(v, 0)$, we consider the auxiliary complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\psi)^n = \frac{v_+}{B}\omega^n, \quad \sup_X \psi = 0, \tag{2.7}$$

with

$$B = \frac{1}{V_\omega}\int_X v_+\omega^n \in \mathbb{R}^+.$$

In fact, $\frac{1}{V_\omega}\int_X |v|\omega^n = 2B$ as $\int_X v\omega^n = 0$. Therefore we can assume $B \geq 1$, otherwise, the proposition automatically holds. One can easily replace v_+ by a positive smoothing of v_+ as in [10] and then take limits. For simplicity, we work with v_+ directly. There exists $C > 0$ such that

$$\|v_+\|_{L^\infty(X)} \leq \|v\|_{L^\infty(X)} \leq CB$$

by applying Lemma 2.1. Consequetially, $\omega + \sqrt{-1}\partial\bar{\partial}\psi \in \mathcal{V}(X, \theta_X, n, A, p, CK)$ for some uniform $C > 0$. We can apply Corollary 4.1 in [10] to derive the L^∞-estimate

$$\|\psi\|_{L^\infty(X)} \leq C. \tag{2.8}$$

for some uniform $C > 0$.

We now consider the following auxiliary function constructed in Lemma 5.1 of [10].

$$\Psi = -E(-\psi + D)^{\frac{n}{n+1}} + (v_+),$$

where $E, D > 0$ are to be determined later. We compute as in [10], at a maximum point p of Ψ, so that

$$\begin{aligned} 0 \geq& \Delta_\omega \Psi \\ \geq& \frac{nE}{n+1}(-\psi + D)^{-\frac{1}{n+1}} \Delta_\omega \psi + \Delta_\omega v_+ \\ \geq& \frac{n^2 E}{n+1}(-\psi + D)^{-\frac{1}{n+1}} \left(\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n}{\omega^n} \right)^{\frac{1}{n}} - 1 - \frac{n^2 E}{n+1}(-\psi + D)^{-\frac{1}{n+1}} \\ \geq& \frac{n^2 E}{n+1}(-\psi + D)^{-\frac{1}{n+1}} \left(\frac{v_+}{B} \right)^{\frac{1}{n}} - 1 - \frac{n^2 E}{(n+1)D^{\frac{1}{n+1}}}. \end{aligned}$$

Therefore at p, we have

$$v_+ \leq \left(1 + \frac{n^2 E}{(n+1)D^{\frac{1}{n+1}}}\right)^n \left(\frac{n+1}{n^2 E}\right)^n B(-\psi + D)^{\frac{n}{n+1}}. \tag{2.9}$$

We set

$$\left(1 + \frac{n^2 E}{(n+1)D^{1/(n+1)}}\right) \frac{(n+1)}{n^2 E} B^{\frac{1}{n}} = E^{\frac{1}{n}}$$

by choosing E and D to satsisfy

$$E = \left(\frac{n+1}{n^2\delta}\right)^{\frac{n}{n+1}} B^{\frac{1}{n+1}}, \quad D = \frac{n^2\delta}{(1-\delta)^{n+1}(n+1)} B \tag{2.10}$$

for some sufficiently small $\delta > 0$ to be determined later. This makes $\Psi \leq 0$ at p, hence $\sup_X \Psi \leq 0$. We now have on $\Omega_+ = \{v > 0\}$,

$$v_+ \leq E(-\psi + D)^{\frac{n}{n+1}} = \left(\frac{n+1}{n^2\delta}\right)^{\frac{n}{n+1}} B^{\frac{1}{n+1}} \left(-\psi + \frac{n^2\delta}{(1-\delta)^{n+1}(n+1)} B\right)^{\frac{n}{n+1}}$$

by plugging the expressions of E and D in (2.10). Integrating over Ω_+, we have

$$B = \frac{1}{V_\omega} \int_{\Omega_+} v_+ \omega^n \leq \frac{1}{2} \left(\frac{n+1}{n^2\delta}\right)^{\frac{n}{n+1}} B^{\frac{1}{n+1}} \left(C + \frac{n^2\delta}{(1-\delta)^{n+1}(n+1)} B\right)^{\frac{n}{n+1}}$$

by (2.8) and (2.6). Therefore

$$B \le 2^{-\frac{n+1}{n}} \left(\frac{n+1}{n^2\delta} C + \frac{1}{(1-\delta)^{n+1}} B \right). \tag{2.11}$$

By choosing $\delta = \delta(n) > 0$ with $\frac{1}{2(1-\delta)^{n+1}} = \frac{2}{3}$, we have $B \le C$ for some uniform constant C.

Finally, as observed earlier, $\frac{1}{V_\omega}\int_X |v|\omega^n = 2B \le 2C$. We now can complete the proof of the proposition by invoking Lemma 2.1 again. □

3 Proof of Theorem 1.1

We will combine the results of [10, 11] for $\mathcal{V}(X, \theta_X, n, p, A, K)$. Due to Proposition 1.1, we can remove the small degeneracy assumption in all the results and applications of [10, 11] on a barrier function γ. As an example, we have the following theorem for smooth Kähler metrics in $\mathcal{V}(X, \theta_X, n, p, A, K)$.

Theorem 3.1 *Let X be an n-dimensional compact Kähler manifold. For any $\omega \in \mathcal{V}(X, \theta_X, n, p, A, K)$, the following hold.*

(1) There exists $C = C(X, \theta_X, n, p, A, K) > 0$ such that

$$\mathrm{diam}(X, \omega) \le C.$$

(2) There exist $q > 1$ and $C_S = C_S(X, \theta_X, n, p, A, K, q) > 0$ such that

$$\left(\frac{1}{V_\omega}\int_X |u|^{2q}\omega^n\right)^{\frac{1}{q}} \le C_S \left(\frac{I_\omega}{V_\omega}\int_X |\nabla u|^2_\omega\, \omega^n + \frac{1}{V_\omega}\int_X u^2\omega^n \right).$$

for all $u \in W^{1,2}(X)$.

(3) There exists $C = C(X, \theta_X, n, p, A, K) > 0$ such that the following trace formula holds for the heat kernel $H(x, y, t)$ of (X, d, ω^n)

$$H(x, x, t) \le \frac{1}{V_\omega} + \frac{C}{V_\omega} I_\omega^{\frac{q}{q-1}} t^{-\frac{q}{q-1}}$$

on $X \times (0, \infty)$.

(4) Let $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$ be the increasing sequence of eigenvalues of the Laplacian $-\Delta_\omega$ on (X, d, ω^n). Then there exists $c = c(X, \theta_X, n, p, A, K) > 0$ such that

$$\lambda_k \ge c I_\omega^{-1} k^{\frac{q-1}{q}}.$$

Here $I_\omega = [\omega] \cdot [\theta_X]^{n-1}$ is the normalization constant.

The proof of Theorem 3.1 follows line by line in the argument of [10, 11] due to Proposition 1.1. Now we can reduce Theorem 1.1 to Theorem 3.1 by the smoothing arguments in Section 7 of [11].

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