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Transferring algebra structures on complexes

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Abstract

With the goal of transferring dg algebra structures on complexes along contractions, we introduce a new condition on the associated homotopy, namely a generalized version of the Leibniz rule. We prove that, with this condition, the transfer works to yield a dg algebra (with vanishing descended higher A_∞ products) and prove that it works also after an application of the Perturbation Lemma even though the new homotopy may no longer satisfy that condition. We also extend these results to the setting of A_∞ algebras. Then we return to our original motivation from commutative algebra. We apply these methods to find a new method for building a dg algebra structure on a well-known resolution, obtaining one that is both concrete and permutation invariant. The naturality of the construction enables us to find dg algebra homomorphisms between these as well, enabling them to be used as inputs for constructing bar resolutions.

Keywords Dg algebras · Contractions · Perturbation lemma · A-infinity algebras

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Introduction

In this paper we prove a new result enabling us to transfer dg algebra structures between complexes along certain homotopy equivalences. Then we give an application to commutative algebra which was, in fact, our original motivation, namely to obtain

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a particularly well-behaved dg algebra structure on a well known free resolution in commutative algebra.

There is indeed a long history of studying the transfer of algebraic structures in different settings, falling under the general rubric of homological perturbation theory (HPT). Consider a contraction, or strong deformation retract, between two complexes. In general, one can use the well-known Homotopy Transfer Theorem (HTT) of Kadeishvili in [32] to transfer a dg algebra structure along it, but this usually yields only a strongly homotopy associative algebra, or A_∞ algebra, structure introduced in [51, 52]. One setting of special interest has been that of a contraction between a complex and its homology; here the transferred operations give the Massey operations on the homology. These techniques were developed further by Gugenheim, Lambe, and Stasheff in [15–19, 41], as well as in papers such as [9, 22, 24, 38, 43], with more recent developments described in [28, 29]; there are also Lie algebra variants.

To add to the story, we prove two results. For each, the key is a condition on the associated homotopy, namely a generalized version of the Leibniz rule that we introduce in this paper.

First, we show that a dg algebra structure on one complex can be transferred along a contraction to yield a dg algebra structure on the other one on the nose, and indeed that *all* the higher A_∞ products obtained via the HTT results vanish, as long as the associated homotopy satisfies the generalized Leibniz rule; see Propositions 1.4 and 5.5. Our extra hypothesis on the homotopy ensures that the product on the retract is, in fact, associative. Similarly, we note that the classical formulas for transferring A_∞ structures become much simpler in the presence of an analogous extension of the generalized Leibniz condition on the homotopy to higher products; see Proposition 5.7.

In our second result, we address the effect on such transfers of a homological tool called the Perturbation Lemma, intimately related to the story above; this scenario is key for our main application to commutative algebra. In [22], Huebschmann and Kadeishvili prove that, for a contraction consisting of dg algebra homomorphisms between two dg algebras, if one performs a multiplicative perturbation of the differential (that is, one for which the complex remains a dg algebra), the Perturbation Lemma produces a contraction in the category of dg algebras with the same product structures. However, one cannot apply this result if either of the maps in the contraction is not a dg algebra homomorphism. Such a situation occurs quite naturally, for example, for the algebra structures resulting from our first transfer result (and in our intended application), where only one of the maps is in general a dg algebra homomorphism; cf. Remark 2.6.

Instead, we prove another transfer result from scratch, namely that transferring a dg algebra structure along a contraction after a multiplicative perturbation still yields a dg algebra structure on the nose as long as the *initial* homotopy (before perturbation) satisfies the generalized Leibniz rule; see Proposition 2.5. This is a useful extension of our first result since the perturbed homotopy may no longer satisfy the generalized Leibniz rule. We also note that transferring after perturbation yields a different dg algebra structure than transferring before would.

In the last section, we provide a more detailed discussion contrasting the results in this paper to similar results in the literature.

In the second portion of the paper, we turn to the application to commutative algebra that motivated us to develop such transfer results. In that setting, one is interested in algebra structures on the *minimal* free resolution of a ring A as an R-module for the regular ring R in a presentation $A = R/I$; this line of investigation was pioneered by Buchsbaum and Eisenbud in [6] (as a shift from the earlier focus on the resolutions of the residue field over A), who also noted that any lift of the multiplication on the quotient ring to the resolution works if it is associative (and hence associativity is really the crux of the matter). However, it is worth noting that few such resolutions are known to carry dg algebra structures. Short resolutions are known to have a dg algebra structure [6, 21, 37, 39] but counterexamples of longer ones can be found in [3, 49, 50]. And yet having such a structure provides one with a powerful tool. We refer the reader to [4] for a full discussion.

We apply our transfer results to obtain a new method of building dg algebra structures on the well-known resolutions constructed by Buchsbaum and Eisenbud in [5] using Schur modules. These are the minimal graded free resolutions $\mathbb{L}_a$ of the quotients $R/\mathfrak{m}^a$ of a polynomial ring R in n variables by the ath powers of the homogeneous maximal ideal. More recently, these resolutions have been used as the main components in constructions of minimal resolutions of general graded Artinian algebras in [12, 44].

The resolutions $\mathbb{L}_a$ were shown to have a dg algebra structure by several authors. The first was Srinivasan in 1989 who constructed an explicit product using Young tableaux; see [48]. Next in 1996 Peeva proved in [45] that one can place a dg algebra structure on the Eliahou–Kervaire resolution, which applies here since the powers of the maximal ideal are Borel-fixed. In [42] Maeda used the representation theory of the symmetric group S_n to show that in characteristic zero any S_n-invariant lift of the multiplication on the quotient ring to the resolution is automatically associative, but did not give any explicit formulas.

In contrast to the other approaches, we use our homotopy transfer results to define a product which is both explicit and very natural in that it is transferred from a truncation of a Koszul complex and is naturally S_n-invariant. Indeed one immediate benefit is that it also enables us to define dg algebra homomorphisms between these resolutions; see Theorem B below. Hence one can use these resolutions as inputs for the bar resolutions given by Iyengar in [30], yielding a free resolution of $R/\mathfrak{m}^a$ as a module over $R/\mathfrak{m}^b$ for any positive integers $b > a$ (in fact, a minimal one as the products are minimal). We note that the only price of the symmetry driving this method is an increase in the number of terms and, relatedly, some non-integer rational coefficients causing us some minor restrictions on the characteristic.

More precisely, we prove the following results. In the first one, which is Theorem 3.11, we first use the Perturbation Lemma to realize the resolution of $\mathbb{L}_a$ of $R/\mathfrak{m}^a$ as a deformation retract of a complex $\mathbb{X}_a$ defined in (3.4) with an obvious dg algebra structure and then transfer the algebra structure over using our perturbed transfer result Proposition 2.5.

Theorem A *Let a be a positive integer. Suppose that k is a field of characteristic zero or positive characteristic $p \geq a + n$. Consider the deformation retract*

$$h_\infty \circlearrowright \mathbb{X}_a \underset{p_\infty}{\overset{i_\infty}{\rightleftarrows}} \mathbb{L}_a$$

obtained in (3.6). *Defining the product of $\alpha, \beta \in \mathbb{L}_a$ by*

$$\alpha\beta = p_\infty\left(i_\infty(\alpha)i_\infty(\beta)\right)$$

yields a dg algebra structure on $\mathbb{L}_a$. Furthermore, with this structure the map i_∞ is a homomorphism of dg algebras.

To prove this, we use a scaled version of the deRham differential to produce a contracting homotopy for the Koszul complex on the variables that satisfies the generalized Leibniz rule; see Lemma 3.10. It is also worth mentioning that the product that we define can be constructed in a basis free fashion; see Remark 3.14.

As a corollary we construct very natural dg algebra homomorphisms between these resolutions as follows; see Theorem 4.1.

Theorem B. *For positive integers $b \geq a$, the natural surjection $\mathbb{X}_b \twoheadrightarrow \mathbb{X}_a$ induces a homomorphism of dg algebras*

$$f_{b,a} \colon \mathbb{L}_b \to \mathbb{L}_a$$

that gives a lifting of the natural surjection $R/\mathfrak{m}^b \longrightarrow R/\mathfrak{m}^a$. In particular, the Koszul complex on the variables, $K = \mathbb{L}_1$, is a dg algebra over $\mathbb{L}_b$ for every positive integer b. Moreover, one has that $f_{c,a} = f_{b,a} f_{c,b}$ whenever $c \geq b \geq a$.

The paper is organized as follows: In Sect. 1, we prove the general statement for transferring dg algebra structures along a strong deformation retract whose homotopy satisfies the generalized Leibniz rule; see Proposition 1.4. In Sect. 2, we recall the well-known Perturbation Lemma and prove Proposition 2.5, a perturbed version of our dg algebra transfer result from Sect. 1. Section 3 contains our main application, which is to obtain, in a new way, dg algebra structures on the resolutions $\mathbb{L}_a$ of Buchsbaum and Eisenbud for all $a \geq 1$. In Sect. 4, we construct dg algebra homomorphisms between these resolutions. Finally, in Sect. 5, we study the implications of the additional hypotheses of generalized Leibniz-type rules for the Homotopy Transfer Theorems, resulting in Propositions 5.5 and 5.7.

Finally, in Sect. 6, we provide a comparison of the results in this paper to similar results in the literature.

In this paper, we assume that the complexes consist of R-modules for some commutative ring R and that they are graded homologically, rather than cohomologically; the reader should be aware that some sources for A_∞ algebras use the latter instead. One more note: All double complexes in this paper are considered to be anticommutative as in [55], and hence their totalizations do not require any change of sign in the differentials. When we speak of a double complex we often mean the totalization of it as a complex (for example, when we discuss a dg algebra structure on it); which way we are viewing it should be clear from the context.

Furthermore, in the body of this paper, we will use the term strong deformation retract, rather than the term contraction.

1 Transfer of dg algebra structures

In this section, we show how to transfer dg algebra structures along certain homotopy equivalences, namely deformation retracts whose associated homotopy behaves well with respect to products. As a further exploration, in Sect. 5 we determine the effect of this condition for the homotopy on the A_∞-algebra structures resulting from the Homotopy Transfer Theorem.

We begin with the classical definition.

1.1 Definition A *differential graded algebra over* R (dg algebra) is a complex (X, ∂) of R-modules lying in nonnegative degrees equipped with a product given by a chain map

$$X \otimes_R X \to X, \ (\alpha, \beta) \to \alpha\beta$$

giving an associative and unitary product with $1 \in X_0$.

The fact that the product is a chain map is equivalent to the differentials of X satisfying the *Leibniz rule*:

$$\partial(\alpha\beta) = \partial(\alpha)\beta + (-1)^{|\alpha|}\alpha\partial(\beta), \text{ for all } \alpha, \beta \in X$$

where $|\alpha|$ denotes the degree of α. In addition, we assume that the product is *strictly graded commutative*, that is, $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$ for all $\alpha, \beta \in X$ and $\alpha^2 = 0$ if the degree of α is odd.

A *homomorphism* of dg algebras is a morphism of complexes $\phi\colon X \to X'$ such that $\phi(1) = 1$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$.

Next we recall the definitions of the main ingredients in transfer of algebra structures.

1.2 Definition A set of *homotopy equivalence data* between two chain complexes is the following set of information: quasi-isomorphisms of complexes

$$(X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

with $ip \simeq 1$ and $pi \simeq 1$.

It is called a *deformation retract* if, in addition, one has $pi = 1$. In this case we use the following notation

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

including only the associated homotopy h with $ip = 1 + \partial^X h + h\partial^X$.

With further hypotheses, these are called *strong deformation retracts* or *contractions*; see Definition 2.3.

Next we introduce a new condition on the associated homotopy that is the key to our results, namely one that is in some sense weakly multiplicative and hence enables associativity to be preserved when transferring algebra structures. This property, a generalization of the Leibniz rule, is crucial in preserving the associativity of a transferred algebra structure.

1.3 Definition Let X be a complex of R-modules equipped with a product, and let $h\colon X \to X$ be a graded map (but not necessarily a chain map).

We say that h satisfies the *scaled Leibniz rule* if for every $\alpha, \beta \in X$ there are $r, s \in R$ depending only on the degrees of α and β, respectively, such that

$$h(\alpha\beta) = rh(\alpha)\beta + s\alpha h(\beta)$$

for every α and β in X.

Although this condition is the natural one for our application, the proofs of our transfer results go through if the homotopy h satisfies a weaker condition, which we call the *generalized Leibniz rule*, namely that

$$h(\alpha\beta) \in h(\alpha)X + Xh(\beta)$$

for every α and β in X.

We now prove the main result of this section. Note that the hypothesis $hi = 0$ holds for strong deformation retract; see Definition 2.3. This is the case in our application in Sect. 3.

1.4 Proposition *Let X be a dg algebra. Consider a deformation retract*

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

with associated homotopy h that satisfies the generalized Leibniz rule and that $hi = 0$ (or, more generally, that $(\partial^X h + h\partial^X)(i(\alpha)i(\beta)) = 0$). The following product defines a dg algebra structure on Y

$$\alpha\beta \overset{def}{=} p\,(i(\alpha)i(\beta)) \text{ for } \alpha, \beta \in Y$$

where the product inside parentheses is the one in X.

Moreover, with this structure on Y, the map i becomes a dg algebra homomorphism.

Proof The Leibniz rule for Y holds without the assumptions that h satisfies the generalized Leibniz rule and $hi = 0$. Indeed, for any elements $\alpha, \beta \in Y$, one has

$$
\begin{aligned}
\partial^Y(\alpha\beta) &= (\partial^Y p)(i(\alpha)i(\beta)) \\
&= (p\partial^X)(i(\alpha)i(\beta)) \\
&= p\left(\partial^X(i(\alpha))i(\beta) + (-1)^{|\alpha|}i(\alpha)\partial^X(i(\beta))\right) \\
&= p\left(i(\partial^Y(\alpha))i(\beta)\right) + (-1)^{|\alpha|}p\left(i(\alpha)i(\partial^Y(\beta))\right) \\
&= \partial^Y(\alpha)\beta + (-1)^{|\alpha|}\alpha\partial^Y(\beta)
\end{aligned}
$$

where the first equality is from the definition of the product, the second one holds since p is a chain map, the third one is from the Leibniz rule for X, the fourth holds since i is a chain map, the fifth is again from the definition of the product.

To prove the associativity and the last assertion, we use the condition that

$$
(\partial^X h + h\partial^X)\,(i(\alpha)i(\beta)) = 0 \tag{1.4.1}
$$

for all $\alpha, \beta \in Y$. As an aside, we verify that this equality holds, in particular, when h satisfies the generalized Leibniz rule: if one expands this expression using this rule and the fact that ∂^X satisfies the Leibniz rule, one sees that every term has a factor with hi, which is zero, or a factor with $h\partial^X i$ which is also zero since $\partial^X i = i\partial^Y$.

To verify associativity, take any elements $\alpha, \beta, \gamma \in Y$. One has

$$
\begin{aligned}
(\alpha\beta)\gamma &= p\,(i(\alpha)i(\beta))\,\gamma \\
&= p\,((ip(i(\alpha)i(\beta))i(\gamma)) \\
&= p\left((1 + \partial^X h + h\partial^X)\,(i(\alpha)i(\beta))\,i(\gamma)\right) \\
&= p\,((i(\alpha)i(\beta))\,i(\gamma))
\end{aligned}
$$

where the first two equalities are from the definition of the product and the third one is by the equality $ip = 1 + \partial^X h + h\partial^X$. A similar argument shows that

$$
\alpha(\beta\gamma) = p\,(i(\alpha)\,(i(\beta)i(\gamma)))
$$

and hence associativity holds since it holds for X.

Finally, one can see that $p(1)$ is the identity element of Y and that the product on Y is graded commutative since p and i are graded maps.

To see that the map i is a dg algebra homomorphism, let $\alpha, \beta \in Y$. One then has

$$
\begin{aligned}
i(\alpha\beta) &= ip\,(i(\alpha)i(\beta)) \\
&= (1 + \partial^X h + h\partial^X)\,(i(\alpha)i(\beta)) \\
&= i(\alpha)i(\beta)
\end{aligned}
$$

where the second equality holds as i and p form a deformation retract and the last one follows from (1.4.1). □

1.5 Remark Although the maps i and p in Proposition 1.4 form a deformation retract, in contrast to i, the map p need not be a homomorphism of dg algebras for the transferred dg algebra structure on Y. We give an example in Remark 3.7.

For the application we have in mind in Sect. 3, we need a slightly stronger result since, after we apply the Perturbation Lemma, the new homotopy need no longer satisfy the generalized Leibniz rule even if the original one does; however we show in this case that the transfer still works as long as the original deformation retract is strong. We give this result in Proposition 2.5.

2 Transfer of dg algebra structures and the Perturbation Lemma

The second aim of this paper, which we address in the next section, is to use transfer along a deformation retract to find a dg algebra structure on a well known complex that we recall in Sect. 3.3. Building this deformation retract involves a homological tool called the Perturbation Lemma. In this section we extend the transfer result in Proposition 1.4 to perturbations of the original setting, resulting in Proposition 2.5. In Remark 2.6 we compare our result with the one in [22].

The Perturbation Lemma generates new homotopy equivalences from initial ones; in general the aim is to modify the differentials of the complexes while maintaining a homotopy equivalence. The Perturbation Lemma first appeared in print in [8], although it had its roots in [7, 20, 47], and unpublished correspondence between Brown and Barratt; it appears again in [18]. Its full early history may be read in [29], as well as the MathSciNet entry for [16] written by R. Brown. Further work applying and extending perturbation-theoretic methods was done in [15–19, 22, 24–29, 41] (as well as recent work in the context of Lie algebras, which we do not mention here). In [11] Dyckerhoff and Murfet develop the lemma for the analogous case of matrix factorizations and in [23] Hogancamp does so for curved algebras. We also found the survey [10] useful.

The Perturbation Lemma is especially useful for double complexes where one can temporarily forget either the horizontal or the vertical differentials and add them back in later as the "perturbation"; this is the context in which we will apply it in Sect. 3.

We define some terminology that is needed in stating the Perturbation Lemma.

2.1 Definition Consider a set of homotopy equivalence data

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

A *perturbation* is a map δ on X of the same degree as the differential ∂^X such that $(\partial^X + \delta)^2 = 0$, that is, $\partial^X + \delta$ is again a differential. The perturbation δ is called *small* if $1 - \delta h$ is invertible. Most commonly, this happens when δh is elementwise nilpotent for then one has

$$(1 - \delta h)^{-1} = \sum_{j=0}^{\infty} (\delta h)^j = 1 + (\delta h) + (\delta h)^2 + \cdots$$

where the sum is finite on each element of X.

In the next definition we recall the the data obtained by perturbing a homotopy equivalence data.

2.2 Definition Consider a set of homotopy equivalence data

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

with associated homotopy h. Let δ be a small perturbation on X, and let $A = (1 - \delta h)^{-1}\delta$. We define the following new data

$$h_\infty \circlearrowright (X, \partial^X_\infty) \underset{p_\infty}{\overset{i_\infty}{\leftrightarrows}} (Y, \partial^Y_\infty)$$

where

$$i_\infty = i + hAi,\ p_\infty = p + pAh,\ \partial^X_\infty = \partial^X + \delta,\ \text{and}\ \partial^Y_\infty = \partial^Y + pAi$$

and set

$$h_\infty = h + hAh$$

Note that when δh is elementwise nilpotent, then the formulas can be rewritten as follows.

$$\begin{aligned} i_\infty &= (1 + (h\delta) + (h\delta)^2 + \cdots)i \\ p_\infty &= p(1 + (\delta h) + (\delta h)^2 + \cdots) \\ h_\infty &= h(1 + (\delta h) + (\delta h)^2 + \cdots) \\ \partial^Y_\infty &= \partial^Y + p\delta i_\infty \\ &= \partial^Y + p_\infty \delta i \end{aligned}$$

2.3 Definition A *strong deformation retract*, often called a *contraction* since its introduction in [13], is a deformation retract that satisfies the following equations

$$hi = 0,\ ph = 0,\ h^2 = 0. \tag{2.3.1}$$

These ensure that the property $pi = 1$ is inherited by the perturbed data.

As described in [10] for example, any deformation retract can be converted into a strong one by modifying the chosen homotopy in several steps. Fortunately, for our application, the deformation retracts involved are all strong.

With this terminology, we are now ready to state the Perturbation Lemma.

2.4 Perturbation Lemma. Given a set of homotopy equivalence data

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

its perturbation via a small perturbation gives a set of homotopy equivalence data

$$h_\infty \circlearrowright (X, \partial^X_\infty) \underset{p_\infty}{\overset{i_\infty}{\leftrightarrows}} (Y, \partial^Y_\infty)$$

If, furthermore, the original homotopy equivalence is a strong deformation retract, then so is the resulting one, that is,

$$p_\infty i_\infty = 1,\ h^2_\infty = 0,\ \ h_\infty i_\infty = 0,\ \text{ and } p_\infty h_\infty = 0,$$

Recall that by Proposition 1.4, given a strong deformation retract whose homotopy satisfies the generalized Leibniz rule, a dg algebra structure can be transferred along it. One might want to use the Perturbation Lemma to obtain new deformation retracts to which one could apply this proposition. However, even if the original homotopy satisfies the generalized Leibniz rule, the new one may no longer satisfy it. We remedy this by proving an extension of the transfer results as follows.

Recall that a perturbation δ on a dg algebra is called *multiplicative* if it satisfies the Leibniz rule (equivalently, if the algebra (X, ∂^X_∞) from Lemma 2.4 is still a dg algebra).

2.5 Proposition *Let X be a dg algebra. Consider a strong deformation retract,*

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

with associated homotopy h that satisfies the generalized Leibniz rule, and let δ be a small perturbation on X.

If δ is multiplicative, then the following product defines a dg algebra structure on the perturbed complex (Y, ∂^Y_∞)

$$\alpha\beta \overset{def}{=} p_\infty\left(i_\infty(\alpha)i_\infty(\beta)\right) = p\left(i_\infty(\alpha)i_\infty(\beta)\right) \text{ for } \alpha, \beta \in Y$$

where the product inside parentheses is the one in X.

Moreover, with this structure on Y, the map i_∞ becomes a dg algebra homomorphism.

Proof The proof is the same as that of Proposition 1.4 with the exception that to prove associativity and that i_∞ is a dg algebra homomorphism, one needs to show that

$$(\partial^X_\infty h_\infty + h_\infty \partial^X_\infty)\left(i_\infty(\alpha)i_\infty(\beta)\right) = 0.$$

Here this follows by similar reasoning due to the facts that one has

$$h_\infty = h + hAh \text{ and } i_\infty = i + hAi \text{ where } A = (1 - \delta h)^{-1}\delta$$

and that $h^2 = 0$ and $hi = 0$. □

2.6 Remark We may contrast this result to the Algebra Perturbation Lemma in [16] and [22, 2.1*]. In their result, Guggenheim, Lambe and Stasheff and, independently, Huebschmann and Kadeishvili assume that X and Y are dg algebras and the maps i and p are both dg algebra homomorphisms. They prove that after an application of the Perturbation Lemma the complex Y with its new differential is still a dg algebra and the maps i_∞ and p_∞ in the resulting deformation retract are dg algebra homomorphisms for this structure. Note that the algebra structures on X and Y have not changed after perturbation.

We, on the other hand, start by assuming that only X is a dg algebra and, even if one were to consider the transferred dg algebra structure on Y using Proposition 1.4, the map p is not necessarily a homomorphism of dg algebras. This is the case in the setting of our application in Sect. 3; see Remark 3.7 for an example. Furthermore, in the same remark we show also that the algebra structure on Y that results from the transfer after perturbation via Proposition 2.5 is in general not the same as the product one gets via transfer without a perturbation. So our results apply in a different setting than the Algebra Perturbation Lemma does and yield different structures.

Connections with the literature are discussed further in Sect. 6.

3 Application to a minimal resolution

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. In [5], Buchsbaum and Eisenbud introduced the minimal free resolution $\mathbb{L}_a$ of the quotient $R/(x_1, \ldots, x_n)^a$ of R by powers of the homogeneous maximal ideal. In [48], Srinivasan gives a dg algebra structure on $\mathbb{L}_a$ using Young tableaux. In this section, we use the Perturbation Lemma in a simple way to obtain a dg algebra structure on $\mathbb{L}_a$ that is S_n-invariant. Our approach works in characteristic zero and in positive characteristic provided that the characteristic is large enough.

We begin by recalling the definition of the resolution $\mathbb{L}_a$ and relating it to (the totalization of) a truncation of a certain double complex in (3.1), (3.3), and (3.4). In (3.5) and (3.6) we use the Perturbation Lemma to form a deformation retract between them, as long as one has an appropriately nice associated homotopy so that one can apply Proposition 2.5. Lastly we define such a homotopy using a scaled de Rham differential in (3.8), proving its properties in Lemmas 3.9 and 3.10, culminating in Theorem 3.11.

3.1 Here we define the double complex we will be working with. This is simply a rearrangement of a Koszul complex as a double complex of free R-modules; see Remark 3.2.

Let $S = R[y_1, \ldots, y_n]$ be a polynomial ring and let $\Lambda = R\langle e_1, \ldots, e_n\rangle$ be an exterior algebra. Consider the following (anticommutative) double complex whose rows are the strands of the Koszul complex $K(y_1, \ldots, y_n; S)$ and whose columns are the tensor product over R of the Koszul complex over R on $x_1, \ldots, x_n$ with the graded pieces S_a of S. We denote both it and its totalization by $\mathbb{S}$, as it is clear everywhere from the context which we mean. All the tensor products in the diagram are over R.

$$\begin{array}{ccccccccc}
 & & & & & & & \Lambda^n\otimes S_a & \xrightarrow{\kappa}\cdots \\
 & & & & & & & \downarrow d & \\
 & & & & & & \cdots\xrightarrow{\kappa} & \Lambda^{n-1}\otimes S_a & \xrightarrow{\kappa}\cdots \\
 & & & & & & & \downarrow & \\
 & & & & \vdots & & & \vdots & \\
 & & & & \downarrow d & & & \downarrow d & \\
 & & \vdots & & \Lambda^a\otimes S_2 & \xrightarrow{\kappa}\cdots & \cdots\xrightarrow{\kappa} & \Lambda^2\otimes S_a & \xrightarrow{\kappa}\cdots \\
 & & \downarrow d & & \downarrow d & & & \downarrow d & \\
\vdots & & \Lambda^a\otimes S_1 & \xrightarrow{\kappa} & \Lambda^{a-1}\otimes S_2 & \xrightarrow{\kappa}\cdots & \cdots\xrightarrow{\kappa} & \Lambda^1\otimes S_a & \xrightarrow{\kappa}\cdots \\
\downarrow d & & \downarrow d & & \downarrow d & & & \downarrow d & \\
\Lambda^a\otimes S_0 & \xrightarrow{\kappa} & \Lambda^{a-1}\otimes S_1 & \xrightarrow{\kappa} & \Lambda^{a-2}\otimes S_2 & \xrightarrow{\kappa}\cdots & \cdots\xrightarrow{\kappa} & \Lambda^0\otimes S_a & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & \vdots & & & & \\
\downarrow d & & \downarrow d & & \downarrow d & & & & \\
\Lambda^3\otimes S_0 & \xrightarrow{\kappa} & \Lambda^2\otimes S_1 & \xrightarrow{\kappa} & \Lambda^1\otimes S_2 & \xrightarrow{\kappa}\cdots & & & \\
\downarrow d & & \downarrow d & & \downarrow d & & & & \\
\Lambda^2\otimes S_0 & \xrightarrow{\kappa} & \Lambda^1\otimes S_1 & \xrightarrow{\kappa} & \Lambda^0\otimes S_2 & & & & \\
\downarrow d & & \downarrow d & & & & & & \\
\Lambda^1\otimes S_0 & \xrightarrow{\kappa} & \Lambda^0\otimes S_1 & & & & & & \\
\downarrow d & & & & & & & & \\
\Lambda^0\otimes S_0 & & & & & & & &
\end{array} \tag{3.1.1}$$

More explicitly, the horizontal differentials $\kappa_{i,a} : \Lambda^i \otimes S_a \to \Lambda^{i-1} \otimes S_{a+1}$ are given by

$$\kappa_{i,a}(e_{i_1} \wedge \cdots \wedge e_{i_i} \otimes p) = \sum_{j=1}^{a} (-1)^{j+1} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_i} \otimes y_{i_j} p \tag{3.1.2}$$

and the vertical differentials $d_{i,a} \colon \Lambda^i \otimes S_a \to \Lambda^{i-1} \otimes S_a$ are given by

$$d_{i,a} = \mathrm{kos}_i \otimes 1$$

where kos_i denotes the ith differential in the Koszul complex $K(x_1, \ldots, x_n; R)$.

For the totalization of this double complex (or of truncations of it), since it is anticommutative, the differentials are defined as

$$\partial_i = \sum_a (\kappa_{i,a} + d_{i,a})$$

without adding any signs. For simplicity we write

$$\partial = \kappa + d.$$

We continue to omit the indices on the maps when there is no ambiguity.

3.2 Remark As an aside, we give a slightly different way of obtaining a double complex which could have been used in this section. It differs only in signs from the one pictured in (3.1.1), but comes from a well known construction as follows.

Let V be a k-vector space with $\dim_k V = n$, and consider the symmetric and exterior algebras

$$\begin{aligned}\overline{S} &= S(V) \cong k[x_1', \dots, x_n'] \cong k[x_1'', \dots, x_n''] \\ \overline{\Lambda} &= \Lambda(V) \cong k\langle e_1, \dots, e_n\rangle\end{aligned}$$

Consider $\overline{S} \cong R$ as a module over its enveloping algebra $\overline{S}^e = \overline{S} \otimes_k \overline{S}$ via the multiplication map. Its minimal graded free resolution, after identifying the two copies of $\overline{S}$ with polynomial rings as in the display above, is the Koszul complex $\overline{\Lambda} \otimes_k \overline{S} \otimes_k \overline{S}$ on the regular sequence $\{x_i' \otimes 1 - 1 \otimes x_i''\}$. Rearranging factors, it can be expressed as

$$\overline{S} \otimes_k \overline{\Lambda} \otimes_k \overline{S} = \underbrace{\overline{S} \otimes_k \overline{\Lambda}}_{\partial'} \otimes_k \overline{S} = \overline{S} \otimes_k \underbrace{\overline{\Lambda} \otimes_k \overline{S}}_{\partial''}$$

with the homological degree being the degree of the middle factor and

$$\partial = \partial' \otimes 1 - 1 \otimes \partial'' = d - \kappa$$

where ∂' is the Koszul differential on $x_1', \dots, x_n'$ and ∂'' is the Koszul differential on $x_1'', \dots, x_n''$. Viewing graded strands, one can write this as a totalization of an anticommutative double complex of free R-modules given by

$$\overline{S} \otimes_k \overline{\Lambda}^i \otimes_k \overline{S}_j \cong R \otimes_k \overline{\Lambda}^i \otimes_k \overline{S}_j \cong (R \otimes_k \overline{\Lambda}^i) \otimes_R (R \otimes_k \overline{S}_j) \cong \Lambda^i \otimes_R S_j$$

Although this double complex differs from the one pictured in (3.1.1) by a sign on the horizontal maps κ, one could equally well use this complex in the rest of this section; similarly, one could obtain the double complex (3.1.1) from a Koszul complex by using $-x_i''$ in place of x_i'' above.

3.3 We introduce the complexes $\mathbb{L}_a$ of Buchsbaum and Eisenbud here. They show that this is a minimal R-free resolution of $R/\mathfrak{m}^a$, where $\mathfrak{m}$ is the homogeneous maximal ideal of R.

It is well known that the rows of the double complex (3.1.1) except the bottom one are exact; in fact, they can be viewed as the result of applying a base change to the strands of the tautological Koszul complex (see, for example, [44]). Hence they are contractible as they consist of free R-modules. So one can define free R-modules

$$L_{i,a} = \operatorname{im} \kappa_{i+1,a-1} = \ker \kappa_{i,a} \cong \operatorname{coker} \kappa_{i+2,a-2},$$

in other words, with split exact sequences

$$\Lambda^{i+2} \otimes S_{a-2} \xrightarrow{\kappa_{i+2,a-2}} \Lambda^{i+1} \otimes S_{a-1} \xrightarrow{\kappa_{i+1,a-1}} L_{i,a} \longrightarrow 0$$
$$0 \longrightarrow L_{i,a} \xrightarrow{\subseteq} \Lambda^{i} \otimes S_{a} \xrightarrow{\kappa_{i,a}} \Lambda^{i-1} \otimes S_{a+1}.$$

The vertical differentials d in the diagram induce maps on these modules, which we again denote by d, to yield a complex

$$\mathbb{L}_a\colon 0 \to L_{n-1,a} \xrightarrow{d_{n-1}} L_{n-2,a} \xrightarrow{d_{n-2}} \cdots \to L_{0,a} \xrightarrow{(-1)^{a-1}\varepsilon} R \to 0$$

augmented by the negative of the evaluation map

$$\varepsilon\colon L_{0,a} = \Lambda^0 \otimes S_a \cong S_a \to R \tag{3.3.1}$$

induced by the evaluation map from $S = R[y_1, \ldots, y_n]$ to $R = k[x_1, \ldots, x_n]$ sending y_i to x_i. We remark that the original complex defined by Buchsbaum and Eisenbud is augmented by ε, not $(-1)^{a-1}\varepsilon$, but we work with this isomorphic complex instead since this convention makes the computations via our method simpler.

3.4 Next we define $\operatorname{tr}_{\geq a}(\mathbb{S})$ and $\operatorname{tr}_{\leq a-1}(\mathbb{S})$ to be the totalizations of the truncations at column a of the anticommutative double complex $\mathbb{S}$

$$\{\Lambda^i \otimes S_j \mid j \geq a\} \text{ and } \{\Lambda^i \otimes S_j \mid j \leq a-1\},$$

respectively, with differentials inherited from $\mathbb{S}$. It is well-known that there is a quasi-isomorphism, and hence a homotopy equivalence,

$$\operatorname{tr}_{\leq a-1}(\mathbb{S}) \simeq \mathbb{L}_a$$

but we will re-derive this via the Perturbation Lemma in order to simultaneously transfer a dg algebra structure from $\operatorname{tr}_{\leq a-1}(\mathbb{S})$ over to $\mathbb{L}_a$ (by obtaining a strong deformation retract rather than just any homotopy equivalence).

To set up for this, we first argue as in [54] that the left truncation $\operatorname{tr}_{\leq a-1}(\mathbb{S})$ itself has a natural dg algebra structure. Indeed, the entire complex $\mathbb{S}$ is a dg algebra with the obvious multiplication: for $\alpha \in \Lambda^i \otimes S_a$ and $\beta \in \Lambda^j \otimes S_b$, the product is obtained by multiplying the factors in Λ and in S independently. It satisfies the Leibniz rule and other properties of a dg algebra because the differentials κ and d do and because homological degree in the totalization of $\mathbb{S}$ is, in fact, given by the degree in Λ. With this multiplication, the right truncation $\mathrm{tr}_{\geq a}(\mathbb{S})$ is clearly a dg ideal and the quotient complex

$$\mathbb{X}_a \stackrel{\text{def}}{=} \mathrm{tr}_{\leq a-1}(\mathbb{S}) \cong \mathbb{S}/\mathrm{tr}_{\geq a}(\mathbb{S})$$

is therefore a dg algebra. Concretely, the resulting product on the left truncation $\mathrm{tr}_{\leq a-1}(\mathbb{S})$ is given by the multiplication on $\mathbb{S}$ with the proviso that any terms landing in $\Lambda^i \otimes S_j$ with $j \geq a$ are taken to be zero.

For the next step, we first need a tool for converting a split exact sequence to a deformation retract from any truncation to the image of the differential at the truncation.

3.5 Let (X, ∂^X) be a contractible complex of R-modules, i.e., one that is homotopy equivalent to zero via a homotopy s, (i.e., one that is split exact). Denote its truncation at position c by

$$\mathrm{tr}_{\geq c}(X) = \cdots \longrightarrow X_n \xrightarrow{\partial^X_{n+1}} \cdots \longrightarrow X_{c+1} \xrightarrow{\partial^X_{c+1}} X_c \longrightarrow 0$$

Let $\mathrm{im}\,\partial^X_c$ denote the stalk complex with this module in degree c and 0 modules elsewhere. The chain maps i and p given in degree c by s_{c-1} and ∂^X_c, respectively, yield a deformation retract

$$\mathrm{tr}_{\geq c}(X) \underset{p}{\overset{i}{\leftrightarrows}} \mathrm{im}\,\partial^X_c$$

with associated homotopy $h = -s|_{\mathrm{tr}_{\geq c} X}$. Indeed one can easily check that $pi = 1$ and $ip \simeq 1$ via the homotopy h. Note that one could also use $\mathrm{coker}\,\partial^X_{c+1}$ instead of $\mathrm{im}\,\partial^X_c$ with appropriate i and p.

If, furthermore, the original contracting homotopy satisfies $s^2 = 0$, then the deformation retract is strong: $h^2 = 0$ and $hi = 0$ and one always has $ph = 0$ due to the fact that $p = 0$ in degrees $n \neq c$.

Next we want to transfer this structure from $\mathbb{X}_a = \mathrm{tr}_{\leq a-1}(\mathbb{S})$, which has a dg algebra structure by (3.4) to the minimal free resolution $\mathbb{L}_a$ of $R/\mathfrak{m}^a$. By Proposition 2.5, it suffices to find a strong deformation retract of the form

$$(\mathbb{X}_a, \partial^{\mathbb{X}_a}) \leftrightarrows (\mathbb{L}_a, d)$$

that comes via a perturbation of a strong deformation retract whose homotopy satisfies generalized Leibniz. Note that the differential of $\mathbb{X}_a$ is exactly $\kappa + d$. In 3.6, we discuss how one can find this deformation retract, given a contracting homotopy on the higher rows of $\mathbb{S}$, which we define in 3.8 and whose required properties we establish in Lemmas 3.9 and 3.10.

3.6 Here is an overview of how we obtain such a deformation retract using the Perturbation Lemma; see (2.4). First we form a deformation retract between two complexes $\mathbb{X}_a^\circ$ and $\mathbb{L}_a^\circ$, where $\mathbb{X}_a^\circ$ is obtained from $\mathbb{X}_a$ by replacing the vertical differentials d by 0 and $\mathbb{L}_a^\circ$ is the complex $\mathbb{L}_a$ with differentials set equal to zero. We do this via (3.5) using the homotopy from 3.8. Second we use the Perturbation Lemma to reinsert the original differentials on each, which has the effect of modifying the maps i and p.

We start by finding a deformation retract of the form

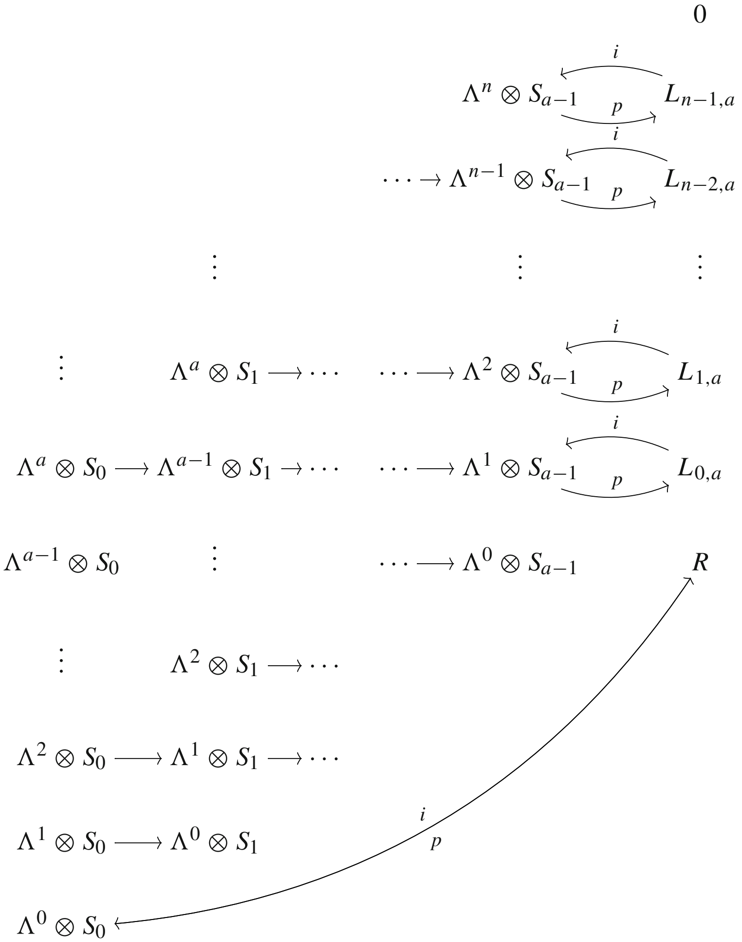
$$h \circlearrowright (\mathbb{X}_a^\circ, \kappa) \underset{p}{\overset{i}{\rightleftarrows}} (\mathbb{L}_a^\circ, 0)$$

For rows of $\mathbb{X}_a^\circ$ except the bottom one, we use (3.5) as follows. Recall that the rows of $\mathbb{S}$ are split exact with a contracting homotopy that we call σ (an explicit one is given in Remark 3.8). So each row that gets truncated has a deformation retract onto the image $L_{i,a}$ of the next horizontal differential; see diagram (3.6.2). Note that some of the lower rows will remain intact and hence are homotopy equivalent to zero; see (3.6.2). On the other hand, the row $\Lambda^0 \otimes S_0$ at the bottom of the diagram is not exact and so needs to be dealt with separately in conjunction with $R = (\mathbb{L}_a)_0$. For this we use that there is an isomorphism $\varepsilon \colon \Lambda^0 \otimes S_0 \to R$ defined in (3.3.1).

Putting this all together, one obtains chain maps given by

$$i = \begin{cases} \sigma & \text{on } L_{i,a} \\ \varepsilon^{-1} & \text{on } R \end{cases}$$
$$p = \begin{cases} \kappa & \text{on } \Lambda^i \otimes S_{a-1} \text{ for } i > 0 \\ \varepsilon & \text{on } \Lambda^0 \otimes S_0 \\ 0 & \text{else,} \end{cases} \tag{3.6.1}$$

with the property that $pi = 1$ and $ip \simeq 1$ via the homotopy $h = -\sigma|_{\mathbb{X}_a^\circ}$. The maps i and p are pictured in following diagram.



(3.6.2)

Next we apply the Pertubation Lemma, adding the missing vertical differentials d of $\mathbb{X}_a$ and d of $\mathbb{L}_a$. More precisely, consider the perturbation $\delta = d$ on $\mathbb{X}_a^\circ$; this is a small perturbation since the double complex $\mathbb{X}_a$ is bounded. First, we check that that the differentials on $\mathbb{L}_a$ obtained in this way are the original differentials on $\mathbb{L}_a$. On $\mathbb{L}_{j,a}$ with $j > 0$, this is because one has

$$\partial^{\mathbb{L}_a^\circ} + p_\infty \delta i = 0 + p(1 + (dh) + (dh)^2 + \cdots)di = pdi = dpi = d$$

where the second equality follows from the fact that p vanishes on most of the diagram, the third one follows from the fact that p is defined using κ and ε, as well as the commutativity of diagram (3.1.1) and the properties of ε, and the last one is because $pi = 1$. On $\mathbb{L}_{0,a}$ a similar computation gives that the inherited differential is $(-1)^{a-1}\varepsilon$

In summary, one gets a homotopy equivalence

$$(\mathbb{X}_a^\circ, \partial^{\mathbb{X}_a} = \kappa + d) \underset{p_\infty}{\overset{i_\infty}{\rightleftarrows}} (\mathbb{L}_a^\circ, \partial^{\mathbb{L}_a} = d) \tag{3.6.3}$$

For later use, we calculate the new chain maps i_∞ and p_∞, as well as the associated homotopy h_∞ using the formulas in Definition 2.2. The map i_∞ is given by

$$i_\infty = (1 + (h\delta) + (h\delta)^2 + \cdots)i$$

where $\delta = d$, and this can be written as

$$i_\infty = \begin{cases} (1 + (-\sigma d) + (-\sigma d)^2 + \cdots)\sigma & \text{on } L_{i,a} \\ \varepsilon^{-1} & \text{on } R. \end{cases} \tag{3.6.4}$$

In contrast, the map p_∞ is remarkably simpler since p equals zero on most of its domain. Indeed it is given by

$$p_\infty = p(1 + (\delta h) + (\delta h)^2 + \cdots)$$

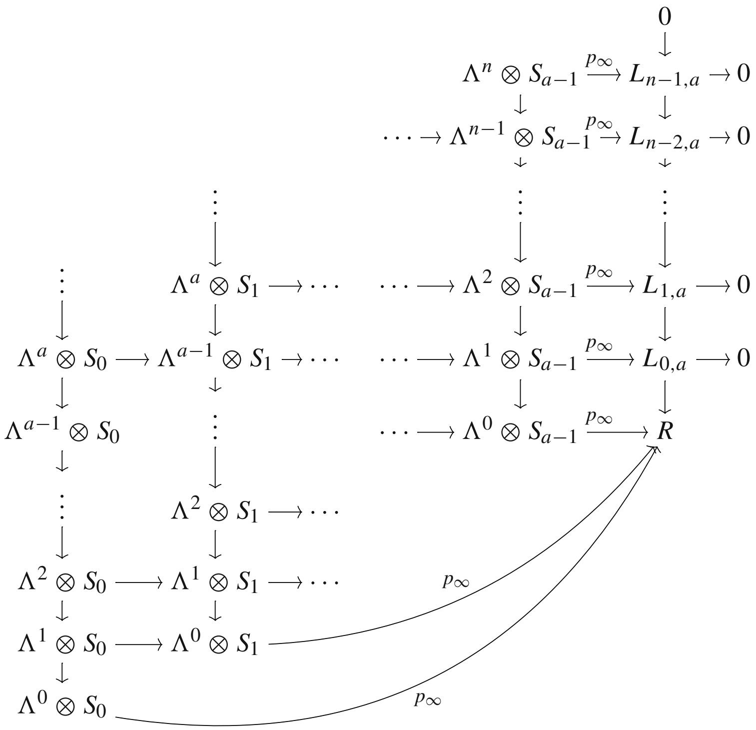
which can be written as

$$p_\infty = \begin{cases} \kappa & \text{on } \Lambda^i \otimes S_{a-1} \text{ for } i > 0 \\ (-1)^j \varepsilon & \text{on } \Lambda^0 \otimes S_j \text{ for all } j \\ 0 & \text{else.} \end{cases} \tag{3.6.5}$$

We record also the resulting homotopy for $i_\infty p_\infty \simeq 1$, which is

$$\begin{aligned} h_\infty &= h(1 + (\delta h) + (\delta h)^2 + \cdots) \\ &= -\sigma(1 + (-d\sigma) + (-d\sigma)^2 + \cdots) \end{aligned} \tag{3.6.6}$$

The map p_∞ has the form pictured in the following diagram.



(3.6.7)

3.7 Remark As an aside, we provide an example to show the claim in Remark 1.5. To see this, consider the dg algebra structure on $\mathbb{L}_2^\circ$ obtained by applying Proposition 1.4 to the deformation retract

$$h \circlearrowright (\mathbb{X}_2^\circ, \kappa) \underset{p}{\overset{i}{\rightleftarrows}} (\mathbb{L}_2^\circ, 0)$$

given in 3.6.1 and 3.6.2. Let $\alpha = e_1 \otimes 1 \in \Lambda^1 \otimes S_0$ and $\beta = 1 \otimes y_1 \in \Lambda^0 \otimes S_1$. Since $p(\alpha) = 0$, one has $p(\alpha)p(\beta) = 0$. However, $p(\alpha\beta) = p(e_1 \otimes y_1) = 1 \otimes y_1^2 \in L_{0,2} = \Lambda^0 \otimes S_2$ is clearly nonzero. Therefore, p is not a homomorphism of dg algebras.

Additionally, one can also see that for $a \geq 2$ the transferred algebra structure on $\mathbb{L}_a$ is trivial except for products involving $(\mathbb{L}_a)_0 = R$. Fortunately, an application of Proposition 2.5 for the perturbation described in 3.6 results in a nontrivial product. Indeed, it is easy to see that a mostly trivial product would not satisfy the Leibniz rule.

We now define an explicit contracting homotopy σ on the rows of $\mathbb{S}$ that can be used to complete the argument in 3.6. This turns out to be nothing but a scaled version of the de Rham differential.

3.8 In view of Proposition 2.5, in order to transfer the dg algebra structure from $\mathbb{X}_a$ to $\mathbb{L}_a$ we need a strong deformation retract. As explained in 3.6 in view of (3.5), this comes down to finding a contracting homotopy σ with $\sigma^2 = 0$ on the rows

$$\cdots \to \Lambda^{i-1} \otimes S_{m+1} \to \Lambda^i \otimes S_m \to \Lambda^{i+1} \otimes S_{m-1} \to \cdots$$

of the entire diagram $\mathbb{S}$ displayed in (3.1.1) with the property that $i + m > 0$.

Assume now that the field k has characteristic zero (for the positive characteristic case, see the end of this portion).

Define $\sigma_{i,m} \colon \Lambda^i \otimes S_m \to \Lambda^{i+1} \otimes S_{m-1}$ as

$$\sigma_{i,m}(e_{t_1} \wedge \cdots \wedge e_{t_i} \otimes y_{p_1} \cdots y_{p_m}) = \frac{1}{i+m} \sum_{j=1}^{m} e_{p_j} \wedge e_{t_1} \wedge \cdots \wedge e_{t_i} \otimes y_{p_1} \cdots \widehat{y}_{p_j} \cdots y_{p_m}$$

where $\widehat{y_t}$ denotes omission of the factor y_t and it is understood that $\sigma_{i,m} = 0$ when the target of the map is the zero module, that is, when $m = 0$ or $i = n$. This can also be written as a scaled de Rham differential

$$\sigma_{i,m} = \frac{1}{i+m} \sum_{j=1}^{n} e_j \otimes \frac{\partial}{\partial y_j}.$$

To address the case of positive characteristic p, note that in general we only need to define a contracting homotopy $\sigma_{i,m}$ for $m \leq a$ when we apply (3.5) to truncate the complex at position $a - 1$, and so it suffices to assume $p \geq n + a$. This ensures that when necessary one has $\frac{1}{i+m} \in k$.

In Lemma 3.9, we show σ is a contracting homotopy with $\sigma^2 = 0$. In view of Lemma 1.4, we also require the homotopy to satisfy the generalized Leibniz property; this also comes down to the same property for σ on the rows of the entire diagram $\mathbb{S}$ in (3.1.1), which we verify in Lemma 3.10.

A contracting homotopy was defined previously by Srinivasan in [48]. However, it does not satisfy either required property. The map σ defined above is more symmetric (it is invariant under permutations of the variables) and hence ends up having its square equal to zero and satisfying the generalized Leibniz rule, in fact, the stronger scaled Leibniz rule, as we see in the next two results.

3.9 Lemma *Consider the maps σ defined in (3.8) on the rows of diagram (3.1.1) in which the indices sum to a positive number.*

If R has characteristic zero, the maps σ give a contracting homotopy on the rows and satisfy $\sigma^2 = 0$.

If R has positive characteristic p, the same conclusions hold for $\sigma_{i,m}$ with $m \leq a-1$ as long as $p \geq n + a$.

Proof First we show that $\kappa\sigma + \sigma\kappa = 1$. At the ends of the rows, one can show this easily, so we may work with basis elements of $\Lambda^i \otimes S_m$ with $m, i > 0$. We compute $\kappa\sigma$ and $\sigma\kappa$ separately. The reader should note that, although some of the terms in the formulas could contain factors of the form $e_j \wedge e_j$, we do not replace any such repeated factors with 0 as leaving them avoids describing complicated cases. This does not affect our computations since the formula for the Koszul differential κ gives the same output for either form of input.

For any $\alpha = e_{t_1} \wedge \cdots \wedge e_{t_i} \otimes y_{p_1} \cdots y_{p_m} \in \Lambda^i \otimes S_m$, one has

$$\begin{aligned}
&\kappa_{i+1,m-1}\sigma_{i,m}(\alpha) \\
&= \frac{1}{i+m}\sum_{j=1}^{m}[\alpha + \sum_{u=1}^{i}(-1)^u e_{p_j} \wedge e_{t_1} \wedge \cdots \widehat{e_{t_u}} \cdots \wedge e_{t_i} \otimes y_{t_u} y_{p_1} \cdots \widehat{y_{p_j}} \cdots y_{p_m}] \\
&= \frac{1}{i+m}[m\alpha + \sum_{j=1}^{m}\sum_{u=1}^{i}(-1)^u e_{p_j} \wedge e_{t_1} \wedge \cdots \widehat{e_{t_u}} \cdots \wedge e_{t_i} \otimes y_{t_u} y_{p_1} \cdots \widehat{y_{p_j}} \cdots y_{p_m}]
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_{i-1,m+1}\kappa_{i,m}(\alpha) \\
&= \sum_{u=1}^{i}(-1)^{u+1}\frac{1}{i+m}[\alpha + \sum_{j=1}^{m} e_{p_j} \wedge e_{t_1} \wedge \cdots \widehat{e_{t_u}} \cdots \wedge e_{t_i} \otimes y_{t_u} y_{p_1} \cdots \widehat{y_{p_j}} \cdots y_{p_m}] \\
&= \frac{1}{i+m}[i\alpha + \sum_{j=1}^{m}\sum_{u=1}^{i}(-1)^{u+1} e_{p_j} \wedge e_{t_1} \wedge \cdots \widehat{e_{t_u}} \cdots \wedge e_{t_i} \otimes y_{t_u} y_{p_1} \cdots \widehat{y_{p_j}} \cdots y_{p_m}]
\end{aligned}$$

Thus for $m, i > 0$ one has

$$(\kappa_{i+1,m-1}\sigma_{i,m} + \sigma_{i-1,m+1}\kappa_{i,m})(\alpha) = \left(\frac{m}{i+m}\right)\alpha + \left(\frac{i}{i+m}\right)\alpha = \alpha$$

Next, to see that $\sigma^2 = 0$, one computes for $m \geq 2$

$$\sigma_{i+1,m-1}\sigma_{i,m}(\alpha) = \frac{1}{(i+m)^2}\sum_{j=1}^{m}\sum_{\substack{u=1 \\ u\neq j}}^{m} e_{p_u} \wedge e_{p_j} \wedge e_{t_1} \wedge \cdots \wedge e_{t_i} \otimes y_{p_1} \cdots \widehat{y_{p_u}} \cdots \widehat{y_{p_j}} \cdots y_{p_m}$$

which is zero since $e_{p_u} \wedge e_{p_j} = -e_{p_j} \wedge e_{p_u}$.

Note that this proof works in positive characteristic as long as the maps $\sigma_{i,m}$ are defined, which is guaranteed by the hypotheses. □

3.10 Lemma *If R has characteristic zero, the maps σ defined in (3.8) satisfy the scaled Leibniz rule. More precisely, when $\alpha \in \Lambda^i \otimes S_a$, $\beta \in \Lambda^j \otimes S_b$ with $i + a + j + b$ positive, the maps σ satisfy*

$$\sigma(\alpha\beta) = \frac{1}{i+a+j+b}\Big((i+a)\sigma(\alpha)\beta + (-1)^i(j+b)\alpha\sigma(\beta)\Big)$$

and when $i = a = j = b = 0$ one has that $\sigma(\alpha\beta)$, $\sigma(\alpha)$, and $\sigma(\beta)$ are all 0.

If R has positive characteristic p, the same conclusion holds as long as $p \geq i + a + j + b$.

Proof Without loss of generality, we may assume that

$$\alpha = e_{t_1} \wedge \cdots \wedge e_{t_i} \otimes y_{p_1} \cdots y_{p_a} \text{ and } \beta = e_{s_1} \wedge \cdots \wedge e_{s_j} \otimes y_{q_1} \cdots y_{q_b}$$

are basis elements with $i + a + j + b > 0$ (the case when $i + a + j + b = 0$ is trivial). Then one has the following equalities

$$\begin{aligned}
\sigma(\alpha\beta) &= \sigma(e_{t_1} \wedge \cdots \wedge e_{t_i} \wedge e_{s_1} \wedge \cdots \wedge e_{s_j} \otimes y_{p_1} \cdots y_{p_a} y_{q_1} \cdots y_{q_b}) \\
&= \frac{1}{i+a+j+b} \Bigg(\sum_{u=1}^{a} e_{p_u} \wedge e_{t_1} \wedge \cdots \wedge e_{s_j} \otimes y_{p_1} \cdots \widehat{y}_{p_j} \cdots y_{p_a} y_{q_1} \cdots y_{q_b} \\
&\qquad + \sum_{v=1}^{b} e_{q_v} \wedge e_{t_1} \wedge \cdots \wedge e_{s_j} \otimes y_{p_1} \cdots y_{p_a} y_{q_1} \cdots \widehat{y}_{q_v} \cdots y_{q_b} \Bigg) \\
&= \frac{1}{i+a+j+b} \Bigg(\sum_{u=1}^{a} (e_{p_u} \wedge e_{t_1} \wedge \cdots \wedge e_{t_i} \otimes y_{p_1} \cdots \widehat{y}_{p_j} \cdots y_{p_a})(\beta) \\
&\qquad + \sum_{v=1}^{b} (-1)^i (\alpha)(e_{q_v} \wedge e_{s_1} \wedge \cdots \wedge e_{s_j} \otimes y_{q_1} \cdots \widehat{y}_{q_v} \cdots y_{q_b}) \Bigg) \\
&= \frac{1}{i+a+j+b} \Big((i+a)\sigma(\alpha)\beta + (-1)^i (j+b)\alpha\sigma(\beta) \Big)
\end{aligned}$$

where an empty sum is considered to be zero. □

Next we put together all the ingredients from this section to obtain our main application of our homotopy transfer results. The proof is an application of Proposition 2.5 to the deformation retract obtained in (3.6) with the homotopy h_∞ defined in (3.6.6) obtained from the homotopy σ on the rows of the diagram (3.1.1) satisfying the scaled Leibniz rule and hence the generalized Leibniz rule; see (3.8), (3.9), and (3.10). See also the overview in the paragraph before (3.6). One note: one need only check the homotopy h on $\mathbb{X}_a$ satisfies the scaled Leibniz rule for products that land in $\Lambda^i \otimes S_m$ for $i \leq n$ and $m < a$ since otherwise the product is zero and the result is trivial; this explains why we need only take $p \geq a + n$ in the statement below.

3.11 Theorem *Let a be a positive integer. Suppose that k is a field of characteristic zero or positive characteristic $p \geq a + n$. Consider the deformation retract obtained in (3.6)*

$$h_\infty \circlearrowright \mathbb{X}_a^\circ, \underset{p_\infty}{\overset{i_\infty}{\rightleftarrows}} \mathbb{L}_a^\circ$$

with the associated homotopy h_∞ defined in (3.6.6) using σ from (3.8), where i_∞ and p_∞ are defined as in (3.6.4) and (3.6.5).

Defining the product of $\alpha, \beta \in \mathbb{L}_a$ by

$$\alpha\beta = p_\infty(i_\infty(\alpha) i_\infty(\beta))$$

yields a dg algebra structure on $\mathbb{L}_a$. Furthermore, with this structure the map i_∞ is a homomorphism of dg algebras.

Proof The proof is is an application of Proposition 2.5 to the deformation retract obtained in (3.6) as described in the paragraph above. □

3.12 Remark The product given in the theorem above can be described explicitly, using the definitions of i_∞ and p_∞, as follows.

Consider elements $\alpha, \beta \in \mathbb{L}_a$. If one of them is in $(\mathbb{L}_a)_0 = R$ then their product is the one coming from the R-module structure of each $(\mathbb{L}_a)_i$. If both have positive degree, then

$$\alpha\beta = \kappa(\widetilde{\alpha}\widetilde{\beta})$$

where κ is defined in (3.1.2) and

$$\widetilde{\alpha} = (1 + (-\sigma d) + (-\sigma d)^2 + \cdots)\sigma(\alpha)$$
$$\widetilde{\beta} = (1 + (-\sigma d) + (-\sigma d)^2 + \cdots)\sigma(\beta)$$

where the scaled de Rham differential σ is defined in 3.8.

3.13 Remark Because of the symmetric way in which the maps κ, d and h are defined, the dg algebra structure defined on $\mathbb{L}_a$ in Theorem 3.11 is invariant under the action of the symmetric group on the polynomial ring.

3.14 Remark One may note that our algebra structure is, in fact, basis free, although we do not describe it in a basis free way because it was more straightforward to show the required properties of the homotopy via an explicit formula. It is well known that the differentials in the complex from which we transfer our structure are so, and one can see that the homotopy is as well, as it is just a scaled version of the de Rham map.

4 Comparison maps

In this section we use the results from the previous sections to obtain comparison maps lifting the natural surjections $R/\mathfrak{m}^b \longrightarrow R/\mathfrak{m}^a$ for any $b \geq a$ to their respective minimal free resolutions $\mathbb{L}_b$ and $\mathbb{L}_a$, and these maps turn out to be dg algebra morphisms (for the dg algebra structures placed on them in the previous section). Since $\mathbb{L}_1$ is simply the Koszul complex K on the variables, this yields that K is a dg algebra over $\mathbb{L}_b$ for each $b \geq 1$.

To set up the statement, recall from (3.6.3) that for any c there is a homotopy equivalence

$$h_\infty \circlearrowright \mathbb{X}_c \underset{p_\infty}{\overset{i_\infty}{\leftrightarrows}} \mathbb{L}_c$$

pictured in (3.6.7) that is used in Theorem 3.11 to place a dg algebra structure on $\mathbb{L}_c$ for which i_∞ is a dg algebra homomorphism. Although the value of c varies below, it should be clear from the context which i_∞ and p_∞ maps are being applied. Recall also from (3.4) that we can view $\mathbb{X}_c$ as the quotient $\mathbb{S}/\mathrm{tr}_{\geq c}(\mathbb{S})$ of the dg algebra $\mathbb{S}$ by the dg ideal

$$\mathrm{tr}_{\geq c}(\mathbb{S}) = \{\Lambda^i \otimes S_j \mid j \geq c\}$$

In this way, $\mathbb{X}_c$ inherits the dg algebra structure from $\mathbb{S}$. Therefore, if $b \geq a$, the inclusion of dg ideals $\mathrm{tr}_{\geq b}(\mathbb{S}) \hookrightarrow \mathrm{tr}_{\geq a}(\mathbb{S})$ gives a natural quotient map

$$\pi_{b,a} \colon \mathbb{X}_b = \mathbb{S}/\mathrm{tr}_{\geq b}(\mathbb{S}) \twoheadrightarrow \mathbb{S}/\mathrm{tr}_{\geq a}(\mathbb{S}) = \mathbb{X}_a$$

which has the effect of sending the columns $\Lambda^i \otimes S_j$ to zero for $a \leq j \leq b-1$. This is clearly a homomorphism of dg algebras.

4.1 Theorem *Let a and b be positive integers with $b \geq a$. The chain map*

$$f_{b,a} = p_\infty \pi_{b,a} i_\infty \colon \mathbb{L}_b \to \mathbb{L}_a$$

is a homomorphism of dg algebras that gives a lifting of the natural surjection $R/\mathfrak{m}^b \longrightarrow R/\mathfrak{m}^a$. In particular, the Koszul complex on the variables, which is $\mathbb{L}_1$, is a dg algebra over $\mathbb{L}_b$ for every positive integer b.

Moreover, if $c \geq b \geq a$ then $f_{c,a} = f_{b,a} f_{c,b}$.

Proof First note that $f_{b,a}$ is a chain map since it is a composition of chain maps. Also, $(f_{b,a})_0$ is the identity map on R; thus $f_{b,a}$ gives a lifting of the natural surjection $R/\mathfrak{m}^b \longrightarrow R/\mathfrak{m}^a$.

Next we show that $f_{b,a}$ is a homomorphism of dg algebras. Clearly, if $b = a$ then $f_{b,a}$ is the identity map. So we may assume that $b > a$. Let α and β be homogeneous elements of $\mathbb{L}_b$. If either sits in degree 0, then $f_{b,a}(\alpha\beta) = f_{b,a}(\alpha) f_{b,a}(\beta)$ as $f_{b,a}$ is a homomorphism of R-modules. So we may assume that $\alpha \in L_{i,b}$ and $\beta \in L_{j,b}$ for some $0 \leq i, j \leq n$. Since $\pi_{b,a}$ and i_∞ are homomorphisms of dg algebras, one has

$$\begin{aligned} f_{b,a}(\alpha\beta) &= p_\infty \pi_{b,a} i_\infty(\alpha\beta) \\ &= p_\infty(\pi_{b,a} i_\infty(\alpha) \mathrm{tr}_{b,a} i_\infty(\beta)) \end{aligned}$$

We pause to compute the composition

$$\begin{aligned} \pi_{b,a} i_\infty &= \pi_{b,a}(1 + (h\delta) + \cdots + (h\delta)^{b-1}) i \\ &= ((h\delta)^{b-a} + \cdots + (h\delta)^{b-1}) i \end{aligned} \tag{4.1.1}$$

and so we have

$$f_{b,a}(\alpha\beta) = p_\infty\Big([(h\delta)^{b-a} + \cdots + (h\delta)^{b-1}] i(\alpha)\, [(h\delta)^{b-a} + \cdots + (h\delta)^{b-1}] i(\beta)\Big)$$

From (4.1.1) we also get an alternate formula for $f_{b,a}$ as follows which we use in the next part

$$\begin{aligned} f_{b,a} &= p_\infty \pi_{b,a} i_\infty \\ &= p_\infty((h\delta)^{b-a} + \cdots + (h\delta)^{b-1})i \\ &= p_\infty (h\delta)^{b-a} i \end{aligned} \tag{4.1.2}$$

where the other terms disappear as they are in the portion of the domain where p_∞ equals 0. Note that in the last line p_∞ can be replaced by p.

Next we compute

$$f_{b,a}(\alpha) f_{b,a}(\beta) = p_\infty(i_\infty(f_{b,a}(\alpha))\, i_\infty(f_{b,a}(\beta)))$$

by the definition of the product in $\mathbb{L}_a$. This is equal to $f_{b,a}(\alpha\beta)$ because

$$\begin{aligned} i_\infty f_{b,a} &= i_\infty p_\infty (h\delta)^{b-a} i \\ &= (1 + (h\delta) + \cdots + (h\delta)^{a-1}) i p [h\delta(h\delta)^{b-a-1}] i \\ &= (1 + (h\delta) + \cdots + (h\delta)^{a-1}) \sigma\kappa(-\sigma)\delta(h\delta)^{b-a-1} i \\ &= (1 + (h\delta) + \cdots + (h\delta)^{a-1})(-\sigma)\delta(h\delta)^{b-a-1} i \\ &= (1 + (h\delta) + \cdots + (h\delta)^{a-1})(h\delta)^{b-a} i \\ &= ((h\delta)^{b-a} + \cdots + (h\delta)^{b-1}) i \end{aligned} \tag{4.1.3}$$

where the first equality is by (4.1.2), the second one is by the definitions of i_∞ and p_∞, the third one is by the definitions of i, p, and h, the fourth one is because $\sigma\kappa\sigma = \sigma$ since the homotopy σ satisfies $\sigma\kappa = 1 - \kappa\sigma$ and $\sigma^2 = 0$, and the fifth is because $h = -\sigma$. Note that when we apply the definitions of i, p, and h we are using that the terms are in $\Lambda^j \otimes S$ for $j > 0$.

Last we compute the composition

$$\begin{aligned} f_{b,a} f_{c,b} &= \left(p_\infty \pi_{b,a} i_\infty\right) f_{c,b} \\ &= p_\infty \pi_{b,a}((h\delta)^{c-b} + \cdots + (h\delta)^{c-1}) i \\ &= p_\infty (h\delta)^{c-a} i = f_{c,a} \end{aligned}$$

where the second equality is by (4.1.3), the third is from the definitions of the maps, and the last is by (4.1.2). as desired. □

Note that for $a = 1$, of course, the map $p_\infty \colon \mathbb{X}_1 \to \mathbb{L}_1$ is an isomorphism of complexes, hence it is trivial that $f_{b,a}$ is a homomorphism of dg algebras since $\pi_{b,a}$ and i_∞ always are.

4.2 Remark In the proof above, the following more explicit formula for the map $f_{b,a} \colon \mathbb{L}_b \to \mathbb{L}_a$ was derived; see 4.1.2.

$$f_{b,a} = p(h\delta)^{b-a} i \tag{4.2.1}$$

As a consequence we see that for $j > 0$

$$\operatorname{im}(f_{b,a})_j \subseteq \mathfrak{m}^{b-a}(L_a)_j \tag{4.2.2}$$

since the map δ which is the same as $d = \text{kos} \otimes 1$ has image in $\mathfrak{m}$ times the next free module as kos is the differential in the Koszul complex on the variables. This may also be seen in an elementary way using long exact sequences of Tor modules.

5 Connections to homotopy transfer theorems

In Proposition 1.4, we found that a dg algebra structure transferred along certain deformation retracts as long as the homotopy satisfies the generalized Leibniz rule, defined in 1.3. In this section, we explore the effect of this condition on the A_∞-algebra structures resulting from the well-known Homotopy Transfer Theorem. The resulting structure via this theorem yields only an A_∞-structure on the retract, which often has nontrivial higher products, even when the transferred structure is associative and hence gives a dg algebra. Under the aforementioned additional hypothesis on the homotopy, we compute the higher operations that arise from the transfer and find them to vanish after all in Proposition 5.5. In Proposition 5.7, we also discuss what happens when the original complex is merely an A_∞-algebra under a similar, but much stronger hypothesis on the homotopy.

We note that most sources for the Homotopy Transfer Theorems work with dg algebras and A_∞-algebras over a field of characteristic zero. However, these are known to hold over a commutative ring R as long as one makes some freeness assumptions. For simplicity we assume in this section that R is a field of characteristic 0 (or, more generally, that we are in a setting in which the Homotopy Transfer Theorems are known to hold). However, we should point out that our transfer results Proposition 1.4 and 2.5 do not require any such hypotheses.

We begin by recalling both the definition of an A_∞-algebra and the Homotopy Transfer Theorem for a dg algebra. The concept of an A_∞-algebra was introduced by Stasheff in [51, 52] in his study of loop spaces, where the natural product is only associative up to homotopy. For some expositions of this topic, see [34–36, 40].

5.1 Definition An A_∞ *-algebra* over a ring R is a complex A of R-modules together with R-multilinear maps of degree $n-2$

$$m_n \colon A^{\otimes n} \to A$$

for each $n \geq 1$, called operations or multiplications, satisfying the following relations, called the Stasheff identities.

- The first operation is simply the differential:

$$m_1 = \partial_A.$$

- The second operation satisfies the Leibniz rule:

$$m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1).$$

- The third one verifies that m_2 is associative up to the homotopy m_3:

$$\begin{aligned} &m_2(1 \otimes m_2 - m_2 \otimes 1) \\ &\quad = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1) \end{aligned}$$

  Note that the left hand side is the obstruction to associativity for m_2 and that the right hand side is the boundary of m_3 in $\mathrm{Hom}_R(A^{\otimes 3}, A)$.
- More generally, for $n \geq 1$, we have

$$\sum_{s=1}^{n} \sum_{r,t \geq 0} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

  where the sums are taken over the values of r, s, t with $r + s + t = n$.

Note that when one applies the maps in each formula above to an element, one should use the Koszul sign rule: For graded maps f and g, one has

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$$

for homogeneous elements x and y, where $|w|$ denotes the degree of w whether it is a map or homogenous element.

Recall that we are using homological notation; in cohomological notation the degree of m_n would be $2 - n$ rather than $n - 2$. Note also that for the signs we follow the conventions in Getzler-Jones [14]; see, for example, the survey by Keller [34].

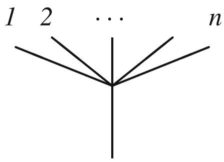
5.2 Remark Note that an A_∞-algebra whose operations m_n are zero for all $n \geq 3$ is a dg algebra where the product is given by m_2. Conversely, a dg algebra can be given the structure of an A_∞-algebra by setting $m_{\geq 3} = 0$.

However, m_1 and m_2 can usually be extended to other A_∞-algebra structures. Indeed, one can have nonzero higher operations for which the boundary of m_3 in $\mathrm{Hom}_R(A^{\otimes 3}, A)$ is equal to zero and hence A is still associative.

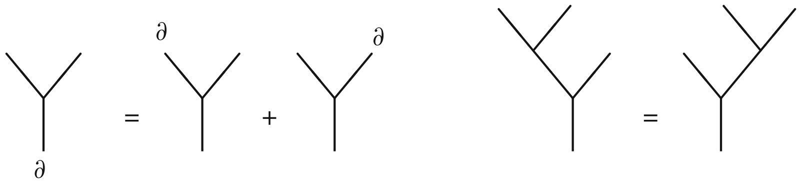
The Homotopy Transfer Theorems were first proved by Kadeishvili in [32] and [33]. We recall them in (5.4) and (5.6). For this, we follow the exposition in Vallette's survey [53]. We note that our signs are the opposite of those in his survey since his homotopy is the negative of ours (he has $1 - ip = \partial^X h + h\partial^X$, rather than $ip - 1$). This should not make a difference as the precise signs do not matter for our proofs.

5.3 Notation *We recall the planar rooted tree notation introduced by Kontsevich and Soibelman in [38] which represents these products pictorially as this will make it easier to describe the Homotopy Transfer Theorems. The notation that we use are from Vallette's survey. All the diagrams are read from the top down, that is, the inputs* *are thought of as being entered on the top and the multi-intersections correspond to the higher products* m_n *being performed. Further, wherever a letter appears in such a diagram, one applies the corresponding map at that point. Again, the sign rule described in Definition 5.1 is understood to be in effect.*

First, the higher operation m_n *is drawn as follows.*



In this notation, the properties of Leibniz rule and associativity can be drawn as follows.



To justify the first equation, note that $\partial \otimes 1$ *will produce no sign when applied to the input* $x \otimes y$ *since* $|1| = 0$*, but* $1 \otimes \partial$ *will have the sign* $(-1)^{|x|}$.

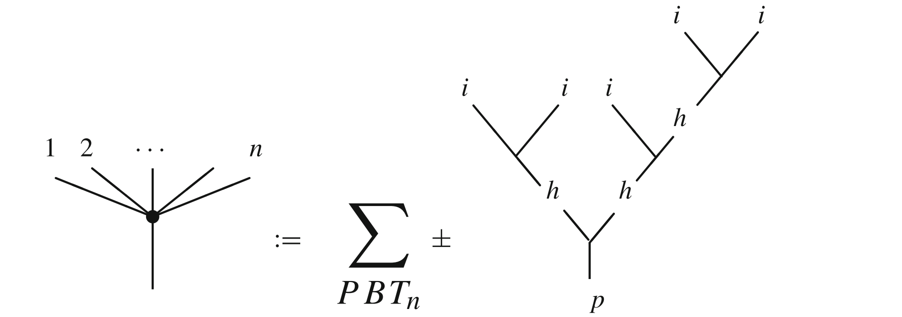
We now recall the Homotopy Transfer Theorem [32] that allows one to transfer a dg algebra structure along a deformation retract yielding an A_∞-structure on the retract. For this, we will define the transferred higher operations using the tree notation introduced above.

5.4 Homotopy Transfer Theorem for dg algebras. Let

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\rightleftarrows}} (Y, \partial^Y)$$

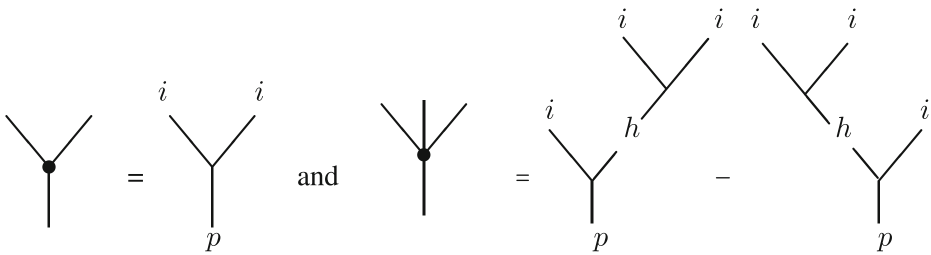
be a deformation retract where X is a dg algebra. As in Remark 5.2, one considers X an A_∞-algebra with m_1 equal to the differential, m_2 equal to the dg algebra product on X, and $m_n^X = 0$ for $n \geq 3$.

The Homotopy Transfer Theorem gives an A_∞-structure on Y as follows: First set $m_1^Y = \partial^Y$. For $n \geq 2$, the nth operation m_n^Y is defined as



where the left hand side is the notation for m_n^Y and where the sum is over PBT_n, the set of all planar binary rooted trees with n leaves, and the tree diagram pictured on the right is just a representative example of such a tree. The pattern of maps appearing on each tree is meant to indicate that every product is followed by an application of h, except for the last one, where instead p is applied. The actual signs, indicated simply as $\pm$ above, are defined in the various sources quoted, but we shall not need them for our results. Again, in applying the maps in trees, the sign rule described in Definition 5.1 is understood to be in effect.

In particular, m_2^Y and m_3^Y are given by



where the signs in the expression for m_3^Y are the opposite of those in [53] since, as we recall, his homotopy is the negative of ours.

Under the hypotheses in this paper, we can show that the transfer actually yields an A_∞-algebra with all higher operations $m_{\geq 3}$ equal to zero. Proposition 1.4 does yield a dg algebra, and so one could extend it to an A_∞-algebra by defining the higher operations $m_{\geq 3}$ equal to zero, but the Homotopy Transfer Theorem (5.4) also gives a set of higher operations, which may not be the same. Here we prove that those vanish as well.

5.5 Proposition *Let X be a dg algebra. Consider a deformation retract*

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

that satisfies the generalized Leibniz rule and $hi = 0$. Then the A_∞-algebra structure on Y obtained from the dg algebra structure on X via 5.4 has trivial higher operations, that is, $m_n^Y = 0$ for all $n \geq 3$.

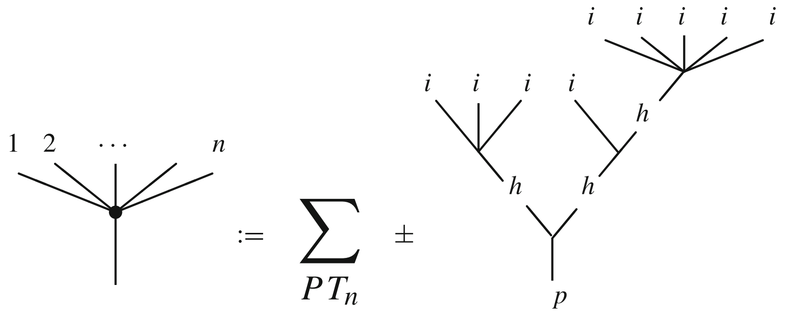
Proof Recall from 5.4 that the operations m_n^Y for $n \geq 3$ transferred from the dg algebra structure on X are signed sums of elements described by planar binary rooted trees with n leaves. The signs do not matter as we prove that every term equals zero. Indeed, each term always includes a factor of the form $h(i(\alpha)i(\beta))$ for some $\alpha, \beta \in Y$ (this will be nested inside other maps i, h and p and products from X). This vanishes as h satisfies the generalized Leibniz rule and $hi = 0$ holds. Hence these higher operations m_n all vanish. □

What if one begins with a complex X that is an A_∞-algebra rather than a dg algebra? First we recall the version of the Homotopy Transfer Theorem for this situation from [33], namely a more general version that allows one to transfer an A_∞-structure along a deformation retract yielding an A_∞-structure on the retract. We again use the tree notation introduced above.

5.6 Homotopy Transfer Theorem for A_∞ -algebras. Let

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

be a deformation retract where X is an A_∞-algebra. The Homotopy Transfer Theorem gives an A_∞-structure on Y as follows: First set $m_1^Y = \partial^Y$. For $n \geq 2$, the operation m_n^Y is defined as



where the left hand side is the notation for m_n^Y and where the sum is over PT_n, the set of all planar (not necessarily binary) rooted trees with n leaves, and the tree diagram pictured on the right is just a representative example of such a tree, where the higher products are all occurring in X. Once again, the pattern is that every such product is followed by an application of h, except for the last one, where instead p is applied. Again, the actual signs are defined in the various sources quoted, but we shall not need them for our results.

Next we impose analogous but much stronger conditions on the homotopy when X is merely an A_∞-algebra, rather than a dg algebra. We do not have any example that satisfies this condition, but include this result for completeness in case it could be useful. We say that the homotopy h on X satisfies the *generalized Leibniz rule for an*

A_∞ *-algebra* if for every $n \geq 2$ one has

$$h(m_n(a_1 \otimes \cdots \otimes a_n)) \in \sum_{i=1}^{n} m_n(X \otimes \cdots \otimes h(a_i) \otimes \cdots \otimes X)$$

where $h(a_i)$ is the ith factor and the other factors are X. Under this hypothesis, we can show that the formulas for the transferred operations via the Homotopy Transfer Theorem for A_∞-algebras (see Sect. 5.6) are much simpler than usual (they are just the ones induced by going back and forth along the homotopy equivalence).

5.7 Proposition *Let X be an A_∞-algebra with operations m_n^X for $n \geq 1$. Consider a deformation retract*

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

that satisfies the generalized Leibniz rule for an A_∞-algebra and $hi = 0$. Then the A_∞-algebra structure on Y obtained from the A_∞-algebra structure on X via Sect. 5.6 has operations given by

$$m_n^Y = p\, m_n^X (i \otimes \cdots \otimes i)$$

for all $n \geq 1$.

Proof Note that in Sect. 5.6, it follows from the construction and the properties of a deformation retract that

$$m_1^Y = \partial^Y = p\, m_1^X\, i \quad \text{and} \quad m_2^Y = p\, m_2^X (i \otimes i).$$

This covers the cases $n = 1, 2$.

Recall from Sect. 5.6 that the operations m_n^Y for $n \geq 3$ transferred from the dg algebra structure on X are signed sums of elements described by planar rooted trees with n leaves. The signs do not matter as all of the terms vanish except one. Indeed, expanding using that $hi = 0$ and the generalized Leibniz rule for an A_∞-algebra leaves only the desired term as that is the only one given by a tree with only one (higher) operation, hence simply followed by an application of p and not involving the homotopy h; this term is known to be positive. □

5.8 Remark We note that one can prove generalizations of Propositions 5.5 and 5.7 to obtain transfer results along the retract obtained after applying the Perturbation Lemma. To see this, one would modify the proofs above similarly to how we extended Proposition 1.4 to Proposition 2.5.

6 Comparison with the literature

We now give a precise comparison with the related results in the literature. Let

$$h \circlearrowright (X, \partial^X) \underset{p}{\overset{i}{\leftrightarrows}} (Y, \partial^Y)$$

be a strong deformation retract, or contraction, as defined in Definition 2.3.

Our basic transfer result, Proposition 1.4, lives in the realm of Homotopy Transfer Theorems (often abbreviated HTT), where X has a certain algebraic structure and one wishes to transfer it to obtain some sort of algebra structure on the retract Y. In Kadeishvili's well known result, [32], he transfers a dg algebra structure on X to the retract Y, potentially losing strict associativity, and obtaining an A_∞-algebra structure, which in particular is associative up to homotopy. In Proposition 1.4, we add a condition on the homotopy h, a weakening of the notion of derivation, namely the generalized Leibniz condition, that $h(\alpha\beta) \in h(\alpha)X + Xh(\beta)$ for all $\alpha, \beta \in X$, or, more generally, that $(\partial^X h + h\partial^X)(i(\alpha)i(\beta)) = 0$. In the process we show that the transfer gives an (associative) dg algebra structure on Y. In Proposition 5.5, we see that the higher multiplication maps $m_{\geq 3}$ from the transfer of Kadeishvili vanish as well. This yields that i becomes an A_∞-algebra homomorphism (and that Kadeishvili's resulting on structure on Y is A_∞-isomorphic to ours).

Our second transfer result, Proposition 2.5, lies in the realm of Homological Perturbation Theory (abbreviated HPT). We have presented our result as a transfer theorem, namely that one can transfer the structure even after using the classic Perturbation Lemma 2.4 to perturb the original contraction via δ and obtain a new contraction

$$h_\infty \circlearrowright (X, \partial^X_\infty) \underset{p_\infty}{\overset{i_\infty}{\leftrightarrows}} (Y, \partial^Y_\infty) \quad \text{where} \quad \partial^X_\infty = \partial^X + \delta$$

We show that the transfer still yields associativity if h satisfies the generalized Leibniz condition (even though h_∞ may not), and hence yields a dg algebra (with the usual assumptions that the perturbation δ is small, that is, locally nilpotent, and is multiplicative, that is, a derivation).

However, in view of our first result, we may assume that Y has *a priori* a dg algebra structure transferred from X along the initial contraction, and then consider that we are perturbing it. In this sense, our second result comes closer to some classical results, as discussed below.

First, in [16, 22], Guggenheim, Lambe and Stasheff, and Huebschmann and Kadeishvili, respectively, proved that if *both* i and p are homomorphisms of dg algebras and h is an algebra homotopy, that is,

$$\mu_X(h \otimes ip + 1 \otimes h) = h\mu_X$$

where μ_X is the product on the algebra X, then perturbing via a small multiplicative perturbation δ yields the same setting for the perturbed contraction. However, for the structures resulting from Proposition 1.4 we only get that i is a dg algebra homomorphism, and indeed p is not one in our application in Sect. 3. A slightly different version is given by Gugenheim and Lambe in [15], in which they assume that only i is a dg algebra homomorphism, but require that $p\delta h = 0$, which again we do not have for our intended application. These results also give a conclusion of a different flavor, as they imply that the transferred product after perturbation is identical to the original product, which is not the case in our application. Finally, we note that if we were to assume that p is a dg algebra homomorphism we would get a product unchanged after perturbation as well.

Second, in [46, Theorem 4.16], Real proves that a weakening of the conditions in the papers discussed above still yields a successful transfer. More precisely, he assumes that i is a dg algebra homomorphism and that

$$h\mu_X \left((h \otimes ip + 1 \otimes h)\delta^{\otimes 2}\right)^n i^{\otimes 2} = 0 \text{ for all } n \geq 1 \tag{6.0.1}$$

and obtains that the perturbed contraction has the property that i_∞ is a dg algebra homomorphism, where the new product Y can be given by either of two equal formulas

$$\begin{aligned}\alpha \cdot \beta &= p_\infty(i_\infty(\alpha)i_\infty(\beta)) \\ &= p\mu_X \sum_{n\geq 0}(-1)^n \left((h \otimes ip + 1 \otimes h)\delta^{\otimes 2}\right)^n i^{\otimes 2}\end{aligned}$$

How does this compare with our result? On the one hand, one can easily check that our condition, namely that h satisfies the generalized Leibniz rule, does indeed yield condition 6.0.1 above. On the other hand, we do not know whether his condition 6.0.1 is stronger or weaker than our condition

$$(\partial^X h + h\partial^X)\,(i(\alpha)i(\beta)) = 0 \text{ for all } \alpha, \beta \in X$$

which is what is really needed for our proof. Note that the proof in [46] seems to boil down to the single condition

$$h_\infty(i_\infty(\alpha)i_\infty(\beta)) = 0 \text{ for all } \alpha, \beta \in X$$

Although our generalized Leibniz condition is stronger than Real's condition 6.0.1, it is often simpler to verify, such as for our application. In addition, our proof is quite simple and direct.

Real's weaker hypotheses were designed for the many applications given in that paper and have also allowed for other applications in which the original perturbation theorems did not suffice, such as those in [1, 2, 31].

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Transfer Theorem, as well as the referee for pointing us to some related results, yielding the detailed discussion in Sect. 6.

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Data availability Not applicable.

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