Transverse measures to infinite type laminations

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Abstract

We study the cone of transverse measures to a fixed geodesic lamination on an infinite type hyperbolic surface. Under simple hypotheses on the metric, we give an explicit description of this cone as an inverse limit of finite-dimensional cones. We study the problem of when the cone of transverse measures admits a base and show that such a base exists for many laminations. Moreover, the base is a (typically infinite-dimensional) simplex (called a *Choquet simplex*) and can be described explicitly as an inverse limit of finite-dimensional simplices. We show that on any fixed infinite type hyperbolic surface, every Choquet simplex arises as a base for *some* lamination. We use our inverse limit description and a new construction of geodesic laminations to give other explicit examples of cones with exotic properties.

1 Introduction

Geodesic laminations on infinite type surfaces are currently poorly understood. However, they promise to be valuable tools in the study of the mapping class groups and Teichmüller theory of infinite type surfaces. As an example, understanding geodesic laminations would help to advance the study of hyperbolic graphs associated to infinite type type surfaces such as the *ray graph* ([4],[5]). What is missing is a structure theory for laminations on infinite type surfaces.

Lacking such a structure theory, one may attempt to understand the *ergodic theory* of infinite type laminations; i.e. the theory of transverse measures to infinite type laminations. Such a goal has been undertaken recently ([8], [22], [23]). Both the structure and the ergodic theory of laminations on finite type surfaces are well understood. Geodesic laminations on finite type surfaces consist of finitely many minimal sub-laminations together with finitely many isolated leaves (see e.g. [12, Theorem I.4.2.8]). The cone of transverse measures to a finite type lamination is a finite-dimensional simplicial cone. Its base is a simplex which embeds projectively into the *Thurston boundary* of Teichmüller space. Moreover, the space of all measured laminations on a fixed closed hyperbolic surface has a natural piecewise-linear structure. Our goals in this paper are to give a very explicit description of the cone of transverse measures to an infinite type lamination and to highlight similarities and differences with the finite type theory as well as connections with ergodic theory and functional analysis.

Fix a complete hyperbolic surface X of infinite type, without boundary, and a geodesic lamination Λ on X. We will assume that X is *of the first kind*, meaning that the limit set of $\pi_1(X)$ acting on the universal cover $\widetilde{X}$ is the entire circle $\partial\widetilde{X}$. A *transverse measure* to Λ assigns to each arc transverse to Λ a finite Borel measure and these measures are invariant under isotopies respecting Λ. Transverse measures may be compared to *invariant measures* of dynamical systems.

The space of all transverse measures to Λ has the structure of a topological convex cone with the *weak* topology*. We denote it by $\mathcal{M}(\Lambda)$.

Our initial result gives a rough description of $\mathcal{M}(\Lambda)$. This will be refined momentarily into a much more explicit description of $\mathcal{M}(\Lambda)$ as an inverse limit of finite-dimensional cones. Here $\mathbb{R}_+ = [0, \infty) \subset \mathbb{R}$.

Theorem A. *The cone $\mathcal{M}(\Lambda)$ is linearly homeomorphic to a closed sub-cone of the product $\mathbb{R}_+^{\mathbb{N}}$ cut out by countably many linear equations.*

This appears to be related to the main result of [23], which describes the cone of measured laminations carried by a train track via equations. However, Theorem A does not follow immediately from this.

To understand the cone $\mathcal{M}(\Lambda)$ better, we fix an *exhaustion* of X, $X_1 \subset X_2 \subset \ldots$, by surfaces with geodesic boundary which are compact, minus finitely many punctures. The intersection $\Lambda \cap X_n$ consists of finitely many compact minimal sub-laminations contained in the interior of X_n, geodesics spiraling onto these minimal sub-laminations, plus some collection of proper arcs. Moreover, there are finitely many proper arcs in $\Lambda \cap X_n$ up to homotopy. We attach to each X_n a finite-dimensional cone $C(X_n)$, which records all the transverse measures to the compact minimal sub-laminations, plus assignments of non-negative numbers to all the homotopy classes of proper arcs. There are natural *transition maps* $\pi_n : C(X_{n+1}) \to C(X_n)$ which record how the arcs and minimal sub-laminations in X_{n+1} traverse those in X_n. This leads to our explicit description:

Theorem B. *The cone $\mathcal{M}(\Lambda)$ is linearly homeomorphic to the inverse limit of the cones $C(X_n)$ together with the transition maps π_n.*

The advantage of Theorem B is that the finite-dimensional cones $C(X_n)$ and transition maps π_n are easily computable in practice, so that the theorem gives a very explicit description of the cone $\mathcal{M}(\Lambda)$. As first examples, we construct a lamination with a single non-zero transverse measure up to scaling (Example 4.6), and another lamination with no non-zero transverse measures at all:

Theorem C. *Let X be a complete, infinite type hyperbolic surface of the first kind. Then there exists a geodesic lamination Λ on X that has no non-zero transverse measures.*

We next study the problem of when the cone $\mathcal{M}(\Lambda)$ admits a convex, compact cross section (a *base*). We show that such bases do exist in many examples and are examples of *Choquet simplices.* Choquet simplices are infinite-dimensional versions of finite-dimensional simplices, familiar from dynamics and functional analysis. As is well known, the space of invariant probability measures of a homeomorphism of a compact metric space is always a Choquet simplex.

Theorem D. *Suppose that there is a compact subsurface of X which intersects every leaf of Λ. Then $\mathcal{M}(\Lambda)$ has a base which is a compact metrizable Choquet simplex. Further, there is an exhaustion $X_1 \subset X_2 \subset \ldots$ of X for which this Choquet simplex is the inverse limit of bases of the cones $C(X_n)$ with the restrictions of the maps π_n.*

In particular, this theorem applies to any *minimal* lamination. Choquet simplices can have exotic spaces of extreme points. For example, in Example 4.4 the space of extreme points is homeomorphic to the ordinal $\omega + 1$. In Example 4.5 the space of extreme points is not closed. An even more exotic example is the *Poulsen simplex* ([19]), which has a dense set of extreme points. Our next results show that cones of transverse measures can be arbitrarily strange. Namely, there is no obstruction whatsoever to the Choquet simplex that can appear as a base:

Theorem E. *Let X be a complete, infinite type hyperbolic surface of the first kind. Let Δ be a compact metrizable Choquet simplex. Then there exists a minimal geodesic lamination Λ on X for which the cone $\mathcal{M}(\Lambda)$ has a base which is affinely homeomorphic to Δ.*

Realization theorems of this type for Choquet simplices are familiar from dynamics and algebra ([14, 16, 7, 17]). For instance, [14] shows that every Choquet simplex arises as the space of invariant probability measures of a minimal compact dynamical system.

Our main tool for proving Theorem E is a construction of laminations as inverse limits of arcs on compact subsurfaces, together with a construction of such inverse limits using planar maps of intervals. These constructions recover *every* geodesic lamination without compact sub-laminations or leaves asymptotic to punctures, and we anticipate that they can be used to construct examples of laminations with other exotic properties.

Unfortunately, the cones $\mathcal{M}(\Lambda)$ do not always admit compact bases, the obstruction being sub-laminations disjoint from any given compact subsurface. We give examples in Section 8.2. One such example is a lamination with cone of transverse measures $\mathbb{R}_+^{\mathbb{N}}$.

It would be interesting to connect the methods of this paper with Teichmüller theory. In [8], Bonahon-Šarić produce a Thurston boundary for the quasi-conformal deformation space of an infinite type hyperbolic surface. This is the space of projective *bounded* measured laminations. It would be interesting to know whether the cone of bounded transverse measures to an infinite type geodesic lamination (with the *uniform* weak* topology) admits an explicit description as an inverse limit, similar to Theorem B. One could then study bases for such cones and ask:

Question 1.1. Is there a Choquet simplex which does not embed projectively into the Thurston boundary of the quasi-conformal deformation space of some infinite type hyperbolic surface X?

Structure of the paper. In Section 2, we study the cone of finite measures on a compact totally disconnected metrizable space. We show that it may be described as a closed sub-cone of $\mathbb{R}_+^{\mathbb{N}}$ cut out by countably many linear equations. This fact is probably well known to the experts but we couldn't find it in the literature. The techniques of Section 2 foreshadow those of Section 3, where we prove Theorem A describing the cone of transverse measures $\mathcal{M}(\Lambda)$ to a lamination Λ via equations. In Section 4 we prove Theorem B describing $\mathcal{M}(\Lambda)$ as an inverse limit. Namely, in Section 4.1 we give a more precise version of Theorem B, in Section 4.2 we give explicit descriptions of certain cones using Theorem B, and in Section 4.3 we complete the proof of Theorem B. In Section 5 we prove Theorem D giving an explicit description of bases for $\mathcal{M}(\Lambda)$ for certain laminations Λ. We also give several explicit examples of bases that arise easily. In Section 6 we give a construction of laminations on infinite type surfaces as "inverse limits" of finite systems of arcs on compact subsurfaces. We use this construction to prove Theorem E in Section 7. Finally in Section 8 we prove Theorem C and give some examples of laminations Λ for which $\mathcal{M}(\Lambda)$ has no compact base.

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2 Borel measures on compact totally disconnected metrizable spaces

Before getting started we set up a few definitions. A *cone* C is a set endowed with operations of addition and multiplication by scalars in $\mathbb{R}_+ = [0, \infty)$ such that addition is associative and commutative and $c \cdot (v + w) = c \cdot v + c \cdot w$ for $c \in \mathbb{R}_+$ and $v, w \in C$. A particular type of cone is a *convex cone*, which is a subset of a real vector space which is closed under the ambient operations of addition and multiplication by scalars in $\mathbb{R}_+$. A map between convex sets $f : C \to D$ is *affine* if $f(rv + sw) = rf(v) + sf(w)$ for $r, s \in \mathbb{R}_+$ with $r + s = 1$ and $v, w \in C$. If C and D are convex cones then $f : C \to D$ is *linear* if additionally $f(0) = 0$. We introduce the following convention:

Convention 2.1. Unless stated otherwise, all cones will be assumed to be convex cones. All maps between cones will be assumed to be linear. All maps between convex subsets of cones will be assumed to be affine.

An *n-dimensional simplicial cone* is a sub-cone of $\mathbb{R}^m$ spanned by n linearly independent vectors.

Our first theorem previews Theorem A, and illustrates many of the techniques that we use to prove it. Let X be a compact, totally disconnected metrizable space (e.g. the Cantor set). The theorem below is presumably known to the experts, but we couldn't find it in the literature. Let $\mathcal{M}(X)$ be the space of finite Borel measures on X with the weak* topology. This is the weakest topology such that for every continuous function $f : X \to \mathbb{R}$, the function $\mathcal{M}(X) \to \mathbb{R}$, $\mu \mapsto \int_X f \, d\mu$ is continuous. By $\mathcal{K}(X)$ denote the (finite or countable) collection of all clopen subsets of X. Applying the definition to the characteristic function of any $K \in \mathcal{K}(X)$, we see that the function $\mathcal{M}(X) \to \mathbb{R}_+$, $\mu \mapsto \mu(K)$ is continuous. Putting all these maps together gives a continuous linear map

$$\Phi : \mathcal{M}(X) \to \prod_{K \in \mathcal{K}(X)} \mathbb{R}_+.$$

This product is homeomorphic to $\mathbb{R}_+^n$ for some $n \geq 0$ or to $\mathbb{R}_+^{\mathbb{N}}$, depending on whether $\mathcal{K}$ is finite or not. However, the map Φ is usually not surjective. For $K \in \mathcal{K}(X)$ and a point $x \in \prod_{\mathcal{K}(X)} \mathbb{R}_+$, we let x_K denote the coordinate of x corresponding to K. If K is the disjoint union of $K_1, \ldots, K_r$ then we have $x_{K_1} + \cdots + x_{K_r} = x_K$ for any x in the image of Φ. We endow $\prod_{\mathcal{K}(X)} \mathbb{R}_+$ with the product topology.

Theorem 2.2. *The map Φ is a linear homeomorphism onto the closed sub-cone $C_X \subset \prod_{K \in \mathcal{K}(X)} \mathbb{R}_+$ cut out by the linear equations $x_K = x_{K_1} + \cdots + x_{K_r}$ whenever $K = \bigsqcup_{j=1}^r K_j$.*

To prove the theorem, we apply the following *Portmanteau Theorem*:

Theorem 2.3 ([6, Theorem 2.3])**.** *Let X be a compact, totally disconnected metrizable space. Let $\{\mu_n\}_{n=1}^\infty$ and μ be finite Borel measures on X. Then $\mu_n \xrightarrow{weak^*} \mu$ if and only if $\mu_n(K) \to \mu(K)$ for every clopen subset K of X.*

Proof of Theorem 2.2. The image is clearly contained in C_X. That Φ is a bijection to this cone follows from the Carathéodory Extension Theorem, which states that any finite, additive measure on the algebra of sets $\mathcal{K}(X)$ uniquely extends to a Borel measure on X (see e.g. [15, Section 7.4.2]). Finally, that $\Phi^{-1} : C_X \to \mathcal{M}(X)$ is continuous follows from the Portmanteau Theorem 2.3. $\square$

In practice, one can use much smaller collections of clopen sets and explicitly compute the cone C_X. We fix a sequence $\mathcal{A}_i$, $i = 1, 2, \ldots$ of finite families of clopen subsets of X so that:

(i) $\mathcal{A}_1 = \{X\}$;

(ii) for each $i > 1$, $\mathcal{A}_i$ forms a finite partition of X that refines $\mathcal{A}_{i-1}$; and

(iii) for some (any) metric on X the mesh of $\mathcal{A}_i$ goes to 0 as $i \to \infty$.

Also let $\mathcal{A} = \bigcup_i \mathcal{A}_i$.

Example 2.4. When X is the middle thirds Cantor set we can take $\mathcal{A}_i$ to consist of the 2^{i-1} clopen sets obtained by intersecting X with the defining intervals at stage i. That is,

$$\mathcal{A}_2 = \left\{ \left[0, \frac{1}{3}\right] \cap X, \left[\frac{2}{3}, 1\right] \cap X \right\}, \quad \mathcal{A}_3 = \left\{ \left[0, \frac{1}{9}\right] \cap X, \left[\frac{2}{9}, \frac{1}{3}\right] \cap X, \left[\frac{2}{3}, \frac{7}{9}\right] \cap X, \left[\frac{8}{9}, 1\right] \cap X \right\}, \quad \text{etc.}$$

When $X = \{1/n : n = 1, 2, \ldots\} \cup \{0\}$ we can take $\mathcal{A}_i$ for $i > 1$ to consist of the singletons $\{1\}, \{1/2\}, \ldots, \{\frac{1}{i-1}\}$ and the set $\{1/n : n = i, i+1, \ldots\} \cup \{0\}$.

Then one obtains a linear map $\mathcal{M}(X) \to \prod_{A \in \mathcal{A}} \mathbb{R}_+$ which is a homeomorphism onto the sub-cone cut out by the equations $x_A = x_{A_1} + \cdots + x_{A_r}$ when $A \in \mathcal{A}_i$, $A_j \in \mathcal{A}_{i+1}$ and $A = \bigsqcup_{j=1}^r A_j$. The proof is the same as that of Theorem 2.2, since both the Carathéodory and Portmanteau theorems hold for $\mathcal{A}$.

Example 2.5. When $X = \{1/n : n = 1, 2, \ldots\} \cup \{0\}$, after removing redundant coordinates and keeping only those corresponding to $\{1/n : n = i, i+1, \ldots\} \cup \{0\}$, we see that $\mathcal{M}(X)$ can be identified with the sub-cone of $\mathbb{R}_+^{\mathbb{N}}$ defined by the inequalities $x_1 \geq x_2 \geq x_3 \geq \ldots$.

2.1 Bases and Choquet simplices

Let B be a compact convex set in a metrizable locally convex topological vector space, such as $\mathbb{R}^{\mathbb{N}}$ with the product topology. Recall that an *extreme point* of B is a point $x \in B$ that is not contained in the interior of any interval in B. The Krein-Milman Theorem states that B is the smallest closed convex set that contains the set $\operatorname{Ext}(B)$ of all extreme points of B (which form a Borel set by [21, Proposition 1.3]). A stronger version of the Krein-Milman Theorem is Choquet's Theorem, that for every point $c \in B$ there is a Borel probability measure ν supported on the set of extreme points such that, formally,

$$c = \int_{\operatorname{Ext}(B)} x \, d\nu.$$

This means that for every affine function $f : B \to \mathbb{R}$, we have

$$f(c) = \int_{\operatorname{Ext}(B)} f(x) \, d\nu$$

(see [21, Sections 3, 4] for all this). This expression is a generalization of a convex combination. In general, this measure ν is not unique. For example, the center of the square can be written as the midpoint of opposite vertices in two ways. A compact convex set B as above is a *Choquet simplex* if the measure ν is unique, for every $c \in B$. A compact convex set in $\mathbb{R}^n$ is a Choquet simplex if and only if it is a simplex.

A *base* of a cone C is a compact convex set that doesn't contain 0 and intersects every ray in C based at the origin in exactly one point. For example, the space of probability measures $\mathcal{P}(X)$ on X (where X is compact, totally disconnected, metrizable, as above) is compact by the Banach-Alaoglu Theorem, and so it is a base for $\mathcal{M}(X)$. A probability measure on X is extreme in $\mathcal{P}(X)$ if and only if it is supported on one point, and the space $\mathrm{Ext}(\mathcal{P}(X))$ can be identified with X. We now see that $\mathcal{P}(X)$ is a Choquet simplex, since for a probability measure μ on X, the required measure ν on $\mathrm{Ext}(\mathcal{P}(X)) = X$ is the measure μ itself. As a simpler example, a base for the simplicial cone $\mathbb{R}_+^{n+1}$ is the standard n-dimensional simplex.

In finite dimensions Choquet simplices are just the standard simplices, but in infinite dimensions they can be quite pathological. The best behaved are *Bauer simplices*, whose extreme points form a closed subset, but there are also *Poulsen simplices*, whose extreme points are dense (see [1, 19] for more on these examples).

3 $\mathcal{M}(\Lambda)$ as a sub-cone of $\mathbb{R}_+^{\mathbb{N}}$

Let Λ be a geodesic lamination on a complete hyperbolic surface X. In the special case that X is finite type we allow X to have geodesic boundary and we allow the leaves of Λ to intersect the boundary transversely. All the definitions below will apply to this special sub-case. Such surfaces with boundary will come up only when we consider an exhaustion of a larger surface. If X is infinite type then we assume that it is without boundary. In the case that X *does not* have boundary, the universal cover $\widetilde{X}$ is homeomorphic to the hyperbolic plane $\mathbb{H}^2$ and $\pi_1(X)$ acts on the compactification $\widetilde{X} \cup \partial_\infty \widetilde{X}$, where $\partial_\infty \widetilde{X}$ is the Gromov boundary, i.e. a circle. Fixing any $x \in \widetilde{X}$, the *limit set* of X is the closure of the orbit of x in $\widetilde{X} \cup \partial_\infty \widetilde{X}$ intersected with the Gromov boundary. I.e. the limit set is $\overline{\pi_1(X) \cdot x} \cap \partial_\infty \widetilde{X}$. We will assume throughout the paper that X is of the *first kind*, meaning that the limit set is all of $\partial_\infty \widetilde{X}$. We have the following theorem of Šarić:

Theorem 3.1 ([23, Theorem 1.1])**.** *Let X be a complete hyperbolic surface of the first kind and Λ a geodesic lamination on X. Then Λ is nowhere dense in X.*

A *transversal* or *transverse arc* is an embedded smooth arc $\tau \subset X$ with endpoints in $X \setminus \Lambda$ such that τ is transverse to every leaf of Λ. Two transversals σ, τ are *homotopic* if there is a smooth map $F : [0,1] \times [0,1] \to X$ so that the restrictions to $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ are diffeomorphisms onto σ and τ, respectively, and the pre-image $F^{-1}(\Lambda)$ consists of horizontal segments $[0,1] \times \{t\}$. Such a map F is a *homotopy* between σ and τ. Denote by $f_\sigma : [0,1] \to \sigma$ the map $f_\sigma(\cdot) = F(0, \cdot)$ and $f_\tau : [0,1] \to \tau$ the map $f_\tau(\cdot) = F(1, \cdot)$. There is an induced diffeomorphism $f = f_\tau \circ f_\sigma^{-1} : \sigma \to \tau$ that preserves intersections with Λ.

A *transverse measure* to Λ is a function μ that to each transversal τ associates a finite Borel measure μ_τ on τ subject to the conditions:

- if $\tau' \subset \tau$ is a subarc which is also a transversal, then $\mu_{\tau'}$ is the restriction of μ_τ, and

- if F is a homotopy from σ to τ and f is the induced diffeomorphism $f = f_\tau \circ f_\sigma^{-1}$, then μ_τ is equal to the push-forward measure $f_*(\mu_\sigma)$.

It follows from the definition that μ_τ is supported on $\Lambda \cap \tau$.

Let $\mathcal{M}(\Lambda)$ be the set of transverse measures to Λ. Transverse measures may be added and multiplied by scalars in $\mathbb{R}_+$ simply by performing these operations to each measure μ_τ. Thus $\mathcal{M}(\Lambda)$ is a cone. We endow $\mathcal{M}(\Lambda)$ with the weakest topology such that the maps $\mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda \cap \tau)$, $\mu \mapsto \mu_\tau$ are continuous for every transversal τ. This is called the *weak* topology*. The addition and scalar multiplication operations are continuous in this topology.

Example 3.2. Let X be a complete hyperbolic surface with finite area and non-empty totally geodesic boundary. Consider a nowhere dense lamination Λ consisting of a family of proper arcs which are homotopic through homotopies preserving ∂X setwise. For instance, Λ could consist of homotopic compact arcs from the boundary ∂X to itself. We may view Λ as an embedding of $A \times I$ where A is compact and totally disconnected, and I is an interval in $\mathbb{R}$ (possibly infinite) with $A \times \partial I$ mapping to ∂X. There is a transversal τ_0 which intersects each leaf of Λ exactly once. We have $\tau_0 \cap \Lambda \cong A$ and thus $\mu \mapsto \mu_{\tau_0}$ defines a linear map $\mathcal{M}(\Lambda) \to \mathcal{M}(A)$. This map is a homeomorphism, since any other transversal to Λ may be partitioned into sub-transversals which are homotopic to sub-transversals of τ_0. Thus, any measure μ_σ is determined entirely by the measure $\mu_{\tau_0} \in \mathcal{M}(A)$. These laminations will turn up extensively in Section 4.

In this section we prove Theorem A from the introduction.

Proof of Theorem A. Fix a family of transversals $\tau_1, \tau_2, \ldots$ such that every leaf intersects at least one τ_j. Recall that for a totally disconnected compact metrizable space X, $\mathcal{K}(X)$ denotes the set of clopen subsets of X. For each τ_j, let $\mathcal{K}_j := \mathcal{K}(\Lambda \cap \tau_j)$. Sending a transverse measure μ to the restrictions μ_{τ_i} defines a map

$$\Phi : \mathcal{M}(\Lambda) \to \prod_j \mathcal{M}(\Lambda \cap \tau_j) \subseteq \prod_j \prod_{K \in \mathcal{K}_j} \mathbb{R}_+ = \mathbb{R}_+^{\mathbb{N}}$$

which is linear and continuous. The image is contained in the sub-cone C_Λ of $\prod_j \mathcal{M}(\Lambda \cap \tau_j)$ cut out by the following linear equations: $x_K = x_L$ whenever there are $K \in \mathcal{K}_i$ and $L \in \mathcal{K}_j$ and transversals $\sigma \subset \tau_i$ and $\tau \subset \tau_j$ with $K = \Lambda \cap \sigma$, $L = \Lambda \cap \tau$, such that σ is homotopic to τ.

We now argue that Φ is a homeomorphism onto C_Λ. We utilize the following basic fact about homotopies. See e.g. [11, Section 4.2] for the argument.

Lemma 3.3. *Let Λ be a geodesic lamination on the hyperbolic surface X of the first kind. Let p_1, p_2 be points lying on a common leaf of Λ. Let σ_i be transversals to Λ through the points p_i. Then there are sub-transversals $\sigma_i' \subset \sigma_i$ containing p_i for $i = 1, 2$ such that σ_1' is homotopic to σ_2'.*

If $\mu, \mu' \in \mathcal{M}(\Lambda)$ with $\mu \neq \mu'$, then there is a transversal τ so that the induced measures μ_τ and μ'_τ are different. By uniqueness in the Carathéodory Extension Theorem, after replacing τ with a sub-transversal, we may assume that the total measures $\mu(\tau)$ and $\mu'(\tau)$ are different. Since every leaf of Λ intersects some τ_i, for each point $p \in \tau \cap \Lambda$ we may apply Lemma 3.3 to find a sub-transversal $\sigma \subset \tau$ containing p which is homotopic into some τ_i. By compactness of $\tau \cap \Lambda$, we can sub-divide τ into finitely many sub-transversals each of which is homotopic to a sub-transversal of some τ_i. It follows that for some i the measures on $\Lambda \cap \tau_i$ induced by μ and μ' are distinct, showing that Φ is injective.

Now, suppose we are given a point in the sub-cone C_Λ. This yields Borel measures on $\Lambda \cap \tau_i$ for every i, satisfying the homotopy invariance. If τ is an arbitrary transversal, we can sub-divide it as above into sub-transversals so that each is homotopic into some τ_i, and we can pull back the measures on τ_i to get a measure on $\Lambda \cap \tau$. If $\tau = \sigma_1 \cup \ldots \cup \sigma_r$ and $\tau = \sigma'_1 \cup \ldots \cup \sigma'_s$ are two different such partitions of τ into sub-transversals, then we may consider their common refinement $\tau = \bigcup_{i,j}(\sigma_i \cap \sigma'_j)$. Using the equations defining C_Λ, we see that the measure on $\sigma_i \cap \sigma'_j$, and thus on τ, is independent of the partition. Independence of the partition yields invariance of the constructed measure under homotopies and passing to sub-transversals. This shows that the image of Φ is the entire sub-cone C_Λ.

Finally, we argue that $\Phi^{-1} : C_\Lambda \to \mathcal{M}(\Lambda)$ is continuous. By the definition of the topology on $\mathcal{M}(\Lambda)$, it suffices to argue that the composition of Φ^{-1} with the restriction to $\mathcal{M}(\Lambda \cap \tau)$ is continuous, for every transversal τ. When $\tau = \tau_i$ for some i this is just a coordinate projection, so it is continuous. For an arbitrary τ, sub-divide and reduce to sub-transversals of τ_i's as above. $\square$

Corollary 3.4. *Let Λ be a geodesic lamination on a complete hyperbolic surface X of the first kind. If there is a compact subsurface of X that intersects every leaf of Λ then $\mathcal{M}(\Lambda)$ admits a base.*

Proof. In this case we can choose a finite collection of transversals that intersect every leaf. Thus, the product $\prod_j \mathcal{M}(\Lambda \cap \tau_j)$ is finite and each factor has its compact base of probability measures. For convex cones the property of having a base passes to finite products and closed sub-cones. $\square$

4 $\mathcal{M}(\Lambda)$ as an inverse limit

As before, X is hyperbolic of the first kind and $\Lambda \subset X$ is a geodesic lamination. Since X is of the first kind, we may fix an exhaustion

$$X_1 \subset X_2 \subset \ldots$$

of X where each X_i is a finite area complete subsurface with totally geodesic boundary (see [3]). Thus, X_i is a compact surface with boundary minus finitely many points. We will sometimes refer to such surfaces as *punctured compact subsurfaces*. In fact the proof of [3, Proposition 3.1] shows that any exhaustion of X by finite type subsurfaces straightens to an exhaustion by complete finite area subsurfaces with geodesic boundary. So we may blur the distinction between topological exhaustions and exhaustions by complete finite area subsurfaces with geodesic boundary.

We will assume for convenience that the boundary components of X_i are transverse to Λ. This may be achieved as follows. Suppose that we have constructed a sequence $Y_1 \subset Y_2 \subset \ldots \subset Y_n$ of punctured compact subsurfaces such that Y_i contains X_i and ∂Y_i is transverse to Λ for each $i \leq n$. Choose m large enough that X_m contains both Y_n and X_{n+1}. Choose $p > m$ large enough that X_p contains ∂X_m in its interior and $q > p$ large enough that X_q contains ∂X_p in its interior. The components of ∂X_p are contained in the interior of $X_q \setminus X_m$. We may apply a mapping class f supported on the components of $X_q \setminus X_m$, so that for any component c of ∂X_p, $f(c)$ intersects Λ transversely (if at all). Then setting $Y_{n+1} = f(X_p)$ yields $Y_n \subset Y_{n+1}$, $X_{n+1} \subset Y_{n+1}$, and ∂Y_{n+1} is transverse to Λ.

We will need the following structure theorem for $\Lambda \cap X_i$. Recall that the *support* of a transverse measure consists of the points p such that every transversal τ containing p has positive measure.

Proposition 4.1. *Consider the lamination $\Lambda_i = \Lambda \cap X_i$. It has a sub-lamination consisting of finitely many compact minimal sub-laminations contained in the interior of X_i, plus finitely many parallel families $A \times I$ of proper arcs, with A compact and totally disconnected and I a closed sub-interval of $\mathbb{R}$. Any transverse measure on Λ_i is supported on this sub-lamination. Any leaf of Λ_i that does not belong to this sub-lamination accumulates on one or more of the compact minimal sub-laminations of Λ_i.*

Proof. The lamination Λ_i contains leaves of three possible types:

(1) arcs which on each end either (a) intersect a boundary component of X_i or (b) are asymptotic to a puncture of X_i;

(2) simple closed geodesics and bi-infinite geodesics which are contained in a compact minimal sub-lamination in the interior of X_i;

(3) rays and bi-infinite geodesics which accumulate onto minimal sub-laminations on at least one end but are not contained in these minimal sub-laminations (we will also say these geodesics *spiral* onto the minimal sub-laminations).

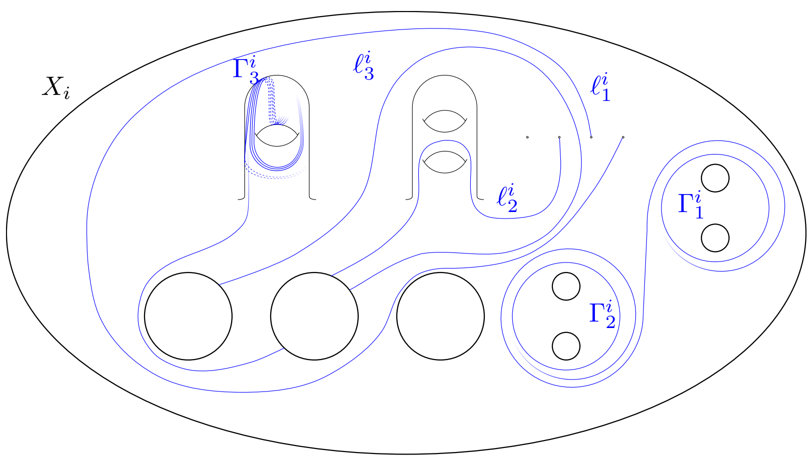


Figure 1: The various geodesics of Λ_i. Boundary components of X_i are denoted by thick black lines while punctures are denoted by small circles. Geodesics of Λ_i are indicated by blue lines. In this case there are three minimal sub-laminations and three homotopy classes of arcs. Homotopy classes ℓ_j^i are drawn as though they consist of a single arc for ease of presentation. Other geodesics of Λ_i accumulate onto compact minimal sub-laminations Γ_j^i on at least one end.

See Figure 1. There are finitely many arcs of type (1) up to homotopies preserving the boundary components of X_i setwise. Moreover, removing the leaves of types (1) and (3) and applying the classification theorem for laminations on finite type surfaces [12, Theorem I.4.2.9] yields that there are finitely minimal sub-laminations in the interior. Considering all of the arcs of type (1) in a single homotopy class yields a clopen subset of leaves of Λ_i which is homeomorphic to $A \times I$ for some closed sub-interval I of $\mathbb{R}$ (possibly all of $\mathbb{R}$ or a ray). The fact that A is totally disconnected follows from Theorem 3.1. This proves the first claim.

Now we show that any transverse measure to Λ_i is supported on the union of the leaves of types (1) and (2). Let μ be a transverse measure to Λ_i and consider a leaf L which accumulates onto a compact minimal sub-lamination Γ of Λ_i but is not contained in Γ. There is a transversal τ through L and a direction such that all the rays of Λ_i through τ in this direction are asymptotic, accumulate onto Γ, and never return to τ. To see the existence of such a τ, we take S to be one of the following surfaces: (i) if Γ is a closed geodesic then S is a collar neighborhood of Γ small enough that any geodesic that intersects S and is disjoint from Γ spirals onto Γ; (ii) if Γ is not a closed geodesic then S is the surface filled by Γ. We may take τ to lie inside of $S \setminus \Gamma$ and then all of the leaves of Λ_i through τ spiral onto Γ. By taking τ even smaller if necessary, such leaves never return to τ. Taking τ smaller again, such leaves all exit the same cusp of $S \setminus \Gamma$ in case (ii) and they are all asymptotic, in either case. Consider a transversal σ which intersects Γ. Considering a point $p \in \sigma \cap \Gamma$, we see that L intersects σ in infinitely many points limiting to p. We see that we may homotope τ to infinitely many disjoint sub-intervals of σ. Since $\mu(\sigma) < \infty$, we must have $\mu(\tau) = 0$. This completes the proof. $\square$

We denote by $\mathcal{M}(\Lambda_i)$ the cone of transverse measures to Λ_i. By Example 3.2, we can write $\mathcal{M}(\Lambda_i)$ as a finite product $\prod_\Gamma \mathcal{M}(\Gamma) \times \prod_{A\times I} \mathcal{M}(A)$ where Γ ranges over the compact minimal sub-laminations of Λ_i and $A \times I$ over the parallel families of proper arcs. There is an associated cone $C_i := \prod_\Gamma \mathcal{M}(\Gamma) \times \prod_{A\times I} \mathbb{R}_+$, which is the quotient of $\mathcal{M}(\Lambda_i)$ obtained by identifying all measures on $A \times I$ with the same total mass. Each cone $\mathcal{M}(\Gamma)$ is finite-dimensional (see e.g. [11, Section 1.9.1]) and thus C_i is a finite-dimensional simplicial cone. We sometimes denote C_i by $C(X_i)$ to make the dependence on the surface X_i clear.

The situation is summarized in the following commutative diagram, where $\mathcal{W}(\Lambda)$ is the inverse limit of the bottom row. The maps $\Psi_i : \mathcal{M}(\Lambda_i) \to C_i$ are the quotient maps just defined. The horizontal arrows ρ_i on the top are restriction maps and on the bottom π_i are the induced maps on the quotient cones. One may check that the maps π_i are linear, since Ψ_{i+1}, ρ_i, and Ψ_i are linear. The map Ψ is $(\Psi_1, \Psi_2, \ldots)$.

$$\begin{tikzcd}
\mathcal{M}(\Lambda_1) \arrow[d, "\Psi_1"] & \mathcal{M}(\Lambda_2) \arrow[l, "\rho_1"'] \arrow[d, "\Psi_2"] & \mathcal{M}(\Lambda_3) \arrow[l, "\rho_2"'] \arrow[d, "\Psi_3"] & \ldots \arrow[l, "\rho_3"'] & \mathcal{M}(\Lambda) \arrow[d, "\Psi"] \\
C_1 & C_2 \arrow[l, "\pi_1"] & C_3 \arrow[l, "\pi_2"] & \ldots \arrow[l, "\pi_3"] & \mathcal{W}(\Lambda)
\end{tikzcd}$$

We now state the main theorem of this section. Theorem B from the introduction will follow immediately from it.

Theorem 4.2. *The map $\Psi : \mathcal{M}(\Lambda) \to \mathcal{W}(\Lambda)$ is a linear homeomorphism.*

The proof has the following outline: (1) $\mathcal{M}(\Lambda)$ is the inverse limit of $\mathcal{M}(\Lambda_i)$. (2) All vertical maps Ψ_i are proper and surjective. (3) Consequently, Ψ is proper and surjective. (4) Since X is of the first kind, Ψ is injective. (5) Consequently, Ψ is a homeomorphism.

Fact (1) follows from the definitions and (2) is a consequence of the Banach-Alaoglu Theorem. Then (3) follows by a diagram chase. The main thing to be proved is (4). Before giving the full proof we pause to consider the cones C_i, the transition maps π_i, and some examples of cones of measures that can be characterized using Theorem B.

4.1 Cones of weights and transition maps

We pause to give a more complete and intuitive description of the maps π_n. Denote by $\ell_1^n, \ldots, \ell_{r(n)}^n$ the homotopy classes of proper arcs in Λ_n and by $\Gamma_1^n, \ldots, \Gamma_{s(n)}^n$ the compact minimal sub-laminations contained in the interior of X_n. Thus, the arcs in the homotopy class ℓ_i^n form some parallel family $A_i^n \times I_i^n$ where A_i^n is compact, totally disconnected, and metrizable, and I_i^n is a closed (possibly infinite) interval in $\mathbb{R}$. There exists $s_0 \geq 0$ such that $\Gamma_1^{n+1}, \ldots, \Gamma_{s_0}^{n+1}$ all intersect X_n in some (possibly empty) collection of arcs, while $\Gamma_{s_0+1}^{n+1}, \ldots, \Gamma_{s(n)}^{n+1}$ are all contained in X_n. We have $\mathcal{M}(\Lambda_n) = \prod_{i=1}^{r(n)} \mathcal{M}(A_i^n) \times \prod_{i=1}^{s(n)} \mathcal{M}(\Gamma_i^n)$ and $C_n = \prod_{i=1}^{r(n)} \mathbb{R}_+ \times \prod_{i=1}^{s(n)} \mathcal{M}(\Gamma_i^n)$. Let e_j^n be the basis element 1 in the j-th factor $\mathbb{R}_+$ in $\prod_{i=1}^{r(n)} \mathbb{R}_+$. Then we may write an element of C_n as

$$w = \sum_{i=1}^{r(n)} b_i^n e_i^n + \sum_{i=1}^{s(n)} \nu_i^n$$

where $b_i^n \geq 0$ and $\nu_i^n \in \mathcal{M}(\Gamma_i^n)$ for each i. If $\nu \in \mathcal{M}(\Lambda_n)$ then $\Psi_n(\nu) = \sum_i b_i^n e_i^n + \sum_i \nu_i^n$ where (1) b_i^n is the measure $\nu(\tau_i^n)$ of a transversal τ_i^n which intersects each arc of ℓ_i^n exactly once and is disjoint from $\Lambda_n \setminus \ell_i^n$; and (2) ν_i^n is the restriction of ν to Γ_i^n: $\nu_i^n := \nu|\Gamma_i^n$. We think of an element of C_n as a *weight*, assigning a number to each homotopy class of arcs ℓ_i^n and a transverse measure to each minimal lamination Γ_i^n. We will refer to C_n as the *cone of weights* for Λ_n. Finally, define τ_i^{n+1} for $1 \leq i \leq r(n+1)$ to be a transversal intersecting each arc of ℓ_i^{n+1} exactly once and disjoint from $\Lambda_{n+1} \setminus \ell_i^{n+1}$.

For $1 \leq j \leq r(n+1)$, choose L to be any arc in ℓ_j^{n+1}. For $1 \leq i \leq r(n)$ we denote by a_{ij} the number of arcs of $L \cap X_n$ which are homotopic to ℓ_i^n. Thus, $A_j^{n+1} \times I_j^{n+1}$ passes through $A_i^n \times I_i^n$ exactly a_{ij} times. Informally, we will say that ℓ_j^{n+1} *traverses* ℓ_i^n a_{ij} times. From this, we see that we may partition τ_i^n into sub-transversals, a_{ij} of which are homotopic to τ_j^{n+1} for each $1 \leq j \leq r(n+1)$, and the remaining of which are disjoint from $\ell_1^{n+1} \cup \ldots \cup \ell_{r(n+1)}^{n+1}$. The sub-transversals of τ_i^n which are disjoint from $\ell_1^{n+1} \cup \ldots \cup \ell_{r(n+1)}^{n+1}$ intersect the various minimal laminations Γ_j^{n+1} for $1 \leq j \leq s_0$ and leaves which spiral onto such Γ_j^{n+1}, but are otherwise disjoint from Λ_{n+1}. Therefore if $\nu \in \mathcal{M}(\Lambda_{n+1})$ then

$$\nu(\tau_i^n) = \sum_{j=1}^{r(n+1)} a_{ij} \nu(\tau_j^{n+1}) + \sum_{j=1}^{s_0} \left(\nu|\Gamma_j^{n+1}\right)(\tau_i^n).$$

Putting this together yields: if $w = \sum_{j=1}^{r(n+1)} b_j^{n+1} e_j^{n+1} + \sum_{j=1}^{s(n+1)} \nu_j^{n+1}$ then

$$\pi_n(w) = \sum_{i=1}^{r(n)} \sum_{j=1}^{r(n+1)} a_{ij} b_j^{n+1} e_i^n + \sum_{i=1}^{r(n)} \sum_{j=1}^{s_0} \nu_j^{n+1}(\tau_i^n) e_i^n + \sum_{j=s_0+1}^{s(n+1)} \nu_j^{n+1}$$

noting that for $s_0 + 1 \leq j \leq s(n+1)$, ν_j^{n+1} lies in C_n since Γ_j^{n+1} is contained in X_n.

The easiest case to understand is when Λ_n and Λ_{n+1} contain no compact minimal sub-laminations Γ^*_*. Then $C_n = \mathbb{R}_+^{r(n)}$, $C_{n+1} = \mathbb{R}_+^{r(n+1)}$, and $\pi_n(w) = \sum_{i=1}^{r(n)} \sum_{j=1}^{r(n+1)} a_{ij} b_j^{n+1} e_i^n$. Thus, π_n is represented by the $r(n) \times r(n+1)$ matrix $(a_{ij})_{i=1,j=1}^{r(n),r(n+1)}$.

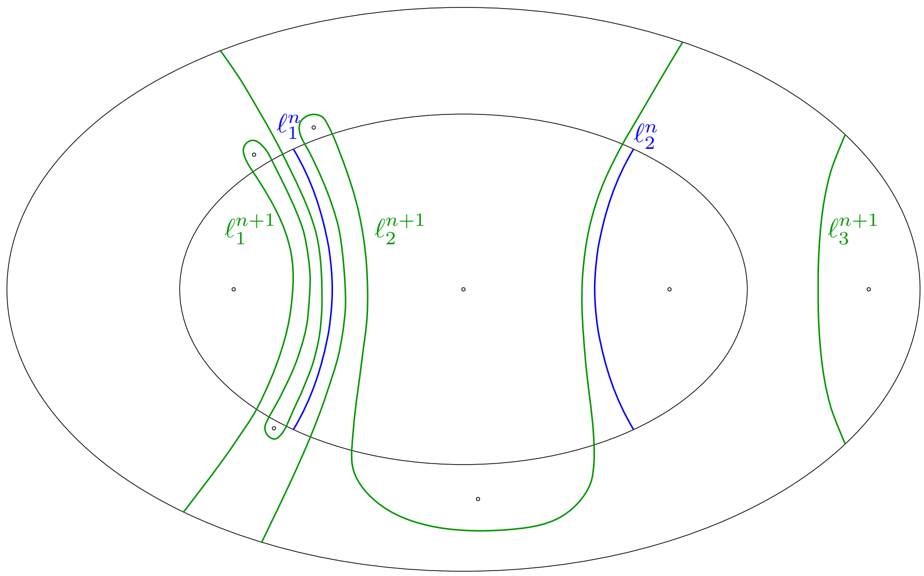


Figure 2: The figure illustrates two punctured disks X_n (the smaller punctured disk) contained inside X_{n+1} (the larger punctured disk). There are two homotopy classes of arcs on X_n, ℓ_i^n, and three homotopy classes of arcs on X_{n+1}, ℓ_j^{n+1}.

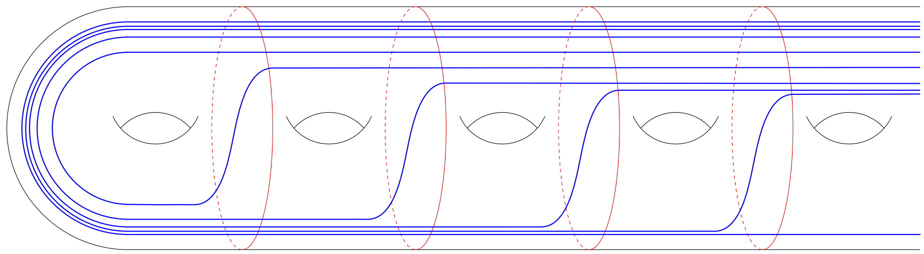
Example 4.3. Consider the punctured disks X_n and X_{n+1} pictured in Figure 2. There are two homotopy classes of arcs ℓ_1^n, ℓ_2^n on X_n and three homotopy classes of arcs $\ell_1^{n+1}, \ell_2^{n+1}, \ell_3^{n+1}$ on X_{n+1}. The class ℓ_1^{n+1} traverses ℓ_1^n three times, the class ℓ_2^{n+1} traverses ℓ_1^n twice and ℓ_2^n once, while ℓ_3^{n+1} doesn't traverse ℓ_1^n or ℓ_2^n. Thus, π_n is represented by the 2×3 matrix $\begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

4.2 Examples of cones of measures

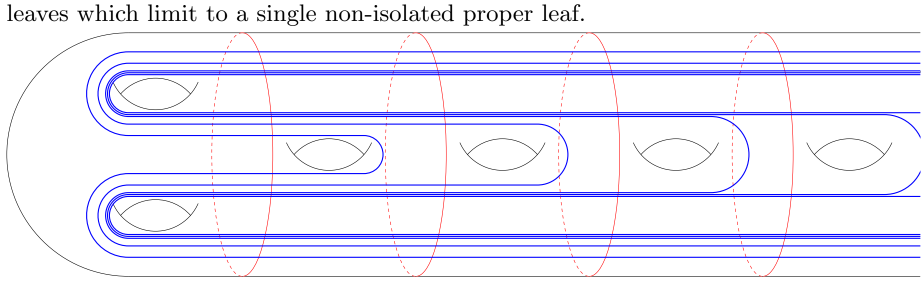
In this section we use Theorem B to give explicit descriptions of cones of transverse measures for certain examples of geodesic laminations.

Example 4.4. Consider the lamination Λ in Figure 3a. Thus Λ consists of a countable collection of isolated proper leaves L_i that converge to a single proper leaf L (which is not isolated). Recall that a leaf is *isolated* when it has an open neighborhood disjoint from the rest of the lamination. There is a transverse arc τ intersecting each leaf exactly once and one may check that $\mathcal{M}(\Lambda)$ is linearly homeomorphic to $\mathcal{M}(\Lambda \cap \tau)$. We verify this using inverse limits.

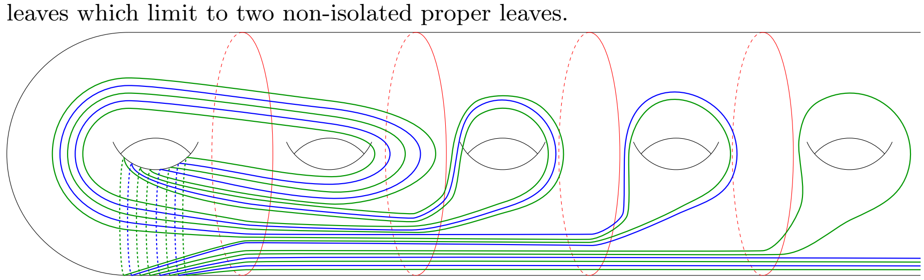
An exhaustion $\{X_n\}$ is given by the surfaces bounded by the red curves. Thus X_n has genus n and one boundary component. There are n homotopy classes of arcs $\ell_1^n, \dots, \ell_n^n$ on X_n. Moreover, choosing the numbering correctly, $\ell_i^{n+1} \cap X_n$ is homotopic to ℓ_i^n for $1 \le i \le n$ whereas $\ell_{n+1}^{n+1} \cap X_n$



(a) The lamination from Example 4.4 is a union of countably many isolated proper leaves which limit to a single non-isolated proper leaf.



(b) The lamination from Example 4.5 is a union of countably many isolated proper leaves which limit to two non-isolated proper leaves.



(c) The lamination from Example 4.6 is the closure of the two pictured non-proper leaves.

Figure 3: The laminations from Examples 4.4-4.6.

is homotopic to ℓ_n^n. Thus $\mathcal{W}(\Lambda)$ is the inverse limit of $\mathbb{R}_+ \xleftarrow{\pi_1} \mathbb{R}_+^2 \xleftarrow{\pi_2} \mathbb{R}_+^3 \xleftarrow{\pi_3} \dots$ where

$$\pi_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \pi_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \dots .$$

We claim that $\mathcal{M}(\Lambda) \cong \mathcal{W}(\Lambda)$ is linearly homeomorphic to the cone

$$C \subset \ell^1 \text{ defined by } C = \left\{ (x, y_1, y_2, \dots) : y_i \geq 0 \text{ for all } i \text{ and } x \geq \sum_{i=1}^{\infty} y_i \right\}$$

where the space ℓ^1 of summable sequences is endowed with its weak* topology as the dual of the space c_0 of sequences convergent to 0, and C is endowed with the subspace topology. We outline the proof. An element of the inverse limit $\mathcal{W}(\Lambda)$ has the form

$$\overleftarrow{x} = \left(\begin{pmatrix} x_0^0 \end{pmatrix}, \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix}, \dots \right)$$

where

$$x_0^0 = x_1^1 + x_2^1 = x_1^1 + x_2^2 + x_3^2 = x_1^1 + x_2^2 + x_3^3 + x_4^3 = \dots \text{ and } x_n^n = x_n^{n+1} = x_n^{n+2} = \dots \text{ for } n \geq 1.$$

Thus, the element is determined by the sequence of non-negative numbers $(x_0^0, x_1^1, x_2^2, \dots)$. Moreover, since $x_0^0 \geq \sum_{i=1}^k x_i^i$ for each $k \geq 0$ we have that $\sum_{i=1}^\infty x_i^i \leq x_0^0 < \infty$. Define a function $\mathcal{W}(\Lambda) \to C$ by sending $\overleftarrow{x}$ to $(x_0^0, x_1^1, \dots)$, which lies in C. The inverse is given by sending an element $(x, y_1, y_2, \dots) \in C$ to the element

$$\left(\begin{pmatrix} x \end{pmatrix}, \begin{pmatrix} y_1 \\ x - y_1 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ x - y_1 - y_2 \end{pmatrix}, \dots \right).$$

One may check that these functions are linear and that the function $C \to \mathcal{W}(\Lambda)$ is continuous since the coordinate functions x and y_n are continuous. One may check that $\mathcal{W}(\Lambda) \to C$ is continuous by using the fact that ℓ^1 is equipped with its weak* topology from its predual c_0.

Example 4.5. Consider the lamination Λ pictured in Figure 3b. Thus Λ consists of a countable collection of isolated proper leaves L_i that converge to the union of two disjoint proper leaves L and L' (neither of which is isolated). An exhaustion is given by the surfaces bounded by the red curves again. Thus X_n has genus $n+1$ and there are $n+1$ homotopy classes of arcs $\ell_1^n, \dots, \ell_{n+1}^n$ on X_n. The numbering can be chosen so that $\ell_i^{n+1} \cap X_n$ is homotopic to ℓ_i^n for $1 \leq i \leq n+1$ and so that $\ell_{n+2}^{n+1} \cap X_n$ is homotopic to the union of ℓ_1^n and ℓ_2^n. Thus, $\mathcal{W}(\Lambda)$ is the inverse limit of $\mathbb{R}_+^2 \xleftarrow{\pi_1} \mathbb{R}_+^3 \xleftarrow{\pi_2} \mathbb{R}_+^4 \xleftarrow{\pi_3} \dots$ where

$$\pi_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \dots.$$

An element of the inverse limit has the form

$$\left(\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}, \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{pmatrix}, \dots \right).$$

where

$$x_1^0 = x_1^1 + x_3^1 = x_1^2 + x_3^1 + x_4^2 = x_1^3 + x_3^1 + x_4^2 + x_5^3 = \dots$$

and

$$x_2^0 = x_2^1 + x_3^1 = x_2^2 + x_3^1 + x_4^2 = x_2^3 + x_3^1 + x_4^2 + x_5^3 = \dots$$

and $x^n_{n+2} = x^{n+1}_{n+2} = x^{n+2}_{n+2} = \ldots$ for $n \geq 1$. Using similar techniques as in Example 4.4 one may show that $\mathcal{M}(\Lambda) \cong \mathcal{W}(\Lambda)$ is isomorphic to the cone

$C \subset \ell^1$ defined by $C = \left\{(x_1, x_2, y_1, y_2, y_3, y_4, \ldots) : y_i \geq 0 \text{ for all } i \text{ and } x_1 \geq \sum y_i \text{ and } x_2 \geq \sum y_i\right\}$

where again ℓ^1 is endowed with its weak* topology as the dual of c_0.

Example 4.6. Consider the lamination Λ pictured in Figure 3c. The figure shows (the beginnings of) two leaves of the lamination. The lamination Λ is the closure of these two leaves. An exhaustion is given by the surfaces X_n bounded by the red curves. Observe that any leaf of the closure Λ intersected with X_n is homotopic to a component of the intersection of either the blue leaf or the green leaf with X_n. Thus, to describe the cone $\mathcal{W}(\Lambda)$ it suffices to study the intersections of the green leaf and the blue leaf with the compact subsurfaces X_n. We refer the reader also to Figure 4. This illustrates the construction of the blue and green leaves from Figure 3c. The reader may continue the construction inductively. Moreover, $\Lambda \cap X_n$ consists of two homotopy classes ℓ^n_1 (the green arcs pictured in Figure 4) and ℓ^n_2 (the blue arcs pictured in Figure 4). At each step, there is a value of i for which $\ell^{n+1}_i \cap X_n$ is homotopic to ℓ^n_i while for the other arc ℓ^{n+1}_{3-i}, $\ell^{n+1}_{3-i} \cap X_n$ consists of one arc homotopic to ℓ^n_1 and another arc homotopic to ℓ^n_2. Moreover, the value of i with this property alternates between 1 and 2 at each step. Thus, $\mathcal{W}(\Lambda)$ is the inverse limit of $\mathbb{R}^2_+ \xleftarrow{\pi_1} \mathbb{R}^2_+ \xleftarrow{\pi_2} \mathbb{R}^2_+ \xleftarrow{\pi_3} \ldots$ where

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \pi_1 = \pi_3 = \pi_5 = \ldots \text{ and } \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \pi_2 = \pi_4 = \pi_6 = \ldots.$$

Observe that

$$M_o := \pi_i \circ \pi_{i+1} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} \text{ for } i \text{ odd and } M_e := \pi_i \circ \pi_{i+1} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} \text{ for } i \text{ even.}$$

Consider the cones $C_n = C(X_n) = \mathbb{R}^2_+$ and the intersection of the images of C_m in C_n. That is, consider $\bigcap_{m=n}^\infty \pi_{nm}(C_m)$ where $\pi_{nm} = \pi_n \circ \pi_{n+1} \circ \cdots \circ \pi_{m-1}$. This is contained in the intersection of cones $\bigcap_{j=0}^\infty M_o^j(\mathbb{R}^2_+)$ or the intersection $\bigcap_{j=0}^\infty M_e^j(\mathbb{R}^2_+)$ depending on whether n is odd or even. In the odd case, this intersection is equal to the ray spanned by v_o where v_o is the (positive) attracting eigenvector $v_o = (\phi, 1)$ of M_o, where ϕ is the golden ratio $(1 + \sqrt{5})/2$. In the even case the intersection is equal to the span $\mathbb{R}_+ v_e$ where v_e is the attracting eigenvector $v_e = (\phi - 1, 1)$ of M_e. Thus, we see that $\mathcal{W}(\Lambda)$ is in fact equal to the limit of an inverse system of rays

$$\mathbb{R}_+ v_o \xleftarrow{\pi_1} \mathbb{R}_+ v_e \xleftarrow{\pi_2} \mathbb{R}_+ v_o \xleftarrow{\pi_3} \mathbb{R}_+ v_e \xleftarrow{\pi_4} \ldots$$

and one may check that each map π_n in this inverse system is surjective. Thus $\mathcal{W}(\Lambda)$ is linearly homeomorphic to a ray $\mathbb{R}_+$. That is, Λ has a single non-zero transverse measure up to scaling.

Example 4.7. Consider the lamination Λ in Figure 5, consisting of countably many isolated proper leaves which exit out the single end of the surface. Using the pictured exhaustion, we see that $\mathcal{M}(\Lambda)$ is the limit of $\mathbb{R}_+ \xleftarrow{\pi_1} \mathbb{R}^2_+ \xleftarrow{\pi_2} \mathbb{R}^3_+ \xleftarrow{\pi_3} \ldots$ where

$$\pi_1 = \begin{pmatrix}1 & 0\end{pmatrix}, \quad \pi_2 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \quad \pi_3 = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \quad \ldots.$$

From this we see that $\mathcal{M}(\Lambda)$ is linearly homeomorphic to $\mathbb{R}^{\mathbb{N}}_+$, a countable product of rays $\mathbb{R}_+$, with the product topology.

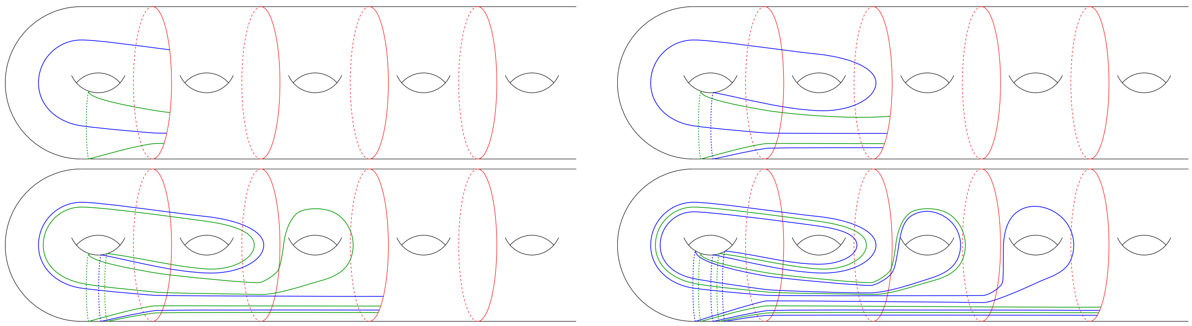


Figure 4: The construction of the two leaves from Example 4.6. At each new step one arc is extended to traverse both arcs from the previous step.

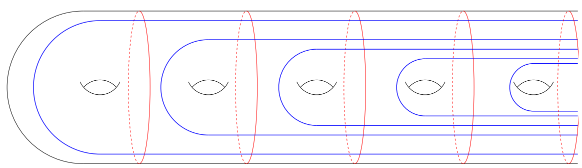


Figure 5: A lamination consisting of countably many isolated proper leaves.

4.3 Completing the proof of Theorem 4.2

In this section we complete the proof of Theorem 4.2. Our main new ingredient will be used to show injectivity of the map Ψ.

Lemma 4.8. *Let Λ be a geodesic lamination on the hyperbolic surface X of the first kind. Let τ be an arc transverse to Λ. Consider the exhaustion $X = \bigcup_{n=1}^{\infty} X_n$ by punctured compact subsurfaces and the homotopy classes of arcs $\{\ell_i^n\}_{i=1}^{r(n)}$ contained in X_n for each n. Then for any m large enough, and any i between 1 and $r(m)$, all of the arcs in the homotopy class ℓ_i^m intersect τ the same number of times.*

Before giving the proof we give an informal description. Each homotopy class ℓ_i^n forms a strip $A_i^n \times I_i^n$ where A_i^n is compact totally disconnected and I_i^n is a closed interval in $\mathbb{R}$. The transversal τ intersects these strips but may not pass all the way through each time. Thus, τ may turn around between two arcs L and M of $A_i^n \times I_i^n$ or have an endpoint between them. For $m \geq n$ the strips $A_j^m \times I_j^m$ traverse the strip $A_i^n \times I_i^n$ and partition it. The partition is eventually fine enough to separate L and M and this fixes the issue.

We also introduce some notation. If $Y \subset \widetilde{X}$ is a closed convex subset, then $\partial_0 Y$ denotes the boundary of Y as a subset of $\widetilde{X}$. The notation $\partial_\infty Y$ denotes the limit set of Y in $\partial_\infty \widetilde{X}$, i.e. the closure of Y in $\widetilde{X} \cup \partial_\infty \widetilde{X}$ intersected with $\partial_\infty \widetilde{X}$. Before proving Lemma 4.8, we prove the following general fact, which will be used several times in the sequel. Note that for a punctured compact subsurface $Y \subset X$, the pre-image of Y in the universal cover $\widetilde{X}$ consists of a family of disjoint closed convex subsets of $\widetilde{X}$.

Lemma 4.9. *Let X be a hyperbolic surface of the first kind. Let $X = \bigcup_{n=1}^{\infty} X_n$ be an exhaustion* *by punctured compact subsurfaces. Let $\widetilde{X}$ be the universal cover of X, choose a basepoint $* \in \widetilde{X}$ in the pre-image of X_1, and let $\widetilde{X_n}$ be the unique component of the pre-image of X_n in $\widetilde{X}$ containing $*$. Then $\bigcup_{n=1}^{\infty} \widetilde{X_n} = \widetilde{X}$. Moreover, for two distinct geodesics $L, M \subset \widetilde{X}$ and any n sufficiently large, the arcs of intersection $L \cap \widetilde{X_n}$ and $M \cap \widetilde{X_n}$ are not homotopic in $\widetilde{X_n}$ through homotopies preserving $\partial_0 \widetilde{X_n}$.*

Proof. The component $\widetilde{X_n}$ is invariant under the fundamental group $\pi_1(X_n)$. Consequently the union $\bigcup_{n=1}^{\infty} \widetilde{X_n}$ is a convex subset of $\widetilde{X}$ which is invariant under $\pi_1(X)$, and therefore we have $\bigcup_{n=1}^{\infty} \widetilde{X_n} = \widetilde{X}$.

For the last sentence of the lemma, note that at least one endpoint of L in $\partial_\infty \widetilde{X}$ is not shared by M. Hence, for any n sufficiently large, either L has an endpoint in $\partial_\infty \widetilde{X_n}$ which is not contained in $M \cap \partial_\infty \widetilde{X_n}$, or $L \cap \widetilde{X_n}$ has an endpoint in $\partial_0 \widetilde{X_n}$ which is not contained in $M \cap \partial_0 \widetilde{X_n}$. In either case, $L \cap \widetilde{X_n}$ is not homotopic to $M \cap \widetilde{X_n}$. $\square$

Proof of Lemma 4.8. Choose n large enough that τ is contained in X_n. We consider the lamination Λ_n and the homotopy classes $\{\ell_i^n\}_{i=1}^{r(n)}$. From Λ_n, remove all of the compact minimal sub-laminations Γ_i^n and all of the geodesics accumulating onto them. Denote by Λ_n' the sub-lamination which remains after this operation, consisting exactly of the arcs in all of the classes ℓ_i^n.

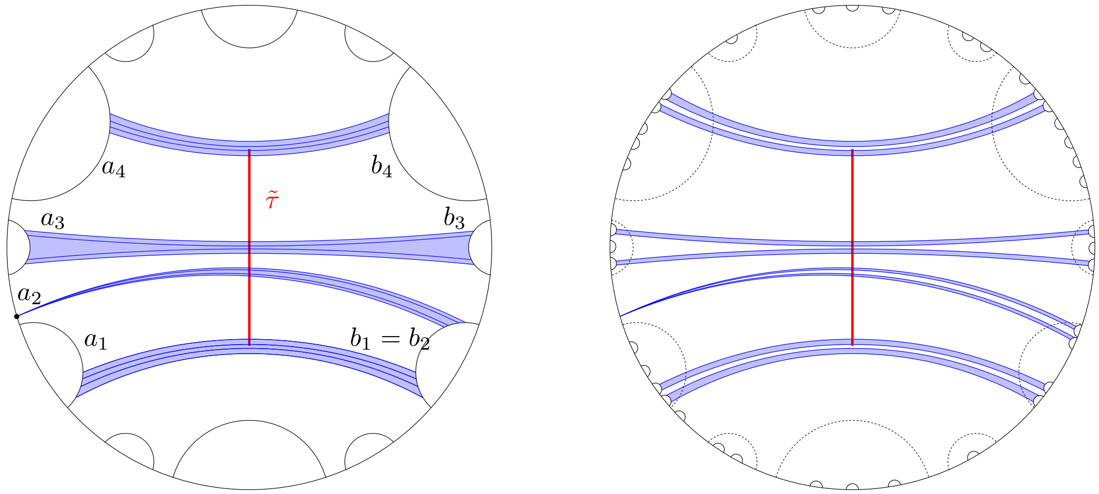
We make one simplifying assumption on τ, which we will remove at the end of the proof: for each ℓ_i^n, each arc in the homotopy class intersects τ exactly 0 or 1 times. This has the following consequence. If τ intersects an arc in the homotopy class ℓ_i^n then either (1) τ crosses every arc in ℓ_i^n exactly once or (2) it intersects some of them once, and has an endpoint between two arcs in ℓ_i^n. In particular, there are some values of i such that τ crosses all arcs in ℓ_i^n once and at most two values of i such that τ crosses some of the arcs in ℓ_i^n once and doesn't cross the others.

Consider the universal cover $\widetilde{X}$. Lift τ to an arc $\widetilde{\tau}$. The arc $\widetilde{\tau}$ is contained in a unique component $\widetilde{X_n}$ of the pre-image of X_n in $\widetilde{X}$. Thus, $\widetilde{X_n}$ is a universal cover for X_n. Define $\widetilde{\Lambda_n'}$ to be the pre-image of Λ_n' in $\widetilde{X_n}$. For each i, the arcs in the pre-image of ℓ_i^n are partitioned into families of arcs which are homotopic in $\widetilde{X_n}$ through homotopies preserving the boundary $\partial_0 \widetilde{X_n}$ setwise. Namely, if ℓ_i^n joins p to q where p, q can each be either punctures or boundary components of X_n, then p and q each either lift to geodesics of $\partial_0 \widetilde{X_n}$ (lifts of boundary components) or to ends of $\partial_\infty \widetilde{X_n}$. A homotopic family of arcs in the pre-image of ℓ_i^n joins a lift of p to a lift of q. See Figure 6a.

Consider the families of homotopic arcs in $\widetilde{\Lambda_n'}$ that intersect $\widetilde{\tau}$. This yields two finite multi-sets $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| = |\mathcal{B}|$, such that:

- each element of $\mathcal{A}$ (or $\mathcal{B}$) is either a boundary component of $\widetilde{X_n}$ in $\partial_0 \widetilde{X_n}$ or an end of $\widetilde{X_n}$ in $\partial_\infty \widetilde{X}$;
- enumerating the elements of $\mathcal{A}$ as $\{a_1, \ldots, a_r\}$ and the elements of $\mathcal{B}$ as $\{b_1, \ldots, b_r\}$, $\widetilde{\tau}$ intersects $\widetilde{\Lambda_n'}$ *only* in the arcs joining a_i to b_i for $i = 1, \ldots, r$.

We allow $\mathcal{A}$ and $\mathcal{B}$ to be multi-sets, since for instance, $\widetilde{\tau}$ may be intersected by two families of arcs which join a common boundary component of $\widetilde{X_n}$ to two different boundary components of $\widetilde{X_n}$. Moreover, we choose the numbering so that $\widetilde{\tau}$ intersects each arc of $\widetilde{\Lambda_n'}$ joining a_i to b_i *exactly once* unless $i = 1$ or r. For $i = 1$ or r, $\widetilde{\Lambda'_n}$ may intersect *only some* of the arcs of $\widetilde{\Lambda'_n}$ joining a_i to b_i (again intersecting each arc at most once).



(a) The cover $\widetilde{X_n}$ is bounded by the solid black geodesics. The lift $\widetilde{\tau}$ is drawn in red along with the strips of $\widetilde{\Lambda_n}$ that intersect it. The elements of $\mathcal{A}$ are the a_i and the elements of $\mathcal{B}$ are the b_i. Note that all a_i and b_i are geodesics of $\partial_0 \widetilde{X_n}$ except for a_2 which is an end of $\widetilde{X_n}$. Note also that $b_1 = b_2$.

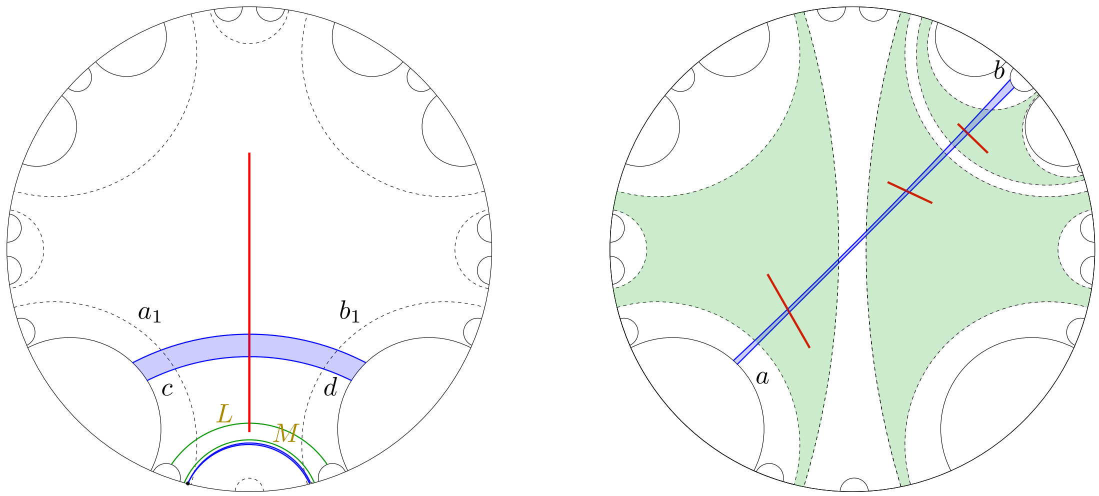
(b) The cover $\widetilde{X_m}$ is bounded by the solid black geodesics. Boundary components of $\widetilde{X_n}$ are indicated by dotted lines. For each strip of geodesics of $\widetilde{\Lambda}$ in $\widetilde{X_m}$, its geodesics all either intersect $\widetilde{\tau}$ exactly once or all intersect it zero times.

Figure 6: Moving to a larger surface $\widetilde{X_m}$ to separate geodesics in $\widetilde{X_n}$.

Now, if the arcs joining a_1 to b_1 do not all intersect $\widetilde{\tau}$ then there is a *last* such arc L which intersects $\widetilde{\tau}$ and a *first* such arc M which does not intersect $\widetilde{\tau}$. That is, L and M separate all the arcs from a_1 to b_1 which do not intersect $\widetilde{\tau}$ from all the arcs from a_1 to b_1 which do. Similarly, among the arcs joining a_r to b_r, there is (possibly) a *last* such arc L' which intersects $\widetilde{\tau}$ and a *first* such arc M' which does not intersect $\widetilde{\tau}$. Now L, M, L', M' may be extended to bi-infinite geodesics in $\widetilde{X}$. By Lemma 4.9 for any m large enough, we have that in $\widetilde{X_m}$, the unique component of the pre-image of X_m containing $\widetilde{\tau}$, the intersections $L \cap \widetilde{X_m}$ and $M \cap \widetilde{X_m}$ are *not* homotopic in $\widetilde{X_m}$ through homotopies preserving $\partial_0 \widetilde{X_m}$ and similarly for L' and M'. See Figure 6b.

We claim that if m is large enough to satisfy this condition, then all arcs in ℓ_i^m for any $i = 1, \ldots, r(m)$ intersect τ the same number of times. To see this, we consider Λ'_m, the sub-lamination of Λ_m consisting of all the arcs in all of the homotopy classes ℓ_i^m. We again consider the pre-image $\widetilde{\Lambda'_m}$ in $\widetilde{X_m}$, the pre-image $\widetilde{\ell_i^m}$ in $\widetilde{X_m}$ for each i, and our fixed lift $\widetilde{\tau}$. Any pre-image $\widetilde{\ell_i^m}$ is partitioned into families of arcs in $\widetilde{X_m}$ which are homotopic through homotopies preserving $\partial_0 \widetilde{X_m}$. Moreover, if c and d are components of $\partial_0 \widetilde{X_m} \cup \partial_\infty \widetilde{X_m}$, then the arcs of $\widetilde{\Lambda'_m}$ joining c to d either *all* intersect $\widetilde{\tau}$ or *all* miss $\widetilde{\tau}$ by our choice of m. See Figure 7a.

Thus, there are finite multi-sets $\mathcal{A}' = \{a'_1, \ldots, a'_s\}$ and $\mathcal{B}' = \{b'_1, \ldots, b'_s\}$, each consisting of boundary components of $\partial_0 \widetilde{X_m}$ and/or ends of $\partial_\infty \widetilde{X_m}$, and such that the arcs of $\widetilde{\Lambda'_m}$ intersecting $\widetilde{\tau}$ are exactly those joining a'_i to b'_i for some i. Moreover, the arcs of $\widetilde{\Lambda'_m}$ joining a'_i to b'_i all intersect



(a) Consider c and d components of $\partial_0\widetilde{X_m} \cup \partial_\infty\widetilde{X_m}$. If the arcs from c to d intersect $\widetilde{\tau}$ and intersect $\partial\widetilde{X_n}$ in arcs from a_1 to b_1 then they all do so exactly once since L separates them from M. Here dotted black lines indicate $\partial_0\widetilde{X_n}$, solid black lines indicate $\partial_0\widetilde{X_m}$, and $\widetilde{\tau}$ is indicated by a solid red line. There is a second (small) strip of geodesics which do not intersect $\widetilde{\tau}$ at all since they are separated from L by M.

(b) Consider a and b which are either boundary components in $\partial_0\widetilde{X_m}$ or ends in $\partial_\infty\widetilde{X_m}$, and the arcs from a to b. They may intersect multiple translates of $\widetilde{X_n}$ and for each translate of $\widetilde{X_n}$ they intersect a translate of $\widetilde{\tau}$ at most once. Here translates of $\widetilde{\tau}$ are indicated by solid red lines. Translates of $\widetilde{X_n}$ are shaded in green.

Figure 7: Intersections of arcs of $\widetilde{\Lambda'_m}$ with lifts of the transversal τ.

$\widetilde{\tau}$ exactly once. For each $i = 1, \ldots, s$ there is a sub-interval of $\widetilde{\tau}$, call it J_i, with endpoints in $\widetilde{\Lambda'_m}$, containing all the intersections with the arcs from a'_i to b'_i and no intersections with arcs from a'_j to b'_j for $j \neq i$. Consider the arcs in a homotopic family in $\widetilde{\Lambda'_m}$ joining a boundary component or end a to a boundary component or end b. Whenever the family intersects a lift of τ, the lift has the form $g\widetilde{\tau}$ for some $g \in \pi_1(X_m)$. Translating by g^{-1}, we see that all of the arcs from a to b intersect the lift $g\widetilde{\tau}$ exactly once, in the interval gJ_i for some i. See Figure 7b.

Finally, we see from this that every arc in a class ℓ_i^m intersects τ a number of time equal to the number of lifts of τ that ℓ_i^m intersects when we lift it to a family of arcs homotopic through homotopies preserving $\partial_0\widetilde{X_m}$. This completes the proof in the special case that every arc in every class ℓ_i^n intersects τ at most once. For the case of a general transversal τ, split τ into transversals $\tau_1, \tau_2, \ldots, \tau_t$ such that each τ_i intersects every arc in every class ℓ_i^n at most once. Apply the previous arguments to the arcs τ_i separately to find numbers $m_i \geq n$ as in the statement of the lemma for each τ_i. Taking $m = \max\{m_1, \ldots, m_t\}$ completes the proof. $\square$

Remark 4.10. Suppose that τ is a transversal and n is large enough that every arc in each equivalence class ℓ_i^n intersects τ the same number of times, for $1 \leq i \leq r(n)$. Let E_i be the number of times that τ intersects any arc in ℓ_i^n. Recall that τ_i^n denotes a transversal intersecting Λ_n only in ℓ_i^n and intersecting each arc of ℓ_i^n exactly once. We may partition τ into (1) some arcs which intersect Λ_n only in ℓ_i^n, for one value of $1 \leq i \leq r(n)$, and intersect each arc in ℓ_i^n at most once, plus (2) some arcs which are disjoint from $\ell_1^n \cup \ldots \cup \ell_{r(n)}^n$. The arcs of type (1) can be homotoped into τ_i^n and then their union covers the points of $\Lambda \cap \tau_i^n$ uniformly E_i times. The arcs of type (2) intersect Λ_n only in the minimal sub-laminations Γ_i^n for $1 \le i \le s(n)$ and leaves spiraling onto them. Thus, for μ any transverse measure,

$$\mu(\tau) = \sum_{i=1}^{r(n)} E_i \mu(\tau_i^n) + \sum_{i=1}^{s(n)} (\mu|\Gamma_i^n)(\tau).$$

Our second preliminary result identifies $\mathcal{M}(\Lambda)$ with the inverse limit of the $\mathcal{M}(\Lambda_n)$'s. Denote by $\rho_{n\infty}$ the linear map $\mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda_n)$ which restricts transverse measures to Λ_n: $\rho_{n\infty}(\mu) = \mu|\Lambda_n$.

Lemma 4.11. *The restriction maps $\rho_{n\infty}$ identify $\mathcal{M}(\Lambda)$ linearly homeomorphically with the limit of the inverse system $\mathcal{M}(\Lambda_1) \xleftarrow{\rho_1} \mathcal{M}(\Lambda_2) \xleftarrow{\rho_2} \ldots$.*

Proof. Since $\rho_{n-1} \circ \rho_{n\infty} = \rho_{(n-1)\infty}$, there is an induced continuous linear map $\mathcal{M}(\Lambda) \to \varprojlim \mathcal{M}(\Lambda_n)$. On the other hand, there is a map $\varprojlim \mathcal{M}(\Lambda_n) \to \mathcal{M}(\Lambda)$ defined as follows. If $(\mu_n)_{n=1}^\infty \in \varprojlim \mathcal{M}(\Lambda_n)$ then its image μ in $\mathcal{M}(\Lambda)$ is the transverse measure defined by $\mu_\tau = (\mu_n)_\tau$ for any n large enough that τ lies in X_n. The map $\varprojlim \mathcal{M}(\Lambda_n) \to \mathcal{M}(\Lambda)$ is linear and continuous since these properties hold for each map $\mu_n \mapsto (\mu_n)_\tau$. One may check that these functions between $\mathcal{M}(\Lambda)$ and $\varprojlim \mathcal{M}(\Lambda_n)$ are mutually inverse. $\square$

Thus, we may define a map $\Psi : \mathcal{M}(\Lambda) = \varprojlim \mathcal{M}(\Lambda_n) \to \varprojlim C_n = \mathcal{W}(\Lambda)$ by $\Psi = (\Psi_1, \Psi_2, \Psi_3, \ldots)$. Our last preliminary result will be used to show that Ψ is proper and surjective.

Lemma 4.12. *Each map $\Psi_n : \mathcal{M}(\Lambda_n) \to C_n$ is proper and surjective.*

Proof. Since $\mathcal{M}(\Lambda_n) = \prod_{i=1}^{r(n)} \mathcal{M}(A_i^n) \times \prod_{i=1}^{s(n)} \mathcal{M}(\Gamma_i^n)$, $C_n = \prod_{i=1}^{r(n)} \mathbb{R}_+ \times \prod_{i=1}^{s(n)} \mathcal{M}(\Gamma_i^n)$, Ψ_n is defined component-wise, and the maps on the $\mathcal{M}(\Gamma_i^n)$ are identities, it suffices to show that the maps $\mathcal{M}(A_i^n) \to \mathbb{R}_+$ defined by taking total mass are proper and surjective. Given any $c \in \mathbb{R}_+$ we may consider $c\delta$ where δ is a point mass at some point of A_i^n. This shows that $\mathcal{M}(A_i^n) \to \mathbb{R}_+$ is surjective. On the other hand, it is proper, since by the Banach-Alaoglu Theorem the space of measures on A_i^n with total mass bounded by some number E is compact in the weak* topology. $\square$

Finally we prove Theorem 4.2.

Proof of Theorem 4.2. The map $\Psi = (\Psi_1, \Psi_2, \ldots)$ is continuous and linear since each Ψ_i is. We now check that Ψ is proper and surjective. If $K \subset \mathcal{W}(\Lambda)$ is compact and non-empty then its images K_i in C_i are each compact and non-empty. Each $\Psi_i^{-1}(K_i)$ is compact and non-empty by Lemma 4.12. Finally, $\Psi^{-1}(K)$ is equal to the inverse limit of the sets $\Psi_i^{-1}(K_i)$ with the transition maps ρ_i ([9, Sec. I.4.4 Corollary to Proposition 9]). An inverse limit of non-empty compact Hausdorff spaces is non-empty and compact ([9, Section I.9.6, Proposition 8]). Thus $\Psi^{-1}(K)$ is compact and non-empty so that Ψ is proper and surjective. A proper map between metrizable spaces is closed, so Ψ is closed since $\mathcal{W}(\Lambda)$ and $\mathcal{M}(\Lambda)$ are metrizable (as subsets of countable products of metrizable spaces). To complete the proof, it suffices to show that Ψ is injective. Suppose that $\mu, \mu' \in \mathcal{M}(\Lambda)$ with $\mu \neq \mu'$. Then choosing a transversal for which $\mu_\tau \neq \mu'_\tau$ and possibly passing to a sub-transversal, we may suppose that $\mu(\tau) \neq \mu'(\tau)$. By Lemma 4.8, we may choose a surface X_n

large enough that for each $1 \leq i \leq r(n)$, each arc in the homotopy class ℓ_i^n intersects τ the same number of times E_i. By Remark 4.10,

$$\mu(\tau) = \sum_{i=1}^{r(n)} E_i \mu(\tau_i^n) + \sum_{i=1}^{s(n)} (\mu|\Gamma_i^n)(\tau)$$

and similarly for μ'. Consequently we must have $\mu(\tau_i^n) \neq \mu'(\tau_i^n)$ or $\mu|\Gamma_i^n \neq \mu'|\Gamma_i^n$ for some i. These are the components of the image of μ in C_n, so $\Psi(\mu) \neq \Psi(\mu')$. This completes the proof. $\square$

As noted earlier, Theorem B follows immediately.

4.4 Effectivizing the linear homeomorphism

We showed that the map $\Psi : \mathcal{M}(\Lambda) \to \mathcal{W}(\Lambda)$ is a linear homeomorphism. However, the inverse Ψ^{-1} remains mysterious from this point of view. Defining Ψ^{-1} would yield a more effective result, in that one could explicitly construct a transverse measure from any element of the inverse limit $\mathcal{W}(\Lambda)$. We outline how to do this, leaving the details to the interested reader.

Consider an element $(w_n)_{n=1}^{\infty} \in \varprojlim C_n$. We wish to construct a transverse measure μ to Λ from $(w_n)_n$. To do this, we construct *approximate measures*. Consider a transversal τ to Λ. It is contained in X_n for all n sufficiently large. Set E_i^n to be the *maximum* number of times that any arc in the homotopy class ℓ_i^n intersects τ (the number of intersection points may vary by arc). If

$$w_n = \sum_{i=1}^{r(n)} b_i^n e_i^n + \sum_{i=1}^{s(n)} \nu_i^n \text{ then we define } w_n(\tau) = \sum_{i=1}^{r(n)} b_i^n E_i^n + \sum_{i=1}^{s(n)} \nu_i^n(\tau)$$

which we think of as an approximate measure of τ. We emphasize that this does not define an actual transverse measure to Λ but only an approximation. We take limits to find honest measures:

Proposition 4.13. *Let τ be a transversal to Λ and $(w_n)_{n=1}^{\infty} \in \mathcal{W}(\Lambda)$. Then the approximate measures $w_n(\tau)$ are decreasing with n and therefore $\lim_{n\to\infty} w_n(\tau)$ exists.*

One now defines a pre-measure μ_τ by setting $\mu(\sigma) = \lim_{n\to\infty} w_n(\sigma)$ for any sub-transversal σ of τ and extending over disjoint unions of such sub-transversals. An application of the Carathéodory Extension Theorem yields an honest measure μ_τ on τ.

Proposition 4.14. *Let $(w_n)_{n=1}^{\infty} \in \mathcal{W}(\Lambda)$ and define the limits μ_τ as above for any transversal τ to Λ. Then the Borel measures μ_τ define a transverse measure to Λ. Moreover, setting $\Psi^{-1}((w_n)_{n=1}^{\infty}) = \mu$ defines the inverse homeomorphism to the homeomorphism $\Psi : \mathcal{M}(\Lambda) \to \mathcal{W}(\Lambda)$.*

5 Bases for cones of measures

In Corollary 3.4 we showed that $\mathcal{M}(\Lambda)$ admits a base whenever there is a compact subsurface of X intersecting every leaf of Λ. This criterion is sufficient but not necessary for the existence of a compact base. Thus, the question of which cones of transverse measures admit bases is not completely straightforward. An example of a cone of transverse measures which has no compact base is the infinite product of rays, $\mathbb{R}_+^{\mathbb{N}}$ (see Example 4.7). This example recurs repeatedly. In Section 8 we give other examples of cones without bases.

Even when a base does exist, its structure is not transparent from Corollary 3.4. In this section we make the structure more transparent by proving Theorem D from the introduction. First consider the case of an inverse system of finite-dimensional simplicial cones

$$C_1 \xleftarrow{f_1} C_2 \xleftarrow{f_2} C_3 \xleftarrow{f_3} \cdots.$$

Here the maps $f_n : C_{n+1} \to C_n$ are linear. For $n \leq m$ we denote by $f_{nm} : C_m \to C_n$ the composition $f_{nm} := f_n \circ f_{n+1} \circ \cdots \circ f_{m-1}$.

Lemma 5.1. *Let $\mathcal{C}$ be the limit of an inverse system*

$$C_1 \xleftarrow{f_1} C_2 \xleftarrow{f_2} C_3 \xleftarrow{f_3} \ldots$$

of finite-dimensional simplicial cones with linear maps f_n. Suppose that the maps f_n satisfy the property that $f_n(v) = 0$ only if $v = 0$. Let B_1 be a base for C_1 and define B_n to be the inverse image $f_{1n}^{-1}(B_1)$. Then the inverse limit of the bases B_n is a base for $\mathcal{C}$ and it is a compact metrizable Choquet simplex.

The condition $f_n(v) = 0$ only if $v = 0$ is *not* equivalent to injectivity of f_n. Rather, f_n may be extended to a linear map on some $\mathbb{R}^m$ and the condition says that the kernel of the extension intersects C_n only at 0. To prove this lemma we use the following important theorem of Davies-Vincent-Smith:

Theorem 5.2 ([13, Theorem 13])**.** *Consider an inverse system $B_1 \xleftarrow{f_1} B_2 \xleftarrow{f_2} B_3 \xleftarrow{f_3} \ldots$ of Choquet simplices with affine maps f_n. Then the limit $\mathcal{B}$ of this inverse system is a Choquet simplex.*

Proof of Lemma 5.1. As in the statement, choose a base B_1 for C_1 and define B_n to be the inverse image $f_{1n}^{-1}(B_1)$ in C_n. One may check that B_n is convex using that B_1 is convex. For $v \in C_n \setminus \{0\}$, there is a unique $r > 0$ with $rf_{1n}(v) \in B_1$ and therefore $r > 0$ is the unique number with $rv \in B_n$; i.e. B_n is a base.

We obtain by restriction an inverse system

$$B_1 \xleftarrow{f_1} B_2 \xleftarrow{f_2} B_3 \xleftarrow{f_3} \ldots$$

of finite-dimensional simplices. Define $\mathcal{B}$ to be the inverse limit of this system. It is a subspace of $\mathcal{C}$. We claim that in fact $\mathcal{B}$ is a base for $\mathcal{C}$. Consider an element $(v_n)_{n=1}^{\infty} \in \mathcal{C} \setminus \{0\}$. We have $v_1 \neq 0$. Thus there is a unique $r > 0$ with $rv_1 \in B_1$. Then for each n we have $rv_n \in f_{1n}^{-1}(B_1) = B_n$. Thus, $r(v_n)_{n=1}^{\infty} \in \mathcal{B}$ and r is the unique number with this property. We may verify that $\mathcal{B}$ is convex by using the convexity of each B_n. This proves that $\mathcal{B}$ is a base, as desired. By Theorem 5.2, $\mathcal{B}$ is a Choquet simplex. As an inverse limit of countably many compact metrizable spaces, $\mathcal{B}$ is compact and metrizable. $\square$

A problem with applying Lemma 5.1 to our cones of measures is that the transition maps π_n do not generally satisfy the condition $\pi_n(w_n) \neq 0$ if $w_n \neq 0$. To utilize Lemma 5.1 it will thus be necessary to modify our inverse system.

5.1 Modifying exhaustions and inverse systems

In this section we wish to prove Theorem D from the introduction. Consider the hyperbolic surface X endowed with an exhaustion $X_1 \subset X_2 \subset \dots$ as considered earlier and a lamination Λ. Thus X_n is a punctured compact subsurface with geodesic boundary. As before we consider the laminations $\Lambda_n = \Lambda \cap X_n$. Each Λ_n contains finitely many homotopy classes of arcs $\{\ell_i^n\}_{i=1}^{r(n)}$ and finitely many compact minimal sub-laminations $\{\Gamma_i^n\}_{i=1}^{s(n)}$ in the interior of X_n. We let C_n be the cone for Λ_n defined in Section 4 and $\pi_n : C_{n+1} \to C_n$ the resulting transition maps. Then $\mathcal{M}(\Lambda)$ is linearly homeomorphic to the inverse limit $\mathcal{W}(\Lambda)$ of the cones C_n with the maps π_n.

Remark 5.3. Note that if each homotopy class of arcs ℓ_i^{n+1} on X_{n+1} and each compact sub-lamination Γ_i^{n+1} on X_{n+1} intersects X_n, then π_n satisfies the property that $\pi_n(w) = 0$ only if $w = 0$. If this property is satisfied for each n, then Lemma 5.1 will show that $\mathcal{M}(\Lambda)$ has a base which is a compact metrizable Choquet simplex. The property may *not* be satisfied for every lamination though, since there may be arcs or minimal sub-laminations of Λ_{n+1} which do not intersect X_n (see e.g. Example 4.7).

Thus, we will attempt to modify our exhaustion $X_1 \subset X_2 \subset \dots$ to have this property. The key lemma to prove is the following:

Lemma 5.4. *Let $U \subset V$ be punctured compact subsurfaces of X with geodesic boundary. Let Λ be a geodesic lamination on X such that every leaf of Λ intersects U. Then there is a larger punctured compact subsurface $W \supset V$ with geodesic boundary such that every geodesic of $\Lambda \cap W$ intersects U.*

Proof. The idea of the proof is that we will construct W by gluing on *strips* $[0,1] \times [0,1]$ to the boundary of V. We will do this as follows: if a geodesic of $\Lambda \cap V$ doesn't hit U then we may extend the geodesic in one direction until it *does* hit U. The extended geodesic leaves V finitely many times and then eventually enters U. We will add on a strip containing each arc where the extended geodesic leaves V. The resulting subsurface may not be essential so we finish the proof by adding on disks and punctured disks and homotoping the boundary components to geodesics.

Every leaf of Λ which is contained entirely in V must intersect U by hypothesis. So we focus on geodesics of $\Lambda \cap V$ which have at least one endpoint on ∂V. Consider $p \in \partial V \cap \Lambda$. It is contained in a leaf L of Λ. Choose an orientation for the geodesic L and denote by $L|[p,\infty)$ and $L|(-\infty, p]$ the rays of L based at p which are oriented *away from* p and *towards* p, respectively. At least one of these two rays intersects U; say $L|[p,\infty)$, without loss of generality. Consider the first intersection point q of $L|[p,\infty)$ with U. This gives rise to a sub-arc $L|[p,q]$ of L from p to q. There is a small open arc I_p of ∂V containing p for which all rays of Λ through I_p in the direction of $L|[p,\infty)$ contain a sub-arc with endpoints in I_p and ∂U which is homotopic to $L|[p,q]$ (through homotopies preserving the boundary components).

The arc $L|[p,q]$ leaves V at most finitely many times and then hits ∂U at q. We may take a small neighborhood of $L|[p,q]$ containing all the homotopic arcs through I_p and ∂U. In the complement of V, this consists of a finite disjoint union S_p of *strips* $[0,1] \times [0,1]$ such that: (1) the *horizontal boundary components* $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ are contained in ∂V; (2) the *vertical boundary components* are disjoint from Λ; and (3) any leaf of Λ through a point in I_p contains a sub-arc in $V \cup S_p$ homotopic to $L|[p,q]$. In particular, any leaf of Λ which passes through I_p contains a sub-arc in $V \cup S_p$ which intersects U.

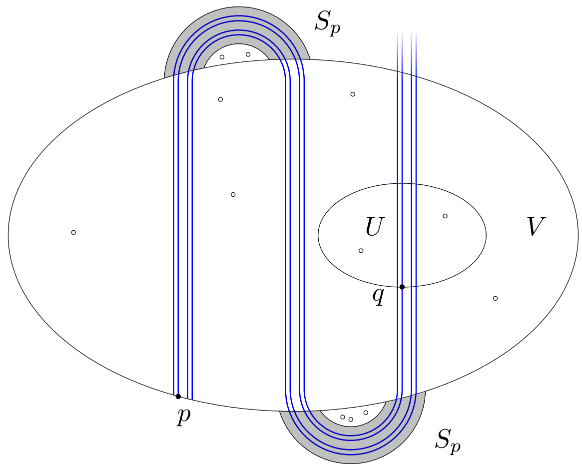


Figure 8: Adding on strips $[0,1]\times[0,1]$ to form the surface V_0. In a small neighborhood of $p \in \partial V \cap \Lambda$, all geodesics fellow travel the ray $L|[p,\infty)$ long enough to have an arc homotopic to $L|[p,q]$. All such homotopic arcs are contained in the union of V with S_p, which consists of two strips.

Now, the arcs I_p form an open cover of the compact set $\Lambda \cap \partial V$. Consider a finite sub-cover $I_{p_1} \cup \ldots \cup I_{p_k}$. We form the subsurface

$$V_0 := V \cup \bigcup_{i=1}^{k} S_{p_i}.$$

Note that the boundary of V_0 consists of some subset of ∂V along with subsets of the *vertical* boundary components of the strips in the finite unions S_{p_i}. Since the vertical boundary components of all the strips are disjoint from Λ, $\Lambda \cap \partial V_0$ is contained in $\Lambda \cap \partial V$. Thus, every point of $\Lambda \cap \partial V_0$ lies in I_{p_i} for some i and therefore the geodesic of $\Lambda \cap V_0$ through such a point intersects U.

Now, the subsurface V_0 may not be essential: some of its boundary components may bound disks or once-punctured disks. Form V_1 by taking the union of V_0 with all disks or once-punctured disks bounded by any of the components of ∂V_0. Any geodesic in $V_1 \cap \Lambda$ contains at least one geodesic (and possibly multiple geodesics) of $V_0 \cap \Lambda$. Hence, any geodesic in $V_1 \cap \Lambda$ intersects U.

Finally, we form the subsurface W by homotoping the boundary components of V_1 to their geodesic representatives. Since V_1 contains V, so does W. Finally, we claim that every geodesic of $\Lambda \cap W$ intersects U. Consider a connected component $\widetilde{W}$ of the pre-image of W in the universal cover $\widetilde{X}$ and let $\pi_1(W)$ act on $\widetilde{X}$ stabilizing this component. Then there is a unique component $\widetilde{V_1}$ of the pre-image of V_1 which is also stabilized by $\pi_1(W)$. Consider a geodesic of $W \cap \Lambda$ with at least one endpoint on ∂W. It is contained in a leaf L of Λ. This geodesic of $W \cap \Lambda$ lifts to a geodesic contained in $\widetilde{W}$ with one endpoint on a geodesic p of $\partial_0 \widetilde{W}$ and the other endpoint in a component of $\partial_0 \widetilde{W} \cup \partial_\infty \widetilde{W}$, which we call q. This lifted geodesic is also contained in a lift $\widetilde{L}$ of L. There is a component of $\partial_0 \widetilde{V_1}$ with the same endpoints as p and similarly a component of $\partial_0 \widetilde{V_1} \cup \partial_\infty \widetilde{V_1}$ corresponding to q. Since any arc of $L \cap V_1$ intersects U, any arc of $\widetilde{L} \cap \widetilde{V_1}$ intersects some lift of U. There is thus a component g of the pre-image of ∂U separating the components of $\partial_0 \widetilde{V_1} \cup \partial_\infty \widetilde{V_1}$ corresponding to p and q. The component g therefore also separates p and q. Thus our lifted geodesic intersects a lift of U. $\square$

Now we may prove Theorem D.

Proof of Theorem D. Begin with an exhaustion $X_1 \subset X_2 \subset \ldots$ of X by punctured compact subsurfaces of X with geodesic boundary. We will modify the X_i to an exhaustion $Y_1 \subset Y_2 \subset Y_3 \subset \ldots$ such that every geodesic of $\Lambda \cap Y_i$ intersects Y_1 for each i (and in particular every geodesic of $\Lambda \cap Y_i$ intersects Y_{i-1}). First choose X_n large enough that every leaf of Λ intersects X_n. Set $Y_1 = X_n$. Now X_{n+1} is a subsurface containing Y_1 and by Lemma 5.4 there is a punctured compact subsurface Y_2 containing X_{n+1} with the property that every geodesic in $\Lambda \cap Y_2$ intersects Y_1. Choose $m > n+1$ large enough that $X_m \supset Y_2$. Again by Lemma 5.4, there is a punctured compact subsurface Y_3 containing X_m such that every geodesic in $\Lambda \cap Y_3$ intersects Y_1. Repeat this process inductively to form the desired exhaustion $Y_1 \subset Y_2 \subset Y_3 \subset \ldots$.

Set C_i to be the cone $C(Y_i)$ of weights on $\Lambda \cap Y_i$ for each i. There is an inverse system

$$C_1 \xleftarrow{\pi_1} C_2 \xleftarrow{\pi_2} C_3 \xleftarrow{\pi_3} \ldots.$$

By Remark 5.3 and Lemma 5.1, the inverse limit $\mathcal{M}(\Lambda) \cong \varprojlim C_i$ has a base $\varprojlim B_i$ where B_i is a base of C_i and $\varprojlim B_i$ is a compact metrizable Choquet simplex. $\square$

5.2 Examples of bases

In this section we re-visit the laminations of Examples 4.4 and 4.5 from Section 4.2 and describe bases for them as Choquet simplices.

Example 5.5. Consider the lamination Λ in Example 4.4. The cone of transverse measures is the cone $C \subset \ell^1$ defined by $C = \{(x, y_1, y_2, \ldots) : y_i \geq 0 \text{ for all } i \text{ and } x \geq \sum y_i\}$ with the weak* topology obtained from the pre-dual c_0. A base for C is given by the convex set B defined by $x = 1$. Thus $B = \{(1, y_1, y_2, \ldots) : y_i \geq 0 \text{ for all } i \text{ and } 1 \geq \sum y_i\}$. The base B is a Choquet simplex. Its extreme points are

$$e = (1, 0, 0, \ldots) \text{ and } e_i = (1, 0, 0, \ldots, 0, 1, 0, \ldots)$$

where e_i has a 1 in the i^{th} position. The points e_i are isolated in the space $\mathrm{Ext}(B)$ of extreme points while $e_i \to e$ as $i \to \infty$. Thus $\mathrm{Ext}(B)$ is homeomorphic to the ordinal $\omega + 1$. We may also identify the extreme points e_i and e with explicit measures on Λ. Namely, denote by L_i the isolated leaves of Λ and by L the non-isolated leaf, so that $L_i \to L$ as $i \to \infty$. Then e_i is identified with the δ-mass on L_i for each i (which assigns to a transversal its number of intersections with L_i), while e is identified with the δ-mass on L.

By choosing bases B_n for the cones $C_n = C(X_n)$ as in Lemma 5.1 (where $X_1 \subset X_2 \subset \ldots$ is the exhaustion chosen in Example 4.4) we may consider B to be the inverse limit $\varprojlim B_n$. The base B_n has n vertices $v_1^n, \ldots, v_n^n$. The map $B_{n+1} \to B_n$ is defined by $v_i^{n+1} \mapsto v_i^n$ for $1 \leq i \leq n$ and $v_{n+1}^{n+1} \mapsto v_n^n$. It is instructive to consider what the extreme points of the inverse limit are. They are:

$$e_i = (v_1^1, v_2^2, \ldots, v_i^i, v_i^{i+1}, v_i^{i+2}, \ldots) \text{ and } e = (v_1^1, v_2^2, \ldots, v_n^n, \ldots).$$

Example 5.6. Consider the lamination Λ in Example 4.5. The cone of transverse measures is now the set $C \subset \ell^1$ consisting of sequences $(x_1, x_2, y_1, y_2, y_3, \ldots)$ with $y_i \geq 0$ for all i and $x_j \geq \sum_i y_i$ for

$j=1,2$. A Choquet simplex base B is defined by $x_1+x_2=1$ (and $y_i\geq 0$, $x_1,x_2\geq\sum_i y_i$). Its extreme points are

$$f_1=(1,0,0,0,\ldots),\quad f_2=(0,1,0,0,\ldots),\text{ and } e_i=\left(\frac{1}{2},\frac{1}{2},0,0,\ldots,0,\frac{1}{2},0\right)$$

where e_i has a $1/2$ in the i^{th} position. The set of extreme points $\mathrm{Ext}(B)$ is not closed: f_1,f_2, and e_i are all isolated in $\mathrm{Ext}(B)$ while $e_i\to\frac{1}{2}f_1+\frac{1}{2}f_2$, which does not lie in $\mathrm{Ext}(B)$. Here f_1 and f_2 are identified with δ-masses on the proper non-isolated leaves L and L', respectively. The points $2e_i$ are identified with δ-masses on the proper isolated leaves L_i, which converge to $L\cup L'$ as $i\to\infty$.

Again consider the cones $C_n=C(X_n)$ and suitable bases B_n for C_n. Then B_n has $n+1$ vertices v_i^n. The map $B_{n+1}\to B_n$ is defined by $v_i^{n+1}\mapsto v_i^n$ for $1\leq i\leq n+1$ and $v_{n+2}^{n+1}\mapsto\frac{1}{2}v_1^n+\frac{1}{2}v_2^n$. The extreme points are $f_1=(v_1^1,v_1^2,v_1^3,\ldots)$, $f_2=(v_2^1,v_2^2,v_2^3,\ldots)$, and

$$e_i=\left(\frac{1}{2}v_1^1+\frac{1}{2}v_2^1,\frac{1}{2}v_1^2+\frac{1}{2}v_2^2,\ldots,\frac{1}{2}v_1^i+\frac{1}{2}v_2^i,v_{i+2}^{i+1},v_{i+2}^{i+2},v_{i+2}^{i+3},\ldots\right).$$

6 Inverse limit laminations

In this section we consider a construction of laminations as "inverse limits" of systems of arcs. We consider the hyperbolic surface X of the first kind and an exhaustion $X_1\subset X_2\subset\ldots$ by punctured compact subsurfaces with geodesic boundary. We consider the universal cover $\widetilde{X}$, which is isometric to the hyperbolic plane and fix a basepoint $*\in\widetilde{X}$ in the pre-image of X_1. By Lemma 4.9, if $\widetilde{X_n}$ is the unique component of the pre-image of X_n containing $*$, we have that $\widetilde{X}=\bigcup_{n=1}^{\infty}\widetilde{X_n}$.

Consider the compactification $\widetilde{X}\cup\partial_\infty\widetilde{X}$ where $\partial_\infty\widetilde{X}\cong S^1$ is the Gromov boundary. Recall that $\partial_\infty\widetilde{X_n}$ denotes the intersection of the closure of $\widetilde{X_n}$ in $\widetilde{X}\cup\partial_\infty\widetilde{X}$ with $\partial_\infty\widetilde{X}$. The complement $\partial_\infty\widetilde{X}\setminus\partial_\infty\widetilde{X_n}$ is a countable collection of open intervals. If $n\leq m$ then the (interval) components of $\partial_\infty\widetilde{X}\setminus\partial_\infty\widetilde{X_m}$ are nested in the components of $\partial_\infty\widetilde{X}\setminus\partial_\infty\widetilde{X_n}$. Since $\bigcup_{n=1}^{\infty}\widetilde{X_n}=\widetilde{X}$, we have the following fact: if I_n are components of $\partial_\infty\widetilde{X}\setminus\partial_\infty\widetilde{X_n}$ with $I_1\supset I_2\supset\ldots$, then the intersection $\bigcap_{n=1}^{\infty}I_n$ consists of a single point of $\partial_\infty\widetilde{X}$. For each component I of $\partial_\infty\widetilde{X}\setminus\partial_\infty\widetilde{X_n}$, there is a unique component of $\partial_0\widetilde{X_n}$ (the topological boundary of $\widetilde{X_n}$ as a subset of $\widetilde{X}$) joining its endpoints.

For each n, fix a (finite) collection A_n of pairwise disjoint, pairwise non-homotopic, homotopically non-trivial arcs in X_n with both endpoints on ∂X_n. We suppose without loss of generality that each $\ell\in A_n$ has been chosen to be a geodesic, so that it intersects X_m minimally for each $m\leq n$. We say that the system (set) $\{A_n\}_{n=1}^{\infty}$ is *directed* if it satisfies the following conditions: (1) for each $\ell\in A_{n+1}$, the arcs of the intersection $\ell\cap X_n$ are homotopic to arcs in A_n; and (2) for each $\ell\in A_n$, there is an arc $\ell'\in A_{n+1}$ such that $\ell'\cap X_n$ contains an arc homotopic to ℓ. Fix a directed system of collections of arcs $\{A_n\}_{n=1}^{\infty}$. We will now construct a lamination Λ on X. The lamination Λ will consist of all geodesics on X which intersect each X_n in a family of arcs homotopic to the arcs in A_n. To verify that this is a lamination will take a bit of work.

Each $\ell\in A_n$ lifts to an infinite set of arcs in $\widetilde{X_n}$. Consider two (interval) components I and J of $\partial_\infty\widetilde{X}\setminus\partial_\infty\widetilde{X_n}$. There are geodesics B and C of $\partial_0\widetilde{X_n}$ with the same endpoints as I and J, respectively. We say that I and J are *joined* by A_n if there is an arc $\ell\in A_n$ and a lift $\widetilde{\ell}$ with one endpoint on B and the other endpoint on C. We say that ℓ *joins* B and C. We define a set of geodesics $\widetilde{\Lambda}$ as follows: if $p, q \in \partial_\infty \widetilde{X}$ and $p \neq q$ then the geodesic $[p, q]$ from p to q in $\widetilde{X}$ lies in $\widetilde{\Lambda}$ if for some $n_0 \geq 1$ we have

$$p = \bigcap_{n=n_0}^{\infty} I_n \text{ and } q = \bigcap_{n=n_0}^{\infty} J_n \text{ where } I_{n_0} \supset I_{n_0+1} \supset I_{n_0+2} \supset \dots \text{ and } J_{n_0} \supset J_{n_0+1} \supset J_{n_0+2} \supset \dots$$

are nested sequences of arcs of $\partial_\infty \widetilde{X} \setminus \partial_\infty \widetilde{X_n}$ with the property that I_n is joined to J_n by A_n for each $n \geq n_0$. See Figure 9.

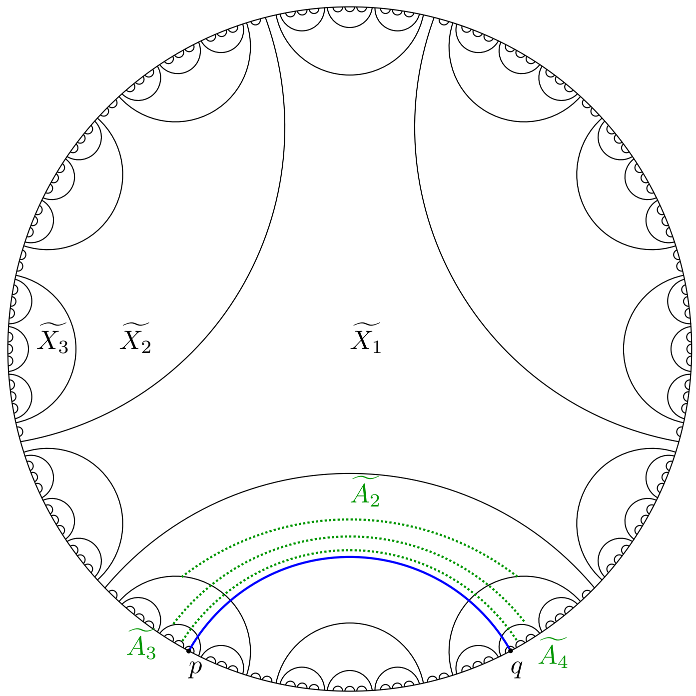


Figure 9: A geodesic $[p, q]$ in $\widetilde{X}$. Dotted green lines denote lifts of arcs in A_n, joining components of $\partial_0 \widetilde{X_n}$. The geodesic $[p, q]$ lies in $\widetilde{\Lambda}$ since there are elements of $\widetilde{A_n}$ joining the pairs of intervals containing p and q, respectively. Here we may take $n_0 = 2$.

It is notable that every lamination on X without compact sub-laminations or leaves asymptotic to isolated punctures arises in this way, as may be seen by considering the arcs of intersection with each subsurface X_n. Thus, the construction is reversible.

We will now investigate the properties of $\widetilde{\Lambda}$ and show that it descends to a lamination on X. By the invariance under $\pi_1(X_n)$ of the property of intervals of $\partial_\infty \widetilde{X} \setminus \partial_\infty \widetilde{X_n}$ being joined by A_n, one sees that $\widetilde{\Lambda}$ is invariant under $\pi_1(X)$. Moreover:

Lemma 6.1. *No two geodesics of $\widetilde{\Lambda}$ cross.*

Proof. Consider $[p, q]$ and $[p', q']$ geodesics of $\widetilde{\Lambda}$. We must show that the pairs $\{p, q\}$ and $\{p', q'\}$ do not separate each other as pairs of points in the circle. For n_0 sufficiently large, we may write

$$p = \bigcap_{n=n_0}^{\infty} I_n, \quad q = \bigcap_{n=n_0}^{\infty} J_n, \quad p' = \bigcap_{n=n_0}^{\infty} I'_n, \quad q' = \bigcap_{n=n_0}^{\infty} J'_n$$

where I_n and J_n are joined by A_n for each n and similarly for I'_n and J'_n. If $\{p, q\}$ separates $\{p', q'\}$ then for all n sufficiently large, I_n and J_n separate I'_n and J'_n. Denote by B_n, C_n, B'_n, and C'_n the components of $\partial_0 \widetilde{X_n}$ with the same endpoints as I_n, J_n, I'_n, and J'_n, respectively. Then there are $\ell, \ell' \in A_n$ with lifts $\widetilde{\ell}, \widetilde{\ell'}$ with endpoints on B_n and C_n and on B'_n and C'_n, respectively. But then for $n \gg 0$, B_n and C_n separate B'_n and C'_n so that $\widetilde{\ell}$ and $\widetilde{\ell'}$ cross, and the same is true for ℓ and ℓ'. This contradicts that the arcs in A_n are pairwise disjoint. □

Hence the image of $\widetilde{\Lambda}$ in X is a family of pairwise non-crossing simple geodesics. Denote this image by Λ. In order to show that Λ is a geodesic lamination, it suffices to show that $\widetilde{\Lambda}$ is closed.

Lemma 6.2. *The set $\widetilde{\Lambda}$ is closed in $\widetilde{X}$. Hence Λ is closed in X.*

Proof. We must show the following: if $p, q \in \partial_\infty \widetilde{X}$, $p \neq q$, and $[p_i, q_i]$ are geodesics of $\widetilde{\Lambda}$ with $p_i \to p$ and $q_i \to q$, then $[p, q] \subset \widetilde{\Lambda}$. First we show that no ray of $[p, q]$ is contained in $\widetilde{X_n}$ for any n.

So suppose that a ray of $[p, q]$ is contained in $\widetilde{X_n}$ for some n. Thus, one of the endpoints, say p, is contained in the closure of $\widetilde{X_n}$ in $\widetilde{X} \cup \partial_\infty \widetilde{X}$. Then p is either an endpoint of a geodesic of $\partial_0 \widetilde{X_n}$ or it is not. We deal with the latter case first. In this case, for all sufficiently large i, $[p_i, q_i]$ intersects $\widetilde{X_n}$. Denote by I_i and J_i the arcs of $\partial_\infty \widetilde{X} \setminus \partial_\infty \widetilde{X}_n$ containing p_i and q_i, respectively. Then the arcs I_i converge to p. The arcs J_i lie in some common neighborhood of q. Let ℓ_i be an arc of A_n joining I_i and J_i. Then we see that the length of ℓ_i goes to infinity as $i \to \infty$. This is a contradiction, since A_n is finite.

Now we consider the case that p is an endpoint of a geodesic of $\partial_0 \widetilde{X_n}$. In this case, for any $m > n$ sufficiently large, p is contained in $\partial_\infty \widetilde{X_m}$ but is not the endpoint of a geodesic of $\partial_0 \widetilde{X_m}$; hence this case reduces to the previous after replacing n by m. Thus we have shown that no ray of $[p, q]$ is contained in $\widetilde{X_n}$ for any n.

Choosing n_0 sufficiently large, $[p, q]$ intersects $\widetilde{X_n}$ for each $n \geq n_0$. Moreover, since neither endpoint lies in $\partial_\infty \widetilde{X_n}$ by what we showed above, we have $p \in I_n$ and $q \in J_n$ for two components I_n and J_n of $\partial_\infty \widetilde{X} \setminus \partial_\infty \widetilde{X}_n$. Consider the sequence $[p_i, q_i]$ of geodesics in $\widetilde{\Lambda}$ for each i. Then for all i sufficiently large, we also have $p_i \in I_n$ and $q_i \in J_n$. Thus, A_n joins I_n to J_n for each $n \geq n_0$. We have $p = \bigcap_{n=n_0}^{\infty} I_n$ and $q = \bigcap_{n=n_0}^{\infty} J_n$ so that $[p, q] \subset \widetilde{\Lambda}$. □

Combining Lemmas 6.1 and 6.2 we have the following:

Theorem 6.3. *Let A_n be a finite collection of homotopically non-trivial, pairwise disjoint, pairwise non-homotopic arcs on X_n. Assume that the system $\{A_n\}_{n=1}^{\infty}$ is directed. Then the set Λ obtained from $\{A_n\}_{n=1}^{\infty}$ is a geodesic lamination on X.*

We call Λ the *inverse limit* of the system $\{A_n\}_{n=1}^{\infty}$. One nice application is that inverse limits allow us to easily construct examples of *minimal* laminations.

Definition 6.4. Let Λ be a geodesic lamination. We say that Λ is *minimal* if it has no proper sub-laminations.

Equivalently, a lamination is minimal exactly when all of its leaves are dense in the lamination.

Proposition 6.5. *Let A_n be a finite set of pairwise non-homotopic, pairwise disjoint, homotopically non-trivial arcs on X_n and suppose that $\{A_n\}_{n=1}^{\infty}$ is directed. Suppose that $\{A_n\}_{n=1}^{\infty}$ has the following property: for each $\ell \in A_n$, there is $m_0 \geq n$, such that if $m \geq m_0$ then for each $\ell' \in A_m$, $\ell' \cap A_n$ contains an arc homotopic to ℓ. Then the inverse limit lamination Λ of $\{A_n\}$ is minimal.*

Proof. Let L be a leaf of Λ. Let $\widetilde{L} = [p, q]$ be a lift of L to $\widetilde{X}$. It suffices to show that for any other leaf M of Λ, there is a lift of M to $\widetilde{X}$ with endpoints on $\partial_\infty \widetilde{X}$ which are arbitrarily close to p and q. By definition of Λ, there is $n_0 \geq 0$ such that $p = \bigcap_{n=n_0}^{\infty} I_n$ and $q = \bigcap_{n=n_0}^{\infty} J_n$ where I_n and J_n are arcs of $\partial_\infty \widetilde{X} \setminus \partial_\infty \widetilde{X_n}$ which are joined by A_n and we have $I_{n_0} \supset I_{n_0+1} \supset \ldots$ and similarly for the J_n. Since the I_n nest down to p and the J_n nest down to q, it suffices to show that there is a lift of M with endpoints in I_n and J_n for any $n \geq n_0$. Fix an $n \geq n_0$. Let B_n and C_n be the geodesics of $\partial_0 \widetilde{X_n}$ with the same endpoints as I_n and J_n, respectively. Then there is an arc $\ell \in A_n$ and a lift $\widetilde{\ell}$ to $\widetilde{X}$ with endpoints on B_n and C_n.

For any m sufficiently large compared to n, each arc of A_m traverses each arc of A_n. Choose any m with this property and with the property that M intersects X_m. Then $M \cap X_m$ contains an arc which is homotopic to some $\ell' \in A_m$. Therefore $M \cap X_n$ contains $\ell' \cap X_n$, which contains an arc homotopic to ℓ. Lifting this arc of $M \cap X_n$ and extending it to a lift of M, we see that there is a lift $\widetilde{M}$ of M which intersects B_n and C_n transversely, and therefore $\widetilde{M}$ has endpoints in I_n and J_n. This completes the proof. $\square$

6.1 Cones of transverse measures for inverse limit laminations

We note that if $\{A_n\}_{n=1}^{\infty}$ is a directed system of arcs and Λ is the inverse limit lamination, then Λ consists of *exactly* the simple bi-infinite geodesics L on X for which each intersection $L \cap X_n$ consists of a (possibly empty) set of arcs all homotopic to arcs in A_n. As already used implicitly earlier, if L is a leaf of Λ then it satisfies this property. On the other hand if, for each n, $L \cap X_n$ consists of arcs homotopic to arcs in A_n, then we may lift L to a geodesics $\widetilde{L} = [p, q]$. Then $\widetilde{L}$ intersects $\widetilde{X_{n_0}}$ for n_0 large enough and for each $n \geq n_0$, $\widetilde{L} \cap \widetilde{X_n}$ is an arc from a component B_n of $\partial_\infty \widetilde{X} \setminus \partial_\infty \widetilde{X_n}$ to a component C_n and B_n is joined to C_n by A_n for each such n. Thus L lies in Λ.

Finally, for each $\ell \in A_n$ there is a leaf L of Λ such that $L \cap X_n$ contains an arc homotopic to ℓ. To see this, set $\ell_n = \ell$, and inductively for $i > n$, set $\ell_i \in A_i$ to be an arc such that $\ell_i \cap X_{i-1}$ contains ℓ_{i-1}. Choose $I_n, J_n \subset \partial_\infty \widetilde{X} \setminus \partial_\infty \widetilde{X_n}$ to be intervals joined by ℓ_n. Inductively, we may choose arcs $I_n \supset I_{n+1} \supset \ldots$ and $J_n \supset J_{n+1} \supset \ldots$ which are joined by ℓ_i for $i \geq n$. Setting $p = \bigcap_{i=n}^{\infty} I_n$ and $q = \bigcap_{i=n}^{\infty} J_n$ we have that $\widetilde{L} = [p, q]$ is a geodesic of $\widetilde{\Lambda}$ and its image L in X is a leaf of Λ satisfying that $L \cap X_n$ contains ℓ.

Using this discussion, we may read off the cone of transverse measures $\mathcal{M}(\Lambda)$. Denote the elements of A_n by $\{\ell_i^n\}_{i=1}^{|A_n|}$. We've shown that $\Lambda \cap X_n$ consists exactly of the homotopy classes of arcs in A_n for each $n \geq 1$. By the discussion in Section 4.1, we have:

Lemma 6.6. *Let $\{A_n\}_{n=1}^{\infty}$ be a directed system of arcs on X_n. Then the cone $\mathcal{M}(\Lambda)$ is linearly homeomorphic to the limit of the inverse system*

$$\mathbb{R}_+^{|A_1|} \xleftarrow{\pi_1} \mathbb{R}_+^{|A_2|} \xleftarrow{\pi_2} \mathbb{R}_+^{|A_3|} \xleftarrow{\pi_3} \dots$$

where π_n is the $|A_n| \times |A_{n+1}|$ matrix whose (i,j)-entry counts the number of arcs of $\ell_j^{n+1} \cap X_n$ which are homotopic to ℓ_i^n.

7 Realizing Choquet simplices

In this section we prove Theorem E from the introduction. To do this, we use several tools. First we have the following realization theorem of Lazar-Lindenstrauss:

Theorem 7.1 ([18, Corollary to Theorem 5.2])**.** *Let Δ be a compact metrizable Choquet simplex. Then there exists a sequence of finite-dimensional simplices Δ_n together with surjective affine maps $f_n : \Delta_{n+1} \to \Delta_n$ such that Δ is affinely homeomorphic to the limit of the inverse system*

$$\Delta_1 \xleftarrow{f_1} \Delta_2 \xleftarrow{f_2} \Delta_3 \xleftarrow{f_3} \dots .$$

Our other main tool is an approximation theorem of Brown ([10]). First we set the notation. If $f : X \to Y$ and $g : X \to Y$ are maps between compact metric spaces then $D(f,g)$ denotes the supremum distance $D(f,g) = \sup\{d(f(x), g(x)) : x \in X\}$. Suppose that

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \dots$$

is an inverse system of topological spaces and X is the inverse limit. Recall that for $i \leq j$, $f_{ij} : X_j \to X_i$ denotes the composition $f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$. We denote by $f_{i\infty} : X \to X_i$ the natural projections, which satisfy $f_{ij} \circ f_{j\infty} = f_{i\infty}$ for each $i \leq j$.

Theorem 7.2 ([10, Theorem 2])**.** *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of compact metric spaces and let $f_i : X_{i+1} \to X_i$ and $g_i : X_{i+1} \to X_i$ be maps. Let X, Y be the inverse limits of (X_i, f_i) and (X_i, g_i), respectively. Then for each i there is a constant $L(g_1, \dots, g_{i-1}) > 0$ depending only on $g_1, \dots, g_{i-1}$ such that if*

$$D(f_i, g_i) < L(g_1, \dots, g_{i-1}) \tag{$*$}$$

for each i then the following properties are satisfied. For each i, the function $F_i : X \to X_i$ defined by $F_i = \lim_{j\to\infty} g_{ij} \circ f_{j\infty}$ is well-defined and continuous. Moreover, the function $F : X \to Y$ defined by $F(s) = (F_1(s), F_2(s), \dots)$ is a homeomorphism.

Our strategy to prove Theorem E will be to choose a Choquet simplex and an inverse system of finite-dimensional simplices Δ_n with that Choquet simplex as an inverse limit, as given by Theorem 7.1. We interpret a simplex Δ_n as the base of a cone of weights of a system of $\dim(\Delta_n) + 1$ arcs on a punctured disk. We may do the same for Δ_{n+1}. The map $f_n : \Delta_{n+1} \to \Delta_n$ may not be realized by including the surface with arcs realizing Δ_n into the surface with arcs realizing Δ_{n+1}. So we perturb f_n by a small amount to be realized by an inclusion of punctured disks. We then take advantage of Theorem 7.2.

The following lemma is the essential part of the inductive step of the proof. For the statement, we define a *punctured disk* to be a closed disk minus at most finitely many interior points. If U is a punctured disk and A is a collection of pairwise disjoint arcs on U with endpoints on ∂U, then we may consider the dual graph T to A. Thus, T has one vertex for each component of $U \setminus A$ and two vertices are joined by an edge if the corresponding components are separated by a single arc of A. One may check that T is a tree.

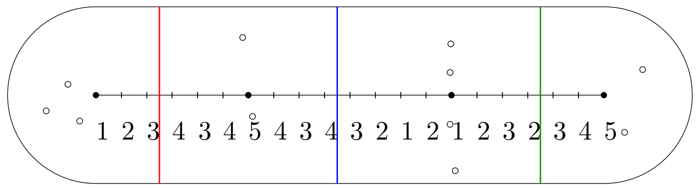
Lemma 7.3. *Let U be a punctured disk. Let A be a finite collection of r disjoint, homotopically distinct, homotopically non-trivial arcs on U such that the dual tree to A is homeomorphic to the interval $[0,1]$. Let $\mathbb{R}^s_+$ be a cone with $s > 0$ and $\pi : \mathbb{R}^s_+ \to \mathbb{R}^r_+$ a linear map whose matrix representative (with respect to the standard bases) has entries which are all positive odd integers. Then U is contained in a punctured disk V, together with a finite collection B of s disjoint, homotopically distinct, homotopically non-trivial arcs such that*

- *the arcs of $B \cap U$ are homotopic to the arcs in A;*
- *the dual tree to B is homeomorphic to an interval; and*
- *the induced map on cones of weights $C(V) \to C(U)$ for A and B is given by π.*

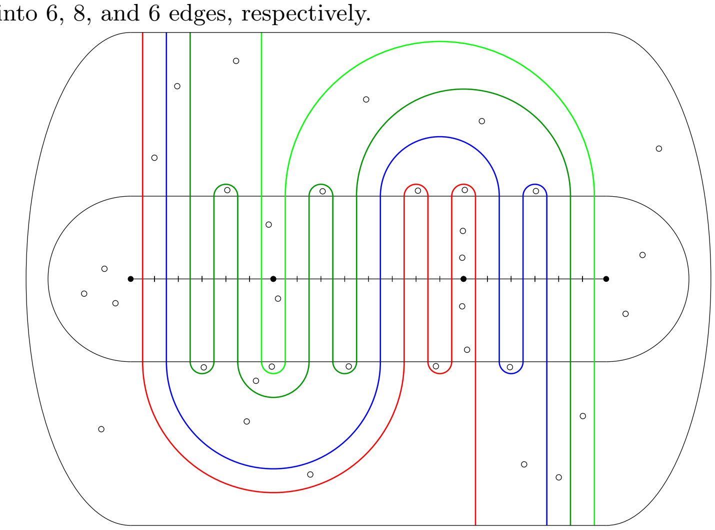
Proof. Set $\pi = (a_{ij})_{i=1,j=1}^{r,s}$. Let T be the dual tree to A. Embed T in U in such a way that a vertex of T lies in the region of $U \setminus A$ that it represents and an edge of T intersects A in exactly one point, which lies on the arc that it represents. Choose an ordering of the vertices $v_0, v_1, \ldots, v_r$ of T such that the edges e_i of T join v_{i-1} to v_i for each i. Denote by $\ell_1^U, \ldots, \ell_r^U$ the arcs in A corresponding to $e_1, \ldots, e_r$. Now, sub-divide each edge e_i into $\sum_{j=1}^s a_{ij}$ edges for each $1 \le i \le r$.

Having sub-divided the edges of T, we label the new vertices of T as follows. We start from the first vertex v_0 of e_1 and traverse the new vertices in order until we reach v_1. We label v_0 with a 1. We label the next vertex of e_1 with a 2. We then continue alternating between labels of 1 and 2 until we have crossed a_{11} (sub-)edges of e_1. Since a_{11} is odd, the final label of a vertex of e_1 that we put down will be a 2. We then consider the next a_{12} edges of e_1, starting with the last vertex labeled 2. We label them alternately 2 and 3 until we have crossed a_{12} edges. Since we started with a 2 and a_{12} is odd, our last label will be a 3. Continue in this way, alternating between 3 and 4, with the next a_{13} edges of e_1 and so on, until we have traversed all $\sum_{j=1}^s a_{1j}$ edges of e_1. The last a_{1s} edges of e_1 will have vertices labeled by s and $s+1$ and the final vertex v_1 of e_1 will be labeled by $s+1$. Now we will label the vertices of the first a_{2s} edges of e_2. The first vertex v_1 of e_2 has already been labeled $s+1$. We label the next a_{2s} vertices alternately by s and $s+1$ and therefore the final vertex labeled in this way will be labeled by s. We then label the vertices of the next $a_{2(s-1)}$ edges of e_2 alternately s and $s-1$, the vertices of the next $a_{2(s-2)}$ edges of e_2 alternately $s-1$ and $s-2$, and so on. The final a_{21} edges of e_2 will be labeled alternately 2 and 1 and the final vertex v_2 of e_2 will be labeled 1. We then repeat the process, labeling the first a_{31} edges of e_3 alternately 1 and 2 and so on, and continue until we have labeled all the vertices of T. See Figure 10a for an example.

Now, embed U into a larger disk V by gluing on a closed annulus to the boundary component of U. We construct a set B of arcs on V with the desired properties. To do this, we construct for each $1 \le j \le s$ an arc ℓ_j^V on V. The arc ℓ_j^V will intersect U in $\sum_{i=1}^r a_{ij}$ arcs. The first time ℓ_j^V traverses U it will cross through the first edge of T (closest to v_0) with endpoints labeled j and $j+1$. The next time it traverses U it will cross through the second edge of T (second closest to



(a) Step 1 of constructing the arcs B from the arcs A and the matrix π. The punctured disk U contains three homotopy classes of arcs drawn from left to right. The dual tree is embedded in U and the edges of T are sub-divided into 6, 8, and 6 edges, respectively.



(b) Step 2 of constructing the arcs B. The punctured disk V is obtained by adding an annulus to U. For each $1 \leq i \leq s$, one arc is drawn through all the edges of T with endpoints labeled by i and $i+1$. There are four homotopy classes in B in this case.

Figure 10: Constructing a punctured disk V and a system of arcs B from the punctured disk U together with the arcs A and the transition matrix $\begin{pmatrix} 1 & 1 & 3 & 1 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 1 \end{pmatrix}$.

v_0) with endpoints labeled j and $j+1$, and moreover do so in the *opposite direction* to the first traversal. We continue this process inductively from the lowest edge with endpoints j and $j+1$ to the highest, switching directions through U each time. Each time ℓ_j^V traverses U it crosses an edge contained in e_i for some i and when it does so we require the arc of intersection of ℓ_j^V with U to be homotopic to ℓ_i^U. See Figure 10b for an example. We claim that one may use this recipe for each $1 \leq j \leq s$ to construct the arcs ℓ_j^V in such a way that they are *pairwise disjoint*.

To see this last claim, construct the arc ℓ_1^V as described. The intersection $\ell_1^V \cap U$ separates the vertices labeled 1 from the vertices labeled $2, \ldots, s+1$ in U. Moreover, between any two vertices labeled 1 there are an *even* number of edges with vertices labeled 2 and 3. This ensures that ℓ_2^V may be constructed, disjoint from ℓ_1^V, using the same recipe. The intersection $\ell_2^V \cap U$ separates the vertices labeled 1 and 2 from the vertices labeled $3, \ldots, s+1$ and between any two vertices labeled

2 there are an *even* number of edges with vertices labeled 3 and 4. This ensures that ℓ_3^V may be constructed disjoint from ℓ_1^V and ℓ_2^V. Inductively, we construct $\ell_4^V, \dots, \ell_s^V$. We denote by B the union of $\ell_1^V, \dots, \ell_s^V$. Then for each j, ℓ_j^V intersects U in a_{ij} arcs homotopic to ℓ_i^U, for each i. By construction, ℓ_i^V separates ℓ_{i-1}^V from ℓ_{i+1}^V. Thus, the dual tree to B is an interval. Finally, we need to ensure that the ℓ_j^V are homotopically non-trivial and pairwise non-homotopic. To do this, we add one puncture to each component of $(V \setminus U) \setminus B$. This completes the proof. □

We will use one other easy technical lemma to prove Theorem E. If Δ is a finite-dimensional simplex then we may endow it with the ℓ^∞-distance d_∞ defined as follows. Denoting by $v_1, \dots, v_k$ the vertices of Δ, we may represent each point $p \in \Delta$ uniquely as a convex combination $p = \sum_{i=1}^k a_i v_i$ where $0 \le a_i \le 1$ for each i and $\sum a_i = 1$. The numbers $a_1, \dots, a_k$ are the *barycentric coordinates* of the point p. If $p = \sum_{i=1}^k a_i v_i$ and $q = \sum_{i=1}^k b_i v_i$ are points of Δ, then their ℓ^∞-distance is

$$d_\infty(p, q) = \sup\{|a_i - b_i| : 1 \le i \le k\}.$$

As in Theorem 7.2, $D(\cdot, \cdot)$ denotes the L^∞-distance between maps with the same domain and codomain. The proof of the following lemma is standard and left to the reader.

Lemma 7.4. *Let Δ_1 and Δ_2 be finite-dimensional simplices endowed with their ℓ^∞ metrics. Let $F, G : \Delta_1 \to \Delta_2$ be affine maps. Suppose that $d_\infty(F(v), G(v)) \le \epsilon$ for all vertices v of Δ_1. Then $D(F, G) \le \epsilon$.*

Let $f : \Delta \to \Delta'$ be an affine map from a $(q-1)$-simplex Δ to a $(p-1)$-simplex Δ'. Choosing an ordering $v_1, \dots, v_q$ of the vertices of Δ and an ordering $w_1, \dots, w_p$ of the vertices of Δ' gives a representation of f by a $p \times q$ matrix M. Namely, the i^{th} column of M gives the barycentric coordinates of $f(v_i)$ with respect to $w_1, \dots, w_p$. In particular, all entries are non-negative and all column sums are 1. We will consider matrices representing affine maps which have nice properties as described below:

Lemma 7.5. *Let M be a $p \times q$ matrix with non-negative entries, all of whose column sums are equal to one. For any $\epsilon > 0$, there is a $p \times q$ matrix M' such that*

- *all entries of M' are positive and all of its column sums are one;*
- *all entries of M' differ from the corresponding entries of M by $< \epsilon$; and*
- *there is a scalar multiple of M' which is a matrix with odd integer entries.*

Proof. If $p = 1$ there is nothing to do, so we assume $p \ge 2$. Let $K > \max\{p, \frac{1}{\epsilon}\}$ be an odd integer. The matrix pKM has column sums pK. The largest entry in each column is $\ge K$. In each column, replace the smaller $p-1$ entries by nearest odd positive integers, and replace the largest entry by the integer that keeps the column sum pK. This last entry is then odd and positive and differs from the original entry by $\le p-1$. After dividing by pK we get the desired matrix M'. □

Proposition 7.6. *Every Choquet simplex Δ is affinely homeomorphic to the limit of an inverse system $(\Delta_n, g_n)_{n=1}^\infty$ where: Δ_n is a finite-dimensional simplex, $g_n : \Delta_{n+1} \to \Delta_n$ is affine, and, choosing orderings for the vertices of Δ_n, the matrix representative for g_n in barycentric coordinates has a scalar multiple with positive odd entries.*

Proof. Represent Δ as the inverse limit of (Δ_n, f_n) given by Theorem 7.1. Using Lemma 7.5 with $\epsilon = L(g_1, \ldots, g_{n-1})$, inductively approximate each bonding map $f_n : \Delta_{n+1} \to \Delta_n$ by an affine map $g_n : \Delta_{n+1} \to \Delta_n$ so that the conditions of Theorem 7.2 are satisfied. Namely, Lemma 7.5 gives $d_\infty(f_n(v), g_n(v)) < L(g_1, \ldots, g_{n-1})$ at the vertices and therefore $D(f_n, g_n) < L(g_1, \ldots, g_{n-1})$ by Lemma 7.4. Let Δ' be the inverse limit of (Δ_n, g_n). It is a Choquet simplex by [13, Theorem 13]. The map $F : \Delta \to \Delta'$ from the conclusion of Theorem 7.2 is an affine homeomorphism. This follows, since each F_n is the limit of $g_{nm} \circ f_{m\infty}$, both maps in the composition are affine, and a limit of affine maps is affine. $\square$

The proof of Theorem E now follows quickly:

Proof of Theorem E. Let Δ be a compact metrizable Choquet simplex. By Proposition 7.6 we may represent Δ as an inverse limit of

$$\Delta_1 \xleftarrow{f_1} \Delta_2 \xleftarrow{f_2} \Delta_3 \xleftarrow{f_3} \ldots$$

where f_n is represented in barycentric coordinates by a matrix which has a scalar multiple whose entries are positive odd integers. Choose K_n a scalar such that $K_n f_n$ has positive odd integer entries. Let D_n be the dimension of Δ_n. We will construct a sequence of punctured disks X_n together with systems of arcs A_n, such that $\{A_n\}_{n=1}^\infty$ is directed and such that the transition map $\pi_n : C(X_{n+1}) \to C(X_n)$ is exactly $K_n f_n$. To do this, we choose X_1 to be any punctured disk containing $D_1 + 1$ pairwise disjoint, homotopically distinct, homotopically non-trivial arcs forming the system A_1. Given the pair (X_n, A_n) for any n, we may use Lemma 7.3 to construct a punctured disk X_{n+1} containing X_n and a system of arcs A_{n+1} on X_{n+1} for which the transition map $\pi_n : C(X_{n+1}) \to C(X_n)$ is exactly $K_n f_n$.

Denote $C_n = C(X_n)$. Now, we choose as a base Σ_1 for the cone $C_1 = \mathbb{R}_+^{D_1+1}$ the convex hull of the standard basis vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. We choose as a base for $C_n = \mathbb{R}_+^{D_n+1}$ the inverse image $\Sigma_n := \pi_{1n}^{-1}(\Sigma_1)$. Since the column sums of π_n are K_n, Σ_n is the convex hull of the multiples $\frac{1}{K_1 \cdots K_n} e_i$ and the induced map $\Sigma_{n+1} \to \Sigma_n$ is exactly f_n, in barycentric coordinates. Thus the inverse limit of $(\Sigma_n, \pi_n)_{n=1}^\infty$ is affinely homeomorphic to Δ.

To finish the proof, for an arbitrary infinite type hyperbolic surface X of the first kind, we need to produce an example of a minimal lamination Λ on X whose cone of transverse measures has a base affinely homeomorphic to Δ. First consider the case that X is the flute surface (genus zero with countably many punctures, exactly one of which is non-isolated). Note that such a hyperbolic metric of the first kind X on the flute surface exists, e.g. by [2, Theorem 4]. We may choose an exhaustion $Y_1 \subset Y_2 \subset Y_3 \subset \ldots$ where Y_n is a disk with the same number of punctures as X_n for each n. By choosing a homeomorphism of X_1 with Y_1, we may embed X_1 in X. By choosing a homeomorphism from the annulus $Y_2 \setminus Y_1$ to the annulus $X_2 \setminus X_1$, we may extend the embedding of X_1 to an embedding of X_2. Continue this process inductively. Thus, $X_1 \subset X_2 \subset \ldots$ is identified with the exhaustion $Y_1 \subset Y_2 \subset \ldots$ and the directed system of arcs $\{A_n\}_{n=1}^\infty$ pushes forward to a directed system of arcs on X. Thus, we may assume that $X_1 \subset X_2 \subset \ldots$ is an exhaustion of X and that $\{A_n\}_{n=1}^\infty$ is a directed system of collections of arcs on the subsurfaces X_n of X. Let $\Lambda \subset X$ be the inverse limit lamination of $\{A_n\}_{n=1}^\infty$. Then $\mathcal{M}(\Lambda)$ is affinely homeomorphic to the limit of the inverse system $C_1 \longleftarrow C_2 \longleftarrow C_3 \longleftarrow \ldots$ described earlier. Hence $\mathcal{M}(\Lambda)$ has a base affinely homeomorphic to Δ. By Proposition 6.5, Λ is minimal.

Finally, we consider the case of an arbitrary X. Replace the isolated punctures of the flute surface Y by boundary components. There is a topological embedding of Y into X (see e.g. [20, Lemma 3.2]). The exhaustion of Y described in the last paragraph pushes forward to an exhaustion $Y_1 \subset Y_2 \subset \ldots$ of the subsurface Y in X. This also embeds the collections of arcs $\{A_n\}_{n=1}^{\infty}$ as collections of arcs on the subsurfaces Y_n of X. Extend the exhaustion $Y_1 \subset Y_2 \subset \ldots$ of Y to an exhaustion of X as follows. The surface X is the union of Y with at most countably many pairwise disjoint subsurfaces $Z^1, Z^2, \ldots$ meeting Y only along their boundary components. Choose an exhaustion $Z^i_1 \subset Z^i_2 \subset \ldots$ of each Z^i by punctured compact subsurfaces. Then we define an exhaustion $X_1 \subset X_2 \subset \ldots$ by taking X_n to be the union of Y_n with Z^i_n for all i such that Z^i_n meets Y_n (along the boundary). One may verify that this does define an exhaustion of X by punctured compact subsurfaces and by [3, Proposition 3.1] we may assume the X_n have geodesic boundary (after a homotopy). Then for each n, A_n is a finite collection of arcs on X_n and $\{A_n\}_{n=1}^{\infty}$ is directed. Taking the inverse limit lamination Λ of $\{A_n\}_{n=1}^{\infty}$, we see that the cones $C_n = C(X_n)$ are unchanged, as are the transition maps $C_{n+1} \to C_n$. Hence, $\mathcal{M}(\Lambda)$ has a base affinely homeomorphic to Δ and Λ is minimal, as desired. □

8 Cones of measures without Choquet simplex bases

In the case that there is no compact subsurface intersecting every leaf of Λ, the cone of transverse measures $\mathcal{M}(\Lambda)$ may be badly behaved. In fact, the cone may have no compact base or no non-zero elements at all. We explore examples of these properties in this section.

8.1 A lamination with no non-zero transverse measures

In this subsection we construct a lamination with no non-zero transverse measures. The idea is to construct an inverse limit lamination with the following properties. Consider an exhaustion $X_1 \subset X_2 \subset \ldots$ by punctured compact subsurfaces together with a system of arcs $\{A_n\}_{n=1}^{\infty}$ which is directed. Going from (X_{n+1}, A_{n+1}) to (X_{n+2}, A_{n+2}), most of the traversals of arcs of A_{n+1} by arcs of A_{n+2} are on the arc of A_{n+1} which is disjoint from X_n. This effect is compounded at later stages so that a huge majority of the transversals of arcs of A_{n+1} by arcs of A_{n+k} for $k \gg 0$ are on the arc of A_{n+1} disjoint from X_n. This will force any transverse measure to assign measure 0 to any leaf of the inverse limit lamination Λ which intersects X_n (for any n).

We will construct nested punctured disks $X_1 \subset X_2 \subset \ldots$ together with systems of arcs A_n such that $\{A_n\}_{n=1}^{\infty}$ is directed and such that the inverse system $C(X_1) \longleftarrow C(X_2) \longleftarrow \ldots$ is exactly

$$\mathbb{R}_+ \xleftarrow{\pi_1} \mathbb{R}_+^2 \xleftarrow{\pi_2} \mathbb{R}_+^3 \xleftarrow{\pi_3} \ldots \text{ where} \tag{***}$$

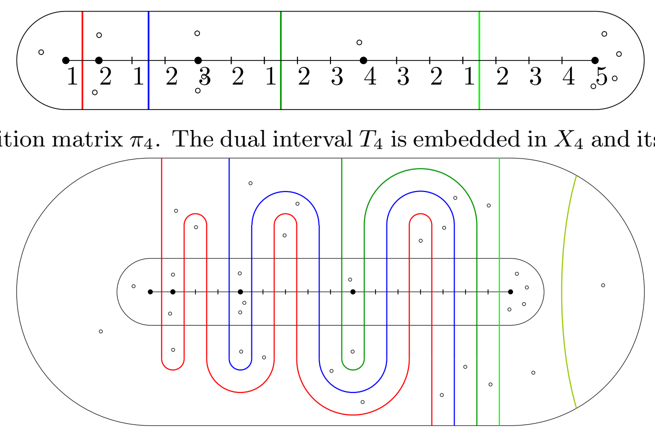
$$\pi_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}, \quad \pi_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}, \quad \ldots.$$

The construction of the disks X_n and arcs A_n is analogous to that of Lemma 7.3 and Theorem E.

We start by defining X_1 to be a punctured disk containing a single homotopically non-trivial arc, which forms the collection A_1. Suppose that $X_1, \ldots, X_n$ and $A_1, \ldots, A_n$ have been constructed such that the transition map from A_{i+1} to A_i is exactly π_i for each $i < n$ and such that the dual tree to A_i is an interval $[0,1]$ for $i \leq n$. We wish to construct X_{n+1} and A_{n+1} such that $\{A_i\}_{i=1}^{n+1}$ is directed and the transition map from A_{n+1} to A_n is π_n. Embed the dual interval T_n to A_n in X_n in such a way that the vertices of T_n lie in the regions of $X_n \setminus A_n$ that they represent and an edge of T_n intersects A_n in exactly one point, which lies on the arc that it represents. Denote by $\ell_1^n, \ldots, \ell_n^n$ the arcs of A_n. Order the vertices $v_0, v_1, \ldots, v_n$ of T_n and edges $e_1, \ldots, e_n$ of T_n such that e_i joins v_{i-1} to v_i for each i and e_i represents ℓ_i^n. See Figure 11a. Sub-divide e_i into $2i-1$ edges for $1 \leq i \leq n$. Label the vertices of e_i, in order from v_{i-1} to v_i, by

$$i, i-1, i-2, \ldots, 2, 1, 2, \ldots, i-1, i, i+1.$$

We form a larger disk by gluing a closed annulus to the boundary component of X_n. On X_n we place one arc ℓ_i^{n+1} for each $1 \leq i \leq n+1$ which crosses through all the edges of T_n with endpoints labeled by $i, i+1$. The arc ℓ_i^{n+1} will alternate directions through X_n each time it crosses it, and whenever it passes through a sub-edge of e_j, it will traverse an arc homotopic to ℓ_j^n. See Figure11b. The $\ell_1^{n+1}, \ldots, \ell_n^{n+1}$ may be constructed such that ℓ_j^{n+1} separates ℓ_{j-1}^{n+1} from ℓ_{j+1}^{n+1}. We additionally construct one more arc ℓ_{n+1}^{n+1} on X_{n+1} which is disjoint from X_n and is separated from ℓ_{n-1}^{n+1} by ℓ_n^{n+1}. We set $A_{n+1} = \{\ell_i^{n+1}\}_{i=1}^{n+1}$ and place a puncture in each component of $(X_{n+1} \setminus X_n) \setminus A_{n+1}$. The arcs of A_{n+1} are then disjoint, homotopically non-trivial, and homotopically distinct, the dual tree to A_{n+1} is an interval, and the transition map is exactly π_n, as desired.



(a) Realizing the transition matrix π_4. The dual interval T_4 is embedded in X_4 and its edges are sub-divided.

(b) The arcs A_5 on X_5. The arc ℓ_j^{n+1} crosses through the edges labeled $[j, j+1]$ for $1 \leq j \leq n$, while ℓ_{n+1}^{n+1} is disjoint from X_n.

Figure 11: Constructing (X_5, A_5) inductively from (X_4, A_4) and the transition matrix π_4.

Now we can prove Theorem C from the introduction:

Proof of Theorem C. Set Λ to be the inverse limit lamination of $\{A_n\}_{n=1}^{\infty}$. The cone $\mathcal{M}(\Lambda)$ is the limit of the inverse system $(***)$. The transition matrix $\pi_{nm} = \pi_n \circ \pi_{n+1} \circ \cdots \circ \pi_{m-1}$ for $m \geq n$

may be determined as follows. Set $i = m - n$. Then

$$\pi_{nm} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_1^i & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_2^i & a_1^i & 1 & \dots & 0 & 0 & \dots & 0 \\ a_3^i & a_2^i & a_1^i & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1}^i & a_{n-2}^i & a_{n-3}^i & \dots & 1 & 0 & \dots & 0 \end{pmatrix}$$

where π_{nm} is $n \times m$, there are $m - n$ columns of zeroes on the right, and a_j^i is the $(j, 1)$-entry of the i^{th} power of the infinite matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots \\ 2 & 2 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We claim that a_j^i is a polynomial in the variable i of degree j. To see this, write $A = I + T$ where I is an infinite identity matrix and $T = A - I$ is lower triangular. Then $A^i = \sum_{k=0}^{i} \binom{i}{k} T^k$ and by examining the powers T^k we see that $a_j^i = c_0\binom{i}{0} + c_1\binom{i}{1} + \dots + c_j\binom{i}{j}$ where c_k is a fixed (positive) entry of T^k. Since $\binom{i}{k}$ is a polynomial in i of degree k, the claim follows.

Hence for $j' > j$ we have $\frac{a_j^i}{a_{j'}^i} = \frac{P_j(i)}{P_{j'}(i)} \to 0$ as $i \to \infty$. Fixing n, we therefore have that the non-zero columns of π_{nm} converge projectively to $(0, 0, \dots, 0, 1) \in \mathbb{R}_+^n$ as $m \to \infty$. Thus, the intersections of the images of the maps π_{nm} in $\mathbb{R}_+^n$ are contained in the sub-cone $0^{n-1} \times \mathbb{R}_+ = \{(0, 0, \dots, 0, x) : x \geq 0\}$. So the inverse limit $\mathcal{M}(\Lambda)$ coincides with the limit of the inverse system

$$\mathbb{R}_+ \xleftarrow{\pi_1} 0 \times \mathbb{R}_+ \xleftarrow{\pi_2} 0^2 \times \mathbb{R}_+ \xleftarrow{\pi_3} 0^3 \times \mathbb{R}_+ \xleftarrow{\pi_4} \dots.$$

However, the map π_n restricted to the cone $0^n \times \mathbb{R}_+$ is just 0. So the inverse limit $\mathcal{M}(\Lambda)$ is 0. As in Theorem E, the lamination Λ may be realized on any infinite type surface. □

We mention an alternative way to prove that $\mathcal{M}(\Lambda) = 0$. Construct Λ as above. By studying the construction, one may notice that Λ consists of a countable collection $L_1, L_2, \dots$ of leaves. The leaf L_i accumulates onto L_{i+1} for each i and L_1 is isolated, L_2 becomes isolated after removing L_1, L_3 becomes isolated after removing L_2, and so on. Since L_1 is isolated but accumulates onto L_2, it lies outside the support of any measure. The same holds inductively for L_i, by removing $L_1, \dots, L_{i-1}$. Thus $\mathcal{M}(\Lambda) = 0$.

8.2 Laminations without compact bases

In this section we give a couple of examples of laminations Λ which support non-zero transverse measures but for which the cones $\mathcal{M}(\Lambda)$ nonetheless lack compact bases. Necessarily, each such lamination Λ contains sub-laminations disjoint from any given finite type subsurface. In particular, Λ cannot be a minimal lamination in any of these cases.

Example 8.1. Consider the lamination Λ of Example 4.7. The cone $\mathcal{M}(\Lambda) \cong \mathbb{R}_+^{\mathbb{N}}$ has no compact base. For if B were a base then B would contain a multiple of $e_i = (0, \dots, 0, 1, 0 \dots)$ for each $i \geq 1$ (where e_i has a 1 in entry i). Thus, $a_i e_i \in B$ for some $a_i > 0$. We see that $a_i e_i \to 0$ as $i \to \infty$, regardless of the values of a_i. Since 0 does not lie in B, it cannot be compact.

Example 8.2. Consider the lamination Λ on the infinite type surface X in Figure 12. Thus, X has a Cantor set of ends, all accumulated by genus. The lamination Λ consists of countably many isolated simple closed curves $\Gamma_1, \Gamma_2, \dots$ plus countably many proper leaves $L_1, L_2, \dots$ such that Γ_j converges to the union $\bigcup_{i=1}^{\infty} L_i$ as $j \to \infty$. Using an appropriate exhaustion of X shows that $\mathcal{M}(\Lambda)$ is affinely homeomorphic to the cone $C \subset \ell^1 \times \mathbb{R}_+^{\mathbb{N}}$ defined by

$$C = \left\{ ((y_i)_{i=1}^{\infty}, x_1, x_2, \dots) \in \ell^1 \times \mathbb{R}_+^{\mathbb{N}} : y_i \geq 0 \text{ for all } i \text{ and } x_j \geq \sum y_i \text{ for all } j \right\}.$$

Here ℓ^1 is endowed with its weak* topology as the dual of c_0, $\ell^1 \times \mathbb{R}_+^{\mathbb{N}}$ with its product topology, and C with the subspace topology. The ℓ^1 coordinates y_i of the cone correspond to weights on the simple closed curves Γ_i while the coordinates x_i correspond to weights on the proper leaves L_i. As in the last example, C has no compact base.

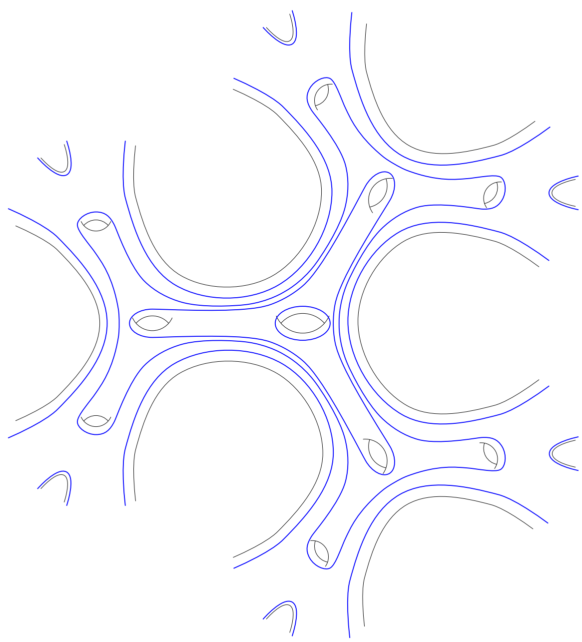


Figure 12: A lamination on the surface with a Cantor set of ends accumulated by genus, consisting of countably many isolated simple closed curves which limit to a countable union of proper leaves.

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