PARTITIO NUMERORUM: SUMS OF A PRIME AND A NUMBER OF k-TH POWERS

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Abstract. Let k be a natural number and let $c = 2.134693\ldots$ be the unique real solution of the equation $2c = 2 + \log(5c - 1)$ in $[1, \infty)$. Then, when $s \geqslant ck + 4$, we establish an asymptotic lower bound of the expected order of magnitude for the number of representations of a large positive integer as the sum of one prime and s positive integral k-th powers.

1. Introduction

Although strongly influenced by an earlier contribution of Hardy and Ramanujan [10], the famous series *Partitio Numerorum* written jointly by Hardy and Littlewood marks the arrival of the circle method. In the speculative section of part III (see [9]), they considered representations of natural numbers n as the sum of a prime and a number of k-th powers. Thus, the equation

$$n = p + x_1^k + \ldots + x_s^k, \tag{1.1}$$

in which $k \geqslant 2$ and $s \geqslant 1$ are given natural numbers, is to be solved in primes p and natural numbers x_j $(1 \leqslant j \leqslant s)$. Although their conjectures H, J and L are concerned with squares and cubes only, it is plain from the discussion that their method supports a conjectural asymptotic formula for the number $r(n) = r_{k,s}(n)$ of solutions of (1.1) in general. If this formula were true for $r_{k,1}(n)$, then all sufficiently large n for which the polynomial $n - x^k$ is irreducible over the rationals would be the sum of a prime and a k-th power. The analogous formula for $r_{k,2}(n)$ suggests that all large n are the sum of a prime and two k-th powers. There is an extensive literature related to the case of squares, in which $k = 2$ (e.g. [8, 21]), culminating in the works of Hooley [12, 13] and Linnik [15, 16] that confirms the Hardy-Littlewood formula for $r_{2,2}(n)$. Miech [19] showed that the formula for $r_{2,1}(n)$ holds for almost all n, in the sense that the exceptional n have density zero.

For $k \geqslant 3$, less is known. There are quantitative results allied to that of Miech counting exceptional n where the conjectured formula for $r_{k,1}(n)$ fails (most recently in work of Brüdern [3]). Other results concern the sparsity of numbers n where $r_{k,1}(n) = 0$, for example [6]. We are not aware of noteworthy unconditional contributions that relate to the cases $s > 1$ of (1.1). There are, of course, results that follow routinely from mean value estimates for k-th

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power Weyl sums. As a first example, we mention the minor arc estimate for the eighth moment of a cubic Weyl sum of Vaughan [22] in the improved form due to Boklan [1]. This is of strength sufficient to decouple the prime from the k-th powers via Schwarz's inequality in a direct application of the Hardy-Littlewood method, and as a result one obtains an asymptotic formula for $r_{3,4}(n)$ (see [14, Theorem 2]). For larger exponents k, one may restrict the variables x_j to be smooth numbers in order to make smaller values of s accessible. If one invokes the best currently known mean value estimates for smooth Weyl sums (see [29, 31] or Lemma 2.2 below) and decouples the prime as before, then for an explicit quantity $s_0(k)$ satisfying

$$s_0(k) = \tfrac{1}{2}k(\log k + \log\log k + 2 + o(1)),$$

one finds that

$$r_{k,s}(n) \gg n^{s/k}/\log n \tag{1.2}$$

whenever $s \geqslant s_0(k)$. The lower bound (1.2) is of the order of magnitude that is suggested by the hypothetical asymptotic formula.

No improvement on the severe condition $s \geqslant s_0(k)$ is known. Our first theorem shows that it suffices for s to grow linearly with k. In order to state this result precisely, let c be the unique solution of the transcendental equation

$$2c = 2 + \log(5c - 1)$$

in the interval $[1, \infty)$. The decimal representation is $c = 2.134693\ldots$.

Theorem 1.1. *Let $k \in \mathbb{N}$ and $s \geqslant ck + 4$. Then $r_{k,s}(n) \gg n^{s/k}/\log n$.*

For small values of k this conclusion is susceptible to some improvement because our proof is tuned to perform optimally for very large k. Based on the methods of [23, 26, 27, 32], the naïve decoupling approach yields (1.2) for the pairs $(k, s) = (4, 6)$, $(5, 9)$, $(6, 12)$ and $(7, 16)$, for example. Our method yields improved results for $k \geqslant 6$.

Theorem 1.2. *Suppose that $6 \leqslant k \leqslant 20$ and $s \geqslant S_0(k)$, where $S_0(k)$ is determined according to the entries of Table 1. Then $r_{k,s}(n) \gg n^{s/k}/\log n$.*

k	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$S_0(k)$		11	13	16	18	20	22	24	27	29	31	33	35	37	40	42
$S_1(k)$	8	10	12	14	17	19	21	23	25	27	29	31	32	34	36	38

TABLE 1. Critical numbers of k-th powers for Theorems 1.2 and 1.4

Our approach to equation (1.1) involves the circle method, and we are therefore limited by the familiar square root cancellation barrier. For the case at hand this says that the range $s \leqslant k$ is outside the scope of the method unless one is able to explore cancellations that get lost after application of the triangle inequality. Thus, we require scarcely more than twice as many k-th powers relative to the limit of the method, and for small k even fewer.

Further progress with problems involving primes is often possible if one assumes that no Dirichlet L-function has a zero in the half-plane $\mathrm{Re}(z) > \frac{1}{2}$. This is the generalised Riemann hypothesis, abbreviated to GRH hereafter (and referred to by Hardy and Littlewood [9] as Hypothesis R*). Hooley's original work [12] on $r_{2,2}(n)$ initially depended on GRH, and in a joint effort with Kawada [5] the authors deduced from GRH the existence of infinitely many primes representable as the sum of $2\lceil 4k/3\rceil$ positive integral k-th powers. The method of [5] is readily adapted to establish the lower bound (1.2) for $s \geqslant 2\lceil 4k/3\rceil$, subject to GRH, but this is now superseded unconditionally by Theorem 1.1. However, assuming the truth of GRH we are able to relax the conditions on s in Theorems 1.1 and 1.2. In particular, we improve on the naïve decoupling approach for fifth powers. Our results feature the unique solution of the transcendental equation

$$2c' = 2 + \log(4c' - 1)$$

in the interval $[1, \infty)$. Its decimal representation is $c' = 1.961969\ldots$.

Theorem 1.3. *Should GRH be true, then when $k \in \mathbb{N}$ and $s \geqslant c'k + 4$, one has $r_{k,s}(n) \gg n^{s/k}/\log n$.*

Theorem 1.4. *Suppose that $5 \leqslant k \leqslant 20$ and $s \geqslant S_1(k)$, where $S_1(k)$ is determined according to the entries of Table 1. Then, should GRH be true, one has $r_{k,s}(n) \gg n^{s/k}/\log n$.*

It should be noted that Theorems 1.3 and 1.4 depend on GRH only in the most indirect way, through the exponential sum estimate (4.2). In fact, the bound (4.2) is also available for a type II exponential sum with both variables of summation near $\sqrt{n}$. The unconditional bound (4.1) requires estimates for type II sums with the shorter of the two variables of summation ranging over $[n^{2/5}, n^{1/2}]$. If one waives Vaughan's identity and works with type II sums directly on the level of Diophantine equations, then prime detecting sieves typically narrow the range for type II sums. This should lead to an improvement of Theorem 1.1, and perhaps one can handle the case $k = 5$, $s = 8$ unconditionally in this way. The extra complications, however, would disguise the simplicity of new elements that we introduce to the circle method here, and we therefore refrain from elaborating on this idea in this paper.

Our methods apply equally well to questions concerning primes representable as sums of positive integral k-th powers. If $R(N) = R_{k,s}(N)$ denotes the number of solutions of the equation

$$p = x_1^k + \ldots + x_s^k \tag{1.3}$$

in primes $p \leqslant N$ and natural numbers x_j, then subject to the conditions in Theorems 1.1, 1.2, 1.3 or 1.4, it can be shewn that $R(N) \gg N^{s/k}(\log N)^{-1}$. Hardy and Littlewood [9] referred to this class of problems as conjugate to those defined by equation (1.1), and made them the subject of their conjectures M and N. It should be said, though, that the equation (1.3) seems to be somewhat easier than its counterpart (1.1). It has been known for centuries that primes of the form $4l + 1$ are the sum of two squares, yet the conjugate problem concerning the numbers that are the sum of two squares and a prime was solved by the generation preceding us. More recent developments with prime detecting sieves led to remarkable progress in this area. Friedlander and Iwaniec [7] showed that the polynomial $x^2 + y^4$ captures its primes. This should be viewed as a result concerning sums of two squares in which one of the variables is itself a square. Soon afterwards Heath-Brown [11] found infinitely many primes of the form $x^3 + 2y^3$, thereby confirming conjecture N of Hardy and Littlewood [9]. Most recently of all, Maynard [18] has considered primes represented by more general incomplete norm forms. These spectacular results depend, in some way or other, on the homogeneity of the polynomial on the right hand side of (1.3). It seems that there are significant obstacles preventing us from extracting lower bounds for $r_{3,3}(n)$, for example, along these lines.

The success of our approach depends on two innovations. The first is a new mean value estimate for moments of smooth Weyl sums over major arcs. Very recently, Liu and Zhao [17] obtained such estimates. Their method rests on the large sieve, and in consequence the width of an individual arc centered at a Farey fraction should be as small as $1/n$ (normalised for applications to equation (1.1)). Any inflation of this width implies an eternal loss, and this is typically not tolerable. In their work, the use of weights and the Poisson summation formula makes it possible to control losses, but in applications with primes, for example, this does not seem possible. In Lemma 2.3 below we describe a result in which the major arcs have their natural shape, and are therefore much wider than $1/n$ if the denominator of the Farey fraction at the center is small. Nonetheless, our estimate performs just as well as one can expect from [17, Lemma 5.6], but it is easier to use, and in some cases, and in particular in the situation considered in this paper, the wider arcs are essential for the success of the method. In contrast to [17, Lemma 5.6], our result neither depends on the large sieve, nor makes reference to Diophantine equations. Thus, again in contrast to the work of Liu and Zhao, we are able to handle fractional moments with ease, and we work with ordinary smooth Weyl sums, avoiding preseeded primes lying in certain intervals. This is important if one wishes to import results depending on breaking convexity devices. Therefore, our approach offers several advantages and extra flexibility. The new lemma has applications well beyond those presented in this paper. In fact, the argument leading to Lemma 2.3 is very direct and simple in spirit. As we shall demonstrate elsewhere, the ideas underpinning the proof can be further developed, and we defer to this future occasion a detailed account of the potential of the method.

Our new major arc mean value estimates provide a versatile pruning device. As we shall see in Section 5, the bounds provided by Lemma 2.3 are of strength sufficient to establish a version of Theorem 1.1 with an inflated value of c. We enhance the power of the new method with our second innovation, a novel large values technique. Drawing inspiration from an argument that occurs *en passant* in the first author's work on a certain quaternary additive problem [4], we explore the consequences of the stipulation that a Weyl sum is large, but not very large, through a comparison of various moments. This new method may be viewed as a pruning device for the minor arcs, and it transpires that in certain cases, it is possible to improve Weyl type bounds in mean over sets that we expect to be small. In applications of the circle method, this works just as if the Weyl bound would be better than currently known. Since the method ultimately rests on mean values, it also cooperates with the new major arc means, and then also helps with the more classical aspects of pruning on major arcs. The technical aspects of our new devices will be explained in the course of the argument, once the notational apparatus has been introduced. We refer the reader to the final part of Section 5, and the proof of Lemma 5.1 below, for details.

This paper is organised as follows. We begin with a discussion of the major arc mean values in Section 2. In Sections 3 and 4 we evaluate the major arcs in a circle method approach to the counting function $r_{k,s}$. This is largely standard. Then, in Section 5, we highlight the potential of major arc mean values for pruning. This section ends with the statement of Lemma 5.1 that in turn is dependent on the new large values technique. In Section 6 we present this as a pruning device on minor arcs, and in Section 7 in a catalytic role to enhance classical pruning. Having made the necessary preparations, the proof of our main theorems is presented in Section 8.

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2. Smooth Weyl sums and their means

In this section we discuss certain mean value estimates for smooth Weyl sums. Our discussion prominently features a certain transcendental function, which we now define. The function $\omega\colon (0,\infty)\to(0,\mathrm{e})$, defined by $t\mapsto \mathrm{e}^{1-t}$, is a strictly decreasing bijection, while the function $\Omega\colon (0,1)\to(0,\mathrm{e})$, defined by $u\mapsto u\,\mathrm{e}^u$, is a strictly increasing bijection. It follows that the equation $\mathrm{H}\,\mathrm{e}^{\mathrm{H}}=\mathrm{e}^{1-t}$ defines a smooth, strictly decreasing bijective function $\mathrm{H}:(0,\infty)\to(0,1)$. We then have

$$\mathrm{H}(t)+\log\mathrm{H}(t)=1-t, \tag{2.1}$$

and we may differentiate to infer that the relation

$$\mathrm{H}'(t)=-\mathrm{H}(t)/(1+\mathrm{H}(t)) \tag{2.2}$$

holds for all $t>0$. Later we require the following simple inequality.

Lemma 2.1. *If* $1\leqslant t\leqslant 3$, *then* $\mathrm{H}(t)>1/(4t-1)$.

Proof. Define $u(t)$ by means of the equation $\mathrm{H}(u(t))=1/(4t-1)$. We show that $u(t)>t$ for $1\leqslant t\leqslant 3$. Since H decreases, the desired inequality follows.

By (2.1), we have

$$\frac{1}{4t-1}-\log(4t-1)=1-u(t),$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t)-t)=\frac{4}{4t-1}+\frac{4}{(4t-1)^2}-1.$$

This function decreases on $[1,3]$ and is positive at $t=1$ but negative at $t=3$. Thus, the minimum of $u(t)-t$ on the interval $[1,3]$ occurs at $t=1$ or $t=3$. However, it is readily checked that $u(1)>1$ and $u(3)>3$, whence $u(t)-t$ is positive throughout $[1,3]$. □

When $1\leqslant R\leqslant P$, let $\mathscr{A}(P,R)$ denote the set of integers $n\in[1,P]$, all of whose prime divisors are at most R. Given an integer $k\geqslant 2$, let

$$f(\alpha;P,R)=\sum_{x\in\mathscr{A}(P,R)}e(\alpha x^k),\tag{2.3}$$

where, as usual, we write $e(z)$ to denote $\mathrm{e}^{2\pi iz}$. In this paper, we refer to the number Δ_t as an *admissible exponent* for the positive real number t if, for any fixed positive real number ε there exists a positive real number η such that, whenever $1\leqslant R\leqslant P^\eta$, one has

$$\int_0^1|f(\alpha;P,R)|^t\,\mathrm{d}\alpha\ll P^{t-k+\Delta_t+\varepsilon}.$$

Admissible exponents Δ_t are always non-negative, and there is no loss of generality in supposing throughout that $\Delta_t\leqslant k$. For large values of k the smallest known admissible exponents are due to Wooley [29, Theorem 3.2]. We reproduce a simplified, slightly weaker version of this conclusion, which appears in [30, Theorem 2.1], in the following lemma.

Lemma 2.2. *Let $k\geqslant 3$ be given. Then, whenever t is an even natural number, the exponent $k\mathrm{H}(t/k)$ is admissible.*

Our next lemma is our development of a related estimate of Liu and Zhao [17]. As pointed out in the introduction, it is indispensible in our approach to Theorem 1.1. From now on we consider $k\geqslant 3$ as fixed. Let Q be a real number with $1\leqslant Q\leqslant \frac12 P^{k/2}$, and let $\mathfrak{M}=\mathfrak{M}(Q)$ denote the union of the intervals

$$\mathfrak{M}(q,a;Q)=\{\alpha\in[0,1]:|q\alpha-a|\leqslant QP^{-k}\}$$

with $0\leqslant a\leqslant q\leqslant Q$ and $(a,q)=1$. Note that the intervals constituting this union are disjoint. Our goal is to estimate the mean value $V=V_t(P,R,Q)$ defined by

$$V_t(P,R,Q)=\int_{\mathfrak{M}(Q)}|f(\alpha;P,R)|^t\,\mathrm{d}\alpha.\tag{2.4}$$

Lemma 2.3. *Let $k\geqslant 3$ be given. Suppose that $t\geqslant k+1$ is a real number and let Δ_t be an admissible exponent for t. Then for each $\varepsilon>0$, there exists a positive number η with the property that whenever $1\leqslant R\leqslant P^\eta$ and $1\leqslant Q\leqslant\frac12 P^{k/2}$, one has the uniform bound*

$$V_t(P,R,Q)\ll P^{t-k+\varepsilon}Q^{2\Delta_t/k}.$$

Proof. We are at liberty to suppose throughout that ε and δ are small positive numbers satisfying $\varepsilon < 10^{-10k}$ and $3\delta k < \varepsilon$. We first dispose of the case where Q is very small. The measure of $\mathfrak{M}(Q)$ is $O(Q^2P^{-k})$, and therefore, the trivial bound $|f(\alpha; P, R)| \leqslant P$ implies that $V \ll Q^2P^{t-k}$. In particular, whenever $1 \leqslant Q \leqslant P^{k\delta/2}$, one has

$$V_t(P, R, Q) \ll P^{t-k+k\delta} \ll P^{t-k+\varepsilon}Q^{2\Delta_t/k}. \tag{2.5}$$

Next, we consider the situation in which $Q^2 > P^{k(1-\delta)}$. In this case, we note that $\mathfrak{M}(Q) \subset [0, 1]$. Thus, the definition of an admissible exponent shows that for some number η_0 with $0 < \eta_0 < \delta/2$, whenever $1 \leqslant R \leqslant P^{\eta_0}$, one has

$$V_t(P, R, Q) \ll P^{t-k+\Delta_t+\delta},$$

and hence

$$V_t(P, R, Q) \ll P^{t-k+\Delta_t(1-\delta)+2k\delta} \ll P^{t-k+\varepsilon}Q^{2\Delta_t/k}. \tag{2.6}$$

The upper bounds (2.5) and (2.6) confirm the conclusion of the lemma in all situations where either $Q^2 < P^{k\delta}$ or $Q^2 > P^{k(1-\delta)}$.

We now launch the main argument that addresses the remaining case, working under the assumption that

$$P^{k\delta} \leqslant Q^2 \leqslant P^{k(1-\delta)}. \tag{2.7}$$

With the parameter M at our disposal, we suppose that $2 \leqslant R \leqslant M \leqslant P$. Throughout, we reserve the letters p and π to denote prime numbers. Then, for each $p \leqslant R$, we define the modified set of smooth numbers

$$\mathscr{B}(M, p, R) = \{m \in \mathscr{A}(Mp, R) : m > M,\ p|m, \text{ and } \pi|m \text{ implies that } \pi \geqslant p\}.$$

By [23, Lemma 10.1], there is a bijection between the numbers $x \in \mathscr{A}(P, R)$ with $x > M$, and triples (p, m, y) with

$$p \leqslant R, \quad m \in \mathscr{B}(M, p, R), \quad y \in \mathscr{A}(P/m, p),$$

in which one has $x = my$. Applied to the exponential sum f defined in (2.3) this provides us with the decomposition

$$f(\alpha; P, R) = \sum_{p,m} f(\alpha m^k; P/m, p) + f(\alpha; M, R),$$

where, both here and in the sequel, the summation over p, m is intended as shorthand for one over $p \leqslant R$ and $m \in \mathscr{B}(M, p, R)$. We write $f(\alpha)$ for $f(\alpha; P, R)$ and $h_{p,m}(\gamma)$ for $f(\gamma; P/m, p)$. In addition, we denote by $\mathfrak{M}_q$ the union of the arcs $\mathfrak{M}(q, a; Q)$ with $0 \leqslant a \leqslant q$ and $(a, q) = 1$. Then, for $\alpha \in \mathfrak{M}_q$, we sort the sum over $m \in \mathscr{B}(M, p, R)$ according to the value of (q, m^k). Thus

$$|f(\alpha)| \leqslant \sum_{d|q} \sum_{\substack{p,m \\ (q,m^k)=d}} |h_{p,m}(\alpha m^k)| + M. \tag{2.8}$$

Taking the t-th power of (2.8), Hölder's inequality combines with the familiar divisor function estimate to give

$$|f(\alpha)|^t \ll q^\delta \sum_{d|q} \Bigl(\sum_{\substack{p,m \\ (q,m^k)=d}} |h_{p,m}(\alpha m^k)| \Bigr)^t + M^t. \tag{2.9}$$

The integer d has a unique factorisation $d = d_1 d_2^2 \cdots d_k^k$ with $d_1 d_2 \cdots d_{k-1}$ square-free. In this notation, we write $d_0 = d_1 d_2 \cdots d_k$. Thus $d|m^k$ if and only if $d_0|m$. By Hölder's inequality again,

$$\Bigl(\sum_{\substack{p,m \\ (q,m^k)=d}} |h_{p,m}(\alpha m^k)| \Bigr)^t \leqslant \Bigl(\sum_{\substack{p,m \\ d_0|m}} 1 \Bigr)^{t-1} \sum_{\substack{p,m \\ (q,m^k)=d}} |h_{p,m}(\alpha m^k)|^t. \tag{2.10}$$

We integrate (2.9) over $\mathfrak{M}_q$. For this we require the mean value

$$J(q,p,m) = \int_{\mathfrak{M}_q} |h_{p,m}(\alpha m^k)|^t \,\mathrm{d}\alpha.$$

The first sum on the right hand side of (2.10) is no larger than MR^2/d_0. Moreover, the measure of $\mathfrak{M}_q$ does not exceed $2QP^{-k}$. Then (2.9) yields

$$\int_{\mathfrak{M}_q} |f(\alpha)|^t \,\mathrm{d}\alpha \ll q^\delta \sum_{d|q} \Bigl(\frac{MR^2}{d_0} \Bigr)^{t-1} \sum_{\substack{p,m \\ (q,m^k)=d}} J(q,p,m) + M^t Q P^{-k}. \tag{2.11}$$

Temporarily, we consider q,p,m as fixed and abbreviate $h_{p,m}$ to h. Also, when $1 \leqslant Z \leqslant P$, we introduce notation to better handle the intervals $I_q(Z) = I_q(Z;Q)$ of interest, writing

$$I_q(Z;Q) = [-Q/(qZ^k), Q/(qZ^k)].$$

Equipped with this notation, if we unfold the definition of $\mathfrak{M}_q$ and apply a change of variables, we see that the mean value $J(q,p,m)$ is equal to

$$\int_{I_q(P)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Bigl| h\Bigl(\Bigl(\frac{a}{q} + \beta \Bigr) m^k \Bigr) \Bigr|^t \,\mathrm{d}\beta = \frac{1}{m^k} \int_{I_q(P/m)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Bigl| h\Bigl(\frac{am^k}{q} + \gamma \Bigr) \Bigr|^t \,\mathrm{d}\gamma.$$

Within (2.11) we need the above relation only when $(q, m^k) = d$. In the latter circumstances one has $(m^k/d, q/d) = 1$, and hence

$$\begin{aligned} J(q,p,m) &\leqslant \frac{1}{m^k} \int_{I_q(P/m)} d \sum_{\substack{a=1 \\ (a,q/d)=1}}^{q/d} \Bigl| h\Bigl(\frac{am^k/d}{q/d} + \gamma \Bigr) \Bigr|^t \,\mathrm{d}\gamma \\ &= \frac{d}{m^k} \sum_{\substack{b=1 \\ (b,q/d)=1}}^{q/d} \int_{I_q(P/m)} \Bigl| h\Bigl(\frac{b}{q/d} + \gamma \Bigr) \Bigr|^t \,\mathrm{d}\gamma. \end{aligned}$$

We apply this formula within (2.11) and sum over $q \leqslant Q$ to obtain the bound

$$\int_{\mathfrak{M}(Q)} |f(\alpha)|^t \,\mathrm{d}\alpha \ll M^t Q^2 P^{-k} + Q^\delta (MR^2)^{t-1}\Xi, \tag{2.12}$$

where

$$\Xi = \sum_{q\leqslant Q}\sum_{d|q}\sum_{\substack{p,m\\ (q,m^k)=d}} \frac{d}{d_0^{t-1}m^k}\sum_{\substack{b=1\\ (b,q/d)=1}}^{q/d}\int_{I_q(P/m)}\Big|h\Big(\frac{b}{q/d}+\gamma\Big)\Big|^t \,\mathrm{d}\gamma.$$

The quantity Ξ may be bounded by first writing $q = dr$, and then replacing the condition $(q, m^k) = d$ by the weaker constraint $d_0|m$. Thus, we find that

$$\Xi \leqslant \sum_{dr\leqslant Q}\sum_{\substack{p,m\\ d_0|m}} \frac{d}{d_0^{t-1}m^k}\sum_{\substack{b=1\\ (b,r)=1}}^{r}\int_{I_{dr}(P/m)}\Big|h\Big(\frac{b}{r}+\gamma\Big)\Big|^t \,\mathrm{d}\gamma. \tag{2.13}$$

We choose M via the relation $(MR)^k = P^k/(3Q^2)$, whence by (2.7) one has

$$P^\delta \ll MR \ll P^{1-\delta}. \tag{2.14}$$

In particular, this choice for M is admissible and for all pairs p, m occurring in the second sum of (2.13), we have $Q^2(P/m)^{-k} < \frac{1}{2}$. It is immediate from this inequality that the intervals $I_{dr}(P/m) + b/r$, with $1 \leqslant b \leqslant r \leqslant Q/d$ and $(b, r) = 1$, are pairwise disjoint, and that all of these sets are contained in the unit interval

$$\left[\frac{Qm^k}{drP^k}, 1 + \frac{Qm^k}{drP^k}\right].$$

We therefore conclude that

$$\sum_{r\leqslant Q/d}\sum_{\substack{b=1\\ (b,r)=1}}^{r}\int_{I_{dr}(P/m)}\Big|h\Big(\frac{b}{r}+\gamma\Big)\Big|^t \,\mathrm{d}\gamma \leqslant \int_0^1 |h_{p,m}(\alpha)|^t \,\mathrm{d}\alpha.$$

But from (2.14) we have $P/m \geqslant P/(MR) \gg P^\delta$. Since $p \leqslant R$ here, it follows from the definition of an admissible exponent that there is a number $\eta > 0$ such that, uniformly in $2 \leqslant R \leqslant P^\eta$ and all p, m in their ranges of summation,

$$\int_0^1 |h_{p,m}(\alpha)|^t \,\mathrm{d}\alpha \ll (P/m)^{t-k+\Delta_t+\delta}.$$

The exponent on the right hand side is positive, so we infer from (2.13) that

$$\Xi \ll \Big(\frac{P}{M}\Big)^{t-k+\Delta_t+\delta}\sum_{d\leqslant Q}\sum_{\substack{p,m\\ d_0|m}} \frac{d}{d_0^{t-1}m^k}. \tag{2.15}$$

One has

$$\sum_{p\leqslant R}\sum_{\substack{m\in\mathscr{B}(M,p,R)\\ d_0|m}} \frac{1}{m^k} \leqslant \sum_{p\leqslant R}\sum_{M/d_0<u\leqslant Mp/d_0} \frac{1}{(d_0u)^k} \ll \frac{R}{d_0M^{k-1}}.$$

Moreover, when $t \geqslant k+1$, one has the simple bound

$$\sum_{d \leqslant Q} \frac{d}{d_0^t} \ll \sum_{d_1 d_2 \cdots d_k \leqslant Q} \frac{1}{d_1 d_2 \cdots d_k} \ll (\log(2Q))^k.$$

Recall that our choice for Q satisfies $(P/M)^k = 3Q^2 R^k$ and $Q \leqslant P^{k/2}$. Then on noting that $MR \gg P^\delta$, it follows from (2.15) that

$$\begin{aligned} \Xi &\ll (P/M)^{t-k+\delta} Q^{2\Delta_t/k} R^{1+\Delta_t} M^{1-k} (\log(2Q))^k \\ &\ll P^{t-k+\delta} Q^{2\Delta_t/k} M^{1-t} R^{2+\Delta_t}. \end{aligned}$$

We may always assume that $\Delta_t \leqslant k < t$, and thus we infer from (2.4) and (2.12), together with the definition of M, the upper bound

$$V_t(P,R,Q) \ll P^{t-k+\delta k} Q^{2\Delta_t/k} R^{3t}. \tag{2.16}$$

Since we are free to diminish the number η implicit in the proof of (2.16), we may arrange that $3t\eta < \delta k$ and $\eta \leqslant \eta_0$, and hence that $\delta k + 3t\eta < \varepsilon$. Consequently, in this final case in which the constraint (2.7) is satisfied, it follows from (2.16) that

$$V_t(P,R,Q) \ll P^{t-k+\delta k+3t\eta} Q^{2\Delta_t/k} \ll P^{t-k+\varepsilon} Q^{2\Delta_t/k},$$

yielding the desired conclusion once more. □

3. The circle method

Our next task is to set up the environment for the proofs of the theorems. The exponent $k \geqslant 3$ is still fixed, and we initially impose the condition $s \geqslant 1$. We gradually import more conditions on s and the smoothness parameter as the argument progresses. Our leading parameter is the number n in (1.1), and we take

$$P = n^{1/k}. \tag{3.1}$$

We adumbrate $f(\alpha; P, R)$ to $f(\alpha)$ and introduce the sum

$$g(\alpha) = \sum_{p \leqslant n} e(\alpha p) \log p.$$

Whenever $\mathfrak{A} \subset [0,1]$ is measurable, we write

$$I(n, \mathfrak{A}) = \int_{\mathfrak{A}} g(\alpha) f(\alpha)^s e(-\alpha n)\, \mathrm{d}\alpha \tag{3.2}$$

and abbreviate $I(n,[0,1])$ to $I(n)$. By orthogonality, the integral counts certain solutions of (1.1) with weight $\log p$. Consequently,

$$r_{k,s}(n) \log n \geqslant I(n). \tag{3.3}$$

The arguments in this section are independent of the theory of admissible exponents. We therefore choose $R = P^\delta$, where $0 < \delta \leqslant 1$ remains at our disposal. It is convenient also to write $\mathscr{Q} = (\log n)^{1/99}$. We begin by extracting a lower bound from the core major arcs $\mathfrak{N}$ that we define as the union of the intervals

$$\mathfrak{N}(q,a) = \{\alpha \in [0,1] : |\alpha - a/q| \leqslant \mathscr{Q} n^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant \mathscr{Q}$ and $(a,q) = 1$. The intervals in this union are again disjoint. Let α be in one of these intervals, say $\mathfrak{N}(q,a)$ with $(a,q) = 1$. On recalling (3.1), arguments that are by now standard in the theory of smooth Weyl sums (see [23, Lemma 5.4]) show in this scenario that there is a positive number $\rho = \rho(\delta)$ such that

$$f(\alpha) = \rho q^{-1} S(q,a) v_k(\alpha - a/q) + O(n^{1/k}(\log n)^{-1/4}),$$

wherein

$$S(q,a) = \sum_{x=1}^{q} e(ax^k/q) \quad \text{and} \quad v_k(\beta) = \frac{1}{k}\sum_{m\leqslant n} m^{-1+1/k} e(\beta m).$$

We take the s-th power and combine the result with Lemma 3.1 of Vaughan [24]. After integrating over $\mathfrak{N}$ we then routinely obtain the asymptotic relation

$$I(n,\mathfrak{N}) = \rho^s \mathfrak{J}(n,\mathscr{Q})\mathfrak{S}(n,\mathscr{Q}) + O(n^{s/k}(\log n)^{-1/5}), \tag{3.4}$$

where, for $1 \leqslant X \leqslant n/2$, we write

$$\mathfrak{J}(n,X) = \int_{-X/n}^{X/n} v_1(\beta) v_k(\beta)^s e(-\beta n)\,\mathrm{d}\beta$$

and

$$\mathfrak{S}(n,X) = \sum_{q\leqslant X} \frac{\mu(q)}{q^s \phi(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)^s e(-an/q). \tag{3.5}$$

By orthogonality, it is immediate that $\mathfrak{J}(n,n/2) \gg n^{s/k}$ (see, for example, the proof of [24, Theorem 2.3]). Moreover, we find via [24, Lemma 2.8] that

$$\mathfrak{J}(n,X) - \mathfrak{J}(n,n/2) \ll n^{s/k} X^{-s/k}.$$

Thus we see that

$$\mathfrak{J}(n,\mathscr{Q}) \gg n^{s/k}. \tag{3.6}$$

Meanwhile, the sum

$$S_n(q) = q^{-s} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)^s e(-an/q)$$

is multiplicative (see [24, Lemma 2.11]), and the presence of the factor $\mu(q)$ in (3.5) reduces our task to bounding $S_n(q)$ when q is a prime. In these circumstances, whenever $s \geqslant 3$, we see from [24, Lemma 4.3] that the bound $S_n(q) \ll q^{-1/2}$ holds uniformly in n. This in turn implies that the series

$$\mathfrak{S}(n) = \lim_{X\to\infty} \mathfrak{S}(n,X)$$

converges absolutely, and that

$$\mathfrak{S}(n) - \mathfrak{S}(n,X) \ll X^{\varepsilon - 1/2}. \tag{3.7}$$

This bound again holds uniformly in n.

Next, we transform $\mathfrak{S}(n)$ into an Euler product $\prod_p \chi_p(n)$, where

$$\chi_p(n) = 1 - (p-1)^{-1} S_n(p) = 1 + O(p^{-3/2}),$$

the last relation holding uniformly in n. Since -1 is equal to the value of the Ramanujan sum $c_p(a)$ for $1 \leqslant a \leqslant p-1$, we find that

$$\begin{aligned}\chi_p(n) &= 1 + \frac{1}{p^s(p-1)} \sum_{a=1}^{p-1} c_p(a) S(p,a)^s e(-an/p) \\ &= \frac{1}{p^s(p-1)} \sum_{a=1}^{p} c_p(a) S(p,a)^s e(-an/p).\end{aligned}$$

Thus, by orthogonality, we have $\chi_p(n) = p^{1-s}(p-1)^{-1} M_p(n)$, where $M_p(n)$ is the number of incongruent solutions of the congruence

$$b + x_1^k + \ldots + x_s^k \equiv n \pmod{p},$$

with $(b,p) = 1$. It is immediate that the lower bound $M_p(n) \geqslant 1$ holds for all n. Hence all factors $\chi_p(n)$ are real with $\chi_p(n) \geqslant p^{-s}$. From this discussion we also see that there is a number p_0 such that the superior lower bound $\chi_p(n) \geqslant 1 - p^{-5/4}$ holds for all n whenever $p \geqslant p_0$. We therefore conclude routinely that $\mathfrak{S}(n) \gg 1$. If we combine this lower bound with (3.4), (3.6) and (3.7), we may conclude as follows.

Lemma 3.1. *Let $k \geqslant 3$ and $s \geqslant 3$. Suppose that $0 < \delta \leqslant 1$, and put $R = n^{\delta/k}$. Then one has $I(n, \mathfrak{N}) \gg n^{s/k}$.*

A slightly weaker version of this lemma remains valid even when $s = 2$. In fact, in this case, the bound $S_n(p) \ll p^{-3/2}$ remains valid for all primes $p \nmid n$. This can been seen by working along the lines of [24, Lemma 4.7]. Further, with more care one can show that $\chi_p(n) \geqslant 1 - k/p$ for primes $p|n$ with $p > k^2$. As this is not used later, we leave the details required to justify these claims to the reader. These bounds show that when $s = 2$, one has

$$\mathfrak{S}(n) \gg \prod_{\substack{p|n \\ p>k^2}} (1 - kp^{-1}) \gg (\log\log n)^{-k},$$

whence $I(n, \mathfrak{N}) \gg n^{s/k}(\log\log n)^{-k}$. This supports our claim in the introduction that all large integers should be the sum of a prime and two k-th powers.

4. Pruning near the root

Our goal in this section is to enlarge the major arcs to the set $\mathfrak{L} = \mathfrak{M}(P^{1/5})$. The choice of height $P^{1/5}$ here is arbitrary, for any small power of P would suffice. As is often the case with competitive applications of smooth Weyl sums, certain estimates are diluted by unwanted factors P^ε. In such circumstances, pruning to the root needs a separate argument that we now present.

We begin by introducing some notation. Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $0 \leqslant a \leqslant q \leqslant \frac{1}{2}\sqrt{n}$ and $(a,q) = 1$. Then, the intervals $\mathfrak{M}(q, a; \frac{1}{2}\sqrt{n})$ are disjoint, and for $\alpha \in \mathfrak{M}(q, a; \frac{1}{2}\sqrt{n})$ we put

$$\Upsilon(\alpha) = (q + n|q\alpha - a|)^{-1}.$$

Meanwhile, for $\alpha \in [0,1] \setminus \mathfrak{M}(\frac{1}{2}\sqrt{n})$ we put $\Upsilon(\alpha) = 0$. This defines a function $\Upsilon : [0,1] \to [0,1]$. In what follows, in the interest of brevity we put

$$L = \log n.$$

Lemma 4.1. *Suppose that $2 \leqslant R \leqslant P^{1/7}$ and $\varepsilon > 0$. Then, uniformly for $\alpha \in \mathfrak{L}$ one has*

$$f(\alpha) \ll PL^3 \Upsilon(\alpha)^{1/(2k)-\varepsilon}.$$

Moreover, when $B > 0$ and $\alpha \in \mathfrak{M}(L^B)$, one has

$$f(\alpha) \ll P\Upsilon(\alpha)^{1/k-\varepsilon}.$$

Proof. The first bound follows by applying [25, Lemma 7.2] with $M = P^{2/3}$. The second bound, on the other hand, is a consequence of [25, Lemma 8.5]. □

Lemma 4.2. *Let $\alpha \in [0,1]$. Then*

$$g(\alpha) \ll \left(n\Upsilon(\alpha)^{1/2} + n^{4/5}\right) L^4. \tag{4.1}$$

Furthermore, should GRH be true, then

$$g(\alpha) \ll (n\Upsilon(\alpha)^{1/2} + n^{3/4})L^2. \tag{4.2}$$

Proof. Both bounds (4.1) and (4.2) are certainly familiar, but perhaps not in this form, and in particular the latter is not easily found in the literature. We therefore provide details for the upper bound (4.2). Let

$$g(\alpha, \nu) = \sum_{p \leqslant \nu} e(\alpha p) \log p.$$

If we suppose that GRH holds, and $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ are coprime, then one finds from [6, Lemma 2], for example, that

$$g(a/q, \nu) \ll (\nu\phi(q)^{-1} + \sqrt{\nu q})(\log \nu)^2. \tag{4.3}$$

Note that $g(\alpha) = g(\alpha, n)$. By partial summation, for all $\beta \in \mathbb{R}$, one has

$$\begin{aligned} |g(\beta + a/q)| &\leqslant |g(a/q)| + 2\pi|\beta| \int_2^n |g(a/q, \nu)| \,\mathrm{d}\nu \\ &\ll (n\phi(q)^{-1} + \sqrt{nq})(1 + n|\beta|)L^2. \end{aligned} \tag{4.4}$$

In order to obtain the bound (4.2), first suppose that $\alpha \in [0,1] \setminus \mathfrak{M}(\frac{1}{2}\sqrt{n})$. Apply Dirichlet's theorem on Diophantine approximation to find integers b and r with $1 \leqslant r \leqslant 2\sqrt{n}$, $(b,r) = 1$ and $|r\alpha - b| \leqslant 1/(2\sqrt{n})$. Then the hypothesis $\alpha \notin \mathfrak{M}(\frac{1}{2}\sqrt{n})$ implies that $r > \frac{1}{2}\sqrt{n}$, and thus we infer from (4.4) that

$$g(\alpha) \ll \left(\frac{n}{\phi(r)} + \sqrt{nr}\right)\left(1 + \frac{\sqrt{n}}{r}\right) L^2 \ll n^{3/4} L^2,$$

as required in (4.2).

Next suppose that $\alpha \in \mathfrak{M}(\frac{1}{2}\sqrt{n})$. For $\frac{1}{2} \leqslant Y \leqslant \frac{1}{4}\sqrt{n}$, we write

$$\mathfrak{P}(Y) = \mathfrak{M}(2Y) \setminus \mathfrak{M}(Y). \tag{4.5}$$

Then since $\mathfrak{M}(\frac{1}{2}\sqrt{n})$ is the union of the sets $\mathfrak{P}(Y)$, there exists a choice for Y in this range with $\alpha \in \mathfrak{P}(Y)$. Note that for α in the latter set, the bound

(4.2) asserts that $g(\alpha) \ll nY^{-1/2}L^2$. To prove this bound, we again apply Dirichlet's theorem to find integers b and r with $1 \leqslant r \leqslant n/Y$, $(b,r)=1$ and $|r\alpha - b| \leqslant Y/n$. We may suppose in this instance that $\alpha \notin \mathfrak{M}(Y)$, and hence that $r > Y$, and thus (4.4) yields the desired bound

$$g(\alpha) \ll \Big(\frac{n}{\phi(r)} + nY^{-1/2}\Big)\Big(1+\frac{Y}{r}\Big)L^2 \ll nY^{-1/2}L^2.$$

The proof of the bound (4.2) is now complete. The proof of (4.1) follows in the same way, save that the initial estimate (4.3) has to be replaced by Vinogradov's unconditional bound [24, Theorem 3.1]. □

We are now well prepared for the pruning.

Lemma 4.3. *Suppose that $s \geqslant k+3$ and $2 \leqslant R \leqslant P^{1/7}$. Then*

$$I(n, \mathfrak{L} \setminus \mathfrak{N}) \ll n^{s/k}(\log n)^{-1/(50k)}.$$

Proof. We proceed in two steps. We first note that an enhanced version (see [20, Lemma 11.1]) of the first author's pruning technique [2, Lemma 2] yields

$$\int_{\mathfrak{L}} \Upsilon(\alpha)^{1+1/(6k)}|f(\alpha)|^2 \,\mathrm{d}\alpha \ll P^2 n^{-1}.$$

By (4.1) and Lemma 4.1, we have

$$g(\alpha)f(\alpha)^{k+1} \ll P^{k+1}n\Upsilon(\alpha)^{1+1/(3k)}L^{7+3k}.$$

Take $B = 6k(8+3k)$. Then, for $\alpha \in \mathfrak{L}\setminus\mathfrak{M}(L^B)$, one has $\Upsilon(\alpha) \ll L^{-B}$, whence

$$I(n, \mathfrak{L}\setminus\mathfrak{M}(L^B)) \ll L^{-1}P^{s-2}n \int_{\mathfrak{L}} \Upsilon(\alpha)^{1+1/(6k)}|f(\alpha)|^2 \,\mathrm{d}\alpha \ll P^s L^{-1}. \tag{4.6}$$

Since $\phi(q) \gg q/\log\log q$, for $\alpha \in \mathfrak{M}(L^B)$, one finds from [24, Lemma 3.1] that

$$g(\alpha) \ll n\Upsilon(\alpha)\log L + nL^{-2B} \ll n\Upsilon(\alpha)\log L.$$

Observe that when $\alpha \in [0,1]\setminus\mathfrak{N}$, one has $\Upsilon(\alpha) \ll L^{-1/99}$. Consequently, by making use of the second estimate of Lemma 4.1, we discern that whenever $\alpha \in \mathfrak{M}(L^B)\setminus\mathfrak{N}$, then

$$g(\alpha)f(\alpha)^s \ll P^s n\Upsilon(\alpha)^{2+1/k}L^{-1/(50k)}.$$

We may therefore integrate routinely to conclude that

$$I(n, \mathfrak{M}(L^B)\setminus\mathfrak{N}) \ll P^s L^{-1/(50k)}. \tag{4.7}$$

The proof of the lemma is completed by collecting together (4.6) and (4.7). □

5. Interlude

Thus far our evaluation of $r_{k,s}(n)$ has followed a well trodden path. Before embarking on the original aspects of our treatment, we pause to outline a simple argument leading to weaker versions of Theorems 1.1 and 1.3, with inflated numerical values for the constants c and c'. This approach rests on Lemma 2.3 alone, and is therefore very flexible. It applies also in other contexts.

From now on, in the remainder of this paper, we apply the following conventions concerning the symbols ε, η and R. If a statement involves the letter ε, then it it is asserted that the statement holds for any positive real number assigned to ε. Implicit constants stemming from Vinogradov or Landau symbols may depend on ε. If a statement also involves the letter R, either implicitly or explicitly, then it is asserted that for any $\varepsilon > 0$ there is a number $\eta > 0$ such that the statement holds uniformly for $2 \leqslant R \leqslant P^\eta$. This will be imported to our arguments through applications of Lemmata 2.2 and 2.3 only. We shall call upon these lemmata only finitely often, and we may therefore pass to the smallest of the numbers η that arise in this way, and then have all estimates with the same positive η in hand.

The Farey dissection that we shall use takes $\mathfrak{K} = \mathfrak{M}(n^{2/5})$ as the set of major arcs, and $\mathfrak{k} = [0,1] \setminus \mathfrak{K}$ as the complementary set of minor arcs. Let c_1 be the positive real number with $\mathrm{H}(c_1) = \frac{1}{5}$. Note from (2.1) that

$$c_1 = \frac{4}{5} + \log 5 = 2.409437\ldots.$$

Let $s_1 = s_1(k)$ be the smallest even integer with $s_1 > c_1 k$. Then, for $s \geqslant s_1$ we have $\mathrm{H}(s/k) < \frac{1}{5}$. By Lemma 2.2 and the definition of an admissible exponent, one sees that for some positive number δ, one has

$$\int_0^1 |f(\alpha)|^s \,\mathrm{d}\alpha \ll P^s n^{-4/5-\delta}. \tag{5.1}$$

Also, as a consequence of Lemma 2.3, whenever $1 \leqslant Q \leqslant \frac{1}{2}\sqrt{n}$, we have

$$\int_{\mathfrak{M}(Q)} |f(\alpha)|^s \,\mathrm{d}\alpha \ll P^s n^{\varepsilon-1} Q^{2/5-\delta}. \tag{5.2}$$

Note here that our conventions concerning the use of ε and R apply. In particular, we may choose $R = P^\eta$ and then the upper bounds (5.1) and (5.2) both hold provided that $\eta > 0$ is sufficiently small. We fix this choice of R now.

We begin by establishing a version of Theorem 1.1. Observe first that when $\alpha \in \mathfrak{k}$, then as a consequence of Dirichlet's approximation theorem, whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, one has $q + n|q\alpha - a| > n^{2/5}$. Hence, by Lemma 4.2, we have $g(\alpha) \ll n^{4/5+\varepsilon}$. We thus deduce from (3.2) and (5.1) that

$$I(n, \mathfrak{k}) \ll P^s n^{-\delta/2}. \tag{5.3}$$

Next recall (4.5) and apply a dyadic dissection to cover the set $\mathfrak{K}\setminus\mathfrak{L}$ by $O(\log n)$ sets $\mathfrak{P}(Q)$, with $P^{1/5} \leqslant Q \leqslant \frac{1}{2}n^{2/5}$. For $\alpha \in \mathfrak{P}(Q)$, we infer from Lemma 4.2

that $g(\alpha) \ll n^{1+\varepsilon}Q^{-1/2}$. Hence, by (3.2) and (5.2), we discern that

$$I(n, \mathfrak{P}(Q)) \ll P^s Q^{-1/10}.$$

Collecting together the contributions from these dyadic intervals, we see that

$$I(n, \mathfrak{K} \setminus \mathfrak{L}) \ll P^s n^{-1/(60k)}. \tag{5.4}$$

Thus, on recalling Lemmata 3.1 and 4.3, we deduce from (5.3) and (5.4) that

$$I(n) = I(n, \mathfrak{k}) + I(n, \mathfrak{K} \setminus \mathfrak{L}) + I(n, \mathfrak{L} \setminus \mathfrak{N}) + I(n, \mathfrak{N}) \gg n^{s/k}.$$

In view of (3.3), we therefore have $r_{k,s}(n) \gg n^{s/k}(\log n)^{-1}$ whenever $s \geqslant c_1 k+1$, thereby delivering a version of Theorem 1.1 with c_1 in place of c.

If GRH is true we adjust the Farey dissection to one comprising the sets $\mathfrak{K}' = \mathfrak{M}(\frac{1}{2}\sqrt{n})$ and $\mathfrak{k}' = [0,1] \setminus \mathfrak{K}'$. Also, we cover the set $\mathfrak{K}' \setminus \mathfrak{L}$ by $O(\log n)$ sets $\mathfrak{P}(Q)$, with $P^{1/5} \leqslant Q \leqslant \frac{1}{4}\sqrt{n}$. In this scenario we take

$$c_1' = \frac{3}{4} + \log 4 = 2.136294\ldots,$$

so that in view of (2.1), one has $\mathrm{H}(c_1') = \frac{1}{4}$. Let $s_1' = s_1'(k)$ be the smallest even integer with $s_1' > c_1' k$. The preceding argument may now be based on (4.2) rather than (4.1), and then shows that $r_{k,s}(n) \gg n^{s/k}(\log n)^{-1}$ whenever $s \geqslant c_1' k + 1$, yielding a version of Theorem 1.3 with c_1' in place of c'. It is perhaps of interest to note that even this conditional conclusion is improved on by Theorem 1.1.

Applications of the Hardy-Littlewood method to the problem at hand start with a minor arc analysis associated with the set $[0,1] \setminus \mathfrak{M}(Q)$, with a choice for Q large enough that estimates of Weyl-type are applicable. Traditional pruning arguments attempt to reduce the size parameter Q so that the associated complementary set of major arcs $\mathfrak{M}(Q)$ becomes workable. Ideally, this parameter Q should not be very large, and indeed $Q = P^{1/5}$ in the set $\mathfrak{L}$ occurring in our argument above. We refer to Q as the *height* of the major arcs, and a pruning argument that reduces Q is referred to as *pruning by height*. There are several models for this kind of argument where one would typically combine pointwise bounds for some generating functions with mean values of others. One standard tool is [2, Lemma 2]. It transpires that Lemma 2.3 is a particularly powerful technique for pruning by height because it is initially based on mean values alone.

In the next section we introduce another new pruning device. In its pure form it is a minor arc technique. We propose a dissection into level sets for several generating functions. If a generating function is very large, then inequalities of Weyl's type will tell us that we are on major arcs. We shall then explore via mean values the proposition that a generating function is large but not very large. In favourable circumstances one ends up with results that have, for the problem at hand, the same effect as an improvement on the Weyl bounds. We refer to this process as *pruning by size*. In our application, the new method performs so well that pure pruning by height on the major arcs would become the bottleneck for the argument. Hence, in the section following the next one, we describe an argument where pruning by height is enhanced by some of the features of pruning by size. It is interesting to note already at this stage that the success of pruning by size and by height ultimately depends on the same inequalities for certain admissible exponents. It is possible to announce the principal conclusion in a form that absorbs potential improvements on admissible exponents easily. This concerns the contribution from the set

$$\mathfrak{l} = [0,1] \setminus \mathfrak{L}.$$

Lemma 5.1. *Let $k \geqslant 3$. Suppose that s and t are real numbers with $0 \leqslant t \leqslant s$, and let Δ_s and Δ_{s+t} be admissible exponents. Should GRH be true, set $\theta = 4$, and otherwise put $\theta = 5$. Then, provided that*

$$2\Delta_s < k \quad \textit{and} \quad \frac{t}{s} + \frac{\theta \Delta_{s+t}}{k} < 1, \tag{5.5}$$

there exists a positive number η such that, uniformly for $2 \leqslant R \leqslant P^\eta$, one has

$$\int_{\mathfrak{l}} |g(\alpha) f(\alpha)^s| \, \mathrm{d}\alpha \ll P^s (\log n)^{-1}.$$

6. Pruning by size

We prove Lemma 5.1 in two steps. In this section we deal with the minor arcs $\mathfrak{k}$ and, subject to GRH, the alternative minor arcs $\mathfrak{k}'$. Throughout we suppose that the hypotheses in Lemma 5.1 are met, with θ depending on and also determining the case under consideration.

We begin by removing from the minor arcs the set

$$\mathscr{T} = \{\alpha \in \mathfrak{k} : |g(\alpha)| \leqslant \sqrt{n}\},$$

where $g(\alpha)$ is tiny. By (5.5) and the definition of an admissible exponent, there is a number $\delta > 0$ with

$$\int_0^1 |f(\alpha)|^s \, \mathrm{d}\alpha \ll P^s n^{-1/2-\delta},$$

whence

$$\int_{\mathscr{T}} |g(\alpha) f(\alpha)^s| \, \mathrm{d}\alpha \ll P^s n^{-\delta}. \tag{6.1}$$

Next, let U be a parameter with $1 \leqslant U \leqslant \sqrt{n}$, and define the level set

$$\mathscr{L}(U) = \{\alpha \in [0,1] : n/U \leqslant |g(\alpha)| \leqslant 2n/U\}. \tag{6.2}$$

In the interest of brevity we revive the use of $L = \log n$. Also, let $U_0 = n^{1/\theta}$ and note that whenever $U \leqslant U_0 L^{-5}$, one must have $\alpha \in \mathfrak{K}$ (if $\theta = 5$), or $\alpha \in \mathfrak{K}'$ (if $\theta = 4$), at least for large n. This follows from Lemma 4.2, which shows that in these respective cases one has

$$\sup_{\alpha \in \mathfrak{k}} |g(\alpha)| \ll n^{4/5} L^4 \quad \text{and} \quad \sup_{\alpha \in \mathfrak{k}'} |g(\alpha)| \ll n^{3/4} L^2.$$

Consequently, we may cover the set $\mathfrak{k} \setminus \mathscr{T}$ by $O(L)$ sets $\mathscr{L}(U)$ with

$$n^{1/5} L^{-5} \leqslant U \leqslant \sqrt{n}. \tag{6.3}$$

Likewise, the set $\mathfrak{k}'$ is the union of $\mathscr{T}$ and $O(L)$ sets $\mathscr{L}(U)$ with

$$n^{1/4}L^{-3} \leqslant U \leqslant \sqrt{n}. \tag{6.4}$$

Hence, we find via (6.1) that there is a number U satisfying (6.3) for which

$$\int_{\mathfrak{k}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^s n^{-\delta} + L\int_{\mathscr{L}(U)} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha. \tag{6.5}$$

This bound also holds with $\mathfrak{k}'$ in place of $\mathfrak{k}$, but then U lies in the narrower interval (6.4).

By orthogonality and Chebychev's bound,

$$\int_{\mathscr{L}(U)} |g(\alpha)|\,\mathrm{d}\alpha \leqslant \frac{U}{n}\int_0^1 |g(\alpha)|^2\,\mathrm{d}\alpha \leqslant UL. \tag{6.6}$$

This suggests the strategy of splitting the set $\mathscr{L}(U)$ into the subsets

$$\mathscr{G} = \{\alpha \in \mathscr{L}(U) : |f(\alpha)|^s \leqslant P^s U^{-1}L^{-3}\},$$
$$\mathscr{H} = \{\alpha \in \mathscr{L}(U) : |f(\alpha)|^s > P^s U^{-1}L^{-3}\},$$

since by (6.6), we have

$$\int_{\mathscr{G}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^s L^{-2}. \tag{6.7}$$

This leaves the set $\mathscr{H}$ for further discussion.

For $\alpha \in \mathscr{H}$ one has $|f(\alpha)|^t > P^t(UL^3)^{-t/s}$. Hence

$$\int_{\mathscr{H}} |f(\alpha)|^s\,\mathrm{d}\alpha < P^{-t}U^{t/s}L^3 \int_0^1 |f(\alpha)|^{s+t}\,\mathrm{d}\alpha.$$

Recalling our convention concerning the use of ε, we deduce via (6.2) that

$$\int_{\mathscr{H}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll U^{t/s-1}P^{s+\Delta_{s+t}+\varepsilon}. \tag{6.8}$$

However, we have $U \geqslant U_0L^{-5}$, and furthermore $t/s-1$ is negative. Then by the second inequality in (5.5) we find that there is a positive number δ with

$$\int_{\mathscr{H}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll U_0^{t/s-1}P^{s+\Delta_{s+t}+\varepsilon} \ll P^{s-\delta}.$$

We now combine this bound with (6.5) and (6.7) and arrive at the final estimate

$$\int_{\mathfrak{k}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^s L^{-1}, \tag{6.9}$$

valid when $\theta = 5$, with the analogue for $\theta = 4$ holding with $\mathfrak{k}'$ in place of $\mathfrak{k}$ subject to GRH.

7. Pruning by height

We treat the intermediate arcs $\mathfrak{K}\setminus\mathfrak{L}$ and $\mathfrak{K}'\setminus\mathfrak{L}$ by a variant of the ideas developed in the previous section. We first slice these intermediate arcs into $O(L)$ pieces $\mathfrak{P}(Q)$, as defined in (4.5), with Q satisfying

$$P^{1/5}\leqslant Q\leqslant \tfrac{1}{2}U_0^2. \tag{7.1}$$

Thus, when $\theta=5$, there is a value of Q constrained by (7.1) with

$$\int_{\mathfrak{K}\setminus\mathfrak{L}}|g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha\ll L\int_{\mathfrak{P}(Q)}|g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha. \tag{7.2}$$

When $\theta=4$ the same conclusion holds, subject to GRH, with $\mathfrak{K}'$ replacing $\mathfrak{K}$.

We are ready to mimic the argument from the previous section, though we invoke Lemma 2.3 rather than Lemma 2.2. We again suppose that the hypotheses in Lemma 5.1 are satisfied. In this setting the first inequality of (5.5) implies that for some positive number δ, one has

$$\int_{\mathfrak{P}(Q)}|f(\alpha)|^s\,\mathrm{d}\alpha\ll P^{s-k+\varepsilon}Q^{1-\delta}. \tag{7.3}$$

The role of the set $\mathscr{T}$ where $g(\alpha)$ is tiny is now played by the set

$$\mathscr{S}=\{\alpha\in\mathfrak{P}(Q):|g(\alpha)|\leqslant nQ^{-1}\}.$$

Thus, by virtue of (7.1) and (7.3), we immediately have

$$\int_{\mathscr{S}}|g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha\ll P^{s+\varepsilon}Q^{-\delta}\ll P^{s-\delta/6}. \tag{7.4}$$

Next we observe that Lemma 4.2 supplies the bound $g(\alpha)\ll nL^4Q^{-1/2}$ for all $\alpha\in\mathfrak{P}(Q)$, provided that Q satisfies (7.1). Hence, the set of arcs $\mathfrak{P}(Q)$ is the union of $\mathscr{S}$ and $O(L)$ level sets

$$\mathscr{K}(V)=\{\alpha\in\mathfrak{P}(Q):n/V\leqslant|g(\alpha)|\leqslant 2n/V\}, \tag{7.5}$$

with

$$Q^{1/2}L^{-5}\leqslant V\leqslant Q, \tag{7.6}$$

at least for large n. In view of the bound (7.4), there is consequently a value of V satisfying (7.6) for which

$$\int_{\mathfrak{P}(Q)}|g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha\ll P^{s-\delta/6}+L\int_{\mathscr{K}(V)}|g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha. \tag{7.7}$$

In the discussion of the previous section, pruning by size first removes from $\mathscr{L}(U)$ a portion $\mathscr{G}$ where $f(\alpha)$ was gentle enough to cooperate with the mean of $|g(\alpha)|$ over $\mathscr{L}(U)$, and the remaining set $\mathscr{H}$ required a slightly harder argument. In the present setting, again, the portion

$$\mathscr{E}=\{\alpha\in\mathscr{K}(V):|f(\alpha)|^s\leqslant P^sV^{-1}L^{-4}\}$$

analogous to $\mathscr{G}$ will be equally easy, but its complement

$$\mathscr{F}=\{\alpha\in\mathscr{K}(V):|f(\alpha)|^s> P^sV^{-1}L^{-4}\}$$

in $\mathscr{K}(V)$ is also straightforward to accommodate by arguing as in our earlier treatment of $\mathscr{H}$. Indeed, as in (6.7), we now have

$$\int_{\mathscr{E}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^s L^{-3}. \tag{7.8}$$

Again following the initial steps of the treatment of $\mathscr{H}$, we find that

$$\int_{\mathscr{F}} |f(\alpha)|^s\,\mathrm{d}\alpha \ll P^{-t}V^{t/s}L^4 \int_{\mathfrak{M}(2Q)} |f(\alpha)|^{s+t}\,\mathrm{d}\alpha.$$

Next, applying Lemma 2.3 together with (7.5), we arrive at the bound

$$\int_{\mathscr{F}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll V^{t/s-1}P^{s+\varepsilon}Q^{2\Delta_{s+t}/k}.$$

This last bound is the counterpart of (6.8). As before, we may suppose that $t/s-1$ is negative, whence (7.6) yields

$$\int_{\mathscr{F}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^{s+\varepsilon}Q^{\omega},$$

where

$$\omega = \frac{1}{2}\Big(\frac{t}{s}-1\Big) + \frac{2\Delta_{s+t}}{k} \leqslant \frac{1}{2}\Big(\frac{t}{s}+\frac{\theta\Delta_{s+t}}{k}-1\Big).$$

The second condition in (5.5) ensures that $\omega < 0$. Hence, by (7.1), it follows that there is a positive number $\delta > 0$ for which

$$\int_{\mathscr{F}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^{s-\delta}. \tag{7.9}$$

It remains to sum up the various contributions required to treat the integrals over $\mathfrak{K}\setminus\mathfrak{L}$ and $\mathfrak{K}'\setminus\mathfrak{L}$. By (7.7), (7.8) and (7.9), we see that

$$\int_{\mathfrak{P}(Q)} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^s L^{-2}.$$

On substituting this estimate into (7.2), we acquire the upper bound

$$\int_{\mathfrak{K}\setminus\mathfrak{L}} |g(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^s L^{-1},$$

valid when $\theta = 5$, with the analogue for $\theta = 4$ once again holding with $\mathfrak{K}'\setminus\mathfrak{L}$ in place of $\mathfrak{K}\setminus\mathfrak{L}$ subject to GRH. Since $\mathfrak{l} = \mathfrak{k}\cup(\mathfrak{K}\setminus\mathfrak{L})$, and likewise $\mathfrak{l} = \mathfrak{k}'\cup(\mathfrak{K}'\setminus\mathfrak{L})$, this bound together with (6.9) proves Lemma 5.1.

8. The theorems

Our work performed thus far allows us to confirm the lower bound (1.2) subject to conditions that involve certain admissible exponents.

Theorem 8.1. *Let $k \geqslant 3$, and suppose that s_0, t_0 are real numbers satisfying both $0 \leqslant t_0 \leqslant s_0$ and the conditions* (5.5) *with $\theta = 5$ and $(s,t) = (s_0,t_0)$. Then, whenever s is a natural number with $s \geqslant \max\{s_0, k+3\}$, one has $r_{k,s}(n) \gg n^{s/k}/\log n$. If* (5.5) *holds only with $\theta = 4$, then this lower bound for $r_{k,s}(n)$ holds subject to GRH.*

Proof. We choose $R = P^\eta$ with a value of η satisfying $0 < \eta < 1/7$ chosen so small that Lemma 5.1 applies with $s = s_0$. Then, for $s \geqslant s_0$, we estimate $|f(\alpha)|^{s-s_0}$ trivially, and conclude via (3.2) that

$$I(n, \mathfrak{l}) \ll n^{s/k}(\log n)^{-1}.$$

Next, since $\mathfrak{L} = \mathfrak{N} \cup (\mathfrak{L} \setminus \mathfrak{N})$, we find by combining Lemmata 3.1 and 4.3 via (3.2) that $I(n, \mathfrak{L}) \gg n^{s/k}$. Recalling that $I(n) = I(n, \mathfrak{L}) + I(n, \mathfrak{l})$, the theorem is now immediate from (3.3). □

We are now equipped to complete the proofs of the theorems presented in the introduction.

The proof of Theorems 1.2 *and* 1.4. We verify the conditions in Theorem 8.1 in the relevant cases by using explicit admissible exponents extracted from the tables in the appendices to [27, 28], suitably rounded up, as we now explain.

Given an exponent k with $5 \leqslant k \leqslant 20$ and a value of $\theta \in \{4, 5\}$, we fix values of s_θ and t_θ according to Table 2. We make use of the tables of permissible exponents λ_u associated to each value of k from the appendices of [27] and [28], using the former for $k = 5, 6$, and the latter for $7 \leqslant k \leqslant 20$. The admissible exponents Δ_u of the present paper are related to these tabulated values when u is even via the relation

$$\Delta_u = \lambda_{u/2} - u + k.$$

When u is odd, we may apply Hölder's inequality to interpolate between even values, so that

$$\Delta_u = \tfrac{1}{2}\left(\lambda_{(u+1)/2} + \lambda_{(u-1)/2}\right) - u + k.$$

When u is either s_θ or $s_\theta + t_\theta$, admissible exponents calculated in this way are presented in Table 2 to 4 decimal places, rounded up in the final place. The final entries recorded in Table 2 are computed values of the quantities

$$\Omega_\theta = \frac{t_\theta}{s_\theta} + \frac{\theta \Delta_{s_\theta + t_\theta}}{k},$$

again presented to 4 decimal places rounded up in the final place. We thus see that both of the conditions (5.5) are met provided that $2\Delta_{s_\theta} < k$ and $\Omega_\theta < 1$, and such is the case for all entries of Table 2. By taking $s_0 = s_\theta$ and $t_0 = t_\theta$ in Theorem 8.1, we confirm the conclusion of Theorem 1.2 in the case $\theta = 5$, and likewise Theorem 1.4 in the case $\theta = 4$. This completes the proof of these theorems. □

The proof of Theorems 1.1 and 1.3 is more involved, and this will require us to establish a technical result along the way. Again, we aim to verify the conditions of Theorem 8.1. The condition $2\Delta_s < k$ in (5.5) is only a moderate constraint on s. Indeed, writing $\sigma_0 = \frac{1}{2} + \log 2$, one finds from (2.1) that $\mathrm{H}(\sigma_0) = \frac{1}{2}$. Hence, there is an even integer s_0 in the interval $(\sigma_0 k, \sigma_0 k + 2]$, and we have $\mathrm{H}(s_0/k) < \frac{1}{2}$. By Lemma 2.2 with $t = s_0$, it follows that whenever $s \geqslant s_0$, there is an admissible exponent $\Delta_s < k/2$, as required. In particular, we now have the first inequality in (5.5) whenever $s \geqslant \sigma_0 k + 2$.

k	s_4	t_4	Δ_{s_4}	$\Delta_{s_4+t_4}$	Ω_4	s_5	t_5	Δ_{s_5}	$\Delta_{s_5+t_5}$	Ω_5
5	8	4	1.4387	0.5418	0.9335					
6	10	4	1.7247	0.8506	0.9671	11	3	1.4782	0.8516	0.9816
7	12	6	2.0144	0.8470	0.9840	13	5	1.7778	0.8470	0.9897
8	14	6	2.3106	1.1284	0.9928	16	8	1.8429	0.6562	0.9102
9	17	7	2.3733	1.1293	0.9137	18	8	2.1426	0.8960	0.9422
10	19	7	2.6661	1.3944	0.9262	20	10	2.4376	0.9304	0.9652
11	21	9	2.9571	1.3941	0.9356	22	10	2.7293	1.1641	0.9837
12	23	9	3.2438	1.6487	0.9409	24	12	3.0175	1.1896	0.9957
13	25	9	3.5299	1.9063	0.9466	27	13	3.0997	1.2191	0.9504
14	27	11	3.8147	1.8961	0.9492	29	13	3.3840	1.4408	0.9629
15	29	11	4.0984	2.1484	0.9523	31	15	3.6673	1.4665	0.9728
16	31	13	4.3819	2.1408	0.9546	33	15	3.9503	1.6884	0.9822
17	32	14	4.8887	2.3897	0.9998	35	17	4.2327	1.7118	0.9892
18	34	14	5.1707	2.6416	0.9988	37	19	4.5147	1.7385	0.9965
19	36	16	5.4518	2.6301	0.9982	40	18	4.5867	1.9556	0.9647
20	38	16	5.7323	2.8788	0.9969	42	20	4.8667	1.9802	0.9713

Table 2. Choice of exponents for $5 \leqslant k \leqslant 20$.

The second inequality in (5.5) is the more restrictive condition. Throughout, let $\theta = 4$ or 5. Note that whenever $s+t$ is an even integer, then by Lemma 2.2 we may assume that the exponent $\Delta_{s+t} = k\mathrm{H}((s+t)/k)$ is admissible. With this choice of Δ_{s+t} we write

$$s = \sigma k, \qquad t = \tau k,$$

and then have

$$\frac{t}{s} + \frac{\theta \Delta_{s+t}}{k} = \frac{\tau}{\sigma} + \theta \mathrm{H}(\sigma + \tau). \tag{8.1}$$

For a given value of s one wishes to minimize this expression, and hence one has to choose τ optimally, or at least nearly so. The constraint $0 \leqslant t \leqslant s$ translates to $0 \leqslant \tau \leqslant \sigma$. We temporarily ignore that (8.1) is available only for even values of $s+t$, a complication soon to be resolved in a technical lemma. Instead, we compute the function $E_\theta : [\frac{5}{4}, 3] \to \mathbb{R}$ defined by

$$E_\theta(\sigma) = \min_{0 \leqslant \tau \leqslant \sigma} h(\tau), \tag{8.2}$$

where for each fixed $\sigma \in [\frac{5}{4}, 3]$, the function $h : [0, \infty) \to \mathbb{R}$ is defined by

$$h(\tau) = \frac{\tau}{\sigma} + \theta \mathrm{H}(\sigma + \tau). \tag{8.3}$$

Notice that in naming the function h, we have suppressed its dependence on both θ and σ in order to simplify our exposition.

The function h is continuously differentiable, and we find from (2.2) that

$$h'(\tau) = \frac{1}{\sigma} - \frac{\theta \mathrm{H}(\sigma + \tau)}{1 + \mathrm{H}(\sigma + \tau)}.$$

Note that $h'(\tau) < 0$ if and only if $\mathrm{H}(\sigma+\tau) > 1/(\theta\sigma - 1)$. In particular, we see from Lemma 2.1 that $h'(0)$ is negative, so that the function h is decreasing for small values of τ. Also, the equation $h'(\tau) = 0$ is equivalent to

$$\mathrm{H}(\sigma+\tau) = \frac{1}{\theta\sigma - 1}. \tag{8.4}$$

Since $\mathrm{H}(\sigma+\tau)$ is continuous in τ and strictly decreases from $\mathrm{H}(\sigma)$ to 0 as τ runs through positive real values, we conclude from Lemma 2.1 that (8.4) has exactly one solution that we denote by $\tau(\sigma)$. We apply (2.1) to $\mathrm{H}(\sigma + \tau(\sigma))$ and insert the identity (8.4), thus finding that

$$\tau(\sigma) = 1 - \sigma - \frac{1}{\theta\sigma - 1} + \log(\theta\sigma - 1). \tag{8.5}$$

In view of (8.3), the function $h(\tau)$ is increasing for large τ, whence $h(\tau(\sigma))$ is the minimum value of h. Further, since $h'(0) < 0$, we must have $\tau(\sigma) > 0$.

With (8.2) in mind, we now show that one has $\tau(\sigma) < \sigma$. By (8.5), this inequality holds if and only if

$$1 + \log(\theta\sigma - 1) < 2\sigma + \frac{1}{\theta\sigma - 1}. \tag{8.6}$$

This is readily checked numerically for $\sigma = \frac{5}{4}$, and amounts to checking that

$$2.658\ldots = 1 + \log\Big(\frac{21}{4}\Big) < \frac{5}{2} + \frac{1}{21/4} = 2.690\ldots.$$

Meanwhile, for $\sigma > \frac{5}{4}$, the derivative of the left hand side of (8.6) is smaller than the derivative of the right hand side. Again, this is easily checked. Thus we discern that the upper bound $\tau(\sigma) < \sigma$ holds for $\sigma \in [\frac{5}{4}, 3]$. Consequently, by (8.2) and (8.4),

$$E_\theta(\sigma) = \frac{\tau(\sigma)}{\sigma} + \theta\mathrm{H}(\sigma + \tau(\sigma)) = \frac{\tau(\sigma)}{\sigma} + \frac{\theta}{\theta\sigma - 1}. \tag{8.7}$$

Next we note that $\tau(\sigma)$ is decreasing on the interval $[\frac{3}{2}, 3]$. Indeed, it follows from (8.5) that throughout the latter interval, the derivative

$$\tau'(\sigma) = -1 + \frac{\theta}{\theta\sigma - 1} + \frac{\theta}{(\theta\sigma - 1)^2} \tag{8.8}$$

is negative. Since $1/\sigma$ also decreases, we see from (8.7) that $E_\theta(\sigma)$ is strictly decreasing on $[\frac{3}{2}, 3]$. One may check numerically from (8.5) and (8.7) that

$$E_\theta(3) < 1 < E_\theta(3/2).$$

Hence there is a unique number c_θ with $E_\theta(c_\theta) = 1$. If we now eliminate $\tau(c_\theta)$ between (8.5) and (8.7), then we obtain

$$1 - c_\theta - \frac{1}{\theta c_\theta - 1} + \log(\theta c_\theta - 1) = c_\theta - \frac{\theta c_\theta}{\theta c_\theta - 1}.$$

This equation shows that c_5 is the real number c occuring in the statement of Theorem 1.1, and that c_4 is the real number c' occuring in the statement of Theorem 1.3.

We now attend to the key technical lemma previously advertised.

Lemma 8.2. *Let $\theta \in \{4,5\}$ and $k \geqslant 17$. Then there exists a number $\sigma_\theta = \sigma_{\theta,k}$, with $\sigma_\theta \in (c_\theta, c_\theta + 4/k)$, satisfying the condition that $k(\sigma_\theta + \tau(\sigma_\theta))$ is an even integer.*

Proof. We begin by noting from (8.8) that the function $k(\sigma+\tau(\sigma))$ is increasing on $(c_\theta, c_\theta + 4/k)$. It follows that this function maps the latter interval onto

$$(kc_\theta + k\tau(c_\theta), kc_\theta + 4 + k\tau(c_\theta + 4/k)).$$

Provided that the inequality

$$k(\tau(c_\theta) - \tau(c_\theta + 4/k)) < 2 \tag{8.9}$$

holds, we see that the set of values of $k\sigma + k\tau(\sigma)$, for $\sigma \in (c_\theta, c_\theta + 4/k)$, covers an interval of length exceeding 2, and therefore contains an even integer $2l$, say. One then has $2l = k(\sigma_\theta + \tau(\sigma_\theta))$ for some $\sigma_\theta \in (c_\theta, c_\theta + 4/k)$, as desired.

It remains to check (8.9). By the mean value theorem and (8.8), there exists a real number $\sigma \in (c_\theta, c_\theta + 4/k)$ with

$$k\left(\tau(c_\theta) - \tau(c_\theta + 4/k)\right) = -4\tau'(\sigma) = 4 - \frac{4\theta}{\theta\sigma - 1} - \frac{4\theta}{(\theta\sigma - 1)^2}.$$

However, we have

$$\frac{\theta}{\theta\sigma - 1} + \frac{\theta}{(\theta\sigma - 1)^2} > \frac{1}{2}$$

provided only that

$$(\theta\sigma - 1 - \theta)^2 < \theta^2 + 2\theta,$$

and this is assured whenever

$$1 + 1/\theta - \sqrt{1 + 2/\theta} < \sigma < 1 + 1/\theta + \sqrt{1 + 2/\theta}.$$

When $\theta = 5$, this last constraint is satisfied for $\sigma \in (0.017, 2.383)$, and hence for $c < \sigma < c + 4/k$ when $k \geqslant 17$. When $\theta = 4$, meanwhile, this last constraint is satisfied for $\sigma \in (0.026, 2.474)$, and hence with ease for $c' < \sigma < c' + 4/k$ under the same condition on k. This establishes (8.9) for $k \geqslant 17$, and completes the proof of the lemma. □

Finally, we establish Theorems 1.1 and 1.3, dividing the natural numbers k into three ranges. In the first range $k \geqslant 17$, we check the conditions of Theorem 8.1 when $\theta \in \{4,5\}$. Let $s_\theta = \sigma_\theta k$ and $t_\theta = \tau(\sigma_\theta)k$. Then $s_\theta + t_\theta$ is the even integer provided by Lemma 8.2. Further, since $E_\theta(\sigma)$ is a strictly decreasing function on $[\frac{3}{2}, 3]$ and

$$3 > c_\theta + 4/k > \sigma_\theta > c_\theta > 3/2,$$

we have $E_\theta(\sigma_\theta) < E_\theta(c_\theta) = 1$. By (8.1), (8.2) and (8.3) we conclude that the second inequality in (5.5) holds with $s = s_\theta$ and $t = t_\theta$. By taking $(s_0, t_0) = (s_\theta, t_\theta)$ in Theorem 8.1, therefore, we obtain Theorem 1.1 when $\theta = 5$, and Theorem 1.3 when $\theta = 4$.

In the second range $\theta + 1 \leqslant k \leqslant 16$, the conclusions of Theorems 1.1 and 1.3 follow, respectively, from Theorems 1.2 and 1.4. The third range $1 \leqslant k \leqslant \theta$, meanwhile, is more or less trivial. Indeed, our remarks concerning the naïve decoupling approach in the preamble to the statement of Theorem 1.2 already establish Theorems 1.1 and 1.3 in the cases $k = 4$ and $k = 5$, and the work of Kawada [14, Theorem 2] confirms both Theorem 1.1 and 1.3 when $k = 3$. This leaves the case $k = 2$ handled by Hooley [12, 13] and Linnik [15, 16], and the trivial case $k = 1$. With these observations, all of the loose ends associated with the confirmation of Theorems 1.1 and 1.3 have been tied up.

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