

# POINTWISE CONVERGENCE OF THE NON-LINEAR FOURIER TRANSFORM

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ABSTRACT. We prove pointwise convergence for the scattering data of a Dirac system of differential equations. Equivalently, we prove an analog of Carleson's theorem on almost everywhere convergence of Fourier series for a version of the non-linear Fourier transform. Our proofs are based on the study of resonances of Dirac systems using families of meromorphic inner functions, generated by a Riccati equation corresponding to the system.

## INTRODUCTION

In this paper we study pointwise convergence of the scattering data for a Dirac system of differential equations. Scattering transforms play an important role in the study of various differential operators and related problems. Extensive evidence generated in this area during the last several decades suggests that scattering can be viewed as a non-linear version of the classical Fourier transform, see for instance [1, 30, 28, 29].

These connections lead to natural problems of establishing versions of the classical results of Fourier analysis in the non-linear settings of scattering. Such problems have been appearing in various forms for most of the last century and remain an object of active research today, see for instance [27] for further references. As an example one can look at the non-linear version of Parseval's identity (5.3), which can be traced as far back as the work of Verblunski in the 1930s, and a non-linear analog of Hausdorff-Young inequality, which appears in more recent work of Christ and Kiselev [4, 3].

One of the fundamental results of classical Fourier analysis is the theorem by L. Carleson (1966, [2]) which says that the Fourier transform

$$\hat{f}_T(x) = \int_{-T}^T f(s) e^{-ixs} ds$$

converges to  $\hat{f}(x) = \hat{f}_\infty(x)$  as  $T \rightarrow \infty$  at almost every point  $x \in \mathbb{R}$  for any  $f \in L^2(\mathbb{R})$ . Answering a question by Luzin from 1915, Carleson's theorem

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finished a long and turbulent story of partial results and counterexamples created by some of the most prominent mathematicians of the 20th century. After more than fifty years the existing proofs are still challenging despite several significant contributions by other analysts, including those by Fefferman (1973, [11]) and by Lacey and Thiele (2000, [13]). The theorem and its proofs opened a variety of directions for further research, see for instance [8, 21] for some of the recent developments and references.

Our main goal is to prove an analog of this classical result in the scattering setting, i.e., to establish pointwise convergence for a version of the non-linear Fourier transform, see Theorem 1 in Section 5.

Our result implies in particular that generalized eigenfunctions of Dirac systems on the half-line with real  $L^2$ -potentials are bounded for almost every real spectral parameter, answering a question by Muscalu, Tao and Thiele [20]. For  $1 \leq p < 2$  this follows from the work of Christ and Kiselev [4, 5].

Convergence in the d-adic model, along with a statement on the maximal operator, was established in [19]. Further discussion of these problems in the context of Ablowitz-Kaup-Newell-Segur (AKNS) systems can be found in the book by Muscalu and Schlag ([18], Vol. 2, Chapter 5).

Our proof is independent from the linear proofs and is based on a study of resonances of Dirac systems using the methods of spectral problems for differential operators and complex function theory. While our tools include the basics of the Krein-de Banges theory and its later developments in [15, 16], we are not using any of the deep results of the theory or any of the recent advances of the non-linear Fourier analysis. The necessary background material is presented in Sections 1, 2 and 3.

Convergence results are usually closely related to estimates of the maximal operator, as was the case with the original version of Carleson's theorem. While some of such estimates for the non-linear transform can be extracted from the results of this paper, sharpness and full extent of such estimates remains unclear and will be studied elsewhere. Let us only mention that pointwise convergence trivially implies finiteness of the maximal function at almost every point, which was unknown up to now.

The plan of our proof is as follows. First we prove universality-type results and show that near almost every point of a Szegő weight the reproducing kernels of the de Branges spaces corresponding to the Dirac system resemble standard sinc functions. Approximations for the reproducing kernels do not imply approximations for the Hermite-Biehler functions generating the space per se, but under an additional assumption of existence of a resonance of the system near a point  $s \in \mathbb{R}$ , those functions can be approximated by sines and cosines. Next we show that to keep a resonance near  $s$  in the case when

the Hermite-Biehler function is a sine or a cosine, or a function close enough to those, requires a large  $L^2$ -norm of the potential function. Interestingly, this step requires us to consider two substantially different cases: when the resonance of the system restricted to the interval  $(0, t)$  moves near  $s$  vertically or diagonally as  $t$  increases. The motion of resonances is studied using families of inner functions satisfying a Riccati equation related to the original Dirac system. Obtaining a contradiction with the finiteness of the  $L^2$ -norm of the potential function, we show that for a.e. point  $s \in \mathbb{R}$  the system restricted to  $(0, t)$  cannot have a resonance within the distance of  $\asymp 1/t$  from  $s$  for large enough  $t$ . From that we deduce that the absolute values of and the angle between the scattering functions, corresponding to Neumann and Dirichlet boundary conditions, converge a.e. on  $\mathbb{R}$ . Our main result then follows.

The contents of the sections:

- In Section 1 we introduce the main object of the paper, a Dirac system on  $\mathbb{R}_+$  with real potential.
- Section 2 contains basics of Krein-de Branges theory for Dirac systems.
- Section 3 contains a definition and a brief discussion of meromorphic inner functions (MIFs).
- In Section 4 we define families of Dirac inner functions satisfying a Riccati equation. These families will play a role in the study of resonance dynamics later in the paper.
- The definition of the scattering matrix and the non-linear Fourier transform, together with their relations with the Hermite-Biehler functions are presented in Section 5.
- In Section 6 we prove universality results characterizing the behavior of reproducing kernels of the de Branges spaces corresponding to the Dirac system near regular points of the spectral measure on the real line.
- Universality results are translated into approximations of the Hermite-Biehler functions by elementary sines and cosines in Section 7.
- Simultaneous approximations for Hermite-Biehler functions corresponding to Neumann and Dirichlet boundary conditions are obtained in Section 8.
- In Section 9 we classify the time intervals according to the motion of the resonances near a fixed point on  $\mathbb{R}$ .
- In Section 10 we estimate the  $L^2$ -norm of the potential function of the Dirac system on the time intervals during which the resonances move 'almost vertically'.
- Estimates for the intervals corresponding to the motion with large horizontal component are obtained in Section 11.
- In Section 12 we finish the proof of the main result, Theorem 1.

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## 1. REAL DIRAC SYSTEMS

We study one of the basic models of scattering corresponding to the 'real' Dirac system on the right half-line  $\mathbb{R}_+$ ,

$$\Omega \dot{X} = zX - QX, \quad (1.1)$$

where  $z \in \mathbb{C}$  is a spectral parameter,

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } Q(t) = \begin{pmatrix} 0 & f(t) \\ f(t) & 0 \end{pmatrix}$$

for some real-valued locally summable function  $f$ . A slightly more general form of the system allows for a locally summable functions  $g$  and  $-g$  on the main diagonal of  $Q$ . The function  $f + ig$  is then called the potential of the system. To simplify our exposition, we keep the potential real, although our methods will work similarly for the general potential.

We will be most interested in the scattering problems corresponding to the case  $f \in L^2(\mathbb{R}_+)$ . For each value of the spectral parameter  $z$  the unknown function

$$X(t, z) = \begin{pmatrix} u(t, z) \\ v(t, z) \end{pmatrix}$$

is assumed to be differentiable on  $\mathbb{R}_+$  with respect to the time variable  $t$  and satisfy a self-adjoint boundary condition at  $t = 0$ .

A special role will be played by solutions satisfying the Neumann,

$$X(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and Dirichlet,

$$X(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

initial conditions. The matrix function  $M$  whose columns are the Neumann and Dirichlet solutions, i.e., the matrix-function which solves (1.1) with the initial condition

$$M(0, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is called the fundamental matrix, or the transfer matrix, of the system.

Any Dirac system can be rewritten in the so-called canonical form and henceforth become a part of the Krein-de Branges theory of canonical systems, see [6, 10, 25, 26]. What follows is a brief outline of the basics of the theory as applicable to the subclass of Dirac systems.

## 2. HERMITE-BIEHLER FUNCTIONS AND SPECTRAL MEASURES

If

$$X(t, z) = \begin{pmatrix} u(t, z) \\ v(t, z) \end{pmatrix}$$

is a solution of (1.1) with a self-adjoint boundary condition at 0 then for each fixed  $t \in \mathbb{R}_+$  the function

$$H(t, z) = u(t, z) - iv(t, z)$$

is an Hermite-Biehler entire function, i.e., an entire function satisfying

$$|H(t, z)| > |H(t, \bar{z})|$$

for all  $z$  from the upper half-plane  $\mathbb{C}_+$ . Throughout this paper we will denote by  $E(t, z)$  and  $\tilde{E}(t, z)$  the functions corresponding to the Neumann and Dirichlet conditions at 0 correspondingly.

According to Krein's formula for the exponential type (see for instance [25], Theorem 11), the functions  $E(t, \cdot), \tilde{E}(t, \cdot)$  have exponential type  $t$ .

If

$$M(t, z) = \begin{pmatrix} A(t, z) & B(t, z) \\ C(t, z) & D(t, z) \end{pmatrix}$$

is the fundamental matrix then

$$E(t, z) = A(t, z) - iC(t, z) \text{ and } \tilde{E}(t, z) = B(t, z) - iD(t, z).$$

Here  $A, B, C, D$  are real (real-valued on the real line) entire functions analogous to sine and cosine. This analogy becomes an equation in the free case  $f \equiv 0$  when

$$\begin{aligned} E(t, z) &= e^{itz}, \tilde{E}(t, z) = -ie^{itz}, \\ A(t, z) &= -D(t, z) = \cos tz \text{ and } -C(t, z) = B(t, z) = \sin tz. \end{aligned}$$

We will use the standard notation  $H^\#(z)$  for the Schwarz reflection of an entire function  $H^\#(z) = \bar{H}(\bar{z})$ . Using this notation,

$$A = \frac{E + E^\#}{2} \text{ and } B = \frac{E - E^\#}{2i}.$$

It follows from (1.1) that

$$\det M(t, z) = 1 \tag{2.1}$$

for all  $t$  and  $z$ . Rewritten in terms of  $E$  and  $\tilde{E}$ , this relation becomes

$$\det \begin{pmatrix} E & \tilde{E} \\ E^\# & \tilde{E}^\# \end{pmatrix} \equiv 2i. \quad (2.2)$$

Associated with every Hermite-Biehler function  $E(z)$  one can consider a de Branges space  $B(E)$ , a Hilbert space of entire functions defined as

$$B(E) = \{F \mid F \text{ is entire, } F/E \in H^2(\mathbb{C}_+), F^\#/E \in H^2(\mathbb{C}_+)\},$$

where  $H^2(\mathbb{C}_+)$  denotes the standard Hardy space in the upper half-plane. The Hilbert structure in  $B(E)$  is inherited from  $H^2$ :

$$\langle F, G \rangle_{B(E)} = \int_{\mathbb{R}} F(x)G(x) \frac{dx}{|E(x)|^2}.$$

Each  $B(E)$  consists of functions of exponential type at most that of  $E$ . In particular, in our settings  $B(E(t, \cdot))$  contains functions of exponential type at most  $t$ .

With this structure  $B(E)$  is a reproducing kernel space: point evaluations are bounded linear functionals on the space and, as follows from the representation theorem, for each  $\lambda \in \mathbb{C}$  there exists  $K(\lambda, \cdot) \in B(E)$  such that for any  $F \in B(E)$ ,

$$F(\lambda) = \langle F, K(\lambda, \cdot) \rangle_{B(E)}.$$

The function  $K(\lambda, z)$  is called the reproducing kernel for the point  $\lambda$ . In the case of the de Branges space  $B(E)$ ,  $K(\lambda, z)$  has the formula

$$K(\lambda, z) = \frac{1}{2\pi i} \frac{E(z)E^\#(\bar{\lambda}) - E^\#(z)E(\bar{\lambda})}{\bar{\lambda} - z} = \frac{1}{\pi} \frac{A(z)C(\bar{\lambda}) - C(z)A(\bar{\lambda})}{\bar{\lambda} - z},$$

where  $A = (E + E^\#)/2$  and  $C = (E^\# - E)/2i$  are real entire functions such that  $E = A - iC$ .

In the free case  $f \equiv 0$  the Hermit-Biehler functions produced by the system (1.1) are the exponential functions  $E(t, z) = e^{-itz}$  ( $\tilde{E}(t, z) = -ie^{-itz}$ ) and  $B(E(t, z))$  is the standard Paley-Wiener space  $PW_t$ . The reproducing kernel of  $PW_t$  is the sinc function

$$\text{Sinc}(t, \lambda, z) = \frac{1}{\pi} \frac{\sin[t(z - \bar{\lambda})]}{z - \bar{\lambda}}. \quad (2.3)$$

It follows from the definition of  $K(\lambda, z)$  that

$$\begin{aligned} \|K(\lambda, \cdot)\|_{B(E)} &= \|K(\lambda, \cdot)/E\|_{H^2} = \sqrt{K(\lambda, \bar{\lambda})} = \\ &= \sup_{F \in B(E), \|F\|_{B(E)} \leq 1} |F(\lambda)|. \end{aligned} \quad (2.4)$$

We denote by  $\Pi$  the Poisson measure on  $\mathbb{R}$ ,  $d\Pi(x) = dx/(1+x^2)$ . We call a measure  $\mu$  on  $\mathbb{R}$  Poisson-finite if

$$\int \frac{d|\mu(x)|}{1+x^2} < \infty.$$

The family of de Branges spaces  $B(E(t, z)), t \in \mathbb{R}_+$ , possesses a unique positive Poisson-finite measure  $\mu$  on  $\mathbb{R}$  such that the embedding  $B(E(t, z)) \rightarrow L^2(\mu)$  is isometric for all  $t \in \mathbb{R}_+$ . Similarly, the family  $B(\tilde{E}(t, z))$  gives rise to a unique measure  $\tilde{\mu}$ . The measures  $\mu$  and  $\tilde{\mu}$  are called the spectral measures for the Dirac system (1.1) corresponding to the Neumann and Dirichlet boundary conditions at 0 correspondingly. Let  $w(x)$  be the density of the absolutely continuous part of  $\mu$ ,  $d\mu_{ac} = w(x)dx$ , and let  $\tilde{w}$  be the density for the absolutely continuous part of  $\tilde{\mu}$ . For  $f \in L^2(\mathbb{R}_+)$  the spectral measures satisfy the Szegő condition

$$\log |w|, \log |\tilde{w}| \in L^1(\Pi),$$

see the paper by Denisov [9] for this and many related results. In particular,  $w, \tilde{w} \neq 0$  a.e. on  $\mathbb{R}$ .

It is well known that in the case of locally summable potentials, and thus in our case when  $f \in L^2(\mathbb{R}_+)$ , the spaces  $B(E(t, z))$  and  $B(\tilde{E}(t, z))$  are equal to the standard Paley-Wiener spaces  $PW_t$  as sets (but with different norms). Indeed, (2.5) below implies

$$|E(t, x)|, |\tilde{E}(t, x)| \leq e^{\int_0^t |f(s)| ds}$$

for  $x \in \mathbb{R}$  and the condition  $\det M = 1$  implies that  $E$  and  $\tilde{E}$  are bounded from below on  $\mathbb{R}$ . Therefore the norms in  $B(E)$  and  $B(\tilde{E})$  are equivalent to the norm in  $L^2(\mathbb{R})$ . Together with the property that  $B(E(t, z))$  and  $B(\tilde{E}(t, z))$  consist of functions of exponential type at most  $t$ , we obtain that they are equal to  $PW_t$  as a sets.

Hence, for the reproducing kernels  $K(t, \lambda, z)$  of  $B(E(t, z))$ , (2.4) can be rewritten as

$$\|K(t, \lambda, \cdot)\|_{B(E(t, z))} = \sup_{f \in PW_t, \|f\|_{L^2(\mu)} \leq 1} |f(\lambda)|$$

and similarly for the reproducing kernels  $\tilde{K}(t, \lambda, z)$  of  $B(\tilde{E}(t, z))$ .

The equation (1.1) rewritten for the Hermite-Biehler function

$$E = E(t, z) = u(t, z) - iv(t, z)$$

becomes

$$\frac{\partial}{\partial t} E(t, z) = -(zv(t, z) - fu(t, z)) - i(zu(t, z) - fv(t, z)),$$

which yields

$$\frac{\partial}{\partial t} E(t, z) = -izE(t, z) + f(t)E^\#(t, z). \quad (2.5)$$

The initial condition is  $E(0, z) = 1$  for  $E$  and  $\tilde{E}(0, z) = -i$  for  $\tilde{E}$ .

Some simple corollaries of the last equation will be useful to us below.

We denote by  $\arg E$  the continuous branch of the argument of  $E$  in the closed upper half-plane satisfying  $\arg E(t, 0) = 0$  ( $\arg \tilde{E}(t, 0) = -\pi/2$  for  $\tilde{E}$ ). If  $E$  satisfies (2.5) then for  $|E(t, x)|$ ,  $x \in \mathbb{R}$ , we have

$$\frac{\partial}{\partial t} |E(t, x)| = f(t) |E(t, x)| \cos[2 \arg E(t, x)],$$

which results in

$$|E(t, x)| = |E(t_0, x)| \exp \left[ \int_{t_0}^t f(t) \cos[2 \arg E(t, x)] dt \right], \quad (2.6)$$

for all  $t > t_0 \geq 0$ .

### 3. MEROMORPHIC INNER FUNCTIONS

Recall that an inner function in the upper half-plane is a bounded analytic function whose non-tangential boundary values are unimodular a.e. on  $\mathbb{R}$ , see for instance [12]. An inner function in  $\mathbb{C}_+$  is called a meromorphic inner function (MIF) if it can be continued meromorphically into the whole complex plane. It can be shown that every MIF has a representation

$$\alpha e^{iaz} \prod \frac{\bar{\lambda}_n z - \lambda_n}{\lambda_n z - \bar{\lambda}_n},$$

where  $\alpha$  is a unimodular complex constant,  $a$  is a positive number,  $\{\lambda_n\}$  is a sequence of points in  $\mathbb{C}_+$  tending to infinity as  $n \rightarrow \infty$  and satisfying the Blaschke condition

$$\sum \frac{\Im \lambda_n}{1 + |\lambda_n|^2} < \infty.$$

The subclass all inner functions consisting of MIFs appears in spectral problems for differential operators with compact resolvents, see [15, 16]. Such functions also appear in problems of Fourier analysis, see [22, 23, 24].

Since every MIF is analytic in a neighborhood of the real line, their boundary values and derivatives are well defined everywhere on  $\mathbb{R}$ . We will need the following simple lemma relating their derivatives and zeros.

For a sequence  $\Lambda = \{\lambda_n\} \subset \mathbb{C}_+$  satisfying the Blaschke condition we denote by  $B_\Lambda$  the corresponding Blaschke product

$$B_\Lambda = \prod \frac{\bar{\lambda}_n z - \lambda_n}{\lambda_n z - \bar{\lambda}_n}.$$

**Lemma 1.** *Let  $\theta$  be a MIF and let  $1 > \varepsilon > 0$ . Let  $x, y \in \mathbb{R}$  be such that*

$$|\theta'(x)|/|\theta'(y)| > 1 + \varepsilon. \quad (3.1)$$

*Then the ball  $\{|z - x| < |y - x|/\varepsilon\}$  contains at least one zero of  $\theta$ .*

*Proof.* As a MIF,  $\theta$  can be represented as

$$\theta(z) = e^{icz} B_\Lambda$$

for some  $c \geq 0$  and a Blaschke sequence  $\Lambda = \{\lambda_n\} \subset \mathbb{C}_+$ . For a Blaschke factor

$$\beta_{\lambda_n} = \frac{\bar{\lambda}_n z - \lambda_n}{\lambda_n z - \bar{\lambda}_n}, \quad \lambda_n = x_n + iy_n$$

the derivative of its argument is

$$\frac{y_n}{(x - x_n)^2 + y_n^2}.$$

For  $|\theta'(x)|$ , which equals to the derivative of the argument of  $\theta$  at  $x$ , we have

$$|\theta'(x)| = c + \sum_n \frac{y_n}{(x - x_n)^2 + y_n^2}.$$

The sum on the right hand side is a sum of positive functions and for (3.1) to hold we need a similar inequality to be satisfied by at least one of the summands, i.e.,

$$\frac{y_n}{(x - x_n)^2 + y_n^2} \cdot \frac{(y - x_n)^2 + y_n^2}{y_n} = \frac{|y - \lambda_n|^2}{|x - \lambda_n|^2} > 1 + \varepsilon$$

for at least one  $n$ . The last inequality holds when  $\lambda_n$  is in (the interior of an Apollonian circle with foci  $x$  and  $y$ , which is contained in) the ball from the statement.

□

#### 4. DIRAC INNER FUNCTIONS

In this section we introduce families of inner functions related to the systems (1.1). We call them Dirac inner functions. In addition to playing a role in our arguments below, such families seem to present independent interest and may prove useful in the studies of further properties of the system. Standard formulas expressing Blaschke products in terms of their zeros establish, in this context, the relation between resonances (poles of the inner function) and spectra (level sets on  $\mathbb{R}$ ) of the system.

If  $H(z)$  is an Hermite-Biehler entire function then the function

$$\theta_H(z) = H^\#(z)/H(z)$$

is a meromorphic inner function in  $\mathbb{C}_+$ . Under the restriction that  $H$  has bounded type, which is the case for functions related to Dirac systems with locally summable potentials,  $H$  can be uniquely, up to a real constant multiple, recovered from  $\theta_H$ .

Recall that to each Dirac system we associate two families of Hermite-Biehler functions  $E(t, z)$  and  $\tilde{E}(t, z)$  corresponding to Neumann and Dirichlet boundary conditions at  $t = 0$ . We will denote the corresponding MIFs by

$$\theta = \theta_E \text{ and } \tilde{\theta} = \theta_{\tilde{E}}.$$

Similarly, families of MIFs can be considered for any self-adjoint boundary condition.

For  $\theta = \theta(t, z)$  we have

$$\theta(t, z) = \frac{u(t, z) + iv(t, z)}{u(t, z) - iv(t, z)}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \theta(t, z) &= \frac{(\frac{\partial}{\partial t} u + i \frac{\partial}{\partial t} v)(u - iv) - (u + iv)(\frac{\partial}{\partial t} u - i \frac{\partial}{\partial t} v)}{(u - iv)^2} = 2i \frac{u \frac{d}{dt} v - v \frac{\partial}{\partial t} u}{(u - iv)^2} = \\ &= 2i \frac{zu^2 + zv^2 - 2fuv}{(u - iv)^2} = 2iz \frac{u + iv}{u - iv} - 4if \frac{uv}{(u - iv)^2} = \\ &= 2iz \frac{u + iv}{u - iv} - f \frac{[(u + iv) - (u - iv)][(u + iv) + (u - iv)]}{(u - iv)^2}, \end{aligned}$$

which produces a Ricatti equation for the family of Dirac inner functions corresponding to the system (1.1):

$$\frac{\partial}{\partial t} \theta = 2iz\theta - f(1 - \theta^2). \quad (4.1)$$

This equation together with its derivatives will be used in our study of the behavior of resonances of the system (1.1).

**Remark 1.** *The Ricatti equation (4.1) and corresponding families of functions pose some interesting questions. In our settings (4.1) is considered with Neumann boundary condition  $\theta(0, z) = 1$  for  $\theta$  and Dirichlet  $\tilde{\theta}(0, z) = -1$  for  $\tilde{\theta}$ . One can however consider other initial conditions. Conditions of the type  $\theta(0, z) = \phi(z)$ , where  $\phi$  is a bounded analytic function in  $\mathbb{C}_+$ ,  $|\phi| < 1$ , present a natural subclass. One can show that with such initial conditions the solutions  $\theta(t, z)$  will remain analytic in  $z$  and satisfy  $|\theta(t, z)| < 1$  for  $z \in \mathbb{C}_+$ . Interpreting the complex values as vectors in  $\mathbb{R}^2$  one can see that at the real points where  $|\theta(t, x)| = 1$  the right-hand side of (4.1) is orthogonal to  $\theta(t, z)$ , which implies that  $\theta$  stays unimodular. Hence, with an inner*

initial condition  $\theta(t, z)$  is a family of inner functions. One can also show that  $\theta(t, z)$  are MIFs if the initial condition is a MIF.

If  $\theta(t, z)$  is a Neumann family of Dirac inner functions, each level set  $\{z \mid \theta(t, z) = \alpha\}$ ,  $|\alpha| = 1$  represents the spectrum of the system (1.1) restricted to the interval  $(0, t)$  with the Neumann boundary condition at 0 and the condition  $u(t) \sin \psi - v(t) \cos \psi = 0$  at  $t$ , where  $\alpha = e^{2i\psi}$ . Analyzing (4.1) one may follow the dynamics of the spectra of the system (1.1) as  $t \rightarrow \infty$ , see Remark 2 below.

Let  $\alpha(t)$  be a continuous curve in  $\mathbb{C}$  such that  $\theta(t, \alpha(t)) = a$  for some constant  $a \in \mathbb{C}$ . Then

$$\frac{d}{dt} \theta(t, \alpha(t)) = 0 = \theta_t(t, \alpha(t)) + \theta_z(t, \alpha(t)) \alpha'(t),$$

which implies

$$\alpha'(t) = -\frac{2i\alpha(t)a + f(t)(1 - a^2)}{\theta_z(t, \alpha(t))}. \quad (4.2)$$

We will be especially interested in the behavior of the zeros of  $\theta$ , whose complex conjugates represent the resonances of the Dirac system (1.1). Let  $z(t)$  be a curve in  $\mathbb{C}_+$  such that  $\theta(t, z(t)) = 0$  for all  $t > 0$ . Then (4.2) becomes

$$z'(t) = -\frac{f(t)}{\theta_z(t, z(t))}. \quad (4.3)$$

**Remark 2.** When  $a = 1$  the point  $\alpha(t)$  belongs to  $\sigma_{NN}(t)$ , the spectrum of the restriction of the Dirac system to the interval  $(0, t)$  with Neumann conditions on both ends. The dynamics of the eigenvalues  $N(t)$  of  $\sigma_{NN}$  is therefore given by

$$N'(t) = -\frac{2iN(t)}{\theta_z(t, N(t))}.$$

Similarly, the eigenvalues of the Neumann-Dirichlet spectrum,  $D(t)$ , obey the equation

$$D'(t) = \frac{2iD(t)}{\theta_z(t, D(t))}.$$

Since for each fixed  $t$ ,  $\theta$  is unimodular on  $\mathbb{R}$  and its argument is an increasing function, the  $z$ -derivative of  $\theta$  is always co-linear with  $i\theta$ . Since

$$\theta(t, N(t)) = 1 \text{ and } \theta(t, D(t)) = -1,$$

the previous two equations imply that  $N'(t)$  and  $D'(t)$  are negative for the positive eigenvalues and positive for the negative ones. This simple observation implies the known fact that the points of  $\sigma_{NN}$  and  $\sigma_{ND}$  tend to zero monotonously as  $t \rightarrow \infty$ .

When  $a$  in (4.2) is unimodular but not equal to  $\pm 1$ , the point  $\alpha(t)$  is an eigenvalue for the Neumann condition at 0 and some other self-adjoint condition at  $t$ , different from Neumann or Dirichlet. Notice that in this case the function  $f$  does not disappear from the numerator in (4.2) and the moving eigenvalue changes direction when it reaches the point  $-f(t)(1-a^2)/2ia$  (note that this number is real for real  $f$  and unimodular  $a$ ). This interesting dynamics of Dirac eigenvalues, viewed as level sets of Dirac inner functions, deserves a separate discussion which we hope to undertake elsewhere.

Let again  $z(t)$  be a continuous curve such that  $\theta(t, z(t)) = 0$ . Our next goal is to study the change of derivative  $\theta_z$  at the zero  $z(t)$ . Using (4.1) and (4.2) we obtain

$$\begin{aligned} \frac{d}{dt}\theta_z(t, z(t)) &= \theta_{zt}(t, z(t)) + \theta_{zz}(t, z(t))z'(t) = \\ &= 2iz\theta_z(t, z(t)) + f(t)\frac{\theta_{zz}(t, z(t))}{\theta_z(t, z(t))}. \end{aligned} \quad (4.4)$$

These formulas will prove useful to us in Section 7.

## 5. SCATTERING FUNCTIONS AND NLFT

Closely related to the Hermite-Biehler functions  $E(t, z)$  and  $\tilde{E}(t, z)$  corresponding to the Dirac system (1.1) are the scattering functions

$$\mathcal{E}(t, z) = e^{itz}E(t, z) \text{ and } \tilde{\mathcal{E}}(t, z) = e^{itz}\tilde{E}(t, z).$$

Without going into a discussion of the physical meaning of the scattering model, let us recall that  $e^{-itz}$  is the Hermite-Biehler function of the free system and note that the functions  $\mathcal{E}(t, z)$  and  $\tilde{\mathcal{E}}(t, z)$  represent the propagation of a wave (signal) up to time  $t$  in the system described by (1.1) and return of the same wave in the free system.

Note that scattering functions  $\mathcal{E}$  and  $\tilde{\mathcal{E}}$  satisfy the equation

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{E}(t, z) &= iz\mathcal{E}(t, z) + e^{izt}\frac{\partial}{\partial t}E(t, z) = \\ &= f(t)e^{2izt}\mathcal{E}^\#(t, z). \end{aligned} \quad (5.1)$$

Further, for each  $t \geq 0$  define entire functions  $a(t, z)$  and  $b(t, z)$  as

$$\begin{aligned} a(t, z) &= \frac{\mathcal{E}(t, z) + i\tilde{\mathcal{E}}(t, z)}{2} = \frac{e^{itz}}{2}(E(t, z) + i\tilde{E}(t, z)), \\ b(t, z) &= \frac{\mathcal{E}(t, z) - i\tilde{\mathcal{E}}(t, z)}{2} = \frac{e^{itz}}{2}(E(t, z) - i\tilde{E}(t, z)). \end{aligned} \quad (5.2)$$

(Note that our notations are slightly different from those in [29] where  $a$  stands for  $a^\#$  in our definitions.)

Using (5.1), or (2.5) for  $E$  and  $\tilde{E}$ , one can show that the matrix

$$G(t, z) = \begin{pmatrix} a^\#(t, z) & b^\#(t, z) \\ b(t, z) & a(t, z) \end{pmatrix}$$

satisfies the differential equation

$$G_t = \begin{pmatrix} 0 & e^{-2izt}f(t) \\ e^{2izt}f(t) & 0 \end{pmatrix} G$$

with the initial condition  $G(0, z) = I$ . One can deduce from the IVP that  $\det G \equiv 1$  for all  $t$  and  $z$ , which also follows from (5.2) and (2.1). Since  $\det G = |a|^2 - |b|^2 = 1$  on  $\mathbb{R}$ ,  $|a| > |b|$  on  $\mathbb{R}$ . Since both  $\mathcal{E}(t, z)$  and  $\tilde{\mathcal{E}}(t, z)$  are functions of Smirnov class in  $\mathbb{C}_+$  for each fixed  $t$ , so are  $a$  and  $b$ . Moreover, it is well known and not difficult to show that  $a$  is outer in  $\mathbb{C}_+$ . Since  $\det G = |a|^2 - |b|^2 = 1$  on  $\mathbb{R}$ ,  $|a/b| < 1$  on  $\mathbb{R}$ . Hence  $a/b$  is a bounded analytic function in  $\mathbb{C}_+$ ,  $|a/b| < 1$ . As was noticed in [29], since  $|a|^2 = |b|^2 + 1$ ,  $a(t, 0) > 0$  and  $a$  is outer in  $\mathbb{C}_+$ ,  $a, b$  and  $G$  can all be uniquely recovered from  $a/b$ .

It is well known that under the restriction  $f \in L^2(\mathbb{R}_+)$  the scattering matrix

$$\widehat{f} = G(\infty, z) = \lim_{t \rightarrow \infty} G(t, z)$$

exists, at least in some sense. In the discrete case discussed in [29] the convergence is proved with respect to a metric on the unit circle (which replaces the line in the discrete situation), see Lecture 2. Normal convergence in the upper half-plane is established in [9] in the equivalent settings of Krein systems in Chapter 12; for the scattering functions  $\mathcal{E}$  and  $\tilde{\mathcal{E}}$  corresponding to Dirac systems it follows from the relations established in Chapter 14 [9]. Normal convergence in  $\mathbb{C}_+$  for  $a$  and  $b$  then follows from (5.2).

Note that since  $\log |a(t, \cdot)|$  is a non-negative function on  $\mathbb{R}$ , convergence of its Poisson integral at  $z = i$  is equivalent to the convergence of  $\log |a(t, \cdot)|$  in  $L^1(\Pi)$ . For  $\log_+ |b| = \max(\log |b|, 0)$ , since  $|b|^2 = |a|^2 - 1$ , it also implies convergence in  $L^1(\Pi)$ . Note that for a family of functions from Smirnov class pointwise convergence in  $\mathbb{C}_+$  to a non-zero function from Smirnov class, under a restriction that the outer part is positive at a fixed point, implies pointwise convergence in  $\mathbb{C}_+$  for their outer and inner parts. Convergence of the outer parts of  $b$  at  $i$  together with convergence of  $\log_+ |b|$  in  $L^1(\Pi)$  implies convergence of  $\log_- |b| = \log |b| - \log_+ |b|$ , and therefore convergence of  $\log |b|$ , in  $L^1(\Pi)$ . For the inner components of  $b$ , pointwise convergence implies convergence in measure on  $\mathbb{R}$ . All in all, we obtain that  $a$  and  $b$ , and therefore  $\mathcal{E}$  and  $\tilde{\mathcal{E}}$ , converge in measure on  $\mathbb{R}$ .

All of the functions  $a(t, z)$ , including the limit function  $a = a(\infty, z)$  for  $t = \infty$ , satisfy the non-linear version of Parseval's identity

$$\|\log |a(t, \cdot)|\|_{L^1(\mathbb{R})} = \|f\|_{L^2(0, t)}^2, \quad (5.3)$$

which was known in various forms for many decades, see [9, 19, 29] for proofs and further references.

In this paper we pay special attention to the function  $b(t, z)/a(t, z)$  and its limit at infinity. Let us denote

$$\overset{\wedge}{f}(z) = \frac{b(\infty, z)}{a(\infty, z)}.$$

If  $f_T$  is the restriction of the potential function  $f$  to the interval  $(0, T)$  (extended by 0 outside of the interval) then

$$\overset{\wedge}{f}_T(z) = \frac{b(T, z)}{a(T, z)}.$$

There is abundant evidence that various versions of the scattering transform, including  $\widehat{f}$  and  $\overset{\wedge}{f}$ , can be viewed as non-linear analogs of the Fourier transform, see for instance [19, 27, 28, 29] for a discussion and further references. The transform  $f \mapsto \overset{\wedge}{f}$  we are about to study shares the modulation/shift property and the rescaling property with its linear predecessor. Parseval's identity in terms of  $\overset{\wedge}{f}$  takes the form

$$\frac{1}{2} \|\log(1 - |\overset{\wedge}{f}|^2)\|_{L^1(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}^2.$$

The analogy extends further by the property that the transform of a function  $f$  supported on a half-line produces a function  $\overset{\wedge}{f}$  holomorphic in the upper half-plane. As was mentioned before, the pair of functions  $(a, b)$ , and therefore the matrix  $G$ , can be uniquely recovered from  $\overset{\wedge}{f}$  since  $a$  is an outer function in  $\mathbb{C}_+$  which has absolute value

$$|a| = \frac{1}{\sqrt{1 - |\overset{\wedge}{f}|^2}}$$

on  $\mathbb{R}$  and is positive at 0, and  $b = a \overset{\wedge}{f}$ .

In this paper we prove the following analog of Carleson's theorem:

**Theorem 1.** *For every real  $f \in L^2(\mathbb{R}_+)$ ,*

$$\overset{\wedge}{f}_T(s) \rightarrow \overset{\wedge}{f}(s) \text{ as } T \rightarrow \infty$$

*for a.e.  $s \in \mathbb{R}$ , where  $f_T$  denotes the restriction of  $f$  to the interval  $(0, T)$ .*

We actually prove a slightly stronger statement that for a.e.  $s \in \mathbb{R}$  and any  $C > 0$ ,

$$\sup_{|z-s| < C/T} |\overset{\wedge}{f}_T(z) - \overset{\wedge}{f}(s)| \rightarrow 0 \text{ as } T \rightarrow \infty,$$

see Section 12. Such convergence is established for  $|E|$ ,  $|\tilde{E}|$  and  $\log |a|$ .

## 6. UNIVERSALITY-TYPE RESULTS

In this section we show that if the spectral measure of a chain of de Branges spaces satisfies the Szegő condition, then near almost every point on the real line the scaling limits of the reproducing kernels are equal to the sinc functions, the reproducing kernels of the Paley-Wiener space. Similar problems, motivated by universality results in random matrix theory, were previously studied by Lubinski [14] for spaces of polynomials and by Mitkovski [17] for de Branges spaces. In particular, an analog of Lemma 4 below was proved in [17] for points  $s$  of continuity of density of a regular spectral measure.

We keep our notations of  $\mu$  and  $\tilde{\mu}$  for the spectral measures of the Dirac system (1.1) with Neumann and Dirichlet boundary conditions correspondingly;  $w$  and  $\tilde{w}$  denote the densities of the absolutely continuous parts of the measures:

$$d\mu = wdx + d\mu_s, \quad d\tilde{\mu} = \tilde{w}dx + d\tilde{\mu}_s.$$

By  $K(t, \lambda, z)$  we denote the reproducing kernel of the space  $B(E(t, z))$  for the point  $\lambda$ . All our statements can be similarly proved for the reproducing kernels  $\tilde{K}(t, \lambda, z)$  of  $\tilde{B}(E(t, z))$ .

We use the notations  $\|\cdot\|_\mu$  and  $\|\cdot\|_2$  for the norms in  $L^2(\mu)$  and in  $L^2(\mathbb{R})$ . For an absolutely continuous measure  $\mu$ ,  $d\mu = w(x)dx$ , we use  $\|\cdot\|_w$  in place of  $\|\cdot\|_\mu$ .

For a Poisson-finite measure  $\mu$  on  $\mathbb{R}$  we denote by  $P\mu$  its Poisson extension to the upper half-plane

$$P\mu(x + iy) = \frac{1}{\pi} \int \frac{y d\mu(t)}{(x - t)^2 + y^2}.$$

For  $x \in \mathbb{R}$  let  $\Gamma(x)$  be the non-tangential sector

$$\Gamma(x) = \{z \in \mathbb{C}_+ \mid |\Re(z - x)| < \Im z\}.$$

For any function  $\phi(z)$  in  $\mathbb{C}_+$  we denote by  $M\phi$  its non-tangential maximal function on  $\mathbb{R}$ :

$$M\phi(x) = \sup_{z \in \Gamma(x)} |\phi(z)|.$$

Thus  $MP\mu$  will stand for the maximal function of the Poisson extension of  $\mu$ .

For  $s \in \mathbb{R}$  and  $C > 0$  we will denote by  $Q(s, C)$  the square box centered at  $s$ :

$$Q(s, C) = \{|\Re(z - s)| \leq C, |\Im z| \leq C\}.$$

The proximity of reproducing kernels  $K(t, z, \cdot)$  to sinc functions  $\text{Sinc}(t, z, \cdot)$ , defined in (2.3), will be studied on boxes  $Q(s, C/t)$  whose size decreases with time.

We start with the following statement.

**Lemma 2.** *For almost all  $s \in \mathbb{R}$  and any  $C > 0$ ,*

$$\sup_{z \in Q(s, C/t)} \left( \frac{w(s) \|K(t, z, \cdot)\|_\mu^2}{\|\text{Sinc}(t, z, \cdot)\|_2^2} - 1 \right) = o(1), \quad (6.1)$$

as  $t \rightarrow \infty$ .

*Proof.* Recall that

$$\|K(t, z, \cdot)\|_\mu = \sup_{f \in PW_t, \|f\|_\mu \leq 1} |f(z)|$$

and

$$\|\text{Sinc}(t, z, \cdot)\|_2 = \sup_{f \in PW_t, \|f\|_2 \leq 1} |f(z)|.$$

The relation we need to establish therefore becomes

$$\sqrt{w(s)} \sup_{f \in PW_t, \|f\|_\mu \leq 1} |f(z)| = (1 + o(1)) \sup_{f \in PW_t, \|f\|_2 \leq 1} |f(z)|. \quad (6.2)$$

Let us first prove that the left hand side of (6.2) is no greater than the right hand side for some choice of  $o(1)$ .

Let  $G$  be an outer function in  $\mathbb{C}_+$  with  $|G|^2 = w$ . Suppose that the non-tangential limit  $G(s)$  exists at  $s$  and  $|G(s)| = \sqrt{w(s)}$ . Multiplying  $G$  by a unimodular constant, we can choose  $G$  so that  $G(s) = \sqrt{w(s)}$ .

Due to the weak-(1,1) type of the non-tangential maximal operator, the function  $R = \sqrt{MP \log w}$  is locally summable and we assume that  $s$  is its Lebesgue point.

In this part we will assume that  $G(s) = \sqrt{w(s)} = 1$  (otherwise, since  $w(s) \neq 0$  for a.e.  $s$ , we can divide  $w$  by  $w(s)$ ). Notice that since  $\|f\|_\mu \geq \|f\|_w$ ,

$$\sup_{f \in PW_t, \|f\|_\mu \leq 1} |f(z)| \leq \sup_{f \in PW_t, \|f\|_w \leq 1} |f(z)|,$$

and it is enough to show that the last supremum is less or equal to the right hand side of (6.2).

Put

$$G_t = G \left( s + \frac{z - s}{t} \right).$$

For each  $t > 0$  choose a point  $z_t \in Q(s, C)$ . Then

$$s + \frac{z_t - s}{t} \in Q(s, C/t).$$

Using that  $\|f\|_w = \|fG\|_2$  and rescaling, we obtain

$$\sup_{f \in PW_t, \|f\|_w \leq 1} \left| f\left(s + \frac{z - s}{t}\right) \right| = \sqrt{t} \sup_{f \in PW_1, \|fG_t\|_2 \leq 1} |f(z_t)|.$$

The inequality we need to establish for every choice of  $z_t \in Q(s, C)$  becomes

$$\sup_{f \in PW_1, \|fG_t\|_2 \leq 1} |f(z_t)| \leq (1 + o(1))D_t, \quad (6.3)$$

where

$$D_t = \sup_{f \in PW_1, \|f\|_2 \leq 1} |f(z_t)|.$$

Suppose that  $f_n$  is a sequence of functions from  $PW_1$  such that

$$\|f_n G_{k_n}\|_2 \leq 1$$

for some  $k_n \rightarrow \infty$  but

$$f_n(z_{k_n}) > D_{k_n} + \varepsilon.$$

Notice that all the points  $z_{k_n}$  belong to  $Q(s, C)$  and therefore, by choosing a subsequence if necessary, we can assume that  $z_{k_n} \rightarrow z_0 \in Q(s, C)$ . Let

$$D = \sup_{f \in PW_1, \|f\|_2 \leq 1} |f(z_0)|.$$

Then  $D_{k_n} \rightarrow D$ .

Let  $g_n = e^{iz} f_n G_{k_n}$ . Then all  $g_n$  are  $H^2(\mathbb{C}_+)$ -functions of norm at most 1. By choosing a subsequence if necessary, we can assume that  $g_n$  converge to some  $g \in H^2$  weakly in  $H^2$  (and therefore pointwise in  $\mathbb{C}_+$ ).

Then  $\|g\|_2 \leq 1$ . Notice that since  $G(z) \rightarrow 1$  as  $z \xrightarrow{\triangleleft} s$ ,  $G_t \rightarrow 1$  as  $t \rightarrow \infty$  normally in  $\mathbb{C}_+$ . Therefore, the sequence  $e^{iz} f_n$  converges to  $g$  normally in  $\mathbb{C}_+$ .

Similarly, by choosing a subsequence if necessary, we can assume that  $e^{-iz} f_n$  converges normally in  $\mathbb{C}_-$  to some analytic function  $g_- \in H^2(\mathbb{C}_-)$ .

Recall that  $s$  is a Lebesgue point of  $\sqrt{MP \log w}$ . Therefore, for an arbitrary large constant  $L$  and every  $n$  we can choose  $c_n$ ,  $L < c_n < 2L$ , such that  $P \log w$  is uniformly bounded on the union of the lines  $x - s = \pm c_n/k_n$ . Let us consider a square  $R_n$  whose sides lie on the lines  $\Im z = \pm c_n$  and  $\Re(z - s) = \pm c_n$ . On the vertical sides of  $R_n$ ,

$$|f_n G_{k_n}| \leq 1/\sqrt{|y|},$$

because

$$\|e^{iz} f_n G_{k_n}\|_{H^2} \leq 1$$

and, by the choice of  $c_n$ ,  $|G_{k_n}| > \delta > 0$ . Hence, on the vertical sides of  $R_n$ ,  $|f_n(s \pm c_n + iy)| \leq 1/\delta\sqrt{|y|}$ . From normal convergence of  $f_n$  in  $\mathbb{C}_\pm$  and these estimates, we obtain dominated convergence for Cauchy integrals for points inside

$$R = \{|\Re z| < L/2, |\Im(z - s)| < L/2\}$$

and conclude that  $f_n$  converges uniformly on any compact inside  $R$ . Since  $L$  can be arbitrarily large, it follows that  $f_n$  converge normally in  $\mathbb{C}$ .

Since a normal limit of a sequence of entire functions is entire, the function  $H$  defined as  $e^{-iz}g$  in  $\mathbb{C}_+$  and as  $e^{iz}g_-$  in  $\mathbb{C}_-$  extends to an entire function. The property that  $g$  and  $g_-$  belong to  $H^2(\mathbb{C}_\pm)$  implies that  $H \in PW_1$ . From normal convergence of  $f_n$  to  $H$  it follows that

$$|H(z_0)| = \lim |f_n(z_{k_n})| \geq \lim D_{k_n} + \varepsilon = D + \varepsilon.$$

Since  $\|H\|_2 = \|g\|_2 \leq 1$  we obtain a contradiction.

To prove that the left hand side of (6.2) is no less than the right hand side, let now  $z_t$  be a point in  $Q(s, C/t)$  for each  $t$ . Notice that

$$s_t(x) = |\text{Sinc}(t, z_t, x)|^2 / \|\text{Sinc}(t, z_t, \cdot)\|_2^2$$

is an approximative unity at the point  $s$  and therefore

$$\left| \int s_t(x) d\mu(x) - w(s) \right| = o(1) \text{ as } t \rightarrow \infty$$

for a.e.  $s$ . Since  $\text{Sinc}(t, z_t, \cdot) \in PW_t$ ,

$$\begin{aligned} \|K(t, z_t, \cdot)\|_\mu^2 &= K(t, z_t, z_t) \geq \\ &\geq [\text{Sinc}(t, z_t, z_t) / \|S(t, z_t, \cdot)\|_\mu]^2 = \\ &= [\text{Sinc}(t, z_t, z_t)]^2 / \left( \|\text{Sinc}(t, z_t, \cdot)\|_2^2 \int s_t d\mu \right) \geq \\ &\geq [\text{Sinc}(t, z_t, z_t)]^2 / (\|\text{Sinc}(t, z_t, \cdot)\|_2^2 (w(s) + o(1))) = \\ &= \|\text{Sinc}(t, z_t, \cdot)\|_2^2 / (w(s) + o(1)). \end{aligned}$$

□

From the asymptotic proximity of norms we can now pass to the proximity of functions themselves.

**Lemma 3.** *For a.e.  $s \in \mathbb{R}$  and any  $C > 0$*

$$\sup_{z \in Q(s, C/t)} \left\| K(t, z, \cdot) - \frac{1}{w(s)} \text{Sinc}(t, z, \cdot) \right\|_\mu^2 = o(t) \quad (6.4)$$

as  $t \rightarrow \infty$ .

Recall that  $w(s) \neq 0$  at a.e.  $s$  and therefore the formula above makes sense for a.e.  $s$ . We denote by  $\langle \cdot, \cdot \rangle_\mu$  the inner product in  $L^2(\mu)$ .

*Proof.* Let  $z_t \in Q(s, C/t)$ . Using the notation  $s_t$  from the proof of Lemma 2,

$$\begin{aligned} & \langle K(t, z_t, \cdot) - \frac{1}{w(s)} \text{Sinc}(t, z_t, \cdot), K(t, z_t, \cdot) - \frac{1}{w(s)} \text{Sinc}(t, z_t, \cdot) \rangle_\mu = \\ & = K(t, z_t, z_t) + \frac{\|\text{Sinc}(t, z_t, \cdot)\|_2^2}{w^2(s)} \int s_t d\mu - \frac{2}{w(s)} \text{Sinc}(t, z_t, z_t) = \\ & = K(t, z_t, z_t) - \frac{1}{w(s)} \text{Sinc}(t, z_t, z_t) + o(\|\text{Sinc}(t, z_t, \cdot)\|_2^2). \end{aligned}$$

Since

$$\text{Sinc}(t, z_t, z_t) = \|\text{Sinc}(t, z_t, \cdot)\|_2^2 \asymp t,$$

the statement follows from Lemma 2.  $\square$

From the  $L^2$ -approximation of the kernels we now pass to the uniform approximation near  $s$ .

If  $I$  is an interval on  $\mathbb{R}$  and  $C > 0$  we denote by  $CI$  the interval with the same center as  $I$  of length  $C|I|$ .

**Lemma 4.** *For a.e.  $s \in \mathbb{R}$  and any  $C > 0$ ,*

$$\sup_{\lambda, z \in Q(s, C/t)} \left| K(t, \lambda, z) - \frac{1}{w(s)} \text{Sinc}(t, \lambda, z) \right| = o(t) \text{ as } t \rightarrow \infty. \quad (6.5)$$

*Proof.* Let  $z_t \in Q(s, C/t)$ . Let  $G$  again be an outer function satisfying  $|G|^2 = w$ . Define  $\Delta(t, z)$  as

$$\Delta(t, z) = K(t, z_t, z) - \frac{1}{w(s)} \text{Sinc}(t, z_t, z).$$

Then, by Lemma 3, for  $F(t, z) = e^{itz} G(z) \Delta(t, z)$  we have

$$\|F(t, \cdot)\|_{H^2}^2 = o(t).$$

From the strong  $L^2$  type of the non-tangential maximal operator  $M$  it follows that

$$\|MF(t, \cdot)\|_2^2 = o(t).$$

We will denote by  $I_t$  the interval  $Q(s, C/t) \cap \mathbb{R}$ . Consider the set  $5I_t \setminus 3I_t$  which is a union of two intervals  $J_t^1$  and  $J_t^2$ . On two-thirds of each of  $J_t^1$  and  $J_t^2$ ,

$$(MF)^2 \leq \frac{3\|MF\|_2^2 t}{C} = o(t^2).$$

We denote by  $S_t^1$  and  $S_t^2$  the subsets of  $J_t^1$  and  $J_t^2$  correspondingly where this inequality is satisfied.

Once again we notice that the function  $R = \sqrt{MP \log w}$  is locally summable and assume that  $s$  is its Lebesgue point.

It follows that for sufficiently large  $t$  there exist points  $x_1, x_2$  in  $S_t^1, S_t^2$  correspondingly such that  $R(x_k) < 2R(s)$  and therefore

$$\max_{z \in \Gamma_{x_k}, \Im z \leq 3C/t} |\Delta(t, z)| \leq 2e^{3C} e^{2R^2(s)} MF(x_k) = o(t),$$

where the right hand side does not depend on a particular choice of  $z_t \in Q(s, C/t)$ . In particular, the last inequality is satisfied on the part of the boundary of the rhombus

$$\mathbb{C} \setminus \cup_{x \notin (x_1, x_2)} (\Gamma_x \cup \bar{\Gamma}_x) \quad (6.6)$$

in  $\mathbb{C}_+$ . Since  $\Delta$  is a real entire function for each fixed  $t$ , the last inequality must also be satisfied on the boundary of the rhombus in  $\mathbb{C}_-$ , and therefore inside the rhombus. It remains to notice that the rhombus contains the box  $Q(s, C/t)$ .  $\square$

## 7. BACK TO $E$

From the estimates of reproducing kernels obtained in the previous section we now obtain estimates for the Hermite-Biehler functions  $E$  and  $\tilde{E}$ . Recall that  $E = A - iC$  and  $\tilde{E} = B - iD$  for

$$A(t, z) = u(t, z), \quad B(t, z) = v(t, z), \quad C(t, z) = \tilde{u}(t, z) \quad \text{and} \quad D(t, z) = \tilde{v}(t, z),$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

are Neumann and Dirichlet solutions of (1.1).

We denote by  $D(t, \lambda, z)$  and  $R(t, \lambda, z)$  the numerators of the kernels  $K(t, \lambda, z)$  and  $\text{Sinc}(t, \lambda, z)$  correspondingly:

$$D(t, \lambda, z) = \det \begin{pmatrix} A(t, z) & \bar{A}(t, \lambda) \\ C(t, z) & \bar{C}(t, \lambda) \end{pmatrix} = \frac{1}{2i} \det \begin{pmatrix} E(t, z) & E(t, \bar{\lambda}) \\ E^\#(t, z) & E^\#(t, \bar{\lambda}) \end{pmatrix}, \quad (7.1)$$

$$\begin{aligned} R(t, \lambda, z) &= \det \begin{pmatrix} \cos tz & \overline{\cos t\lambda} \\ \sin tz & \overline{\sin t\lambda} \end{pmatrix} = \\ &= \cos tz \sin t\bar{\lambda} - \sin tz \cos t\bar{\lambda} = \sin[t(\bar{\lambda} - z)]. \end{aligned}$$

**Lemma 5.** *For a.e.  $s \in \mathbb{R}$  and any  $C > 0$ ,*

$$\sup_{\lambda, z \in Q(s, C/t)} \left| D(t, \lambda, z) - \frac{1}{w(s)} R(t, \lambda, z) \right| = \quad (7.2)$$

$$= \sup_{\lambda, z \in Q(s, C/t)} \left| D(t, \lambda, z) - \frac{1}{w(s)} \sin[t(\bar{\lambda} - z)] \right| = o(1)$$

as  $t \rightarrow \infty$ .

*Proof.* Note that

$$\left| D(t, \lambda, z) - \frac{1}{w(s)} R(t, \lambda, z) \right| = \left| K(t, z, \lambda) - \frac{1}{w(s)} \text{Sinc}(t, z, \lambda) \right| |\bar{\lambda} - z|.$$

Now the statement follows from Lemma 4 because  $|\bar{\lambda} - z| \lesssim 1/t$  for  $\lambda, z \in Q(s, C/t)$ .

□

The last lemma admits the obvious self-improvement: one can allow the size of the box  $Q$  tend to zero slower than  $1/t$ .

**Corollary 1.** *For a.e.  $s \in \mathbb{R}$  there exists a function  $C(t) > 0$ ,  $C(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , such that*

$$\begin{aligned} & \sup_{\lambda, z \in Q(s, C(t)/t)} \left| D(t, \lambda, z) - \frac{1}{w(s)} R(t, \lambda, z) \right| = \quad (7.3) \\ & = \sup_{\lambda, z \in Q(s, C(t)/t)} \left| D(t, \lambda, z) - \frac{1}{w(s)} \sin[t(\bar{\lambda} - z)] \right| = o(1) \end{aligned}$$

as  $t \rightarrow \infty$ .

We can now proceed to the approximation of the Hermite-Biehler function  $E$  near  $s$  in the case when  $s$  is close to a resonance.

We define  $T_0(s, C) \subset \mathbb{R}_+$  as the set of all  $t$  for which  $Q(s, C/t)$  contains a zero of  $E(t, \cdot)$ . Here  $C$  can be a constant or a function of  $t$ .

**Lemma 6.** *For a.e.  $s \in \mathbb{R}$  such that  $T_0(s, D)$  is unbounded for some constant  $D > 0$  there exists  $C(t) > D$ ,  $C(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , with the following properties.*

*Consider a continuous function  $z(t) = x(t) - iy(t)$  on  $T_0(s, C)$  such that for each  $t \in T_0(s, C)$ ,  $z(t)$  is one of the zeros of  $E(t, \cdot)$  in  $Q(s, C(t)/t)$ . Then*

$$\sup_{z \in Q(s, C(t)/t)} \left| E(t, z) - \frac{\alpha \sqrt{2}}{\sqrt{w(s) \sinh[2ty(t)]}} \sin[t(z - z(t))] \right| = o(1), \quad (7.4)$$

as  $t \rightarrow \infty$ ,  $t \in T_0(s, C)$  for some unimodular continuous function  $\alpha = \alpha(s, t)$ .

*Proof.* Let  $s$  and  $C_1(t)$  be such that (7.3) is satisfied (with  $C = C_1$ ).

Since  $E = A - iC$  vanishes at  $z(t)$ ,

$$A(t, z(t)) = iC(t, z(t)) = \beta$$

and

$$D(t, z(t), w) = \beta C(t, \bar{w}) + i\beta A(t, \bar{w}) = i\beta E(t, \bar{w}).$$

Hence,  $i\beta E(t, w)$  satisfies

$$\sup_{w \in Q(t, C_1/t)} \left| i\beta E(t, w) - \frac{1}{w(s)} \sin[t(w - z(t))] \right| = o(1)$$

by Corollary 1.

Let us first establish (7.4) for  $Q(s, L/t)$  with a constant  $L > 2D$  in place of  $C(t)$ . On one hand, from the last equation for all  $z, w \in Q(s, L/t)$ ,

$$\begin{aligned} & \det \begin{pmatrix} i\beta E(t, z) & i\beta E(t, \bar{w}) \\ -i\bar{\beta} E^\#(t, z) & -i\bar{\beta} E^\#(t, \bar{w}) \end{pmatrix} = \\ & \frac{1}{w(s)^2} \det \begin{pmatrix} \sin[t(z - z(t))] & \sin[t(\bar{w} - z(t))] \\ \sin[t(z - \bar{z}(t))] & \sin[t(\bar{w} - \bar{z}(t))] \end{pmatrix} + o(1)\psi_1(t, z, w) = \\ & \frac{1}{2w(s)^2} [\cos[t((z - \bar{w}) - 2iy(t))] - \cos[t((z - \bar{w}) + 2iy(t))] + \\ & + o(1)\psi_1(t, z, w) = \frac{1}{w(s)^2} \sin[2ity(t)] \sin[t(\bar{w} - z)] + o(1)\psi_1(t, z, w), \end{aligned} \quad (7.5)$$

as  $t \rightarrow \infty$  for some bounded function  $\psi_1$ . On the other hand, by (7.1) and Lemma 5, for a.e.  $s$ , any  $z, w \in Q(s, L/t)$  and  $t \in T_0(s, L)$ ,

$$\frac{2i}{w(s)} \sin[t(\bar{w} - z)] = \det \begin{pmatrix} E(t, z) & E(t, \bar{w}) \\ E^\#(t, z) & E^\#(t, \bar{w}) \end{pmatrix} + o(1)\psi_2(t, z, w)$$

as  $t \rightarrow \infty$  for some bounded function  $\psi_2$ .

Comparing the last two equations we obtain

$$2|\beta|^2 w(s) = (1 + o(1)) \sinh[2ty(t)]$$

as  $t \rightarrow \infty$ ,  $t \in T_0(s, L)$ .

Altogether,

$$\sup_{z \in Q(s, L/t)} \left| E(t, z) - \frac{\alpha\sqrt{2}}{\sqrt{w(s) \sinh[2ty(t)]}} \sin[t(z - z(t))] \right| = o(1)$$

for some continuous unimodular  $\alpha$  as  $t \rightarrow \infty$ ,  $t \in T_0(s, L)$ .

Once again, considering larger  $L$  the statement can be improved from constant  $L$  to  $L(t) \rightarrow \infty$ . The function  $C$  from the statement can be chosen as  $L/2$ .

□

Let  $Q_{\pm}(s, C) = Q(s, C) \cap \mathbb{C}_{\pm}$ . In terms of Dirac inner functions  $\theta(t, z)$  the last Lemma can be reformulated as follows

**Corollary 2.** *If  $s, C, \alpha$  and  $z(t)$  are from (7.4) then*

$$\sup_{z \in Q_+(s, C/t)} \left| \theta(t, z) - \bar{\alpha}^2 \frac{\sin[t(z - \bar{z}(t))]}{\sin[t(z - z(t))]} \right| = o(1) \quad (7.6)$$

as  $t \rightarrow \infty, t \in T_0(s, C)$ .

We obtain the following property of resonances for Dirac systems (1.1).

**Corollary 3.** *Let  $C$  be an arbitrary positive constant. The set of all  $s \in \mathbb{R}$  for which  $Q(s, C/t)$  contains a zero of  $E$  for all sufficiently large  $t$  has Lebesgue measure zero.*

Although we will need this statement as an intermediate step, it will eventually be improved from 'for all sufficiently large  $t$ ' to just 'for arbitrarily large  $t$ '.

*Proof.* Let  $\Sigma, |\Sigma| > 0$ , be a set of  $s$  such that for some  $T(s) > 0$ ,  $Q(s, C/t)$  contains a zero of  $E$  for all  $t > T$  and (7.4) holds for all  $s \in \Sigma$ . By Lusin's theorem we can assume that  $T(s) = T$  and  $o(1)$  are the same for all  $s \in \Sigma$ . Suppose that for all  $s \in \Sigma$  the left hand side of (7.4) is less than  $\varepsilon$  for  $t > T_0$  for some  $T_0 > T$  and some small  $\varepsilon > 0$  to be specified later. We can also assume that  $w(s) < D < \infty$  for all  $s \in \Sigma$ .

Let  $n$  be such that  $2^n > T_0$ . Let us consider a cover of index 2 of the set  $\Sigma$  with intervals of length  $2^{-(n+2)}$ . The equation (7.4) shows that for  $t = 2^n$  and for  $t = 2^{n+1}$  the functions  $E(2^n, x)$  and  $E(2^{n+1}, x)$  are approximated on those intervals by sines with frequencies  $2^n$  and  $2^{n+1}$  respectively. It follows that on each of the intervals

$$||E(2^n, x) - E(2^{n+1}, x)|| > \delta > 0$$

on at least 1/100-th of the interval for some small  $\delta$ , if  $\varepsilon$  is small enough. Since the intervals cover  $\Sigma$  with index 2, the last inequality holds on a set of measure at least  $\frac{1}{200}|\Sigma|$ . Since  $n$  can be chosen arbitrarily large, this contradicts to the convergence of  $|E(t, x)|$  in measure on  $\mathbb{R}$ . □

For two complex numbers  $z = x + iy$  and  $w = u + iv$ ,  $\text{Proj}_z w$  stands for the complex number  $p + iq$  such that the vector  $(p, q)$  is the orthogonal projection of  $(u, v)$  onto the direction of  $(x, y)$ .

**Lemma 7.** *Let  $s, C(t), D$  and  $z(t) = x(t) - iy(t)$  be like in Lemma 6.*

*There exists  $T > 0$  such that for every interval  $[t_1, t_2] \subset T_0(s, D) \cap [T, \infty)$ ,  $\alpha = \alpha(s, t)$  from (7.4) satisfies*

$$\alpha(s, t) = e^{ist + \psi(t)}, \quad (7.7)$$

*on  $[t_1, t_2]$ , where  $\psi$  is a function on  $[t_1, t_2]$  such that*

$$|\psi(x_2) - \psi(x_1)| < 2 \cosh D \int_{x_1}^{x_2} |f(t)| dt$$

*for any  $t_1 \leq x_1 \leq x_2 \leq t_2$ .*

*Proof.* Since the two functions in the left hand side of (7.4) are close on  $Q(s, 2D/t)$  for large enough  $t$ , by the Cauchy estimates their derivatives inside  $Q(s, D/t)$  must be within  $o(t)$  from each other. Their absolute values at  $\bar{z}(t)$  are  $\equiv 1$  and within  $o(1)$  from each other. The argument of the second function at  $\bar{z}(t)$  is  $-\pi/2$  and the argument of  $E$  is within  $o(1)$  from  $-\pi/2 \pmod{2\pi}$ . Noticing that

$$\frac{\partial}{\partial z}(\sin[t(z - z(t))])|_{z=z(t)} = t > 0,$$

we see that  $\arg i\alpha^2\theta_z(t, z(t)) = o(1) \pmod{2\pi}$ . Hence  $\alpha = \alpha(s, t)$  can be changed to satisfy

$$i\alpha^2\theta_z(t, z(t)) = 1. \quad (7.8)$$

The rest follows from the equation (4.4) for  $\theta_z(t, z(t))$ . Indeed, first note that

$$\text{Proj}_{i\theta_z(t, z(t))} \frac{\partial}{\partial t} \theta_z(t, z(t))$$

is the component of the vector

$$\frac{\partial}{\partial t} \theta_z(t, z(t))$$

responsible for the change of the argument of  $\theta_z$ . The equation (4.4) shows that

$$\begin{aligned} \text{Proj}_{i\theta_z(t, z(t))} \frac{\partial}{\partial t} \theta_z(t, z(t)) &= \\ si\theta_z(t, z(t)) + \text{Proj}_{i\theta_z(t, z(t))} f(t) \frac{\theta_{zz}(t, z(t))}{\theta_z(t, z(t))} &= \\ = (s + f(t)A(t))i\theta_z(t, z(t)) \end{aligned}$$

for some real function  $A(t)$ . It follows from (7.6) that

$$\theta_z(t, z(t)) = (1 + o(1))t \frac{1}{\sin[2it\Im z(t)]},$$

and

$$\theta_{zz}(t, z(t)) = (1 + o(1))t^2 \frac{-2 \cos[2it\Im z(t)]}{\sin^2[2it\Im z(t)]}.$$

Hence

$$|A(t)| \leq 2 \cosh[ty(t)] + o(1) \leq 3 \cosh D$$

for large enough  $t$ . Now the statement follows from the equation

$$\frac{\pi}{2} + 2 \arg \alpha(s, t) = - \arg \theta_z(t, z(t))$$

which is implied, for properly chosen continuous branches of argument, by (7.8).  $\square$

Consider again the box  $Q(s, C/t)$  from Corollary 1. Recall that we denote by  $T_0(s, C)$  the set of  $t$  such that  $Q(s, C/t)$  contains a zero of  $E(t, \cdot)$ . In Lemma 6 we used Corollary 1 to obtain approximations for  $E$  for  $t \in T_0(s, C)$ . Let us now discuss the case when (7.3) holds but  $t \notin T_0(s, C)$ , i.e.,  $Q(s, C/t)$  does not contain a zero of  $E(t, \cdot)$ . Our goal is to show that then  $E$  can be approximated by an exponential near  $s$ , see Corollary 4 below.

Let  $I_t = \mathbb{R} \cap Q(s, C/t)$ . For  $x, w \in I_t$  one can interpret  $D(t, x, w)$  as a scalar product of two  $\mathbb{R}^2$ -vectors and write (7.3) as

$$\begin{aligned} D(t, x, w) &= (A(t, x), C(t, x))^T \cdot (C(t, w), -A(t, w)) = \\ &= \frac{1}{w(s)} \sin[t(w - x)] + o(1)\psi(t, x), \end{aligned} \quad (7.9)$$

as  $t \rightarrow 0$  for some uniformly bounded  $\psi$ . Let us consider two fixed values of  $w$  in  $I_t$ ,  $w_1$  and  $w_2 = w_1 + \pi/2t$ . Since the last formula must hold for both  $w_{1,2}$  and every  $x \in I_t$ , we obtain that the vector-function  $(A(t, x), C(t, x))^T$  is within  $o(1)$  from a  $2\pi/t$ -periodic function whose modulus is bounded away from zero for large enough  $t$  on  $I_t$ .

Then there exist points  $x_0$  and  $x_1$  on  $I_t$  such that  $E(t, x_0)$  is positive and  $E(t, x_1)$  is negative imaginary. Then  $A(t, x_0) = 1/\sqrt{w(s)} + o(1)$ ,  $C(t, x_0) = 0$ ,  $A(t, x_1) = 0$  and  $C(t, x_1) = 1/\sqrt{w(s)} + o(1)$ . Using (7.3) with  $D(t, z, x_0)$  and  $D(t, z, x_1)$  we see that  $C(t, z)$  is within  $o(1)$  from

$$\frac{1}{\sqrt{w(s)}} \sin[t(z - x_0)]$$

and  $A(t, z)$  from

$$\frac{1}{\sqrt{w(s)}} \cos[t(z - x_2)]$$

on  $Q(t, C/t)$ , where  $x_2 = x_1 - \pi/2$ . Therefore  $E(t, z)$  is within  $o(1)$  from

$$\frac{1}{\sqrt{w(s)}} (\cos[t(z - x_2)] - i \sin[t(z - x_0)])$$

on  $Q(s, C/t)$ .

Let  $\phi(x) = \arg E(t, x)$  and let  $J_t = (s - 4\pi/t, s + 4\pi/t)$ . Notice that if  $|x_2 - x_0| > \delta \pmod{2\pi}$ , then

$$\frac{\sup_{J_t} \phi'}{\inf_{J_t} \phi'} > 1 + \varepsilon$$

for some  $\varepsilon = \varepsilon(\delta) > 0$ . Since  $C(t) \rightarrow \infty$ , Lemma 1 implies that for large  $t$  there is a zero of  $\theta_E$ , in  $Q(s, C/t)$ , which contradicts our assumption that  $t \notin T_0(s, C)$ . Hence,  $|x_2 - x_0| = o(1) \pmod{2\pi}$  and  $E(t, z)$  is within  $o(1)$  from

$$\frac{\beta(t)}{\sqrt{w(s)}} e^{-itz}$$

on some  $Q(s, C_1/t)$ ,  $C_1(t) \rightarrow \infty$ , for some  $\beta(t)$ ,  $|\beta(t)| = 1$ .

Let  $z(t) = u(t) - ip(t)$ . Then for the second function in (7.4) we have

$$\begin{aligned} & \frac{\sqrt{2}}{\sqrt{w(s) \sinh[2tp(t)]}} \sin[t(z - z(t))] = \\ &= \frac{\sqrt{2}}{\sqrt{w(s) \sinh[2tp(t)]}} \sin[t((z - u(t)) + ip(t))] = \\ & \frac{\sqrt{2}}{\sqrt{w(s) \sinh[2tp(t)]}} [\sin[t(z - u(t))] \cos[tp(t)] + \sin[tp(t)] \cos[t(z - u(t))]]. \end{aligned}$$

Notice that

$$\frac{\sqrt{2}}{\sqrt{\sinh[2tp(t)]}} \cos[tp(t)] \rightarrow 1 \text{ and } \frac{\sqrt{2}}{\sqrt{\sinh[2tp(t)]}} \sin[tp(t)] \rightarrow -i$$

as  $p(t) \rightarrow \infty$  and therefore the second function in (7.4) tends to

$$\frac{-i\alpha(t)}{\sqrt{w(s)}} e^{itz},$$

which is within  $o(1)$  from  $E$  on  $Q(t, C_1/t)$  if we put  $\alpha = i\beta$ .

Summarizing the above discussion we see that (7.4) holds not only for  $t \in T_0(s, C)$ , for which  $z(t)$  in (7.4) can be chosen as a zero of  $E$  in  $Q(s, C/t)$  (or a point close to zero as in Lemmas 9 and 10 below), but for all  $t$  with some  $z(t)$ . When  $t \notin T_0(s, C)$ ,  $z(t)$  in (7.4) needs to satisfy  $\Im z(t) > C/t$ , in which case the approximating function is close to an exponential on a smaller box.

We obtain the following

**Corollary 4.** *1) For a.e.  $s$  there exists  $C(t) > 0$ ,  $C(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , and  $z(t) \in \mathbb{C}_-$  such that*

$$\sup_{z \in Q(s, C/t)} \left| E(t, z) - \frac{\alpha\sqrt{2}}{\sqrt{w(s) \sinh[2ty(t)]}} \sin[t(z - z(t))] \right| = o(1) \quad (7.10)$$

for some  $\alpha = \alpha(s, t)$ ,  $|\alpha| = 1$  as  $t \rightarrow \infty$ . For  $t \in T_0(s, C)$ ,  $z(t)$  can be chosen as a zero of  $E(t, \cdot)$ .

2) If (7.10) holds for some  $s \in \mathbb{R}$  and some  $C(t) > 0, C(t) \rightarrow \infty$  as  $t \rightarrow \infty$  then for any constant  $D > 0$ ,

$$\sup_{z \in Q(s, D/t)} \left| E(t, z) - \frac{-i\alpha(t)}{\sqrt{w(s)}} e^{itz} \right| = o(1) \quad (7.11)$$

as  $t \rightarrow \infty$ ,  $t \notin T_0(s, C)$ .

## 8. JOINT APPROXIMATIONS FOR $E$ AND $\tilde{E}$

Approximations obtained for the Neumann family of Hermite-Biehler functions  $E(t, z)$  in the previous section are also valid for the Dirichlet family  $\tilde{E}(t, z)$ . It will be more convenient for us to use cosines instead of sines for  $\tilde{E}$ , which corresponds to the substitution of  $z(t)$  with  $z(t) + \pi/2t$  in the last statement.

In this section we establish relations between the parameters of the two approximating functions.

We denote by  $\gamma(p)$  the function

$$\gamma(p) = \sqrt{2}/\sqrt{\sinh[2p]}.$$

**Lemma 8.** *For a.e.  $s$  there exists  $C(t) > 0$ ,  $C(t) \rightarrow \infty$  with the following properties.*

*For every  $t > 0$  there exist  $z(t) = u(t) - ip(t)$ ,  $\tilde{z}(t) = \tilde{u}(t) - ip(t)$  and  $\alpha(t) = \alpha(s, t)$  such that  $p(t) > 0$ ,  $|\alpha(t)| = 1$  and*

$$\sup_{z \in Q(s, C/t)} \left| E(t, z) - \alpha(t) \frac{\gamma(ty(t))}{\sqrt{w(s)}} \sin[t(z - z(t))] \right| = o(1)$$

and

$$\sup_{z \in Q(s, C/t)} \left| \tilde{E}(t, z) - \alpha(t) \frac{\gamma(ty(t))}{\sqrt{\tilde{w}(s)}} \cos[t(z - \tilde{z}(t))] \right| = o(1)$$

as  $t \rightarrow \infty$ . The function  $p(t)$  satisfies  $p(t) > C(t)$  for  $t \notin T_0(s, C)$  and  $p(t) \leq C(t)$  for  $t \in T_0(s, C)$ ;  $u(t)$  and  $\tilde{u}(t)$  satisfy

$$\cos[t(\tilde{u}(t) - u(t))] = \sqrt{w(s)\tilde{w}(s)}.$$

*Proof.* Applying Corollary 4 to  $E$  and then to  $\tilde{E}$  we obtain that for a.e.  $s$ ,

$$\begin{aligned} \sup_{z \in Q(s, D/t)} \left| E(t, z) - \beta(t) \frac{\gamma(ty(t))}{\sqrt{w(s)}} \sin[t(z - \xi(t))] \right| &= o(1) \text{ and} \\ \sup_{z \in Q(s, D/t)} \left| \tilde{E}(t, z) - \delta(t) \frac{\gamma(t\tilde{y}(t))}{\sqrt{\tilde{w}(s)}} \cos[t(z - \tilde{\xi}(t))] \right| &= o(1) \end{aligned} \quad (8.1)$$

on  $Q(s, D/t)$  for some  $\xi(t) = x(t) - iy(t)$ ,  $\tilde{\xi}(t) = \tilde{x}(t) - i\tilde{y}(t)$ , any fixed constant  $D > 4\pi$  and unimodular  $\beta(t), \delta(t)$ .

If the approximating functions from (8.1) are plugged into the determinant (2.2) in place of  $E$  and  $\tilde{E}$  we get

$$\begin{aligned} &\frac{\gamma(ty(t))\gamma(t\tilde{y}(t))}{\sqrt{w(s)\tilde{w}(s)}} (\beta\bar{\delta} \sin[t(z - \xi(t))] \cos[t(z - \tilde{\xi}(t))] - \\ &\quad \bar{\beta}\delta \sin[t(z - \bar{\xi}(t))] \cos[t(z - \tilde{\xi}(t))]). \end{aligned} \quad (8.2)$$

Notice that (8.1) imply that  $|E|, |\tilde{E}|$  are bounded on  $Q(s, D/t)$  uniformly with respect to  $t$ , which implies that the expression in (8.2) is within  $o(1)$  from the determinant in (2.2) on  $Q(s, D/t)$ .

For  $z = x \in \mathbb{R}$ , (8.2) becomes

$$\begin{aligned} &= 2i\Im \left( \frac{\gamma(ty(t))\gamma(t\tilde{y}(t))}{\sqrt{w(s)\tilde{w}(s)}} \beta\bar{\delta} \sin[t(x - \xi(t))] \cos[t(x - \tilde{\xi}(t))] \right) = \\ &= 2i\Im \left( \frac{\gamma(ty(t))\gamma(t\tilde{y}(t))}{\sqrt{w(s)\tilde{w}(s)}} \beta\bar{\delta} \frac{1}{2} \left[ \sin[t(\xi(t) - \tilde{\xi}(t))] + \sin[t(2x - (\xi(t) + \tilde{\xi}(t)))] \right] \right). \end{aligned}$$

Suppose that  $t|\tilde{y}(t) - y(t)| > \Delta > 0$ . Then for a fixed  $t$ , with the first sine being constant, the second has absolute value  $\geq \sinh \Delta$  and its argument grows by more than  $2\pi$  on  $I_t = Q(s, D/t) \cap \mathbb{R}$ . Hence the expression cannot be within  $o(1)$  from  $2i$  on  $I_t$ . This shows that  $t|\tilde{y}(t) - y(t)| = o(1)$ . Hence we can replace  $\tilde{y}(t)$  with  $y(t)$  so that (8.1) still holds (with a different  $o(1)$ ).

In the case  $\xi(t) \in Q(s, D/t)$ , setting  $z = \xi(t), \bar{\xi}(t)$ , we obtain the following equations from (8.2):

$$\begin{aligned} &\frac{2i\beta\bar{\delta}}{\sqrt{w(s)\tilde{w}(s)}} \cos[t(\tilde{x}(t) - x(t))] = 2i + o(1) \text{ and} \\ &\frac{2i\beta\bar{\delta}}{\sqrt{w(s)\tilde{w}(s)}} \cos[t(\tilde{x}(t) - x(t))] = 2i + o(1). \end{aligned}$$

From these equations we see that the unimodular constants must satisfy  $\beta\bar{\delta} = 1 + o(1)$  and put  $\alpha(t) = \beta(t) = \delta(t) + o(1)$ .

Calculating the absolute values on each side of either of the equations we obtain

$$\cos[t(\tilde{x}(t) - x(t))] = \sqrt{w(s)\tilde{w}(s)} + o(1)$$

to see that  $\tilde{x}(t)$  and  $x(t)$  can be changed into  $\tilde{u}(t)$  and  $u(t)$  respectively to satisfy the equations of the Lemma.

We obtain the statement with a constant  $D$  in place of  $C(t)$ . Since  $D$  can be chosen arbitrarily large, standard argument allows us to improve the statement to  $C(t) \rightarrow \infty$ .  $\square$

**Remark 3.** *It is well known (Alexandrov-Clark formulas) that for a certain bounded analytic function  $\phi$  in  $\mathbb{C}_+$ ,  $\|\phi\|_{H^\infty} \leq 1$ ,*

$$w(s) = \frac{1 - |\phi(s)|^2}{|1 - \phi(s)|^2}, \quad \tilde{w}(s) = \frac{1 - |\phi(s)|^2}{|1 + \phi(s)|^2} \quad \text{and} \quad \sqrt{w(s)\tilde{w}(s)} = \frac{1 - |\phi(s)|^2}{|1 - \phi^2(s)|} \leq 1.$$

*One can see that there always exists  $\delta$  such that  $\cos \delta = \sqrt{w(s)\tilde{w}(s)}$ . In the last statement  $u_t$  and  $\tilde{u}_t = u_t + \delta/t$  satisfy*

$$\cos[t(\tilde{u}_t - u_t)] = \cos \delta = \sqrt{w(s)\tilde{w}(s)}.$$

*Because of periodicity one can assume that  $|\delta| \leq \frac{\pi}{2}$ . Recall that for almost all  $s \in \mathbb{R}$ ,  $\sqrt{w(s)\tilde{w}(s)} \neq 0$  and  $\delta$  can be chosen so that  $|\delta| < \frac{\pi}{2}$ .*

In Lemma 6 the approximation of  $E$  by a sine was constructed so that one of the zeros of  $E$  was also a zero of the approximating function. In some of our future calculations it will be more convenient for us to choose the approximating functions so that their values at  $s$  coincided with the value of the approximant. Our next Lemma states that it is possible to achieve such an approximation simultaneously for  $E$  and  $\tilde{E}$  while keeping the relations between the parameters of the approximating functions from Lemma 8.

**Lemma 9.** *Suppose that  $s \in \mathbb{R}$  and  $C > 1$  is a constant. Suppose that  $\sqrt{w(s)\tilde{w}(s)} \neq 0$ .*

*There exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$  the following holds.*

*If*

$$\left| E(t, z) - \frac{\alpha\gamma(y)}{\sqrt{w(s)}} \sin[t(z - (x - iy))] \right| < \varepsilon$$

*and*

$$\left| \tilde{E}(t, z) - \frac{\alpha\gamma(y)}{\sqrt{\tilde{w}(s)}} \cos[t(z - (\tilde{x} - iy))] \right| < \varepsilon$$

on  $Q(s, 3C/t)$  for some  $t > 0, x, \tilde{x}, y \in \mathbb{R}, \alpha \in \mathbb{C}$ , satisfying  $1 < ty < 2C, |\alpha| = 1$  and

$$-\frac{\pi}{2} < t(\tilde{x} - x) < \frac{\pi}{2}, \quad \cos[t(\tilde{x} - x)] = \sqrt{w(s)\tilde{w}(s)}, \quad (8.3)$$

then there exist  $y', x', \tilde{x}' \in \mathbb{R}$  and  $\alpha' \in \mathbb{C}, |\alpha'| = 1$ , satisfying

$$|tx' - tx| + |t\tilde{x}' - t\tilde{x}| + |ty' - ty| + |\alpha' - \alpha| < D\varepsilon, \quad \tilde{x} - x = \tilde{x}' - x',$$

for some constant  $D = D(C, s)$  and such that

$$\begin{aligned} E(t, s) &= \alpha' \frac{\gamma(y)}{\sqrt{w(s)}} \sin[t(s - (x' - iy'))] \quad \text{and} \\ \tilde{E}(t, s) &= \alpha' \frac{\gamma(y)}{\sqrt{\tilde{w}(s)}} \cos[t(s - (\tilde{x}' - iy'))]. \end{aligned} \quad (8.4)$$

*Proof.* Put

$$f(z) = \sqrt{\frac{\tilde{w}(s)}{w(s)}} \cdot \frac{\sin[t(z - (x - iy))]}{\cos[t(z - (\tilde{x} - iy))]}.$$

Note that under the restriction  $t|\tilde{x} - x| < \pi/2$ ,  $f$  is not constant. Denote by  $J$  the middle third of  $I = Q(s, 3C/t) \cap \mathbb{R}$ .

Note that

$$|E(t, s)/\tilde{E}(t, s) - f(s)| < 2\varepsilon$$

for all  $s \in J$ , if  $\varepsilon_0$  is small enough. Under the restriction imposed on  $y$ ,  $|f'(z)/t|$  and  $|f''(z)/t^2|$  are bounded and bounded away from zero in  $1/2t$ -neighborhood of  $J$  for large enough  $t$  by constants depending only on  $\sqrt{w(s)/\tilde{w}(s)}$ . Hence, for small enough  $\varepsilon$ , there exists  $D_1 > 0$  such that in the disk  $B(s, D_1\varepsilon/t)$ ,  $f$  takes all values from  $B(f(s), 2\varepsilon)$ . Let  $a \in B(f(s), 2\varepsilon)$  be such that  $f(a) = E(s)/\tilde{E}(s)$ . Then  $\tilde{x}' = \tilde{x} + \Re(a - s)$ ,  $x' = x + \Re(a - s)$ ,  $y' = y - \Im(a - s)$  will satisfy

$$E(s)/\tilde{E}(s) = \sqrt{\frac{\tilde{w}(s)}{w(s)}} \frac{\sin[t(s + (x' + iy'))]}{\cos[t(s + (\tilde{x}' + iy'))]}.$$

Recalling that  $E, \tilde{E}$  satisfy (2.2), this implies that

$$|E(s)| = \left| \frac{\gamma(y)}{\sqrt{w(s)}} \sin[t(s + (x + iy))] \right|$$

and

$$|\tilde{E}(s)| = \left| \frac{\gamma(y)}{\sqrt{\tilde{w}(s)}} \cos[t(s + (\tilde{x} + iy))] \right|$$

in addition to

$$\arg \frac{E(s)}{\tilde{E}(s)} = \arg \frac{\sin[t(s + (x + iy))]}{\cos[t(s + (\tilde{x} + iy))]}.$$

Hence (8.4) will hold with some  $\alpha'$ ,  $|\alpha'| = 1$ . Then  $|\alpha - \alpha'| \lesssim \varepsilon$  will follow automatically from the inequalities in the statement.

□

It will be convenient for us to restate the last lemma without ' $\varepsilon$ ':

**Corollary 5.** *For a.e.  $s \in \mathbb{R}$  and  $t > 0$  there exist positive functions  $C(t)$ ,  $C(t) \rightarrow \infty$ ,  $\psi(t), \psi(t) \rightarrow 0$  as  $t \rightarrow \infty$ , real functions  $x(t), \tilde{x}(t), y(t)$  and a complex function  $\alpha(t) \in \mathbb{C}$ , satisfying*

1)

$$\sup_{z \in Q(s, 3C(t)/t)} \left| E(t, z) - \frac{\alpha(t)\gamma(y(t))}{\sqrt{w(s)}} \sin[t(z - (x(t) - iy(t)))] \right| < \psi(t) \quad (8.5)$$

and

$$\sup_{z \in Q(s, 3C(t)/t)} \left| \tilde{E}(t, z) - \frac{\alpha(t)\gamma(y(t))}{\sqrt{\tilde{w}(s)}} \cos[t(z - (\tilde{x}(t) - iy(t)))] \right| < \psi(t). \quad (8.6)$$

2)

$$-\frac{\pi}{2} \leq t(\tilde{x}(t) - x(t)) \leq \frac{\pi}{2}, \quad \cos[t(\tilde{x}(t) - x(t))] = \sqrt{w(s)\tilde{w}(s)}, \quad \text{and } |\alpha(t)| = 1. \quad (8.7)$$

3) For large enough  $t$  for which  $1 < ty(t) < 2C(t)$ ,

$$E(t, s) = \frac{\alpha(t)\gamma(y(t))}{\sqrt{w(s)}} \sin[t(s - (x(t) - iy(t)))] \quad (8.8)$$

and

$$\tilde{E}(t, s) = \frac{\alpha(t)\gamma(y(t))}{\sqrt{\tilde{w}(s)}} \cos[t(s - (\tilde{x}(t) - iy(t)))] \quad (8.9)$$

Let  $0 < \sigma < 1$  be a small number. We will say that an interval  $I \subset \mathbb{R}_+$  is a  $\sigma$ -interval for  $f$  if

$$\left| \int_I f \right| \geq (1 - \sigma) \int_I |f|.$$

As one can see from the definition, a  $\sigma$ -interval is an interval where  $f$  has 'almost' the same sign, which leads to limited cancellation in  $\int_I f$ . Note that for a locally summable  $f$  almost every point where  $f \neq 0$  is a Lebesgue point and therefore has a neighborhood which is a  $\sigma$ -interval.

For the rest of the paper we consider  $\sigma$ -intervals for the potential  $f$  with a fixed a small positive  $\sigma < 1/100$ , whose exact value is unimportant.

In our next statement we show that the approximating functions can be chosen so that their zeros move in the same way as the zeros of  $E$  and  $\tilde{E}$  as  $t$  changes over a  $\sigma$ -interval, while the values of approximating functions at  $s$  remain equal to  $E(s)$  and  $\tilde{E}(s)$ .

**Lemma 10.** *For almost every  $s \in \mathbb{R}$  there exists  $C(t) > 0, C(t) \rightarrow \infty$  and  $\psi(t) > 0, \psi(t) \rightarrow 0$  as  $t \rightarrow \infty$  such that the following holds.*

*Let  $(t_1, t_2) \subset T_0(s, C(t)/t)$ ,  $t_2 - t_1 \leq \frac{1}{|s|+1}$  be a  $\sigma$ -interval such that*

$$\int_{t_1}^{t_2} |f| < \frac{1}{100},$$

*and  $C(t) > 10\pi$  for  $t > t_1$ . Let  $A$  be a constant  $8\pi < A < C(t)$  for  $t > t_1$ . Let  $\xi_1$  be a zero of  $E(t_1, z)$  in  $Q(s, A/t_1)$  which moves inside  $Q(t, A/t)$  continuously to  $\xi_2$  as  $t$  changes from  $t_1$  to  $t_2$  ( $\xi_2$  is a zero of  $E(t_2, z)$  in  $Q(s, A/t_2)$ ). Let  $\tilde{\xi}_1, \tilde{\xi}_2$  be similar zeros of  $\tilde{E}$  inside  $Q(t, A/t)$ . Assume that  $t_k \Im \xi_k, t_k \Im \tilde{\xi}_k > 2$  for  $k = 1, 2$ .*

*Then the zeros of  $E$  and  $\tilde{E}$  change in similar ways as  $t$  changes from  $t_1$  to  $t_2$ ,*

$$|(\xi_2 - \xi_1) - (\tilde{\xi}_2 - \tilde{\xi}_1)| \leq \psi(t_1)|\xi_2 - \xi_1|,$$

*and there exist real constants  $y_{t_k}, x_{t_k}, \tilde{x}_{t_k}$  and complex  $\alpha_{t_k}$ ,  $k = 1, 2$ , such that*

*1)  $t_k y_{t_k} > 1$  and (8.7) holds for  $t = t_k$ , for  $k = 1, 2$ .*

*2) (8.5) and (8.6) hold for  $t = t_1$ .*

*3) (8.8) and (8.9) hold for  $t = t_1$  and  $t = t_2$ .*

*4) As  $t$  changes from  $t_1$  to  $t_2$  the zeros of approximating functions change similarly to the zeros of  $E$  and  $\tilde{E}$ , i.e.,*

$$|(x_{t_2} + iy_{t_2}) - (x_{t_1} + iy_{t_1})| - |\xi_2 - \xi_1| \leq \psi(t_1)|\xi_2 - \xi_1|$$

*and*

$$\left| [(\tilde{x}_{t_2} + iy_{t_2}) - (\tilde{x}_{t_1} + iy_{t_1})] - [\tilde{\xi}_2 - \tilde{\xi}_1] \right| \leq \psi(t_1)|\xi_2 - \xi_1|.$$

*Proof.* Assume that  $s$  is such that the conclusion of Corollary 5 holds. Then  $y_{t_k}, x_{t_k}, \tilde{x}_{t_k}$  and  $\alpha_{t_k}$  satisfying 1), 2) 3) exist.

Let  $S(t, z)$  be the solution to the equation (2.5) for  $t \in [t_1, t_2]$  with the initial condition

$$S(t_1, z) = \alpha_{t_1} \frac{\gamma(y_{t_1})}{\sqrt{w(s)}} \sin[t_1(z - (x_{t_1} - iy_{t_1}))].$$

Let  $I(t, z)$  be the corresponding Dirac inner functions which solve (4.1) with the initial condition

$$I(t_1, z) = I_{S(t_1, z)}.$$

Since  $\mathcal{S}(t, z) = e^{itz} S(t, z)$  satisfies (5.1) it follows that for  $x \in \mathbb{R}$ ,

$$\frac{\partial}{\partial t} |\mathcal{S}(t, x)| \leq |f(t)| |\mathcal{S}(t, x)|$$

and therefore

$$|\mathcal{S}(t, x)| \leq |\mathcal{S}(t_1, x)| e^{\int_{t_1}^t |f|} \leq \frac{4}{\sqrt{w(s)}} \left(1 + \int_{t_1}^t |f|\right) \leq \frac{8}{\sqrt{w(s)}}. \quad (8.10)$$

Using (5.1) once again we obtain

$$\left| \mathcal{S}(t_1, x) - e^{-ix(t-t_1)} \mathcal{S}(t, x) \right| \leq \frac{8}{\sqrt{w(s)}} \int_{t_1}^t |f| \quad (8.11)$$

for all  $t \in (t_1, t_2)$  and  $x \in \mathbb{R}$ .

Let us denote by  $\zeta_{t_1} = \zeta_1$  the zero of  $S(t_1, z)$  and by  $\zeta_t$  the zero of  $S(t, z)$  evolved from  $\zeta_1$ . We will assume that  $\zeta_1$  is the zero of  $S(t, z)$  closest to  $\xi_1$  (or one of the closest zeros if there are more than one at the same distance). Then  $\xi_1 = \zeta_1 + o(1/t)$  because of 2). Let  $\xi_t$  be the zero of  $E(t, z)$  evolving from  $\xi_1 = \xi_{t_1}$ .

Since  $\mathcal{S}(t, z)$  is outer in  $\mathbb{C}_+$ , the previous inequality and the maximum principle implies

$$|\mathcal{S}(t_1, z) - \mathcal{S}(t, z)| < \frac{8}{\sqrt{w(s)}} \int_{t_1}^t |f|$$

for all  $z \in \mathbb{C}_+$ . It follows that

$$|\mathcal{S}(t_1, \zeta_t) - \mathcal{S}(t, \zeta_t)| = |\mathcal{S}(t_1, \zeta_t)| < \frac{8}{\sqrt{w(s)}} \int_{t_1}^t |f|$$

and therefore

$$|\zeta_1 - \zeta_t| < C_1 \int_{t_1}^{t_2} |f|/t_2. \quad (8.12)$$

where the constant  $C_1$  depends only on  $w(s)$  and  $A$ .

The same inequality holds for every zero  $\nu_1$  of  $S(t_1, z)$  and its evolution  $\nu_t$ . We also have

$$|z_1 - z_t| < C_2 \int_{t_1}^{t_2} |f|/t_2$$

for  $t \in (t_1, t_2)$  for the zeros of  $E$  in  $Q(s, 3C(t_1)/t_1)$  (including  $z_t = \xi_t$ ). Indeed, by (8.5),  $|\theta_z(t, z_t)| \asymp t$  for  $t \in [t_1, t_2]$ . Hence (4.3) implies the last inequality.

Note that since, as follows from (8.5),  $\xi_1 = \zeta_1 + o(1/t)$  the last two inequalities imply

$$|\xi_t - \zeta_t| < 2C_3 \int_{t_1}^{t_2} |f|/t_2 + o(1/t_2).$$

Since  $E$  and  $S$  satisfy (8.5) at  $t = t_1$ ,

$$|\theta_z(t_1, \xi_1) - I_z(t_1, \zeta_1)| < o(1)t_1.$$

Since  $E$  satisfies (8.5) for all  $t \in (t_1, t_2)$ ,

$$|\theta_z(t, \xi_t)|, |\theta_{zz}(t, \xi_t)/\theta_z(t, \xi_t)| < C_4 t.$$

Since the zeros of  $S(t, z)$  are close to the zeros of  $S(t_1, z)$  (see (8.12)),

$$|I_z(t, \zeta_t)|, |I_{zz}(t, \zeta_t)/I_z(t, \zeta_t)| < C_5 t$$

for  $t \in (t_1, t_2)$ . From (4.4) we now obtain

$$\begin{aligned} & |I_z(t, \zeta_t) - \theta_z(t, \xi_t)| < \\ & < \int_{t_1}^t 2|\zeta_t - \xi_t|(C_4 + C_5)t + (C_4 + C_5)t|f(t)| < \\ & < (C_4 + C_5)t \left( \int_{t_1}^{t_2} |f| + o(1) \right). \end{aligned}$$

for all  $t \in (t_1, t_2)$ . Since  $|\theta_z(t, z_t)|, |I_z(t, \zeta_t)| > C_6 t$  for some  $C_6 > 0$ , (4.3) now implies that

$$|(\zeta_t)'(t) - \xi_t'(t)| \leq C_7 \left( \int_{t_1}^{t_2} |f| + o(1) \right) |f(t)| \cdot |\xi_t'(t)|.$$

(Here and throughout the proof we use the notation  $(\zeta_t)'(t)$ , and similar notations, for the rate of change of  $\zeta_l$  with respect to  $l$  at the time  $t$ .)

Notice that (4.4) implies that the argument of  $\theta_z$  satisfies

$$\begin{aligned} & \arg \theta_z(t, \xi_t) - \arg \theta_z(t_1, \xi_{t_1}) \leq \\ & (t_2 - t_1)s + \int_{t_1}^{t_2} |f| + o(1) \leq 1 + o(1) < \frac{\pi}{2} \end{aligned} \tag{8.13}$$

for large enough  $t$ . Since by (4.3),

$$\xi_2 - \xi_1 = \int_{t_1}^{t_2} f(t)/\theta_z(t, \xi_t) \text{ and } \zeta_2 - \zeta_1 = \int_{t_1}^{t_2} f(t)/I_z(t, \zeta_t),$$

together with the property that  $(t_1, t_2)$  is a  $\sigma$ -interval, the last two inequalities imply

$$(\zeta_2 - \zeta_1) - (\xi_2 - \xi_1) = o(\xi_2 - \xi_1).$$

Since  $(t_1, t_2)$  is a  $\sigma$ -interval, by (4.3),

$$|\xi_2 - \xi_1| = \left| \int_{t_1}^{t_2} f(t)/\theta_z(t, \xi_t) \right| \gtrsim \int_{t_1}^{t_2} |f(t)|/t_2.$$

Together with (8.12) we obtain that for all zeros of  $I(t, z)$

$$|\nu_{t_2} - \nu_{t_1}| \lesssim |\xi_2 - \xi_1|. \quad (8.14)$$

Without loss of generality we can assume that  $C(t) < \sqrt{t}$ . Then all zeros of  $I(t_1, z)$  in  $Q(s, C/t_1)$  have the same imaginary parts and their real parts differ by at most  $1/\sqrt{t_1}$ . Since all zeros satisfy (4.4), their velocities  $(\nu_t)'$  may differ by at most

$$C_8 t \left( 1/\sqrt{t} + \int_{t_1}^{t_2} |f| \right) = o(|(\zeta_t)'|).$$

Similarly to (8.13), their arguments  $\arg[(\nu_t)']$  change by less than  $\pi/2$ , which implies that for any such zero  $\nu_t$  for which  $\nu_{t_1}$  belonged to  $Q(s, C(t_1)/t_1)$ , we have, like for  $\zeta_1, \zeta_2$ ,

$$(\nu_{t_2} - \nu_{t_1}) - (\xi_2 - \xi_1) = o(\xi_2 - \xi_1).$$

This estimate together with (8.14), which holds for the zeros of  $S$  which are far away from  $Q(s, A/t)$ , imply that

$$|S(t_2, z) - D_1 \sin[t_2(z - (\zeta_1 + (\xi_2 - \xi_1)))]| = o(\xi_2 - \xi_1)$$

for  $z \in Q(s, A/t)$  for some complex constant  $D_1$ . Once again, the constant  $A$  in  $Q(s, A/t)$  can now be replaced with a slowly growing function and we will assume that the last equation holds for  $z \in Q(s, C(t)/t)$  (making  $C$  grow slower if necessary).

Analogously, the solution  $\tilde{S}(t, z)$  of (2.5) with the initial condition

$$\tilde{S}(t_1, z) = \alpha_1 \frac{\gamma(y_1)}{\sqrt{\tilde{w}(s)}} \cos[t_1(z - \tilde{\zeta}_1)],$$

where  $\tilde{\zeta}_1 = \tilde{x}_{t_1} - iy_{t_1}$ , will satisfy

$$|\tilde{S}(t_2, z) - D_2 \cos[t_2(z - (\tilde{\zeta}_1 + (\tilde{\xi}_2 - \tilde{\xi}_1)))]| = o(\tilde{\xi}_2 - \tilde{\xi}_1)$$

on  $Q(s, C(t)/t)$  for some complex constant  $D_2$  (if  $C(t)$  is different from the one above, we choose the minimum of the two for each  $t$ ).

Since  $E, \tilde{E}$  satisfy (8.5) and (8.6),

$$|\theta_z(t_1, \xi_1) - \tilde{\theta}_z(t_1, \tilde{\xi}_1)| = o(t).$$

Since  $|\xi_1 - \tilde{\xi}_1| \lesssim 1/t$  and  $\theta_z(t, \xi_t), \tilde{\theta}_z(t, \tilde{\xi}_t)$  satisfy (4.4), via the argument similar to the one used above, we get

$$|\theta_z(t, \xi_t) - \tilde{\theta}_z(t, \tilde{\xi}_t)| = o(t) \quad (= o(\theta_z(t, \xi_t)))$$

for all  $t \in [t_1, t_2]$ .

Since the velocities of zeros of  $E$  and  $\tilde{E}$  satisfy (4.3),

$$(\xi_2 - \xi_1) = (\tilde{\xi}_2 - \tilde{\xi}_1) + o(\xi_2 - \xi_1).$$

Therefore,

$$|\tilde{S}(t_2, z) - D_2 \cos[t_2(z - (\tilde{\zeta}_2 + (\xi_2 - \xi_1)))]| = o((\xi_2 - \xi_1)).$$

Because of the determinant equation (2.2), which must be satisfied by the solutions  $S$  and  $\tilde{S}$  in place of  $E$  and  $\tilde{E}$ ,  $D_1$  and  $D_2$  can be chosen so that

$$D_1 \bar{D}_2 > 0, \quad |D_1 D_2| = \frac{\gamma^2(y_{t_2})}{\sqrt{w(s)} \tilde{w}(s)}. \quad (8.15)$$

If the point  $s_1$  is such that  $t_1 s_1 = t_1(s + (\tilde{z}_1 - x_1)) - \pi/2$  then

$$\tilde{S}(t_1, s_1) = S(t_1, s).$$

Notice that since  $|s - s_1| < 2\pi/t_2$ , (2.5) implies

$$\phi(t) = \arg(\tilde{S}(t, s_1)/S(t, s)) < |s - s_1|(t_2 - t_1) + 2 \int_{t_1}^{t_2} |f| < 2\pi/t_2 + 2 \int_{t_1}^{t_2} |f|$$

for  $t \in (t_1, t_2)$ . For the absolute values, if we take into account the initial condition

$$|\tilde{S}(t_1, s_1)| = |S(t_1, s)| = D \leq 2,$$

(2.6) implies

$$\begin{aligned} & |\tilde{S}(t, s_1)| - |S(t, s)| = \\ & D \left( e^{\int_{t_1}^t f(t) \cos[2 \arg \tilde{S}(t, s_1)]} - e^{\int_{t_1}^t f(t) \cos[2 \arg S(t, s)]} \right) \lesssim \\ & \lesssim (1/t_2 + \int_{t_1}^{t_2} |f|) \int_{t_1}^{t_2} |f|. \end{aligned}$$

Therefore,

$$|\tilde{S}(t_2, s_1)| - |S(t_2, s)| < C_9 \left( \left( \int_{t_1}^{t_2} |f| \right)^2 + \int_{t_1}^{t_2} |f|/t_2 \right).$$

Combining the last relation with (8.15) we obtain that the constants  $D_1, D_2$  can be chosen so that

$$D_1 = \alpha_2 \frac{\gamma(y_2)}{\sqrt{w(s)}}, \quad D_2 = \alpha_2 \frac{\gamma(y_2)}{\sqrt{\tilde{w}(s)}}$$

for some unimodular constant  $\alpha_2$ .

Next, notice that the uniqueness of solution for the differential equation (2.5) implies that

$$S(t_2, s) = E(t_2, s), \quad \tilde{S}(t_2, s) = \tilde{E}(t_2, s).$$

Combining this with earlier estimates, it follows that

$$\begin{aligned} & \sin[t_2(s - (\zeta_1 + (\xi_2 - \xi_1)))] / \cos[t_2(s - (\tilde{\zeta}_1 + (\xi_2 - \xi_1)))] = \\ & = E(t_2, s) / \tilde{E}(t_2, s) + o(\xi_2 - \xi_1). \end{aligned}$$

Using the same argument as in the proof of Lemma 9, one can find a constant  $\Delta = o(1)(\xi_2 - \xi_1)/t_2$  such that

$$\begin{aligned} \sin[t_2(s - (\zeta_1 + (\xi_2 - \xi_1)) + \Delta)] / \cos[t_2(s - (\tilde{\zeta}_1 + (\xi_2 - \xi_1)) + \Delta)] &= \\ &= E(t_2, s) / \tilde{E}(t_2, s). \end{aligned}$$

Using trigonometric identities one can show that

$$\left| \begin{array}{cc} \frac{\gamma(y_2)}{\sqrt{w(s)}} \sin[t_2(s - (\zeta_1 + (\xi_2 - \xi_1)) + \Delta)] & \frac{\gamma(y_2)}{\sqrt{w(s)}} \cos[t_2(s - (\tilde{\zeta}_1 + (\xi_2 - \xi_1)) + \Delta)] \\ \frac{\gamma(y_2)}{\sqrt{w(s)}} \sin[t_2(s - (\tilde{\zeta}_1 + (\xi_2 - \xi_1)) + \tilde{\Delta})] & \frac{\gamma(y_2)}{\sqrt{w(s)}} \cos[t_2(s - (\tilde{\zeta}_1 + (\xi_2 - \xi_1)) + \tilde{\Delta})] \end{array} \right| = 2i.$$

Since  $E, \tilde{E}$  must satisfy the same relation (2.2) it follows that

$$\left| \frac{\gamma(y_2)}{\sqrt{w(s)}} \sin[t_2(s - (\zeta_1 + (\xi_2 - \xi_1)) + \Delta)] \right| = |E(t_2, s)|$$

and

$$\left| \frac{\gamma(y_2)}{\sqrt{w(s)}} \cos[t_2(s - (\tilde{\zeta}_1 + (\xi_2 - \xi_1)) + \Delta)] \right| = |\tilde{E}(t_2, s)|.$$

Finally the unimodular constant  $\alpha_2$  can be adjusted to satisfy 3).  $\square$

## 9. CLASSIFICATION OF TIME INTERVALS ACCORDING TO THE MOVEMENT OF RESONANCES

In this section let  $C > 0$  be a fixed constant. Recall that for  $s \in \mathbb{R}$ ,  $T_0(s, C)$  is the set of  $t \in \mathbb{R}_+$  such that the box  $Q(s, C/t)$  contains a zero of  $E(s, \cdot)$ . Suppose that there exists a set  $S$  such that  $|S| > 0$  and the set  $T_0(s, C)$  is unbounded for every  $s \in S$ . It follows from Corollary 3 that  $T_0(s, 3C)$  cannot cover a half-line except possibly for a zero set of  $s$ . We will assume that for all  $s \in S$ ,  $T_0(s, C)$  is unbounded but  $T_0(s, 3C)$  does not cover a half-line. Then for each  $s \in S$  there will exist time intervals  $(t_1, t_2)$  arbitrarily far in time such that a zero  $z(s, t)$  of  $E(t, z)$  enters  $Q(s, 3C/t)$  at  $t = t_1$  and enters  $Q(s, C/t)$  at  $t = t_2$ , while moving continuously inside  $Q(s, 3C/t)$  during the whole time period  $t \in (t_1, t_2)$ .

In particular, for every  $s \in S$  there will exist infinitely many intervals with the following properties, arbitrarily far in time.

Let  $L_s = (\tau_1, \tau_2) \subset T_0(s, 3C)$  be a time interval satisfying

- 1) There exists a zero  $z(s, t)$  of  $E(t, z)$  in  $Q(s, 3C/t)$  for all  $t \in L_s$ ;  $t \mapsto z(s, t)$  is a continuous curve for  $t \in L_s$ .
- 2)  $\Im z(s, \tau_1) = 3C/\tau_1$ ;
- 3) if  $2^n \leq \tau_1 < 2^{n+1}$  then  $\tau_2$  is such that either  $\Im z(s, \tau_2) = C/\tau_2$  or  $\tau_2 = 2^{n+2}$  (if the zero enters the box  $Q(s, C/t)$  after  $t = 2^{n+2}$ ).

As was mentioned before, for every  $s \in S$  we can find infinitely many disjoint intervals  $L_s$  with the above properties, arbitrarily far in time. We will denote such an interval by  $L_s^n$  if  $2^n \leq \tau_1 < 2^{n+1}$ . (If there are more than one such interval for a given  $s$  and  $n$ , we will choose one of them.) We will denote by  $S^n$  the set of all  $s$  such that there exists an interval  $L_s^n = (\tau_1, \tau_2)$  with the above properties. Then each  $s \in S$  belongs to infinitely many  $S^n$ .

We will also assume that the approximation formulas from our previous lemmas and corollaries, which hold for a.e.  $s$ , hold for all  $s \in S$ . By Lusin's theorem, we can also assume that all  $o(\cdot)$  appearing in those statements are majorated by uniform  $o(\cdot)$  over  $s \in S$ .

Next, observe that there exists  $\Delta > 0$ , depending only on  $C$ , such for each  $L_s^n = (\tau_1, \tau_2)$ ,

$$\int_{\tau_1}^{\tau_2} |f(t)| dt > \Delta. \quad (9.1)$$

This follows from the property that

$$|\theta_z(t, z(s, t))| \gtrsim t$$

for all  $t \in T_0(s, 3C)$  (which is implied by (7.6)), from (4.3) and the property that  $\Im z(s, \tau_1) - \Im z(s, \tau_2) \gtrsim 1/t$ .

We will assume that  $0 < \Delta < \frac{1}{100 \cosh 3C}$  and adjust  $L_s^n = (\tau_1, \tau_2)$  in such a way that

$$2\Delta > \int_{\tau_1}^{\tau_2} |f(t)| dt > \Delta > 0$$

by decreasing  $\tau_2$  if necessary.

For  $t \in T_0(s, 3C)$  let  $\alpha(s, t)$  be the continuous function from (7.4) in Lemma 6 satisfying the conclusion of Lemma 7. An interval  $I \subset T_0(s, 3C)$  is a  $V_s$ -interval if  $|\Im \alpha^2(s, t)| < 1/100$  for all  $t \in I$ . It is an  $H_s$ -interval if  $|\Im \alpha^2(s, t)| \geq 1/200$  for all  $t \in I$ .

Note that since  $\alpha(s, t)$  is continuous on  $L_s^n$ , the whole interval  $L_s^n$  can be covered, up to countably many points, by disjoint intervals each of which is either  $V_s$ - or  $H_s$ -interval (or both). We will denote the first collection of intervals  $VL_s^n$  and the second  $HL_s^n$ .

In our notations above, 'V' stands for 'vertical' and 'H' for 'horizontal'. Indeed, according to (4.3), when  $|\Im \alpha^2(s, t)|$  is small the resonance moves almost vertically and when it is bounded away from zero the motion has a horizontal component, comparable with the total increment. These two cases of motion will require different estimates in the next two sections.

The set  $S^n$  can be split into disjoint subsets  $S_V^n$  and  $S_H^n$  defined as:

$$S_V^n = \left\{ s \in S^n \left| \int_{\cup_{I \in V L_s^n} I} |f| \geq \frac{99}{100} \int_{L_s^n} |f| \right. \right\}$$

and

$$S_H^n = \left\{ s \in S^n \left| \int_{\cup_{I \in V L_s^n} I} |f| < \frac{99}{100} \int_{L_s^n} |f| \right. \right\}.$$

Note that then for each  $s \in S_H^n$ ,

$$\int_{\cup_{I \in H L_s^n} I} |f| > \frac{1}{100} \int_{L_s^n} |f|.$$

## 10. VERTICAL INTERVALS

Let  $s \in S_V^n$ . Since  $s \in S^n$ , for each such  $s$  there exists the interval  $L_s^n = (\tau_1, \tau_2)$  with  $2^n \leq \tau_1 < 2^{n+1}$ , defined in the last section. Split the interval  $L_s^n$  into three intervals,  $L_s^n = T_-^n \cup T_s^n \cup T_+^n$ , where  $T_-^n$  is the interval to the left from  $T_s^n$  and  $T_+^n$  is the interval to the right, so that

$$\int_{T_s^n} |f| = \Delta/3, \quad \int_{T_-^n} |f| > \Delta/3, \quad \int_{T_+^n} |f| > \Delta/3. \quad (10.1)$$

Without loss of generality we can assume that each interval from  $V L_s^n$  and  $H L_s^n$  is either contained in  $T_s^n$  or is disjoint from  $T_s^n$ . Let  $f_{\pm}$  denote the two positive functions with disjoint essential supports such that  $f = f_+ - f_-$ . Since either  $\int_{T_s^n} f_+ > \Delta/6$  or  $\int_{T_s^n} f_- > \Delta/6$ , we can also assume that the inequality with  $f_+$  holds for all  $T_s^n$ . Note that (10.1) implies that if  $T_s^n$  intersects  $T_q^n$  for some other  $q \in S_V^n$  then  $T_q^n \subset L_s^n$ .

Consider the set  $W = \cup_{s \in S_V^n} T_s^n$ . One can choose a finite collection  $\mathcal{T}^k = T_{s_k}^n, k = 1, 2, \dots, N$  such that

$$\text{A1) } \int_{W \setminus \cup \mathcal{T}^k} |f| < \Delta/100$$

A2) Each  $\mathcal{T}^k$  intersects at most 2 other  $\mathcal{T}^k$  and each point in  $\cup \mathcal{T}^k$  is covered by at most two intervals.

Note that, as follows from A1) and A2), for each  $T_s^n, s \in S_V^n$  there exists an interval from our collection,  $\mathcal{T}^m$ , such that

$$\int_{T_s^n \cap \mathcal{T}^m} f_+ \geq \Delta/12.$$

Furthermore, if the last inequality is satisfied then

$$\left| \int_{\mathcal{T}^m} e^{4its} f_+(t) dt \right| > \Delta/40.$$

Indeed,  $s \in S_V^n$  and  $\mathcal{T}^m \subset L_s^n$  which implies

$$\text{B1)} \quad \int_{\mathcal{T}^m \setminus \cup_{I \in VL_s^n} I} f_+ < \Delta/100$$

$$\text{B2)} \quad \int_{\mathcal{T}^m \cap (\cup_{I \in VL_s} I)} f_+ \geq \Delta/6 - \Delta/100 \geq \Delta/10$$

B3)

$$\begin{aligned} \int_{\mathcal{T}^m \cap (\cup_{I \in VL_s^n} I)} f_+ &\leq 2 \left| \int_{\mathcal{T}^m \cap (\cup_{I \in VL_s} I)} f_+ \alpha^4(s, t) \right| \leq \\ &\leq 4 \left| \int_{\mathcal{T}^m \cap (\cup_{I \in VL_s} I)} f_+ e^{4its} dt \right|. \end{aligned}$$

The first inequality in B3) follows from the definition of vertical intervals from  $VL_s^n$  and the second inequality holds because on  $L_s = (\tau_1, \tau_2)$ , and therefore on  $\cup_{I \in VL_s} I$ ,  $\alpha(s, t)$  satisfies the relation from Lemma 7, which yields

$$|\alpha(s, t) \bar{\alpha}(s, \tau_1) - e^{st}| < 4\Delta \cosh 3C < 1/10$$

for  $t \in L_s^n \subset T_0(s, 3C)$ .

Let  $\mathcal{S}^m$  be the set of all  $s \in S_V^n$  such that  $\int_{T_s^n \cap \mathcal{T}^m} f_+ \geq \Delta/12$ . Then, as we discussed before,  $\cup \mathcal{S}^m = S_V^n$ .

We obtain that for all  $s \in \mathcal{S}^m$ ,

$$\int_{\mathcal{T}^m} e^{4its} f_+(t) dt > \Delta/40.$$

Therefore,  $\|f_+\|_{L^2(\mathcal{T}^m)}^2 > D|\mathcal{S}^m|$  and

$$\sum \|f_+\|_{L^2(\mathcal{T}^m)}^2 \geq D \sum |\mathcal{S}^m| \geq D|S_V^n|$$

for some  $D > 0$ . Since each point of  $\cup \mathcal{T}^m$  is covered by at most two intervals (see A2), this implies

**Claim 1.**

$$2\|f\|_{L^2([2^n, 2^{n+2}])}^2 \geq 2\|f\|_{L^2(\cup \mathcal{T}^m)}^2 \geq D|S_V^n|.$$

## 11. HORIZONTAL INTERVALS

In this section we will estimate the scattering function  $a_{t_1 \rightarrow t_2}$  corresponding to an interval  $(t_1, t_2) \subset \mathbb{R}_+$ . The shortest way to define such a function is to say that it is the function  $a$ , defined as in Section 5, corresponding to the system (1.1) whose potential function is equal to  $f$  on  $(t_1, t_2)$  and to 0 elsewhere.

More constructively, if  $M(t, z)$  is the transfer matrix of the system (1.1) then one can define the transfer matrix from  $t = t_1$  to  $t = t_2$  as

$$M_{t_1 \rightarrow t_2}(z) = \begin{pmatrix} A_{t_1 \rightarrow t_2}(z) & B_{t_1 \rightarrow t_2}(z) \\ C_{t_1 \rightarrow t_2}(z) & D_{t_1 \rightarrow t_2}(z) \end{pmatrix} = M(t_2, z)M^{-1}(t_1, z).$$

After that the Hermite-Biehler functions  $E_{t_1 \rightarrow t_2}(z)$  and  $\tilde{E}_{t_1 \rightarrow t_2}(z)$  can be defined as

$$E_{t_1 \rightarrow t_2} = A_{t_1 \rightarrow t_2} - iC_{t_1 \rightarrow t_2}, \quad \tilde{E}_{t_1 \rightarrow t_2} = B_{t_1 \rightarrow t_2} - iD_{t_1 \rightarrow t_2},$$

and

$$a_{t_1 \rightarrow t_2}(z) = \frac{1}{2}e^{i(t_2 - t_1)z}(E_{t_1 \rightarrow t_2}(z) + i\tilde{E}_{t_1 \rightarrow t_2}(z)).$$

Note that  $M_{t_1 \rightarrow t_2}$  is equal to the transfer matrix  $M^*(t_2 - t_1, z)$  of the real Dirac system whose potential function  $f^*$  is equal to  $f(t - t_1)$  for  $0 \leq t \leq t_2 - t_1$  and to 0 for  $t > t_2 - t_1$ . Similarly, the functions  $E_{t_1 \rightarrow t_2}$ ,  $\tilde{E}_{t_1 \rightarrow t_2}$  and  $a_{t_1 \rightarrow t_2}$  are equal to the functions  $E^*(t_2 - t_1, z)$ ,  $\tilde{E}^*(t_2 - t_1, z)$  and  $a^*(t_2 - t_1, z)$  generated by that system. In particular, Parseval's identity for  $a_{t_1 \rightarrow t_2}$  becomes

$$\|\log |a_{t_1 \rightarrow t_2}|\|_{L^1(\mathbb{R})} = \|f\|_{L^2((t_1, t_2))}.$$

One of the main tools in our estimates for the  $H_s$ -intervals will be the following lemma.

**Lemma 11.** *Let  $s$  be such that the conclusion of Lemma 10 holds. Let the interval  $(t_1, t_2) \subset T_0(s, A)$ ,  $C(t)$ ,  $A$ ,  $x_t$  and  $y_t$  be like in Lemma 10. Let  $a_{t_1 \rightarrow t_2}$  be the scattering function and let  $I_{t_2}$  be the interval  $(s - \frac{1}{t_2}, s + \frac{1}{t_2})$ . Then*

$$\begin{aligned} & \|\log |a_{t_1 \rightarrow t_2}|\|_{L^1(I_{t_2})} \geq \\ & \geq \frac{D}{t_2} [|t_2(x_{t_2} - s) - t_1(x_{t_1} - s)| + O(|t_2 y_{t_2} - t_1 y_{t_1}|^2 + \\ & \quad + |t_2(x_{t_2} - s) - t_1(x_{t_1} - s)|^2)]. \end{aligned}$$

for some constant  $D > 0$ .

*Proof.* By Lemma 10

$$E(s, t_k) = \alpha_{t_k} \frac{\gamma(p)}{\sqrt{w(s)}} \sin[t_k(s - (x_{t_k} - i y_{t_k}))]$$

and

$$\tilde{E}(s, t_k) = \alpha_{t_k} \frac{\gamma(p)}{\sqrt{\tilde{w}(s)}} \cos[t_k(s - (\tilde{x}_{t_k} - iy_{t_k}))],$$

for  $k = 1, 2$ . Then

$$\begin{aligned} M(t_k, s) &= \begin{pmatrix} 1/2 & 1/2 \\ -1/2i & 1/2i \end{pmatrix} \begin{pmatrix} E(t_k, s) & \tilde{E}(t_k, s) \\ E^\#(t_k, s) & \tilde{E}^\#(t_k, s) \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2i & 1/2i \end{pmatrix} \times \\ &\times \begin{pmatrix} \alpha_{t_k} \frac{\gamma(t_k y_{t_k})}{\sqrt{w(s)}} \sin[t_k(s - (x_{t_k} - iy_{t_k}))] & \alpha_{t_k} \frac{\gamma(t_k y_{t_k})}{\sqrt{\tilde{w}(s)}} \cos[t_k(s - (\tilde{x}_{t_k} - iy_{t_k}))] \\ \bar{\alpha}_{t_k} \frac{\gamma(t_k y_{t_k})}{\sqrt{w(s)}} \sin[t_k(s - (x_{t_k} + iy_{t_k}))] & \bar{\alpha}_{t_k} \frac{\gamma(t_k y_{t_k})}{\sqrt{\tilde{w}(s)}} \cos[t_k(s - (\tilde{x}_{t_k} + iy_{t_k}))] \end{pmatrix} \end{aligned}$$

for  $k = 1, 2$ .

The transfer matrix  $M_{t_1 \rightarrow t_2}$  can be calculated as

$$\begin{aligned} M_{t_1 \rightarrow t_2}(z) &= M(t_2, z) M(t_1, z)^{-1} = \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} E(t_2, z) & \tilde{E}(t_2, z) \\ E^\#(t_2, z) & \tilde{E}^\#(t_2, z) \end{pmatrix} \begin{pmatrix} E(t_1, z) & \tilde{E}(t_1, z) \\ E^\#(t_1, z) & \tilde{E}^\#(t_1, z) \end{pmatrix}^{-1} 2 \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}. \end{aligned}$$

Also

$$\begin{aligned} &\begin{pmatrix} E_{t_1 \rightarrow t_2} & \tilde{E}_{t_1 \rightarrow t_2} \\ E_{t_1 \rightarrow t_2}^\# & \tilde{E}_{t_1 \rightarrow t_2}^\# \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} M_{t_1 \rightarrow t_2} \\ &= \begin{pmatrix} E(t_2, z) & \tilde{E}(t_2, z) \\ E^\#(t_2, z) & \tilde{E}^\#(t_2, z) \end{pmatrix} \frac{1}{2i} \begin{pmatrix} \tilde{E}^\#(t_1, z) & -\tilde{E}(t_1, z) \\ -E^\#(t_1, z) & E(t_1, z) \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \end{aligned}$$

After recalling that

$$|a_{t_1 \rightarrow t_2}| = \frac{1}{2} |E_{t_1 \rightarrow t_2} + i \tilde{E}_{t_1 \rightarrow t_2}|$$

we obtain

$$\begin{aligned} &|a_{t_1 \rightarrow t_2}(z)| = \\ &= \frac{1}{4} |[(E(t_2, z) \tilde{E}^\#(t_1, z) - \tilde{E}(t_2, z) E^\#(t_1, z)) + \\ &\quad + (-E(t_2, z) \tilde{E}(t_1, z) + \tilde{E}(t_2, z) E(t_1, z))] \\ &\quad + i[-i(E(t_2, z) \tilde{E}^\#(t_1, z) - \tilde{E}(t_2, z) E^\#(t_1, z)) + \\ &\quad + i(-E(t_2, z) \tilde{E}(t_1, z) + \tilde{E}(t_2, z) E(t_1, z))]| \\ &= \frac{1}{2} |E(t_2, z) \tilde{E}^\#(t_1, z) - \tilde{E}(t_2, z) E^\#(t_1, z)|. \end{aligned}$$

To shorten our next series of formulas we will use the notations  $x_k = x_{t_k} - s$ ,  $\tilde{x}_k = \tilde{x}_{t_k} - s$  and  $y_k = y_{t_k}$  for  $k = 1, 2$ .

The last equation leads to

$$|a_{t_1 \rightarrow t_2}(s)| = \frac{1}{2} |\alpha_{t_2} \frac{\gamma(t_2 y_2)}{\sqrt{w(s)}} \sin[t_2(x_2 - iy_2)] \bar{\alpha}_{t_1} \frac{\gamma(t_1 y_1)}{\sqrt{\tilde{w}(s)}} \cos[t_1(\tilde{x}_1 + iy_1)] -$$

$$\begin{aligned}
& \alpha_{t_2} \frac{\gamma(t_2 y_2)}{\sqrt{\tilde{w}(s)}} \cos[t_2(\tilde{x}_2 - iy_2)] \bar{\alpha}_{t_1} \frac{\gamma(t_1 y_1)}{\sqrt{w(s)}} \sin[t_1(x_1 + iy_1)] = \\
& = \frac{\gamma(t_1 y_1) \gamma(t_2 y_2)}{2 \sqrt{w(s) \tilde{w}(s)}} |\sin[t_2(x_2 - iy_2)] \cos[t_1(\tilde{x}_1 + iy_1)] - \\
& \quad - \cos[t_2(\tilde{x}_2 - iy_2)] \sin[t_1(x_1 + iy_1)]|.
\end{aligned}$$

Using several trigonometric identities and condition 1) from Lemma 10, the last expression can be further simplified:

$$\begin{aligned}
& \sin[t_2(x_2 - iy_2)] \cos[t_1(\tilde{x}_1 + iy_1)] - \cos[t_2(\tilde{x}_2 - iy_2)] \sin[t_1(x_1 + iy_1)] = \\
& = \frac{1}{2} (\sin[t_2(x_2 - iy_2) + t_1(\tilde{x}_1 + iy_1)] + \sin[t_2(x_2 - iy_2) - t_1(\tilde{x}_1 + iy_1)] - \\
& \quad - \sin[t_1(x_1 + iy_1) + t_2(\tilde{x}_2 - iy_2)] - \sin[t_1(x_1 + iy_1) - t_2(\tilde{x}_2 - iy_2)]) = \\
& = \cos \left[ \frac{1}{2} ((t_2(x_2 - iy_2) + t_1(\tilde{x}_1 + iy_1)) + (t_1(x_1 + iy_1) + t_2(\tilde{x}_2 - iy_2))) \right] \times \\
& \times \sin \left[ \frac{1}{2} ((t_2(x_2 - iy_2) + t_1(\tilde{x}_1 + iy_1)) - (t_1(x_1 + iy_1) + t_2(\tilde{x}_2 - iy_2))) \right] + \\
& + \cos \left[ \frac{1}{2} ((t_2(x_2 - iy_2) - t_1(\tilde{x}_1 + iy_1)) + (t_1(x_1 + iy_1) - t_2(\tilde{x}_2 - iy_2))) \right] \times \\
& \times \sin \left[ \frac{1}{2} ((t_2(x_2 - iy_2) - t_1(\tilde{x}_1 + iy_1)) - (t_1(x_1 + iy_1) - t_2(\tilde{x}_2 - iy_2))) \right] = \\
& = \cos \left[ \frac{1}{2} (t_2(x_2 + \tilde{x}_2) + t_1(\tilde{x}_1 + x_1)) - i(t_2 y_2 - t_1 y_1) \right] \times \\
& \quad \times \sin \left[ \frac{1}{2} (t_2(x_2 - \tilde{x}_2) - t_1(x_1 - \tilde{x}_1)) \right] + \\
& \quad + \cos \left[ \frac{1}{2} (t_2(x_2 - \tilde{x}_2) + t_1(x_1 - \tilde{x}_1)) \right] \times \\
& \quad \times \sin \left[ \frac{1}{2} (t_2(x_2 + \tilde{x}_2) - t_1(\tilde{x}_1 + x_1)) - i(t_2 y_2 + t_1 y_1) \right]
\end{aligned}$$

Recall that by (8.7),  $\cos[t(x_t - \tilde{x}_t)] = \sqrt{w(s)\tilde{w}(s)}$ . Since  $x_t$  changes continuously with  $t$ , it follows that  $t_2(x_2 - \tilde{x}_2) = t_1(x_1 - \tilde{x}_1)$  and the last expression is equal to

$$\sqrt{w(s)\tilde{w}(s)} \sin \left[ \frac{1}{2} (t_2(x_2 + \tilde{x}_2) - t_1(\tilde{x}_1 + x_1)) - i(t_2 y_2 + t_1 y_1) \right].$$

Altogether we obtain

$$\begin{aligned}
|a_{t_1 \rightarrow t_2}| &= \frac{\gamma(t_1 y_1) \gamma(t_2 y_2)}{2} \left| \sin \left[ \frac{1}{2} (t_2(x_2 + \tilde{x}_2) - t_1(\tilde{x}_1 + x_1)) - i(t_2 y_2 + t_1 y_1) \right] \right| = \\
&= \frac{1}{\sqrt{|\sin[2it_1 y_1] \sin[2it_2 y_2]|}} \left| \sin \left[ \frac{1}{2} (t_2(x_2 + \tilde{x}_2) - t_1(\tilde{x}_1 + x_1)) - i(t_2 y_2 + t_1 y_1) \right] \right| = \\
&= \frac{1}{\sqrt{|\sin[2it_1 y_1] \sin[2it_2 y_2]|}} |\sin[(t_2 x_2 - t_1 x_1) - i(t_2 y_2 + t_1 y_1)]|.
\end{aligned}$$

If we put  $t_1x_1 = u, t_2x_2 = u + \varepsilon_1, t_1y_1 = v, t_2y_2 = v + \varepsilon_2$  then the last equation becomes

$$\begin{aligned} |a_{t_1 \rightarrow t_2}| &= \frac{1}{\sqrt{|\sin[2iv] \sin[2i(v + \varepsilon_2)]|}} |\sin[\varepsilon_1 - 2iv + i\varepsilon_2]| = \\ &= |1 + \frac{\varepsilon_1}{2} \coth[2v] + O(|\varepsilon_2^2 + \varepsilon_1^2|)|, \end{aligned}$$

where  $O(\cdot)$  depends only on  $A$  (recall that  $v = t_1y_1$  satisfies  $2 < v < A$  in Lemma 10).

Thus we obtain that

$$|\log |a_{t_1 \rightarrow t_2}(s)|| \geq \frac{1}{2}|\varepsilon_1| + O(|\varepsilon_2^2 + \varepsilon_1^2|).$$

Note that Lemma 10 could be formulated with any any point  $u$  in  $I_{t_2} = (s-1/t_2, s+1/t_2)$  in place of  $s$ . Repeating our argument we could then obtain the last inequality for any such  $u$  in place of  $s$  to estimate  $\int_{I_{t_2}} |\log |a_{t_1 \rightarrow t_2}(s)||$  from below and deduce the inequality in the statement.

□

As was mentioned before, every Lebesgue point  $s$  of  $f$  such that  $f(s) \neq 0$  has a neighborhood, which is a  $\sigma$ -interval for  $f$ . Therefore, for each  $H_s$ -interval  $I$  the set  $I \cap \{f \neq 0\}$  can be covered, up to a set of arbitrarily small mass, by finitely many disjoint  $\sigma$ -intervals contained in  $I$ . These intervals will also be  $H_s$ -intervals.

**Claim 2.** *If the interval  $J = (t_1, t_2)$  in the last statement is an  $H_s$  and a  $\sigma$ -interval then*

$$\|\log |a_{t_1 \rightarrow t_2}|\|_{L^1(s-1/t_2, s+1/t_2)} \geq D \int_{t_1}^{t_2} |f|/t_2$$

for large enough  $t_1$  and some absolute constant  $D > 0$ .

*Proof.* Since  $J$  is a  $\sigma$ -interval,  $|\int_J f| \geq \frac{9}{10} \int_J |f|$ . Since  $J \subset T_0(s, 3C)$ ,  $|\theta_z(t, z_t)| \asymp t$  for all  $t \in J$ . Also, by the definition of  $H_s$  intervals, by Corollary 2 and (7.8),

$$|\Re \theta_z(t, z_t)| \geq |\theta_z(t, z_t)|/300$$

for all  $t \in J$  if  $t_1$  is large enough. By continuity it implies that

$$\Re \theta_z(t, z_t) \geq |\theta_z(t, z_t)|/300 \text{ or } \Re \theta_z(t, z_t) \leq -|\theta_z(t, z_t)|/300$$

on  $J$ . Assume that

$$\Re \int_J \frac{f(t)}{\theta_z(t, z_t)} \geq 0$$

(the case  $< 0$  can be proved similarly). Then by (4.3),  $x_{t_2} - s \geq x_{t_1} - s$ . Since in Lemma 10,  $x_t$  is the real part of any zero in  $Q(s, A/t_1)$  and  $A > 8\pi$ , one can pick the zero with  $x_{t_2} > s$ . Recall that  $t_2 - t_1 < 1$ . Then

$$\begin{aligned} (t_2(x_{t_2} - s) - t_1(x_{t_1} - s)) &\geq t_1(x_{t_2} - x_{t_1}) \geq \\ &\geq D_1 t_1 \left| \Re \int_J \frac{f(t)}{\theta_z(t, z_t)} \right| \geq D_2 \int_J |f|, \end{aligned}$$

whereas

$$|t_2 y_{t_2} - t_1 y_{t_1}|^2 < D_3 \left( \int_J |f| \right)^2.$$

Now the statement follows from the last lemma.  $\square$

Recall that we are considering the set  $S \subset \mathbb{R}$  as defined in Section 9. We can assume that  $|S| < \infty$ . Once again, we can assume the condition 'for large enough  $t$ ' in the last statement is uniform over  $S$  and that the inequality holds for all  $s \in S$  for large enough  $t$ .

The sets  $S_H^n$  were also defined in Section 9.

**Claim 3.**

$$|S_H^n| < D \|f\|_{L^2([2^n, 2^{n+2}])}^2$$

for some constant  $D > 0$ .

*Proof.* Note that  $|S_H^n| \leq |S| < \infty$ . Consider a finite collection of intervals  $I_1, \dots, I_N$  centered at  $s_1, \dots, s_N \in S_H$  of the size  $|I_k| = C2^{-n}$  which covers at least one half of  $S_H^n$  so that no point on  $\mathbb{R}$  is covered by more than two of  $I_k$ . Consider the intervals  $L_{s_k}^n, k = 1, 2, \dots, N$  as defined in Section 9.

Notice that if  $(t_1, t_2)$  from Claim 2 is inside  $L_{s_k}^n$  then each  $I_{t_2}(s_k)$  from Claim 2 is inside  $I_k$ .

Consider a collection of disjoint intervals  $\mathcal{T}_1, \dots, \mathcal{T}_M, \mathcal{T}_k = (\tau_1^k, \tau_2^k)$  with the following properties.

- 1) Each  $\mathcal{T}_l$  belongs to  $HL_{s_k}$  for some  $k$ ;
- 1) All  $\mathcal{T}_l$  are  $\sigma$ -intervals;
- 2) for each  $k, \int_{\cup \mathcal{T}_l \in HL_{s_k}} |f| > \Delta/500$ , where  $\Delta > 0$  is from (9.1).

Existence of such intervals  $\mathcal{T}_l$  follows from the property that  $s_k \in S_H^n$  and from the fact that for every Lebesgue point  $t$  of  $f, f(t) \neq 0$ , all small enough intervals containing  $t$  are  $\sigma$ -intervals.

Then by Claim 2

$$\begin{aligned} \sum_{l=1}^M \|\log |a_{\tau_1^l \rightarrow \tau_2^l}| \|_{L^1(\mathbb{R})} &\geq \sum_{l=1}^M \left( \int_{T_l} |f| \sum_{k, T_l \in HL_{s_k}} |I_k| \right) = \\ &= \sum_{k=1}^N \left( |I_k| \sum_{l, T_l \in HL_{s_k}} \int_{T_l} |f| \right) > \frac{\Delta}{500} \sum_{k=1}^N |I_k| \geq \frac{\Delta}{2000} |S_H|. \end{aligned}$$

Now applying Parseval's identity to each  $a_{\tau_1^l \rightarrow \tau_2^l}$  in the above inequality we obtain the statement.  $\square$

## 12. PROOF OF MAIN THEOREM

Recall that the set  $S \subset \mathbb{R}$  was defined as the set of  $s$  such that the set  $T_0(s, C)$  is unbounded, i.e., such  $s$  for which there exist arbitrarily large  $t$  such that the box  $Q(s, C/t)$  contains a zero of  $E(t, \cdot)$ . As was discussed in Section 9, every  $s \in S$  must belong to infinitely many subsets  $S_n$  consisting of those  $s$  for which the zero enters the box during the time period  $t \in [2^n, 2^{n+1})$ .

Since by Claims 1 and 3,

$$\sum_n |S^n| = \sum_n |S_V^n| + |S_H^n| \lesssim \sum_n \|f\|_{L^2([2^n, 2^{n+2}])}^2 < \infty,$$

the set of  $s$  that belong to infinitely many  $S^n$  has measure zero. Therefore,  $|S| = 0$  for any  $C$ . Hence, for a.e.  $s \in \mathbb{R}$  there exists  $C(t) > 0$ ,  $C(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , such that  $Q(s, C(t)/t)$  has no zero of  $E(t, \cdot)$ . Corollary 4, part 2), now implies that for a.e.  $s$  and any  $D > 0$ ,

$$\sup_{Q(s, D/t)} \left| E(t, z) - \frac{-i\alpha(s, t)}{\sqrt{w(s)}} e^{itz} \right| = o(1) \quad (12.1)$$

as  $t \rightarrow \infty$ . Since  $|\alpha| = 1$ , it follows that

$$|E(t, s)| \rightarrow \frac{1}{\sqrt{w(s)}}$$

for a.e.  $s$ .

Similarly, for a.e.  $s$ ,

$$\sup_{Q(s, D/t)} \left| \tilde{E}(t, z) - \frac{-i\beta\alpha}{\sqrt{\tilde{w}(s)}} e^{itz} \right| = o(1) \quad (12.2)$$

where  $\alpha$  is the same as in the previous formula. The equation (2.2) implies that  $\beta(t)$  can be chosen as a constant  $\beta = e^{-i\phi}$ ,  $\phi = \arcsin \sqrt{w(s)\tilde{w}(s)}$ . Therefore

$$|\tilde{E}(t, s)| \rightarrow \frac{1}{\sqrt{\tilde{w}(s)}}$$

and

$$|a(t, s)| \rightarrow \frac{1}{2} \left| \frac{1}{\sqrt{w(s)}} + \frac{\cos[\phi + \pi/2]}{\sqrt{\tilde{w}(s)}} \right| = \frac{1}{2} \left| \frac{1}{\sqrt{w(s)}} + \sqrt{w(s)} \right|$$

at a.e.  $s$  and

$$|b(s, t)| \rightarrow \frac{1}{2} \left| \frac{1}{\sqrt{w(s)}} - \frac{\cos[\phi + \pi/2]}{\sqrt{\tilde{w}(s)}} \right| = \frac{1}{2} \left| \frac{1}{\sqrt{w(s)}} - \sqrt{w(s)} \right|$$

at a.e.  $s$ . Note that convergence of  $|a|$  a.e. and convergence of  $\|\log |a|\|_1$ , which follows from Parseval's identity, implies convergence of  $\log |a|$  in  $L^1$ .

Also,

$$\sup_{Q(s, D/t)} \left| \frac{b}{a} - \frac{\frac{1}{\sqrt{w(s)}} + \frac{\beta}{\sqrt{\tilde{w}(s)}}}{\frac{1}{\sqrt{w(s)}} - \frac{\beta}{\sqrt{\tilde{w}(s)}}} \right| = o(1)$$

and therefore  $\overset{\wedge}{f} = b/a$  converges pointwise a.e. on  $\mathbb{R}$ .

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