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Resolutions of Differential Operators of Low Order for an Isolated Hypersurface Singularity

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Abstract. In this paper we develop a new approach for studying differential operators of an isolated singularity graded hypersurface ring R defining a surface in affine three-space over a field of characteristic zero. With this method, we construct an explicit minimal generating set for the modules of differential operators of order two and three, as well as their minimal free resolutions; this expands the results of Bernstein, Gel'fand, and Gel'fand and of Vigué. Our construction relies, in part, on a description of these modules that we derive in the singularity category of R. Namely, we build explicit matrix factorizations starting from that of the residue field.

1. Introduction

For an algebra R over a field k, its module of derivations and its module of Kähler differentials are fundamental classical objects that inform on the singularities of R. The derivations are part of the ring of k-linear differential operators on R, denoted $D_{R|k}$; this ring is filtered by the submodules $D^i_{R|k}$, the differential operators of order at most i, for $i \geq 0$. The ring of differential operators has proved invaluable in both algebra and geometry [9; 22; 29; 30].

Throughout this article, k is of characteristic zero. In this setting, the ring of differential operators of a polynomial ring $R = k[x_1, \dots, x_n]$ is the Weyl algebra $D_{R|k} = R\langle \partial_1, \dots, \partial_n \rangle$ where the differential operators of order i are the R-linear combinations of partial derivatives of order, in the familiar sense, at most i; see Definition 2.1.1 for a precise definition. More generally, Grothendieck showed that $D_{R|k}$ is generated as an R-algebra by the derivations under composition whenever R is smooth. In 1961, Nakai [33] conjectured that the converse should hold; this is now known as *Nakai's conjecture* and implies the well-known *Lipman–Zariski conjecture* [28]. Nakai's conjecture is still wide open outside of a handful of cases [12; 23; 32; 35; 37; 39].

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When R is singular, even in specific examples, it is extremely difficult to determine the differential operators of each order that are not compositions of lower order operators; the behavior is radically different from the smooth case. Nevertheless, studying $D_{R|k}$ when R is singular is an old and interesting problem that has seen a revival of interest lately, especially with its connections with simplicity of D-modules [5; 7; 11; 14; 15; 24; 31; 39; 40; 41]. Many approaches for studying differential operators are either algebraic or via sheaf cohomology.

The singular rings under consideration in the present article are those isolated singularity hypersurface rings considered by Vigué [41]. He established that $D_{R|k}$ is not generated by the operators of any bounded order and has no differential operators of negative degree when $R = k[x, y, z]/(f)$ is an isolated singularity hypersurface with f homogeneous of degree at least 3; this generalizes the work of Bernstein, Gel'fand, and Gel'fand on the cubic cone $k[x, y, z]/(x^3 + y^3 + z^3)$ in [8]. Moreover, Vigué showed that in each order i, the module $D^i_{R|k}$ has at least three generators that are not in the R-subalgebra of $D_{R|k}$ generated by lower order operators. These operators were identified abstractly by an analysis of sheaf cohomology.

In this paper we develop a method for gleaning new insights on these objects, including the explicit operators of a fixed order. Our approach, which is via the *homological algebra* of their resolutions, is in some ways similar in spirit to Herzog–Martsinkovsky's work on modules of derivations [21]. Surprisingly, a crucial ingredient for discovering the differential operators of low orders, which can be described as syzygies (cf. Section 2.2), was formulating a hypothesized structure for their resolutions, rather than calculating generators as a usual first step in building the resolutions. Our method is a way to work forward in the resolution to discover the generators. In fact, neither previous work in the literature nor our extensive `Macaulay2` [19] experiments identified viable candidates for a possible set of generators. We describe this method in Section 1.1.2 but omit the extensive work involved and concentrate on proving that these are indeed the generators and resolutions by localization and depth counting methods.

Indeed our methods reveal an unexpectedly beautiful structure to the resolutions: They come from matrix factorizations built in a very simple way from several copies of two different Koszul complexes on the ambient polynomial ring Q, namely $\mathrm{Kos}^Q(x, y, z)$ on the variables and $\mathrm{Kos}^Q(f_x, f_y, f_z)$ on the partial derivatives of f, connected by natural maps constructed from the Hessian matrix of second derivatives of f and its exterior powers. See Section 1.1.1 for details.

It should be stressed that the computations involved (although they look complex) stem from the familiar Euler identity, and it is really the homological underpinnings that allow for success.

The generating operators can be expressed in terms of foundational derivations. Namely, the generators can be expressed in terms of the Euler and Hamiltonian derivations

$$\mathcal{E} := x\partial_x + y\partial_y + z\partial_z, \qquad \mathcal{H}_{yz} := f_z\partial_y - f_y\partial_z,$$
$$\mathcal{H}_{zx} := f_x\partial_z - f_z\partial_x, \quad \text{and} \quad \mathcal{H}_{xy} := f_y\partial_x - f_x\partial_y.$$

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These four operators form a minimal generating set for the module of derivations; its minimal resolution is recalled in 2.4.1. Our main result is the following, contained in Theorems 4.4.3, 4.5.1, 5.4.3, and 5.5.1.

THEOREM A. *Assume that $R = k[x, y, z]/(f)$ is an isolated hypersurface singularity where f is homogeneous of degree $d \geq 3$ and k is a field of characteristic zero. A minimal set of generators for $D^2_{R|k}$ is given by*

$$\{1, \mathcal{E}, \mathcal{H}_{yz}, \mathcal{H}_{zx}, \mathcal{H}_{xy}, \mathcal{E}^2, \mathcal{E}\mathcal{H}_{yz}, \mathcal{E}\mathcal{H}_{zx}, \mathcal{E}\mathcal{H}_{xy}, \mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z\},$$

with

$$\mathcal{A}_x = \frac{1}{x}\left[\mathcal{H}^2_{yz} + \frac{1}{(d-1)^2}\Delta_{xx}\mathcal{E}^2 + \frac{d-2}{(d-1)^2}\Delta_{xx}\mathcal{E}\right],$$

where Δ_{xx} is the 2×2-minor obtained by deleting the 1st row and 1st column of the Hessian matrix of f, and similar formulas hold for $\mathcal{A}_y$, $\mathcal{A}_z$ as described in Theorem 4.5.1.

A minimal set of generators for $D^3_{R|k}$ is the union of the generating set above with

$$\{\mathcal{E}^3, \mathcal{E}^2\mathcal{H}_{yz}, \mathcal{E}^2\mathcal{H}_{zx}, \mathcal{E}^2\mathcal{H}_{xy}, \mathcal{E}\mathcal{A}_x, \mathcal{E}\mathcal{A}_y, \mathcal{E}\mathcal{A}_z, \mathcal{Z}_x, \mathcal{Z}_y, \mathcal{Z}_z\},$$

where $\mathcal{Z}_x$, $\mathcal{Z}_y$, and $\mathcal{Z}_z$ are defined in Theorem 5.5.1.

Furthermore, the augmented minimal R-free resolutions of $D^2_{R|k}$ and $D^3_{R|k}$ have the forms

$$\cdots \xrightarrow{\psi} R^{11} \xrightarrow{\varphi} R^{11} \xrightarrow{\psi} R^{11} \xrightarrow{\varphi} R^{11} \xrightarrow{\left[\begin{smallmatrix}\psi\\0\end{smallmatrix}\right]} R^{12} \to D^2_{R|k} \to 0$$

and

$$\cdots \xrightarrow{\psi'} R^{21} \xrightarrow{\varphi'} R^{21} \xrightarrow{\psi'} R^{21} \xrightarrow{\varphi'} R^{21} \xrightarrow{\left[\begin{smallmatrix}\psi'\\0\end{smallmatrix}\right]} R^{22} \to D^3_{R|k} \to 0,$$

where the differentials are described explicitly in Theorems 4.4.3 and 5.4.3, respectively.

The resolutions in Theorem A are minimal graded free resolutions, where R is viewed as a standard graded k-algebra. We record the graded Betti numbers of $D^2_{R|k}$ and $D^3_{R|k}$ in Corollaries 4.4.5 and 5.4.6, respectively.

In the process of proving Theorem A we also establish, in Corollaries 4.4.4 and 5.4.4, the following result.

COROLLARY B. *For $i = 2, 3$, there are short exact sequences of R-modules*

$$0 \to D^{i-1}_{R|k} \to D^i_{R|k} \to \operatorname{Hom}_R(\operatorname{Sym}^i(\Omega_{R|k}), R) \to 0,$$

where $D^{i-1}_{R|k} \to D^i_{R|k}$ is the inclusion; in particular, $D^i_{R|k}/D^{i-1}_{R|k}$ is maximal Cohen–Macaulay.

We note that there are always left exact sequences of the form above; surjectivity on the right is the novel content. Some of the results above are naturally stated in terms of the singularity category $\mathsf{D}_{\mathsf{sg}}(R)$ of R introduced by Buchweitz [13];

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see 2.3.8. The singularity category is a triangulated category that records the tails of resolutions over R. Since R is an isolated singularity, it is well known that every object in $\mathsf{D}_{\mathsf{sg}}(R)$ can be built from k in finitely many steps using shifts, mapping cones, and retracts. The number of mapping cones $(+1)$ it takes to build M from k is called the level of M with respect to k, denoted $\mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)} M$; this invariant was introduced in [3]. Since R is a hypersurface, the tail of any R-resolution is necessarily given by a matrix factorization, in the sense of [17], of f over $k[x, y, z]$. In constructing the resolutions of $\mathrm{Hom}_R(\mathrm{Sym}^i(\Omega_{R|k}), R)$, for $i = 2, 3$, we describe a procedure to build its corresponding matrix factorization starting from the matrix factorization of k; see Sections 4.2 and 5.2. As a consequence, we obtain the following; see Corollaries 4.4.6 and 5.4.7.

Corollary C. *For $i = 2, 3$,*

$$\mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(\mathrm{Hom}_R(\mathrm{Sym}^i(\Omega_{R|k}), R)) \le i \quad \text{and} \quad \mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(D^i_{R|k}) \le i(i+1)/2.$$

1.1. Methods

Our general approach to Theorem A is as follows: Focusing on the order filtration on the ring of differential operators

$$D^0_{R|k} \subseteq D^1_{R|k} \subseteq D^2_{R|k} \subseteq \cdots,$$

we proceed inductively. For $i > 1$, we first find the minimal R-free resolution and a minimal set of generators of $\mathrm{Hom}_R(\mathrm{Sym}^i(\Omega_{R|k}), R)$, the R-linear dual of the ith symmetric power of the modules of Kähler differentials. Next, we build a degree one chain map from this resolution to our, inductively constructed, resolution of $D^{i-1}_{R|k}$. We then show its mapping cone is a minimal resolution of $D^i_{R|k}$, which yields Corollary B.

We employ two key techniques in building the resolution of $\mathrm{Hom}_R(\mathrm{Sym}^i(\Omega_{R|k}), R)$.

1.1.1. Matrix Factorizations from Diagrams of Koszul Complexes. Here we describe the structure of the matrix factorizations associated with the minimal resolutions of each D^n/D^{n-1} for various orders n. They are obtained from the $\mathbb{Z}/2$-graded totalizations of the diagrams below by adding in nullhomotopies for multiplication by f. Equivalently to find homotopies, we instead describe a structure of dg module over the Koszul complex $\mathrm{Kos}(f)$ on these totalized complexes, as described in 2.3.5 and 2.3.6. The diagram for D^1/D^0 is shown in Figure 1.

The diagram for D^2/D^1 is shown in Figure 2. Notice that it is the diagram for D^1/D^0 with a new first row glued in. The new maps are described in Section 3.

The diagram for D^3/D^2 (see Figure 3) is then obtained by gluing a new first row into the diagram for D^2/D^1. The diagonal maps are obtained from the Koszul

$$\mathrm{Kos}(x, y, z) \qquad\qquad Q \xrightarrow{\partial_3} Q^3 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q\,.$$

Figure 1 Diagram for matrix factorization of D^1/D^0.

$$
\begin{tikzcd}
\mathrm{Kos}(f_x, f_y, f_z) \arrow[d, "\bigwedge \mathrm{Hessian}(f)"] \\
\mathrm{Kos}(x, y, z)
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "D_3"] \arrow[d, "\alpha_3"] & Q^3 \arrow[r, "D_2"] \arrow[d, "\alpha_2"] & Q^3 \arrow[r, "D_1"] \arrow[d, "\alpha_1"] & Q \arrow[d, equal, "\alpha_0"] \\
Q \arrow[r, "\partial_3"] & Q^3 \arrow[r, "\partial_2"] & Q^3 \arrow[r, "\partial_1"] & Q
\end{tikzcd}
$$

Figure 2 Diagram for matrix factorization of D^2/D^1.

$$
\begin{tikzcd}
\mathrm{Kos}(x, y, z) \arrow[d, "(\bigwedge \mathrm{Hessian}(f))^*"] \arrow[dd, bend right, dashed] \\
\mathrm{Kos}(f_x, f_y, f_z) \arrow[d, "\bigwedge \mathrm{Hessian}(f)"] \\
\mathrm{Kos}(x, y, z)
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "\partial_3"] \arrow[d, equal, "\alpha_0"] & Q^3 \arrow[r, "\partial_2"] \arrow[d, "-\alpha_1"] \arrow[ddl] & Q^3 \arrow[r, "\partial_1"] \arrow[d, "\alpha_2"] \arrow[ddl] & Q \arrow[d, "-\alpha_3"] \arrow[ddl] \\
Q \arrow[r, "D_3"] \arrow[d, "\alpha_3"] & Q^3 \arrow[r, "D_2"] \arrow[d, "\alpha_2"] & Q^3 \arrow[r, "D_1"] \arrow[d, "\alpha_1"] & Q \arrow[d, equal, "\alpha_0"] \\
Q \arrow[r, "\partial_3"] & Q^3 \arrow[r, "\partial_2"] & Q^3 \arrow[r, "\partial_1"] & Q
\end{tikzcd}
$$

Figure 3 Diagram for matrix factorization of D^3/D^2.

$$
\begin{tikzcd}
\mathrm{Kos}(f_x, f_y, f_z) \arrow[d, "\bigwedge \mathrm{Hessian}(f)"] \arrow[dd, bend right, dashed] \arrow[ddd, bend right=60, dashed, "?"'] \\
\mathrm{Kos}(x, y, z) \arrow[d, "(\bigwedge \mathrm{Hessian}(f))^*"] \arrow[dd, bend right, dashed] \\
\mathrm{Kos}(f_x, f_y, f_z) \arrow[d, "\bigwedge \mathrm{Hessian}(f)"] \\
\mathrm{Kos}(x, y, z)
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "D_3"] \arrow[d, "\alpha_3"] \arrow[ddr] & Q^3 \arrow[r, "D_2"] \arrow[d, "\alpha_2"] \arrow[ddr] & Q^3 \arrow[r, "D_1"] \arrow[d, "\alpha_1"] \arrow[ddr] & Q \arrow[d, equal, "\alpha_0"] \\
Q \arrow[r, "\partial_3"] \arrow[d, equal, "\alpha_0"] & Q^3 \arrow[r, "\partial_2"] \arrow[d, "-\alpha_1"] \arrow[ddl] & Q^3 \arrow[r, "\partial_1"] \arrow[d, "\alpha_2"] \arrow[ddl] & Q \arrow[d, "-\alpha_3"] \arrow[ddl] \\
Q \arrow[r, "D_3"] \arrow[d, "\alpha_3"] & Q^3 \arrow[r, "D_2"] \arrow[d, "\alpha_2"] & Q^3 \arrow[r, "D_1"] \arrow[d, "\alpha_1"] & Q \arrow[d, equal, "\alpha_0"] \\
Q \arrow[r, "\partial_3"] & Q^3 \arrow[r, "\partial_2"] & Q^3 \arrow[r, "\partial_1"] & Q
\end{tikzcd}
$$

Figure 4 Conjectured diagram for matrix factorization of D^4/D^3.

complex on the partial derivatives of the Hessian determinant of f; these are described as σ_i for $i = 1, 2, 3$ in Section 3.

For order 4 we conjecture that the diagram on the right in Figure 4, together with additional maps from the top row to the bottom that are as yet undetermined (the orange maps indicated on the left), supports the matrix factorization for the resolution. The Betti numbers and the displayed maps certainly agree with our computations on `Macaulay2`. Similarly, we believe that analogous diagrams with n rows support the factorizations for higher orders n.

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1.1.2. Working Forward from Matrix Factorizations to Relations and Generators. We briefly describe our method for discovering the generators of D^i/D^{i-1} for $i = 2, 3$. However, the details of these calculations do not appear in this paper; instead we concentrate only on proving exactness of the augmented complexes that arise from the method.

We realize $\mathrm{Hom}_R(\mathrm{Sym}^i(\Omega_{R|k}), R)$ as the kernel of a matrix $J_{i,i-1}$ obtained from the Jacobian matrix; see Remark 2.2.3. Its minimal resolution fits into a diagram as follows:

$$\begin{tikzcd} F = \cdots \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[rr, "\partial_0"] \arrow[dr] & & F_{-1} \arrow[r, "\partial_{-1}=J_{i,i-1}"] & F_{-2} \\ & & & \ker J_{i,i-1} \arrow[ur] & & \end{tikzcd} \tag{1}$$

We conjecture that the form of $F_{\geq 0}$ is given by the complex coming from the matrix factorization described in Section 1.1.1. Then we work forward to discover a suitable map ∂_0 that completes the diagram. This gives the generators of D^i/D^{i-1}.

To discover the map ∂_0, first note that the dual of the complex F in (1) is a complex. Therefore, the image of ∂_0^* is contained in the kernel of ∂_1^*. But the dual $F_{\geq 0}^*$ of the 2-periodic portion of the complex $F_{\geq 0}$ is exact since it arises from the transposed matrix factorization. Therefore, the kernel of ∂_1^* is the image of ∂_0^*. Choosing bases, this means that the rows of ∂_0 are combinations of the rows of ∂_2. We construct the rows of ∂_0 from suitable R-linear combinations of the rows of ∂_2 so that the columns of ∂_0 are in the kernel of $J_{i,i-1}$ and match the expected degrees of the operators.

1.2. Convention

Before sketching an outline for the paper, we wish to highlight an important bit of notation here. In the body of the paper, given a differential operator $\mathcal{F}$ in $D^i_{R|k}$, we will write its images in $D^i_{R|k}/D^{i-1}_{R|k}$ and $\mathrm{Hom}_R(\mathrm{Sym}^i(\Omega_{R|k}), R)$ in roman font F. In 2.2.4, we also identify $D^i_{R|k}/D^{i-1}_{R|k}$ as a submodule of a free module, and so F will also be identified with its corresponding column vector. The basis of the column space is indexed by the divided power partial derivatives of order exactly i (listed in lexicographic order); see 2.1.4 for further details. For example, EH_{xy} denotes the image of the differential operator $\mathcal{E} \circ \mathcal{H}_{xy}$ in $D^2_{R|k}/D^1_{R|k}$:

$$EH_{xy} = xf_y\partial_x^2 + (yf_y - xf_x)\partial_x\partial_y + zf_y\partial_x\partial_z - yf_x\partial_y^2 - zf_x\partial_y\partial_z \mod D^1_{R|k},$$

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as well as the column vector

$$EH_{xy} = \begin{bmatrix} 2xf_y \\ yf_y - xf_x \\ zf_y \\ -2yf_x \\ -zf_x \\ 0 \end{bmatrix}.$$

It turns out that working in the quotients $D^i_{R|k}/D^{i-1}_{R|k}$, and this slight abuse of notation, simplifies several calculations. See Remark 4.1.1, 2.2.4, Section 4.5, and Section 5.5 for more details on these notational conventions.

1.3. Outline

Finally, we end the introduction with a brief outline of the paper. Section 2 contains background information on differential operators, homological constructions (such as matrix factorizations), and a minimal free resolution of $D^1_{R|k}$. Section 3 contains a glossary of matrices that are used throughout the remainder of the document.

Sections 4 and 5 are the central parts of the paper. We briefly describe the former, and the latter follows the identical outline: In Section 4.1, a set of generators for $\operatorname{Hom}_R(\operatorname{Sym}^2(\Omega_{R|k}), R)$ is proposed; a matrix factorization corresponding to this module is introduced in Section 4.2; in Section 4.3, the pieces from the previous subsections are stitched together to identify a minimal free resolution of $\operatorname{Hom}_R(\operatorname{Sym}^2(\Omega_{R|k}), R)$; it is in Section 4.4 that the degree one map from this resolution to the resolution of $D^1_{R|k}$ is constructed; as a consequence, the minimal R-free resolution of $D^2_{R|k}$ is obtained in Theorem 4.4.3; finally, in Section 4.5, we identify a minimal set of generators for $D^2_{R|k}$ written as honest differential operators in $D_{R|k}$.

The article is concluded by several appendices containing formulas and calculations that are utilized in the arguments discussed before.

2. Background

In this paper, we work in the following setting unless specified otherwise.

NOTATION 2.0.1. Throughout Q is the polynomial ring $k[x, y, z]$ over a field k, let f be a homogeneous polynomial in Q of a fixed degree $d \geq 3$, and set

$$R := Q/(f).$$

We assume that the characteristic of k is zero. Since $f \in (x, y, z)^3$, it follows that R is a singular hypersurface ring. Finally, we make the assumption that R is an isolated singularity, meaning—in the usual sense—that $R_{\mathfrak{p}}$ is a regular local ring for all primes $\mathfrak{p}$ different from the maximal ideal $\mathfrak{m} = (x, y, z)R$ of R. This implies, since R is therefore a normal graded ring, that R is a domain.

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2.1. Generalities on Differential Operators

We recall some basics on Grothendieck's notion of differential operators; see [20]. We do not restrict ourselves to the setting of Notation 2.0.1 in this subsection.

DEFINITION 2.1.1. Let $\phi\colon k \to A$ be a homomorphism of commutative rings. The module of k-linear *differential operators* on A of order at most i, denoted $D^i_{A|k}$, is defined inductively as follows:

- $D^0_{A|k} = \operatorname{Hom}_A(A, A) \cong A$;
- $D^i_{A|k} = \{\delta \in \operatorname{Hom}_k(A, A) \mid \delta \circ \mu - \mu \circ \delta \in D^{i-1}_{A|k} \text{ for all } \mu \in D^0_{A|k}\}$ for all $i > 0$.

Each $D^i_{A|k}$ is an A-module where A acts via postmultiplication. The union of the modules $D^i_{A|k}$ over all natural numbers i is the ring of k-linear differential operators on A, denoted $D_{A|k}$.

2.1.2. Given two differential operators $\alpha \in D^i_{A|k}$ and $\beta \in D^j_{A|k}$, the composition $\alpha \circ \beta$ is an element of $D^{i+j}_{A|k}$; the union $D_{A|k}$ obtains the structure of a noncommutative ring with composition as the multiplication operation. We write compositions with product notation in the sequel. Because multiplication is compatible with the order filtration, $D_{A|k}$ equipped with the order filtration is a filtered ring. In general, the associated graded ring with respect to this filtration is a commutative ring, which we denote as $\operatorname{gr}(D_{A|k})$. We employ the tautological exact sequences

$$0 \to D^{i-1}_{A|k} \to D^i_{A|k} \to \operatorname{gr}(D_{A|k})_i \to 0$$

in what follows.

2.1.3. In general, the inclusion $D^{i-1}_{A|k} \subseteq D^i_{A|k}$ does not split as A-modules. However, the inclusion $D^0_{A|k} \subseteq D^i_{A|k}$ has an A-linear left inverse given by evaluation at $1 \in A$; the kernel of this evaluation map is called the module of ith order derivations. For $i = 1$, this is the usual module of derivations $\operatorname{Der}_{A|k}$.

2.1.4. If k is a commutative ring and $P = k[x_1, \dots, x_n]$ is a polynomial ring over k, then $D^i_{P|k}$ is a free P-module with basis

$$\{\partial^{(a_1)}_{x_1} \cdots \partial^{(a_n)}_{x_n} \mid a_1 + \cdots + a_n \leq i\},$$

where $\partial^{(a_i)}_{x_i}$ is the k-linear map determined by

$$\partial^{(a_i)}_{x_i}(x_1^{b_1} \cdots x_n^{b_n}) = \binom{b_i}{a_i} x_1^{b_1} \cdots x_i^{b_i - a_i} \cdots x_n^{b_n};$$

if $\mathbb{Q} \subseteq k$, then we can identify $\partial^{(a_i)}_{x_i}$ with $\frac{1}{a_i!}$ times the a_i-iterate of the derivation $\frac{d}{dx_i}$. By convention, we set $\partial^{(a_i)}_{x_i}$ to be the zero map if $a_i < 0$.

We also have that $\operatorname{gr}(D_{P|k})_i$ is a free P-module with basis

$$\{\overline{\partial^{(a_1)}_{x_1} \cdots \partial^{(a_n)}_{x_n}} \mid a_1 + \cdots + a_n = i\},$$

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where overline denotes congruence class modulo $D^{i-1}_{P|k}$. The ring $\operatorname{gr}(D_{P|k})$ is a divided power algebra over B in n divided power variables generated by the classes of the derivations $\frac{d}{dx_i}$; if $\mathbb{Q} \subseteq k$, it is isomorphic to a polynomial ring on these variables or, equivalently, the symmetric algebra on $\operatorname{Der}_{P|k}$.

2.1.5. Let A be a commutative ring, $P = k[x_1, \dots, x_n]$ be a polynomial ring over k, and set $A = P/J$ with $J = (f_1, \dots, f_m)$ an ideal of P. There are isomorphisms

$$D^i_{A|k} \cong \frac{\{\delta \in D^i_{P|k} \mid \delta(J) \subseteq J\}}{J D^i_{P|k}}.$$

To identify these elements in the numerator, we introduce the following notation: for a differential operator δ and sequence of elements $\mathbf{g} = (g_1, \dots, g_m)$, we set

$$[\delta, \mathbf{g}] := [\cdots[[\delta, g_1], g_2], \dots, g_m].$$

Note that $[\delta, \mathbf{g}]$ is unaffected by permuting the elements of $\mathbf{g}$. If $\mathbf{g}$ is the empty sequence, we take $[\delta, \mathbf{g}] := \delta$.

LEMMA 2.1.6. *In the notation of 2.1.5, an element $\delta \in D^i_{P|k}$ maps J into J if and only if $[\delta, \mathbf{g}](f_j) \in J$ for all $j = 1, \dots, m$ and all tuples $\mathbf{g} = (g_1, \dots, g_\ell)$ such that $0 \le \ell < i$ and $g_t \in \{x_1, \dots, x_n\}$ for all $t = 1, \dots, \ell$.*

Proof. The forward implication is straightforward. For the reverse implication, it suffices to show that if $\delta \in D^i_{P|k}$ satisfies the condition in the statement, then

$$\delta(x_1^{a_1} \cdots x_n^{a_n} f_j) \in J \quad \text{for all } a_1, \dots, a_n \ge 0.$$

We proceed by induction on i. For the case $i = 0$, an operator of order zero stabilizes every ideal, so the claim is clear. For the inductive step, we proceed by induction on $\ell = a_1 + \cdots + a_n$. For the case $\ell = 0$, this follows by the hypothesis $[\delta, \emptyset](f_j) \in J$. For $\ell > 0$, we have $a_k > 0$ for some k; without loss of generality, we can take $k = 1$. For any sequence $\mathbf{g}$ of variables of length at most $i - 1$, let $\mathbf{g}'$ be the sequence obtained from $\mathbf{g}$ by appending x_1. Then $[\delta, x_1] \in D^{i-1}_{B|A}$ and $[[\delta, x_1], \mathbf{g}](f_j) = [\delta, \mathbf{g}'](f_j) \in J$ by hypothesis, so by the induction hypothesis on i, $[\delta, x_1]$ maps J into J. By the induction hypothesis on ℓ, $\delta(x_1^{a_1-1} \cdots x_n^{a_n} f_j) \in J$. Thus,

$$\begin{aligned} \delta(x_1^{a_1} \cdots x_n^{a_n} f_j) &= \delta x_1 (x_1^{a_1-1} \cdots x_n^{a_n} f_j) \\ &= x_1 \delta(x_1^{a_1-1} \cdots x_n^{a_n} f_j) + [\delta, x_1](x_1^{a_1-1} \cdots x_n^{a_n} f_j) \in J, \end{aligned}$$

as required. ☐

We may identify $D^i_{A|k}$ with a submodule of the free A-module $A \otimes_P D^i_{P|k}$ via the isomorphism above, where an element of $A \otimes_P D^i_{P|k}$ corresponds to an operator taken modulo J. Under this identification, we have the following.

PROPOSITION 2.1.7. *In the notation of 2.1.5, an element of the free A-module $A \otimes_P D^i_{P|k}$ corresponds to an element of $D^i_{A|k}$ if and only if it is in the kernel of the homomorphism of free A-modules $\phi\colon A \otimes_P D^i_{P|k} \to F_{<i}$ given by*

$$\phi_i(\partial_{x_1}^{(a_1)} \cdots \partial_{x_n}^{(a_n)}) = \sum_{\substack{0 \le b_1 + \cdots + b_n < i \\ 1 \le j \le m}} \partial_{x_1}^{(a_1 - b_1)} \cdots \partial_{x_n}^{(a_n - b_n)}(f_j) e_{(b_1, \ldots, b_n), j},$$

where $F_{<i}$ is the free A-module with basis

$$\{e_{(b_1,\ldots,b_n),j} \mid 0 \le b_1 + \cdots + b_n < i, 1 \le j \le m\}.$$

Proof. This follows from Lemma 2.1.6 plus the observation that, for the sequence $\mathbf{g}$ in which each x_i appears b_i times, we have $[\partial_{x_1}^{(a_1)} \cdots \partial_{x_n}^{(a_n)}, \mathbf{g}] = \partial_{x_1}^{(a_1 - b_1)} \cdots \partial_{x_n}^{(a_n - b_n)}$. $\square$

We note that this description also follows from the presentations of modules of principal parts in [6] and [11].

2.1.8. The maps ϕ_i from Proposition 2.1.7 fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \otimes_P D^{i-1}_{P|k} \arrow[r] \arrow[d, "\phi_{i-1}"] & A \otimes_P D^i_{P|k} \arrow[r] \arrow[d, "\phi_i"] & A \otimes_P \operatorname{gr}(D^i_{P|k})_i \arrow[r] \arrow[d, "\psi_i"] & 0 \\ 0 \arrow[r] & F_{<i-1} \arrow[r] & F_{<i} \arrow[r] & F_{=i-1} \arrow[r] & 0 \end{tikzcd} \tag{2}$$

with exact rows, where $F_{=i-1}$ is the free A-module with basis

$$\{e_{(b_1,\ldots,b_n),j} \mid b_1 + \cdots + b_n = i - 1, 1 \le j \le m\}$$

and

$$\psi_i(\overline{\partial_{x_1}^{(a_1)} \cdots \partial_{x_n}^{(a_n)}}) = \sum_{\substack{b_1 + \cdots + b_n = i-1 \\ 1 \le j \le m}} \partial_{x_1}^{(a_1 - b_1)} \cdots \partial_{x_n}^{(a_n - b_n)}(f_j) e_{(b_1, \ldots, b_n), j}.$$

Note that $\partial_{x_1}^{(a_1 - b_1)} \cdots \partial_{x_n}^{(a_n - b_n)}(f_j)$ is nonzero only when, for some $\ell \in \{1, \ldots, n\}$, we have $a_\ell = b_\ell + 1$ and $a_m = b_m$ for $m \neq \ell$; for this tuple $b_1, \ldots, b_n$, we have

$$\partial_{x_1}^{(a_1 - b_1)} \cdots \partial_{x_n}^{(a_n - b_n)}(f_j) = \sum_j \partial_{x_\ell}(f_j) e_{(b_1, \ldots, b_n), j}.$$

From this description, one sees that ψ_i can be identified with the ith symmetric power of the map ψ_1, which is concretely given by

$$\psi_1(\partial_{x_i}) = \sum_j \partial_{x_i}(f_j) e_{(0,\ldots,0),j};$$

that is, ψ_1 is the transpose of the Jacobian morphism that presents the module of Kahler differentials.

2.1.9. By considering kernels in (2), we obtain a left exact sequence

$$0 \to D^{i-1}_{A|k} \to D^{i}_{A|k} \to \ker(\psi_i) \tag{3}$$

induced by the natural inclusion and projection maps. In general, this does not need to be a short exact sequence—that is, the last map need not be surjective—however, when $i = 1$, this is the case, and the kernel of ψ_1 is the usual module of derivations.

From (3), and the definition, we obtain injective morphisms

$$\operatorname{gr}(D_{A|k})_i \to \ker(\psi_i), \tag{4}$$

which are isomorphisms when (3) forms a short exact sequence.

For an A-linear derivation $\eta \in D^1_{A|k}$ and $\mu \in D^i_{A|k}$, we have $\eta \circ \mu \in D^{i+1}_{A|k}$. Thus, composition with η gives a well-defined map:

$$\operatorname{gr}(D_{A|k})_i \xrightarrow{\eta} \operatorname{gr}(D_{A|k})_{i+1}. \tag{5}$$

If the morphism (4) is an isomorphism for some i, then the morphism (5) of composition by η yields a morphism

$$\ker(\psi_i) \xrightarrow{\eta} \ker(\psi_{i+1}). \tag{6}$$

2.2. *Concrete Considerations on Differential Operators*

We now realize the objects and maps discussed in Section 2.1 concretely in the setting of Notation 2.0.1.

2.2.1. We have that $D^i_{Q|k}$ is a free Q-module with basis

$$\{\partial_x^{(a)}\partial_y^{(b)}\partial_z^{(c)} \mid 0 \le a+b+c \le i\}.$$

We order our bases with the graded lexicographic order throughout. In particular, the ordered basis for $F_{\le 3}$ we use throughout is

$$\begin{aligned}
&\partial_x^{(3)}, \partial_x^{(2)}\partial_y, \partial_x^{(2)}\partial_z, \partial_x\partial_y^{(2)}, \partial_x\partial_y\partial_z, \partial_x\partial_z^{(2)}, \partial_y^{(3)}, \partial_y^{(2)}\partial_z, \partial_y\partial_z^{(2)}, \partial_z^{(3)},\\
&\partial_x^{(2)}, \partial_x\partial_y, \partial_x\partial_z, \partial_y^{(2)}, \partial_y\partial_z, \partial_z^{(2)},\\
&\partial_x, \partial_y, \partial_z,\\
&1;
\end{aligned}$$

likewise, the first row above gives the ordered basis of $\operatorname{gr}(D_{Q|k})_3$, the last three rows give the ordered basis of $D^2_{Q|k}$, and so on. We order the bases of $F_{<i}$ similarly.

2.2.2. We now write the matrices corresponding to the maps ϕ_i and ψ_i from Proposition 2.1.7 and 2.1.8 with respect to the bases in 2.2.1. First, the matrices for ψ_1, ψ_2, and ψ_3, respectively, are

$$J_{1,0} := \begin{array}{c c} & \begin{array}{ccc} \partial_x & \partial_y & \partial_z \end{array} \\ e_1 & \left[\begin{array}{ccc} f_x & f_y & f_z \end{array}\right] \end{array},$$

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$$J_{2,1} \coloneqq \begin{array}{cc} & \begin{array}{cccccc} \partial_x^{(2)} & \partial_x\partial_y & \partial_x\partial_z & \partial_y^{(2)} & \partial_y\partial_z & \partial_z^{(2)} \end{array} \\ \begin{array}{c} e_x \\ e_y \\ e_z \end{array} & \begin{bmatrix} f_x & f_y & f_z & 0 & 0 & 0 \\ 0 & f_x & 0 & f_y & f_z & 0 \\ 0 & 0 & f_x & 0 & f_y & f_z \end{bmatrix} \end{array},$$

$$J_{3,2} \coloneqq \begin{array}{cc} & \begin{array}{cccccccccc} \partial_x^{(3)} & \partial_x^{(2)}\partial_y & \partial_x^{(2)}\partial_z & \partial_x\partial_y^{(2)} & \partial_x\partial_y\partial_z & \partial_x\partial_z^{(2)} & \partial_y^{(3)} & \partial_y^{(2)}\partial_z & \partial_y\partial_z^{(2)} & \partial_z^{(3)} \end{array} \\ \begin{array}{c} e_{xx} \\ e_{xy} \\ e_{xz} \\ e_{yy} \\ e_{yz} \\ e_{zz} \end{array} & \begin{bmatrix} f_x & f_y & f_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_x & 0 & f_y & f_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_x & 0 & f_y & f_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_x & 0 & 0 & f_y & f_z & 0 & 0 \\ 0 & 0 & 0 & 0 & f_x & 0 & 0 & f_y & f_z & 0 \\ 0 & 0 & 0 & 0 & 0 & f_x & 0 & 0 & f_y & f_z \end{bmatrix} \end{array}.$$

Above, we have rewritten the target bases as monomials; for example, e_{xy} corresponds to $e_{(1,1,0),1}$ in the notation of 2.1.8.

In particular, for $i = 2, 3$, the left exact sequences (3) take the form

$$0 \to D^{i-1}_{R|k} \to D^{i}_{R|k} \xrightarrow{\nu_i} \ker(J_{i,i-1}), \tag{7}$$

inducing injective maps as in (4),

$$\operatorname{gr}(D_{R|k})_i \to \ker(J_{i,i-1}). \tag{8}$$

Under these identifications, the maps

$$D^{i}_{R|k} \to \ker(J_{i,i-1}) \quad \text{and} \quad \operatorname{gr}(D_{R|k})_i \to \ker(J_{i,i-1})$$

as in (3) are simply given by restricting coordinates.

We describe the matrices for ϕ_i as block matrices via grouping the basis elements in the source and the target by the sums of their indices. Block components include the following:

$$J_{2,0} \coloneqq \begin{array}{cc} & \begin{array}{cccccc} \partial_x^{(2)} & \partial_x\partial_y & \partial_x\partial_z & \partial_y^{(2)} & \partial_y\partial_z & \partial_z^{(2)} \end{array} \\ e_1 & \begin{bmatrix} \frac{1}{2}f_{xx} & f_{xy} & f_{xz} & \frac{1}{2}f_{yy} & f_{yz} & \frac{1}{2}f_{zz} \end{bmatrix} \end{array},$$

$$J_{3,0} \coloneqq \begin{array}{cc} & \begin{array}{cccccccccc} \partial_x^{(3)} & \partial_x^{(2)}\partial_y & \partial_x^{(2)}\partial_z & \partial_x\partial_y^{(2)} & \partial_x\partial_y\partial_z & \partial_x\partial_z^{(2)} & \partial_y^{(3)} & \partial_y^{(2)}\partial_z & \partial_y\partial_z^{(2)} & \partial_z^{(3)} \end{array} \\ e_1 & \begin{bmatrix} \frac{1}{6}f_{xxx} & \frac{1}{2}f_{xxy} & \frac{1}{2}f_{xxz} & \frac{1}{2}f_{xyy} & f_{xyz} & \frac{1}{2}f_{xzz} & \frac{1}{6}f_{yyy} & \frac{1}{2}f_{yyz} & \frac{1}{2}f_{yzz} & \frac{1}{6}f_{zzz} \end{bmatrix} \end{array},$$

$$J_{3,1} \coloneqq \begin{array}{cc} & \begin{array}{cccccccccc} \partial_x^{(3)} & \partial_x^{(2)}\partial_y & \partial_x^{(2)}\partial_z & \partial_x\partial_y^{(2)} & \partial_x\partial_y\partial_z & \partial_x\partial_z^{(2)} & \partial_y^{(3)} & \partial_y^{(2)}\partial_z & \partial_y\partial_z^{(2)} & \partial_z^{(3)} \end{array} \\ \begin{array}{c} e_x \\ e_y \\ e_z \end{array} & \begin{bmatrix} \frac{1}{2}f_{xx} & f_{xy} & f_{xz} & \frac{1}{2}f_{yy} & f_{yz} & \frac{1}{2}f_{zz} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}f_{xx} & 0 & f_{xy} & f_{xz} & 0 & \frac{1}{2}f_{yy} & f_{yz} & \frac{1}{2}f_{zz} & 0 \\ 0 & 0 & \frac{1}{2}f_{xx} & 0 & f_{xy} & f_{xz} & 0 & \frac{1}{2}f_{yy} & f_{yz} & \frac{1}{2}f_{zz} \end{bmatrix} \end{array}.$$

Consider the following block matrices:

$$P_1 \coloneqq \begin{bmatrix} J_{1,0} \end{bmatrix}, \qquad P_2 \coloneqq \begin{bmatrix} J_{1,0} & J_{2,0} \\ 0 & J_{2,1} \end{bmatrix}, \quad \text{and} \quad P_3 \coloneqq \begin{bmatrix} J_{1,0} & J_{2,0} & J_{3,0} \\ 0 & J_{2,1} & J_{3,1} \\ 0 & 0 & J_{3,2} \end{bmatrix}.$$

Then, for $i = 1, 2, 3$, the matrix for ϕ_i is given by P_i with an additional column of zeroes corresponding to the R-basis element $1 \in R \otimes_Q D^i_{Q|k}$. The column of zeroes corresponds to a free cyclic summand of ϕ_i with basis 1, which corresponds to the image of $D^0_{R|k}$ in $D^i_{R|k}$.

REMARK 2.2.3. For a matrix M, we take $\mathrm{S}^i(M) = \mathrm{Sym}^i(M)$ to ith symmetric power of a matrix: that is, the matrix minimally presenting the ith symmetric power of the cokernel of M.

Let J be the usual Jacobian matrix. Observe that $J_{1,0} = J$, $J_{2,1} = \mathrm{S}^2(J^T)^T$, and $J_{3,2} = \mathrm{S}^3(J^T)^T$, and more generally $J_{i,i-1} = \mathrm{S}^i(J^T)^T$. In particular, we have

$$\begin{aligned}\ker(J_{i,i-1}) &= \ker(\mathrm{S}^i(J^T)^T) \cong \mathrm{Hom}_R(\operatorname{coker} \mathrm{S}^i(J^T), R)\\ &\cong \mathrm{Hom}_R(\mathrm{S}^i(\operatorname{coker} J^T), R) \cong \mathrm{Hom}_R(\mathrm{S}^i\Omega_R, R).\end{aligned}$$

2.2.4. In light of the discussion before, we identify elements of $D^2_{R|k}$ as constant operators in $D^0_{R|k}$ plus the collection of vectors $v \in R^9$ such that $P_2 v = 0$, that is,

$$D^2_{R|k} \cong R \oplus \ker P_2.$$

Likewise, we identify elements of $D^3_{R|k}$ as constant operators in $D^0_{R|k}$ plus the collection of vectors $v \in R^{19}$ such that $P_3 v = 0$, that is,

$$D^3_{R|k} \cong R \oplus \ker P_3.$$

As in (6), for any derivation η, we obtain a well-defined map induced by composition with η:

$$\ker(J) \to \ker(J_{2,1}).$$

If the restriction map $D^2_{R|k} \to \ker(J_{2,1})$ is surjective, then we also obtain a well-defined map induced by composition with η:

$$\ker(J_{2,1}) \to \ker(J_{3,2}).$$

2.3. Koszul Complexes and Matrix Factorizations

In this subsection we make use of differential graded (henceforth abbreviated to dg) algebras and their dg modules. A suitable reference for background on these topics is [1].

2.3.1. Let $\boldsymbol{f} = f_1, \dots, f_n$ be a list of elements in Q. We write $\mathrm{Kos}^Q(\boldsymbol{f})$ for the Koszul complex on Q over $\boldsymbol{f}$, regarded as a dg Q-algebra in the usual way. That is, $\mathrm{Kos}^Q(\boldsymbol{f})$ is the exterior algebra on the free module $\bigoplus_{i=1}^n Qe_i$, where e_i has homological degree one, and differential that is completely determined by $\partial(e_i) = f_i$ and the Leibniz rule.

Of particular interest will be the case in which $n = 3$, where we order the bases of homological degrees 1 and 2 according to

$$\mathrm{Kos}^Q_1(\boldsymbol{f}) = Qe_1 \oplus Qe_2 \oplus Qe_3 \quad \text{and} \quad \mathrm{Kos}^Q_2(\boldsymbol{f}) = Qe_3e_2 \oplus Qe_1e_3 \oplus Qe_2e_1.$$

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With this convention the differentials in $\mathrm{Kos}^Q(\boldsymbol{f})$ can be expressed using the following matrices:

$$0 \to Q \xrightarrow{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} 0 & -f_3 & f_2 \\ f_3 & 0 & -f_1 \\ -f_2 & f_1 & 0 \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}} Q \to 0.$$

NOTATION 2.3.2. Consider the Koszul complexes $\mathrm{Kos}^Q(x, y, z)$ and $\mathrm{Kos}^Q(f_x, f_y, f_z)$. The differential of the first Koszul complex is denoted by ∂ and the differential of the second Koszul complex is written as D. Explicitly,

$$\partial_2 = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}, \qquad \partial_1 = \partial_3^T = \begin{bmatrix} x & y & z \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0 & -f_z & f_y \\ f_z & 0 & -f_x \\ -f_y & f_x & 0 \end{bmatrix}, \qquad D_1 = D_3^T = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}.$$

REMARK 2.3.3. The assumption that R has an isolated singularity implies f_x, f_y, f_z forms a Q-regular sequence; this is explained at the beginning of Section 2.4. Hence, $\mathrm{Kos}^Q(f_x, f_y, f_z)$ is a dg Q-algebra resolution of $Q/(f_x, f_y, f_z)$. Furthermore, $\mathrm{Kos}^Q(x, y, z)$ is a dg Q-algebra resolution of k.

2.3.4. Throughout we will consider the dg Q-algebra $E = \mathrm{Kos}^Q(d \cdot f)$ with e the degree one generator. Since f is a Q-regular element, the augmentation map

$$E \xrightarrow{\simeq} Q/(d \cdot f) = R$$

is a quasi-isomorphism of dg Q-algebras.

Next, the Euler identity $d \cdot f = x f_x + y f_y + z f_z$ in Q defines dg E-module structures on $\mathrm{Kos}^Q(f_x, f_y, f_z)$ and $\mathrm{Kos}^Q(x, y, z)$. Left multiplication by e on $\mathrm{Kos}^Q(f_x, f_y, f_z)$ is described by the following matrices:

$$0 \leftarrow Q \xleftarrow{\partial_1} Q^3 \xleftarrow{-\partial_2} Q^3 \xleftarrow{\partial_3} Q \leftarrow 0;$$

recall that e has homological degree 1, and so left multiplication by e increases homological degree by 1. In fact, $e\cdot$ is a square-zero nullhomotopy for multiplication by $d \cdot f$ on $\mathrm{id}^{\mathrm{Kos}^Q(f_x, f_y, f_z)}$, which is exactly the data of a dg E-module; see [1, Remark 2.2.1].

Similarly, left multiplication by e on $\mathrm{Kos}^Q(x, y, z)$ is described by the following matrices:

$$0 \leftarrow Q \xleftarrow{D_1} Q^3 \xleftarrow{-D_2} Q^3 \xleftarrow{D_3} Q \leftarrow 0.$$

When we refer to $\mathrm{Kos}^Q(f_x, f_y, f_z)$ and $\mathrm{Kos}^Q(x, y, z)$ as dg E-modules, it is with the structures prescribed before, respectively. The fact that left multiplication by e on these complexes defines square-zero nullhomotopies for multiplication by $d \cdot f$ can be checked from Appendix A.

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In [17], Eisenbud showed that any R-module's minimal free resolution is eventually two-periodic; this data can be described completely in terms of his theory of matrix factorizations as introduced in *loc. cit.* We briefly recall these points and their connections with certain triangulated categories associated with R. For ease of exposition, we describe the former over regular rings; see [17; 27; 43] for more general treatments of this topic.

For the rest of the subsection, let A be a regular ring and $a \in A$ be an A-regular element.

2.3.5. A *matrix factorization of a (over A)* is a pair of maps between finite rank free A-modules

$$F_0 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} F_0$$

such that $\alpha\beta = a \cdot \mathrm{id}_{F_1}$ and $\beta\alpha = a \cdot \mathrm{id}_{F_0}$. It follows that F_0, F_1 have the same rank, and so α and β can be represented by square matrices of the same size.

The collection of matrix factorizations of a (over A), with morphisms defined in the obvious way, forms a Frobenius exact category. Hence its stable category, denoted $[\mathsf{mf}(A, a)]$, is a triangulated category where the suspension functor is given by

$$\Sigma(F_0 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} F_0) := F_1 \xrightarrow{-\alpha} F_0 \xrightarrow{-\beta} F_1.$$

2.3.6. Set $K := \mathrm{Kos}^A(a)$. Given a bounded complex G of finite rank free A-modules

$$0 \to G_n \to G_{n-1} \to \cdots \to G_m \to 0$$

with a dg K-module structure, one can define a matrix factorization of a over A as follows: Set

$$F_0 := \bigoplus_{i \text{ even}} G_i, \qquad F_1 := \bigoplus_{i \text{ odd}} G_i,$$

and, slightly abusing notation, set α and β to be $\partial^G + \sigma$ where σ denotes left multiplication by the exterior generator of K. This matrix factorization of a is denoted by $\mathsf{Fold}(G)$.

2.3.7. In [17, Section 6], the machinery above is applied to show that every minimal free resolution over a hypersurface ring is eventually two-periodic. A fundamental, yet key idea behind this is that given a matrix factorization $F_0 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} F_0$ of a the complex

$$\cdots \to F_0 \otimes_A A/(a) \xrightarrow{\alpha\otimes 1} F_1 \otimes_A A/(a) \xrightarrow{\beta\otimes 1} F_0 \otimes_A A/(a) \xrightarrow{\alpha\otimes 1} \cdots$$

is exact; see [17, Proposition 5.1] or [36].

2.3.8. For a commutative Noetherian ring B, we let $\mathsf{D}^{\mathsf{b}}(B)$ denote the bounded derived category of finitely generated B-modules. The objects are those B-complexes whose homology is concentrated in finitely many degrees and each homology module is finitely generated over B. This is a triangulated category in the standard way; see [26] and details on thick subcategories.

We let $\mathsf{D}_{\mathsf{sg}}(B)$ denote the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(B)$ by the perfect B-complexes. This was independently introduced by Buchweitz in [13] and Orlov in [34]. This is a triangulated category that records the singularity of B, and more generally the asymptotics of free resolutions over B. Assume $B = A/(a)$ a singular hypersurface ring. By *loc. cit.,* it follows that there is an equivalence of triangulated categories

$$\mathsf{D}_{\mathsf{sg}}(B) \equiv [\mathsf{mf}(A, a)].$$

2.3.9. Let T be a triangulated category. Here we recall the notion of level from [3]; this is an invariant that records how many "steps" it takes to build on object in T from another only utilizing the triangulated structure of T.

Recall that a thick subcategory of T is a triangulated subcategory that is closed under retracts. For an object X in T, we let $\mathsf{thick}_{\mathsf{T}} X$ denote the smallest thick subcategory of T containing X. There is an inductive construction of $\mathsf{thick}_{\mathsf{T}} X$ from [10], see also [3]. Set $\mathsf{thick}_{\mathsf{T}}^{1} X$ to be the smallest full subcategory of T containing X and closed under (de-)suspensions, finite sums, and retracts. Now inductively, $\mathsf{thick}_{\mathsf{T}}^{n} X$ is the smallest full subcategory of T containing objects C fitting into an exact triangle

$$A \to B \to C \to$$

with A in $\mathsf{thick}_{\mathsf{T}}^{n-1} X$ and B in $\mathsf{thick}_{\mathsf{T}}^{1} X$, which is closed under (de-)suspensions, finite sums, and retracts. By [3, Section 2.2],

$$\mathsf{thick}_{\mathsf{T}} X = \bigcup_{n \geq 1} \mathsf{thick}_{\mathsf{T}}^{n} X.$$

Following [3], the *level of an object Y with respect to X* is

$$\mathsf{level}_{\mathsf{T}}^{X} Y = \inf\{n \geq 0 : Y \text{ is in } \mathsf{thick}_{\mathsf{T}}^{n} X\}.$$

Now assume that B is an isolated singularity hypersurface ring with residue field k. It is well known that $\mathsf{level}_{\mathsf{D}_{\mathsf{sg}}(R)}^{k} M < \infty$ for any M in $\mathsf{D}_{\mathsf{sg}}(R)$; see, for example, [16; 25]. We give bounds on $\mathsf{level}_{\mathsf{D}_{\mathsf{sg}}(R)}^{k} D_{R|k}^{i}$ for $i = 2, 3$ in Corollaries 4.4.6 and 5.4.7, respectively. See also Question 5.4.8 for a proposed upper bound on $\mathsf{level}_{\mathsf{D}_{\mathsf{sg}}(R)}^{k} D_{R|k}^{i} < \infty$ for arbitrary i.

2.4. *First Order Differential Operators*

Let $P = k[x_1, \ldots, x_n]$ and $A = P/(f)$ where k is a perfect field and f is a homogeneous element of degree $d \geq 2$. We also impose that A is an isolated singularity. Thus, $(f_{x_1}, \ldots, f_{x_n}, f) = (f_{x_1}, \ldots, f_{x_n})$ has height n, so the partial derivatives form a P-regular sequence. Set $B := A/(f_{x_1}, \ldots, f_{x_n})$, which is an Artinian complete intersection quotient of A. In this subsection we recall the relationship between B and the first order differential operators on A, as well as the minimal A-free resolution of B. We refer the reader to 2.4.5 for the 3-variable case that examined in this paper.

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2.4.1. We recall from 2.1.3 that $D^1_{A|k} = A \oplus \mathrm{Der}_{A|k}$, where the copy of A is the set of constant operators, and $\mathrm{Der}_{A|k}$ is the collection of derivations; in particular $\mathrm{Der}_{A|k} \cong [\mathrm{gr}(D_{A|k})]_1$.

As in 2.1.8, we have

$$\mathrm{Der}_{A|k} \cong \ker(J) = \ker\begin{bmatrix} f_{x_1} & \cdots & f_{x_n}\end{bmatrix}.$$

To find the generators and minimal free resolution of this R-module, we construct a minimal free resolution of $B = \operatorname{coker} J$

$$\cdots \to F_3 \xrightarrow{\partial_3^F} F_2 \xrightarrow{\partial_2^F} R^n \xrightarrow{J} R \to 0$$

and adding a contractible subcomplex $R \xrightarrow{=} R$ in degrees 1 and 2 yields $\mathrm{Der}_{A|k}$ as the image of the second differential. That is, in

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[r, "{\left[\begin{smallmatrix} \partial_3^F \\ 0 \end{smallmatrix}\right]}"] & F_2 \oplus R \arrow[rr, "{\left[\begin{smallmatrix} \partial_2^F & 0 \\ 0 & 1 \end{smallmatrix}\right]}"] \arrow[dr, "\partial"'] & & R^n \oplus R \arrow[r, "J"] & R \arrow[dr, "\pi"'] & \\
& & & \mathrm{Der}_{A|k} \arrow[ur, "i"'] & & & B
\end{tikzcd} \tag{9}$$

we obtain the generators of $\mathrm{Der}_{A|k}$ from the image of $\left[\begin{smallmatrix} \partial_2^F & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and the minimal free resolution of $\mathrm{Der}_{A|k}$ is procured by truncating the complex in (9) at homological degree two and shifting by two.

Next we explain how to obtain the minimal R-free resolution of B, and hence of $D^1_{A|k}$, in light of 2.4.1.

2.4.2. Let $C = \mathrm{Kos}^B(d \cdot f)$ be the Koszul complex on the homogeneous element $d \cdot f$ over P, where recall d is the degree of f. Since $f_{x_1}, \ldots, f_{x_n}$ is a P-regular sequence,

$$K := \mathrm{Kos}^P(f_{x_1}, \ldots, f_{x_n})$$

is a P-free resolution of B. Also, set

$$K' := \mathrm{Kos}^P(x_1, \ldots, x_n),$$

which is a P-free resolution of the residue field k. In fact, K and K' are dg C-algebras. Indeed, let the standard bases of K_1 and K'_1 be denoted $\xi_1, \ldots, \xi_n$ and $\xi'_1, \ldots, \xi'_n$, respectively. The Euler identity in P,

$$d \cdot f = \sum_{i=1}^n x_i f_{x_i}$$

defines dg C-module structures on K and K': Namely, if ξ is an exterior variable of C, then left multiplication by ξ is given by left multiplication by

$$\sum_{i=1}^n x_i \xi_i \quad \text{on } K \quad \text{and} \quad \sum_{i=1}^n f_{x_i} \xi'_i \quad \text{on } K', \tag{10}$$

respectively. As a consequence, using [38, Theorem 4], the minimal A-free resolutions of B and k are given by

$$A \otimes_P K\left\langle y \;\middle|\; \partial y = \sum_{i=1}^{n} x_i \otimes \xi_i \right\rangle \quad \text{and} \quad A \otimes_P K'\left\langle y \;\middle|\; \partial y = \sum_{i=1}^{n} f_{x_i} \otimes \xi_i' \right\rangle,$$

respectively; as underlying graded A-modules these have the form

$$A \otimes_P K \otimes_P \bigoplus_{i \geq 0} P y^{(i)} \quad \text{and} \quad A \otimes_P K' \otimes_P \bigoplus_{i \geq 0} P y^{(i)},$$

respectively, where $y^{(i)}$ is a basis element living in homological degree $2i$ and the differential on each complex is uniquely extended by the rule specified and the formula $\partial(y^{(i)}) = \partial y \cdot y^{(i-1)}$. See [2] for a detailed account of this construction. Each of these is also the Shamash resolution from [36] (see also [17]). Note that upon fixing the lexicographically ordered bases on K and K', respectively, left multiplication by $\sum_{i=1}^{n} x_i \xi_i$ on K is given by the appropriate matrix $\partial^{K'}$ representing the differential of K'. Similarly, left multiplication by $\sum_{i=1}^{n} f_{x_i} \xi_i'$ on K' is given by the appropriate matrix ∂^{K} representing the differential of K.

2.4.3. Continue with the notation set in 2.4.2. By inspection of the resolutions in (10), minding that y is a divided power variable of degree two, the matrix factorizations describing the minimal A-resolutions of B and k, respectively, have the following forms:

$$P^{2^{n-1}} \xrightarrow{\alpha} P^{2^{n-1}} \xrightarrow{\beta} P^{2^{n-1}} \quad \text{and} \quad P^{2^{n-1}} \xrightarrow{\beta} P^{2^{n-1}} \xrightarrow{\alpha} P^{2^{n-1}},$$

respectively, where

$$\alpha = \begin{bmatrix} (\partial_1^{K'})^T & \partial_2^K & 0 & 0 & \cdots \\ 0 & (\partial_3^{K'})^T & \partial_4^K & 0 & \cdots \\ 0 & 0 & (\partial_5^{K'})^T & \partial_6^K & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{and}$$

$$\beta = \begin{bmatrix} \partial_1^K & 0 & 0 & 0 & \cdots \\ (\partial_2^{K'})^T & \partial_3^K & 0 & 0 & \cdots \\ 0 & (\partial_4^{K'})^T & \partial_5^K & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix};$$

here the self-duality of the Koszul complex is being used. Therefore, there are the following isomorphisms in $\mathsf{D}_{\mathsf{sg}}(A)$:

$$D^1_{A|k} \simeq B \simeq \Sigma k.$$

REMARK 2.4.4. In [21], Herzog and Martsinkovsky show that $D^1_{A|k} \simeq \Sigma k$ in $\mathsf{D}_{\mathsf{sg}}(A)$ whenever A is an isolated singularity complete intersection ring of positive Krull dimension. The strategy employed in *loc. cit.* is the so-called "gluing" method to produce complete resolutions; this differs from the dg-method

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described above. We opt for the latter as it keeps the present discussion mostly self-contained and in line with the strategy employed in Sections 4 and 5.

Note that in the case that A is an isolated singularity complete intersection ring, we trivially have

$$\mathsf{level}^k_{\mathsf{D_{sg}}(A)}(D^1_{A|k}) = 1$$

from the isomorphism above. This equality should be compared with Corollaries 4.4.6 and 5.4.7 for $D^2_{A|k}$ and $D^3_{A|k}$ when A is a hypersurface of Krull dimension two; see Question 5.4.8 as well.

2.4.5. Now returning to Notation 2.0.1, following 2.4.2, we obtain the following minimal R-free resolution of $B = R/(f_x, f_y, f_z)$:

$$\cdots \xrightarrow{\begin{bmatrix} \partial_3 & D_2 \\ 0 & \partial_1 \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} D_1 & 0 \\ -\partial_2 & D_3 \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} \partial_3 & D_2 \\ 0 & \partial_1 \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} D_1 & 0 \\ -\partial_2 & D_3 \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} \partial_3 & D_2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} f_x & f_y & f_z \end{bmatrix}} R \to 0,$$

where the matrices are defined in Notation 2.3.2. Therefore, again using 2.4.1, the (augmented) minimal R-free resolution of $D^1_{R|k}$ has the form

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r, "{\begin{bmatrix} \partial_3 & D_2 \\ 0 & \partial_1 \end{bmatrix}}"] & R^4 \arrow[r, "{\begin{bmatrix} D_1 & 0 \\ -\partial_2 & D_3 \end{bmatrix}}"] & R^4 \arrow[r, "{\begin{bmatrix} \partial_3 & D_2 \\ 0 & \partial_1 \end{bmatrix}}"] & R^4 \arrow[r, "{\begin{bmatrix} D_1 & 0 \\ -\partial_2 & D_3 \\ 0 & 0 \end{bmatrix}}"] & R^4 \oplus R \arrow[r, "{\begin{bmatrix} \partial_3 & D_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}"] \arrow[dr, two heads] & & R^3 \oplus R \arrow[r, "{[f_x \; f_y \; f_z \; 0]}"] & R \arrow[dr, two heads, "\pi"'] & \\
& & & & & D^1_{R|k} \arrow[ur, "i"] & & & B
\end{tikzcd}$$

and a minimal set of generators $D^1_{R|k}$ is given by the columns of $\left[\begin{smallmatrix} \partial_3 & D_2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$, where the rows correspond to the basis $\partial_x, \partial_y, \partial_z, 1$ as described in 2.2.1. The following notation for the matrices above will be used in the sequel:

$$M_0(1) = \begin{bmatrix} \partial_3 & D_2 \end{bmatrix}, \qquad M_1(1) = \begin{bmatrix} D_1 & 0 \\ -\partial_2 & D_3 \end{bmatrix}, \qquad M_2(1) = \begin{bmatrix} \partial_3 & D_2 \\ 0 & \partial_1 \end{bmatrix}.$$

Recall that $D^1_{R|k} = R \oplus \mathrm{Der}_{R|k}$. In particular, a minimal set of generators of $\mathrm{Der}_{R|k}$ is given by the columns of $[\partial_3 \;\; D_2]$. We name these derivations:

$$\mathcal{E} := x\partial_x + y\partial_y + z\partial_z \tag{11}$$

is the Euler derivation, and

$$\begin{aligned} &\mathcal{H}_{yz} := f_z\partial_y - f_y\partial_z, \qquad \mathcal{H}_{zx} := f_x\partial_z - f_z\partial_x, \quad \text{and} \\ &\mathcal{H}_{xy} := f_y\partial_x - f_x\partial_y \end{aligned} \tag{12}$$

are the Hamiltonian derivations. The relations on these derivations are given by the columns of $\left[\begin{smallmatrix} D_1 & 0 \\ -\partial_2 & D_3 \end{smallmatrix}\right]$, or more explicitly as

$$\begin{aligned} f_x\mathcal{E} + y\mathcal{H}_{xy} - z\mathcal{H}_{zx} &= 0, \\ f_y\mathcal{E} - x\mathcal{H}_{xy} + z\mathcal{H}_{yz} &= 0, \\ f_z\mathcal{E} + x\mathcal{H}_{zx} - y\mathcal{H}_{yz} &= 0. \end{aligned} \tag{13}$$

As previously mentioned, a similar method will be employed in Sections 4 and 5 for calculating the minimal free resolutions of $D^2_{R|k}$ and $D^3_{R|k}$. We end this subsection by recording the graded Betti numbers of $D^1_{R|k}$ regarded as a graded module over the standard k-algebra R.

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2.4.6. Regarding R as a standard graded k-algebra, it follows that each $D^i_{R|k}$ is a graded R-submodule of the graded k-linear endomorphisms of R. For a k-linear graded endomorphism g of R, we let g denote the degree of g. That is, $|g|$ is the unique integer satisfying $g(R_j) \subseteq R_{j+|g|}$ for each $j \in \mathbb{Z}$. Under these conventions,

$$|\mathcal{E}| = 0 \quad \text{and} \quad |\mathcal{H}_{yz}| = |\mathcal{H}_{zx}| = |\mathcal{H}_{xy}| = d - 2$$

using that $|\partial_x| = |\partial_y| = |\partial_z| = -1$.

2.4.7. Let N be a finitely generated graded R-module. Its i, jth-graded Betti number is

$$\beta^R_{ij}(N) := \operatorname{rank}_k \operatorname{Tor}^R_i(N, k)_j;$$

that is, $\beta^R_{i,j}(N)$ is the rank of the free module in homological degree i and basis in internal degree $-j$.

It is straightforward to check that the differentials in the free resolution of $D^1 = D^1_{R|k}$, in 2.4.5, are homogeneous with respect to the internal grading of R. Therefore, the graded Betti numbers of D^1 are

$$\beta^R_{0,j}(D^1) = \begin{cases} 2, & j = 0, \\ 3, & j = d-2, \\ 0, & \text{otherwise,} \end{cases}$$

and for $n \geq 1$,

$$\beta^R_{2n-1,j}(D^1) = \begin{cases} 1, & j = nd-1, \\ 3, & j = nd+d-3, \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$$\beta^R_{2n,j}(D^1) = \begin{cases} 3, & j = nd, \\ 1, & j = nd+d-2, \\ 0, & \text{otherwise.} \end{cases}$$

3. Glossary of Matrices

All of the following are matrices with entries in R; see Notation 2.0.1. When writing a block matrix, $0_{m\times n}$ will denote the $m \times n$-matrix whose entries are all zero.

3.1. *Matrices Needed in Section 4*

$$\partial_1 = \begin{bmatrix} x & y & z \end{bmatrix}, \qquad \partial_2 = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}, \qquad \partial_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$D_1 = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0 & -f_z & f_y \\ f_z & 0 & -f_x \\ -f_y & f_x & 0 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix},$$

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$$
q = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}, \qquad \Delta = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}, \qquad \delta = \det(\Delta),
$$

$$
\begin{aligned}
&\Delta_{xx} = f_{yy} f_{zz} - f_{yz}^2, \qquad \Delta_{yy} = f_{xx} f_{zz} - f_{xz}^2, \qquad \Delta_{zz} = f_{xx} f_{yy} - f_{xy}^2, \\
&\Delta_{xy} = f_{xz} f_{yz} - f_{xy} f_{zz}, \qquad \Delta_{xz} = f_{xy} f_{yz} - f_{xz} f_{yy}, \\
&\Delta_{yz} = f_{xy} f_{xz} - f_{xx} f_{yz}, \\
&\delta_x = \partial_x(\delta), \qquad \delta_y = \partial_y(\delta), \qquad \delta_z = \partial_z(\delta), \\
&\alpha_1 = \frac{1}{d-1}\Delta, \qquad \alpha_2 = \frac{1}{(d-1)^2} \begin{bmatrix} \Delta_{xx} & \Delta_{xy} & \Delta_{xz} \\ \Delta_{xy} & \Delta_{yy} & \Delta_{yz} \\ \Delta_{xz} & \Delta_{yz} & \Delta_{zz} \end{bmatrix}, \\
&\alpha_3 = \frac{1}{(d-1)^3}\delta, \\
\theta_0(2) &= J_{2,0} = \begin{bmatrix} \frac{1}{2} f_{xx} & f_{xy} & f_{xz} & \frac{1}{2} f_{yy} & f_{yz} & \frac{1}{2} f_{zz} \end{bmatrix}, \\
\theta_1(2) &= (d-1) \begin{bmatrix} 0_{3\times 4} & \alpha_2 \end{bmatrix}, \\
\theta_{2i}(2) &= (d-1) \begin{bmatrix} 0_{1\times 3} & 0_{1\times 3} & -\alpha_3 \\ 0_{3\times 3} & \alpha_1 & 0_{3\times 1} \end{bmatrix} \quad \text{for } i \geq 1, \\
\theta_{2i+1}(2) &= \frac{-(d-1)}{2} \begin{bmatrix} 0_{3\times 1} & 0_{3\times 3} & -2\alpha_2 \\ 0 & \partial_1 & 0_{1\times 3} \end{bmatrix} \quad \text{for } i \geq 1.
\end{aligned}
$$

3.2. Additional Matrices Needed in Section 5

$$
\begin{aligned}
\sigma_1 &= \sigma_3^T = \frac{1}{(d-1)^3(d-2)} \begin{bmatrix} \delta_x & \delta_y & \delta_z \end{bmatrix}, \\
\sigma_2 &= \frac{1}{(d-1)^3(d-2)} \begin{bmatrix} 0 & -\delta_z & \delta_y \\ \delta_z & 0 & -\delta_x \\ -\delta_y & \delta_x & 0 \end{bmatrix}, \\
B_1 &= \frac{1}{d-1} \begin{bmatrix} x\Delta_{xx} & y\Delta_{xx} & z\Delta_{xx} \\ x\Delta_{xy} & y\Delta_{xy} & z\Delta_{xy} \\ x\Delta_{xz} & y\Delta_{xz} & z\Delta_{xz} \\ x\Delta_{yy} & y\Delta_{yy} & z\Delta_{yy} \\ x\Delta_{yz} & y\Delta_{yz} & z\Delta_{yz} \\ x\Delta_{zz} & y\Delta_{zz} & z\Delta_{zz} \end{bmatrix}, \\
B_2 &= \begin{bmatrix} H_{yz}(\Delta_{xx}) & H_{yz}(\Delta_{xy}) - \frac{x\delta_z}{d-1} & H_{yz}(\Delta_{xz}) + \frac{x\delta_y}{d-1} \\ H_{zx}(\Delta_{xx}) & H_{yz}(\Delta_{yy}) & \frac{1}{2}(H_{yz}(\Delta_{yz}) + H_{zx}(\Delta_{xz}) + \frac{y\delta_y - x\delta_x}{d-1}) \\ H_{xy}(\Delta_{xx}) & \frac{1}{2}(H_{yz}(\Delta_{yz}) + H_{xy}(\Delta_{xy}) + \frac{x\delta_x - z\delta_z}{d-1}) & H_{yz}(\Delta_{zz}) \\ H_{zx}(\Delta_{xy}) + \frac{y\delta_z}{d-1} & H_{zx}(\Delta_{yy} & H_{zx}(\Delta_{yz}) - \frac{y\delta_x}{d-1} \\ \frac{1}{2}(H_{zx}(\Delta_{xz}) + H_{xy}(\Delta_{xy}) + \frac{z\delta_z - y\delta_y}{d-1}) & H_{xy}(\Delta_{yy}) & H_{zx}(\Delta_{zz}) \\ H_{xy}(\Delta_{xz}) - \frac{z\delta_y}{d-1} & H_{xy}(\Delta_{yz}) + \frac{z\delta_x}{d-1} & H_{xy}(\Delta_{zz}) \end{bmatrix}, \\
\theta_0(3) &= \begin{bmatrix} J_{3,0} \\ J_{3,1} \end{bmatrix}
\end{aligned}
$$

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$$
= \begin{bmatrix} \frac{1}{6} f_{xxx} & \frac{1}{2} f_{xxy} & \frac{1}{2} f_{xxz} & \frac{1}{2} f_{xyy} & f_{xyz} & \frac{1}{2} f_{xzz} & \frac{1}{6} f_{yyy} & \frac{1}{2} f_{yyz} & \frac{1}{2} f_{yzz} & \frac{1}{6} f_{zzz} \\ \frac{1}{2} f_{xx} & f_{xy} & f_{xz} & \frac{1}{2} f_{yy} & f_{yz} & \frac{1}{2} f_{zz} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} f_{xx} & 0 & f_{xy} & f_{xz} & 0 & \frac{1}{2} f_{yy} & f_{yz} & \frac{1}{2} f_{zz} & 0 \\ 0 & 0 & \frac{1}{2} f_{xx} & 0 & f_{xy} & f_{xz} & 0 & \frac{1}{2} f_{yy} & f_{yz} & \frac{1}{2} f_{zz} \end{bmatrix},
$$

$$
\theta_1(3) = \begin{bmatrix} 0_{3\times 4} & -\frac{d^2-1}{2}\alpha_2 & -(d-1)(d-2)\sigma_2 \\ 0_{6\times 4} & B_1 & \frac{-3}{(d-1)(d-2)} B_2 \end{bmatrix},
$$

$$
\theta_{2i}(3) = \begin{bmatrix} 0_{1\times 3} & 0_{1\times 3} & -(d-1)(d-2)\sigma_1 & 0 \\ 0_{3\times 3} & \frac{-(d+1)}{2}\Delta & 0_{3\times 3} & -(d-1)(d-2)\sigma_3 \\ 0_{1\times 3} & 0_{1\times 3} & -3(d-1)\sigma_1 & 0 \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & -3(d-1)\sigma_3 \\ 0_{3\times 3} & 0_{3\times 3} & \frac{9}{2}\Delta & 0_{3\times 1} \end{bmatrix}
$$

for $i \geq 1$,

$$
\theta_{2i+1}(3) = \begin{bmatrix} 0_{3\times 1} & 0_{3\times 3} & \frac{-(d^2-1)}{2}\alpha_2 & -(d-1)(d-2)\sigma_2 \\ 0 & \frac{d^2-1}{4}\partial_1 & 0_{1\times 3} & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 3} & 0_{3\times 3} & -3(d-1)\sigma_2 \\ 0_{3\times 1} & 0_{3\times 3} & 0_{3\times 3} & \frac{9(d-1)}{2}\alpha_2 \\ 0 & 0_{1\times 3} & \frac{-3(d-1)}{2}\partial_1 & 0_{1\times 3} \end{bmatrix}
$$

for $i \geq 1$.

4. Differential Operators of Order 2

In this subsection we adopt Notation 2.0.1 and construct the minimal R-free resolution of $D^2_{R|k}$; see Theorem 4.4.3. The bulk of the work is in Sections 4.1 to 4.3, and it is assembled in Section 4.4.

4.1. *A Set of Generators of* $\ker J_{2,1}$

In this subsection we introduce what we later show, see Proposition 4.3.4, to be the generators of $\ker J_{2,1}$. We have $\mathrm{Der}_{R|K} = R\langle \mathcal{E}, \mathcal{H}_{yz}, \mathcal{H}_{zx}, \mathcal{H}_{xy}\rangle$, where $\mathcal{E} = x\partial_x + y\partial_y + z\partial_z$ is the Euler operator and $\mathcal{H}_{yz} = f_z\partial_y - f_y\partial_z$, $\mathcal{H}_{zx} = f_x\partial_z - f_z\partial_x$, and $\mathcal{H}_{xy} = f_y\partial_x - f_x\partial_y$ are the Hamiltonians. Composing $\mathcal{E}$ with each of these to obtain the elements $\mathcal{E}\circ\mathcal{E}$, $\mathcal{E}\circ\mathcal{H}_{yz}$, $\mathcal{E}\circ\mathcal{H}_{zx}$, and $\mathcal{E}\circ\mathcal{H}_{xy}$ in $D^2_{R|k}$. Their images in $\ker(J_{2,1}) \subseteq R^6$ under the map ν_2 from (7) are of particular interest, and so we introduce the following notation:

$$
\begin{aligned}
E^2 &:= \nu_2(\mathcal{E}\circ\mathcal{E}) \\
&= x^2\partial_x^2 + 2xy\partial_x\partial_y + 2xz\partial_x\partial_z + y^2\partial_y^2 + 2yz\partial_y\partial_z + z^2\partial_z^2, \\
EH_{yz} &:= \nu_2(\mathcal{E}\circ\mathcal{H}_{yz}) \\
&= xf_z\partial_x\partial_y - xf_z\partial_x\partial_z + yf_z\partial_y^2 + (zf_z - yf_y)\partial_y\partial_z - zf_y\partial_z^2, \\
EH_{zx} &:= \nu_2(\mathcal{E}\circ\mathcal{H}_{zx}) \\
&= -xf_z\partial_x^2 - yf_z\partial_x\partial_y + (xf_x - zf_z)\partial_x\partial_z + yf_x\partial_y\partial_z + zf_x\partial_z^2,
\end{aligned}
$$

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$$\begin{aligned}EH_{xy} &\coloneqq \nu_2(\mathcal{E}\circ\mathcal{H}_{xy})\\ &= xf_y\partial_x^2 + (yf_y - xf_x)\partial_x\partial_y + zf_y\partial_x\partial_z - yf_x\partial_y^2 - zf_x\partial_y\partial_z;\end{aligned}$$

note that lower order terms are dropped since the map ν_2 factors through $D^2_{R|k}/D^1_{R|k}$. These correspond to vectors in R^6 with the basis $\{\partial_x^{(2)}, \partial_x\partial_y, \partial_x\partial_z, \partial_y^{(2)}, \partial_y\partial_z, \partial_z^{(2)}\}$:

$$E^2 = 2\begin{bmatrix} x^2\\ xy\\ xz\\ y^2\\ yz\\ z^2\end{bmatrix}, \qquad EH_{yz} = \begin{bmatrix} 0\\ xf_z\\ -xf_y\\ 2yf_z\\ zf_z - yf_y\\ -2zf_y\end{bmatrix},$$

$$EH_{zx} = \begin{bmatrix} -2xf_z\\ -yf_z\\ xf_x - zf_z\\ 0\\ yf_x\\ 2zf_x\end{bmatrix}, \qquad EH_{xy} = \begin{bmatrix} 2xf_y\\ yf_y - xf_x\\ zf_y\\ -2yf_x\\ -zf_x\\ 0\end{bmatrix}.$$

Similarly, by composing each Hamiltonian with itself and considering their images under ν_2, we have the following elements in R^6:

$$H_{yz}^2 = 2\begin{bmatrix} 0\\ 0\\ 0\\ f_z^2\\ -f_yf_z\\ f_y^2\end{bmatrix}, \qquad H_{zx}^2 = 2\begin{bmatrix} f_z^2\\ 0\\ -f_xf_z\\ 0\\ 0\\ f_x^2\end{bmatrix}, \qquad H_{xy}^2 = 2\begin{bmatrix} f_y^2\\ -f_xf_y\\ 0\\ f_x^2\\ 0\\ 0\end{bmatrix}.$$

REMARK 4.1.1. In Section 4.5, we identify elements of $\ker(J_{2,1})$ with elements in $R\otimes_Q F_{\le 2}$, the free module with basis

$$\partial_x^{(2)}, \partial_x\partial_y, \partial_x\partial_z, \partial_y^{(2)}, \partial_y\partial_z, \partial_z^{(2)}, \partial_x, \partial_y, \partial_z, 1$$

as in 2.2.1, by taking the naive lift: the vector with zeroes in the last four coordinates.

We now introduce the remaining three generators of a minimal generating set of $\ker J_{2,1}$; see Proposition 5.3.1.

LEMMA 4.1.2. *The elements*

$$\alpha_x = \frac{1}{x}\left[H_{yz}^2 + \frac{1}{(d-1)^2}\Delta_{xx}E^2\right],$$

$$\alpha_y = \frac{1}{y}\left[H_{zx}^2 + \frac{1}{(d-1)^2}\Delta_{yy}E^2\right],$$

$$\alpha_z = \frac{1}{z}\left[H_{xy}^2 + \frac{1}{(d-1)^2}\Delta_{zz}E^2\right]$$

are well defined in R^6.

Proof. We verify the statement for α_z and the other two arguments are similar. Consider

$$H_{xy}^2 + \frac{1}{(d-1)^2}\Delta_{zz}E^2 = \begin{array}{c} \partial_x^{(2)} \\ \partial_x\partial_y \\ \partial_x\partial_z \\ \partial_y^{(2)} \\ \partial_y\partial_z \\ \partial_z^{(2)} \end{array}\begin{bmatrix} 2f_y^2 + \frac{2}{(d-1)^2}x^2\Delta_{zz} \\ -2f_x f_y + \frac{2}{(d-1)^2}xy\Delta_{zz} \\ \frac{2}{(d-1)^2}xz\Delta_{zz} \\ 2f_x^2 + \frac{2}{(d-1)^2}y^2\Delta_{zz} \\ \frac{2}{(d-1)^2}yz\Delta_{zz} \\ \frac{2}{(d-1)^2}z^2\Delta_{zz} \end{bmatrix}, \tag{14}$$

which we claim is divisible by z. This is clear for the $\partial_x\partial_z$, $\partial_y\partial_z$, and $\partial_z^{(2)}$ entries.

For the $\partial_x^{(2)}$ entry, we have

$$\begin{aligned} 2f_y^2 + \frac{2}{(d-1)^2}x^2\Delta_{zz} &= \frac{2}{(d-1)^2}(-x^2\Delta_{zz} - z^2\Delta_{xx} + 2xz\Delta_{xz} + x^2\Delta_{zz}) \\ &= \frac{2z}{(d-1)^2}(2x\Delta_{xz} - z\Delta_{xx}), \end{aligned}$$

where the first equality follows from identity (59).

For the $\partial_x\partial_y$ entry, we have

$$\begin{aligned} &-2f_x f_y + \frac{2}{(d-1)^2}xy\Delta_{zz} \\ &\quad= \frac{2}{(d-1)^2}(-z^2\Delta_{xy} - xy\Delta_{zz} + xz\Delta_{yz} + yz\Delta_{xz} + xy\Delta_{zz}) \\ &\quad= \frac{2z}{(d-1)^2}(x\Delta_{yz} + y\Delta_{xz} - z\Delta_{xy}), \end{aligned}$$

where the first equality follows from identity (58).

Similarly, applying identity (59) to the $\partial_y^{(2)}$ entry (or swapping x and y in the $\partial_x^{(2)}$ entry), we find that vector (14) is given by

$$\frac{2z}{(d-1)^2}\begin{array}{c} \partial_x^{(2)} \\ \partial_x\partial_y \\ \partial_x\partial_z \\ \partial_y^{(2)} \\ \partial_y\partial_z \\ \partial_z^{(2)} \end{array}\begin{bmatrix} 2x\Delta_{xz} - z\Delta_{xx} \\ x\Delta_{yz} + y\Delta_{xz} - z\Delta_{xy} \\ x\Delta_{zz} \\ 2y\Delta_{yz} - z\Delta_{yy} \\ y\Delta_{zz} \\ z\Delta_{zz} \end{bmatrix}, \tag{15}$$

which is divisible by z, as desired. □

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We will show that $\{E^2, EH_{yz}, EH_{zx}, EH_{xy}, \alpha_x, \alpha_y, \alpha_z\}$ is a minimal generating set for $\ker J_{2,1}$ in Section 4.3.

4.1.3. From the computations above, writing these generators in terms of the basis $\{\partial_x^{(2)}, \partial_x\partial_y, \partial_x\partial_z, \partial_y^{(2)}, \partial_y\partial_z, \partial_z^{(2)}\}$ gives the columns of the following matrix:

$$M_0(2) := \Big[E^2 \quad EH_{yz} \quad EH_{zx} \quad EH_{xy} \quad \tfrac{2}{(d-1)^2}A(2)\Big],$$

where

$$A(2) := \begin{array}{c} \begin{array}{ccc} \qquad\qquad \alpha_x \qquad\qquad & \qquad\qquad \alpha_y \qquad\qquad & \qquad\qquad \alpha_z \qquad\qquad \end{array} \\ \begin{bmatrix} x\Delta_{xx} & 2x\Delta_{xy} - y\Delta_{xx} & 2x\Delta_{xz} - z\Delta_{xx} \\ y\Delta_{xx} & x\Delta_{yy} & x\Delta_{yz} + y\Delta_{xz} - z\Delta_{xy} \\ z\Delta_{xx} & z\Delta_{xy} + x\Delta_{yz} - y\Delta_{xz} & x\Delta_{zz} \\ 2y\Delta_{xy} - x\Delta_{yy} & y\Delta_{yy} & 2y\Delta_{yz} - z\Delta_{yy} \\ y\Delta_{xz} + z\Delta_{xy} - x\Delta_{yz} & z\Delta_{yy} & y\Delta_{zz} \\ 2z\Delta_{xz} - x\Delta_{zz} & 2z\Delta_{yz} - y\Delta_{zz} & z\Delta_{zz} \end{bmatrix} \end{array}.$$

4.2. *Matrix Factorization*

In this subsection we construct a matrix factorization that is associated with $\ker J_{2,1}$; see 2.3.7. Adopt the notation and conventions from Section 2.3.

4.2.1. We first define a dg E-module map $\alpha\colon \operatorname{Kos}^Q(f_x, f_y, f_z) \to \operatorname{Kos}^Q(x, y, z)$:

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "D_3"] \arrow[d, "\alpha_3"] & Q^3 \arrow[r, "D_2"] \arrow[d, "\alpha_2"] & Q^3 \arrow[r, "D_1"] \arrow[d, "\alpha_1"] & Q \arrow[r] \arrow[d, equal, "\alpha_0"] & 0 \\ 0 \arrow[r] & Q \arrow[r, "\partial_3"] & Q^3 \arrow[r, "\partial_2"] & Q^3 \arrow[r, "\partial_1"] & Q \arrow[r] & 0 \end{tikzcd}$$

with α_i defined in Section 3. Using Appendix D, namely equations (86), α is a map of complexes. Furthermore, combining the fact that each α_i is symmetric and the E-actions in 2.3.4, it follows from (86) that α is a dg E-module map.

Next, since α is a dg E-module map, its mapping cone $\operatorname{cone}(\alpha)$ is a dg E-module; see, for example, [2, Section 1.1]. Explicitly, $\operatorname{cone}(\alpha)$ is the complex of free Q-modules

$$0 \to Q \xrightarrow{\begin{bmatrix} -D_3 \\ \alpha_3 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q \end{matrix} \xrightarrow{\begin{bmatrix} -D_2 & 0 \\ \alpha_2 & \partial_3 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q^3 \end{matrix} \xrightarrow{\begin{bmatrix} -D_1 & 0 \\ \alpha_1 & \partial_2 \end{bmatrix}} \begin{matrix} Q \\ \oplus \\ Q^3 \end{matrix} \xrightarrow{\begin{bmatrix} \alpha_0 & \partial_1 \end{bmatrix}} Q \to 0$$

with $e\cdot$ given by

$$0 \leftarrow Q \xleftarrow{\begin{bmatrix} -\partial_1 & 0 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q \end{matrix} \xleftarrow{\begin{bmatrix} \partial_2 & 0 \\ 0 & D_1 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q^3 \end{matrix} \xleftarrow{\begin{bmatrix} -\partial_3 & 0 \\ 0 & -D_2 \end{bmatrix}} \begin{matrix} Q \\ \oplus \\ Q^3 \end{matrix} \xleftarrow{\begin{bmatrix} 0 \\ D_3 \end{bmatrix}} Q \leftarrow 0.$$

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The Q-linear quotient complex L of $\Sigma^{-1}\operatorname{cone}(\alpha)$, obtained by contracting away the copy of $Q \xrightarrow{\equiv} Q$, given by

$$L: \quad 0 \to Q \xrightarrow{\begin{bmatrix} -D_3 \\ \alpha_3 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q \end{matrix} \xrightarrow{\begin{bmatrix} -D_2 & 0 \\ \alpha_2 & \partial_3 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q^3 \end{matrix} \xrightarrow{\begin{bmatrix} \alpha_1 & \partial_2 \end{bmatrix}} Q^3 \to 0 \tag{16}$$

can be equipped with a dg E-module structure by defining $e\cdot$ as

$$0 \leftarrow Q \xleftarrow{\begin{bmatrix} -\partial_1 & 0 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q \end{matrix} \xleftarrow{\begin{bmatrix} \partial_2 & 0 \\ 0 & D_1 \end{bmatrix}} \begin{matrix} Q^3 \\ \oplus \\ Q^3 \end{matrix} \xleftarrow{\begin{bmatrix} q \\ -D_2 \end{bmatrix}} Q^3 \leftarrow 0.$$

The dg E-module structure is obtained from the quasi-isomorphism $\iota\colon L \to \Sigma^{-1}\operatorname{cone}(\alpha)$ given by

$$\iota_i = \begin{cases} \begin{bmatrix} -\partial_1 \\ 1 \end{bmatrix}, & i = 0, \\ \mathrm{id}, & i = 1, 2, 3, \\ 0, & \text{else}. \end{cases}$$

4.2.2. Let L be as constructed in 4.2.1. Following 2.3.6, from L we obtain the matrix factorization $\mathsf{Fold}(L)$ of $d \cdot f$ over Q:

$$Q^7 \xrightarrow{\begin{bmatrix} q & -D_2 & 0 \\ -D_2 & \alpha_2 & \partial_3 \\ 0 & -\partial_1 & 0 \end{bmatrix}} Q^7 \xrightarrow{\begin{bmatrix} \alpha_1 & \partial_2 & 0 \\ \partial_2 & 0 & -D_3 \\ 0 & D_1 & \alpha_3 \end{bmatrix}} Q^7.$$

Changing bases yields the equivalent matrix factorization:

$$Q^7 \xrightarrow{\begin{bmatrix} \partial_3 & D_2 & 2\alpha_2 \\ 0 & \frac{1}{2}q & D_2 \\ 0 & 0 & \partial_1 \end{bmatrix}} Q^7 \xrightarrow{\begin{bmatrix} D_1 & 0 & -2\alpha_3 \\ -\partial_2 & 2\alpha_1 & 0 \\ 0 & -\partial_2 & D_3 \end{bmatrix}} Q^7.$$

Finally, we set $M_2(2)$ and $M_1(2)$ to be these matrices tensored down to R, that is,

$$M_2(2) := \begin{bmatrix} \partial_3 & D_2 & 2\alpha_2 \\ 0 & \frac{1}{2}q & D_2 \\ 0 & 0 & \partial_1 \end{bmatrix} \quad \text{and} \quad M_1(2) := \begin{bmatrix} D_1 & 0 & -2\alpha_3 \\ -\partial_2 & 2\alpha_1 & 0 \\ 0 & -\partial_2 & D_3 \end{bmatrix},$$

where both matrices have entries in R. Recall from 2.3.7 that the two-periodic complex

$$\cdots \to R^7 \xrightarrow{M_2(2)} R^7 \xrightarrow{M_1(2)} R^7 \xrightarrow{M_2(2)} R^7 \xrightarrow{M_1(2)} R^7 \to \cdots$$

is exact.

4.3. *Resolution of* $\ker J_{2,1}$

In this subsection we establish exactness of the following sequence of R-modules:

$$\cdots \xrightarrow{M_1(2)} R^7 \xrightarrow{M_2(2)} R^7 \xrightarrow{M_1(2)} R^7 \xrightarrow{M_0(2)} R^6 \xrightarrow{J_{2,1}} R^3,$$

where the definitions of the matrices $M_i(2)$ can be found in 4.1.3 and 4.2.2.

LEMMA 4.3.1. *The R_x-module $(\ker_R(J_{2,1}))_x = \ker_{R_x}(J_{2,1})$ is freely generated by E^2, EH_{yz}, and H_{yz}^2.*

Proof. We show that the given set generates $\ker J_{2,1}$. For this, recall that since R has an isolated singularity, we have that f_y, f_z forms a (possibly improper) regular sequence in R_x: indeed, R_x is a domain so f_y and f_z are nonzerodivisors, and by the Euler relation, $f_x \in (f_y, f_z)R_x$, so $(f_y, f_z)R_x = (f_x, f_y, f_z)R_x = R_x$ by the isolated singularity hypothesis, and then f_y, f_z is an (improper) regular sequence by the Chinese remainder theorem. To compute $\ker J_{2,1}$, we let $[a_{xx}, a_{xy}, a_{xz}, a_{yy}, a_{yz}, a_{zz}]^T$ be an element in $\ker_{R_x}(J_{2,1})$.

Rewriting each $f_x = -(\frac{y}{x} f_y + \frac{z}{x} f_z)$ in the $J_{2,1}$ matrix, and multiplying it by $[a_{xx}, a_{xy}, a_{xz}, a_{yy}, a_{yz}, a_{zz}]^T$, we get the following equalities:

$$\begin{aligned} 0 &= -\left(\frac{y}{x} f_y + \frac{z}{x} f_z\right) a_{xx} + f_y a_{xy} + f_z a_{xz} \\ &= f_y\left(a_{xy} - \frac{y}{x} a_{xx}\right) + f_z\left(a_{xz} - \frac{z}{x} a_{xx}\right) \end{aligned}$$

and hence as f_y, f_z is R_x-regular, there exists b_1 in R_x such that

$$a_{xy} = b_1 f_z + \frac{y}{x} a_{xx}, \qquad a_{xz} = -b_1 f_y + \frac{z}{x} a_{xx}. \tag{17}$$

Similarly, from

$$\begin{aligned} -(y/x f_y + z/x f_z) a_{xy} + f_y a_{yy} + f_z a_{yz} = 0 \quad \text{and} \\ -(y/x f_y + z/x f_z) a_{xz} + f_y a_{yz} + f_z a_{zz} = 0 \end{aligned}$$

we conclude that there exist $b_2 \in R_x$ from the first equation and $b_3 \in R_x$ from the second equation, such that

$$\begin{aligned} a_{yy} &= b_2 f_z + \frac{y}{x} a_{xy}, \qquad a_{yz} = -b_2 f_y + \frac{z}{x} a_{xy}, \\ a_{yz} &= b_3 f_z + \frac{y}{x} a_{xz}, \qquad a_{zz} = -b_3 f_y + \frac{z}{x} a_{xz}. \end{aligned}$$

Substituting (17) into these equations yields

$$\begin{aligned} a_{yy} &= b_2 f_z + b_1 \frac{y}{x} f_z + \frac{y^2}{x^2} a_{xx}, \qquad a_{yz} = -b_2 f_y + b_1 \frac{z}{x} f_z + \frac{yz}{x^2} a_{xx}, \\ a_{yz} &= b_3 f_z - b_1 \frac{y}{x} f_y + \frac{yz}{x^2} a_{xx}, \qquad a_{zz} = -b_3 f_y - b_1 \frac{z}{x} f_y + \frac{z^2}{x^2} a_{xx}. \end{aligned}$$

Equating the expressions for a_{yz}, one can see that

$$f_y\left(b_2 - \frac{y}{x} b_1\right) + f_z\left(b_3 - \frac{z}{x} b_1\right) = 0,$$

and so finally there exists $c \in R_x$ such that

$$b_2 = c f_z + \frac{y}{x} b_1, \qquad b_3 = -c f_y + \frac{z}{x} b_1.$$

Therefore, from the equalities above and substituting in $a = a_{xx}/2x^2$, $b = b_1/x$, we get that

$$\begin{bmatrix} a_{xx} \\ a_{xy} \\ a_{xz} \\ a_{yy} \\ a_{yz} \\ a_{zz} \end{bmatrix} = \begin{bmatrix} 2ax^2 \\ bxf_z + 2axy \\ -bxf_y + 2axz \\ cf_z^2 + 2byf_z + 2ay^2 \\ -cf_y f_z - b(zf_z - yf_y) + 2ayz \\ cf_y^2 - 2bzf_y + 2az^2 \end{bmatrix} = aE^2 + bEH_{yz} + cH_{yz}^2.$$

To see linear independence over R_x, suppose that $aE^2 + bEH_{yz} + cH_{yz}^2 = 0$ with $a, b, c \in R_x$. The $\partial_x^{(2)}$-coordinate of the left-hand side is $2ax^2$, so $a = 0$. After substituting $a = 0$, the $\partial_x\partial_y$-coordinate of the left-hand side is bxf_z, so $b = 0$. Then, substituting $c = 0$, the $\partial_y^{(2)}$-coordinate is cf_z^2, so $c = 0$, and the linear independence follows. □

LEMMA 4.3.2. *We have $M_0(2)M_1(2) = 0_{6\times 7}$.*

Proof. We verify that each column of $M_1(2)$ yields a relation on the columns of $M_0(2)$, which correspond to the generators of $\ker(J_{2,1})$.

Composing the relations in (13) with $\mathcal{E}$ (cf. 2.2.4) yields relations

$$\begin{aligned} f_x E^2 + yEH_{xy} - zEH_{zx} &= 0, \\ f_y E^2 - xEH_{xy} + zEH_{yz} &= 0, \\ f_z E^2 + xEH_{zx} - yEH_{yz} &= 0, \end{aligned}$$

which correspond to the first three columns of $M_1(2)$.

There is another relation of the form

$$y\alpha_x - x\alpha_y = \frac{2}{d-1}(f_{xz}EH_{yz} + f_{yz}EH_{zx} + f_{zz}EH_{xy}). \tag{18}$$

We check coordinate by coordinate. In the $\partial_x^{(2)}$ coordinate, we have

$$\begin{aligned} \frac{2}{(d-1)^2}(xy\Delta_{xx} - 2x^2\Delta_{xy} + xy\Delta_{xx}) &= \frac{4x}{(d-1)^2}(y\Delta_{xx} - x\Delta_{xy}) \\ &= \frac{2}{d-1}(-2xf_z f_{yz} + 2xf_y f_{zz}), \end{aligned}$$

where the second equality uses (53). In the $\partial_x\partial_y$ coordinate, we have the following, where (53) is again applied for the second equality:

$$\begin{aligned} &\frac{2}{(d-1)^2}(y^2\Delta_{xx} - x^2\Delta_{yy}) \\ &\quad = \frac{2y}{(d-1)^2}(y\Delta_{xx} - x\Delta_{xy}) - \frac{2x}{(d-1)^2}(x\Delta_{yy} + y\Delta_{xy}) \\ &\quad = \frac{2}{d-1}(xf_z f_{xz} - yf_z f_{yz} + yf_y f_{zz} - xf_x f_{zz}). \end{aligned}$$

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In the $\partial_x\partial_z$ coordinate, another application of (53) yields the second equality below

$$\begin{aligned}
&\frac{2}{(d-1)^2}(yz\Delta_{xx} - xz\Delta_{xy} - x^2\Delta_{yz} + xy\Delta_{xz})\\
&\quad = \frac{2z}{(d-1)^2}(y\Delta_{xx} - x\Delta_{xy}) - \frac{2x}{(d-1)^2}(x\Delta_{yz} - y\Delta_{xz})\\
&\quad = \frac{2}{d-1}(-xf_y f_{xz} - zf_z f_{yz} + xf_x f_{yz} + zf_y f_{zz}).
\end{aligned}$$

The $\partial_y^{(2)}$ coordinate follows from the $\partial_x^{(2)}$ coordinate by symmetry, and likewise the $\partial_y\partial_z$ coordinate follows from the $\partial_x\partial_z$ coordinate. In the $\partial_z^{(2)}$ coordinate, a final application of (53) establishes

$$\begin{aligned}
&\frac{2}{(d-1)^2}(2yz\Delta_{xz} - xy\Delta_{zz} - 2xz\Delta_{yz} + xy\Delta_{zz})\\
&\quad = \frac{2}{d-1}(-2zf_y f_{xz} + 2zf_x f_{yz}).
\end{aligned}$$

Thus we have established the relation in (18) that corresponds to the sixth column of $M_1(2)$; the relations coming from the fourth and fifth columns of $M_1(2)$ follow from this one by symmetry.

The last relation, corresponding to the seventh column of $M_1(2)$ is

$$-\frac{2\delta}{(d-1)^3}E^2 + f_x\alpha_x + f_y\alpha_y + f_z\alpha_z = 0.$$

To see this, by symmetry, it suffices to check for the $\partial_x^{(2)}$ and $\partial_x\partial_y$ coordinates. In the $\partial_x^{(2)}$ coordinate, we have

$$\begin{aligned}
&-\frac{2}{(d-1)^3}2x^2\delta\\
&\qquad + \frac{2}{(d-1)^2}(xf_x\Delta_{xx} + 2xf_y\Delta_{xy} - yf_y\Delta_{xx} + 2xf_z\Delta_{xz} - zf_z\Delta_{xx})\\
&\quad = \frac{4x}{(d-1)^3}(-x\delta + (d-1)(f_x\Delta_{xx} + f_y\Delta_{xy} + f_z\Delta_{xz}))\\
&\quad = 0,
\end{aligned}$$

where the last equality is (52).

In the $\partial_x\partial_y$ coordinate, we have

$$\begin{aligned}
&-\frac{2}{(d-1)^3}2xy\delta + \frac{2}{(d-1)^2}(yf_x\Delta_{xx} + xf_y\Delta_{yy} + xf_z\Delta_{yz} + yf_z\Delta_{xz} - zf_z\Delta_{xy})\\
&\quad = \frac{-4xy}{(d-1)^3}\delta + \frac{2x}{(d-1)^2}(f_y\Delta_{yy} + f_z\Delta_{yz}) + \frac{2y}{(d-1)^2}(f_x\Delta_{xx} + f_z\Delta_{xz})\\
&\qquad - \frac{2z}{(d-1)^2}f_z\Delta_{xy}\\
&\quad = \frac{-4xy}{(d-1)^3}\delta + \frac{2x}{(d-1)^2}\left(\frac{y\delta}{d-1} - f_x\Delta_{xy}\right) + \frac{2y}{(d-1)^2}\left(\frac{x\delta}{d-1} - f_y\Delta_{xy}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2z}{(d-1)^2} f_z \Delta_{xy} \\
&= \frac{2\Delta_{xy}}{(d-1)^2}(-xf_x - yf_y - zf_z) = 0,
\end{aligned}
$$

where the second equality uses (52). □

LEMMA 4.3.3. *We have* $(\ker M_0(2))_x = (\operatorname{im} M_1(2))_x$.

Proof. Let $v \in (\ker M_0(2))_x$; write $v = (a, b_x, b_y, b_z, c_x, c_y, c_z) \in R_x^7$ so that

$$aE^2 + b_x H_{yz} + b_y H_{zx} + b_z H_{xy} + c_x\alpha_x + c_y\alpha_y + c_z\alpha_z = 0. \tag{19}$$

From Lemma 4.3.1 and symmetry we have that E^2, EH_{xy}, and H_{xy}^2 freely generate

$$R_x\langle E^2, EH_{yz}, EH_{zx}, EH_{xy}, \alpha_x, \alpha_y, \alpha_z\rangle.$$

Using the relations on the generators, we have that

$$
\begin{aligned}
\alpha_x &= \frac{x}{z^2}H_{xy}^2 - 2\frac{f_y}{z^2}EH_{xy} + \frac{1}{x}\left(\frac{f_y^2}{z^2} + \frac{\Delta_{xx}}{(d-1)^2}\right)E^2, \\
\alpha_y &= \frac{y}{z^2}H_{xy}^2 + 2\frac{f_x}{z^2}EH_{xy} + \frac{1}{y}\left(\frac{f_x^2}{z^2} + \frac{\Delta_{yy}}{(d-1)^2}\right)E^2, \\
\alpha_z &= \frac{1}{z}H_{xy}^2 + \frac{\Delta_{zz}}{(d-1)^2 z}E^2.
\end{aligned}
$$

By considering the H_{yz}^2 coefficient on relation (19), we get that $xc_x + yc_y + zc_z = 0$. Observe that $D_3 = \partial_2[0, f_z/x, f_y/x]^T$, so $\ker[x \;\; y \;\; z]R_x$ is generated by the image of ∂_2^T. Thus, using columns 4–6 of $M_1(2)$, there exists a relation $v' \in (\operatorname{im} M_1(2))_x$ such that the c_x, c_y, c_z coordinates of v' agree with v. Replacing v with $v - v'$, we may assume that $c_x = c_y = c_z = 0$.

Now, we have

$$EH_{yz} = \frac{x}{z}EH_{xy} - \frac{f_y}{z^2}E^2, \qquad EH_{zx} = \frac{y}{z}EH_{xy} + \frac{f_x}{z^2}E^2, \qquad EH_{xy} = EH_{xy},$$

and by considering the EH_{yz} coefficient on relation (19), we get that $xb_x + yb_y + zb_z = 0$. Using columns 1–3 of $M_1(2)$, we can find a relation $v'' \in (\operatorname{im} M_1(2))_x$ such that the b_x, b_y, b_z-coordinates of v'' agree with those of v and the c_x, c_y, c_z coordinates are zero. Replacing v with $v - v''$, we can assume that $b_x = b_y = b_z = c_x = c_y = c_z = 0$. But $aE^2 = 0$ implies $a = 0$, and the assertion follows. □

We are now ready to prove the main result of this subsection.

PROPOSITION 4.3.4. *The sequence of* R*-modules*

$$\cdots \xrightarrow{M_1(2)} R^7 \xrightarrow{M_2(2)} R^7 \xrightarrow{M_1(2)} R^7 \xrightarrow{M_0(2)} R^6 \xrightarrow{J_{2,1}} R^3$$

is exact. In particular, the R*-module* $\ker_R J_{2,1}$ *is generated by the operators*

$$\{E^2, EH_{yz}, EH_{zx}, EH_{xy}, \alpha_x, \alpha_y, \alpha_z\}.$$

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Proof. First note that since $M_1(2)$ and $M_2(2)$ are the images over R of a matrix factorization of f over Q as proved in 4.2.2, one obtains that the (infinite) portion of the sequence involving those matrices is exact; see 4.2.2. Furthermore, the initial portion of the sequence is also a complex by 4.3.2 and the facts that the columns of $M_0(2)$ correspond locally to compositions of lower order operators and that x, y, and z are each nonzerodivisors over R.

For the initial portion of the sequence, we argue by localization. We begin by using the lemmas above to establish that the homologies of the sequence are supported at the homogeneous maximal ideal (that is, the sequence is exact on the punctured spectrum). Let $\mathfrak{p} \neq (x, y, z)$, so that x, y, or z is a unit after localizing at $\mathfrak{p}$. By Lemma 4.3.3 and symmetry, we have $(\ker M_0(2))_\mathfrak{p} = (\operatorname{im} M_1(2))_\mathfrak{p}$. Furthermore, to see $\ker(J_{2,1})_\mathfrak{p} = \operatorname{im} M_0(2)$, note that, since H^2_{yz} is generated by α_x and E^2 over R_x, $\ker(J_{2,1})_x$ is generated by E^2, EH_{yz}, and α_x. By symmetry, we have that $\ker(J_{2,1})_y$ is generated by E^2, EH_{zx}, and α_y; and $\ker(J_{2,1})_z$ is generated by E^2, EH_{xy}, and α_z. Thus, if $\mathfrak{p} \neq (x, y, z)$, we have that $\ker(J_{2,1})_\mathfrak{p}$ is generated by E^2, EH_{yz}, EH_{zx}, EH_{xy}, α_x, α_y, α_z.

Next we use depth to argue that the initial portion of the sequence is, in fact, exact. For this, we use the lemma below in stages. First consider the inclusion

$$i\colon \operatorname{im} M_1(2) \hookrightarrow \ker M_0(2).$$

From the exactness of the infinite periodic portion of the complex above, we see that $\operatorname{im} M_1(2)$ is an infinite syzygy, hence maximal Cohen–Macaulay, and so it satisfies S_2 as the ring R is a 2-dimensional hypersurface. Furthermore, since $\ker M_0(2)$ is contained in R^7, it satisfies S_1. By exactness on the punctured spectrum as explained before, the inclusion i is isomorphism when localized at each prime of height at most 1, and so by Lemma 4.3.5 it is an isomorphism.

Next consider the inclusion

$$j\colon \operatorname{im} M_0(2) \hookrightarrow \ker J_{2,1}.$$

To see that $\operatorname{im} M_0(2)$ satisfies S_2, we note that by the previous step, one has an isomorphism $\operatorname{coker} M_1(2) \cong \operatorname{im} M_0(2)$, and again from the exactness of the infinite periodic portion of the complex above $\operatorname{coker} M_1(2) = \ker M_2(2)$ is an infinite syzygy and hence maximal Cohen–Macaulay. The module $\ker J_{2,1}$ satisfies S_2 because it is contained in a free R-module, and hence j is an isomorphism by Lemma 4.3.5 and exactness on the punctured spectrum as explained before. □

The following is a standard useful tool for proving that maps are isomorphisms. It can be verified by depth-counting arguments after localization at minimal primes in the support of the cokernel and the kernel.

LEMMA 4.3.5. *Let $\varphi\colon M \to N$ be a homomorphism of finitely generated modules over a Noetherian ring. Suppose that M and N satisfy Serre's property S_2 and S_1, respectively. If the localization $\varphi_\mathfrak{p}$ is an isomorphism for all primes of height at most 1, then φ is an isomorphism.*

4.4. *Resolution of* $D^2_{R|k}$

We now construct the minimal R-free resolution of $D^2_{R|k}$.

4.4.1. Recall the R-free resolution of $R/(f_x, f_y, f_z)$ constructed in 2.4.5:

$$G(1) = \cdots \xrightarrow{M_2(1)} R^4 \xrightarrow{M_1(1)} R^4 \xrightarrow{M_2(1)} R^4 \xrightarrow{M_1(1)} R^4 \xrightarrow{M_0(1)} R^3 \xrightarrow{J} R \to 0.$$

Also, consider the sequence

$$G(2) = \cdots \xrightarrow{M_2(2)} R^7 \xrightarrow{M_1(2)} R^7 \xrightarrow{M_2(2)} R^7 \xrightarrow{M_1(2)} R^7 \xrightarrow{M_0(2)} R^6 \xrightarrow{-J_{2,1}} R^3 \to 0,$$

which is an R-free resolution of $\operatorname{coker} J_{2,1}$ by Proposition 4.3.4. Define $\theta(2) : \Sigma^{-1}G(2) \to G(1)$ according to the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "-M_1(2)"] & R^7 \arrow[r, "-M_2(2)"] \arrow[d, "\theta_3(2)"] & R^7 \arrow[r, "-M_1(2)"] \arrow[d, "\theta_2(2)"] & R^7 \arrow[r, "-M_0(2)"] \arrow[d, "\theta_1(2)"] & R^6 \arrow[r, "-J_{2,1}"] \arrow[d, "\theta_0(2)"] & R^3 \arrow[r] & 0 \\
\cdots \arrow[r, "M_2(1)"] & R^4 \arrow[r, "M_1(1)"] & R^4 \arrow[r, "M_0(1)"] & R^3 \arrow[r, "J"] & R \arrow[r] & 0 &
\end{tikzcd}$$

where each $\theta_i(2)$ is defined in Section 3.

LEMMA 4.4.2. *The map* $\theta(2)\colon \Sigma^{-1}G(2) \to G(1)$, *defined in* 4.4.1, *is a morphism of complexes. In particular, its cone, which we denote by* (C, ∂^C), *is the following nonnegatively graded complex of free* R*-modules*:

$$C := \operatorname{cone}(\theta(2)) = \cdots \to \begin{matrix} R^4 \\ \oplus \\ R^7 \end{matrix} \xrightarrow{\begin{bmatrix} M_1(1) & \theta_2(2) \\ 0 & M_1(2) \end{bmatrix}} \begin{matrix} R^4 \\ \oplus \\ R^7 \end{matrix} \xrightarrow{\begin{bmatrix} M_0(1) & \theta_1(2) \\ 0 & M_0(2) \end{bmatrix}} \begin{matrix} R^3 \\ \oplus \\ R^6 \end{matrix} \xrightarrow{\begin{bmatrix} D_1 & \theta_0(2) \\ 0 & J_{2,1} \end{bmatrix}} \begin{matrix} R \\ \oplus \\ R^3 \end{matrix} \to 0.$$

Proof. Adopt the notation from 4.1.3. For the rightmost square in 4.4.1, observe that

$$\begin{aligned}
&-\theta_0(2)M_0(2) \\
&\quad = \frac{1}{d-1}\begin{bmatrix} 0_{1\times 4} & f_x\Delta_{xx} + f_y\Delta_{xy} + f_z\Delta_{xz} & f_x\Delta_{xy} + f_y\Delta_{yy} + f_z\Delta_{yz} & f_x\Delta_{xz} + f_y\Delta_{yz} + f_z\Delta_{zz} \end{bmatrix} \\
&\quad = \begin{bmatrix} 0_{1\times 4} & J\alpha_2 \end{bmatrix} = J\theta_1(2);
\end{aligned}$$

we justify the first equality, and the remaining ones are evident.

For the first equality, first use (49) and (50) from Appendix A to deduce that the first four columns of $M_0(2)$ are in the kernel of $\theta_0(2)$. Now note that

$$\begin{aligned}
\theta_0(2)(\alpha_z) &= (d-1)(f_x\Delta_{xz} + f_y\Delta_{yz} + f_z\Delta_{zz}) - z\theta_0(2)\begin{bmatrix} \Delta_{xx} \\ \Delta_{xy} \\ \Delta_{xz} \\ \Delta_{yy} \\ \Delta_{yz} \\ \Delta_{zz} \end{bmatrix} \\
&= (d-1)(f_x\Delta_{xz} + f_y\Delta_{yz} + f_z\Delta_{zz}) - \frac{3}{2}z\delta
\end{aligned}$$

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$$
\begin{aligned}
&= (d-1)(f_x\Delta_{xz} + f_y\Delta_{yz} + f_z\Delta_{zz}) \\
&\quad - \frac{3}{2}(d-1)(f_x\Delta_{xz} + f_y\Delta_{yz} + f_z\Delta_{zz}) \\
&= -\frac{d-1}{2}(f_x\Delta_{xz} + f_y\Delta_{yz} + f_z\Delta_{zz});
\end{aligned}
$$

the first equality follows from (49), the second equality uses (51), the third equality follows from (52). This shows that the last entries are equal and the equality in the two remaining entries can be shown similarly.

The second square commutes as

$$
\begin{aligned}
-\theta_1(2)M_1(2) &= (d-1)\begin{bmatrix} 0_{3\times3} & \alpha_2\partial_2 & -\alpha_2 D_3 \end{bmatrix} \\
&= (d-1)\begin{bmatrix} 0_{3\times3} & D_2\alpha_1 & -\partial_3\alpha_3 \end{bmatrix} \\
&= M_0(1)\theta_2(2),
\end{aligned}
$$

where the second equality uses (86). Also, to see that the third square commutes observe that

$$
\begin{aligned}
-\theta_2(2)M_2(2) &= -(d-1)\begin{bmatrix} 0_{3\times3} & 0_{3\times3} & -\alpha_3\partial_1 \\ 0_{1\times3} & \frac{1}{2}\alpha_1 q & \alpha_1 D_2 \end{bmatrix} \\
&= -\frac{d-1}{2}\begin{bmatrix} 0_{3\times3} & 0_{3\times3} & -2D_1\alpha_2 \\ 0_{1\times3} & D_3\partial_1 & \partial_2\alpha_2 \end{bmatrix} \\
&= M_1(1)\theta_3(2),
\end{aligned}
$$

where the second equality uses the symmetry of α_i, (86), and (89).

Finally, since $(M_2(i), M_1(i))$ for $i = 1, 2$ correspond to matrix factorizations of $d \cdot f$ over Q and

$$-\theta_2(2)M_2(2) = M_1(1)\theta_3(2),$$

it is standard that

$$-\theta_3(2)M_2(1) = M_2(1)\theta_2(2).$$

Now, using the two-periodicity of $G(1)$ and $\Sigma^{-1}G(2)$ after homological degree two, all the remaining squares in the diagram commute. Thus $\theta(2)$ is a map of complexes, so taking its cone yields the desired complex C defined before. $\square$

THEOREM 4.4.3. *Assume that $R = k[x, y, z]/(f)$ is an isolated singularity hypersurface where k is a field of characteristic zero. The augmented minimal R-free resolution of $D^2_{R|k} \subseteq R^3 \oplus R^6 \oplus R$ has the form*

$$
\cdots \xrightarrow{\begin{bmatrix} M_2(1) & \theta_3(2) \\ 0 & M_2(2) \end{bmatrix}} \begin{matrix} R^4 \\ \oplus \\ R^7 \end{matrix} \xrightarrow{\begin{bmatrix} M_1(1) & \theta_2(2) \\ 0 & M_1(2) \end{bmatrix}} \begin{matrix} R^4 \\ \oplus \\ R^7 \end{matrix} \xrightarrow{\begin{bmatrix} M_2(1) & \theta_3(2) \\ 0 & M_2(2) \end{bmatrix}} \begin{matrix} R^4 \\ \oplus \\ R^7 \end{matrix} \xrightarrow{\begin{bmatrix} M_1(1) & \theta_2(2) \\ 0 & M_1(2) \\ 0 & 0 \end{bmatrix}} \begin{matrix} R^4 \\ \oplus \\ R^7 \\ \oplus \\ R \end{matrix} \xrightarrow{\varepsilon} D^2_{R|k} \to 0,
$$

where a minimal set of generators for $D^2_{R|k}$ is given by the columns of

$$
\varepsilon = \begin{bmatrix} M_0(1) & \theta_1(2) & 0 \\ 0 & M_0(2) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

and the block matrices are defined in 4.1.3, 4.2.2, and Section 3.

Proof. By Lemma 4.4.2, C is a bounded below complex of finite rank free R-modules. Moreover, there is the cone exact sequence of complexes

$$0 \to G(1) \to C \to G(2) \to 0. \tag{20}$$

Since $G(1)$ and $G(2)$ have homology concentrated in degree zero by 2.4.5 and Proposition 4.3.4, examining the induced long exact sequence in homology forces C to also have homology concentrated only in degree zero. Thus C is a minimal free resolution of $\operatorname{coker} \partial_1^C$. The desired conclusion now follows from the R-module isomorphism

$$D^2_{R|k} \cong \ker \partial_1^C \oplus R = \operatorname{im} \partial_2^C \oplus R = \operatorname{im} \varepsilon,$$

which follows from 2.2.4, observing that ∂_1^C equals P_2 in 2.2.2 and noting that $D_1 = J_{1,0}$ and $\theta_0(2) = J_{2,0}$. □

By truncating the exact sequence of complexes in (20) in homological degrees one and higher, the resulting long exact sequence in homology yields the following corollary to Theorem 4.4.3.

COROLLARY 4.4.4. *There is a short exact sequence of R-modules*

$$0 \to D^1_{R|k} \to D^2_{R|k} \to \ker J_{2,1} \to 0,$$

where $D^1_{R|k} \to D^2_{R|k}$ is the inclusion.

Similar to 2.4.7, the differentials in the resolution of $D^2_{R|k}$ are homogeneous with respect to the internal grading of R where

$$|\alpha_x| = |\alpha_y| = |\alpha_z| = 2d - 5.$$

Hence, from Theorem 4.4.3 we can read off the graded Betti numbers of $D^2_{R|k}$.

COROLLARY 4.4.5. *The graded Betti numbers of $D^2 = D^2_{R|k}$ are given by*

$$\beta^R_{0,j}(D^2) = \begin{cases} 3, & j = 0, \\ 6, & j = d-2, \\ 3, & j = 2d-5, \\ 0, & \textit{otherwise}, \end{cases}$$

and for $n \geq 1$,

$$\beta^R_{2n-1,j}(D^2) = \begin{cases} 6, & j = nd-1, \\ 1, & j = nd+d-3, \\ 3, & j = nd+d-4, \\ 1, & j = nd+2d-6, \\ 0, & \textit{otherwise} \end{cases} \qquad \textit{and}$$

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$$\beta^R_{2n,j}(D^2) = \begin{cases} 2, & j = nd, \\ 6, & j = nd + d - 2, \\ 3, & j = nd + 2d - 5, \\ 0, & \textit{otherwise}. \end{cases}$$

COROLLARY 4.4.6. *The following inequalities are both satisfied*

$$\mathsf{level}^k_{\mathsf{D_{sg}}(R)}(\ker J_{2,1}) \leq 2 \quad \textit{and} \quad \mathsf{level}^k_{\mathsf{D_{sg}}(R)}(D^2_{R|k}) \leq 3.$$

Proof. The second inequality follows from the first as

$$\mathsf{level}^k_{\mathsf{D_{sg}}(R)}(D^2_{R|k}) \leq \mathsf{level}^k_{\mathsf{D_{sg}}(R)}(\ker J_{2,1}) + \mathsf{level}^k_{\mathsf{D_{sg}}(R)}(k)$$

using Corollary 4.4.4, as well as Remark 2.4.4. For the first inequality, from 4.2.1 and 4.2.2 it follows that the matrix factorization modeling the tail of the minimal free resolution of $\ker J_{2,1}$ is a mapping cone on the matrix factorization of k (up to a shift), and as a consequence the desired inequality is satisfied. ☐

REMARK 4.4.7. Since R is an isolated singularity,

$$A \coloneqq Q/(f_x, f_y, f_z) = R/(f_x, f_y, f_z)$$

is an Artinian complete intersection. Its socle is δ as defined in Section 3. It follows from [18, Exercise 21.23] that the matrix factorization corresponding to the R-resolution of $A/(\delta)$ is the same as the matrix factorization constructed in 4.2.2. Therefore there is an isomorphism $\ker J_{2,1} \simeq A/(\delta)$ in $\mathsf{D_{sg}}(R)$, see 2.3.8. Hence, $D^2_{R|k}$ is a mapping cone of a morphism $\Sigma^{-1}A/(\delta) \to k$ in $\mathsf{D_{sg}}(R)$. The isomorphism above does not necessarily hold when Q has more than three variables; indeed, even for straightforward examples in four variables their corresponding matrix factorizations have different ranks.

4.5. *A Set of Generators of $D^2_{R|k}$*

Now we write the generators of $D^2_{R|k}$ given in Theorem 4.4.3 as differential operators. By abuse of notation, we consider elements of $\ker(J_{2,1})$ as elements of the ambient free module R^{10} of $D^2_{R|k}$ as in Remark 4.1.1. We begin by writing some relevant operators of order two in terms of lifts of their images in $\ker(J_{2,1})$ and operators of order one, as follows:

$$\mathcal{E}^2 = E^2 + \mathcal{E}, \tag{21}$$

$$\begin{aligned} &\mathcal{E}\mathcal{H}_{yz} = EH_{yz} + (d-1)\mathcal{H}_{yz}, \qquad \mathcal{E}\mathcal{H}_{zx} = EH_{zx} + (d-1)\mathcal{H}_{zx}, \\ &\mathcal{E}\mathcal{H}_{xy} = EH_{xy} + (d-1)\mathcal{H}_{xy}, \end{aligned} \tag{22}$$

$$\begin{aligned} \mathcal{H}^2_{yz} &= H^2_{yz} + \frac{1}{d-1} x(\Delta_{xx}\partial_x + \Delta_{xy}\partial_y + \Delta_{xz}\partial_z) - \frac{1}{d-1}\Delta_{xx}\mathcal{E}, \\ \mathcal{H}^2_{zx} &= H^2_{zx} + \frac{1}{d-1} y(\Delta_{xy}\partial_x + \Delta_{yy}\partial_y + \Delta_{yz}\partial_z) - \frac{1}{d-1}\Delta_{yy}\mathcal{E}, \\ \mathcal{H}^2_{xy} &= H^2_{xy} + \frac{1}{d-1} z(\Delta_{xz}\partial_x + \Delta_{yz}\partial_y + \Delta_{zz}\partial_z) - \frac{1}{d-1}\Delta_{zz}\mathcal{E}. \end{aligned} \tag{23}$$

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Note that above we are lifting the representative for E^2 in Section 4.1 from $\ker(J_{2,1}) \cong D^2_{R|k}/D^1_{R|k}$ to $D^2_{R|k}$, and so

$$E^2 = x^2\partial_x^2 + 2xy\partial_x\partial_y + 2xz\partial_x\partial_z + y^2\partial_x + 2yz\partial_y\partial_z + z^2\partial_z^2;$$

the same convention is adopted for EH_{ij}.

THEOREM 4.5.1. *A minimal set of generators for $D^2_{R|k}$ is given by $G_0 \cup G_1 \cup G_2$, where $G_0 = \{1\}$, $G_1 = \{\mathcal{E}, \mathcal{H}_{yz}, \mathcal{H}_{zx}, \mathcal{H}_{xy}\}$ and*

$$G_2 = \{\mathcal{E}^2,\ \mathcal{E}\mathcal{H}_{yz},\ \mathcal{E}\mathcal{H}_{zx},\ \mathcal{E}\mathcal{H}_{xy},\ \mathcal{A}_x,\ \mathcal{A}_y,\ \mathcal{A}_z\},$$

with

$$\begin{aligned}
\mathcal{A}_x &= \frac{1}{x}\left[\mathcal{H}_{yz}^2 + \frac{1}{(d-1)^2}\Delta_{xx}\mathcal{E}^2 + \frac{d-2}{(d-1)^2}\Delta_{xx}\mathcal{E}\right],\\
\mathcal{A}_y &= \frac{1}{y}\left[\mathcal{H}_{zx}^2 + \frac{1}{(d-1)^2}\Delta_{yy}\mathcal{E}^2 + \frac{d-2}{(d-1)^2}\Delta_{yy}\mathcal{E}\right],\\
\mathcal{A}_z &= \frac{1}{z}\left[\mathcal{H}_{xy}^2 + \frac{1}{(d-1)^2}\Delta_{zz}\mathcal{E}^2 + \frac{d-2}{(d-1)^2}\Delta_{zz}\mathcal{E}\right].
\end{aligned}$$

Proof. First note that a generating set for $D^2_{R|k}$ can be given by the union of a minimal set of generators for $D^1_{R|k}$ and any set of lifts to $D^2_{R|k}$ of a minimal set of generators for the quotient $D^2_{R|k}/D^1_{R|k}$. First, from Section 2.4, especially (11) and (12), the union $G_0 \cup G_1$ is a minimal generating set for $D^1_{R|k}$. Second, we claim that G_2 is the desired set of lifts from the quotient $D^2_{R|k}/D^1_{R|k}$. As compositions of lower order operators, the first seven elements of G_2 are clearly in $D^2_{R|k}$. Furthermore, $\mathcal{A}_x$, $\mathcal{A}_y$, $\mathcal{A}_z$ are also well-defined operators, so elements of $D^2_{R|k}$: Indeed, using (21) and (23), one can check that $\mathcal{A}_x$, $\mathcal{A}_y$, and $\mathcal{A}_z$ agree with the generators corresponding to last three columns of the augmentation map ε in Theorem 4.4.3. Now, since the images of the elements of G_2 in the quotient $D^2_{R|k}/D^1_{R|k}$ are the minimal set of generators $\{E^2, EH_{yz}, EH_{zx}, EH_{xy}, \alpha_x, \alpha_y, \alpha_z\}$ from Lemma 4.3.1, we get that $G_0 \cup G_1 \cup G_2$ is a set of generators for $D^2_{R|k}$. This is in fact a minimal set of generators of $D^2_{R|k}$ since its cardinality agrees with $\beta_0^R(D_{R|k})$ calculated in Theorem 4.4.3. □

REMARK 4.5.2. Note that except for $\mathcal{A}_x$, $\mathcal{A}_y$, and A_z, the generators in Theorem 4.5.1 do not agree with the lifts of the generators of $\ker J_{2,1}$, that is, the generators of $D^2_{R|k}$ given in the augmentation map ε in Theorem 4.4.3. However, one can subtract lower order operators so that they do agree. For example, one could replace $\mathcal{E}^2$ in the generating set with $\mathcal{E}^2 - \mathcal{E}$ and $\mathcal{E}\mathcal{H}_{yz}$ with $\mathcal{E}\mathcal{H}_{yz} - (d-1)\mathcal{H}_{yz}$ so that they agree with the corresponding generators from Theorem 4.4.3. One can replace the others similarly using (22).

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5. Differential Operators of Order 3

We continue with Notation 2.0.1. In this section, we construct the minimal R-free resolution of $D^3_{R|k}$; see Theorem 5.4.3. Following the model in the previous section, the majority of the work in the present section is contained in Sections 5.1 to 5.3.

5.1. A Set of Generators of $\ker J_{3,2}$

In this subsection we introduce a set of generators for $\ker J_{3,2}$; see Proposition 5.3.4. We begin by composing $\mathcal{E}$ with each of the generators of $\ker J_{2,1}$ (see 2.2.4)

$$\{E^2, EH_{yz}, EH_{zx}, EH_{xy}, \alpha_x, \alpha_y, \alpha_z\}$$

from Proposition 4.3.4 as well as each Hamiltonian with itself three times, and we compute their coefficients in the basis

$$\{\partial_x^{(3)}, \partial_x^{(2)}\partial_y, \partial_x^{(2)}\partial_z, \partial_x\partial_y^{(2)}, \partial_x\partial_y\partial_z, \partial_x\partial_z^{(2)}, \partial_y^{(3)}, \partial_y^{(2)}\partial_z, \partial_y\partial_z^{(2)}, \partial_z^{(3)}\}$$

as follows:

$$E^3 = 6\begin{bmatrix} x^3 \\ x^2y \\ x^2z \\ xy^2 \\ xyz \\ xz^2 \\ y^3 \\ y^2z \\ yz^2 \\ z^3 \end{bmatrix}, \qquad E^2H_{yz} = \begin{bmatrix} 0 \\ 2x^2f_z \\ -2x^2f_y \\ 4xyf_z \\ -2xyf_y + 2xzf_z \\ -4xzf_y \\ 6y^2f_z \\ -2y^2f_y + 4yzf_z \\ -4yzf_y + 2z^2f_z \\ -6z^2f_y \end{bmatrix},$$

$$E^2H_{zx} = \begin{bmatrix} -6x^2f_z \\ -4xyf_z \\ 2x^2f_x - 4xzf_z \\ -2y^2f_z \\ 2xyf_x - 2yzf_z \\ 4xzf_x - 2z^2f_z \\ 0 \\ 2y^2f_x \\ 4yzf_x \\ 6z^2f_x \end{bmatrix}, \qquad E^2H_{xy} = \begin{bmatrix} 6x^2f_y \\ -2x^2f_x + 4xyf_y \\ 4xzf_y \\ -4xyf_x + 2y^2f_y \\ -2xzf_x + 2yzf_y \\ 2z^2f_y \\ -6y^2f_x \\ -4yzf_x \\ -2z^2f_x \\ 0 \end{bmatrix},$$

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$$E\alpha_x = \frac{2}{(d-1)^2}\begin{bmatrix} 3x^2\Delta_{xx} \\ 3xy\Delta_{xx} \\ 3xz\Delta_{xx} \\ 2xy\Delta_{xy} - x^2\Delta_{yy} + 2y^2\Delta_{xx} \\ xy\Delta_{xz} + xz\Delta_{xy} - x^2\Delta_{yz} + 2yz\Delta_{xx} \\ 2xz\Delta_{xz} - x^2\Delta_{zz} + 2z^2\Delta_{xx} \\ 6y^2\Delta_{xy} - 3xy\Delta_{yy} \\ 2y^2\Delta_{xz} + 4yz\Delta_{xy} - 2xy\Delta_{yz} - xz\Delta_{yy} \\ 4yz\Delta_{xz} - xy\Delta_{zz} + 2z^2\Delta_{xy} - 2xz\Delta_{yz} \\ 6z^2\Delta_{xz} - 3xz\Delta_{zz} \end{bmatrix},$$

$$E\alpha_y = \frac{2}{(d-1)^2}\begin{bmatrix} 6x^2\Delta_{xy} - 3xy\Delta_{xx} \\ 2xy\Delta_{xy} - y^2\Delta_{xx} + 2x^2\Delta_{yy} \\ 4xz\Delta_{xy} - yz\Delta_{xx} + 2x^2\Delta_{yz} - 2xy\Delta_{xz} \\ 3xy\Delta_{yy} \\ yz\Delta_{xy} + xy\Delta_{yz} - y^2\Delta_{xy} + 2xz\Delta_{yy} \\ 2z^2\Delta_{xy} + 4xz\Delta_{yz} - 2yz\Delta_{xz} - xy\Delta_{zz} \\ 3y^2\Delta_{yy} \\ 3yz\Delta_{yy} \\ 2yz\Delta_{yz} - y^2\Delta_{zz} + 2z^2\Delta_{yy} \\ 6z^2\Delta_{yz} - 3yz\Delta_{zz} \end{bmatrix},$$

$$E\alpha_z = \frac{2}{(d-1)^2}\begin{bmatrix} 6x^2\Delta_{xz} - 3xz\Delta_{xx} \\ 2x^2\Delta_{yz} + 4xy\Delta_{xz} - 2xz\Delta_{xy} - yz\Delta_{xx} \\ 2xz\Delta_{xz} - z^2\Delta_{xx} + 2x^2\Delta_{zz} \\ 4xy\Delta_{yz} - xz\Delta_{yy} + 2y^2\Delta_{xz} - 2yz\Delta_{xy} \\ xz\Delta_{yz} + yz\Delta_{xz} - z^2\Delta_{yz} + 2xy\Delta_{zz} \\ 3xz\Delta_{zz} \\ 6y^2\Delta_{yz} - 3yz\Delta_{yy} \\ 2yz\Delta_{yz} - z^2\Delta_{yy} + 2y^2\Delta_{zz} \\ 3yz\Delta_{zz} \\ 3z^2\Delta_{zz} \end{bmatrix},$$

$$H^3_{yz} = 6\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_z^3 \\ -f_y f_z^2 \\ f_y^2 f_z \\ -f_y^3 \end{bmatrix}, \qquad H^3_{zx} = 6\begin{bmatrix} -f_z^3 \\ 0 \\ f_x f_z^2 \\ 0 \\ 0 \\ -f_x^2 f_z \\ 0 \\ 0 \\ 0 \\ f_x^3 \end{bmatrix}, \qquad H^3_{xy} = 6\begin{bmatrix} f_y^3 \\ -f_x f_y^2 \\ 0 \\ f_x^2 f_y \\ 0 \\ 0 \\ -f_x^3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rest of the subsection is spent constructing the remaining generators of a minimal generating set of $\ker J_{3,2}$; see Proposition 5.3.1.

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LEMMA 5.1.1. *The elements*

$$\begin{aligned}
\zeta_x &= \frac{1}{x^2}\left[H_{yz}^3 + \frac{3}{(d-1)^2}\Delta_{xx}E^2H_{yz} - \frac{1}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})E^3\right],\\
\zeta_y &= \frac{1}{y^2}\left[H_{zx}^3 + \frac{3}{(d-1)^2}\Delta_{yy}E^2H_{zx} - \frac{1}{(d-1)^2(d-2)}H_{zx}(\Delta_{yy})E^3\right],\\
\zeta_z &= \frac{1}{z^2}\left[H_{xy}^3 + \frac{3}{(d-1)^2}\Delta_{zz}E^2H_{xy} - \frac{1}{(d-1)^2(d-2)}H_{xy}(\Delta_{zz})E^3\right]
\end{aligned}$$

are well defined in R^{10}.

Proof. We verify that ζ_x is well defined in R^{10}, and the other two calculations follow by symmetry. That is, we claim

$$H_{yz}^3 + \frac{3}{(d-1)^2}\Delta_{xx}E^2H_{yz} - \frac{1}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})E^3,$$

which is given by the matrix

$$\begin{array}{c}
\partial_x^{(3)}\\ \partial_x^{(2)}\partial_y\\ \partial_x^{(2)}\partial_z\\ \partial_x\partial_y^{(2)}\\ \partial_x\partial_y\partial_z\\ \partial_x\partial_z^{(2)}\\ \partial_y^{(3)}\\ \partial_y^{(2)}\partial_z\\ \partial_y\partial_z^{(2)}\\ \partial_z^{(3)}
\end{array}
\begin{bmatrix}
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})x^3\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})x^2y + \frac{6}{(d-1)^2}x^2f_z\Delta_{xx}\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})x^2z - \frac{6}{(d-1)^2}x^2f_y\Delta_{xx}\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})xy^2 + \frac{12}{(d-1)^2}xyf_z\Delta_{xx}\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})xyz - \frac{6}{(d-1)^2}(xyf_y - xzf_z)\Delta_{xx}\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})xz^2 - \frac{12}{(d-1)^2}xzf_y\Delta_{xx}\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})y^3 + \frac{18}{(d-1)^2}y^2f_z\Delta_{xx} + 6f_z^3\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})y^2z - \frac{6}{(d-1)^2}(y^2f_y - 2yzf_z)\Delta_{xx} - 6f_yf_z^2\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})yz^2 - \frac{6}{(d-1)^2}(2yzf_y - z^2f_z)\Delta_{xx} + 6f_y^2f_z\\
-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})z^3 - \frac{18}{(d-1)^2}z^2f_y\Delta_{xx} - 6f_y^3
\end{bmatrix} \tag{24}$$

is divisible by x^2. We show that the $\partial_x\partial_z^{(2)}$ and $\partial_z^{(3)}$ entries are divisible by x^2; the calculations for the other entries are similar.

For the $\partial_x\partial_z^{(2)}$ entry, we have

$$\begin{aligned}
&-\frac{6}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})xz^2 - \frac{12}{(d-1)^2}xzf_y\Delta_{xx}\\
&\quad= -\frac{6}{(d-1)^2(d-2)}\\
&\qquad\times[(f_z\Delta_{xx,y} - f_y\Delta_{xx,z})xz^2 + xzf_y(x\Delta_{xx,x} + y\Delta_{xx,y} + z\Delta_{xx,z})]\\
&\quad= -\frac{6}{(d-1)^2(d-2)}[xz^2f_z\Delta_{xx,y} + x^2zf_y\Delta_{xx,x} + xyzf_y\Delta_{xx,y}]\\
&\quad= -\frac{6}{(d-1)^2(d-2)}[x^2zf_y\Delta_{xx,x} - x^2zf_x\Delta_{xx,y}]
\end{aligned}$$

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$$= -\frac{6x^2 z}{(d-1)^2(d-2)} H_{xy}(\Delta_{xx}),$$

where the third equality follows from the Euler identity (48).

For the $\partial_z^{(3)}$ entry, we have

$$\begin{aligned}
&-\frac{6}{(d-1)^2(d-2)} H_{yz}(\Delta_{xx}) z^3 - \frac{18}{(d-1)^2} z^2 f_y \Delta_{xx} - 6 f_y^3 \\
&\quad = -\frac{6}{(d-1)^2(d-2)} [H_{yz}(\Delta_{xx}) z^3 + 3(d-2) z^2 f_y \Delta_{xx} + (d-2) f_y^3] \\
&\quad = -\frac{6}{(d-1)^2(d-2)} \\
&\qquad \times [H_{yz}(\Delta_{xx}) z^3 + (d-2) f_y (3z^2 \Delta_{xx} + 2xz \Delta_{xz} - x^2 \Delta_{zz} - z^2 \Delta_{xx})] \\
&\quad = -\frac{6}{(d-1)^2(d-2)} \\
&\qquad \times [H_{yz}(\Delta_{xx}) z^3 + 2(d-2) f_y (z^2 \Delta_{xx} + xz \Delta_{xz}) - (d-2) x^2 f_y \Delta_{zz}] \\
&\quad = -\frac{6}{(d-1)^2(d-2)} \\
&\qquad \times [H_{yz}(\Delta_{xx}) z^3 + f_y (z^2 E(\Delta_{xx}) + xz E(\Delta_{xz})) - (d-2) x^2 f_y \Delta_{zz}],
\end{aligned}$$

where the second equality follows from identity (59) in Appendix A.3.

Applying identity (66b) from Appendix A.5 to rewrite $f_y E(\Delta_{xx})$ and $f_y E(\Delta_{xz})$, we have

$$\begin{aligned}
&f_y (z^2 E(\Delta_{xx}) + xz E(\Delta_{xz})) \\
&\quad = z^2 (x H_{xy}(\Delta_{xx}) - z H_{yz}(\Delta_{xx})) + xz (x H_{xy}(\Delta_{xz}) - z H_{yz}(\Delta_{xz}))
\end{aligned}$$

so that the $\partial_z^{(3)}$ entry becomes

$$\begin{aligned}
&-\frac{6}{(d-1)^2(d-2)} H_{yz}(\Delta_{xx}) z^3 - \frac{18}{(d-1)^2} z^2 f_y \Delta_{xx} - 6 f_y^3 \\
&\quad = -\frac{6}{(d-1)^2(d-2)} \\
&\qquad \times [xz^2 (H_{xy}(\Delta_{xx}) - H_{yz}(\Delta_{xz})) + x^2 (z H_{xy}(\Delta_{xz}) - (d-2) f_y \Delta_{zz})] \\
&\quad = -\frac{6}{(d-1)^2(d-2)} \\
&\qquad \times \left[xz^2 \left(-\frac{1}{d-1} x \delta_y \right) + x^2 (z H_{xy}(\Delta_{xz}) - (d-2) f_y \Delta_{zz}) \right] \\
&\quad = -\frac{6x^2}{(d-1)^2(d-2)} \\
&\qquad \times \left[z H_{xy}(\Delta_{xz}) - (d-2) f_y \Delta_{zz} - \frac{1}{d-1} z^2 \delta_y \right],
\end{aligned}$$

where the second equality follows from identity (68) in Appendix B.1.

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By similar calculations for the remaining entries, we find that matrix (24) is given by

$$-\frac{6x^2}{(d-1)^2(d-2)}\begin{array}{r} \partial_x^{(3)} \\ \partial_x^{(2)}\partial_y \\ \partial_x^{(2)}\partial_z \\ \partial_x\partial_y^{(2)} \\ \partial_x\partial_y\partial_z \\ \partial_x\partial_z^{(2)} \\ \partial_y^{(3)} \\ \partial_y^{(2)}\partial_z \\ \partial_y\partial_z^{(2)} \\ \partial_z^{(3)} \end{array}\begin{bmatrix} xH_{yz}(\Delta_{xx}) \\ yH_{yz}(\Delta_{xx}) - (d-2)f_z\Delta_{xx} \\ zH_{yz}(\Delta_{xx}) + (d-2)f_y\Delta_{xx} \\ yH_{zx}(\Delta_{xx}) \\ \frac{1}{2}(yH_{xy}(\Delta_{xx}) + zH_{zx}(\Delta_{xx})) \\ zH_{xy}(\Delta_{xx}) \\ yH_{zx}(\Delta_{xy}) + \frac{1}{d-1}y^2\delta_z + (d-2)f_z\Delta_{yy} \\ yH_{xy}(\Delta_{xy}) - \frac{1}{d-1}yz\delta_z - (d-2)f_y\Delta_{yy} \\ zH_{zx}(\Delta_{xz}) + \frac{1}{d-1}yz\delta_y + (d-2)f_z\Delta_{zz} \\ zH_{xy}(\Delta_{xz}) - \frac{1}{d-1}z^2\delta_y - (d-2)f_y\Delta_{zz} \end{bmatrix},$$

which is divisible by x^2, as desired. The corresponding statements about divisibility for ζ_y and ζ_z follow from x, y, z-symmetry. □

We will show that $\{E^3, E^2H_{yz}, E^2H_{zx}, E^2H_{xy}, E\alpha_x, E\alpha_y, E\alpha_z, \zeta_x, \zeta_y, \zeta_z\}$ is a minimal generating set for $\ker J_{3,2}$ in Section 5.3; see Lemma 5.3.1.

5.1.2. From the computations above, writing these generators in terms of the basis $\{\partial_x^{(2)}, \partial_x\partial_y, \partial_x\partial_z, \partial_y^{(2)}, \partial_y\partial_z, \partial_z^{(2)}\}$ gives the columns of the following matrix:

$$M_0(3) := \begin{bmatrix} E^3 & E^2H_{yz} & E^2H_{zx} & E^2H_{xy} & E\alpha_x & E\alpha_y & E\alpha_z & -\frac{6}{(d-1)^2(d-2)}Z \end{bmatrix},$$

where

$$Z := \begin{array}{c} \begin{array}{ccc} \zeta_x \qquad\qquad\qquad & \qquad\qquad\qquad \zeta_y \qquad\qquad\qquad & \qquad\qquad\qquad \zeta_z \end{array} \\ \begin{bmatrix} xH_{yz}(\Delta_{xx}) & xH_{yz}(\Delta_{xy}) - \frac{1}{d-1}x^2\delta_z - (d-2)f_z\Delta_{xx} & xH_{yz}(\Delta_{xz}) + \frac{1}{d-1}x^2\delta_y + (d-2)f_y\Delta_{xx} \\ yH_{yz}(\Delta_{xx}) - (d-2)f_z\Delta_{xx} & xH_{yz}(\Delta_{yy}) & xH_{zx}(\Delta_{xz}) - \frac{1}{d-1}xy\delta_y - (d-2)f_x\Delta_{xx} \\ zH_{yz}(\Delta_{xx}) + (d-2)f_y\Delta_{xx} & xH_{xy}(\Delta_{xy}) + \frac{1}{d-1}xz\delta_z + (d-2)f_x\Delta_{xx} & xH_{yz}(\Delta_{zz}) \\ yH_{zx}(\Delta_{xx}) & xH_{zx}(\Delta_{yy}) + (d-2)f_z\Delta_{yy} & yH_{yz}(\Delta_{yz}) + \frac{1}{d-1}xy\delta_x + (d-2)f_y\Delta_{yy} \\ \frac{1}{2}(yH_{xy}(\Delta_{xx}) + zH_{zx}(\Delta_{xx})) & \frac{1}{2}(zH_{yz}(\Delta_{yy}) + xH_{xy}(\Delta_{yy})) & \frac{1}{2}(xH_{zx}(\Delta_{zz}) + yH_{yz}(\Delta_{zz})) \\ zH_{xy}(\Delta_{xx}) & zH_{yz}(\Delta_{yz}) - \frac{1}{d-1}xz\delta_x - (d-2)f_z\Delta_{zz} & xH_{xy}(\Delta_{zz}) - (d-2)f_y\Delta_{zz} \\ yH_{zx}(\Delta_{xy}) + \frac{1}{d-1}y^2\delta_z + (d-2)f_z\Delta_{yy} & yH_{zx}(\Delta_{yy}) & yH_{zx}(\Delta_{yz}) - \frac{1}{d-1}y^2\delta_x - (d-2)f_x\Delta_{yy} \\ yH_{xy}(\Delta_{xy}) - \frac{1}{d-1}yz\delta_z - (d-2)f_y\Delta_{yy} & zH_{zx}(\Delta_{yy}) - (d-2)f_x\Delta_{yy} & yH_{zx}(\Delta_{zz}) \\ zH_{zx}(\Delta_{xz}) + \frac{1}{d-1}yz\delta_y + (d-2)f_z\Delta_{zz} & zH_{xy}(\Delta_{yy}) & yH_{xy}(\Delta_{zz}) + (d-2)f_x\Delta_{zz} \\ zH_{xy}(\Delta_{xz}) - \frac{1}{d-1}z^2\delta_y - (d-2)f_y\Delta_{zz} & zH_{xy}(\Delta_{yz}) + \frac{1}{d-1}z^2\delta_x + (d-2)f_x\Delta_{zz} & zH_{xy}(\Delta_{zz}) \end{bmatrix} \end{array}.$$

5.2. Matrix Factorization

Throughout this subsection we recall the dg E-module L constructed in 4.2.1. In particular, its differentials ∂_i^L are defined in (16).

5.2.1. Consider the map $\beta\colon \operatorname{Kos}^Q(x, y, z) \to L$ given by

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "\partial_3"] \arrow[d, equal, "\beta_3"] & Q^3 \arrow[r, "\partial_2"] \arrow[d, "\beta_2"] & Q^3 \arrow[r, "\partial_1"] \arrow[d, "\beta_1"] & Q \arrow[r] \arrow[d, "\beta_0"] & 0 \\ 0 \arrow[r] & Q \arrow[r, "\partial_3^L"] & Q^4 \arrow[r, "\partial_2^L"] & Q^6 \arrow[r, "\partial_1^L"] & Q^3 \arrow[r] & 0 \end{tikzcd}$$

where

$$\beta_2 = \begin{bmatrix} -\alpha_1 \\ \frac{1}{3}\sigma_1 \end{bmatrix}, \qquad \beta_1 = \begin{bmatrix} \alpha_2 \\ \frac{1}{3}\sigma_2 \end{bmatrix}, \qquad \beta_0 = \frac{1}{3}\sigma_3,$$

and the matrices σ_i are defined in Section 3.2. First note that this is a chain map: Using Appendix D, the rightmost square commutes by matrix identity (88), the middle square commutes by matrix identities (89) and (86), and the leftmost square commutes by Euler (48) applied to f and δ.

Next note that β is a morphism of dg E-modules. Indeed, the E-action commutes with β_0, β_1 using (88). The E-action commutes with β_1, β_2 using (86) and (91). The E-action commutes with β_2, β_3 just using the Euler identity. Therefore β is a dg E-module map, and so $\operatorname{cone}(\beta)$ naturally inherits a dg E-module structure. Consider the projection of complexes

$$\begin{tikzcd}
\operatorname{cone}(\beta): & 0 \arrow[r] & Q \arrow[r] \arrow[d] & Q^4 \arrow[r] \arrow[d, "\pi"] & Q^7 \arrow[r] \arrow[d, equal] & Q^7 \arrow[r] \arrow[d, equal] & Q^3 \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & 0 \arrow[r] & Q^3 \arrow[r] & Q^7 \arrow[r] & Q^7 \arrow[r] & Q^3 \arrow[r] & 0
\end{tikzcd}$$

with $\pi = [1 \ \ \partial_3]$; this is a quasi-isomorphism that defines a dg E-module on the target. Namely, if we let G denote the source of the projection, then G is the dg E-module whose underlying complex is

$$G: \quad 0 \to Q^3 \xrightarrow{\begin{bmatrix} -\partial_2 \\ -\alpha_1 \\ \frac{1}{3}\sigma_1 \end{bmatrix}} Q^7 \xrightarrow{\begin{bmatrix} -\partial_1 & 0 & 0 \\ \alpha_2 & -D_2 & 0 \\ \frac{1}{3}\sigma_2 & \alpha_2 & \partial_3 \end{bmatrix}} Q^7 \xrightarrow{\begin{bmatrix} \frac{1}{3}\sigma_3 & \alpha_1 & \partial_2 \end{bmatrix}} Q^3 \to 0$$

with $e\cdot$ given by

$$0 \leftarrow Q^3 \xleftarrow{\begin{bmatrix} D_2 & -q & 0 \end{bmatrix}} Q^7 \xleftarrow{\begin{bmatrix} -D_3 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & D_1 \end{bmatrix}} Q^7 \xleftarrow{\begin{bmatrix} 0 \\ q \\ -D_2 \end{bmatrix}} Q^3 \leftarrow 0.$$

5.2.2. Let G be as constructed in 5.2.1. Applying 2.3.6 to it produces the matrix factorization $\mathsf{Fold}(G)$ of $d \cdot f$ over Q:

$$Q^{10} \xrightarrow{\begin{bmatrix} 0 & -\partial_1 & 0 & 0 \\ q & \alpha_2 & -D_2 & 0 \\ -D_2 & \frac{1}{3}\sigma_2 & \alpha_2 & \partial_3 \\ 0 & -D_2 & -q & 0 \end{bmatrix}} Q^{10} \xrightarrow{\begin{bmatrix} \frac{1}{3}\sigma_3 & \alpha_1 & \partial_2 & 0 \\ -D_3 & 0 & 0 & -\partial_2 \\ 0 & \partial_2 & 0 & -\alpha_1 \\ 0 & 0 & D_1 & \frac{1}{3}\sigma_1 \end{bmatrix}} Q^{10}.$$

Changing bases yields the equivalent matrix factorization:

$$Q^{10} \xrightarrow{\begin{bmatrix} \partial_3 & D_2 & 2\alpha_2 & -2\sigma_2 \\ 0 & \frac{1}{2}q & D_2 & 3\alpha_2 \\ 0 & 0 & \frac{1}{3}q & D_2 \\ 0 & 0 & 0 & \partial_1 \end{bmatrix}} Q^{10} \xrightarrow{\begin{bmatrix} D_1 & 0 & -2\sigma_1 & 0 \\ -\partial_2 & 2\alpha_1 & 0 & -2\sigma_3 \\ 0 & -\partial_2 & 3\alpha_1 & 0 \\ 0 & 0 & -\partial_2 & D_3 \end{bmatrix}} Q^{10}.$$

Therefore, from 2.3.7, the complex

$$\cdots \to R^{10} \xrightarrow{M_2(3)} R^{10} \xrightarrow{M_1(3)} R^{10} \xrightarrow{M_2(3)} R^{10} \xrightarrow{M_1(3)} R^{10} \to \cdots$$

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is exact, where

$$M_2(3) := \begin{bmatrix} \partial_3 & D_2 & 2\alpha_2 & -2\sigma_2 \\ 0 & \frac{1}{2}q & D_2 & 3\alpha_2 \\ 0 & 0 & \frac{1}{3}q & D_2 \\ 0 & 0 & 0 & \partial_1 \end{bmatrix} \quad \text{and}$$

$$M_1(3) := \begin{bmatrix} D_1 & 0 & -2\sigma_1 & 0 \\ -\partial_2 & 2\alpha_1 & 0 & -2\sigma_3 \\ 0 & -\partial_2 & 3\alpha_1 & 0 \\ 0 & 0 & -\partial_2 & D_3 \end{bmatrix}.$$

5.3. *Resolution of* ker $J_{3,2}$

In this subsection we establish exactness of the following sequence of R-modules:

$$\cdots \xrightarrow{M_1(3)} R^{10} \xrightarrow{M_2(3)} R^{10} \xrightarrow{M_1(3)} R^{10} \xrightarrow{M_0(3)} R^{10} \xrightarrow{J_{3,2}} R^6,$$

where the definitions of the matrices $M_i(3)$ can be found in 5.1.2 and 5.2.2.

The proof of the following lemma is similar to Lemma 4.3.1.

LEMMA 5.3.1. *The R_x-module* $(\ker_R(J_{3,2}))_x = \ker_{R_x}(J_{3,2})$ *is freely generated by the operators E^3, E^2H_{yz}, $E\alpha_x$, and H^3_{yz}.*

LEMMA 5.3.2. *We have* $M_0(3)M_1(3) = 0_{10\times 10}$.

Proof. First note that the relations on the generators of ker $J_{2,1}$ give seven relations on the first seven columns of $M_0(3)$. These are simply obtained by composing the generators of ker $J_{2,1}$ by E (and viewing the images in the filtration factor D_3/D_2); see Appendix C.1 for the details. To see that the remaining columns of $M_1(3)$ give relations, see Proposition C.1.1 and Lemma C.2.1. □

LEMMA 5.3.3. *We have* $(\ker M_0(3))_x = (\operatorname{im} M_1(3))_x$.

Proof. The proof follows along the same lines as Lemma 4.3.3. Namely, one can express the columns of $M_0(3)$ in terms of the free basis E^3, E^2H_{yz}, $E\alpha_x$, H^3_{yz} above. Considering the H^3_{yz} coefficients, the coefficients of ζ_x, ζ_y, ζ_z in the relation must be in the span of ∂_2. Subtracting off a suitable linear combination of columns 7–9 of $M_1(3)$, one can replace the given relation with one in which the ζ_x, ζ_y, ζ_z coefficients are zero. Repeating like so with the α_x, α_y, α_z coefficients, and then the E^2H_{yz}, E^2H_{zx}, E^2H_{xy} coefficients, one obtains a relation of the form $aE^3 = 0$, which must be the zero relation. □

We are now ready to prove the main result of this subsection.

PROPOSITION 5.3.4. *The sequence of R-modules*

$$\cdots \xrightarrow{M_1(3)} R^{10} \xrightarrow{M_2(3)} R^{10} \xrightarrow{M_1(3)} R^{10} \xrightarrow{M_0(3)} R^{10} \xrightarrow{J_{3,2}} R^6$$

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is exact. In particular, the R-module $\ker_R J_{3,2}$ is generated by the elements

$$\{E^3, E^2H_{yz}, E^2H_{zx}, E^2H_{xy}, E\alpha_x, E\alpha_y, E\alpha_z, \zeta_x, \zeta_y, \zeta_z\}.$$

Proof. In view of 5.2.2 and Lemmas 5.3.1, 5.3.2, and 5.3.3, a proof parallel to that of Proposition 4.3.4 can be used to prove the statement. $\square$

5.4. *Resolution of $D^3_{R|k}$*

In this subsection we construct the resolution of $D^3_{R|k}$ by defining a "lift" of $\theta_0(3)$; here it is interpreted as a map from the degree one part of the resolution of coker $J_{3,2}$, constructed in Section 5.3, to the complex $C = \operatorname{cone}(\theta(2))$ defined in Lemma 4.4.2.

5.4.1. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & R^{10} \arrow[r, "-M_2(3)"] \arrow[d, "\theta_3(3)"] & R^{10} \arrow[r, "-M_1(3)"] \arrow[d, "\theta_2(2)"] & R^{10} \arrow[r, "-M_0(3)"] \arrow[d, "\theta_1(3)"] & R^{10} \arrow[r, "-J_{3,2}"] \arrow[d, "\theta_0(3)"] & R^6 \arrow[r] & 0 \\
\cdots \arrow[r] & R^{11} \arrow[r, "\partial_3^C"] & R^{11} \arrow[r, "\partial_2^C"] & R^9 \arrow[r, "\partial_1^C"] & R^4 \arrow[r] & 0
\end{tikzcd}$$

where $M_i(3)$ are defined in 5.1.2 and 5.2.2, and the $\theta_i(3)$ are declared in Section 3.

The proof of the next lemma is similar to that of Lemma 4.4.2. In light of this, and the lengthy computations involved, its proof is Appendix E.

LEMMA 5.4.2. *The diagram in 5.4.1 commutes. That is, $\theta(3)$ is a morphism of complexes.*

THEOREM 5.4.3. *Assume that $R = k[x, y, z]/(f)$ is an isolated singularity hypersurface where k is a field of characteristic zero. The augmented minimal R-free resolution of $D^3_{R|k} \subseteq R^9 \oplus R^6 \oplus R$ has the form*

$$\cdots \to \begin{matrix} R^{11} \\ \oplus \\ R^{10} \end{matrix} \xrightarrow{\begin{bmatrix} \partial_4^C & \theta_3(3) \\ 0 & M_2(3) \end{bmatrix}} \begin{matrix} R^{11} \\ \oplus \\ R^{10} \end{matrix} \xrightarrow{\begin{bmatrix} \partial_3^C & \theta_2(3) \\ 0 & M_1(3) \end{bmatrix}} \begin{matrix} R^{11} \\ \oplus \\ R^{10} \end{matrix} \xrightarrow{\begin{bmatrix} \partial_4^C & \theta_3(3) \\ 0 & M_2(3) \end{bmatrix}} \begin{matrix} R^{11} \\ \oplus \\ R^{10} \end{matrix} \xrightarrow{\begin{bmatrix} \partial_3^C & \theta_2(3) \\ 0 & M_1(3) \\ 0 & 0 \end{bmatrix}} \begin{matrix} R^{11} \\ \oplus \\ R^{10} \\ \oplus \\ R \end{matrix} \xrightarrow{\varepsilon} D^3_{R|k} \to 0,$$

where a minimal set of generators for $D^3_{R|k}$ is given by the columns of

$$\varepsilon = \begin{bmatrix} \partial_2^C & \theta_1(3) & 0 \\ 0 & M_0(3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the block matrices making up the differentials are defined in Lemma 4.4.2, 5.1.2, and 5.2.2; see also Section 3.

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Proof. The proof is essentially the same as that of Theorem 4.4.3; one uses Proposition 5.3.4 and Lemma 5.4.2 in lieu of Proposition 4.3.4 and Lemma 4.4.2, respectively. □

The theorem yields the following corollary, which is analogous to Corollary 4.4.4.

COROLLARY 5.4.4. *There is a short exact sequence of R-modules*

$$0 \to D^2_{R|k} \to D^3_{R|k} \to \ker J_{3,2} \to 0$$

where $D^2_{R|k} \to D^3_{R|k}$ is the inclusion.

QUESTION 5.4.5. Is there a short exact sequence of R-modules

$$0 \to D^{i-1}_{R|k} \to D^i_{R|k} \to \ker J_{i,i-1} \to 0$$

where $D^{i-1}_{R|k} \to D^i_{R|k}$ is the inclusion for all $i > 0$?

In contrast, the reader can find short exact sequences relating the cokernel of the inclusion $D^{i-1}_{R|k} \to D^i_{R|k}$ to global sections of certain sheaves in [42].

Similar to 2.4.7 and Corollary 4.4.5, the differentials in the resolution of $D^3_{R|k}$ are homogeneous with respect to the internal grading of R where

$$|\zeta_x| = |\zeta_y| = |\zeta_z| = 3d - 8.$$

Hence, from Theorem 5.4.3 we can read off the graded Betti numbers of $D^3_{R|k}$.

COROLLARY 5.4.6. *The graded Betti numbers of $D^3 = D^3_{R|k}$ are given by*

$$\beta^R_{0,j}(D^3) = \begin{cases} 4, & j = 0, \\ 9, & j = d-2, \\ 6, & j = 2d-5, \\ 3, & j = 3d-8, \\ 0, & \textit{otherwise} \end{cases}$$

and for $n \geq 1$

$$\beta^R_{2n-1,j}(D^3) = \begin{cases} 9, & j = nd-1, \\ 1, & j = nd+d-3, \\ 6, & j = nd+d-4, \\ 1, & j = nd+2d-6, \\ 3, & j = nd+2d-7, \\ 1, & j = nd+3d-9, \\ 0, & \textit{otherwise} \end{cases} \quad \textit{and}$$

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$$\beta^R_{2n,j}(D^3) = \begin{cases} 3, & j = nd, \\ 9, & j = nd + d - 2, \\ 6, & j = nd + 2d - 5, \\ 3, & j = nd + 3d - 8, \\ 0, & \textit{otherwise}. \end{cases}$$

COROLLARY 5.4.7. *The following inequalities are satisfied*:

$$\mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(\ker J_{3,2}) \leq 3 \quad \textit{and} \quad \mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(D^3_{R|k}) \leq 6.$$

Proof. Using Corollary 5.4.4 and Corollary 4.4.6, we have the following:

$$\begin{aligned}\mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(D^3_{R|k}) &\leq \mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(\ker J_{3,2}) + \mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(D^2_{R|k}) \\ &\leq \mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(\ker J_{3,2}) + 3,\end{aligned}$$

and so the first inequality implies the second. The first inequality is from 5.2.1 and Proposition 5.3.4. □

Based on the (partial) evidence in Corollaries 4.4.6 and 5.4.7, as well as Remark 2.4.4, we ask the following.

QUESTION 5.4.8. Let R be an isolated singularity hypersurface ring with residue field k of characteristic zero. For every positive integer i, do the following inequalities hold:

$$\mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(\ker J_{i,i-1}) \leq i \quad \text{and} \quad \mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(D^i_{R|k}) \leq i(i+1)/2?$$

REMARK 5.4.9. By [4, Proposition 4.11], for any N in $\mathsf{D}_{\mathsf{sg}}(R)$, the following inequality is satisfied:

$$\mathsf{level}^k_{\mathsf{D}_{\mathsf{sg}}(R)}(N) \leq 2\ell\ell(R/(f_x, f_y, f_z));$$

here $\ell\ell(-)$ denotes the Loewy length of an R-module. So Question 5.4.8 has a positive answer whenever n is at least $2\ell\ell(R/(f_x, f_y, f_z))$. However, it would still be enlightening if $\ker J_{i,i-1}$ can be built from k using i explicitly described mapping cones as was the case for $i = 2, 3$ in Corollaries 4.4.6 and 5.4.7, respectively.

5.5. *A Set of Generators of $D^3_{R|k}$*

Now we write the generators of $D^3_{R|k}$ given in Theorem 5.4.3 as differential operators. In what follows, as in Section 4.5, we write E^2, E^3, EH_{ij}, E^2H_{ij}, H^3_{ij} for the representatives in $D^3_{R|k}$ specified in Sections 4.1 and 5.1. For example,

$$\begin{aligned}E^3 = x^3\partial_x^3 &+ 3x^2y\partial_x^2\partial_y + 3x^2z\partial_x^2\partial_z + 3xy^2\partial_x\partial_y^2 + 6xyz\partial_x\partial_y\partial_z + 3xz^2\partial_x\partial_z^2 \\ &+ y^3\partial_z^3 + 3y^2z\partial_y^2\partial_z + 3yz^2\partial_y\partial_z^2 + z^3\partial_z^3.\end{aligned}$$

Finally, we also adopt the notation from Section 4.5.

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We begin by writing some relevant operators of order at most three in terms of their images in the $\ker(J_{3,2}) \cong D^3_{R|k}/D^2_{R|k}$ and lower order operators. Composing $\mathcal{E}$ with (21) and (22), we have

$$\mathcal{E}^3 = E^3 + 3E^2 + E = E^3 + 3\mathcal{E}^2 - 2\mathcal{E}, \tag{25}$$

$$\begin{aligned}\mathcal{E}^2\mathcal{H} &= E^2H + dEH + (d-1)\mathcal{E}\mathcal{H}\\ &= E^2H + (2d-1)EH + (d-1)^2\mathcal{H}\\ &= E^2H + (2d-1)\mathcal{E}\mathcal{H} - d(d-1)\mathcal{H},\end{aligned} \tag{26}$$

where $\mathcal{H}$ denotes any one of the Hamiltonians, $\mathcal{H}_{yz}$, $\mathcal{H}_{zx}$, or $\mathcal{H}_{xy}$. Composing $\mathcal{H}$ with (23), we have

$$\mathcal{H}^3_{yz} = H^3_{yz} + \mathcal{H}_{yz}(f_z^2)\partial_y^2 - 2\mathcal{H}_{yz}(f_y f_z)\partial_y\partial_z + \mathcal{H}_{yz}(f_y^2)\partial_z^2 \tag{27}$$

$$- \frac{1}{d-1}\mathcal{H}_{yz}(\Delta_{xx})\mathcal{E} - \frac{1}{d-1}\Delta_{xx}\mathcal{H}_{yz}\mathcal{E} \tag{28}$$

$$+ \frac{1}{d-1}(\mathcal{H}_{yz}(x\Delta_{xx})\partial_x + \mathcal{H}_{yz}(x\Delta_{xy})\partial_y + \mathcal{H}_{yz}(x\Delta_{xz})\partial_z) \tag{29}$$

$$+ \frac{1}{d-1}(x\Delta_{xx}\mathcal{H}_{yz}\partial_x + x\Delta_{xy}\mathcal{H}_{yz}\partial_y + x\Delta_{xz}\mathcal{H}_{yz}\partial_z). \tag{30}$$

Noting that, by (22), we have $\mathcal{H}_{yz}\mathcal{E} = EH_{yz} + H_{yz} = \mathcal{E}\mathcal{H}_{yz} - (d-2)\mathcal{H}_{yz}$, simplifying (29) and (30), and using identities (53) to rewrite (27), we find that

$$\begin{aligned}\mathcal{H}^3_{yz} = H^3_{yz} &+ \frac{2}{d-1}((d-1)\Delta_{xx}\mathcal{H}_{yz} - \Delta_{xx}\mathcal{E}\mathcal{H}_{yz} + x\mathcal{D}_x\mathcal{H}_{yz})\\ &- \frac{1}{d-1}\mathcal{H}_{yz}(\Delta_{xx})\mathcal{E} - \frac{1}{d-1}\Delta_{xx}(\mathcal{E}\mathcal{H}_{yz} - (d-2)\mathcal{H}_{yz})\\ &+ \frac{1}{d-1}x(\mathcal{H}_{yz}(\Delta_{xx})\partial_x + \mathcal{H}_{yz}(\Delta_{xy})\partial_y + \mathcal{H}_{yz}(\Delta_{xz})\partial_z)\\ &+ \frac{1}{d-1}x\mathcal{D}_x\mathcal{H}_{yz},\end{aligned}$$

where $\mathcal{D}_x = \Delta_{xx}\partial_x + \Delta_{xy}\partial_y + \Delta_{xz}\partial_z$. Simplifying once more, we obtain the following expression for H^3_{yz}:

$$\begin{aligned}\mathcal{H}^3_{yz} = H^3_{yz} - \frac{1}{d-1}[&3\Delta_{xx}\mathcal{E}\mathcal{H}_{yz} + \mathcal{H}_{yz}(\Delta_{xx})\mathcal{E} - (3d-4)\Delta_{xx}\mathcal{H}_{yz} - 3x\mathcal{D}_x\mathcal{H}_{yz}\\ &- x(\mathcal{H}_{yz}(\Delta_{xx})\partial_x + \mathcal{H}_{yz}(\Delta_{xy})\partial_y + \mathcal{H}_{yz}(\Delta_{xz})\partial_z)].\end{aligned} \tag{31}$$

We obtain similar expressions for the cubes of the other Hamiltonians below.

$$\begin{aligned}\mathcal{H}^3_{zx} = H^3_{zx} - \frac{1}{d-1}[&3\Delta_{yy}\mathcal{E}\mathcal{H}_{zx} + \mathcal{H}_{zx}(\Delta_{yy})\mathcal{E} - (3d-4)\Delta_{yy}\mathcal{H}_{zx} - 3y\mathcal{D}_yH_{zx}\\ &- y(\mathcal{H}_{zx}(\Delta_{xy})\partial_x + \mathcal{H}_{zx}(\Delta_{yy})\partial_y + \mathcal{H}_{zx}(\Delta_{yz})\partial_z)],\end{aligned} \tag{32}$$

$$\begin{aligned}\mathcal{H}^3_{xy} = H^3_{xy} - \frac{1}{d-1}[&3\Delta_{zz}\mathcal{E}\mathcal{H}_{xy} + \mathcal{H}_{xy}(\Delta_{zz})\mathcal{E} - (3d-4)\Delta_{zz}\mathcal{H}_{xy} - 3z\mathcal{D}_z\mathcal{H}_{xy}\\ &- z(\mathcal{H}_{xy}(\Delta_{xz})\partial_x + \mathcal{H}_{xy}(\Delta_{yz})\partial_y + \mathcal{H}_{xy}(\Delta_{zz})\partial_z)].\end{aligned} \tag{33}$$

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THEOREM 5.5.1. *A minimal set of generators for $D^3_{R|k}$ is given by $G_0 \cup G_1 \cup G_2 \cup G_3$ where G_0, G_1, and G_2 are defined in Theorem 4.5.1 and*

$$G_3 = \{\mathcal{E}^3,\ \mathcal{E}^2\mathcal{H}_{yz},\ \mathcal{E}^2\mathcal{H}_{zx},\ \mathcal{E}^2\mathcal{H}_{xy},\ \mathcal{E}\mathcal{A}_x,\ \mathcal{E}\mathcal{A}_y\ \mathcal{E}\mathcal{A}_z,\ \mathcal{Z}_x,\ \mathcal{Z}_y,\ \mathcal{Z}_z\},$$

with

$$\begin{aligned}\mathcal{Z}_x = \frac{1}{x^2}\Bigg[\mathcal{H}^3_{yz} + \frac{1}{2(d-1)^2(d-2)}&(6(d-2)\Delta_{xx}\mathcal{E}^2\mathcal{H}_{yz} - 2\mathcal{H}_{yz}(\Delta_{xx})\mathcal{E}^3\\ &- 6(d-2)\Delta_{xx}\mathcal{E}\mathcal{H}_{yz} - 3(d-3)\mathcal{H}_{yz}(\Delta_{xx})\mathcal{E}^2\\ &- 2(d-1)(d-2)(d-3)\Delta_{xx}\mathcal{H}_{yz} + (3d-7)\mathcal{H}_{yz}(\Delta_{xx})\mathcal{E})\Bigg],\\ \mathcal{Z}_y = \frac{1}{y^2}\Bigg[\mathcal{H}^3_{zx} + \frac{1}{2(d-1)^2(d-2)}&(6(d-2)\Delta_{yy}\mathcal{E}^2\mathcal{H}_{zx} - 2\mathcal{H}_{zx}(\Delta_{yy})\mathcal{E}^3\\ &- 6(d-2)\Delta_{yy}\mathcal{E}\mathcal{H}_{zx} - 3(d-3)\mathcal{H}_{zx}(\Delta_{yy})\mathcal{E}^2\\ &- 2(d-1)(d-2)(d-3)\Delta_{yy}\mathcal{H}_{zx} + (3d-7)\mathcal{H}_{zx}(\Delta_{yy})\mathcal{E})\Bigg],\\ \mathcal{Z}_z = \frac{1}{z^2}\Bigg[\mathcal{H}^3_{xy} + \frac{1}{2(d-1)^2(d-2)}&(6(d-2)\Delta_{zz}\mathcal{E}^2\mathcal{H}_{xy} - 2\mathcal{H}_{xy}(\Delta_{zz})\mathcal{E}^3\\ &- 6(d-2)\Delta_{zz}\mathcal{E}\mathcal{H}_{xy} - 3(d-3)\mathcal{H}_{xy}(\Delta_{zz})\mathcal{E}^2\\ &- 2(d-1)(d-2)(d-3)\Delta_{zz}\mathcal{H}_{xy} + (3d-7)\mathcal{H}_{xy}(\Delta_{zz})\mathcal{E})\Bigg].\end{aligned}$$

Proof. The proof is similar to Theorem 4.5.1, so it suffices to show that $\mathcal{Z}_x$, $\mathcal{Z}_y$, and $\mathcal{Z}_z$ agree with the generators corresponding to the last three columns of ε in Theorem 5.4.3, and thus they are well-defined operators in $D^3_{R|k}$.

Analyzing the column of the augmentation map ε in Theorem 5.4.3 that lifts ζ_x, we see that the corresponding operator is given by

$$\begin{aligned}\mathcal{Z}_x = \frac{1}{x^2}\Bigg[H^3_{yz} + \frac{3}{(d-1)^2}\Delta_{xx}E^2H_{yz} - \frac{1}{(d-1)^2(d-2)}H_{yz}(\Delta_{xx})E^3\Bigg] \quad &(34)\\ + \frac{1}{(d-1)^2}(\delta_y\partial_z - \delta_z\partial_y) \quad &(35)\\ - \frac{3}{(d-1)(d-2)}\Bigg[H_{yz}(\Delta_{xx})\partial_x^{(2)} + H_{zx}(\Delta_{xx})\partial_x\partial_y&\\ + H_{xy}(\Delta_{xx})\partial_x\partial_z + \left(H_{zx}(\Delta_{xy}) + \frac{y\delta_z}{d-1}\right)\partial_y^{(2)} \quad &(36)\\ + \frac{1}{2}\left(H_{zx}(\Delta_{xz}) + H_{xy}(\Delta_{xy}) + \frac{z\delta_z - y\delta_y}{d-1}\right)\partial_y\partial_z&\\ + \left(H_{xy}(\Delta_{xz}) - \frac{z\delta_y}{d-1}\right)\partial_z^{(2)}\Bigg], \quad &(37)\end{aligned}$$

where (34) comes from the formula for ζ_x in Lemma 5.1.1, and (35), (36), and (37) come from the lift $\theta_1(3)$ defined in Section 3.

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Next we use (25), (26), and (31) to rewrite (34), as follows:

$$\begin{aligned}
\mathcal{Z}_x = \frac{1}{x^2}\Bigg[&\mathcal{H}_{yz}^3 + \frac{1}{d-1}[3\Delta_{xx}\mathcal{E}\mathcal{H}_{yz} - (3d-4)\Delta_{xx}\mathcal{H}_{yz} &(38)\\
&- 3x\mathcal{D}_x\mathcal{H}_{yz} &(39)\\
&+ \mathcal{H}_{yz}(\Delta_{xx})\mathcal{E} - x(\mathcal{H}_{yz}(\Delta_{xx})\partial_x + \mathcal{H}_{yz}(\Delta_{xy})\partial_y + \mathcal{H}_{yz}(\Delta_{xz})\partial_z)] &(40)\\
&+ \frac{3}{(d-1)^2}\Delta_{xx}(\mathcal{E}^2\mathcal{H}_{yz} - (2d-1)\mathcal{E}\mathcal{H}_{yz} + d(d-1)\mathcal{H}_{yz}) &(41)\\
&- \frac{1}{(d-1)^2(d-2)}\mathcal{H}_{yz}(\Delta_{xx})(\mathcal{E}^3 - 3\mathcal{E}^2 + 2\mathcal{E})\Bigg] &(42)\\
&+ \frac{1}{(d-1)^2}(\delta_y\partial_z - \delta_z\partial_y) &(43)\\
&- \frac{3}{(d-1)(d-2)}\Bigg[\mathcal{H}_{yz}(\Delta_{xx})\partial_x^{(2)} + \mathcal{H}_{zx}(\Delta_{xx})\partial_x\partial_y\\
&+ \mathcal{H}_{xy}(\Delta_{xx})\partial_x\partial_z + \left(H_{zx}(\Delta_{xy}) + \frac{y\delta_z}{d-1}\right)\partial_y^{(2)} &(44)\\
&+ \frac{1}{2}\left(\mathcal{H}_{zx}(\Delta_{xz}) + \mathcal{H}_{xy}(\Delta_{xy}) + \frac{z\delta_z - y\delta_y}{d-1}\right)\partial_y\partial_z\\
&+ \left(\mathcal{H}_{xy}(\Delta_{xz}) - \frac{z\delta_y}{d-1}\right)\partial_z^{(2)}\Bigg]. &(45)
\end{aligned}$$

One can check that using identities (67) and (68) to rewrite (40), combining them with (43), and simplifying using the Euler identity (48) yields

$$\begin{aligned}
&\frac{1}{x^2}\Bigg[\frac{x^2}{(d-1)^2}(\delta_y\partial_z - \delta_z\partial_y)\\
&\quad + \frac{1}{d-1}(\mathcal{H}_{yz}(\Delta_{xx})\mathcal{E} - x(\mathcal{H}_{yz}(\Delta_{xx})\partial_x + \mathcal{H}_{yz}(\Delta_{xy})\partial_y + \mathcal{H}_{yz}(\Delta_{xz})\partial_z))\Bigg]\\
&= \frac{2(d-2)}{x^2(d-1)}\Delta_{xx}\mathcal{H}_{yz}. \qquad (46)
\end{aligned}$$

One can also check that combining (39) with (44) and (45) and using the Euler identity (48) to simplify yields

$$\begin{aligned}
-\frac{3}{x^2(d-1)}\Bigg[&x\mathcal{D}_x\mathcal{H}_{yz}\\
&+ \frac{x^2}{d-2}\Bigg[\mathcal{H}_{yz}(\Delta_{xx})\partial_x^{(2)} + \mathcal{H}_{zx}(\Delta_{xx})\partial_x\partial_y\\
&+ \mathcal{H}_{xy}(\Delta_{xx})\partial_x\partial_z + \left(H_{zx}(\Delta_{xy}) + \frac{y\delta_z}{d-1}\right)\partial_y^{(2)}\\
&+ \frac{1}{2}\left(\mathcal{H}_{zx}(\Delta_{xz}) + \mathcal{H}_{xy}(\Delta_{xy}) + \frac{z\delta_z - y\delta_y}{d-1}\right)\partial_y\partial_z
\end{aligned}$$

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$$
\begin{aligned}
&\quad + \left(\mathcal{H}_{xy}(\Delta_{xz}) - \frac{z\delta_y}{d-1}\right)\partial_z^{(2)}\Big]\Big] \\
&= \frac{3}{2x^2(d-1)(d-2)}(2(d-2)\Delta_{xx} E H_{yz} - \mathcal{H}_{yz}(\Delta_{xx})E^2) \\
&= \frac{3}{2x^2(d-1)(d-2)} \\
&\quad \times (2(d-2)\Delta_{xx}(\mathcal{E}\mathcal{H}_{yz} - (d-1)\mathcal{H}_{yz}) - \mathcal{H}_{yz}(\Delta_{xx})(\mathcal{E}^2 - \mathcal{E})), \qquad (47)
\end{aligned}
$$

where the last equality follows from (21) and (22). Finally, using (46) and (47) to rewrite the formula for $\mathcal{Z}_x$ in (38) through (45) gives the desired formula for $\mathcal{Z}_x$. The proofs for $\mathcal{Z}_y$ and $\mathcal{Z}_z$ are similar. □

REMARK 5.5.2. Similar to Remark 4.5.2, except for $\mathcal{Z}_x$, $\mathcal{Z}_y$, and $\mathcal{Z}_z$, the generators in Theorem 5.5.1 do not agree with the generators of $D^3_{R|k}$ given in the augmentation map ε in Theorem 5.4.3. However, one can subtract lower order operators to obtain generators that do agree. For the first four generators, such formulas follow directly from (25) and (26). One can check that the operator given by subtracting $\frac{5d-7}{2}\mathcal{A}_x$ from $\mathcal{E}\mathcal{A}_x$ agrees with the corresponding generator in Theorem 5.4.3, and similarly for $\mathcal{E}\mathcal{A}_y$ and $\mathcal{E}\mathcal{A}_z$.

A. Identities

In this appendix we collect identities that are used extensively throughout the paper. Let $f \in k[x, y, z]$ be homogeneous of degree $d \geq 3$, where k is a field of characteristic zero. Recall that

$$E(f) = xf_x + yf_y + zf_z = d \cdot f \qquad (48)$$

in $k[x, y, z]$ (and 0 in R), where $E = x\partial_x + y\partial_y + z\partial_z$ is the *Euler operator.* Differentiating (48) with respect to x, y, z produces the following system where $\{x, y, z\} = \{a, b, c\}$:

$$af_{aa} + bf_{ab} + cf_{ac} = (d-1)f_a. \qquad (49)$$

A.1. *Identities Involving Second Order Derivatives*

Multiplying the first equation in the system above by x, the second one by y, and the third by z and then adding them all together, we obtain a second order Eulerian identity

$$x^2 f_{xx} + y^2 f_{yy} + z^2 f_{zz} + 2xyf_{xy} + 2xzf_{xz} + 2yzf_{yz} = 0. \qquad (50)$$

Importing Δ from Section 3, we have

$$\operatorname{adj}(\Delta) = \begin{bmatrix} \Delta_{xx} & \Delta_{xy} & \Delta_{xz} \\ \Delta_{xy} & \Delta_{yy} & \Delta_{yz} \\ \Delta_{xz} & \Delta_{yz} & \Delta_{zz} \end{bmatrix},$$

and from the matrix identity $\Delta\,\mathrm{adj}(\Delta) = \delta I_{3\times 3}$, where $I_{3\times 3}$ is the identity 3×3 matrix, we obtain the following identities with $\{x, y, z\} = \{a, b, c\}$:

$$\delta = f_{aa}\Delta_{aa} + f_{ab}\Delta_{ab} + f_{ac}\Delta_{ac}, \tag{51a}$$

$$0 = f_{aa}\Delta_{ab} + f_{ab}\Delta_{bb} + f_{ac}\Delta_{bc}. \tag{51b}$$

An application of Cramer's rule to system (49) yields for $\{x, y, z\} = \{a, b, c\}$

$$a\delta = (d-1)(f_a\Delta_{aa} + f_b\Delta_{ab} + f_c\Delta_{ac}). \tag{52}$$

Using equations (49), one can easily obtain the following useful identities:

$$\begin{aligned}
x\Delta_{xy} - y\Delta_{xx} &= (d-1)(f_z f_{yz} - f_y f_{zz}),\\
y\Delta_{xy} - x\Delta_{yy} &= (d-1)(f_z f_{xz} - f_x f_{zz}),\\
x\Delta_{yz} - y\Delta_{xz} &= (d-1)(f_y f_{xz} - f_x f_{yz}),\\
z\Delta_{yy} - y\Delta_{zy} &= (d-1)(f_z f_{xx} - f_x f_{xz}),\\
y\Delta_{zz} - z\Delta_{zy} &= (d-1)(f_y f_{xx} - f_x f_{xy}),\\
z\Delta_{xx} - x\Delta_{xz} &= (d-1)(f_z f_{yy} - f_y f_{zy}),\\
z\Delta_{xy} - x\Delta_{yz} &= (d-1)(f_x f_{zy} - f_z f_{xy}),\\
z\Delta_{xz} - x\Delta_{zz} &= (d-1)(f_y f_{xy} - f_x f_{yy}),\\
z\Delta_{xy} - y\Delta_{xz} &= (d-1)(f_y f_{xz} - f_z f_{xy}).
\end{aligned} \tag{53}$$

To prove these, one starts by the right-hand side of the equations above and replaces $(d-1)f_x$, $(d-1)f_y$ and $(d-1)f_z$ with (49) respectively and then simplifies the resulting equations.

A.2. *Identities Involving Third Order Derivatives*

Differentiating (49) in various manners yields the following system of equations for $\{a, b, c\} = \{x, y, z\}$:

$$af_{aaa} + bf_{aab} + cf_{aac} = (d-2)f_{aa}, \tag{54}$$

$$af_{aab} + bf_{abb} + cf_{abc} = (d-2)f_{ab}. \tag{55}$$

From (54) and (55), one obtains the following identities with $\{a, b, c\} = \{x, y, z\}$:

$$\begin{aligned}
&a^2 f_{aaa} + b^2 f_{abb} + c^2 f_{acc} + 2abf_{aab} + 2acf_{acc} + 2bcf_{abc}\\
&\quad = (d-2)(d-1)f_a.
\end{aligned} \tag{56}$$

From (56) we obtain the following *third order* Eulerian identity:

$$\begin{aligned}
&x^3 f_{xxx} + y^3 f_{yyy} + z^3 f_{zzz} + 3(x^2yf_{xxy} + xy^2 f_{xyy} + y^2zf_{yyz}\\
&\quad + yz^2 f_{yzz} + x^2zf_{xxz} + xz^2 f_{xzz}) + 6xyzf_{xyz} = 0.
\end{aligned} \tag{57}$$

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A.3. *Applications of the Eulerian Identities*

We can express $f_x f_y$, $f_x f_z$, $f_y f_z$ as linear combinations of the cofactors of Δ with coefficients certain monomials in x, y, z. More precisely, we have for $\{a, b, c\} = \{x, y, z\}$:

$$f_a f_b = \frac{1}{(d-1)^2}\{c^2\Delta_{ab} + ab\Delta_{cc} - ac\Delta_{bc} - bc\Delta_{ac}\} + \frac{d}{(d-1)} f_{ab} f, \tag{58}$$

$$f_a^2 = \frac{1}{(d-1)^2}\{-b^2\Delta_{cc} - c^2\Delta_{bb} + 2bc\Delta_{bc}\} + \frac{d}{(d-1)} f_{aa} f. \tag{59}$$

We shall prove (58) with $a = y$, $b = z$, and $c = x$. The others follow similarly.

Proof of (58). From (49) we obtain

$$\begin{aligned} f_y f_z &= \frac{1}{(d-1)^2}[(xf_{xy} + yf_{yy} + zf_{yz})(xf_{xz} + yf_{yz} + zf_{zz})] \\ &= \frac{1}{(d-1)^2}[x^2 f_{xz} f_{xy} + xyf_{yz} f_{xy} + xzf_{zz} f_{xy} + xyf_{xz} f_{yy} \\ &\quad + y^2 f_{yz} f_{yy} + yzf_{zz} f_{yy} + xzf_{yz} f_{xz} + yzf_{yz}^2 + z^2 f_{zz} f_{yz}] \\ &= \frac{1}{(d-1)^2}[f_{yz}(y^2 f_{yy} + z^2 f_{zz} + yzf_{yz} + xzf_{xz} + xyf_{xy}) \\ &\quad + f_{xy}(x^2 f_{xz} + xzf_{zz}) + xyf_{xz} f_{yy} + yzf_{zz} f_{yy}] \\ &= \frac{1}{(d-1)^2}[f_{yz}(-x^2 f_{xx} - yzf_{yz} - xzf_{xz} - xyf_{xy} + d(d-1)f) \\ &\quad + f_{xy}(x^2 f_{xz} + xzf_{zz}) + xyf_{xz} f_{yy} + yzf_{zz} f_{yy}]. \end{aligned}$$

Using (50) to replace $(y^2 f_{yy} + z^2 f_{zz} + yzf_{yz} + xzf_{xz} + xyf_{xy})$ with $-x^2 f_{xx} - yzf_{yz} - xzf_{xz} - xyf_{xy} + d(d-1)f$, we obtain

$$f_y f_z = \frac{1}{(d-1)^2}[x^2\Delta_{yz} - xy\Delta_{xz} - xz\Delta_{xy} + yz\Delta_{xx} + d(d-1) f_{yz} f]. \qquad \square$$

A.4. *Some Facts on Partial Derivatives*

Now, if we differentiate (52) with respect to y and z, (52) with respect to x and z, (52) with respect to x and y and use the identity (51b), we obtain the following with $\{x, y, z\} = \{a, b, c\}$:

$$a\delta_b = (d-1)(f_a\Delta_{aa,b} + f_b\Delta_{ab,b} + f_c\Delta_{ac,b}). \tag{60}$$

Here $\Delta_{ab,c}$ denotes the derivative of Δ_{ab} with respect to the variable c. Moreover, if we differentiate (52) with respect to x, (52) with respect to y, (52) with respect to z, use the identity (51a) and then simplify, we obtain for $\{x, y, z\} = \{a, b, c\}$

$$a\delta_a = (d-2)\delta + (d-1)(f_a\Delta_{aa,a} + f_b\Delta_{ab,a} + f_c\Delta_{ac,a}). \tag{61}$$

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Let

$$F := \begin{bmatrix} f_{xxx} & f_{xyx} & f_{xzx} \\ f_{xyx} & f_{yyx} & f_{yzx} \\ f_{xzx} & f_{yzx} & f_{zzx} \end{bmatrix}, \qquad G := \begin{bmatrix} f_{xxy} & f_{xyy} & f_{xzy} \\ f_{xyy} & f_{yyy} & f_{yzy} \\ f_{xzy} & f_{yzy} & f_{zzy} \end{bmatrix},$$
$$H := \begin{bmatrix} f_{xxz} & f_{xyz} & f_{xzz} \\ f_{xyz} & f_{yyz} & f_{yzz} \\ f_{xzz} & f_{yzz} & f_{zzz} \end{bmatrix}$$

denote the matrices obtained by differentiating the columns of the matrix Δ with respect to x, y, z respectively. If we differentiate (51a) with respect to the variables x, y, z respectively and simplify, then we can see that

$$\delta_x = \operatorname{tr}(\operatorname{adj}(\Delta)F), \qquad \delta_y = \operatorname{tr}(\operatorname{adj}(\Delta)G), \qquad \delta_z = \operatorname{tr}(\operatorname{adj}(\Delta)H), \tag{62}$$

where $\operatorname{tr} A$ denotes the trace of a matrix A. We will prove the first equality in (62). The others follow similarly.

To see this, recall $\delta = f_{xx}\Delta_{xx} + f_{xy}\Delta_{xy} + f_{xz}\Delta_{xz}$. If we differentiate with respect to x, then we obtain

$$\begin{aligned} \delta_x = {} & (f_{xxx}\Delta_{xx} + f_{xyx}\Delta_{xy} + f_{xzx}\Delta_{xz}) \\ & + f_{xx}\Delta_{xx,x} + f_{xy}\Delta_{xy,x} + f_{xz}\Delta_{xz,x}. \end{aligned} \tag{63}$$

Differentiating the following subdeterminants

$$\Delta_{xx} = f_{yy}f_{zz} - f_{yz}^2, \qquad \Delta_{xy} = f_{xz}f_{yz} - f_{xy}f_{zz}, \qquad \Delta_{xz} = f_{xy}f_{yz} - f_{xz}f_{yy}$$

with respect to x, we obtain the following equations:

$$\Delta_{xx,x} = f_{yyx}f_{zz} + f_{yy}f_{zzx} - 2f_{yz}f_{yzx}, \tag{64a}$$
$$\Delta_{xy,x} = f_{xzx}f_{yz} + f_{zx}f_{yzx} - f_{xyx}f_{zz} - f_{xy}f_{zzx}, \tag{64b}$$
$$\Delta_{xz,x} = f_{xyx}f_{yz} + f_{xy}f_{yzx} - f_{xzx}f_{yy} - f_{xz}f_{yyx}. \tag{64c}$$

Now we see that

$$\begin{aligned} & f_{xx}\Delta_{xx,x} + f_{xy}\Delta_{xy,x} + f_{xx}\Delta_{xz,x} \\ & \quad = f_{xx}(f_{yyx}f_{zz} + f_{yy}f_{zzx} - 2f_{yz}f_{yzx}) \\ & \quad\quad + f_{xy}(f_{xzx}f_{yz} + f_{zx}f_{yzx} - f_{xyx}f_{zz} - f_{xy}f_{zzx}) \\ & \quad\quad + f_{xz}(f_{xyx}f_{yz} + f_{xy}f_{yzx} - f_{xzx}f_{yy} - f_{xz}f_{yyx}) \\ & \quad = f_{xyx}(f_{xz}f_{yz} - f_{xy}f_{zz}) + f_{yyx}(f_{xx}f_{zz} - f_{xy}^2) + f_{xyz}(f_{zx}f_{xy} - f_{yz}f_{xx}) \\ & \quad\quad + f_{xyz}(f_{zx}f_{xy} - f_{yz}f_{xx}) + f_{xzx}(f_{xy}f_{yz} - f_{yy}f_{xz}) + f_{xzz}(f_{xx}f_{yy} - f_{xy}^2) \\ & \quad = (\Delta_{xy}f_{xyx} + \Delta_{yy}f_{yyx} + \Delta_{yz}f_{yzx}) + (\Delta_{xz}f_{xzx} + \Delta_{yz}f_{yzx} + \Delta_{zz}f_{zzx}). \end{aligned}$$

So we have

$$\begin{aligned} & f_{xx}\Delta_{xx,x} + f_{xy}\Delta_{xy,x} + f_{xx}\Delta_{xz,x} \\ & \quad = (\Delta_{xy}f_{xyx} + \Delta_{yy}f_{yyx} + \Delta_{yz}f_{yzx}) + (\Delta_{xz}f_{xzx} + \Delta_{yz}f_{yzx} + \Delta_{zz}f_{zzx}). \end{aligned}$$

Putting this together with (63), we see that $\delta_x = \operatorname{tr}(\operatorname{adj}(\Delta)F)$, as claimed.

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A.5. Identities Involving the Derivatives of the Subdeterminants Δ_{ab}

We collect some useful identities involving the derivatives of the (i, j) cofactors of Δ:

$$\Delta_{aa,a} + \Delta_{ab,b} + \Delta_{ac,c} = 0 \quad \text{with } \{x, y, z\} = \{a, b, c\}. \tag{65}$$

Proof of (65). We show this for one choice of a, b, c, but the others follow from symmetry. Differentiating the equation

$$\Delta_{xy} = f_{xz} f_{yz} - f_{xy} f_{zz}$$

with respect to x, we obtain the following equation:

$$\Delta_{xy,x} = f_{xzx} f_{yz} + f_{xz} f_{yzx} - f_{xyx} f_{zz} - f_{xy} f_{zzx}.$$

Now, we have that $\Delta_{yy,y} = f_{xxy} f_{zz} + f_{xx} f_{zzy} - 2 f_{xz} f_{xzy}$ and $\Delta_{yz,z} = f_{xyz} f_{xz} + f_{xy} f_{xzz} - f_{xxz} f_{yz} - f_{xx} f_{yzz}$. Adding the last two equalities together, we obtain

$$\Delta_{yy,y} + \Delta_{yz,z} = f_{xxy} f_{zz} - f_{xyz} f_{xz} - f_{xxz} f_{yz} + f_{xzz} f_{xy} = -\Delta_{xy,x}. \qquad \square$$

B. Hamiltonian Identities

In this appendix, we collect some identities involving the Hamiltonians acting on various cofactors of the matrix Δ. These identities are used in the calculation of the entries of the matrix product $\theta_1(3) M_1(3)$ and thus in construction of the lift $\theta_2(3)$ in Section 5.4. Recall

$$H_{yz} = f_z \partial_y - f_y \partial_z, \qquad H_{zx} = f_x \partial_z - f_z \partial_x, \qquad H_{xy} = f_y \partial_x - f_x \partial_y,$$

and we define $H_{zy} = -H_{yz}$, $H_{xz} = -H_{zx}$, $H_{yx} = -H_{xy}$. There are some fundamental relations among all the Hamiltonians and the Euler derivations:

$$f_x H_{yz} + f_y H_{zx} + f_z H_{xy} = 0, \tag{66a}$$

$$f_a E - c H_{ca} + b H_{ab} = 0 \quad \text{with } \{a, b, c\} = \{x, y, z\}. \tag{66b}$$

B.1. Relations Among $H_{ij}(\Delta_{kl})$

We have the following identities with $\{a, b, c\} = \{x, y, z\}$:

$$H_{ca}(\Delta_{aa}) - H_{bc}(\Delta_{ab}) = \frac{a\delta_c}{(d-1)}, \tag{67}$$

$$H_{bc}(\Delta_{bc}) - H_{ab}(\Delta_{ab}) = \frac{b\delta_b - (d-2)\delta}{(d-1)}. \tag{68}$$

Using (67) we obtain

$$2H_{yz}(\Delta_{yz}) = H_{zx}(\Delta_{xz}) + H_{xy}(\Delta_{xy}) + \frac{y\delta_y - z\delta_z}{(d-1)}, \tag{69}$$

$$0 = H_{xy}(\Delta_{xy}) - H_{zx}(\Delta_{xz}) + \frac{y\delta_y + z\delta_z}{(d-1)} - \frac{2(d-2)\delta}{(d-1)}. \tag{70}$$

Additionally, if we replace $y\delta_y + z\delta_z$ with its equal $3(d-2)\delta - x\delta_x$, then (70) becomes

$$0 = H_{xy}(\Delta_{xy}) - H_{zx}(\Delta_{xz}) + \frac{(d-2)\delta - x\delta_x}{(d-1)}. \tag{71}$$

We present the proofs (67) with $a = x$, $b = y$, and $c = z$, and the remaining follow along the same lines.

Proof. We use the identities in (60) from Appendix A to verify the identity:

$$\begin{aligned} H_{yz}(\Delta_{xy}) &= f_z\Delta_{xy,y} - f_y\Delta_{xy,z} \\ &= -f_z\Delta_{xx,x} - f_z\Delta_{xz,z} - f_y\Delta_{xy,z} \\ &= H_{zx}(\Delta_{xx}) - \frac{x\delta_z}{d-1}; \end{aligned}$$

the second equality uses (65), whereas the third uses (60). □

B.2. *Identities on How H_{ij} Act on Some Elements in $k[x, y, z]$*

Here we collect some useful identities on how H_{ij} act on (49), (52). Note that $H_{yz}(x) = 0$, $H_{yz}(y) = f_z$, and $H_{yz}(z) = -f_y$. Moreover, using the equalities in (53), we have

$$H_{yz}(f_x) = f_z f_{yx} - f_y f_{zx} = \frac{1}{(d-1)}(y\Delta_{xz} - z\Delta_{xy}), \tag{72a}$$

$$H_{yz}(f_y) = f_z f_{yy} - f_y f_{zy} = \frac{1}{(d-1)}(z\Delta_{xx} - x\Delta_{xz}), \tag{72b}$$

$$H_{yz}(f_z) = f_z f_{yz} - f_y f_{zz} = \frac{1}{(d-1)}(x\Delta_{xy} - y\Delta_{xx}). \tag{72c}$$

If we apply the various Hamiltonians to (52), then we obtain the following with $\{a, b, c\} = \{x, y, z\}$:

$$f_a H_{bc}(\Delta_{aa}) + f_b H_{bc}(\Delta_{ab}) + f_c H_{bc}(\Delta_{bc}) = \frac{a(f_c\delta_b - f_b\delta_c)}{d-1}, \tag{73a}$$

$$f_a H_{bc}(\Delta_{ab}) + f_b H_{bc}(\Delta_{bb}) + f_c H_{bc}(\Delta_{bc}) = \frac{b(f_c\delta_b - f_b\delta_c)}{d-1} - \frac{(d-2)\delta}{d-1} f_c, \tag{73b}$$

$$f_a H_{bc}(\Delta_{ac}) + f_b H_{bc}(\Delta_{bc}) + f_c H_{bc}(\Delta_{cc}) = \frac{c(f_c\delta_b - f_b\delta_c)}{d-1} + \frac{(d-2)\delta}{d-1} f_b, \tag{73c}$$

$$H_{bc}(f_a)\Delta_{aa} + H_{bc}(f_b)\Delta_{ab} + H_{bc}(f_c)\Delta_{ac} = 0. \tag{73d}$$

Remark B.2.1. Some of the equations above can be rewritten in different ways. For example, if we look at equation (73a) using the basic identity $x\delta_x + y\delta_y + z\delta_z = 3(d-2)\delta$ and rewriting $xf_x = -yf_y - zf_z$ (over R) or $xf_x = d \cdot f - yf_y -$

zf_z on $k[x, y, z]$, then we can write

$$\begin{aligned}
&\frac{x(f_x\delta_z - f_z\delta_x)}{d-1} + \frac{(d-2)\delta}{d-1} f_z \\
&\quad = \frac{\delta_z(d \cdot f - yf_y - zf_z) - xf_z\delta_x + (d-2)\delta f_z}{d-1} \\
&\quad = \frac{-f_z(x\delta_x + z\delta_z) - yf_y\delta_z + (d-2)\delta f_z}{d-1} + d\frac{d\delta_z}{d-1} f \\
&\quad = \frac{-f_z(3(d-2)\delta - y\delta_y) - yf_y\delta_z + (d-2)\delta f_z}{d-1} + d\frac{d\delta_z}{d-1} f \\
&\quad = \frac{y(f_z\delta_y - f_y\delta_z)}{d-1} - \frac{2(d-2)\delta}{d-1} f_z + d\frac{d\delta_z}{d-1} f.
\end{aligned}$$

We prove the identities in (73) for $a = x$, $b = y$, and $c = z$. The remaining identities follow similarly.

Proof of (73). We first prove $H_{yz}(f_x)\Delta_{xx} + H_{yz}(f_y)\Delta_{xy} + H_{yz}(f_z)\Delta_{xz} = 0$. We have

$$\begin{aligned}
&H_{yz}(f_x)\Delta_{xx} + H_{yz}(f_y)\Delta_{xy} + H_{yz}(f_z)\Delta_{xz} \\
&\quad = (f_z f_{xy} - f_y f_{zx})\Delta_{xx} + (f_z f_{yy} - f_y f_{yz})\Delta_{xy} + (f_z f_{yz} - f_y f_{zz})\Delta_{xz} \\
&\quad = f_z(f_{xy}\Delta_{xx} + f_{yy}\Delta_{xy} + f_{yz}\Delta_{xz}) - f_y(f_{xz}\Delta_{xx} + f_{yz}\Delta_{xy} + f_{zz}\Delta_{zz}) \\
&\quad = 0,
\end{aligned}$$

where the last equality uses (51). Applying H_{yz} to (52) yields

$$\begin{aligned}
xH_{yz}(\delta) &= (d-1)[f_x H_{yz}(\Delta_{xx}) + f_y H_{yz}(\Delta_{xy}) + f_z H_{yz}(\Delta_{xz}) \\
&\qquad + H_{yz}(f_x)\Delta_{xx} + H_{yz}(f_y)\Delta_{xy} + H_{yz}(f_z)\Delta_{xz}] \\
&= (d-1)[f_x H_{yz}(\Delta_{xx}) + f_y H_{yz}(\Delta_{xy}) + f_z H_{yz}(\Delta_{xz})],
\end{aligned}$$

where the second equality is from (73d).

Hence

$$f_x H_{yz}(\Delta_{xx}) + f_y H_{yz}(\Delta_{xy}) + f_z H_{yz}(\Delta_{xz}) = \frac{xH_{yz}(\delta)}{d-1},$$

and thus (73a) is proved.

For (73b) we use (67) and (68), rewrite $H_{yz}(\Delta_{yy})$ in terms of $H_{zx}(\Delta_{xy})$, rewrite $H_{yz}(\Delta_{yz})$ in terms of $H_{xy}(\Delta_{xy})$, and then use the fundamental relation among the Hamiltonians (66a) applied to Δ_{xy}. So we have

$$\begin{aligned}
&f_x H_{yz}(\Delta_{xy}) + f_y H_{yz}(\Delta_{yy}) + f_z H_{yz}(\Delta_{yz}) \\
&\quad = f_x H_{yz}(\Delta_{xy}) + f_y\left(H_{zx}(\Delta_{xy}) - \frac{y\delta_z}{d-1}\right) \\
&\qquad + f_z\left(H_{xy}(\Delta_{xy}) + \frac{y\delta_y - (d-2)\delta}{d-1}\right) \\
&\quad = f_x H_{yz}(\Delta_{xy}) + f_y H_{zx}(\Delta_{xy}) + f_z H_{xy}(\Delta_{xy})
\end{aligned}$$

$$+\frac{y(f_z\delta_y - f_y\delta_z)}{d-1} - \frac{(d-2)\delta}{d-1} f_z$$
$$= \frac{y(f_z\delta_y - f_y\delta_z)}{d-1} - \frac{(d-2)\delta}{d-1} f_z,$$

where the last equality uses (66a). Finally, for (73c) we will use (67) and (68), rewrite $H_{yz}(\Delta_{yz})$ in terms of $H_{zx}(\Delta_{xz})$, rewrite $H_{yz}(\Delta_{zz})$ in terms of $H_{xy}(\Delta_{xz})$, and then use the fundamental relation among the Hamiltonians (66a) applied to Δ_{xz}.

So we have

$$\begin{aligned} & f_x H_{yz}(\Delta_{xz}) + f_y H_{yz}(\Delta_{yz}) + f_z H_{yz}(\Delta_{zz}) \\ &= f_x H_{yz}(\Delta_{xz}) + f_y \left(H_{zx}(\Delta_{xz}) - \frac{z\delta_z - (d-2)\delta}{d-1} \right) \\ &+ f_z \left(H_{xy}(\Delta_{xz}) + \frac{z\delta_y}{d-1} \right) \\ &= f_x H_{yz}(\Delta_{xz}) + f_y H_{zx}(\Delta_{xz}) + f_z H_{xy}(\Delta_{xz}) \\ &+ \frac{z(f_z\delta_y - f_y\delta_z)}{d-1} + \frac{(d-2)\delta}{d-1} f_y \\ &\overset{(66a)}{=} \frac{z(f_z\delta_y - f_y\delta_z)}{d-1} + \frac{(d-2)\delta}{d-1} f_z. \end{aligned}$$ □

The remaining identities are used in Appendix E.

By direct calculation one can show

$$\begin{aligned} H_{yz}(f_{xx}) + H_{zx}(f_{xy}) + H_{xy}(f_{xz}) &= 0, \\ H_{yz}(f_{xy}) + H_{zx}(f_{yy}) + H_{xy}(f_{yz}) &= 0, \\ H_{yz}(f_{xz}) + H_{zx}(f_{yz}) + H_{xy}(f_{zz}) &= 0. \end{aligned} \tag{74}$$

We also see how the Hamiltonians H_{ij} act on (51a) and (51b). Applying H_{yz} to (51a) yields

$$\begin{aligned} H_{yz}(\delta) &= H_{yz}(f_{xx})\Delta_{xx} + H_{yz}(f_{xy})\Delta_{xy} + H_{yz}(f_{xz})\Delta_{xz} \\ &\quad + [f_{xx} H_{yz}(\Delta_{xx}) + f_{xy} H_{yz}(\Delta_{xy}) + f_{xz} H_{yz}(\Delta_{xz})], \end{aligned}$$

and hence

$$\begin{aligned} & f_{xx} H_{yz}(\Delta_{xx}) + f_{xy} H_{yz}(\Delta_{xy}) + f_{xz} H_{yz}(\Delta_{xz}) \\ &= H_{yz}(\delta) - [H_{yz}(f_{xx})\Delta_{xx} + H_{yz}(f_{xy})\Delta_{xy} + H_{yz}(f_{xz})\Delta_{xz}]. \end{aligned} \tag{75}$$

Similarly, if we apply H_{zx} to (51b), then we have

$$\begin{aligned} 0 &= H_{zx}(f_{xy})\Delta_{xx} + H_{zx}(f_{yy})\Delta_{xy} + H_{zx}(f_{yz})\Delta_{xz} \\ &\quad + [f_{xy} H_{zx}(\Delta_{xx}) + f_{yy} H_{zx}(\Delta_{xy}) + f_{yz} H_{zx}(\Delta_{xz})], \end{aligned}$$

and hence

$$\begin{aligned} & f_{xy} H_{zx}(\Delta_{xx}) + f_{yy} H_{zx}(\Delta_{xy}) + f_{yz} H_{zx}(\Delta_{xz}) \\ &= -[H_{zx}(f_{xy})\Delta_{xx} + H_{zx}(f_{yy})\Delta_{xy} + H_{zx}(f_{yz})\Delta_{xz}]. \end{aligned} \tag{76}$$

And finally, if we apply H_{xy} to (51b), then we have

$$\begin{aligned}0 = {} & H_{xy}(f_{xz})\Delta_{xx} + H_{xy}(f_{yz})\Delta_{xy} + H_{xy}(f_{zz})\Delta_{xz} \\ & + [f_{xz}H_{xy}(\Delta_{xx}) + f_{yy}H_{xy}(\Delta_{xy}) + f_{zz}H_{xy}(\Delta_{xz})],\end{aligned}$$

and hence

$$\begin{aligned}& f_{xz}H_{xy}(\Delta_{xx}) + f_{yz}H_{xy}(\Delta_{xy}) + f_{zz}H_{xy}(\Delta_{xz}) \\ & \quad = -[H_{xy}(f_{xz})\Delta_{xx} + H_{xy}(f_{yz})\Delta_{xy} + H_{xy}(f_{zz})\Delta_{xz}].\end{aligned} \tag{77}$$

REMARK B.2.2. Analogous equations hold if we apply the other Hamiltonians to the equations in (51).

C. Computations for $M_0(3)M_1(3) = 0$

C.1. From Relations Among Generators of $\ker(J_{2,1})$ *to Relations Among Generators of* $\ker(J_{3,2})$

Recall from Lemma 4.3.2 that the following relations hold for the generators of $\ker(J_{2,1})$:

$$f_a E^2 - cEH_{ca} + bEH_{ab} = 0, \quad \text{for } \{a, b, c\} = \{x, y, z\}, \tag{78}$$

$$\begin{aligned}& \frac{2}{d-1}(f_{aa}EH_{bc} + f_{ab}EH_{ca} + f_{ac}EH_{ab}) - c\alpha_b + b\alpha_c \\ & \quad = 0 \quad \text{for } \{a, b, c\} = \{x, y, z\}\end{aligned} \tag{79}$$

$$-\frac{2\delta}{(d-1)^3}E^2 + f_x\alpha_x + f_y\alpha_y + f_z\alpha_z = 0. \tag{80}$$

Precomposing with E we obtain seven relations amongst the generators of $\ker(J_{3,2})$, corresponding to the first seven columns of $M_1(3)$.

We shall now describe what will turn out to be the remaining relations among the generators. They correspond to columns 8–10 of $M_1(3)$.

PROPOSITION C.1.1. *We have over* R *that*

$$-\frac{2\delta_x}{d'^3(d-2)}E^3 + \frac{3}{d-1}(f_{xx}E\alpha_x + f_{xy}E\alpha_y + f_{xz}E\alpha_z) - z\zeta_y + y\zeta_z = 0, \tag{81a}$$

$$-\frac{2\delta_y}{d'^3(d-2)}E^3 + \frac{3}{d-1}(f_{xy}E\alpha_x + f_{yy}E\alpha_y + f_{yz}E\alpha_z) + z\zeta_x - x\zeta_z = 0, \tag{81b}$$

$$-\frac{2\delta_z}{d'^3(d-2)}E^3 + \frac{3}{d-1}(f_{xz}E\alpha_x + f_{yz}E\alpha_y + f_{zz}E\alpha_z) - y\zeta_x + x\zeta_y = 0. \tag{81c}$$

Proof. We shall prove (81a). The rest follow similarly. We will use $d' := d - 1$.

Recall that $H_{xy}^2 + \frac{\Delta_{zz}}{d'^2}E^2 = z\alpha_z$. If we apply E to it and look at the "cubic order terms", then we obtain

$$EH_{xy}^2 = zE\alpha_z - \frac{\Delta_{zz}}{d'^2}E^3. \tag{82}$$

We have

$$\begin{aligned}
-z\zeta_y + y\zeta_z &= -\frac{z}{y^2}\left(\frac{1}{z^3}(zH_{zx})^3 + \frac{3\Delta_{yy}}{d'^2}E^2H_{zx} - \frac{H_{zx}(\Delta_{yy})}{d'^2(d-2)}E^3\right) \\
&\quad + \frac{y}{z^2}\left(H_{xy}^3 + \frac{3\Delta_{zz}}{d'^2}E^2H_{xy} - \frac{H_{xy}(\Delta_{zz})}{d'^2(d-2)}E^3\right) \\
&= -\frac{z}{y^2}\left[\frac{1}{z^3}(f_x^3E^3 + 3yf_x^2E^2H_{xy} + 3y^2f_xEH_{xy}^2 + y^3H_{xy}^3)\right. \\
&\quad \left. + \frac{3\Delta_{yy}}{d'^2}E^2H_{zx} - \frac{H_{zx}(\Delta_{yy})}{d'^2(d-2)}E^3\right] \\
&\quad + \frac{y}{z^2}\left(H_{xy}^3 + \frac{3\Delta_{zz}}{d'^2}E^2H_{xy} - \frac{H_{xy}(\Delta_{zz})}{d'^2(d-2)}E^3\right) \\
&= -\frac{z}{y^2}\left[\frac{1}{z^3}\left(f_x^3E^3 + 3yf_x^2E^2H_{xy}\right.\right. \\
&\quad \left. + 3y^2f_x\left(zE\alpha_z - \frac{\Delta_{zz}}{d'^2}E^3\right) + y^3H_{xy}^3\right) \\
&\quad \left. + \frac{3\Delta_{yy}}{d'^2}E^2H_{zx} - \frac{H_{zx}(\Delta_{yy})}{d'^2(d-2)}E^3\right] \\
&\quad + \frac{y}{z^2}\left(H_{xy}^3 + \frac{3\Delta_{zz}}{d'^2}E^2H_{xy} - \frac{H_{xy}(\Delta_{zz})}{d'^2(d-2)}E^3\right) \\
&= \left[-\frac{f_x^3}{y^2z^2} + \frac{3f_x\Delta_{zz}}{d'^2z^2} + \frac{zH_{zx}(\Delta_{yy})}{d'^2(d-2)y^2} - \frac{yH_{xy}(\Delta_{zz})}{d'^2(d-2)z^2}\right]E^3 \\
&\quad + \left(-\frac{3f_x^2}{yz^2} + \frac{3y\Delta_{zz}}{d'^2z^2}\right)E^2H_{xy} - \frac{3\Delta_{yy}}{d'^2y^2}zE^2H_{zx} - \frac{3f_x}{z}E\alpha_z \\
&= \left[-\frac{f_x^3}{y^2z^2} + \frac{3f_x\Delta_{zz}}{d'^2z^2} - \frac{3f_x\Delta_{yy}}{d'^2y^2}\right. \\
&\quad \left. + \frac{1}{d'^2(d-2)}\left(\frac{zH_{zx}(\Delta_{yy})}{y^2} - \frac{yH_{xy}(\Delta_{zz})}{z^2}\right)\right]E^3 \\
&\quad + \left(-\frac{3f_x^2}{yz^2} + \frac{3y\Delta_{zz}}{d'^2z^2} - \frac{3\Delta_{yy}}{d'^2y}\right)E^2H_{xy} - \frac{3f_x}{z}E\alpha_z,
\end{aligned}$$

where the second equality uses $zH_{zx} = f_xE + yH_{xy}$, and in order to get the last equality, we use (78) and rewrite zE^2H_{zx} in terms of E^3, E^2H_{xy} via the formula $zE^2H_{zx} = f_xE^3 + yE^2H_{xy}$. To conclude the proof of (81a), we need the following lemma.

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LEMMA C.1.2. *The following equality holds*:

$$\frac{3}{d'}(f_{xx}E\alpha_x + f_{xy}E\alpha_y + f_{xz}E\alpha_z) = \frac{3f_x}{z}E\alpha_z + \frac{6}{d'^2z^2}(z\Delta_{yz} - y\Delta_{zz})E^2H_{xy} - \frac{6f_x}{d'^2z^2}\Delta_{zz}E^3. \tag{83}$$

Proof. We have

$$\begin{aligned}
&\frac{3}{d'}(f_{xx}E\alpha_x + f_{xy}E\alpha_y + f_{xz}E\alpha_z) \\
&\quad = \frac{3}{d'z}[f_{xx}(zE\alpha_x) + f_{xy}(zE\alpha_y) + f_{xz}(zE\alpha_z)] \\
&\quad = \frac{3}{d'z}\left[(xf_{xx} + yf_{xy} + zf_{xz})E\alpha_z + \frac{2}{d'}(\Delta_{yz}E^2H_{xy} - \Delta_{zz}E^2H_{zx})\right] \\
&\quad = \frac{3f_x}{z}E\alpha_z + \left(\frac{6}{d'^2z}\Delta_{yz} - \frac{6y}{d'^2z^2}\Delta_{zz}\right)E^2H_{xy} - \frac{6f_x}{d'^2z^2}\Delta_{zz}E^3 \\
&\quad = \frac{3f_x}{z}E\alpha_z + \frac{6}{d'^2z^2}(z\Delta_{yz} - y\Delta_{zz})E^2H_{xy} - \frac{6f_x}{d'^2z^2}\Delta_{zz}E^3,
\end{aligned}$$

where the second equality follows from (79), the third one follows from (49), and the last one holds by (78). □

To return to the proof of (81a), we combine the equalities in the proof of Proposition C.1.1 and Lemma C.1.2 to obtain

$$\begin{aligned}
&\frac{3}{d'}(f_{xx}E\alpha_x + f_{xy}E\alpha_y + f_{xz}E\alpha_z) - z\zeta_y + y\zeta_z \\
&\quad = \Bigg[-\frac{f_x^3}{y^2z^2} + \frac{3f_x\Delta_{zz}}{d'^2z^2} - \frac{3f_x\Delta_{yy}}{d'^2y^2} \\
&\qquad + \frac{1}{d'^2(d-2)}\left(\frac{zH_{zx}(\Delta_{yy})}{y^2} - \frac{yH_{xy}(\Delta_{zz})}{z^2}\right)\Bigg]E^3 \\
&\qquad + \left(-\frac{3f_x^2}{yz^2} + \frac{3y\Delta_{zz}}{d'^2z^2} - \frac{3\Delta_{yy}}{d'^2y}\right)E^2H_{xy} - \frac{3f_x}{z}E\alpha_z \\
&\qquad + \frac{3f_x}{z}E\alpha_z + \frac{6}{d'^2z^2}(z\Delta_{yz} - y\Delta_{zz})E^2H_{xy} - \frac{6f_x}{d'^2z^2}\Delta_{zz}E^3 \\
&\quad = \Bigg[\frac{f_x}{d'^2y^2z^2}(y^2\Delta_{zz} + z^2\Delta_{yy} - 2yz\Delta_{yz} + 3y^2\Delta_{zz} - 3z^2\Delta_{yy} - 6y^2\Delta_{zz}) \\
&\qquad + \frac{1}{d'^2(d-2)y^2z^2}(z^3H_{zx}(\Delta_{yy}) - y^3H_{xy}(\Delta_{zz})\Bigg]E^3 \\
&\qquad + \Bigg[\frac{3}{d'^2yz^2}(y^2\Delta_{zz} + z^2\Delta_{yy} - 2yz\Delta_{yz}) + \frac{3y}{d'^2z^2}\Delta_{zz} \\
&\qquad - \frac{3\Delta_{yy}}{d'^2y} + \frac{6\Delta_{yz}}{d'^2z} - \frac{6y\Delta_{zz}}{d'^2z^2}\Bigg]E^2H_{xy}
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{d'^2(d-2)y^2z^2}[-y^2 f_x E(\Delta_{zz}) - z^2 f_x E(\Delta_{yy}) - yz f_x E(\Delta_{yz}) \\
&\quad + z^3 H_{zx}(\Delta_{yy}) - y^3 H_{xy}(\Delta_{zz})]E^3 + 0E^2 H_{xy} \\
&= \frac{1}{d'^2(d-2)y^2z^2}[-y^2(zH_{zx}(\Delta_{zz}) - yH_{xy}(\Delta_{zz})) \\
&\quad - z^2(zH_{zx}(\Delta_{yy}) - yH_{xy}(\Delta_{yy})) - yz(zH_{zx}(\Delta_{yz} - yH_{xy}(\Delta_{yz}))) \\
&\quad + z^3 H_{zx}(\Delta_y y) - y^3 H_{xy}(\Delta_{zz})]E^3 \\
&= \frac{1}{d'^2(d-2)y^2z^2} \\
&\quad \times [y^2 z(H_{xy}(\Delta_{yz}) - H_{zx}(\Delta_{zz})) + yz^2(H_{xy}(\Delta_{yy}) - H_{zx}(\Delta_{yz}))]E^3 \\
&= \frac{2}{d'^3(d-2)}\delta_x E^3,
\end{aligned}
$$

where the second equality follows from (59); to obtain the fifth equality, we repeatedly apply the fundamental identity $f_x E - zH_{zx} + yH_{xy} = 0$ to each of Δ_{zz}, Δ_{yy}, and Δ_{yz} and use the fact that $E(\Delta_{ij}) = 2(d-2)\Delta_{ij}$; and the last equality follows from (67) and (68). We have thus proven (81a). $\square$

C.2. Relation Among ζ_x, ζ_y, ζ_z and the Various $E^2 H_{ij}$

In this subsection we establish a relation that corresponds to the last column of $M_1(3)$. We use the following notation: $d' := (d-1)$, $d'' := (d-1)^3(d-2) = d'^3(d-2)$.

Lemma C.2.1. *We have*

$$
f_x\zeta_x + f_y\zeta_y + f_z\zeta_z - \frac{2\delta_x}{d''}E^2 H_{yz} - \frac{2\delta_y}{d''}E^2 H_{zx} - \frac{2\delta_z}{d''}E^2 H_{xy} = 0. \tag{84}
$$

Proof. Observe that

$$
\begin{aligned}
f_x\zeta_x &= \frac{1}{x}(-yf_y - zf_z)\zeta_x \\
&= \frac{f_y}{x}(-y\zeta_x) + \frac{f_z}{x}(-z\zeta_x) \\
&= \frac{f_y}{x}\left[\frac{2\delta_z}{d''}E^3 - \frac{3}{d'}(f_{xz}E\alpha_x + f_{yz}E\alpha_y + f_{zz}E\alpha_z) - x\zeta_y\right] \\
&\quad + \frac{f_z}{x}\left[-\frac{2\delta_y}{d''}E^3 + \frac{3}{d'}(f_{xy}E\alpha_x + f_{yy}E\alpha_y + f_{yz}E\alpha_z) - x\zeta_z\right] \\
&= -f_y\zeta_y - f_z\zeta_z + \frac{2(f_y\delta_z - f_z\delta_y)}{d''x}E^3 + \frac{3}{d'x}[(f_z f_{xy} - f_y f_{xz})E\alpha_x \\
&\quad + (f_z f_{yy} - f_y f_{yz})E\alpha_y + (f_z f_{yz} - f_y f_{zz})E\alpha_z]
\end{aligned}
$$

$$
\begin{aligned}
&= -f_y\zeta_y - f_z\zeta_z + \frac{2(f_y\delta_z - f_z\delta_y)}{d''x}E^3 \\
&\quad + \frac{3}{d'^2x}[(y\Delta_{xz} - z\Delta_{xy})E\alpha_x + (z\Delta_{xx} - x\Delta_{xz})E\alpha_y \\
&\quad + (x\Delta_{xy} - y\Delta_{xx})E\alpha_z)] \\
&= -f_y\zeta_y - f_z\zeta_z + \frac{2}{d''x}(\delta_z f_y E^3 - \delta_y f_z E^3) \\
&\quad + \frac{3}{d'^2x}[\Delta_{xx}(zE\alpha_y - yE\alpha_x) + \Delta_{xy}(xE\alpha_z - zE\alpha_x) \\
&\quad + \Delta_{xz}(yE\alpha_x - xE\alpha_y)] \\
&= -f_y\zeta_y - f_z\zeta_z \\
&\quad + \frac{2}{d''x}[\delta_z(-zE^2H_{yz} + xE^2H_{xy}) - \delta_y(yE^2H_{yz} - xE^2H_{zx})] \\
&\quad + \frac{3}{d'^2x}[\Delta_{xx}(zE\alpha_y - yE\alpha_x) + \Delta_{xy}(xE\alpha_z - zE\alpha_x) \\
&\quad + \Delta_{xz}(yE\alpha_x - xE\alpha_y)] \\
&= -f_y\zeta_y - f_z\zeta_z \\
&\quad + \frac{2}{d''x}[(-3(d-2)\delta + x\delta_x)E^2H_{yz} + x\delta_y E^2H_{zx} + x\delta_z E^2H_{xy}] \\
&\quad + \frac{6}{d'^3x}[\Delta_{xx}(f_{xx}E^2H_{yz} + f_{xy}E^2H_{zx} + f_{xz}E^2H_{xy}) \\
&\quad + \Delta_{xy}(f_{xy}E^2H_{yz} + f_{yy}E^2H_{zx} + f_{yz}E^2H_{xy}) \\
&\quad + \Delta_{xz}(f_{xz}E^2H_{yz} + f_{yz}E^2H_{zx} + f_{zz}E^2H_{xy})] \\
&= -f_y\zeta_y - f_z\zeta_z - \frac{6\delta}{d'^3x}E^2H_{yz} + \frac{2}{d''}(\delta_x E^2H_{yz} + \delta_y E^2H_{zx} + \delta_z E^2H_{xy}) \\
&\quad + \frac{6}{d'^3x}[(f_{xx}\Delta_{xx} + f_{xy}\Delta_{xy} + f_{xz}\Delta_{xz})E^2H_{yz} \\
&\quad + (f_{xy}\Delta_{xx} + f_{yy}\Delta_{xy} + f_{yz}\Delta_{xz})E^2H_{zx} \\
&\quad + (f_{xz}\Delta_{xx} + f_{yz}\Delta_{xy} + f_{zz}\Delta_{xz})E^2H_{xy}] \\
&= -f_y\zeta_y - f_z\zeta_z + \frac{2}{d''}(\delta_x E^2H_{yz} + \delta_y E^2H_{zx} + \delta_z E^2H_{xy}) \\
&\quad - \frac{6\delta}{d'^3x}E^2H_{yz} + \frac{6\delta}{d'^3x}E^2H_{yz},
\end{aligned}
$$

where the first equality follows from (48), the third from (81c) and (81b), the fifth from (53), the seventh from (78), the eighth from (79), and the ninth from (51). Hence we have shown

$$f_x\zeta_x + f_y\zeta_y + f_z\zeta_z - \frac{2}{d''}\delta_x E^2H_{yz} - \frac{2}{d''}\delta_y E^2H_{zx} - \frac{2}{d''}\delta_z E^2H_{xy} = 0. \qquad \square$$

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D. Matrix Identities

In this appendix we collect matrix identities involving the matrices, with entries in R, from Section 3.

$$\begin{gathered}\partial_1 = \partial_3^T, \qquad \partial_2^T = -\partial_2, \qquad D_1 = D_3^T, \\ D_2^T = -D_2, \qquad \sigma_1 = \sigma_3^T, \qquad \sigma_2^T = -\sigma_2,\end{gathered} \tag{85}$$

$$\partial_1 D_3 = 0_{3\times 3}, \qquad \partial_3 D_1 = D_2\partial_2, \qquad \partial_i\alpha_i = \alpha_{i-1}D_i, \tag{86}$$

$$\alpha_1\alpha_2 = \alpha_2\alpha_1 = \frac{\delta}{(d-1)^3} I_{3\times 3}, \tag{87}$$

$$\frac{1}{3}\sigma_3\partial_1 = \alpha_1\alpha_2 + \frac{1}{3}\partial_2\sigma_2, \qquad q\sigma_3 = 3\alpha_2 D_3, \tag{88}$$

$$q = \partial_3\partial_1, \qquad \alpha_1 q = \partial_2 D_2, \qquad \alpha_1 q = D_3\partial_1, \qquad q\partial_2 = 0, \tag{89}$$

$$\sigma_3\partial_1 - \partial_2\sigma_2 = \frac{3\delta}{(d-1)^3} I_{3\times 3}, \tag{90}$$

$$D_2\sigma_3 + \sigma_2 D_3 = 0, \qquad D_1\sigma_2 = -\sigma_1 D_2. \tag{91}$$

E. Proof of Lemma 5.4.2

E.1. First Square Commutes

Write

$$\theta_0(3) = \begin{bmatrix} \frac{1}{6}f_{xxx} & \frac{1}{2}f_{xxy} & \frac{1}{2}f_{xxz} & \frac{1}{2}f_{xyy} & f_{xyz} & \frac{1}{2}f_{xzz} & \frac{1}{6}f_{yyy} & \frac{1}{2}f_{yyz} & \frac{1}{2}f_{yzz} & \frac{1}{6}f_{zzz} \\ \frac{1}{2}f_{xx} & f_{xy} & f_{xz} & \frac{1}{2}f_{yy} & f_{yz} & \frac{1}{2}f_{zz} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}f_{xx} & 0 & f_{xy} & f_{xz} & 0 & \frac{1}{2}f_{yy} & f_{yz} & \frac{1}{2}f_{zz} & 0 \\ 0 & 0 & \frac{1}{2}f_{xx} & 0 & f_{xy} & f_{xz} & 0 & \frac{1}{2}f_{yy} & f_{yz} & \frac{1}{2}f_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} < -r_1- > \\ < -r_2- > \\ < -r_3- > \\ < -r_4- > \end{bmatrix},$$

where r_i denote the rows of the matrix $\theta_0(3)$.

PROPOSITION E.1.1. *The following equality is satisfied*: $-\theta_0(3)M_0(3) = \partial_1^C\theta_1(3)$.

The proof is given at the end of the subsection after a series of preparatory lemmas. We make use of the description of $M_0(3)$ as

$$M_0(3) = \begin{bmatrix} E^3 & E^2H_{yz} & E^2H_{zx} & E^2H_{xy} & E\alpha_x & E\alpha_y & E\alpha_z & \zeta_x & \zeta_y & \zeta_z \end{bmatrix}$$

discussed in 5.1.2.

LEMMA E.1.2. *The products* $\theta_0(3)E^3$, $\theta_0(3)E^2H_{yz}$, $\theta_0(3)E^2H_{zx}$, *and* $\theta_0(3)E^2H_{xy}$ *all vanish.*

Proof. Using (57), one shows that $r_1E^3 = 0$. Also,

$$r_2E^3 = 3x(x^2f_{xx} + y^2f_{yy} + z^2f_{zz} + 2xyf_{xy} + 2xzf_{xz} + 2yzf_{yz}),$$

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$$
\begin{aligned}
r_3 E^3 &= 3y(x^2 f_{xx} + y^2 f_{yy} + z^2 f_{zz} + 2xyf_{xy} + 2xzf_{xz} + 2yzf_{yz}),\\
r_4 E^3 &= 3z(x^2 f_{xx} + y^2 f_{yy} + z^2 f_{zz} + 2xyf_{xy} + 2xzf_{xz} + 2yzf_{yz}),
\end{aligned}
$$

and note that the right-hand sides of the equations above are zero in R by (50), and hence $\theta_0(3)E^3 = 0_{4\times 1}$. Next we show that $\theta_0(3)E^2 H_{yz} = 0_{4\times 1}$. Similarly, one can show that $\theta_0(3)E^2 H_{zx} = 0_{4\times 1}$, $\theta_0(3)E^2 H_{xy} = 0_{4\times 1}$. We have

$$
\begin{aligned}
r_1 E^2 H_{yz} &= f_z(x^2 f_{xxy} + 2xyf_{xyy} + 2xzf_{xyz} + y^2 f_{yyy} + 2yzf_{yyz} + z^2 f_{yzz})\\
&\quad - f_y(x^2 f_{xxz} + 2xyf_{xyz} + 2xzf_{xzz} + y^2 f_{yyz} + 2yzf_{yzz} + z^2 f_{zzz})\\
&= (d-2)(d-1)(f_z f_y - f_y f_z)\\
&= 0,
\end{aligned}
$$

where the second equality uses (56). The calculations that $r_i E^2 H_{yz} = 0$ for $i = 2, 3, 4$ follow similarly using (49) and (50). Hence $\theta_0(3)E^2 H_{yz} = 0_{4\times 1}$, as claimed. □

LEMMA E.1.3. *The following equalities are satisfied*:

$$
\theta_0(3)E\alpha_x = \frac{-x}{(d-1)^2}\begin{bmatrix}(d-2)\delta\\ x\delta\\ y\delta\\ z\delta\end{bmatrix}, \qquad \theta_0(3)E\alpha_y = \frac{-y}{(d-1)^2}\begin{bmatrix}(d-2)\delta\\ x\delta\\ y\delta\\ z\delta\end{bmatrix},
$$

$$
\theta_0(3)E\alpha_z = \frac{-z}{(d-1)^2}\begin{bmatrix}(d-2)\delta\\ x\delta\\ y\delta\\ z\delta\end{bmatrix}.
$$

Proof. We verify that the first equality is satisfied in the first and third entry; the rest are similar and left to the reader.

Observe that

$$
\begin{aligned}
\frac{(d-1)^2}{2} r_1 E\alpha_x &= \Delta_{xx}\left((d-2)(d-1)f_x - \frac{x}{2}(xf_{xxx} + yf_{xxy} + zf_{xxz})\right)\\
&\quad + \Delta_{xy}(xyf_{xyy} + xzf_{xyz} + y^2 f_{yyy} + 2yzf_{yyz} + z^2 f_{yzz})\\
&\quad + \Delta_{xz}(xyf_{xyz} + xzf_{xzz} + y^2 f_{yyz} + 2yzf_{yzz} + z^2 f_{zzz})\\
&\quad + \Delta_{yy}\left(-\frac{x^2}{2} f_{xyy} - \frac{xy}{2} f_{yyy} - \frac{xz}{2} f_{yyz}\right)\\
&\quad + \Delta_{yz}(-x^2 f_{xyz} - xyf_{yyz} - xzf_{yzz})\\
&\quad + \Delta_{zz}\left(-\frac{x^2}{2} f_{xzz} - \frac{xy}{2} f_{yzz} - \frac{xz}{2} f_{zzz}\right)\\
&= (d-2)(d-1)f_x\Delta_{xx} - \frac{(d-2)xf_{xx}\Delta_{xx}}{2}\\
&\quad + \Delta_{xy}((d-2)(d-1)f_y - x(xf_{xxy} + yf_{xyy} + zf_{xyz}))\\
&\quad + \Delta_{xz}((d-2)(d-1)f_z - xyf_{xyz} - xzf_{xzz})
\end{aligned}
$$

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$$
\begin{aligned}
&- \frac{x}{2}\Delta_{yy}(xf_{xyy} + yf_{yyy} + zf_{yyz}) \\
&- x\Delta_{yz}(xf_{xyz} + yf_{yyz} + zf_{yzz}) \\
&- \frac{x}{2}\Delta_{zz}(xf_{xzz} + yf_{yzz} + zf_{zzz}) \\
= {}& (d-2)(d-1)(f_x\Delta_{xx} + f_y\Delta_{xy} + f_z\Delta_{xz}) \\
&- \frac{(d-2)x}{2}(f_{xx}\Delta_{xx} + f_{xy}\Delta_{xy} + f_{xz}\Delta_{xz}) \\
&- \frac{(d-2)x}{2}(f_{xy}\Delta_{xy} + f_{yy}\Delta_{yy} + f_{yz}\Delta_{yz}) \\
&- \frac{(d-2)x}{2}(f_{xz}\Delta_{xz} + f_{yz}\Delta_{yz} + f_{zz}\Delta_{zz}) \\
= {}& -\frac{d-2}{2}x\delta.
\end{aligned}
$$

The first equality uses (56), the second follows from (54) and (56), the third equality is from (54) and (55), and the fourth equality is from (51) and (52). Hence $r_1 E\alpha_x = -d\frac{(d-2)}{(d-1)^2}x\delta$, as needed.

Next consider

$$
\begin{aligned}
\frac{(d-1)^2}{2}r_3 E\alpha_x = {}& 2y\Delta_{xx}(xf_{xx} + yf_{xy} + zf_{xz}) - \frac{xy}{2}f_{xx}\Delta_{xx} \\
&+ \Delta_{xy}(2xyf_{xy} + xzf_{xz} + 3y^2 f_{yy} + 4yzf_{yz} + z^2 f_{zz}) \\
&+ 2y\Delta_{xz}(xf_{xz} + yf_{yz} + zf_{zz}) \\
&- xyf_{xz}\Delta_{xz} - x\Delta_{yy}(xf_{xy} + yf_{yy} + zf_{yz}) \\
&- \frac{xy}{2}f_{yy}\Delta_{yy} - x\Delta_{yz}(xf_{xz} + yf_{xz} + zf_{zz}) - \frac{xy}{2}f_{zz}\Delta_{zz} \\
= {}& 2(d-1)yf_x\Delta_{xx} - \frac{xy}{2}f_{xx}\Delta_{xx} \\
&+ \Delta_{xy}(-x^2 f_{xx} - xzf_{xz} + 2y^2 f_{yy} + 2yzf_{yz}) \\
&+ 2(d-1)yf_z\Delta_{xz} - xyf_{xz}\Delta_{xz} - (d-1)xf_y\Delta_{yy} - \frac{xy}{2}\Delta_{yy} \\
&- (d-1)xf_z\Delta_{yz} - \frac{xy}{2}f_{zz}\Delta_{zz} \\
= {}& 2(d-1)yf_x\Delta_{xx} - \frac{xy}{2}f_{xx}\Delta_{xx} + 2(d-1)yf_y\Delta_{xy} \\
&- (d-1)xf_x\Delta_{xy} - xyf_{xy}\Delta_{xy} \\
&+ 2(d-1)yf_z\Delta_{xz} - xyf_{xz}\Delta_{xz} - (d-1)xf_y\Delta_{yy} \\
&- \frac{xy}{2}\Delta_{yy} - (d-1)xf_z\Delta_{yz} - \frac{xy}{2}f_{zz}\Delta_{zz} \\
= {}& 2(d-1)y(f_x\Delta_{xx} + f_y\Delta_{xy} + f_z\Delta_{xz}) \\
&- (d-1)x(f_x\Delta_{xy} + f_y\Delta_{yy} + f_z\Delta_{yz}) \\
&- \frac{xy}{2}(f_{xx}\Delta_{xx} + f_{xy}\Delta_{xy} + f_{xz}\Delta_{xz})
\end{aligned}
$$

$$\begin{aligned}
&- \frac{xy}{2}(f_{xy}\Delta_{xy} + f_{yy}\Delta_{yy} + f_{yz}\Delta_{yz}) \\
&- \frac{xy}{2}(f_{xz}\Delta_{xz} + f_{yz}\Delta_{yz} + f_{zz}\Delta_{zz}) \\
&= 2yx\delta - xy\delta - \frac{3xy}{2}\delta,
\end{aligned}$$

where the second equality uses (49) and (50), and the last equality uses (51) and (52). Hence $r_3 E\alpha_x = -\frac{x\delta}{(d-1)^2}y$, as claimed. □

LEMMA E.1.4. *There is an equality*

$$\theta_0(3)\begin{bmatrix}\zeta_z & \zeta_y & \zeta_z\end{bmatrix} = \frac{1}{(d-1)^2}\begin{bmatrix} \delta_z f_y - \delta_y f_z & \delta_x f_z - \delta_z f_x & \delta_y f_x - \delta_x f_y \\ 0 & 3\delta f_z & -3\delta f_y \\ -3\delta f_z & 0 & 3\delta f_x \\ 3\delta f_y & -3\delta f_x & 0 \end{bmatrix}.$$

Proof. We compute first $\theta_0(3)\zeta_x$. Recall that

$$\zeta_x = \frac{1}{x^2}\left[H_{yz}^3 + \frac{3}{(d-1)^2}\Delta_{xx}E^2 H_{yz} - \frac{1}{(d-2)(d-1)^2}H_{yz}(\Delta_{xx})E^3\right],$$

and since by Lemma E.1.2

$$\theta_0(3)E^3 = 0_{4\times 1} = \theta_0(3)E^2 H_{yz},$$

it suffices to compute $\theta_0(3)H_{yz}^3$. We claim that

$$\theta_0(3)H_{yz}^3 = \frac{x^2}{(d-1)^2}\begin{bmatrix} \delta_z f_y - \delta_y f_z \\ 0 \\ -3\delta f_z \\ 3\delta f_y \end{bmatrix}. \tag{92}$$

Indeed, since the first six entries of H_{yz}^3 are zero, we see that $r_2 H_{yz}^3 = 0$. Next, observe

$$\begin{aligned}
r_3 H_{yz}^3 &= 3f_{yy}f_z^3 - 6f_{yz}f_z^2 f_y + 3f_{zz}f_z f_y^2 \\
&= 3f_z^2(f_{yy}f_z - f_{yz}f_y) - 3f_z f_y(f_{yz}f_z - f_{zz}f_y) \\
&= \frac{3f_z}{(d-1)}[f_z(z\Delta_{xx} - x\Delta_{xz}) - f_y(x\Delta_{xy} - y\Delta_{xx})] \\
&= \frac{3f_z}{(d-1)}[\Delta_{xx}(zf_z + yf_y) - x(f_y\Delta_{xy} + f_z\Delta_{xz})] \\
&= \frac{3f_z}{(d-1)}[-x(f_x\Delta_{xx} + f_y\Delta_{xy} + f_z\Delta_{xz})] \\
&= x^2\frac{-3\delta f_z}{(d-1)^2},
\end{aligned}$$

where the third equality uses (53) and the last equality is from (52). Similarly, one can verify that $r_4 H_{yz}^3 = x^2\frac{3\delta f_y}{(d-1)^2}$.

To compute $r_1 H_{yz}^3$, notice that

$$H_{yz}(f_{yy}) = f_z f_{yyy} - f_y f_{yyz}, \qquad H_{yz}(f_{yz}) = f_z f_{yyz} - f_y f_{yzz},$$
$$H_{yz}(f_{zz}) = f_z f_{yzz} - f_y f_{zzz}.$$

As a consequence, with $H = H_{yz}$, the following equalities hold:

$$\begin{aligned} r_1 H^3 &= f_z^3 f_{yyy} - 3 f_z^2 f_y f_{yyz} + 3 f_z f_y^2 f_{yzz} - f_y^3 f_{zzz} \\ &= f_z^2 (f_z f_{yyy} - f_y f_{yyz}) - 2 f_z f_y (f_z f_{yyz} - f_y f_{yzz}) + f_y^2 (f_z f_{yzz} - f_y f_{zzz}) \\ &= f_z^2 H(f_{yy}) - 2 f_z f_y H(f_{yz}) + f_y^2 H(f_{zz}) \\ &= f_z [f_z H(f_{yy}) - f_y H(f_{yz})] - f_y [f_z H(f_{yz}) - f_y H(f_{zz})]. \end{aligned}$$

From (53) we have

$$H(f_y) = f_z f_{yy} - f_y f_{yz} = \frac{1}{(d-1)}(z\Delta_{xx} - x\Delta_{xz}) \quad \text{and}$$
$$H(f_z) = f_z f_{yz} - f_y f_{zz} = \frac{1}{d-1}(x\Delta_{xy} - y\Delta_{xx}).$$

Applying H to the first and second equation, respectively, yields

$$\begin{aligned} f_z H(f_{yy}) - f_y H(f_{yz}) = \frac{1}{d-1}&[-f_y \Delta_{xx} + zH(\Delta_{xx}) - xH(\Delta_{xz})] \\ &- f_{yy} H(f_z) + f_{yz} H(f_y), \end{aligned} \tag{93}$$

$$\begin{aligned} f_z H(f_{yz}) - f_y H(f_{zz}) = \frac{1}{d-1}&[xH(\Delta_{xy}) - f_z \Delta_{xx} - yH(\Delta_{xx})] \\ &- f_{yz} H(f_z) + f_{zz} H(f_y). \end{aligned} \tag{94}$$

Multiplying (93) with f_z, (94) with f_y and subtracting establishes the following:

$$\begin{aligned} r_1 H^3 &= f_z [f_z H(f_{yy}) - f_y H(f_{yz})] - f_y [f_z H(f_{yz}) - f_y H(f_{zz})] \\ &= \frac{1}{d-1}[H(\Delta_{xx})(z f_z + y f_y) - x(f_z H(\Delta_{xz}) + f_y H(\Delta_{xy}))] \\ &\quad + H(f_z)(f_y f_{yz} - f_{yy} f_z) + H(f_y)(f_z f_{yz} - f_y f_{zz}) \\ &= \frac{-x}{d-1}(f_x H(\Delta_{xx}) + f_y H(\Delta_{xy}) + f_z H(\Delta_{xz})) \\ &= x^2 \, \frac{\delta_z f_y - \delta_y f_z}{(d-1)^2}, \end{aligned}$$

where the first equality was established above and the last equality is from (73a).

Similarly, one can check that

$$\theta_0(3) H_{zx}^3 = \frac{y^2}{(d-1)^2} \begin{bmatrix} \delta_x f_z - \delta_z f_x \\ 3\delta f_z \\ 0 \\ -3\delta f_x \end{bmatrix} \quad \text{and}$$

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$$\theta_0(3)H^3_{xy} = \frac{z^2}{(d-1)^2}\begin{bmatrix} \delta_y f_x - \delta_x f_y \\ -3\delta f_y \\ 3\delta f_x \\ 0 \end{bmatrix},$$

from which

$$\theta_0(3)\zeta_y = \frac{1}{(d-1)^2}\begin{bmatrix} \delta_x f_z - \delta_z f_x \\ 3\delta f_z \\ 0 \\ -3\delta f_x \end{bmatrix} \quad \text{and}$$

$$\theta_0(3)\zeta_z = \frac{1}{(d-1)^2}\begin{bmatrix} \delta_y f_x - \delta_x f_y \\ -3\delta f_y \\ 3\delta f_x \\ 0 \end{bmatrix}$$

follow, respectively. $\square$

E.1.5. Next consider the vectors

$$V_x = \left[\begin{array}{c} 0 \\ \frac{\delta_z}{(d-1)^2} \\ -\frac{\delta_y}{(d-1)^2} \\ \hline A_x \end{array}\right]$$

$$\text{where } A_x := \frac{3}{(d-1)(d-2)}\begin{bmatrix} H_{yz}(\Delta_{xx}) \\ H_{zx}(\Delta_{xx}) \\ H_{xy}(\Delta_{xx}) \\ H_{zx}(\Delta_{xy}) + \frac{y\delta_z}{d-1} \\ \frac{1}{2}(H_{zx}(\Delta_{xz}) + H_{xy}(\Delta_{xy}) + \frac{z\delta_z - y\delta_y}{d-1}) \\ H_{xy}(\Delta_{xz}) - \frac{z\delta_y}{d-1} \end{bmatrix},$$

$$V_y = \left[\begin{array}{c} -\frac{\delta_z}{(d-1)^2} \\ 0 \\ \frac{\delta_x}{(d-1)^2} \\ \hline A_y \end{array}\right]$$

$$\text{where } A_y := \frac{3}{(d-1)(d-2)}\begin{bmatrix} H_{yz}(\Delta_{xy}) - \frac{x\delta_z}{d-1} \\ H_{yz}(\Delta_{yy}) \\ \frac{1}{2}(H_{yz}(\Delta_{yz}) + H_{xy}(\Delta_{xy}) + \frac{x\delta_x - z\delta_z}{d-1}) \\ H_{zx}(\Delta_{yy}) \\ H_{xy}(\Delta_{yy}) \\ H_{xy}(\Delta_{yz}) + \frac{z\delta_x}{d-1} \end{bmatrix},$$

$$V_z = \left[\begin{array}{c} \frac{\delta_y}{(d-1)^2} \\ -\frac{\delta_x}{(d-1)^2} \\ 0 \\ \hline A_z \end{array}\right]$$

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$$\text{where } A_z := \frac{3}{(d-1)(d-2)} \begin{bmatrix} H_{yz}(\Delta_{xz}) + \frac{x\delta_y}{d-1} \\ \frac{1}{2}(H_{yz}(\Delta_{yz}) + H_{zx}(\Delta_{xz}) + \frac{y\delta_y - x\delta_x}{d-1}) \\ H_{yz}(\Delta_{zz}) \\ H_{zx}(\Delta_{yz}) - \frac{y\delta_x}{d-1} \\ H_{zx}(\Delta_{zz}) \\ H_{xy}(\Delta_{zz}) \end{bmatrix}.$$

LEMMA E.1.6. *The following equality is satisfied*:

$$\partial_1^C \begin{bmatrix} V_x & V_y & V_z \end{bmatrix} = \frac{1}{(d-1)^2} \begin{bmatrix} \delta_z f_y - \delta_y f_z & \delta_x f_z - \delta_z f_x & \delta_y f_x - \delta_x f_y \\ 0 & 3\delta f_z & -3\delta f_y \\ -3\delta f_z & 0 & 3\delta f_x \\ 3\delta f_y & -3\delta f_x & 0 \end{bmatrix}.$$

Proof. Using (67), (68), (75), (76), (77), and (74), it follows that $A_x \in \ker\theta_0(2)$. Now applying (66a), (73b), and (70), one can verify that $\partial_1^C V_x$ is the first column of the matrix on the right-hand side. Similar computations show that the other columns agree. □

Proof of Proposition E.1.1. Using identities (51) and (52), and Lemma E.1.6, it is straightforward to compute the entries of $\partial_1^C\theta_1(3)$. Comparing these with the entries computed in Lemmas E.1.2 to E.1.4, one concludes

$$\partial_1^C\theta_1(3) = -\theta_0(3)M_0(3). \qquad \square$$

E.2. *The Second Square Commutes*

Next, we show the following.

PROPOSITION E.2.1. *The equality* $-\theta_1(3)M_1(3) = \partial_2^C\theta_2(3)$ *holds.*

Proof. Let $d' = d-1$. Using the identities in Appendices A and B, one can show that $-\theta_1(3)M_1(3)$ is of the form

$$\left[\begin{array}{ccc|c} 0_{3\times 3} & \frac{d+1}{2}K & \Lambda & \multirow{2}{*}{W} \\ 0_{6\times 3} & 0_{6\times 3} & \Pi & \end{array}\right],$$

where

$$K = \begin{bmatrix} f_z f_{xy} - f_y f_{xz} & f_z f_{yy} - f_y f_{yz} & f_z f_{yz} - f_y f_{zz} \\ f_x f_{xz} - f_z f_{xx} & f_x f_{yz} - f_z f_{xy} & f_x f_{zz} - f_z f_{xz} \\ f_y f_{xx} - f_x f_{xy} & f_y f_{xy} - f_x f_{yy} & f_y f_{xz} - f_x f_{yz} \end{bmatrix},$$

$$\Lambda = \frac{9\delta}{2d'} I_{3\times 3} - \frac{1}{d'^2} \begin{bmatrix} x\delta_x & x\delta_y & x\delta_z \\ y\delta_x & y\delta_y & y\delta_z \\ z\delta_x & z\delta_y & z\delta_z \end{bmatrix},$$

$$W = \frac{1}{d'^2}\begin{bmatrix} H_{yz}(\delta) \\ H_{zx}(\delta) \\ H_{xy}(\delta) \\ \frac{6x}{d-2}H_{yz}(\delta) \\ \frac{3}{d-2}(xH_{zx}(\delta)+yH(\delta)) \\ \frac{3}{d-2}(xH_{xy}(\delta)+zH(\delta)) \\ \frac{6y}{d-2}H_{zx}(\delta) \\ \frac{3}{d-2}(yH_{xy}(\delta)+zH_{zx}(\delta)) \\ \frac{6z}{d-2}H_{xy}(\delta) \end{bmatrix}$$

and where Π is the matrix described as

$$\Pi = -\frac{9}{d'}\begin{bmatrix} \Delta_{xx} \\ \Delta_{xy} \\ \Delta_{xz} \\ \Delta_{yy} \\ \Delta_{yz} \\ \Delta_{zz} \end{bmatrix}\begin{bmatrix} f_x & f_y & f_z \end{bmatrix} + \frac{9}{d'^2}\begin{bmatrix} 2x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & 2y & 0 \\ 0 & z & y \\ 0 & 0 & 2z \end{bmatrix}$$
$$-\frac{3}{d'^2(d-2)}\begin{bmatrix} 2x^2 \\ 2xy \\ 2xz \\ 2y^2 \\ 2yz \\ 2z^2 \end{bmatrix}\begin{bmatrix} \delta_x & \delta_y & \delta_z \end{bmatrix}.$$

Note that the first three columns of $-\theta_1(3)M_1(3)$ are zero. Since the first three columns of $\theta_2(3)$ are also zero, one has

$$\mathrm{Col}_i(-\theta_1(3)M_1(3)) = \mathrm{Col}_i(\partial_2^C\theta_2(3)) \quad \text{for } i = 1, 2, 3.$$

Next the fourth column of $-\theta_1(3)M_1(3)$ is of the form

$$\left[\begin{array}{c} \frac{d+1}{2}(f_z f_{xy} - f_y f_{xz}) \\ \frac{d+1}{2}(f_x f_{xz} - f_z f_{xx}) \\ \frac{d+1}{2}(f_y f_{xx} - f_x f_{xy}) \\ \hline 0_{6\times 1} \end{array}\right],$$

and we have the following equality:

$$\begin{aligned} &\frac{d+1}{2}\begin{bmatrix} f_z f_{xy} - f_y f_{xz} \\ f_x f_{xz} - f_z f_{xx} \\ f_y f_{xx} - f_x f_{xy} \end{bmatrix} \\ &\quad = -\frac{d+1}{2}\left(f_{xx}\begin{bmatrix} 0 \\ f_z \\ -f_y \end{bmatrix} + f_{xy}\begin{bmatrix} -f_z \\ 0 \\ f_x \end{bmatrix} + f_{xz}\begin{bmatrix} f_y \\ -f_x \\ 0 \end{bmatrix}\right). \end{aligned}$$

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Therefore,

$$\left[\begin{array}{c} \frac{d+1}{2}(f_z f_{xy} - f_y f_{xz}) \\ \frac{d+1}{2}(f_x f_{xz} - f_z f_{xx}) \\ \frac{d+1}{2}(f_y f_{xx} - f_x f_{xy}) \\ \hline 0_{6\times 1} \end{array}\right] = \mathrm{Col}_4(\partial_2^C \theta_2(3)).$$

Similar calculations show that the fifth and sixth columns of $-\theta_1(3)M_1(3)$ are the same as those of $\partial_2^C\theta_2(3)$, respectively. The last column of $-\theta_1(3)M_1(3)$ is given by

$$\begin{aligned}
W &= \begin{bmatrix} \frac{1}{d'^2}(f_z\delta_y - f_y\delta_z) \\ \frac{1}{d'^2}(f_x\delta_z - f_z\delta_x) \\ \frac{1}{d'^2}(f_y\delta_x - f_x\delta_y) \\ \frac{3}{d'^2(d-2)}\ \ 2x(f_z\delta_y - f_y\delta_z) \\ \frac{3}{d'^2(d-2)}\ \ (x(f_x\delta_z - f_z\delta_x) + y(f_z\delta_y - f_y\delta_z)) \\ \frac{3}{d'^2(d-2)}\ \ (x(f_y\delta_x - f_x\delta_y) + z(f_z\delta_y - f_y\delta_z)) \\ \frac{3}{d'^2(d-2)}\ \ 2y(f_x\delta_z - f_z\delta_x) \\ \frac{3}{d'^2(d-2)}\ \ (y(f_y\delta_x - f_x\delta_y) + z(f_x\delta_z - f_z\delta_x)) \\ \frac{3}{d'^2(d-2)}\ \ 2z(f_y\delta_x - f_x\delta_y) \end{bmatrix} \\
&= \frac{\delta_x}{d'^2}\left[\begin{array}{c} 0 \\ -f_z \\ f_y \\ \hline 0_{6\times 1} \end{array}\right] + \frac{\delta_y}{d'^2}\left[\begin{array}{c} f_z \\ 0 \\ -f_x \\ \hline 0_{6\times 1} \end{array}\right] + \frac{\delta_z}{d'^2}\left[\begin{array}{c} -f_y \\ f_x \\ 0 \\ \hline 0_{6\times 1} \end{array}\right] \\
&\quad + \frac{3\delta_x}{d'^2(d-2)}\left[\begin{array}{c} 0_{3\times 1} \\ \hline 0 \\ -xf_z \\ xf_y \\ -2yf_z \\ (yf_y - zf_z) \\ 2zf_y \end{array}\right] + \frac{3\delta_y}{d'^2(d-2)}\left[\begin{array}{c} 0_{3\times 1} \\ \hline 2xf_z \\ yf_z \\ zf_z - xf_x \\ 0 \\ -yf_x \\ -2zf_x \end{array}\right] \\
&\quad + \frac{3\delta_z}{d'^2(d-2)}\left[\begin{array}{c} 0_{3\times 1} \\ \hline -2xf_y \\ xf_x - yf_y \\ -zf_y \\ 2yf_x \\ zf_x \\ 0 \end{array}\right].
\end{aligned}$$

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Hence

$$W = \frac{-1}{d'^2}\Big(\delta_x \operatorname{Col}_2(\partial_2^C) + \delta_y \operatorname{Col}_3(\partial_2^C) + \delta_z \operatorname{Col}_4(\partial_2^C) \\ + \frac{3}{d-2}[\delta_x \operatorname{Col}_6(\partial_2^C) + \delta_y \operatorname{Col}_7(\partial_2^C) + \delta_z \operatorname{Col}_8(\partial_2^C)]\Big).$$

It then follows that

$$\operatorname{Col}_{10}(-\theta_1(3)M_1(3)) = \operatorname{Col}_{10}(\partial_2^C\theta_2(3)).$$

To verify equalities in columns 7–9, we need the following observations:

Observation 1. The following equalities are satisfied:

$$\begin{aligned}\frac{9}{2}f_{xx}\alpha_x + \frac{9}{2}f_{xy}\alpha_y + \frac{9}{2}f_{xz}\alpha_z &= \frac{9}{d'^2}\begin{bmatrix} 2x\delta - (d-1)f_x\Delta_{xx} \\ y\delta - (d-1)f_x\Delta_{xy} \\ z\delta - (d-1)f_x\Delta_{xz} \\ -(d-1)f_x\Delta_{yy} \\ -(d-1)f_x\Delta_{yz} \\ -(d-1)f_x\Delta_{zz}\end{bmatrix} \\ &= \frac{9\delta}{d'^2}\begin{bmatrix}2x\\ y\\ z\\ 0\\ 0\\ 0\end{bmatrix} - \frac{9f_x}{d'}\begin{bmatrix}\Delta_{xx}\\ \Delta_{xy}\\ \Delta_{xz}\\ \Delta_{yy}\\ \Delta_{yz}\\ \Delta_{zz}\end{bmatrix}.\end{aligned}$$

Similarly, one can show

$$\frac{9}{2}f_{xy}\alpha_x + \frac{9}{2}f_{yy}\alpha_y + \frac{9}{2}f_{yz}\alpha_z = \frac{9\delta}{d'^2}\begin{bmatrix}0\\ x\\ 0\\ 2y\\ z\\ 0\end{bmatrix} - \frac{9f_y}{d'}\begin{bmatrix}\Delta_{xx}\\ \Delta_{xy}\\ \Delta_{xz}\\ \Delta_{yy}\\ \Delta_{yz}\\ \Delta_{zz}\end{bmatrix},$$

$$\frac{9}{2}f_{xz}\alpha_x + \frac{9}{2}f_{yz}\alpha_y + \frac{9}{2}f_{zz}\alpha_z = \frac{9\delta}{d'^2}\begin{bmatrix}0\\ 0\\ x\\ 0\\ y\\ 2z\end{bmatrix} - \frac{9f_z}{d'}\begin{bmatrix}\Delta_{xx}\\ \Delta_{xy}\\ \Delta_{xz}\\ \Delta_{yy}\\ \Delta_{yz}\\ \Delta_{zz}\end{bmatrix}.$$

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Observation 2. One has an equality

$$\frac{9}{2} f_{xx} \operatorname{Col}_9(\partial_2^C) + \frac{9}{2} f_{xy} \operatorname{Col}_{10}(\partial_2^C) + \frac{9}{2} f_{xz} \operatorname{Col}_{11}(\partial_2^C)$$
$$= \frac{9\delta}{d'^2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2x \\ y \\ z \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{9}{d'} \begin{bmatrix} \frac{\delta}{2} \\ 0 \\ 0 \\ f_x \Delta_{xx} \\ f_x \Delta_{xy} \\ f_x \Delta_{xz} \\ f_x \Delta_{yy} \\ f_x \Delta_{yz} \\ f_x \Delta_{zz} \end{bmatrix}.$$

Using the observations above, we conclude that the seventh column of $-\theta_1(3) M_1(3)$ can be written as

$$-\frac{\delta_x}{d'^2} \operatorname{Col}_1(\partial_2^C) - \frac{3\delta_x}{d'^2(d-2)} \operatorname{Col}_5(\partial_2^C) + \frac{9}{2}(f_{xx} \operatorname{Col}_9(\partial_2^C)$$
$$+ f_{xy} \operatorname{Col}_{10}(\partial_2^C) + f_{xz} \operatorname{Col}_{11}(\partial_2^C)),$$

which is the same as the seventh column of $\partial_2^C \theta_2(3)$. Similar arguments take care of the eighth and ninth columns. □

E.3. The Third Square Commutes

PROPOSITION E.3.1. *The following equality is satisfied*: $-\theta_2(3) M_2(3) = \partial_3^C \theta_3(3)$.

Proof. Let K and P denote the 3×3 matrices defined by

$$K := \Delta D_2 \quad \text{and} \quad P := D_1 \partial_1.$$

Let τ be the 3×3 matrix defined by

$$\tau := \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}.$$

The matrix $-\theta_2(3)M_2(3)$ is of the form

$$\left[\begin{array}{c|ccc|ccc|ccc}
0 & 0 & 0 & 0 & \frac{(d-2)}{d'^2}x\delta & \frac{(d-2)}{d'^2}y\delta & \frac{(d-2)}{d'^2}z\delta & \frac{H(\delta)}{d'^2} & \frac{H_{zx}(\delta)}{d'^2} & \frac{H_{xy}(\delta)}{d'^2} \\
\hline
0 & & & & & & & & & \\
0 & & \frac{d^2-1}{4}P & & & \frac{(d+1)}{2}K & & & \frac{3(d+1)\delta}{2d'^2}I_{3\times3}+\frac{1}{d'^2}\tau & \\
0 & & & & & & & & & \\
\hline
0 & 0 & 0 & 0 & \frac{3x\delta}{d'^2} & \frac{3y\delta}{d'^2} & \frac{3z\delta}{d'^2} & \frac{3H(\delta)}{d'^2(d-2)} & \frac{3H_{zx}(\delta)}{d'^2(d-2)} & \frac{3H_{xy}(\delta)}{d'^2(d-2)} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & \frac{3}{d'^2(d-2)}\tau & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
\hline
0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & & \frac{-3d'}{2}P & & & \frac{-9}{2}K & \\
0 & 0 & 0 & 0 & & & & & &
\end{array}\right].$$

Using the identities in Appendix A is a direct check, similar to those in Proposition E.1.1 and Appendix E.2, that $\partial_3^C\theta_3(3)$ agrees with the matrix above. Alternatively, expressing each matrix as a block matrix, one can use the identities from Appendix D. □

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