THE BGMN CONJECTURE VIA STABLE PAIRS

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In memory of M. S. Narasimhan

Abstract

Let C be a smooth projective curve of genus $g \geq 2$, and let N be the moduli space of stable vector bundles on C of rank 2 and fixed determinant of odd degree. We construct a semiorthogonal decomposition of $D^b(N)$ conjectured by Narasimhan and by Belmans, Galkin, and Mukhopadhyay. It has two blocks for each ith symmetric power of C for $i = 0, \dots, g-2$ and one block for the $(g-1)$st symmetric power. We conjecture that the subcategory generated by our blocks has a trivial semiorthogonal complement, proving the full BGMN conjecture. Our proof is based on an analysis of wall-crossing between moduli spaces of stable pairs, combining classical vector bundles techniques with the method of windows.

Contents

1. Introduction

Let C be a smooth complex projective curve of genus $g \geq 2$. Let $N = M_C(2, \Lambda)$ be the moduli space of stable vector bundles on C of rank 2 and fixed determinant Λ of

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odd degree. It is a smooth Fano variety of index 2, with $\operatorname{Pic} N = \mathbb{Z} \cdot \theta$ for some ample line bundle θ.

THEOREM 1.1
$D^b(N)$ has a semiorthogonal decomposition $\langle \mathcal{P}, \mathcal{A} \rangle$, where

$$\begin{array}{rlllll} \mathcal{A} = \langle \theta^* \otimes \mathcal{G}_0, & (\theta^*)^2 \otimes \mathcal{G}_2, & (\theta^*)^3 \otimes \mathcal{G}_4, & (\theta^*)^4 \otimes \mathcal{G}_6, & \dots, \\ \dots, & (\theta^*)^4 \otimes \overline{\mathcal{G}}_7, & (\theta^*)^3 \otimes \overline{\mathcal{G}}_5, & (\theta^*)^2 \otimes \overline{\mathcal{G}}_3, & \theta^* \otimes \overline{\mathcal{G}}_1, \\ \mathcal{G}_0, & \theta^* \otimes \mathcal{G}_2, & (\theta^*)^2 \otimes \mathcal{G}_4, & (\theta^*)^3 \otimes \mathcal{G}_6, & \dots, \\ \dots, & (\theta^*)^3 \otimes \overline{\mathcal{G}}_7, & (\theta^*)^2 \otimes \overline{\mathcal{G}}_5, & \theta^* \otimes \overline{\mathcal{G}}_3, & \overline{\mathcal{G}}_1 \rangle. \end{array} \tag{1.1}$$

Each subcategory $\mathcal{G}_i \simeq D^b(\operatorname{Sym}^i C)$ (resp., $\overline{\mathcal{G}}_i \simeq D^b(\operatorname{Sym}^i C)$) is embedded in $D^b(N)$ by a fully faithful Fourier–Mukai functor with kernel given by the ith tensor bundle $\mathcal{E}^{\boxtimes i}$ (resp., $\overline{\mathcal{E}}^{\boxtimes i}$) (see Section 2) of the Poincaré bundle $\mathcal{E}$ on $C \times N$ normalized so that $\det \mathcal{E}_x \simeq \theta$ for every $x \in C$.

There are two blocks isomorphic to $D^b(\operatorname{Sym}^i C)$ for each $i = 0, \dots, g-2$ and one block isomorphic to $D^b(\operatorname{Sym}^{g-1} C)$, which appears on the first or second line of (1.1), depending on the parity of g.

The blocks appearing in (1.1) cannot be further decomposed (see [26]). Remarkably, our decomposition is compatible with the results of Muñoz [29]–[31] (cf. [5, Proposition 3.2]), that the operator of the quantum multiplication by $c_1(N)$ on the quantum cohomology $QH^\bullet(N)$ has eigenvalues 8λ, where

$$\lambda = (1-g),\ (2-g)\sqrt{-1},\ (3-g), \dots, (g-3),\ (g-2)\sqrt{-1},\ (g-1)$$

and the eigenspace of 8λ is isomorphic to $H^\bullet(\operatorname{Sym}^{g-1-|\lambda|} C)$. There are many other results (e.g., [12], [24]) on cohomology and motivic decomposition of N compatible with (1.1). This provides ample evidence toward the expectation that $\mathcal{P} = 0$. We hope to address this question in the future, as well as to use our methods to study properties of analogous Fourier–Mukai functors for moduli spaces of vector bundles of higher rank on curves and for moduli spaces of sheaves with 1-dimensional support on K3 surfaces.

Partial results toward Theorem 1.1 have appeared in the literature. The case $g = 2$ is a classical theorem of Bondal and Orlov [7, Theorem 2.9], who also proved that $\mathcal{P} = 0$ in that case. Fonarev and Kuznetsov [15] proved that $D^b(C) \hookrightarrow D^b(N)$ if C is a hyperelliptic curve using an explicit description of N due to Desale and Ramanan [13]. They also proved that $D^b(C) \hookrightarrow D^b(N)$ for a general curve C by a deformation argument. Narasimhan [32], [33] proved that $D^b(C) \hookrightarrow D^b(N)$ for all curves using Hecke correspondences. He also showed that one can add the line bundles $\mathcal{O}$ and θ^* to $D^b(C)$ to start a semiorthogonal decomposition of $D^b(N)$.

In [6], Belmans and Mukhopadhyay work with the moduli space $M_C(r,\Lambda)$ of vector bundles of rank r and determinant Λ, where $r \geq 2$ and $\deg\Lambda = 1$. They show that there is a fully faithful embedding $D^b(C) \hookrightarrow D^b(M_C(r,\Lambda))$ provided that the genus is sufficiently high. Moreover, they use this embedding to find the start of a semiorthogonal decomposition of $D^b(M_C(r,\Lambda))$ of the form θ^*, $D^b(C)$, $\mathcal{O}$, $\theta^* \otimes D^b(C)$, this way extending the decomposition on $N = M_C(2,\Lambda)$ present in [33]. Belmans, Galkin, and Mukhopadhyay have conjectured, independently of Narasimhan, that $D^b(N)$ should have a semiorthogonal decomposition with blocks $D^b(\mathrm{Sym}^i C)$ (see [3], [24]), and have collected additional evidence toward this conjecture in [5]. Lee and Narasimhan [25] proved using Hecke correspondences that, if C is non-hyperelliptic and $g \geq 16$, there is a fully faithful functor $D^b(\mathrm{Sym}^2 C) \hookrightarrow D^b(N)$ whose image is left semiorthogonal to the copy of $D^b(C)$ obtained earlier. They also introduced tensor bundles $\mathcal{E}^{\boxtimes i}$ of the Poincaré bundle (see Section 2), which we discovered independently. If $D \in \mathrm{Sym}^i C$ is a reduced sum of points $x_1 + \cdots + x_i$, then the fiber $(\mathcal{E}^{\boxtimes i})_D$ is a vector bundle on N isomorphic to the tensor product $\mathcal{E}_{x_1} \otimes \cdots \otimes \mathcal{E}_{x_i}$. If the points have multiplicities, then $(\mathcal{E}^{\boxtimes i})_D$ is a deformation of the tensor product over $\mathbb{A}^1$ (see Corollary 2.9).

Instead of using Hecke correspondences (although they do make a guest appearance in Section 6), we prove Theorem 1.1 by analyzing Fourier–Mukai functors given by tensor bundles $F^{\boxtimes i}$ of the universal bundle F on the moduli space of stable pairs (E,ϕ), where E is a rank-2 vector bundle on C with fixed odd determinant line bundle of degree d and $\phi \in H^0(E)$ is a nonzero section. The stability condition on these spaces depends on a parameter, and we use extensively results of Thaddeus [39] on wall-crossing. If $d = 2g-1$, then there is a well-known diagram of flips

$$\begin{array}{ccccccccc} & \tilde{M}_2 & & \tilde{M}_3 & & \tilde{M}_{g-1} & \\ \swarrow & & \searrow \ \swarrow & & \searrow \ \swarrow & & \searrow \\ M_1 & & M_2 & & \cdots & & M_{g-1} \\ \downarrow & & & & & & \downarrow \\ M_0 & & & & & & N \end{array} \tag{1.2}$$

where $M_0 = \mathbb{P}^{3g-3}$, M_1 is the blowup of M_0 in C, the rational map $M_{i-1} \dashrightarrow M_i$ is a standard flip of projective bundles over $\mathrm{Sym}^i C$, and $\xi : M_{g-1} \to N$ is a birational Abel–Jacobi map with fiber $\mathbb{P}H^0(E)$ over a stable vector bundle E. Accordingly, $D^b(M_i)$ has a semiorthogonal decomposition into $D^b(M_{i-1})$ and several blocks equivalent to $D^b(\mathrm{Sym}^i C)$ with torsion supports (see Proposition 3.18 or [4]). While these decompositions do not descend to N and are not associated with the universal bundle, they are useful. Philosophically, tensor bundles on $\mathrm{Sym}^i C \times N$ are related to exterior powers of the tautological bundle of the universal bundle, which appear in the Koszul complex of the tautological section that vanishes on the flipped locus. One can try to connect two Fourier–Mukai functors via mutations. In practice, this Koszul complex is difficult to analyze except for M_1 (see Section 5). We followed another strategy toward proving Theorem 1.1.

In order to prove semiorthogonality in (1.1) and full faithfulness of the Fourier–Mukai functors via the Bondal–Orlov criterion, we had to investigate coherent cohomology for a large class of vector bundles. The main difficulty in this kind of analysis is to find a priori numerical bounds on the class of acyclic vector bundles to get the induction going.

Definition 1.2
For an object $\mathcal{F}$ in the derived category of a scheme M, we say that $\mathcal{F}$ is Γ*-acyclic* if $R\Gamma(\mathcal{F}) = 0$. That is, for us Γ-acyclicity will mean the vanishing of *all* cohomology groups, including $H^0(\mathcal{F})$. Other authors have used the term *immaculate* for this property (cf. [1]).

Theorem 1.1 then requires the proof of Γ-acyclicity for several vector bundles. It is worth emphasizing that the moduli space N depends on the complex structure of the curve C by a classical theorem of Mumford and Newstead [28] later extended by Narasimhan and Ramanan [34]. The uniform shape of Theorem 1.1 is thus a surprisingly strong statement about coherent cohomology of vector bundles on N that does not involve any conditions of the Brill–Noether type. Our approach utilizes the method of windows into derived categories of geometric invariant theory (GIT) quotients of Teleman [38], Halpern-Leistner [18], and Ballard, Favero, and Katzarkov [2] to systematically predict behavior of coherent cohomology under wall-crossing. This dramatically reduces otherwise unwieldy cohomological computations to a few key cases, which can be analyzed using other techniques. Rather unexpectedly, one of the difficult ingredients in the proof is acyclicity of certain line bundles (see Section 6). While cohomology of line bundles on the space of stable pairs was extensively studied in [39] in order to prove the Verlinde formula, the line bundle that we need is outside the scope of that paper.

Analogous recent applications of windows to moduli spaces include the proof of the Manin–Orlov conjecture on $\bar{M}_{0,n}$ by Castravet and Tevelev [9]–[11] and analysis of Bott vanishing on GIT quotients by Torres [40].

2. Tensor vector bundles

Let C be a smooth projective curve over $\mathbb{C}$. For integers $\alpha \geq 1$ and $1 \leq j \leq \alpha$, let $\pi_j : C^\alpha \to C$ be the jth projection, and let $\tau : C^\alpha \to \mathrm{Sym}^\alpha C$ be the categorical

S_α-quotient, where S_α is the symmetric group. Since C^α is Cohen–Macaulay (in fact smooth), $\mathrm{Sym}^\alpha C$ is smooth, and τ is equidimensional, we conclude that τ is flat by miracle flatness. Therefore, any base change $\tau : C^\alpha \times M \to \mathrm{Sym}^\alpha C \times M$ is also a finite and flat categorical S_α-quotient, where M is any scheme over $\mathbb{C}$. The constructions in this section are functorial in M. In the following sections, M will be one of the moduli spaces we consider.

Notation 2.1
For an S_α-equivariant vector bundle $\mathcal{E}$ on $C^\alpha \times M$, we will denote by $\tau_*^{S_\alpha} \mathcal{E}$ the S_α-invariant part of the pushforward $\tau_* \mathcal{E}$.

LEMMA 2.2
Let $\mathcal{E}$ be an S_α-equivariant locally free sheaf on $C^\alpha \times M$. Then $\tau_* \mathcal{E}$ and $\tau_*^{S_\alpha} \mathcal{E}$ are locally free sheaves on $\mathrm{Sym}^\alpha C \times M$.

Proof
The scheme $C^\alpha \times M$ is covered by S_α-equivariant affine charts $\operatorname{Spec} R$ and τ^* is given by the inclusion of invariants $R^{S_\alpha} \subset R$. Since R is a finitely generated and flat R^{S_α}-module, it is also a projective R^{S_α}-module. Let $E = H^0(\operatorname{Spec} R, \mathcal{E})$. Since E is a projective R-module, it is a direct summand of R^s for some s. It follows that E is a projective R^{S_α}-module; that is, $\tau_* \mathcal{E}$ is locally free. Since E^{S_α} is a direct summand of E as an R^{S_α}-module, it is also a projective R^{S_α}-module. Therefore, $\tau_*^{S_\alpha} \mathcal{E}$ is a locally free sheaf as well. □

Definition 2.3
For any vector bundle $\mathcal{F}$ on $C \times M$, we define the following *tensor vector bundles* on $\mathrm{Sym}^\alpha C \times M$,

$$\mathcal{F}^{\boxtimes \alpha} = \tau_*^{S_\alpha}\Big(\bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F}\Big) \qquad \text{and} \qquad \overline{\mathcal{F}}^{\boxtimes \alpha} = \tau_*^{S_\alpha}\Big(\bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \otimes \mathrm{sgn}\Big),$$

where S_α acts on C^α and also permutes the factors of the corresponding vector bundle on C^α. Here sgn is the sign representation of S_α.

LEMMA 2.4
The formation of tensor vector bundles is functorial in M; that is, given a morphism $f : M' \to M$ and its base changes $C \times M' \to C \times M$ and $\mathrm{Sym}^\alpha C \times M' \to \mathrm{Sym}^\alpha C \times M$*, which we also denote by f, we have*

$$f^*(\mathcal{F}^{\boxtimes \alpha}) = (f^* \mathcal{F})^{\boxtimes \alpha} \qquad \text{and} \qquad f^*(\overline{\mathcal{F}}^{\boxtimes \alpha}) = \overline{(f^* \mathcal{F})}^{\boxtimes \alpha}.$$

Proof
Since τ is flat, this follows from cohomology and base change. □

For a divisor $D \in \mathrm{Sym}^\alpha C$ and a vector bundle $\mathcal{G}$ on $\mathrm{Sym}^\alpha C \times M$, we write $\mathcal{G}_D := \mathcal{G}|_{\{D\}\times M}$. We usually view $\mathcal{G}_D$ as a vector bundle on M.

Remark 2.5
For the empty divisor $D = 0$, we have $\mathcal{G}_0 \simeq \mathcal{O}_M$.

LEMMA 2.6
If $D = \sum \alpha_k x_k$ with $x_k \neq x_l$ for $k \neq l$, then we have

$$(\mathcal{F}^{\boxtimes\alpha})_D = \bigotimes(\mathcal{F}^{\boxtimes\alpha_k})_{\alpha_k x_k}, \qquad (\overline{\mathcal{F}}^{\boxtimes\alpha})_D = \bigotimes(\overline{\mathcal{F}}^{\boxtimes\alpha_k})_{\alpha_k x_k}. \tag{2.1}$$

Proof
Indeed, the quotient $\tau : C^\alpha \to \mathrm{Sym}^\alpha C$ is étale locally near $D \in \mathrm{Sym}^\alpha C$ isomorphic to the product of quotients $\prod C^{\alpha_k} \to \prod \mathrm{Sym}^{\alpha_k} C$. Moreover, the stabilizer of the point D under the S_α-action is $\prod S_{\alpha_k}$, and sgn restricts to the tensor product of sign representations of $\prod S_{\alpha_k}$. □

Consider the nonreduced scheme $\mathbb{D}_\alpha = \mathrm{Spec}\,\mathbb{C}[t]/t^\alpha$, with maps $\mathrm{pt} \overset{\iota}{\hookrightarrow} \mathbb{D}_\alpha \overset{\rho}{\to} \mathrm{pt}$ given by the obvious pullbacks $\mathbb{C} \overset{\rho^\#}{\longrightarrow} \mathbb{C}[t]/t^\alpha \overset{\iota^\#}{\longrightarrow} \mathbb{C}$. We still write ι and ρ for the base changes to M of these morphisms, that is, $M \overset{\iota}{\to} \mathbb{D}_\alpha \times M \overset{\rho}{\to} M$. For a locally free sheaf $\mathcal{F}$ on $\mathbb{D}_\alpha \times M$, we denote by $\mathcal{F}_0 = \iota^*\mathcal{F}$ its restriction to M.

Definition 2.7
For two vector bundles $\mathcal{F}, \mathcal{G}$ on a scheme M, we will say that $\mathcal{F}$ is a *deformation of $\mathcal{G}$ over $\mathbb{A}^1$* if there is a coherent sheaf $\tilde{\mathcal{F}}$ on $M \times \mathbb{A}^1$, flat over $\mathbb{A}^1$, with $\tilde{\mathcal{F}}|_t \simeq \mathcal{F}$ for $t \neq 0$, while $\tilde{\mathcal{F}}|_0 \simeq \mathcal{G}$.

LEMMA 2.8
Every locally free sheaf $\mathcal{F}$ on $\mathbb{D}_\alpha \times M$ is a deformation of $\rho^*\mathcal{F}_0$ over $\mathbb{A}^1$. In particular, $\rho_*\mathcal{F}$ is a deformation of $\mathcal{F}_0^{\oplus\alpha}$ over $\mathbb{A}^1$.

Proof
Let $\lambda : \mathbb{A}^1_s \times \mathbb{D}_\alpha \to \mathbb{D}_\alpha$ be the map defined by its pullback $\lambda^\# : t \mapsto ts$, and also denote by λ its base change to M. We claim that the locally free sheaf $\lambda^*\mathcal{F}$ gives the required deformation. Indeed, the restriction of $\lambda^*\mathcal{F}$ to $\{s_0\} \in \mathbb{A}^1_s$ is the pullback of $\mathcal{F}$ along the composition $b_{s_0} = \lambda \circ j_{s_0}$,

$$\mathbb{D}_\alpha \times M \xhookrightarrow{j_{s_0}} \mathbb{D}_\alpha \times \mathbb{A}^1_s \times M \xrightarrow{\lambda} \mathbb{D}_\alpha \times M,$$

determined by its pullback $b^{\#}_{s_0} : t \mapsto s_0 t$. When $s_0 \neq 0$, $b^*_{s_0}\mathcal{F} \simeq \mathcal{F}$. On the other hand, when $s_0 = 0$, the map b_0 factors as the composition

$$\mathbb{D}_\alpha \times M \xrightarrow{\rho} M \xrightarrow{\iota} \mathbb{D}_\alpha \times M,$$

so $b_0^*\mathcal{F} = \rho^*\iota^*\mathcal{F} = \rho^*\mathcal{F}_0$, as desired. The last statement follows from projection formula and the fact that $\rho_*\rho^*\mathcal{O}_M \simeq \mathcal{O}_M^{\oplus\alpha}$. ☐

Suppose that $D = \alpha x$ is a fat point, that is, a divisor given by a single point x with multiplicity α, and let t be a local parameter on C at x. Note that the notation $\mathcal{O}_D$ is unfortunately ambiguous, because it can denote both the structure sheaf of the subscheme $D \subset C$ and the skyscraper sheaf of the point $\{D\} \in \operatorname{Sym}^\alpha C$. When confusion is possible, we denote the latter sheaf by $\mathcal{O}_{\{D\}}$. Then

$$\tau^*\mathcal{O}_{\{D\}} \simeq \frac{\mathbb{C}[t_1, \dots, t_\alpha]}{(\sigma_1, \dots, \sigma_\alpha)} \tag{2.2}$$

is the so-called *covariant algebra*, where $\sigma_1, \dots, \sigma_\alpha$ are the elementary symmetric functions in variables $t_j = \pi_j^*(t)$. Call $\mathbb{B}_\alpha = \operatorname{Spec} \tau^*\mathcal{O}_{\{D\}}$. By the Newton formulas, $t_j^\alpha = 0$ for every $j = 1, \dots, \alpha$, and in particular, every map $\pi_j : \mathbb{B}_\alpha \to C$ factors through $\mathbb{D}_\alpha$. By abuse of notation, we have a diagram of morphisms

$$\begin{array}{ccccc} \mathbb{B}_\alpha \times M & \xrightarrow{\pi_j} & \mathbb{D}_\alpha \times M & \xrightarrow{q} & C \times M \\ & {\scriptstyle\tau}\searrow & \iota\uparrow\downarrow\rho & \swarrow & \\ & & M & & \end{array} \tag{2.3}$$

COROLLARY 2.9

Let $D = x_1 + \cdots + x_\alpha$ (possibly with repetitions). Then both $(\mathcal{F}^{\boxtimes\alpha})_D$ and $(\overline{\mathcal{F}}^{\boxtimes\alpha})_D$ are deformations of $\mathcal{F}_{x_1} \otimes \cdots \otimes \mathcal{F}_{x_\alpha}$ over $\mathbb{A}^1$.

Proof

By (2.1), it suffices to consider the case when $D = \alpha x$. Using the notation as in the diagram (2.3), the restriction $(\mathcal{F}^{\boxtimes\alpha})_D$ can be written as $\tau_*^{S_\alpha}(\bigotimes \pi_j^* q^*\mathcal{F})$, by flatness of τ. The construction of Lemma 2.8 commutes with the S_α-action, so $\tau_*^{S_\alpha}(\bigotimes \pi_j^* q^*\mathcal{F})$ is a deformation of $\tau_*^{S_\alpha}(\bigotimes \pi_j^*\rho^*\mathcal{F}_x)$ over $\mathbb{A}^1$, since $(q^*\mathcal{F})_0 = \mathcal{F}_x = \mathcal{F}|_{\{x\}\times M}$. Note that $\pi_j^*\rho^* = \tau^*$, so using the projection formula, we get that $(\mathcal{F}^{\boxtimes\alpha})_D$ is a deformation of $(\bigotimes_{j=1}^\alpha \mathcal{F}_x) \otimes \tau_*^{S_\alpha}(\mathcal{O}_{\mathbb{B}_\alpha \times M})$, and similarly, $(\overline{\mathcal{F}}^{\boxtimes\alpha})_D$

is a deformation of $(\bigotimes_{j=1}^{\alpha} \mathcal{F}_x) \otimes \tau_*^{S_\alpha}(\mathcal{O}_{\mathbb{B}_\alpha \times M} \otimes \mathrm{sgn})$. By flatness of the quotient $C^\alpha \to \mathrm{Sym}^\alpha C$, the covariant algebra $\mathcal{O}_{\mathbb{B}_\alpha}$ (2.2) is the regular representation $\mathbb{C}[S_\alpha]$ of S_α. It follows that it contains the trivial and the sign representations each with multiplicity 1, and therefore $\tau_*^{S_\alpha}(\mathcal{O}_{\mathbb{B}_\alpha \times M}) = \tau_*^{S_\alpha}(\mathcal{O}_{\mathbb{B}_\alpha \times M} \otimes \mathrm{sgn}) = \mathcal{O}_M$. This concludes the proof. □

Remark 2.10

If we have a G-action on M and a G-equivariant bundle $\mathcal{F}$, then the deformations constructed in the proofs of Lemma 2.8 and Corollary 2.9 are also G-equivariant, that is, given by a G-equivariant bundle on $\mathbb{A}^1 \times M$. This is because the map $\lambda : \mathbb{A}_s^1 \times \mathbb{D}_\alpha \times M \to \mathbb{D}_\alpha \times M$ is given by the identity on the factor M, and hence λ is G-invariant. Thus, the pullback $\lambda^* \mathcal{F}$ of a G-equivariant sheaf is naturally again a G-equivariant sheaf.

Definition 2.11

A vector bundle $\mathcal{F}$ on a scheme M is said to be a *stable deformation* of a vector bundle $\mathcal{G}$ *over* $\mathbb{A}^1$ if there is some vector bundle $\mathcal{K}$ such that $\mathcal{F} \oplus \mathcal{K}$ is a deformation over $\mathbb{A}^1$ of a direct sum $\mathcal{G}^{\oplus r}$ for some $r > 0$.

PROPOSITION 2.12

Let $D = x + \tilde{D}$. Then the vector bundle $(\mathcal{F}^{\boxtimes \alpha})_D$ is a stable deformation of the vector bundle $\mathcal{F}_x \otimes (\mathcal{F}^{\boxtimes(\alpha-1)})_{\tilde{D}}$ over $\mathbb{A}^1$.

Proof

By Lemma 2.6, it suffices to consider the case $D = \alpha x$. Let $W_\alpha = \mathbb{C}^\alpha$ be the tautological representation of S_α, which splits as a sum of the trivial and the standard representations, $W_\alpha = \mathbb{C} \oplus V_\alpha$. For any S_α-equivariant vector bundle $\mathcal{E}$ on $\mathbb{B}_\alpha \times M$, we have

$$\tau_*^{S_\alpha}(\mathcal{E} \otimes W_\alpha) = \tau_*^{S_\alpha}(\mathcal{E}) \oplus \tau_*^{S_\alpha}(\mathcal{E} \otimes V_\alpha). \tag{2.4}$$

On the other hand, we have $W_\alpha = \mathbb{C}[S_\alpha / S_{\alpha-1}]$, where $S_{\alpha-1} \hookrightarrow S_\alpha$ is the inclusion given by fixing the αth element. Then by Frobenius reciprocity, $\tau_*^{S_\alpha}(\mathcal{E} \otimes W_\alpha) = \tau_*^{S_{\alpha-1}}(\mathcal{E}) = \rho_* \circ (\pi_\alpha)_*^{S_{\alpha-1}}(\mathcal{E})$, where π_α is the αth projection. By Lemma 2.8, this bundle is a deformation of $((\pi_\alpha)_*^{S_{\alpha-1}} \mathcal{E})_0^{\oplus \alpha}$ over $\mathbb{A}^1$. Now let $\mathcal{E}$ be $\bigotimes \pi_j^* q^* \mathcal{F}$. Then $\tau_*^{S_\alpha}(\mathcal{E})$ is precisely $(\mathcal{F}^{\boxtimes \alpha})_D$ and, by projection formula,

$$\big((\pi_\alpha)_*^{S_{\alpha-1}} \mathcal{E}\big)_0 = \mathcal{F}_x \otimes \Big((\pi_\alpha)_*^{S_{\alpha-1}} \Big(\bigotimes_{j=1}^{\alpha-1} \pi_j^* q^* \mathcal{F}\Big)\Big)_0$$

$$= \mathcal{F}_x \otimes (\pi_\alpha)_*^{S_{\alpha-1}} \bigotimes_{j=1}^{\alpha-1} (\pi_j^* q^* \mathcal{F})\Big|_{t_\alpha=0} = \mathcal{F}_x \otimes (\mathcal{F}^{\boxtimes(\alpha-1)})_{(\alpha-1)x}$$

since the subscheme $(t_\alpha = 0) \subset \mathbb{B}_\alpha$ is isomorphic to $\mathbb{B}_{\alpha-1}$ and the restriction of π_α to it is isomorphic to the quotient τ (for the group $S_{\alpha-1}$). □

Remark 2.13
We will use stable deformations for semicontinuity arguments. If $\mathcal{F}$ is a stable deformation of $\mathcal{G}$, M is proper, and $H^p(\mathcal{G}) = 0$, then, by the semicontinuity theorem, $H^p(\mathcal{F}) = 0$, too. In particular, if $\mathcal{G}$ is Γ-acyclic, then so is $\mathcal{F}$.

Remark 2.14
Let $D = x_1 + \tilde{D}$, $\tilde{D} = x_2 + \cdots + x_\alpha$ (possibly with repetitions). Suppose that M is proper. Since $(\mathcal{F}^{\boxtimes\alpha})_D$ and $\mathcal{F}_{x_1} \otimes (\mathcal{F}^{\boxtimes(\alpha-1)})_{\tilde{D}}$ are both deformations of $\mathcal{F}_{x_1} \otimes \cdots \otimes \mathcal{F}_{x_\alpha}$ over $\mathbb{A}^1$ by Corollary 2.9, they have the same Euler characteristic. Combining this with Remark 2.13, if $H^p(\mathcal{F}_x \otimes (\mathcal{F}^{\boxtimes(\alpha-1)})_{\tilde{D}}) = 0$ for $p > 0$, then both $H^p((\mathcal{F}^{\boxtimes\alpha})_D) = 0$ for $p > 0$ and $H^0((\mathcal{F}^{\boxtimes\alpha})_D) = H^0(\mathcal{F}_x \otimes (\mathcal{F}^{\boxtimes(\alpha-1)})_{\tilde{D}})$. The same results hold for $(\overline{\mathcal{F}}^{\boxtimes\alpha})_D$ and $\mathcal{F}_x \otimes (\overline{\mathcal{F}}^{\boxtimes(\alpha-1)})_{\tilde{D}}$.

3. Wall-crossing on moduli spaces of stable pairs

Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. In [39], Thaddeus studies moduli spaces of pairs (E, ϕ), where E is a rank-2 vector bundle on C with fixed determinant line bundle Λ and $\phi \in H^0(E)$ is a nonzero section. We use these results extensively and so, for ease of reference, try to follow the notation in [39] as closely as possible. We always assume that $d = \deg E > 0$. For a given choice of a parameter $\sigma \in \mathbb{Q}$, the following stability condition is imposed: for every line subbundle $L \subset E$, one must have

$$\deg L \leq \begin{cases} \frac{d}{2} - \sigma & \text{if } \phi \in H^0(L), \\ \frac{d}{2} + \sigma & \text{if } \phi \notin H^0(L). \end{cases}$$

Throughout the text, we work with the general assumption $\sigma \in (0, d/2]$, which guarantees the existence of stable pairs (see [39, 1.3]). The next lemma follows the ideas of [39, 2.1].

LEMMA 3.1
For a given line bundle Λ of degree d, the moduli stack $\mathcal{M}_\sigma(\Lambda)$ of semistable pairs is a smooth algebraic stack.

Proof
$\mathcal{M}_\sigma(\Lambda)$ is a fiber of the morphism $\mathcal{M}_\sigma^d \to \mathrm{Pic}^d(C)$, $(E,\phi) \mapsto \det E$, from the stack of semistable pairs (E,ϕ), where E is a degree d vector bundle. We first show that $\mathcal{M}_\sigma^d$ is smooth. Obstructions to deformations of a morphism of sheaves ϕ from a fixed source $\mathcal{O}_C$ to a varying target E lie in $\mathrm{Ext}^1([\mathcal{O}_C \xrightarrow{\phi} E], E)$. The truncation exact triangle of the complex $[\mathcal{O}_C \xrightarrow{\phi} E]$ yields an exact sequence

$$\mathrm{Ext}^1(E,E) \xrightarrow{\phi} \mathrm{Ext}^1(\mathcal{O}_C, E) \to \mathrm{Ext}^1\big([\mathcal{O}_C \xrightarrow{\phi} E], E\big) \to 0.$$

We claim that the first map is surjective, so obstructions vanish. By Serre duality, it suffices to prove injectivity of the map of sheaves $E^*(K_C) \xrightarrow{\phi} E^* \otimes E(K_C)$ and this follows from $\phi \neq 0$ (cf. the proof of [39, 2.1]). Next we consider obstructions to deformations of (E,ϕ) fixing the determinant, which amounts to studying the map $\mathrm{Ext}^1(E,E)_0 \xrightarrow{\phi} \mathrm{Ext}^1(\mathcal{O}_C, E)$, where $\mathrm{Ext}^1(E,E)_0$ denotes traceless endomorphisms. However, this map is also surjective because the Serre-dual map is induced by the map of sheaves $E^*(K_C) \xrightarrow{\phi} \mathcal{E}nd(E)_0(K_C)$, where $\mathcal{E}nd(E)_0$ is identified with the quotient of $\mathcal{E}nd(E)$ by the subspace of scalar multiples of the identity. This map is still injective, as a nonzero scalar multiple of the identity cannot have rank 1. □

The moduli space $M_\sigma(\Lambda)$ of S-equivalence classes of stable pairs exists as a projective variety and, in the case there is no strictly semistable locus, it is smooth, isomorphic to the stack $\mathcal{M}_\sigma(\Lambda)$ and carries a universal bundle F with a universal section $\tilde{\phi} : \mathcal{O}_{C \times M_\sigma(\Lambda)} \to F$. A salient point is that stable pairs, unlike stable vector bundles, do not have any automorphisms besides the identity (see [39, 1.6]). Note that nontrivial multiples of the identity are not automorphisms, as they do not preserve the section ϕ.

The spaces $M_\sigma(\Lambda)$ can be obtained as GIT quotients as follows (see [39, Section 1] for further details). Let $\chi = \chi(E) = d + 2 - 2g$. For $d \gg 0$, every bundle E of rank 2 and $\det E = \Lambda$ is generated by global sections, and $\chi = h^0(E)$. Then $M_\sigma(\Lambda)$ is a GIT quotient of $U \times \mathbb{P}\mathbb{C}^\chi$ by SL_χ, where $U \subset \mathrm{Quot}$ is the locally closed subscheme of the Quot scheme (see [17]) corresponding to locally free quotients $\mathcal{O}_C^\chi \twoheadrightarrow E$ inducing an isomorphism $s : \mathbb{C}^\chi \xrightarrow{\sim} H^0(E)$ and such that $\wedge^2 E = \Lambda$. The isomorphism s induces a map $\wedge^2 \mathbb{C}^\chi \to H^0(\Lambda)$, and we get an inclusion $U \times \mathbb{P}\mathbb{C}^\chi \hookrightarrow \mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$, where we write $\mathbb{P}\mathrm{Hom}$ for $\mathbb{P}\mathrm{Hom}(\wedge^2\mathbb{C}^\chi, H^0(\Lambda))$, and a quotient $s : \mathcal{O}_C^\chi \twoheadrightarrow E$ on the left is sent to the induced map in the first coordinate. Then $M_\sigma(\Lambda)$ can be seen as the GIT quotient of a closed subset of $\mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$ by SL_χ, where the linearization is given by $\mathcal{O}(\chi + 2\sigma, 4\sigma)$.

For arbitrary d, we pick any effective divisor D on C with $\deg D \gg 0$, and $M_\sigma(\Lambda)$ can be seen as the closed subset of $M_\sigma(\Lambda(2D))$ consisting of pairs (E,ϕ)

such that $\phi|_D = 0$. This way, $M_\sigma(\Lambda)$ is a GIT quotient by $\mathrm{SL}_{\chi'}$, with $\chi' = d + 2 - 2g + 2\deg D$, of the closed subset $X \subset U' \times \mathbb{P}\mathbb{C}^{\chi'}$ determined by the condition that ϕ vanishes along D (see [39, 1.9, 1.20]). Regardless of the GIT, the embedding $M_\sigma(\Lambda) \subset M_\sigma(\Lambda(2D))$ will play an important role in our induction arguments.

Remark 3.2
Scalar matrices in $\mathrm{SL}_{\chi'}$ act trivially on $U \times \mathbb{P}\mathbb{C}^{\chi'}$, so the action factors through the quotient $\mathrm{SL}_{\chi'} \to \mathrm{PGL}_{\chi'}$. If we replace $\mathcal{O}(\chi' + 2\sigma, 4\sigma)$ by its χ'th power, this line bundle carries a $\mathrm{PGL}_{\chi'}$-linearization and $M_\sigma(\Lambda)$ can also be written as a GIT quotient $X /\!\!/ \mathrm{PGL}_{\chi'}$. Moreover, the moduli stack $\mathcal{M}_\sigma(\Lambda)$ is isomorphic to the corresponding GIT quotient stack $[X^{\mathrm{ss}}/\mathrm{PGL}_{\chi'}]$.

For fixed Λ but varying σ, the spaces $M_\sigma(\Lambda)$ are all GIT quotients of the same scheme, with different stability conditions. The GIT walls occur when $\sigma \in d/2 + \mathbb{Z}$, and for $0 \le i \le v = \lfloor (d-1)/2 \rfloor$ we have different GIT chambers with moduli spaces $M_0, M_1, \ldots, M_v$, where $M_i = M_i(\Lambda) = M_\sigma(\Lambda)$ for $\sigma \in (\max(0, d/2 - i - 1), d/2 - i)$. These M_i are smooth projective rational varieties of dimension $d + g - 2$ (see [39, 2.2, 3.6]). Indeed, $M_0 = \mathbb{P}H^1(C, \Lambda^{-1})$ is a projective space, M_1 is a blowup of M_0 along a copy of C embedded by the complete linear system of $\omega_C \otimes \Lambda$, and the remaining ones are small modifications of M_1. More precisely, for each $0 \le i \le v = \lfloor (d-1)/2 \rfloor$ there are projective bundles $\mathbb{P}W_i^+$ and $\mathbb{P}W_i^-$ over the symmetric product $\mathrm{Sym}^i C$, of (projective) ranks $d + g - 2i - 2$, $i - 1$, respectively, with embeddings $\mathbb{P}W_i^+ \hookrightarrow M_i$ and $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$, and such that $\mathbb{P}W_i^+$ parameterizes the pairs (E, ϕ) appearing in M_i but not in M_{i-1}, while $\mathbb{P}W_i^-$ parameterizes those appearing in M_{i-1} but not in M_i.

We have a diagram of flips (3.1), where $\tilde{M}_i$ is the blowup of M_{i-1} along $\mathbb{P}W_i^-$ and also the blowup of M_i along $\mathbb{P}W_i^+$. Here N is the moduli space of ordinary slope-semistable vector bundles as in the introduction and the map $M_v \to N$ is an "Abel–Jacobi" map with fiber $\mathbb{P}H^0(C, E)$ over a vector bundle E. If $d \ge 2g - 1$ the Abel–Jacobi map is surjective, and if $d = 2g - 1$ it is a birational morphism (see [39, Section 3] for details).

$$
\begin{array}{ccccccccc}
 & \tilde{M}_2 & & \tilde{M}_3 & & \tilde{M}_v & \\
\swarrow & & \searrow \ \swarrow & & \searrow \ \swarrow & & \searrow \\
M_1 & & M_2 & & \cdots & & M_v \\
\downarrow & & & & & & \downarrow \\
M_0 & & & & & & N
\end{array}
\tag{3.1}
$$

Notation 3.3
By abuse of notation, we will sometimes write $M_i(d)$ to denote the moduli space $M_i = M_i(\Lambda)$, where $d = \deg \Lambda$.

Notation 3.4
In what follows, v will always denote $\lfloor (d-1)/2 \rfloor$.

The Picard group of $M_1 = \mathrm{Bl}_C M_0$ is generated by a hyperplane section H in $M_0 = \mathbb{P}^{d+g-2}$ and the exceptional divisor E_1 of the morphism $M_1 \to M_0$. Since the maps $M_i \dashrightarrow M_{i+1}$ are small birational modifications for each $i \geq 1$, there are natural isomorphisms $\operatorname{Pic} M_1 \simeq \operatorname{Pic} M_i$, $i \geq 1$. The following notation is taken from [39, Section 5].

Definition 3.5
For each m, n, we denote the line bundle $\mathcal{O}_{M_1}((m+n)H - nE_1)$ by $\mathcal{O}_1(m,n)$, while $\mathcal{O}_i(m,n)$ will denote the image of $\mathcal{O}_{M_1}(m,n)$ under the isomorphism $\operatorname{Pic} M_1 \simeq \operatorname{Pic} M_i$.

Remark 3.6
By [39, 5.3], the ample cone of M_i is bounded by $\mathcal{O}_i(1, i-1)$ and $\mathcal{O}_i(1,i)$ for $0 < i < v$, while the ample cone of M_v is bounded below by $\mathcal{O}_v(1, v-1)$ and contains the cone bounded on the other side by $\mathcal{O}_v(2, d-2)$. In other words, the ray bounding the cone above has slope at least $(d-2)/2$.

Remark 3.7
For any effective divisor D on C of $\deg D = \alpha$, we have a closed immersion $M_{i-\alpha}(\Lambda(-2D)) \hookrightarrow M_i(\Lambda)$, as the locus of pairs (E, ϕ) where the section ϕ vanishes along D (see [39, 1.9]). The restriction of $\mathcal{O}_i(m,n)$ to $M_{i-\alpha}(\Lambda(-2D))$ is $\mathcal{O}_{i-\alpha}(m, n - m\alpha)$ (see [39, 5.7]). If $i - \alpha = 0$, the restriction of $\mathcal{O}_i(m,n)$ to $M_0(\Lambda(-2D)) = \mathbb{P}^r$ is $\mathcal{O}_{\mathbb{P}^r}(n + m(1-i))$. This follows from [39, 5.7] together with the fact that, for an embedding $\mathbb{P}^r = M_0(\Lambda(-2x)) \hookrightarrow M_1(\Lambda)$, $\mathcal{O}_{M_1}(E_1)$ restricts to $\mathcal{O}_{\mathbb{P}^r}(-1)$ while $\mathcal{O}_{M_1}(H)$ restricts to $\mathcal{O}_{\mathbb{P}^r}$.

Suppose that $d \gg 0$. Then the universal bundle F on $M_i \times C$ is the descent from the equivariant vector bundle $\mathcal{F}(1)$ on $X \times C \subset U \times \mathbb{P}\mathbb{C}^{\chi} \times C$, where $\mathcal{O}^{\chi} \twoheadrightarrow \mathcal{F}$ is the universal quotient bundle over $U \times C$, and the universal section $\tilde{\phi}$ descends from the universal section of $\mathcal{F}(1)$ (see [39, 1.19]). Let $\pi : C \times M_i \to M_i$ be the projection. For every $i \geq 1$, the determinant of cohomology line bundle $\det \pi_! F$ (cf. [21]) descends from $\mathcal{O}(0, \chi)$ on $\mathbb{P}\operatorname{Hom} \times \mathbb{P}\mathbb{C}^{\chi}$ (see [39, 5.4, proof of 5.14]). On M_1,

$\det \pi_! F$ corresponds to $\mathcal{O}_{M_1}((g-d-1)H-(g-d)E_1)=\mathcal{O}_1(-1,g-d)$. For $x \in C$, call $F_x = F|_{\{x\}\times M}$. The line bundle $\det F_x = \wedge^2 F_x$ does not depend on x, and it is the descent of $\mathcal{O}(1,2)$ on $\mathbb{P}\operatorname{Hom}\times\mathbb{P}\mathbb{C}^\chi$. It corresponds to $\mathcal{O}_{M_1}(E_1-H)=\mathcal{O}_i(0,-1)$ (see [39, 5.4, proof of 5.14]).

For arbitrary d, consider an embedding $\iota : M_i \hookrightarrow M' = M_\sigma(\Lambda(2D))$, $\deg D \gg 0$, as above, and let F' be the universal bundle on M'. Then we have a short exact sequence (see [39, 1.20])

$$0 \to F \to \iota^* F' \to \iota^* F'|_{D\times M_i} \to 0. \tag{3.2}$$

In particular, F is the descent from an object on $X \times C \subset U' \times \mathbb{P}\mathbb{C}^{\chi'} \times C$. The same is true for $\det \pi_! F$ and $\wedge^2 F_x$.

LEMMA 3.8
$F_x \simeq \iota^* F'_x$ for every $x \in C$.

Proof
We tensor (3.2) with $\mathcal{O}_{\{x\}\times M_i}$, which gives an exact sequence

$$\begin{aligned} 0 &\to \mathcal{T}or^1_{C\times M_i}(\iota^* F'|_{D\times M_i}, \mathcal{O}_{\{x\}\times M_i}) \to F_x \to \iota^* F'_x \\ &\to \iota^* F'|_{D\times M_i} \otimes_{C\times M_i} \mathcal{O}_{\{x\}\times M_i} \to 0. \end{aligned}$$

If $x \notin D$, then $\mathcal{T}or^1(\iota^* F'|_{D\times M_i}, \mathcal{O}_{\{x\}\times M_i}) = \iota^* F'|_{D\times M_i} \otimes \mathcal{O}_{\{x\}\times M_i} = 0$, and we get $F_x \simeq \iota^* F'_x$. If $x \in D$, then $\mathcal{T}or^1_C(\mathcal{O}_D, \mathcal{O}_x) \simeq \mathcal{O}_D \otimes_C \mathcal{O}_x \simeq \mathcal{O}_x$, and the sequence splits into two isomorphisms, $\iota^* F'_x \simeq F_x$ and $\iota^* F'_x \simeq \iota^* F'_x$. □

LEMMA 3.9
On $M_0 = \mathbb{P}^r$, $F_x \simeq \mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^r}(-1)$.

Proof
In fact, F_x is a rank-2 bundle on $\mathbb{P}^r$, carrying a nowhere vanishing section, and with determinant $\mathcal{O}_{\mathbb{P}^r}(-1)$. Hence, F_x must be isomorphic to $\mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^r}(-1)$. □

Definition 3.10
We introduce notation for some important line bundles:

$$\begin{aligned} \psi^{-1} &:= \det \pi_! F = \mathcal{O}_i(-1, g-d), \\ \Lambda_M &:= \wedge^2 F_x = \mathcal{O}_i(0,-1), \\ \zeta &:= \psi \otimes \Lambda_M^{d-2g+1} = \mathcal{O}_i(1, g-1) \end{aligned}$$

and

$$\theta := \psi^2 \otimes \Lambda_M^{\chi} = \mathcal{O}_i(2, d-2),$$

where $\chi = d + 2 - 2g$ (cf. [32, Proposition 2.1]).

LEMMA 3.11
For a point $x \in C$ and every $i \geq 1$, we have exact sequences

$$0 \to \Lambda_M^{-1} \to F_x^{\vee} \to \mathcal{O}_{M_i(\Lambda)} \to \mathcal{O}_{M_{i-1}(\Lambda(-2x))} \to 0 \tag{3.3}$$

and

$$0 \to \mathcal{O}_{M_i(\Lambda)} \to F_x \to \Lambda_M \to \Lambda_M|_{M_{i-1}(\Lambda(-2x))} \to 0. \tag{3.4}$$

Proof
By Remark 3.7, the zero locus of the section ϕ_x of F_x is smooth and has codimension 2. Therefore, the Koszul complex and the dual Koszul complex of (F_x, ϕ_x) are exact. □

Definition 3.12
Let $M = M_i(\Lambda)$ be a moduli space in the interior of a GIT chamber, as above, and let F be the universal bundle on $C \times M$. We apply the constructions of Section 2 to F. In particular, for a divisor $D \in \operatorname{Sym}^{\alpha} C$, we will denote

$$G_D = (F^{\boxtimes \alpha})_D \qquad \text{and} \qquad \overline{G}_D = (\overline{F}^{\boxtimes \alpha})_D.$$

We write $G_D^{\vee}, \overline{G}_D^{\vee}$ for their respective duals.

LEMMA 3.13
We have the following formulas:

$$(F^{\vee})^{\boxtimes \alpha} \simeq \big((\Lambda^{\vee})^{\boxtimes \alpha} \boxtimes \Lambda_M^{-\alpha}\big) \otimes F^{\boxtimes \alpha},$$
$$G_D^{\vee} \simeq (\overline{F^{\vee}}^{\boxtimes \alpha})_D, \qquad \overline{G}_D^{\vee} \simeq \big((F^{\vee})^{\boxtimes \alpha}\big)_D.$$

Proof
Let us denote

$$\widehat{\Lambda^{\vee}}^{\boxtimes \alpha} := \bigotimes_{j=1}^{\alpha} \pi_j^*(\Lambda^{\vee}), \qquad \widehat{F}^{\boxtimes \alpha} := \bigotimes_{j=1}^{\alpha} \pi_j^* F,$$

which are bundles on C^α and $C^\alpha \times M$, respectively. By [14, Théorème 2.3], $(\Lambda^\vee)^{\boxtimes\alpha}$ is the descent of $\widehat{\Lambda^\vee}^{\boxtimes\alpha}$. Using this together with the fact that $F^\vee \simeq F \otimes (\det F)^{-1} \simeq F \otimes (\Lambda \boxtimes \Lambda_M)^{-1}$, we get

$$(F^\vee)^{\boxtimes\alpha} \simeq \tau_*^{S_\alpha}\big((\widehat{\Lambda^\vee}^{\boxtimes\alpha} \boxtimes \Lambda_M^{-\alpha}) \otimes \widehat{F}^{\boxtimes\alpha}\big) \simeq \big((\Lambda^\vee)^{\boxtimes\alpha} \boxtimes \Lambda_M^{-\alpha}\big) \otimes \tau_*^{S_\alpha}(\widehat{F}^{\boxtimes\alpha}).$$

The latter expression is precisely $((\Lambda^\vee)^{\boxtimes\alpha} \boxtimes \Lambda_M^{-\alpha}) \otimes F^{\boxtimes\alpha}$.

We write $\mathcal{O}_{\mathrm{Sym}^\alpha C}(-\Delta/2) := \tau_*^{S_\alpha}(\mathcal{O}_{C^\alpha} \otimes \mathrm{sgn})$, a line bundle on $\mathrm{Sym}^\alpha C$ such that $\mathcal{O}_{\mathrm{Sym}^\alpha C}(-\Delta/2)^{\otimes 2} \simeq \mathcal{O}_{\mathrm{Sym}^\alpha C}(-\Delta)$, where $\Delta \subset \mathrm{Sym}^\alpha C$ is the diagonal divisor. The morphism τ is ramified along $B = \tau^{-1}(\Delta)$ generically of order 2, so $\mathcal{O}_{C^\alpha}(B)$ is a relative dualizing sheaf for τ. The equivariant structure on $\mathcal{O}_{C^\alpha}(B)$ is dual to the equivariant structure of the ideal sheaf $\mathcal{O}_{C^\alpha}(-B) \subset \mathcal{O}_{C^\alpha}$. Since the local equation of B is anti-invariant, $\mathcal{O}_{C^\alpha}(B) \simeq \tau^* \mathcal{O}_{\mathrm{Sym}^\alpha C}(\Delta/2) \otimes \mathrm{sgn}$.

By duality,

$$\big((F^\vee)^{\boxtimes\alpha}\big)^\vee \simeq \tau_*^{S_\alpha}\big(\widehat{F}^{\boxtimes\alpha}(B)\big) \simeq \tau_*^{S_\alpha}(\widehat{F}^{\boxtimes\alpha} \otimes \mathrm{sgn})(\Delta/2) \simeq \overline{F}^{\boxtimes\alpha}(\Delta/2).$$

Restricting to a divisor $D \in \mathrm{Sym}^\alpha C$, we obtain

$$\big((F^\vee)^{\boxtimes\alpha}\big)_D^\vee \simeq (\overline{F}^{\boxtimes\alpha})_D,$$

and similarly, arguing with $F^\vee$ in place of F, we get

$$(F^{\boxtimes\alpha})_D^\vee \simeq (\overline{F^\vee}^{\boxtimes\alpha})_D.$$

This completes the proof. □

COROLLARY 3.14
We have $G_D^\vee \simeq \overline{G}_D \otimes \Lambda_M^{-\deg D}$ *and* $G_D \simeq \overline{G}_D^\vee \otimes \Lambda_M^{\deg D}$.

Proof
This follows from restricting $(F^\vee)^{\boxtimes\alpha} \simeq ((\Lambda^\vee)^{\boxtimes\alpha} \boxtimes \Lambda_M^{-\alpha}) \otimes F^{\boxtimes\alpha}$ to $\{D\} \times M$. □

Consider again the diagram (3.1). The wall between two consecutive chambers M_{i-1} and M_i occurs at $\sigma = d/2 - i$. The birational transformation $M_{i-1} \dashrightarrow M_i$ is an isomorphism outside of the loci $\mathbb{P}W_i^- \subset M_{i-1}$, $\mathbb{P}W_i^+ \subset M_i$, where W_i^- and W_i^+ are vector bundles over the symmetric product $\mathrm{Sym}^i C$ of rank i and $d + g - 1 - 2i$, respectively. We have a diagram

$$\begin{array}{ccccc} & & \tilde{M} & & \\ & \swarrow & & \searrow & \\ M_{i-1} = M_{\sigma+\epsilon} & & & & M_{\sigma-\epsilon} = M_i \\ & \searrow & & \swarrow & \\ & & M_\sigma & & \end{array}$$

where $\tilde{M}$ is both the blowup of $M_{\sigma+\epsilon} = M_{i-1}$ along $\mathbb{P}W_i^-$ and the blowup of $M_{\sigma-\epsilon} = M_i$ along $\mathbb{P}W_i^+$. The variety M_σ is singular, obtained from the contraction to $\operatorname{Sym}^i C$ of the exceptional locus $\mathbb{P}W_i^- \times_{\operatorname{Sym}^i C} \mathbb{P}W_i^+ \subset \tilde{M}$.

When $d \gg 0$, $M_{\sigma\pm\epsilon}(\Lambda)$ and $M_\sigma(\Lambda)$ are obtained as GIT quotients of $U \times \mathbb{P}\mathbb{C}^\chi$, with $\chi = d+2-2g$. When d is arbitrary, take an effective divisor D' of large degree, so that $M_\sigma \hookrightarrow M'_\sigma := M_\sigma(\Lambda(2D'))$, where M'_σ is a GIT quotient with a semistable locus $X' \subset U' \times \mathbb{P}\mathbb{C}^{\chi'}$, $\chi' = d+2-2g+2\deg D'$. The spaces $M_{\sigma\pm\epsilon}(\Lambda)$ and $M_\sigma(\Lambda)$ are then GIT quotients by $\mathrm{SL}_{\chi'}$ of a closed subset of $U' \times \mathbb{P}\mathbb{C}^{\chi'}$ determined by the condition that in the pair (E', ϕ'), the section ϕ' vanishes along D'. If we call $\mathcal{L}_\pm$, $\mathcal{L}_0$ the corresponding linearizations, then we can write $X \subset X'$, the semistable locus of $\mathcal{L}_0$, as the union $X = X^{\mathrm{ss}}(\mathcal{L}_+) \cup X^{\mathrm{ss}}(\mathcal{L}_-) \sqcup Z$, where the locus $Z = X^u(\mathcal{L}_+) \cap X^u(\mathcal{L}_-)$ corresponds to pairs (E', ϕ'), such that E' splits as

$$E' = L' \oplus K',$$

with $\deg L' = i + \deg D'$, $\deg K' = d - i + \deg D'$, and $\phi' \in H^0(L')$ vanishes along D' (see [39, 1.4]). The map $\mathcal{O}_C^{\chi'} \twoheadrightarrow E'$ is then given by a block-diagonal matrix $(\mathcal{O}_C^a \twoheadrightarrow L') \oplus (\mathcal{O}_C^b \twoheadrightarrow K')$, where $a = h^0(L')$, $b = h^0(K')$, and $a + b = h^0(L' \oplus K') = \chi'$. The strictly semistable locus $X^{\mathrm{sss}}(\mathcal{L}_0) = X^u(\mathcal{L}_+) \cup X^u(\mathcal{L}_-)$ consists of the orbits whose closure intersects Z (cf. [36, Remark 7.4]).

Using techniques from [18] and [2], we compare the derived categories on either side of the wall M_σ. We write $M_{\sigma\pm\epsilon} = X /\!\!/_{\mathcal{L}_\pm} \mathrm{PGL}_{\chi'}$ (cf. Remark 3.2) and take Kempf–Ness (KN) stratifications of the unstable loci $X^u(\mathcal{L}_\pm)$ with strata $S_\pm^j$ determined by pairs $(Z^j, \lambda_\pm^j)$, where $\lambda_-^j(t) = \lambda_+^j(t)^{-1}$ are one-parameter subgroups and Z^j is the fixed locus of $\lambda^j = \lambda_+^j$ (see [18, Section 2.1] for details).

Remark 3.15
From the discussion above, it follows that in this case the KN stratification of the unstable locus in X with respect to $\mathcal{L}_\pm$ has only one stratum $S_\pm$, parameterizing framed extensions as in [39, (3.2), (3.3)]. In the notation of [18, Section 2], the stratum $S_\pm$ is determined by the pair (Z, λ), where $\lambda = \lambda_+ = \mathbb{G}_m$ is the stabilizer of Z, and some power of λ acts on a split bundle $E' = L' \oplus K'$ by (t^b, t^{-a}).

Remark 3.16
Let $\mathfrak{Z}$ be the stack $[Z/\mathbb{L}]$, where $\mathbb{L}$ is the Levi subgroup, that is, the centralizer of λ in $\mathrm{PGL}_{\chi'}$. We have a short exact sequence of groups $1 \to \mathbb{G}_m \to \mathbb{L} \to \mathrm{PGL}_a \times \mathrm{PGL}_b \to 1$ with $\mathbb{G}_m = \lambda$ acting on Z trivially and $[Z/\mathrm{PGL}_a \times \mathrm{PGL}_b] \simeq \mathrm{Sym}^i C$. Indeed, the action of $\mathrm{PGL}_a \times \mathrm{PGL}_b$ on Z is free, and each orbit is determined by a divisor $D \in \mathrm{Sym}^i C$, where $D + D'$ is the zero locus of the section $\phi' \in H^0(L')$. Therefore $\mathfrak{Z} \simeq [\mathrm{Sym}^i C/\mathbb{G}_m]$, with the trivial action of $\mathbb{G}_m$.

For $\sigma = d/2 - i$ with $1 < i \leq v$, $M_{\sigma\pm\epsilon}$ $(\epsilon > 0)$ is isomorphic to the corresponding quotient stack, since the action of $\mathrm{PGL}_{\chi'}$ is free on the stable locus by [39, 1.6]. Let $\eta_\pm = \mathrm{weight}_{\lambda_\pm} \det \mathcal{N}^\vee_{S_\pm/X}|_Z$. For any choice of an integer w, $D^b(M_{\sigma\pm\epsilon})$ is equivalent to the window subcategory $G^\pm_w \subset D^b([X/\mathrm{PGL}_{\chi'}])$ determined by objects having $\lambda_\pm$-weights in the range $[w,\ w+\eta_\pm)$ for the unique stratum $S_\pm$ (see [18, Theorem 2.10]). If $\mathrm{weight}_\lambda\, \omega_X|_Z = \eta_- - \eta_+ > 0$, then we get an embedding $D^b(M_{\sigma+\epsilon}) \subset D^b(M_{\sigma-\epsilon})$ (see [18, Proposition 4.5] and the Remark following it).

LEMMA 3.17
In the wall-crossing between the spaces $M_{\sigma+\epsilon}(\Lambda) = M_{i-1}$ and $M_{\sigma-\epsilon}(\Lambda) = M_i$, the window has width $\eta_+ = i$, $\eta_- = d + g - 1 - 2i$.

Proof
We use the notation as in the discussion above, with $M_\sigma \hookrightarrow M'_\sigma := M_\sigma(\Lambda(2D'))$, D' effective with $\deg D' \gg 0$. For $\mathcal{L}_\pm$, there is no strictly semistable locus and in fact $\mathrm{PGL}_{\chi'}$ acts freely on the semistable locus (see [39, 1.6]), so $M_{i-1} = M_{\sigma+\epsilon}(\Lambda) = X /\!\!/_{\mathcal{L}_+} \mathrm{SL}_{\chi'}$ and $M_i = M_{\sigma-\epsilon}(\Lambda) = X /\!\!/_{\mathcal{L}_-} \mathrm{SL}_{\chi'}$ are isomorphic to the quotient stacks $[X^{\mathrm{ss}}(\mathcal{L}_\pm)/\mathrm{PGL}_{\chi'}]$ (cf. Remark 3.2). By Lemma 3.1, both $[X/\mathrm{PGL}_{\chi'}]$ and $[X'/\mathrm{PGL}_{\chi'}]$ are smooth quotient stacks of dimension $d + g - 2$ and $d + g - 2 + 2\deg D'$, respectively, and thus X and X' are both smooth and $X \subset X'$ is a local complete intersection cut out precisely by the $2\deg D'$ conditions imposed by the vanishing of a section along D'.

Recall that the unique KN stratum of $X^u(\mathcal{L}_\pm)$ is determined by (Z, λ) (cf. Remark 3.15), where for a pair $(E', \phi') \in Z$, the bundle $E' = L' \oplus K'$ is acted on by (some power of) $\lambda = \mathbb{G}_m$ by (t^b, t^{-a}). We will first compute the weights with respect to this action, and later rescale according to the parameterization that describes the whole one-parameter subgroup. By [36, Lemma 7.6] and its proof, the λ-weights of $\mathcal{N}^\vee_{S_\pm/X'}$ on Z are all $\pm(a + b) = \pm\chi'$ or 0. Then the weights of $\mathcal{N}^\vee_{S_\pm/X}$ are all $\pm\chi'$, and $\eta_\pm = \mathrm{weight}_{\lambda_\pm} \det \mathcal{N}^\vee_{S_\pm/X}|_Z$ is just the codimension of $S_\pm \subset X$. Since $S_\pm$ is the bundle $W_i^\pm$ on Z, we have $\mathrm{codim}(S_\pm \subset X) = \mathrm{rk}\, W_i^\mp$, so that $\eta_+ = i\chi'$ and $\eta_- = (d + g - 1 - 2i)\chi'$.

As a one-parameter subgroup of $\mathrm{PGL}_{\chi'}$, λ is given by sending $t \mapsto \mathrm{diag}(s^b, \dots, s^b, s^{-a}, \dots, s^{-a})$, where $s^{\chi'} = t$. Indeed, note that this is well defined, since when replacing s by ξs, with ξ a χ'th root of unity, the matrix $\lambda(t)$ gets scaled by $\xi^b = \xi^{-a}$. Also, note that this λ is injective, whereas its χ'th power $t \mapsto \mathrm{diag}(t^b, \dots, t^b, t^{-a}, \dots, t^{-a})$ is not. Therefore, all weights computed above need to be rescaled by $1/\chi'$. This gives the formulas in the statement. □

Using this we obtain the following result.

PROPOSITION 3.18
For $1 \le i \le \frac{d+g-1}{3}$ (resp., $i \ge \frac{d+g-1}{3}$) there is an admissible embedding $D^b(M_{i-1}) \hookrightarrow D^b(M_i)$ (resp., $D^b(M_i) \hookrightarrow D^b(M_{i-1})$). When $1 < i \le \frac{d+g-1}{3}$, the admissible embedding can be chosen to be the window subcategory $G_0^+ \subset D^b(M_i)$ determined by the range of weights $[0, i) \subset [0, d+g-1-2i)$ (cf. [18]) and moreover there is a semiorthogonal decomposition

$$D^b(M_i) = \langle D^b(M_{i-1}), D^b(\mathrm{Sym}^i C), \dots, D^b(\mathrm{Sym}^i C)\rangle \tag{3.5}$$

with $\mu = d + g - 3i - 1$ copies of $D^b(\mathrm{Sym}^i C)$ given by the fully faithful images of functors $Rj_*(L\pi^*(\cdot) \otimes^L \mathcal{O}_\pi(l)) : D^b(\mathrm{Sym}^i C) \to D^b(M_i)$ for $l = 0, \dots, \mu - 1$, where $\pi : \mathbb{P}W_i^+ \to \mathrm{Sym}^i C$ is the projection and $j : \mathbb{P}W_i^+ \hookrightarrow M_i$ the inclusion.

The semiorthogonal decomposition (3.5) follows from [4], as the birational transformation between M_{i-1} and M_i is a standard flip of projective bundles over $\mathrm{Sym}^i C$. Here we provide an alternative proof for this case. We also note that [36, Corollary 8.1] shows the admissible embeddings $D^b(M_{i-1}) \hookrightarrow D^b(M_i)$ when i is in the specified range.

As explained in the introduction, Proposition 3.18 does not provide a semiorthogonal decomposition with Fourier–Mukai functors associated with Poincaré bundles and it is not used in our paper. However, we find this result relevant.

Proof
If $i = 1$, this follows from Orlov's blowup formula in [35]. Let $i > 1$. From Lemma 3.17, $\mathrm{weight}_\lambda\, \omega_X|_Z = \eta_- - \eta_+ = (d + g - 1 - 3i)$. By [18, Proposition 4.5, Remark 4.6], and since $M_{\sigma\pm\epsilon} \simeq [X^{\mathrm{ss}}(\mathcal{L}_\pm)/\mathrm{PGL}_{\chi'}]$, we get a window embedding $D^b(M_{\sigma+\epsilon}) \subset D^b(M_{\sigma-\epsilon})$ if $\eta_+ \le \eta_-$ and the other way around if $\eta_+ \ge \eta_-$. Moreover, if $G_w^+ = D^b(M_{\sigma+\epsilon})$ is a window, determined by the range of weights $[w,\ w + \eta_+) \subset [w,\ w + \eta_-)$, then [18, Theorem 2.11] and [2, Theorem 1] give semiorthogonal blocks $D^b(\mathfrak{Z})_k$, so that

$$D^b(M_{\sigma-\epsilon}) = \langle G_w^+, D^b(\mathfrak{Z})_w, \dots, D^b(\mathfrak{Z})_{w+\mu-1}\rangle, \tag{3.6}$$

where $\mu = \eta_- - \eta_+$ and $\mathfrak{Z} = [Z/\mathbb{L}]$ is the quotient stack by the Levi subgroup. By Remark 3.16, $D^b(\mathfrak{Z}) = D^b_{\mathbb{G}_m}(\mathrm{Sym}^i C)$, so the blocks in (3.6) are given by the fully faithful images of $Rj_*(L\pi^*(\cdot) \otimes^L \mathcal{O}_\pi(l)) : D^b(\mathrm{Sym}^i C) \to D^b(M_i)$ for $l \in [w, w + \mu)$, where $\pi : \mathbb{P}W_i^+ \to \mathrm{Sym}^i C$ is the projection and $j : \mathbb{P}W_i^+ \hookrightarrow M_i$ the inclusion. Taking $w = 0$ gives the claim. ☐

COROLLARY 3.19
If $d \leq 2g - 1$, then $D^b(M_{i-1}) \subset D^b(M_i)$ for any $1 \leq i \leq v$.

Proof
In this case $i \leq (d-1)/2 \leq g - 1$, so the inequality $i < (d + g - 1)/3$ holds for every i. ☐

Consider an object G in $D^b([X/\mathrm{PGL}_{\chi'}])$ descending to some objects on $D^b(M_{i-1})$ and $D^b(M_i)$. We can use windows to determine when such object can "cross the wall." Namely, if the weights of G are in the required range, cohomology groups will be the same on either side. By abuse of notation, we often denote in the same way both the object on $D^b([X/\mathrm{PGL}_{\chi'}])$ and the objects it descends to in $M_{\sigma\pm\epsilon}(\Lambda)$.

THEOREM 3.20
Let $\sigma = d/2 - i$, $1 < i \leq v$. If A, B are objects in $D^b([X/\mathrm{PGL}_{\chi'}])$, with $\lambda = \lambda_+$-weights satisfying the inequalities

$$1 + 2i - d - g < \mathrm{weight}_\lambda B|_Z - \mathrm{weight}_\lambda A|_Z < i, \tag{3.7}$$

then $R\,\mathrm{Hom}_{M_{\sigma+\epsilon}}(A, B) = R\,\mathrm{Hom}_{M_{\sigma-\epsilon}}(A, B)$. In particular, if $1 + 2i - d - g <$ $\mathrm{weight}_\lambda B|_Z < i$, then $R\Gamma_{M_{i-1}}(B) = R\Gamma_{M_i}(B)$.

Proof
By Lemma 3.17, (3.7) is equivalent to the inequalities

$$-\eta_- < \mathrm{weight}_\lambda B|_Z - \mathrm{weight}_\lambda A|_Z < \eta_+,$$

so the quantization theorem (see [18, Theorem 3.29]) implies that

$$R\,\mathrm{Hom}_{M_{\sigma+\epsilon}}(A, B) = R\,\mathrm{Hom}_{[X/\mathrm{PGL}_{\chi'}]}(A, B) = R\,\mathrm{Hom}_{M_{\sigma-\epsilon}}(A, B).$$

Indeed, the first equality follows directly from [18, Theorem 3.29] applied on $M_{\sigma+\epsilon}$, while the second is the same theorem applied on $M_{\sigma-\epsilon}$, using the fact that $\mathrm{weight}_{\lambda\_} B|_Z - \mathrm{weight}_{\lambda\_} A|_Z = -(\mathrm{weight}_\lambda B|_Z - \mathrm{weight}_\lambda A|_Z)$. ☐

We finish this section with the computation of all weights that we need in order to construct the semiorthogonal decompositions.

THEOREM 3.21
The objects of the form F_x, Λ_M, ψ, ζ, G_D on both M_{i-1} and M_i are the descents of objects $\tilde{F}_x$, $\tilde{\Lambda}_M$, $\tilde{\psi}$, $\tilde{\zeta}$, $\tilde{G}_D$ on $D^b([X/\mathrm{PGL}_{\chi'}])$ that have λ-weights

$$\begin{aligned}\mathrm{weights}_\lambda\, \tilde{F}_x|_Z &= \{0,-1\},\\ \mathrm{weight}_\lambda\, \tilde{\Lambda}_M|_Z &= -1,\\ \mathrm{weight}_\lambda\, \tilde{\psi}|_Z &= d+1-g-i,\\ \mathrm{weight}_\lambda\, \tilde{\zeta}|_Z &= g-i,\\ \mathrm{weights}_\lambda\, \tilde{G}_D|_Z &= \{0,-1,\dots,-\deg D\}.\end{aligned}$$

Proof
Let $\sigma = d/2 - i$, and embed $\iota : M_\sigma(\Lambda) \hookrightarrow M'_\sigma = M_\sigma(\Lambda(2D'))$ for an effective divisor D', $\deg D' \gg 0$, as usual. Recall that the universal bundle F' on $C \times M'_{\sigma\pm\epsilon}$ is the descent of $\mathcal{F}'(1)$ on $C \times X' \subset C \times U' \times \mathbb{PC}^{\chi'}$, where $\mathcal{F}'$ is the universal family on $C \times U'$ (see [39, 1.19]). Let us compute the λ-weights of $\mathcal{F}'_x(1)$ on the σ-strictly semistable locus, for a point $x \in C$. The fiber of $\mathcal{F}'_x$ over $L' \oplus K'$ is $L'_x \oplus K'_x$, which is acted on with weights b in the first component and $-a$ in the second. Since the λ-weight of $\mathcal{O}_{\mathbb{PC}^{\chi'}}(1)$ over the section $(\phi', 0)$ is $-b$, the weights of $\mathcal{F}'_x(1)$ are 0 and $-a-b = -\chi'$. By Lemma 3.8, we have $F_x \simeq \iota^* F'_x$. Hence, F_x also is the descent of an object with weights 0 and $-\chi'$.

The bundle $\det \pi_! F'$ descends from $\det \pi_! \mathcal{F}'(1)$. On the fiber of $\pi_! \mathcal{F}'$ over $L' \oplus K'$, λ acts on $H^0(L') \oplus H^0(K')$ with weights b and $-a$, with multiplicities $h^0(L') = a$ and $h^0(K') = b$, respectively. Taking the tensor product with $\mathcal{O}_{\mathbb{PC}^{\chi'}}(1)$ shifts each weight by $-b$, and then taking the determinant we get $\mathrm{weight}_\lambda \det \pi_! \mathcal{F}'(1)|_{Z'} = 0 \cdot a + (-a-b)\cdot b = -b\chi'$. For $\det F'_x$, which is the descent of $\det \mathcal{F}'_x(1)$, we see that λ acts with weights $b, -a$ on $L'_x \oplus K'_x$ and then shifting by $-b$ and taking determinants we get $\mathrm{weight}_\lambda \det \mathcal{F}'_x(1)|_{Z'} = -a-b = -\chi'$.

Now for the universal bundle F on $C \times M_{\sigma\pm\epsilon}(\Lambda)$, we use the short exact sequence (3.2). From this we see that $\Lambda_M = \det F_x \simeq \det F'_x$ is the descent of an object with λ-weight equal to $-\chi'$. Also, since $\det \pi_! F'|_{D' \times M_{\sigma\pm\epsilon}} = \det \bigoplus_{x \in D'} F'_x = (\det F'_x)^{\deg D'}$, we obtain that $\psi^{-1} = \det \pi_! F = \det \pi_! F' \otimes (\det F'_x)^{-\deg D'}$ is the descent of an object with λ-weight equal to $-b\chi' + \deg D' \chi'$. Recall that $\deg L' = i + \deg D'$, $\deg K' = d - i + \deg D'$ (see the discussion before Remark 3.15), so by Riemann–Roch $b = h^0(K') = d - i + \deg D' + 1 - g$ and the weight of ψ is $-\chi'(\deg D' - b) = \chi'(d+1-g-i)$. As for $\zeta = \psi \otimes \Lambda_M^{d-2g+1}$, the weights must be $(d+1-g-i-(d-2g+1))\chi' = (g-i)\chi'$. Rescaling everything by $1/\chi'$ as in Lemma 3.17, we get the weights as in the statement.

Finally, we consider G_D. Let $D = x_1 + \cdots + x_\alpha$. Since by construction tensor bundles are functorial in M, the bundle G_D is the descent of a vector bundle $(\mathcal{E}^{\boxtimes\alpha})_D$ on X, where $M = X /\!\!/ \mathrm{SL}_{\chi'}$ and $\mathcal{E}$ descends to F. By Lemma 2.9, $(\mathcal{E}^{\boxtimes\alpha})_D$ is a deformation of $\mathcal{E}_{x_1} \otimes \cdots \otimes \mathcal{E}_{x_\alpha}$, and the deformation can be chosen to be $\mathrm{SL}_{\chi'}$-equivariant (see Remark 2.10). Therefore, $(\mathcal{E}^{\boxtimes\alpha})_D$ has the same weights as the tensor product $\mathcal{E}_{x_1} \otimes \cdots \otimes \mathcal{E}_{x_\alpha}$, that is, $0, -1, \ldots, -\alpha$. □

Remark 3.22
Observe that $\mathcal{O}_i(1,0) = \psi \otimes \Lambda_M^{d-g}$ and $\mathcal{O}_i(0,1) = \Lambda_M^{-1}$, so we can use the previous theorem to see that in general, a line bundle $\mathcal{O}_i(m,n)$ is the descent on both M_{i-1} and M_i of an object having λ-weight $m(1-i)+n$ on the strictly semistable locus of the wall.

4. Acyclic vector bundles on M_i—easy cases

In order to prove Theorem 1.1, we will first construct fully faithful functors $\Phi^i_\alpha : D^b(\mathrm{Sym}^\alpha C) \hookrightarrow D^b(M_i)$ for $\alpha \le i$ and show that, after suitable twists, the essential images of these functors are semiorthogonal to each other in the required way (see Theorem 9.3, Definition 10.1, and Theorem 10.4 below). By means of Bondal–Orlov's criterion in [7], this reduces to the computation of $R\Gamma$ for a large class of vector bundles on M_i. In particular, we will need to prove Γ-acyclicity for several of these vector bundles.

THEOREM 4.1
Let $d > 2$ and $1 \le i \le v$. Let $D = x_1 + \cdots + x_\alpha$, $D' = y_1 + \cdots + y_\beta$ (possibly with repetitions). Suppose that

$$\deg D - g < t < d - \deg D' - i - 1.$$

Then

$$R\Gamma_{M_i(d)}\Big(\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}\Big) = 0. \tag{4.1}$$

Remark 4.2
By Corollary 2.9 and semicontinuity, the same vanishing holds if in (4.1) we replace $\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee$ by either $G_D^\vee$ or $\overline{G}_D^\vee$ and $\bigotimes_{k=1}^{\beta} F_{y_k}$ by either $G_{D'}$ or $\overline{G}_{D'}$.

We start with a lemma.

LEMMA 4.3

$R\Gamma_{M_1(d)}(\mathcal{O}_{M_1(d)}(-kH+lE_1))=0$ for $0<k\le d+g-2$ and $0\le l\le d+g-4$. In particular, taking $t=k=l$ we get $R\Gamma_{M_1(d)}(\Lambda_M^t)=0$ for $0<t\le d+g-4$.

Proof

Consider the short exact sequence

$$0\to\mathcal{O}_{M_1(d)}\to\mathcal{O}_{M_1(d)}(E_1)\to\mathcal{O}_\pi(-1)\to 0, \tag{4.2}$$

where $E_1=\mathbb{P}W_1^+$ and $\pi: E_1\to C$ is the $\mathbb{P}^r$-bundle, $r=d+g-4$; $\mathcal{O}_{M_1(d)}(-kH)$ is Γ-acyclic provided that $0<k\le d+g-2=\dim M_1(d)$. Then twisting (4.2) by $\mathcal{O}_{M_1(d)}(-kH)$ and taking a long exact sequence in cohomology gives Γ-acyclicity of $\mathcal{O}_{M_1(d)}(-kH+E_1)$ for such k. Similarly, twisting by powers of $\mathcal{O}_{M_1(d)}(E_1)$ and using induction, we get that $R\Gamma_{M_1(d)}(\mathcal{O}_{M_1(d)}(-kH+lE))=0$ as well, since $\mathcal{O}_\pi(-l)$ is Γ-acyclic for $0<l\le d+g-4$. □

We will prove Theorem 4.1 by induction, starting with the base case $i=1$.

LEMMA 4.4

The statement of Theorem 4.1 holds for $i=1$.

Proof

Let $\alpha=\deg D$, $\beta=\deg D'$. We are given that $\alpha-g<t<d-\beta-2$. We do induction on $\alpha+\beta$. If $\alpha=\beta=0$, then we have to check that $\Lambda_M^t\otimes\zeta^{-1}=-(t+g)H+(g+t-1)E_1$ is Γ-acyclic on $M_1(d)$. By Lemma 4.3, this holds provided that $0<t+g\le d+g-2$ and $0\le g+t-1\le d+g-4$, which is true by hypothesis.

If $\alpha>0$, then we write $D=\tilde{D}+x_\alpha$. Consider the exact sequence (3.3) from Lemma 3.11 and twist it by $U:=\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee\otimes\bigotimes_{k=1}^{\beta}F_{y_k}\otimes\Lambda_M^t\otimes\zeta^{-1}$ to get

$$0\to\Lambda_M^{-1}\otimes U\to\bigotimes_{k=1}^{\alpha}F_{x_k}^\vee\otimes\bigotimes_{k=1}^{\beta}F_{y_k}\otimes\Lambda_M^t\otimes\zeta^{-1}\to U\to U|_{M_0(d-2)}\to 0. \tag{4.3}$$

The restriction of F_y to $M_0(d-2)=\mathbb{P}^r$, $r=d+g-4$, is equal to $\mathcal{O}_{\mathbb{P}^r}\oplus\mathcal{O}_{\mathbb{P}^r}(-1)$ by Lemma 3.9. Therefore, we see that the restriction of the bundle $U=\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee\otimes\bigotimes_{k=1}^{\beta}F_{y_k}\otimes\Lambda_M^t\otimes\zeta^{-1}$ to $M_0(d-2)$ is a sum of bundles $\bigoplus\mathcal{O}_{\mathbb{P}^r}(s_j)\otimes\mathcal{O}_{\mathbb{P}^r}(1-t-g)$, with $-\beta\le s_j\le\alpha-1$ (cf. Remark 3.7). These are all Γ-acyclic on $\mathbb{P}^{d+g-4}$, since by hypothesis

$$\alpha-t-g<0,\qquad -\beta+1-t-g\ge-(d+g-4). \tag{4.4}$$

Now we look at the first and third terms from the sequence (4.3), which are $\bigotimes_{k=1}^{\alpha-1} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}$ and $\bigotimes_{k=1}^{\alpha-1} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^{t-1} \otimes \zeta^{-1}$. We observe that they both satisfy the inequalities of the hypothesis, so by induction they are Γ-acyclic on $M_1(d)$. Therefore, applying $R\Gamma$ to (4.3) we see that the second term must also be Γ-acyclic, as desired.

Similarly, if $\beta > 0$, then we write $D' = \tilde{D}' + y_\beta$ and use the exact sequence

$$0 \to \mathcal{O}_{M_1(d)} \to F_{y_\beta} \to \Lambda_M \to \Lambda_M|_{M_0(d-2)} \to 0,$$

twisted by $\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta-1} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}$. The resulting term on the right is a sum $\bigoplus \mathcal{O}_{\mathbb{P}^r}(s_j) \otimes \mathcal{O}_{\mathbb{P}^r}(-t-g)$, with $-\beta + 1 \le s_j \le \alpha$, and it is again Γ-acyclic by the same inequalities (4.4). Finally, the remaining two terms are Γ-acyclic by induction, and we conclude that $R\Gamma_{M_1(d)}(\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}) = 0$ as well. □

Proof of Theorem 4.1
Let $\alpha = \deg D$ and $\beta = \deg D'$. We do induction on i. If $i = 1$, then this is Lemma 4.4. Let $i > 1$, and suppose that the statement holds for $i - 1$. For t in the given range, we have

$$R\Gamma_{M_{i-1}(d)}\Big(\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}\Big) = 0$$

by the induction hypothesis. Consider the wall-crossing between M_{i-1} and M_i. Here, the bundle $\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}$ descends from an object with weights $\{-\beta - t + i - g, \ldots, \alpha - t + i - g\}$ (see Theorem 3.21). Our hypothesis guarantees that $\alpha - t + i - g < i = \eta_+$ and $-\beta - t + i - g > 1 + 2i - d - g = -\eta_-$, that is, all these weights live in the range $(-\eta_-, \eta_+)$. By Theorem 3.20, this implies $R\Gamma_{M_i(d)}(\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}) = R\Gamma_{M_{i-1}(d)}(\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee \otimes \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t \otimes \zeta^{-1}) = 0$, as desired. □

5. A fully faithful embedding $D^b(C) \subset D^b(M_1)$

The following Theorem 5.1 is a special case of Theorem 9.3 and will be needed for our proof of the latter. Namely, the results of the present section will be used in Sections 7 and 9, in results that are necessary for Theorem 9.3. While Theorem 5.1 could be avoided by including it as a step of a more complicated inductive proof, we find it more convenient to prove it first, both to make the inductions less cumbersome and to introduce some ideas that will help understand the general picture.

We assume that $v \ge 1$, that is, $d \ge 3$. As before, let $E_1 \subset M_1$ be the exceptional locus of the blowup $M_1 \to M_0$ along $C \subset M_0$. By Orlov's blowup formula in

[35], we have a fully faithful functor $\Psi : D^b(C) \hookrightarrow D^b(M_1)$, corresponding to the Fourier–Mukai transform given by $\mathcal{O}_Z(E_1)$, where $Z = C \times_C E_1$. Now consider the Fourier–Mukai transform

$$\Phi_F = Rp_*\big(Lq^*(\cdot) \otimes^L F\big) : D^b(C) \to D^b(M_1)$$

determined by the universal bundle F on $C \times M_1$.

THEOREM 5.1
The functor Φ_F is fully faithful.

We need a few constructions and lemmas first. Observe that $Z = C \times_C E_1$ is supported precisely on the zero locus of the universal section $\tilde{\phi} : \mathcal{O}_{C \times M_1} \to F$. Indeed, pairs (E, ϕ) in $\mathbb{P}W_1^+ = E_1$ parameterize extensions

$$0 \to \mathcal{O}_C(x) \to E \to \Lambda(-x) \to 0$$

with the canonical section $\phi \in H^0(C, \mathcal{O}_C(x))$ vanishing on $x \in C$ (see [39, 3.2]), and in fact $\tilde{\phi}$ has no zeros outside this locus, since $M_1 \backslash E_1$ consists of extensions $0 \to \mathcal{O}_C \to E \to \Lambda \to 0$ together with a (constant) section $\phi \in H^0(C, \mathcal{O}_C)$ (see [39, 3.1]). Since Z has codimension 2, we have a Koszul resolution

$$[\wedge^2 F^\vee \to F^\vee \xrightarrow{\tilde{\phi}} \mathcal{O}_{C \times M_1}] \xrightarrow{\sim} \mathcal{O}_Z. \tag{5.1}$$

LEMMA 5.2
We have $R\Gamma_{M_1}(\Lambda_M^{-1}) = 0$.

Proof
Recall that $\Lambda_M^{-1} = \mathcal{O}_{M_1}(H - E_1)$. We have an exact sequence

$$0 \to \mathcal{O}_{M_1}(H - E_1) \to \mathcal{O}_{M_1}(H) \to \mathcal{O}_{E_1}(H) \to 0,$$

so it suffices to show that $j^* : H^p(M_1, \mathcal{O}_{M_1}(H)) \xrightarrow{\sim} H^p(E_1, \mathcal{O}_{E_1}(H))$ for every p, where $j : E_1 \hookrightarrow M_1$ is the inclusion. For each p, consider the commutative diagram

$$\begin{array}{ccc} H^p(M_1, \mathcal{O}_{M_1}(H)) & \xrightarrow{j^*} & H^p(E_1, \mathcal{O}_{E_1}(H)) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle q^*} \\ H^p(M_0, \mathcal{O}_{M_0}(H)) & \xrightarrow{\imath^*} & H^p(C, \mathcal{O}_C(H)) \end{array} \tag{5.2}$$

where $\iota : C \hookrightarrow M_0 = \mathbb{P}^{d+g-2}$ is the inclusion, $\pi : M_1 = \mathrm{Bl}_C M_0 \to M_0$ is the blowup along C, and $q = \pi|_{E_1} : E_1 \to C$, which is a $\mathbb{P}^r$-bundle. Hence, both vertical arrows in (5.2) are isomorphisms. Indeed, these pullbacks are fully faithful at the level of derived categories. Moreover, $\iota : C \hookrightarrow M_0$ is the embedding by the complete linear system $|\omega_C \otimes \Lambda|$ (see [39, 3.4]). Therefore, $\mathcal{O}_C(H) \simeq \omega_C \otimes \Lambda$ and $\iota^* : H^0(M_0, \mathcal{O}_{M_0}(H)) \to H^0(C, \mathcal{O}_C(H))$ is an isomorphism. For $p > 0$, $H^p(M_0, \mathcal{O}_{M_0}(H)) = 0$ because M_0 is a projective space. On the other hand, since $\deg \omega_C \otimes \Lambda > \deg \omega_C$, we also have $H^p(C, \mathcal{O}_C(H)) = 0$ for $p > 0$. In summary, the two vertical maps and the lower horizontal map in the commutative diagram are isomorphisms for all p. Hence, the same holds for the upper horizontal map. □

LEMMA 5.3
Let $x \in C$. Then $R\Gamma_{M_1}(F_x^\vee) = 0$, while $R\Gamma_{M_1}(F_x) = \mathbb{C}$, with $H^0(M_1, F_x) = \mathbb{C}$ given by restriction of the universal section $\tilde{\phi}$ of F to $\{x\} \times M_1$.

Proof
Consider the resolution (5.1) and restrict to $\{x\} \times M_1$ to get

$$[\Lambda_M^{-1} \to F_x^\vee \to \mathcal{O}_{M_1}] \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_x^r}, \tag{5.3}$$

where $\mathbb{P}_x^r = M_0(\Lambda(-2x))$ is the fiber over $x \in C \subset M_0$ along the blowup $\pi : M_1 \to M_0$. We twist by $\Lambda_M = \mathcal{O}_{M_1}(E_1 - H)$ to get

$$[\mathcal{O}_{M_1} \xrightarrow{\tilde{\phi}} F_x \to \Lambda_M] \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_x^r}(-1), \tag{5.4}$$

using that $F_x^\vee \otimes \Lambda_M = F_x^\vee \otimes (\wedge^2 F_x) \simeq F_x$ and that $\mathcal{O}_{M_1}(H)$ restricts trivially to $\mathcal{O}_{\mathbb{P}_x^r}$ (see Lemma 3.11 for a generalization of (5.3) and (5.4)). It is well known that $R\Gamma(\mathcal{O}_{\mathbb{P}_x^r}(-1)) = 0$. By Lemma 4.2, we also have $R\Gamma(\Lambda_M) = 0$. Hence, by (5.4), $\tilde{\phi}$ induces an isomorphism $R\Gamma(\mathcal{O}_{M_1}) \simeq R\Gamma(F_x)$. As M_1 is a blowup of a projective space along a smooth center, we get $R\Gamma(F_x) \simeq R\Gamma(\mathcal{O}_{M_1}) \simeq \mathbb{C}$, with $H^0(M_1, F_x) = \mathbb{C}$ given by restriction of $\tilde{\phi}$ to $\{x\} \times M_1$.

To show that $R\Gamma_{M_1}(F_x^\vee) = 0$, we apply $R\Gamma$ to (5.3). We already know that $R\Gamma_{M_1}(\Lambda_M^{-1}) = 0$ by Lemma 5.2, so it suffices to show that that the restriction map $H^p(M_1, \mathcal{O}_{M_1}) \to H^p(\mathbb{P}_x^r, \mathcal{O}_{\mathbb{P}_x^r})$ is an isomorphism for every p. For $p > 0$, both vector spaces vanish, because we have a projective space and a blowup of a projective space. For $p = 0$, we have an isomorphism of one-dimensional vector spaces because this is just restriction of constant sections. □

Proof of Theorem 5.1
By Bondal–Orlov's criterion (see [7]), in order to show full faithfulness of Φ_F we only need to consider the sheaves $\Phi_F(\mathcal{O}_x) = F_x$ for closed points $x \in C$. On the

other hand, consider the functor Ψ from Orlov's blowup formula, with Fourier–Mukai kernel $\mathcal{O}_Z(E_1)$, $Z = C \times_C E_1$. We can compute $\Psi(\mathcal{O}_x) = \Phi_{\mathcal{O}_Z(E_1)}(\mathcal{O}_x)$ for a point $x \in C$ using (5.1) as follows. As before, let $\mathbb{P}^r_x = M_0(\Lambda(-2x))$ denote the fiber over $x \in C \subset M_0$ along the blowup. The fact that $\mathcal{O}_{M_1}(H)$ restricts trivially to this fiber implies that both Λ_M and $\mathcal{O}_{M_1}(E_1)$ restrict to $\mathcal{O}_{\mathbb{P}^r_x}(-1)$ there. Now we restrict (5.1) to $\{x\} \times M_1$ and twist it by Λ_M to get $\Phi_{\mathcal{O}_Z(E_1)}(\mathcal{O}_x) \simeq [\mathcal{O}_{M_1} \to F_x \to \Lambda_M] \simeq \mathcal{O}_{\mathbb{P}^r_x}(-1)$, as in (5.4). Since we already know that Ψ is fully faithful, we have

$$\operatorname{Hom}_{D^b(M_1)}\big(\Psi(\mathcal{O}_x), \Psi(\mathcal{O}_y)[k]\big) = \begin{cases} 0 & \text{if } x \neq y, \\ 0 & \text{if } x = y \text{ and } k \neq 0, 1, \\ \mathbb{C} & \text{if } x = y \text{ and } k = 0, 1. \end{cases} \tag{5.5}$$

But $R\operatorname{Hom}_{M_1}(\Psi(\mathcal{O}_x), \Psi(\mathcal{O}_y)) \simeq R\Gamma \circ R\mathcal{H}om(\Psi(\mathcal{O}_x), \Psi(\mathcal{O}_y))$ can also be obtained as follows: take $R\mathcal{H}om(\Psi(\mathcal{O}_x), \Psi(\mathcal{O}_y)) \simeq \Psi(\mathcal{O}_x)^\vee \otimes^L \Psi(\mathcal{O}_y)$ as an inner tensor product obtained from the double complex

$$\begin{array}{ccccc} \mathcal{O}_{M_1} & \longrightarrow & F_x^\vee \otimes \Lambda_M & \longrightarrow & \Lambda_M \\ \uparrow & & \uparrow & & \uparrow \\ \Lambda_M^{-1} \otimes F_y & \longrightarrow & F_x^\vee \otimes F_y & \longrightarrow & F_y \\ \uparrow & & \uparrow & & \uparrow \\ \Lambda_M^{-1} & \longrightarrow & F_x^\vee & \longrightarrow & \mathcal{O}_{M_1} \end{array} \tag{5.6}$$

which produces the total complex

$$\begin{aligned} &\big[\Lambda_M^{-1} \to F_x^\vee \oplus F_y^\vee \to \mathcal{O}_{M_1}^{\oplus 2} \oplus (F_x^\vee \otimes F_y) \to F_x \oplus F_y \to \Lambda_M\big] \\ &\quad \simeq \Psi(\mathcal{O}_x)^\vee \otimes^L \Psi(\mathcal{O}_y), \end{aligned} \tag{5.7}$$

again using $F_x \simeq F_x^\vee \otimes \Lambda_M$. Recall that our descriptions of $\Psi(\mathcal{O}_y)$ and $\Psi(\mathcal{O}_x)^\vee$ were obtained from the Koszul resolution (5.4) and its dual. In particular, the maps $\mathcal{O}_{M_1} \to F_x^\vee \otimes \Lambda_M \simeq F_x$ and $\mathcal{O}_{M_1} \to F_y$ appearing in (5.6) correspond to the restriction of the universal section $\tilde{\phi}$ to $\{x\} \times M_1$ and $\{y\} \times M_1$, respectively.

The hypercohomology $R\Gamma$ of (5.7) can be computed by taking the spectral sequence with first page $E_1^{p,q} = H^q(X, \mathcal{F}^p) \Rightarrow H^{p+q}(X, \mathcal{F}^\bullet)$. On the other hand, we know that the $R\Gamma$ of this complex is given by (5.5). We will combine these to show that

$$R\Gamma_{M_1}(F_x^\vee \otimes F_y) = \begin{cases} 0 & \text{if } x \neq y, \\ \mathbb{C} \oplus \mathbb{C}[-1] & \text{if } x = y. \end{cases} \tag{5.8}$$

By Lemma 4.3, $R\Gamma_{M_1}(\Lambda_M) = 0$, and by Lemma 5.2, $R\Gamma_{M_1}(\Lambda_M^{-1}) = 0$. Also, Lemma 5.3 computes the hypercohomology of both F_x and $F_x^\vee$. Summing up, applying $R\Gamma$ to (5.7) yields a spectral sequence $E_1^{p,q}$ of the form

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & 0 & \longrightarrow & H^1(F_x^\vee \otimes F_y) & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & H^0(\mathcal{O}_{M_1})^{\oplus 2} \oplus H^0(F_x^\vee \otimes F_y) & \longrightarrow & H^0(F_x) \oplus H^0(F_y) & \longrightarrow & 0, \end{array}$$

where the map $H^0(\mathcal{O}_{M_1})^{\oplus 2} \to H^0(F_x) \oplus H^0(F_y)$ is the isomorphism $\mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2$ given by the universal section in each coordinate, by Lemma 5.3 and the discussion above. Since this spectral sequence converges to (5.5), we obtain (5.8). □

6. Acyclicity of powers of $\Lambda_M^\vee$

The goal of the present section is to prove the following generalization of Lemma 5.2.

THEOREM 6.1
Suppose that $2 < d \leq 2g + 1$ and $1 \leq k \leq l \leq v$. Then

$$R\Gamma_{M_l(d)}(\Lambda_M^{-k}) = 0.$$

Γ-acyclicity of these negative powers of Λ_M will be crucial for the cohomology computations in the upcoming sections.

LEMMA 6.2
Under the assumptions of Theorem 6.1, $H^0(M_l(d), \Lambda_M^{-k}) = 0$.

Proof
Since M_l is isomorphic to M_1 in codimension 1, it suffices to prove that $H^0(M_1, \Lambda_M^{-k}) = H^0(M_1, kH - kE_1) = 0$. Recall that M_1 is the blowup of $\mathbb{P}^r$ in C embedded by a complete linear system of $K_C + \Lambda$, $r = d + g - 2$, E_1 is the exceptional divisor, and H is a hyperplane divisor. The claim is that there is no hypersurface $D \subset \mathbb{P}^r$ of degree k that vanishes along C with multiplicity at least k. We argue by contradiction. Choose $r + 1$ points $x_1, \ldots, x_{r+1} \in C$ in linearly general position. Then D vanishes at these points with multiplicity $\geq k$. Let R be a rational normal curve passing through $x_1, \ldots, x_{r+1}$. Let $\tilde{R}$ and $\tilde{D}$ be the proper transforms of R and D in $\mathrm{Bl}_{x_1,\ldots,x_{r+1}} \mathbb{P}^r$.

Then $\tilde{D} \cdot \tilde{R} \leq kr - k(r+1) < 0$. It follows that $R \subset D$. But we can choose R passing through a general point of $\mathbb{P}^r$, which is a contradiction. □

LEMMA 6.3
Under the assumptions of Theorem 6.1, if $R\Gamma_{M_k(d)}(\Lambda_M^{-k}) = 0$*, then*

$$R\Gamma_{M_l(d)}(\Lambda_M^{-k}) = 0.$$

Proof
By Theorem 3.21, in the wall between M_{l-1} and M_l, Λ_M^{-k} descends from an object of weight k, with $-\eta_- < k < \eta_+$ when $k < l \leq v$, that is, $1 + 2l - d - g < k < l$ for l in that range. This way, $0 = R\Gamma_{M_k}(\Lambda_M^{-k}) = R\Gamma_{M_l}(\Lambda_M^{-k})$ for $l \geq k$ by Theorem 3.20. □

Definition 6.4
For $0 \leq \alpha \leq i$, we introduce the following loci:

$$E_i^\alpha := \{(E,s) \mid Z(s) \subset C \text{ has degree } \geq \alpha\} \subset M_i,$$
$$\mathcal{D}_i^\alpha := \{(D,E,s) \mid s|_D = 0\} \subset \operatorname{Sym}^\alpha C \times M_i,$$
$$R_i^\alpha := \{(D,E,s) \mid s|_D = 0 \text{ and } Z(s) \text{ has degree } \geq \alpha + 1\} \subset \mathcal{D}_i^\alpha,$$

where $Z(s)$ denotes the zero locus subscheme of the section s.

Note that E_i^i is precisely $\mathbb{P}W_i^+$ (see [39, proof of 3.2]), while $E_i^1 = E_i$ is the proper transform of E_1 under the birational equivalence given by (3.1). Recall that $\mathcal{O}_{M_i}(E_i) = \mathcal{O}_i(1,-1)$ according to Definition 3.5. For a divisor $D \in \operatorname{Sym}^\alpha C$, we observe that the fiber $(\mathcal{D}_i^\alpha)_D$ along the projection $\operatorname{Sym}^\alpha C \times M_i \to \operatorname{Sym}^\alpha C$ is isomorphic to $M_{i-\alpha}(\Lambda(-2D))$ (see Remark 3.7 or [39, 1.9]). Similarly, $(R_i^\alpha)_D \simeq E_{i-\alpha}(\Lambda(-2D))$. In particular, $\mathcal{D}_i^\alpha$ is smooth, and we have a diagram

$$\begin{array}{ccc} R_i^\alpha & \hookrightarrow & \mathcal{D}_i^\alpha \\ \downarrow & & \downarrow \nu \\ E_i^{\alpha+1} & \hookrightarrow & E_i^\alpha \end{array} \tag{6.1}$$

where ν is the normalization morphism.

LEMMA 6.5
We have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \nu_* \mathcal{O}_{D_i^\alpha}(-R_i^\alpha) & \longrightarrow & \nu_* \mathcal{O}_{D_i^\alpha} & \longrightarrow & \nu_* \mathcal{O}_{R_i^\alpha} & \longrightarrow & 0 \\
 & & \wr\big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & \mathcal{I}_{E_i^{\alpha+1}} & \longrightarrow & \mathcal{O}_{E_i^\alpha} & \xrightarrow{\beta} & \mathcal{O}_{E_i^{\alpha+1}} & \longrightarrow & 0
\end{array}
\tag{6.2}
$$

where $\mathcal{I}_{E_i^{\alpha+1}} \simeq \nu_* \mathcal{O}_{D_i^\alpha}(-R_i^\alpha)$ *is the conductor sheaf of the normalization (6.1) and* R_i^α *(resp.,* $E_i^{\alpha+1}$*) is a conductor subscheme in* $\mathcal{D}_i^\alpha$ *(resp.,* E_i^α*).*

Proof
From the flipping diagram (3.1), $E_\alpha^\alpha \subset M_\alpha$ is the projective bundle $\mathbb{P}\, W_\alpha^+$ and $E_{\alpha+1}^\alpha \subset M_{\alpha+1}$ is isomorphic to E_α^α away from $E_{\alpha+1}^{\alpha+1} \simeq \mathbb{P}\, W_{\alpha+1}^+$.

CLAIM 6.6
$E_{\alpha+1}^\alpha$ *has a multicross singularity generically along* $E_{\alpha+1}^{\alpha+1}$ *(concretely, this means that a general section of* $E_{\alpha+1}^\alpha$ *that intersects* $E_{\alpha+1}^{\alpha+1}$ *in a point is étale locally isomorphic to the union of coordinate axes in* $\mathbb{A}^{\alpha+1}$*).*

Given the claim, and since multicross singularities are seminormal (see [23]), $E_{\alpha+1}^\alpha$ has seminormal singularities in codimension 1. For $i > \alpha+1$, E_i^α is isomorphic to $E_{\alpha+1}^\alpha$ in codimension 2, and so also has seminormal singularities in codimension 1. Next we argue by induction on α that $\mathcal{D}_i^\alpha \to E_i^\alpha$ has reduced conductor subschemes $E_i^{\alpha+1} \subset E_i^\alpha$ and $R_i^\alpha \subset \mathcal{D}_i^\alpha$ and $E_i^{\alpha+1}$ is Cohen–Macaulay and seminormal, and in particular that we have a commutative diagram (6.2).

Indeed, $E_i^1 \subset M_i$ is Cohen–Macaulay as a hypersurface in a smooth variety. Suppose that E_i^α is Cohen–Macaulay. Since it is seminormal in codimension 1 by the above, it is seminormal everywhere (see [16, Corollary 2.7]). Therefore, its conductor subschemes in E_i^α and $\mathcal{D}_i^\alpha$ are both reduced (see [42, Lemma 1.3]) and all of their associated primes have height 1 in E_i^α and $\mathcal{D}_i^\alpha$, respectively (see [16, Lemma 7.4]). It follows that these conductor subschemes are equal to $E_i^{\alpha+1}$ and R_i^α, respectively. Finally, $R_i^\alpha \subset \mathcal{D}_i^\alpha$ is Cohen–Macaulay as a hypersurface in a smooth variety and therefore $E_i^{\alpha+1} \subset E_i^\alpha$ is also Cohen–Macaulay (see [37, Theorem 2.2]), and we can proceed with induction.

It remains to prove the claim. We analyze the flipping diagram (3.1) between the spaces M_α and $M_{\alpha+1}$, where M_α contains projective bundles $\mathbb{P}\, W_{\alpha+1}^-$ (over $\mathrm{Sym}^{\alpha+1} C$) and $\mathbb{P}\, W_\alpha^+ \simeq E_\alpha^\alpha$ (over $\mathrm{Sym}^\alpha C$) of dimensions $2\alpha+1$ and $d+g-2-\alpha$, respectively. What is their intersection over a point $D' \in \mathrm{Sym}^{\alpha+1} C$, for simplicity a reduced sum of points? By [39, 3.3], $\mathbb{P}\, W_{\alpha+1}^-$ parameterizes pairs (E,ϕ) that appear in extensions

$$0 \to L \to E \to \Lambda \otimes L^{-1} \to 0$$

with $\deg L = d - \alpha - 1$ and $\phi \notin H^0(L)$. Projecting ϕ to $\Lambda \otimes L^{-1}$ gives a nonzero vector $\gamma \in H^0(\Lambda \otimes L^{-1})$ with $Z(\gamma) = D'$, so that $\Lambda \otimes L^{-1} = \mathcal{O}_C(D')$, where $\deg D' = \alpha + 1$ (this gives the map from $\mathbb{P} W^-_{\alpha+1}$ to $\operatorname{Sym}^{\alpha+1} C$). Moreover, at D' the section lifts to a section of $\mathcal{O}_{D'} \otimes L \simeq \mathcal{O}_{D'} \otimes \Lambda(-D')$, and this vector $u \in H^0(\mathcal{O}_{D'} \otimes \Lambda(-D'))$ (determined uniquely up to a scalar) determines (E, ϕ) uniquely (see [39, 3.3]).

The same pair (E, ϕ) belongs to $\mathbb{P} W^+_\alpha$ if it can be given by an extension

$$0 \to \mathcal{O}_C(D) \to E \to \Lambda(-D) \to 0$$

with $\phi \in H^0(\mathcal{O}_C(D))$ and $\deg D = \alpha$ (see [39, 3.2]). Since ϕ vanishes at D and its image in $\mathcal{O}_C(D')$ vanishes at D', we have $D \subset D'$. Since we assume that D' is a reduced divisor, there are exactly $\alpha + 1$ choices for D. Since u has to vanish at points of $D \subset D'$, there is exactly one vector $u \in H^0(\mathcal{O}_{D'} \otimes \Lambda(-D'))$ (up to a multiple) that works for a given choice of D. Moreover, in this way we get a basis of $H^0(\mathcal{O}_{D'} \otimes \Lambda(-D')) \simeq \mathbb{C}^{\alpha+1}$. It follows that, over $D' \in \operatorname{Sym}^{\alpha+1} C$, $\mathbb{P} W^-_{\alpha+1}$ and $\mathbb{P} W^+_\alpha \simeq E^\alpha_\alpha$ intersect in $\alpha + 1$ reduced points which form a basis of the projective space $(\mathbb{P} W^-_{\alpha+1})_{D'} \simeq \mathbb{P}^\alpha$.

The strict transform of $\mathbb{P} W^+_\alpha$ in $M_{\alpha+1}$ is $E^\alpha_{\alpha+1}$, which contains the bundle $\mathbb{P} W^+_{\alpha+1}$ of dimension $d+g-3-\alpha$ (the flipped locus). After the flip, linearly independent intersection points in $(\mathbb{P} W^-_{\alpha+1})_{D'} \cap \mathbb{P} W^+_\alpha$ become linearly independent normal directions of branches of $E^\alpha_{\alpha+1}$ along $\mathbb{P} W^+_{\alpha+1}$, that is, $E^\alpha_{\alpha+1}$ has a multicross singularity in codimension 1, as claimed. We illustrate the geometry of M_α, $M_{\alpha+1}$ and the common resolution $\tilde{M}_{\alpha+1}$ in Figure 1. ☐

COROLLARY 6.7

If the claim of Theorem 6.1 is proved for $1 \le k = l \le i-1$, *then, for* $1 \le \alpha \le i-1$, $R\Gamma_{M_i}(\mathcal{O}_{E^\alpha_i}(1, i-1)) \simeq R\Gamma_{M_i}(\mathcal{O}_{E^{\alpha+1}_i}(1, i-1))$ *via* $R\Gamma(\beta)$.

Proof

Twisting by $\mathcal{O}_i(1, i-1)$ and applying $R\Gamma$ to the bottom sequence in (6.2), we see that it suffices to show that $\mathcal{I}_{E^{\alpha+1}_i}(1, i-1) \simeq \nu_* \mathcal{O}_{\mathcal{D}^\alpha_i}(-R^\alpha_i)(1, i-1)$ is Γ-acyclic. But ν is a finite map, so this is equivalent to Γ-acyclicity of $\mathcal{O}_{\mathcal{D}^\alpha_i}(-R^\alpha_i)(1, i-1)$. Using the Leray spectral sequence for the fibration $p : \mathcal{D}^\alpha_i \to \operatorname{Sym}^\alpha C$, it suffices to prove that $R\Gamma(\mathcal{O}_{\mathcal{D}^\alpha_{i,D}}(-R^\alpha_{i,D})(1, i-1)) = 0$. Under the isomorphism $(\mathcal{D}^\alpha_i)_D \simeq M_{i-\alpha}(\Lambda(-2D))$, $R^\alpha_i \subset \operatorname{Sym}^\alpha C \times M_i$ restricts to $E^1_{i-\alpha}$ on $M_{i-\alpha}(\Lambda(-2D))$, while $\mathcal{O}_i(m, n)$ on $M_i(\Lambda)$ restricts to $\mathcal{O}_{i-\alpha}(m, n - m\alpha)$ on $M_{i-\alpha}(\Lambda(-2D))$ (cf. Remark 3.7). Therefore,

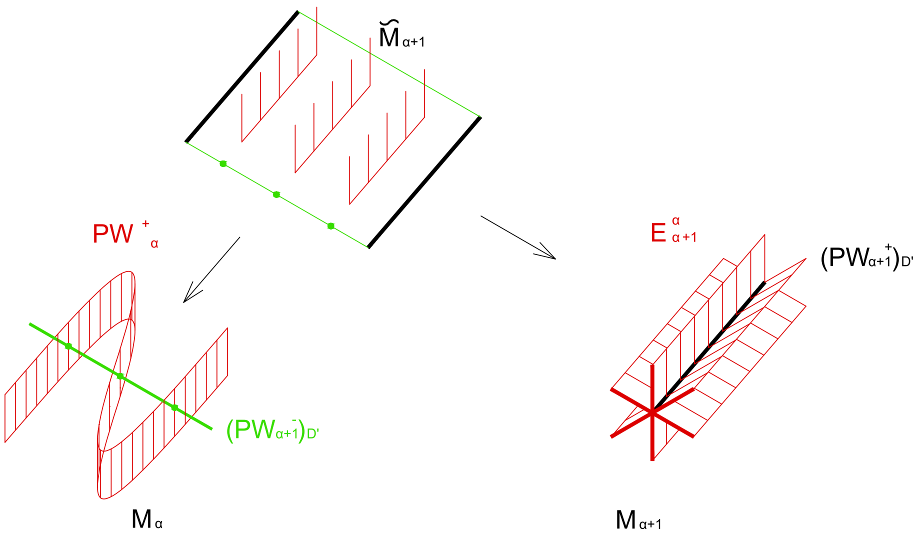

Figure 1. (Color online) Common resolution $\tilde{M}_{\alpha+1}$ of M_α and $M_{\alpha+1}$.

$$R\Gamma_{M_i(d)}\big(\mathcal{O}_{\mathcal{D}_i^\alpha,D}(-R_{i,D}^\alpha)(1,i-1)\big) = R\Gamma_{M_{i-\alpha}(d-2\alpha)}(\Lambda_M^{\alpha-i})$$

which is zero by hypothesis. □

LEMMA 6.8
Suppose that $d \leq 2g+1$. Then for $1 \leq i \leq d+1-g$, $i \leq v$, we have $H^p(M_i(d), \mathcal{O}_i(1,i-1)) = 0$ for any $p > 0$.

Proof
Recall that $\omega_{M_k} = \mathcal{O}_{M_k}(-3, 4-d-g)$ for every $1 \leq k \leq v$ (see [39, 6.1]). First, we see that there is some $i \leq k \leq v$ such that the bundle $\mathcal{O}_{M_k}(1,i-1) \otimes \omega_{M_k}^{-1} = \mathcal{O}_{M_k}(4, d+g+i-5)$ is big and nef. By the description of the ample cones in Remark 3.6, it suffices to check that $(4, d+g+i-5) \in \mathbb{R}^2$ lies in the closed cone bounded below by the ray through $(1, i-1)$ and above by the ray through $(2, d-2)$. Considering the slopes, this is equivalent to $i-1 \leq \frac{d+g+i-5}{4} \leq \frac{d-2}{2}$. The inequality on the left is equivalent to $3i \leq d+g-1$, which is guaranteed by the fact that $i \leq v = \lfloor (d-1)/2 \rfloor$ and $d \leq 2g+1$. The other inequality is equivalent to $i \leq d+1-g$, which is given as a hypothesis. Therefore, there is some $k \geq i$, $k \leq v$ such that $\mathcal{O}_{M_k}(1,i-1) \otimes \omega_{M_k}^{-1}$ is big and nef. By the Kawamata–Viehweg vanishing theorem, $H^p(M_k, \mathcal{O}_k(1,i-1)) = 0$ for $p > 0$.

Now, we claim that in fact

$$\begin{aligned}R\Gamma_{M_i}\big(\mathcal{O}_i(1,i-1)\big) &= R\Gamma_{M_{i+1}}\big(\mathcal{O}_{i+1}(1,i-1)\big) = \cdots \\ &= R\Gamma_{M_k}\big(\mathcal{O}_k(1,i-1)\big).\end{aligned} \tag{6.3}$$

Indeed, in the wall-crossing between M_{l-1} and M_l, there are windows of width $\eta_+ = l$ and $\eta_- = d+g-1-2l$ and $\mathcal{O}_l(1,i-1)$, $\mathcal{O}_{l-1}(1,i-1)$ both descend from the same object that has λ-weight $i-l$ (see Theorem 3.21, Remark 3.22). By Theorem 3.20, we will have $R\Gamma_{M_{l-1}}(\mathcal{O}_{l-1}(1,i-1)) = R\Gamma_{M_l}(\mathcal{O}_l(1,i-1))$ whenever

$$1+2l-d-g < i-l < l. \tag{6.4}$$

But (6.4) holds for any $i < l \le k$, because then $i < 2l$, while $3l \le 3(d-1)/2 < i+d+g-1$ provided that $d \le 2g+1$. Therefore, (6.3) holds and in particular $H^p(M_i, \mathcal{O}_i(1,i-1)) = 0$ for $p > 0$. □

Remark 6.9
Suppose that $d \le 2g+1$. Then (6.4) holds for $l \in (i/2, v]$, and the same reasoning shows that $R\Gamma_{M_i}(\mathcal{O}_i(1,i-1)) = R\Gamma_{M_l}(\mathcal{O}_l(1,i-1))$ for every $\lfloor i/2 \rfloor \le l \le v$. In particular, under the same hypotheses of Lemma 6.8, $\mathcal{O}_l(1,i-1)$ has no higher cohomology whenever $\lfloor i/2 \rfloor \le l \le v$.

Definition 6.10
Let L_i be the line bundle on $\operatorname{Sym}^i C$ defined by

$$L_i = \det{}^{-1} \pi_! \Lambda(-\Delta) \otimes \det{}^{-1} \pi_! \mathcal{O}(\Delta), \tag{6.5}$$

where $\Delta \subset \operatorname{Sym}^i C \times C$ is the universal divisor (cf. [39, 6.5]). To emphasize the degree d, sometimes we denote this line bundle by $L_i(d)$.

LEMMA 6.11
$H^p(\operatorname{Sym}^i C, L_i(d)) = 0$ if $p > 0$, $1 \le i \le d-g$.

Proof
By [39, 7.5] (see also [27]), and mixing notation for line bundles and divisors,

$$L_i(d) = (d-2i)\eta + 2\sigma \qquad \text{and} \qquad K_{\operatorname{Sym}^i C} = (g-i-1)\eta + \sigma, \tag{6.6}$$

where $\eta = x + \operatorname{Sym}^{i-1} C \subset \operatorname{Sym}^i C$ is an ample divisor for any fixed $x \in C$ and $\sigma \subset \operatorname{Sym}^i C$ is a pullback of a theta divisor via the Abel–Jacobi map; in particular, σ is nef. It follows that $L_i(d) - K_{\operatorname{Sym}^i C} = (d-i-g+1)\eta + \sigma$ is ample if $i \le d-g$, and the result follows by the Kodaira vanishing theorem. □

LEMMA 6.12
Suppose that $i+g \le d \le 2g+1$. *Then* $\chi(M_i(d), \mathcal{O}_i(1, i-1)) = \chi(\operatorname{Sym}^i C, L_i(d))$.

Proof
Since $i \le d-g$, we can use Lemma 6.8 together with [39, 7.8] to compute

$$\begin{aligned}\chi\big(\mathcal{O}_i(1,i-1)\big)& \\ &= \operatorname*{Res}_{t=0}\Big\{\frac{(1-t^3)^{2i-d-1}(1-t^2)^{2d+1-2i-2g}}{t^{i+1}(1-t)^{d+g-1}}\big(1-5(1-t)t^2-t^5\big)^g\,dt\Big\} \\ &= \operatorname*{Res}_{t=0}\Big\{\frac{(1+t)^{2d+1-2i-2g}(1+3t+t^2)^g(1-t)}{t^{i+1}(1+t+t^2)^{d+1-2i}}\,dt\Big\}. \qquad (6.7)\end{aligned}$$

On the other hand, we use the Hirzebruch–Riemann–Roch theorem to compute, using the formulas (see [39, Section 7])

$$\operatorname{ch}(L_i) = e^{(d-2i)\eta+2\sigma}, \qquad \operatorname{td}(\operatorname{Sym}^i C) = \Big(\frac{\eta}{1-e^{-\eta}}\Big)^{i-g+1}\exp\Big(\frac{\sigma}{e^\eta-1}-\frac{\sigma}{\eta}\Big)$$

and notation from the proof of Lemma 6.11, that

$$\chi(L_i) = \operatorname*{Res}_{\eta=0}\Big\{\frac{e^{\eta(d-2i)}}{(1-e^{-\eta})^{i-g+1}}\Big(2+\frac{1}{e^\eta-1}\Big)^g\,d\eta\Big\},$$

where we have used [39, 7.2] with

$$A(\eta) = e^{\eta(d-2i)}\Big(\frac{\eta}{1-e^{-\eta}}\Big)^{i-g+1}, \qquad B(\eta) = 2+\frac{1}{e^\eta-1}-\frac{1}{\eta}.$$

If we let $u(\eta) = e^\eta - 1$, then u is biholomorphic near $\eta = 0$, with $u(0) = 0$, $u'(0) = 1$, so we can do a change of variables $u = e^\eta - 1$, $du = e^\eta\,d\eta$ to obtain

$$\chi(L_i) = \operatorname*{Res}_{u=0}\Big\{\frac{(1+u)^{d-i-g}(2u+1)^g}{u^{i+1}}\,du\Big\}. \qquad (6.8)$$

Next, we apply an *ad hoc* change of variables

$$u = \frac{t}{t^2+t+1}, \qquad du = \frac{1-t^2}{(t^2+t+1)^2}\,dt$$

to (6.8) and we get precisely (6.7) after some algebraic manipulations. □

For what follows, we need some geometric constructions. Fix a point $x \in C$, and consider a subvariety $M_{i-1}(d-1) \subset M_i(d+1)$ of codimension 2 as in Remark 3.7, with $D = x$. Let B be the blowup of $M_i(d+1)$ in $M_{i-1}(d-1)$ with exceptional divisor $\mathcal{E}$.

Consider the $\mathbb{P}^1$-bundle $\mathbb{P}F_x$ over $M_i(d+1)$ that parameterizes triples (E,ϕ,l), where ϕ is a nonzero section of E and $l \subset E_x$ is a line, subject to the usual stability condition (see Section 3) that for every line subbundle $L \subset E$, one must have

$$\deg L \leq \begin{cases} i + \frac{1}{2} & \text{if } \phi \in H^0(L), \\ d - i + \frac{1}{2} & \text{if } \phi \notin H^0(L). \end{cases} \tag{6.9}$$

LEMMA 6.13
With the notation as above, the blowup B of $M_i(d+1)$ in $M_{i-1}(d-1)$ is isomorphic to the following locus:

$$Z = \big\{(E,\phi,l) : \phi(x) \in l\big\} \subset \mathbb{P}F_x.$$

Proof
Indeed, the projection of Z onto $M_i(d+1)$ is clearly an isomorphism outside of $M_{i-1}(d-1)$, since the latter is precisely the locus where $\phi(x) = 0$. Over $M_{i-1}(d-1)$, the fiber of this projection is $\mathbb{P}^1$. By the universal property of the blowup, it suffices to check that Z is the blowup of $M_i(d+1)$ in $M_{i-1}(d-1)$ locally near $(E,\phi) \in M_{i-1}(d-1)$, where we can trivialize $F_x \simeq \mathcal{O} \oplus \mathcal{O}$. Its universal section can be written as $s = (a,b)$, where $a,b \in \mathcal{O}$ is a regular sequence (its vanishing locus is $M_{i-1}(d-1)$ locally near (E,ϕ)). Then Z is locally given by the equation $ay - bx = 0$, where $[x : y]$ are homogeneous coordinates of the $\mathbb{P}^1$-bundle $\mathbb{P}F_x$ given by the trivialization $F_x \simeq \mathcal{O} \oplus \mathcal{O}$. Thus Z is indeed isomorphic to the blowup B. ☐

Now we can prove the main result of this section.

Proof of Theorem 6.1
By Lemma 6.3, it suffices to prove that $R\Gamma_{M_i}(\Lambda_M^{-i})$ is zero for every $i = 1,\dots,v$, which we will do by induction on i. The base case $i = 1$ is Lemma 5.2. Recall that $\mathcal{O}_{M_i}(E_i) = \mathcal{O}_i(1,-1)$. Twist the tautological short exact sequence for $E_i \subset M_i$ by $\mathcal{O}_i(1,i-1)$ to get

$$0 \to \Lambda_M^{-i} \to \mathcal{O}_i(1,i-1) \xrightarrow{\gamma} \mathcal{O}_{E_i}(1,i-1) \to 0.$$

It suffices to prove that $R\Gamma_{M_i}(\mathcal{O}_i(1,i-1)) \simeq R\Gamma_{E_i}(\mathcal{O}_{E_i}(1,i-1))$ via $R\gamma$. By the induction hypothesis, we can apply Corollary 6.7 to see that

$$R\Gamma\big(\mathcal{O}_{E_i}(1,i-1)\big) \simeq \cdots \simeq R\Gamma\big(\mathcal{O}_{E_i^i}(1,i-1)\big) = R\Gamma\big(\mathcal{O}_{\mathbb{P}W_i^+}(1,i-1)\big).$$

But $\mathcal{O}_{\mathbb{P}W_i^+}(1,i-1)$ restricts trivially to each fiber of $\mathbb{P}W_i^+$. Arguing as in [39, 6.5], where an analogous statement is proved for $\mathcal{O}_{\mathbb{P}W_i^-}(1,i-1)$ (but using [39, 3.2] instead of [39, 3.3]), the restriction $\mathcal{O}_{\mathbb{P}W_i^+}(1, i-1)$ is a pullback of the line bundle L_i on $\operatorname{Sym}^i C$ defined in (6.5). Alternatively, it is clear that $\mathcal{O}_{\mathbb{P}W_i^+}(1, i-1)$ and $\mathcal{O}_{\mathbb{P}W_i^-}(1, i-1)$ are pullbacks of the same line bundle on $\operatorname{Sym}^i C$ because these projective bundles are contracted to their base $\operatorname{Sym}^i C$ by birational morphisms from $M_i(d)$ and $M_{i-1}(d)$ to the (singular) GIT quotient $M_\sigma(d)$, where $\sigma = \frac{d}{2} - i$ is the slope of the wall between the moduli spaces $M_i(d)$ and $M_{i-1}(d)$. Furthermore, $\mathcal{O}_i(1, i-1)$ is a pullback of a line bundle from that GIT quotient.

This implies that $R\Gamma(\mathcal{O}_{\mathbb{P}W_i^+}(1, i-1)) \simeq R\Gamma(\operatorname{Sym}^i C, L_i)$. Therefore, it suffices to show that

$$R\Gamma_{M_i(d)}\big(\mathcal{O}_i(1, i-1)\big) \simeq R\Gamma_{\operatorname{Sym}^i C}\big(L_i(d)\big) \tag{6.10}$$

via the composition of functors as above.

CLAIM 6.14
If $d \geq i + g$, then (6.10) holds.

Proof
In this case, $H^p(M_i, \mathcal{O}_i(1, i-1)) = H^p(\operatorname{Sym}^i C, L_i) = 0$ for $p > 0$ by Lemmas 6.8 and 6.11. Using this together with the fact that $\Lambda_M^{-i} = \mathcal{O}_i(0, i)$ has no global sections by Lemma 6.2, it suffices to prove that $h^0(M_i, \mathcal{O}_i(1, i-1)) = h^0(\operatorname{Sym}^i C, L_i)$ or, equivalently, that $\chi(M_i, \mathcal{O}_i(1, i-1)) = \chi(\operatorname{Sym}^i C, L_i)$. Thus, Lemma 6.12 proves the claim. □

We now proceed by a downward induction on d, starting with any d such that $d \geq i + g$. For such d, we have the result by the claim above.

Next we perform a step of the downward induction assuming that the theorem holds for degree $d + 1$. As above, we fix a point $x \in C$ and consider the subvariety $M_{i-1}(d-1) \subset M_i(d+1)$ of codimension 2 described in Remark 3.7. Let $\mathcal{J} \subset \mathcal{O}_{M_i(d+1)}$ be its ideal sheaf. As in the proof of Lemma 6.11, we denote the divisor $x + \operatorname{Sym}^{i-1} C \subset \operatorname{Sym}^i C$ by η and, by abuse of notation, we denote its pullback to the projective bundle $\mathbb{P}W_i^+$ by η as well. Note that $M_{i-1}(d-1) \cap \mathbb{P}W_i^+ = \mathbb{P}W_{i-1}^+$. To summarize, we have a commutative diagram of sheaves on $M_i(d+1)$ with exact rows, where we suppress closed embeddings from notation:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{M_i(d+1)} & \longrightarrow & \mathcal{O}_{M_{i-1}(d-1)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}W_i^+}(-\eta) & \longrightarrow & \mathcal{O}_{\mathbb{P}W_i^+} & \longrightarrow & \mathcal{O}_{\mathbb{P}W_{i-1}^+} & \longrightarrow & 0 \end{array} \tag{6.11}$$

We tensor (6.11) with $\mathcal{O}_i(1, i-1)$. Recall that the restriction of $\mathcal{O}_i(1, i-1)$ to $M_{i-1}(d-1)$ is $\mathcal{O}_{i-1}(1, i-2)$, to $\mathbb{P} W_i^+$ is the pullback of $L_i(d+1)$ from $\operatorname{Sym}^i C$, and to $\mathbb{P} W_{i-1}^+$ is the pullback of $L_{i-1}(d-1)$ from $\operatorname{Sym}^{i-1} C$. By inductive hypothesis on i, the arrow γ in (6.11) gives an isomorphism in cohomology after tensoring with $\mathcal{O}_i(1, i-1)$. The same is true for β by our inductive assumption on d. By the 5-lemma, we conclude that we have an isomorphism

$$R\Gamma\big(\mathcal{I}(1, i-1)\big) \simeq R\Gamma\big(\mathcal{O}_{\mathbb{P} W_i^+}(-\eta)(1, i-1)\big). \tag{6.12}$$

As $\mathcal{O}_{\mathbb{P} W_i^+}(1, i-1)$ is the pullback of $L_i(d+1)$, we see that $\mathcal{O}_{\mathbb{P} W_i^+}(-\eta)(1, i-1)$ is the pullback of $L_i(d)$ to the projective bundle (see (6.6)). Hence, we can rewrite (6.12) as

$$R\Gamma_B\big(\mathcal{O}_B(1, i-1)(-\mathcal{E})\big) \simeq R\Gamma_{\operatorname{Sym}^i C}\big(L_i(d)\big), \tag{6.13}$$

where B is the blowup of $M_i(d+1)$ in $M_{i-1}(d-1)$ and $\mathcal{E}$ its exceptional divisor.

Recall that the goal is to prove (6.10). We can do one extra simplification. Let $\sigma = \frac{d}{2} - i$ be the slope on the wall between the moduli spaces $M_i(d)$ and $M_{i-1}(d)$, and let $M_\sigma(d)$ be the corresponding (singular) GIT quotient. The birational morphism $M_i(d) \to M_\sigma(d)$ contracts the projective bundle $\mathbb{P} W_i^+$ to its base $\operatorname{Sym}^i C$, and in particular proving (6.10) is equivalent to proving that

$$R\Gamma_{M_\sigma(d)}\big(\mathcal{O}_i(1, i-1)\big) \simeq R\Gamma_{\operatorname{Sym}^i C}\big(L_i(d)\big) \tag{6.14}$$

by projection formula and Boutot's theorem [8, Corollaire]. To show how (6.13) implies (6.14), we need a geometric construction, a variant of the Hecke correspondence, relating B to $M_\sigma(d)$.

By Lemma 6.13, B carries a family of parabolic (at $x \in C$) rank-2 vector bundles E with a section ϕ. The parabolic line at x defines a quotient $E \to \mathcal{O}_x$, and we define a rank-2 vector bundle E' as an elementary transformation by the formula

$$0 \to E' \to E \to \mathcal{O}_x \to 0. \tag{6.15}$$

Our condition $\phi(x) \in l$ implies that the section ϕ lifts to a section ϕ' of E'. Elementary transformation is well known to be a functorial construction (see [34, Section 4]), and in fact we claim that (E', ϕ') is a σ-semistable pair; that is, we have a morphism

$$h : B \to M_\sigma(d), \qquad (E, \phi, l) \mapsto (E', \phi').$$

Indeed, we need to check that

$$\deg L' \le \begin{cases} i & \text{if } \phi' \in H^0(L'), \\ d - i & \text{if } \phi' \notin H^0(L') \end{cases}$$

for every line subbundle $L' \subset E'$, which follows from (6.9) applied to L'.

By the Kollár vanishing theorem in [22, Theorem 7.1], $Rh_* \mathcal{O}_B = \mathcal{O}_{M_\sigma(d)}$. Indeed, B is smooth, $M_\sigma(d)$ has rational singularities, and a general geometric fiber of h is isomorphic to $\mathbb{P}^1$ (given by extensions (6.15) with fixed E'). By projection formula, (6.13) implies (6.14) if we can show that

$$h^* \mathcal{O}_i(1, i-1) \simeq \mathcal{O}_B(1, i-1)(-\mathcal{E}).$$

Outside of $\mathcal{E}$ and for any $q \in C$, the bundle F_q over the stack of the σ-semistable pairs (resp., its determinant Λ'), pulls back to the bundle F_q over $B \setminus \mathcal{E}$ (resp., its determinant Λ), by (6.15). On the other hand, the divisor E_i' of σ-semistable stable pairs (E', ϕ') such that ϕ' has a zero, pulls back to the analogous divisor E_i of $B \setminus \mathcal{E}$, because the section ϕ of E is the same as the section ϕ' of E'. Since E and Λ generate the Picard group of $B \setminus \mathcal{E}$, it follows that $h^* \mathcal{O}_i(1, i-1) \simeq \mathcal{O}_B(1, i-1)(-c\,\mathcal{E})$ for some integer c. It remains to show that $c = 1$. To this end, we re-examine the diagram (6.11). Note that the proper transform $\tilde{\mathbb{P}}$ of $\mathbb{P}\, W_i^+$ in B is isomorphic to its blowup in $\mathbb{P}\, W_{i-1}^+$, which is the Cartier divisor η. Therefore, $\tilde{\mathbb{P}} \simeq \mathbb{P}\, W_i^+$. However, the restriction $h^* \mathcal{O}_i(1, i-1)|_{\tilde{\mathbb{P}}}$ is isomorphic to the pullback of $L_i(d)$ from $\operatorname{Sym}^i C$, while the restriction $\mathcal{O}_B(1, i-1)|_{\tilde{\mathbb{P}}}$ is isomorphic to the pullback of $L_i(d+1)$. Since $L_i(d) \simeq L_i(d+1)(-\eta)$, and $\mathcal{E}$ restricts to $\tilde{\mathbb{P}}$ as η, the claim follows. $\square$

7. Acyclic vector bundles on M_i—hard cases

The main goal of the present section is to prove the following result.

THEOREM 7.1

Suppose that $2 < d \leq 2g+1$ *and* $1 \leq i \leq v$. *Let* $D = x_1 + \cdots + x_\alpha$, $D' = y_1 + \cdots + y_\beta$ *(possibly with repetitions), and let* t *be an integer satisfying*

$$\deg D - i - 1 < t < d + g - 2i - 1 - \deg D'. \tag{7.1}$$

If $t \notin [0, \deg D]$, *then we have*

$$R\Gamma_{M_i(d)}\Big(\Big(\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee\Big) \otimes \overline{G}_{D'} \otimes \Lambda_M^t\Big) = 0.$$

Equivalently, if $\deg D \notin [t,\ t + \deg D']$, *then*

$$R\Gamma_{M_i(d)}\Big(G_D^\vee \otimes \Big(\bigotimes_{k=1}^{\beta} F_{y_k}\Big) \otimes \Lambda_M^t\Big) = 0.$$

Remark 7.2

In the vanishings of Theorem 7.1, we can write $G_D^\vee$ or $\overline{G}_D^\vee$ in place of $\bigotimes_{k=1}^{\alpha} F_{x_k}^\vee$, and $G_{D'}$ or $\overline{G}_{D'}$ in place of $\bigotimes_{k=1}^{\beta} F_{y_k}$. This follows from Corollary 2.9 and semicontinuity.

These computations will allow us to verify both the Bondal–Orlov conditions for the fully faithful embeddings of $D^b(\operatorname{Sym}^{\alpha} C)$ into $D^b(M_i)$, for $\alpha \le i$, as well as the vanishings needed in order to show semiorthogonality between the corresponding subcategories of $D^b(M_i)$ thus defined.

We start with a lemma on $M_0(d)$.

LEMMA 7.3
Let $d > 0$ and $i = 0$. Let $D = x_1 + \cdots + x_\alpha$, $D' = y_1 + \cdots + y_\beta$ (possibly with repetitions), and let t be an integer satisfying $\deg D < t < d + g - 1 - \deg D'$. *Then* $R\Gamma_{M_0(d)}((\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^t) = 0.$

Proof
The vector bundle $(\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^t|_{M_0}$ has the form $\bigoplus \mathcal{O}_{\mathbb{P}^{d+g-2}}(s_j - t)$ on $M_0 = \mathbb{P}^{d+g-2}$, where $-\beta \le s_j \le \alpha$ (see Lemma 3.9). By hypothesis, $\alpha - t < 0$ and $-\beta - t \ge -(d + g - 2)$, so this bundle is Γ-acyclic. $\square$

THEOREM 7.4
Let $d > 2$ and $1 \le i \le v$. Let $D = x_1 + \cdots + x_\alpha$, $D' = y_1 + \cdots + y_\beta$ (possibly with repetitions), and let t be an integer satisfying

$$\deg D < t < d + g - 1 - 2i - \deg D'.$$

Then $R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^t) = 0.$

Proof
By Theorem 3.21, the bundle $(\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^t$ descends from an object with weights in $[-\beta - t, \alpha - t]$. For every $1 < j \le i$, these weights live in the window between M_{j-1} and M_j, since by hypothesis $1 + 2j - d - g < -\beta - t$ and $\alpha - t < 0 < j$. Then using Theorem 3.20, $R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^t) = R\Gamma_{M_1(d)}((\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^t)$, so it suffices to show the theorem for the case $i = 1$.

Also, using $(\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^t \simeq (\bigotimes_{k=1}^{\alpha} F_{x_k}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^{t-\alpha}$, it is easy to see that it suffices to show the theorem for the case $\alpha = 0$. So we assume that $\alpha = 0$ and do induction on β. If $\beta = 0$, then $0 < t \le d + g - 4$ and the result follows from Lemma 4.3. If $\beta > 0$, write $D' = \tilde{D}' + y_\beta$. We use the sequence (3.4) from Lemma 3.11 with F_{y_β} and twist it by $(\bigotimes_{k=1}^{\beta-1} F_{y_k}) \otimes \Lambda_M^t$ to obtain an exact sequence

$$0 \to \bigotimes_{k=1}^{\beta-1} F_{y_k} \otimes \Lambda_M^t \to \bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t$$
$$\to \bigotimes_{k=1}^{\beta-1} F_{y_k} \otimes \Lambda_M^{t+1} \to \bigotimes_{k=1}^{\beta-1} F_{y_k} \otimes \Lambda_M^{t+1}\Big|_{M_0(\Lambda(-2y_\beta))} \to 0.$$

Of these terms, $R\Gamma_{M_0(d-2)}(\bigotimes_{k=1}^{\beta-1} F_{y_k} \otimes \Lambda_M^{t+1}) = 0$ by Lemma 7.3, since $0 < t+1 < (d-2)+g-1-(\beta-1)$, while by induction $R\Gamma_{M_1(d)}(\bigotimes_{k=1}^{\beta-1} F_{y_k} \otimes \Lambda_M^t) = R\Gamma_{M_1(d)}(\bigotimes_{k=1}^{\beta-1} F_{y_k} \otimes \Lambda_M^{t+1}) = 0$. Therefore, we obtain $R\Gamma_{M_1(d)}(\bigotimes_{k=1}^{\beta} F_{y_k} \otimes \Lambda_M^t) = 0$ as well. ☐

COROLLARY 7.5

Suppose that $d > 0$ and $0 \le i \le v$. Let $D = x_1 + \cdots + x_\alpha$ (possibly with repetitions), with $\alpha = \deg D < d + g - 2i - 1$. Then

$$R\Gamma_{M_i}\Big(\bigotimes_{k=1}^{\alpha} F_{x_k}\Big) = R\Gamma_{M_i}(G_D) = R\Gamma_{M_i}(\overline{G}_D) = \mathbb{C}. \tag{7.2}$$

Moreover, if $i \ge 1$, the unique (up to a scalar) global section of these bundles vanishes precisely along the union of codimension 2 loci $M_{i-1}(\Lambda(-2x_k))$, for $k \in \{1, \ldots, \alpha\}$.

Proof

When $i = 0$, $F_{x_k} = \mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^r}(-1)$ on $M_0 = \mathbb{P}^r$, $r = d + g - 2$ (see Lemma 3.9), and $\bigotimes F_{x_k}$ splits as a sum of line bundles $\bigoplus \mathcal{O}_{\mathbb{P}^r}(s_j)$, where $-\alpha \le s_j \le 0$ and exactly one of the summands is $\mathcal{O}_{\mathbb{P}^r}$. Since $\alpha \le d + g - 2$, $R\Gamma_{M_i}(\bigotimes_{k=1}^{\alpha} F_{x_k}) = \mathbb{C}$ in this case. Since G_D and $\overline{G}_D$ are deformations of $\bigotimes_{k=1}^{\alpha} F_{x_k}$ over $\mathbb{A}^1$, we have (7.2) by semicontinuity and equality of the Euler characteristic.

Let $i \ge 1$. We see that, using Theorem 3.20, it suffices to prove (7.2) on $M_1(d)$. In fact, by Theorem 3.21, $\bigotimes_{k=1}^{\alpha} F_{x_k}$ descends from an object with weights within $[-\alpha,\ 0]$, all of which live in the window $(1 + 2j - d - g,\ j)$ for $1 < j \le i$, since $1 + 2j - d - g \le 1 + 2i - d - g < -\alpha$ by hypothesis. This way we get $R\Gamma_{M_i}(\bigotimes_{k=1}^{\alpha} F_{x_k}) = R\Gamma_{M_1}(\bigotimes_{k=1}^{\alpha} F_{x_k})$. Similarly, $R\Gamma_{M_i}(G_D) = R\Gamma_{M_1}(G_D)$ and $R\Gamma_{M_i}(\overline{G}_D) = R\Gamma_{M_1}(\overline{G}_D)$.

Hence, we take $i = 1$ and $\alpha < d + g - 3$. In this case, $d > 2$. Let us show that $R\Gamma_{M_1}(\bigotimes F_{x_k}) \simeq \mathbb{C}$ first. We do induction on α. If $D = 0$, then the result is trivial. Otherwise, use the sequence (3.4) from Lemma 3.11 on F_{x_α} to obtain an exact sequence

$$0 \to \bigotimes_{k=1}^{\alpha-1} F_{x_k} \to \bigotimes_{k=1}^{\alpha} F_{x_k} \to \bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M \to \bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M\Big|_{M_0(d-2)} \to 0.$$

Of these terms, we get $R\Gamma_{M_1(d)}(\bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M) = 0$ from Theorem 7.4. Also, we have $R\Gamma_{M_0(d-2)}(\bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M) = 0$ from Lemma 7.3, given that here $t = 1$ and $0 < 1 < (d-2) + g - 1 - (\alpha - 1)$. Using the hypercohomology spectral sequence $E_1^{p,q} = H^q(X, \mathcal{F}^p)$ and induction, we obtain

$$R\Gamma_{M_1}\Big(\bigotimes_{k=1}^{\alpha} F_{x_k}\Big) = R\Gamma_{M_1}\Big(\bigotimes_{k=1}^{\alpha-1} F_{x_k}\Big) = \mathbb{C}.$$

Finally, by Corollary 2.9 both G_D and $\overline{G}_D$ are deformations over $\mathbb{A}^1$ of $\bigotimes_{k=1}^{\alpha} F_{x_k}$, so we have (7.2) by semicontinuity and equality of the Euler characteristic. It also follows that the global section of G_D (resp., $\overline{G}_D$) is a deformation of the global section of $\bigotimes_{k=1}^{\alpha} F_{x_k}$ over $\mathbb{A}^1$, which does not vanish outside of the union of loci $M_{i-1}(\Lambda(-2x_k))$ for $k = 1, \dots, \alpha$. On the other hand, the tautological sections of these bundles, that is, the descent of the tensor product of tautological sections of $\bigotimes \pi_j^* \mathcal{F}_k$ (resp., this tensor product tensored with the sign representation) for G_D (resp., $\overline{G}_D$), vanish precisely along these loci. □

A key step in the proof of Theorem 7.1 will be the following proposition.

PROPOSITION 7.6
Suppose that $d > 2$ and $1 \le i \le v$. Let D be an effective divisor on C, and suppose that $\deg D \le d + g - 2i - 1$. *Then*

$$R\Gamma_{M_i(d)}(G_D^\vee \otimes \Lambda_M^{\deg D - 1}) = R\Gamma_{M_i(d)}(\overline{G}_D \otimes \Lambda_M^{-1}) = 0. \tag{7.3}$$

We will first show how Theorem 7.1 follows from Proposition 7.6 and then proceed with the proof of Proposition 7.6.

Proof of Theorem 7.1
Note that, by rewriting $\overline{G}_{D'}$ in terms of $G_{D'}^\vee$ using Corollary 3.14, both statements can be seen to be equivalent, so we will only prove the first one.

We first suppose that $D = 0$ and do induction on $\deg D'$. If $D = D' = 0$, then we need to show that for $t \ne 0$ with $-i - 1 < t < d + g - 2i - 1$ we have $R\Gamma_{M_i(d)}(\Lambda_M^t) = 0$. If $t > 0$, then Lemma 4.3 ensures that $R\Gamma_{M_1(d)}(\Lambda_M^t) = 0$, since $i \ge 1$ and so $t \le d + g - 4$. But also for every $1 < j \le i$ we have $1 + 2j - d - g < -t < 0 < j$, that is, the weight of Λ_M^t lives in the window between M_{j-1} and M_j, so we conclude that $R\Gamma_{M_i(d)}(\Lambda_M^t) = R\Gamma_{M_1(d)}(\Lambda_M^t) = 0$ by Theorem 3.20. Suppose now that $t < 0$, so that $-i \le t < 0$. By Theorem 6.1, $R\Gamma_{M_i(d)}(\Lambda_M^t) = 0$.

Let $D = 0$ and $\deg D' \ge 1$. By induction, we may assume that the result holds for divisors $\tilde{D}'$ with $\deg \tilde{D}' < \deg D'$. We need to show that $R\Gamma_{M_i(d)}(\overline{G}_{D'} \otimes \Lambda_M^t) = 0$

for $-i-1<t<d+g-2i-1-\deg D'$ and $t\neq 0$. The case $t=-1$ follows directly from Proposition 7.6, since here $\deg D'\leq d+g-2i-1$, so we may assume that $t\notin\{-1,0\}$. We write $D'=\tilde{D}'+y$ and use the fact that $\overline{G}_{D'}$ is a stable deformation of $F_y\otimes\overline{G}_{\tilde{D}'}$ over $\mathbb{A}^1$ (see Proposition 2.12). If we take the second sequence of Lemma 3.11 twisted by $\overline{G}_{\tilde{D}'}\otimes\Lambda_M^t$, then we get an exact sequence

$$0\to\overline{G}_{\tilde{D}'}\otimes\Lambda_M^t\to F_y\otimes\overline{G}_{\tilde{D}'}\otimes\Lambda_M^t\to\overline{G}_{\tilde{D}'}\otimes\Lambda_M^{t+1}\to\overline{G}_{\tilde{D}'}\otimes\Lambda_M^{t+1}|_{M_{i-1}}\to 0.$$

Observe that this is an acyclic chain complex involving $F_y\otimes\overline{G}_{\tilde{D}'}\otimes\Lambda_M^t$ and where the remaining three terms satisfy the corresponding inequalities from (7.1): $-i-1<t<d+g-2i-1-\deg\tilde{D}'$, $-i-1<t+1<d+g-2i-1-\deg\tilde{D}'$, $-(i-1)-1<t+1<d-2+g-2(i-1)-1-\deg\tilde{D}'$. Given that $t\notin\{-1,0\}$, we have both $t,\ t+1\neq 0$ so by induction we see that $R\Gamma_{M_i(d)}(\overline{G}_{\tilde{D}'}\otimes\Lambda_M^t)=R\Gamma_{M_i(d)}(\overline{G}_{\tilde{D}'}\otimes\Lambda_M^{t+1})=0$. On the other hand, we obtain $R\Gamma_{M_{i-1}(d-2)}(\overline{G}_{\tilde{D}'}\otimes\Lambda_M^{t+1})=0$ either by induction if $i>1$, or from Lemma 7.3 if $i=1$. Therefore, we get the desired vanishing from the corresponding hypercohomology spectral sequence and semicontinuity.

Next we do induction on $\alpha=\deg D$. If $\alpha\geq 1$, then we write $D=\tilde{D}+x_\alpha$ and take the first sequence of Lemma 3.11 with $F_{x_\alpha}^\vee$, twisted by $(\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^t$. This way we get an exact sequence involving $(\bigotimes_{k=1}^{\alpha}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^t$, and where the remaining terms are $(\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^{t-1}$ and $(\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^t$ on $M_i(d)$, and $(\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^t$ on $M_{i-1}(d-2)$. All three still satisfy the inequalities (7.1): $\deg\tilde{D}-i-1<t-1<d+g-2i-1-\deg D'$, $\deg\tilde{D}-i-1<t<d+g-2i-1-\deg D'$, $\deg\tilde{D}-(i-1)-1<t<d-2+g-2(i-1)-1-\deg D'$. Further, $t,\ t-1\notin[0,\ \deg\tilde{D}]$, so by induction $R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^{t-1})=R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^t)=0$, while $R\Gamma_{M_{i-1}(d-2)}((\bigotimes_{k=1}^{\alpha-1}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^t)=0$ either by induction when $i>1$ or by Lemma 7.3 when $i=1$ (observe that when $i=1$ we must have $t>\deg\tilde{D}$). By looking at the corresponding hypercohomology spectral sequence, we obtain the vanishing $R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha}F_{x_k}^\vee)\otimes\overline{G}_{D'}\otimes\Lambda_M^t)=0$. □

It remains to prove Proposition 7.6, which will take the rest of this section and require several steps. First, we see that it reduces to showing that $\overline{G}_D\otimes\Lambda_M^{-1}$ has no global sections on $M_1(d)$.

LEMMA 7.7
Under the assumptions of Proposition 7.6, (7.3) is equivalent to proving that

$$H^0\big(M_1(d),\overline{G}_D\otimes\Lambda_M^{-1}\big)=0 \tag{7.4}$$

for the case that every point in D has multiplicity at least 2.

Proof
First, we see that (7.4) is clearly necessary, so we need to show that it is sufficient. Note that $G_D^\vee \otimes \Lambda_M^{\deg D-1} \simeq \overline{G}_D \otimes \Lambda_M^{-1}$ by Corollary 3.14. We know by Theorem 3.21 that for $1 < j \le i$ this bundle descends from an object with weights within $[-\deg D + 1,\ 1]$, where $1 < j$ and $-\deg D + 1 > 1 + 2j - d - g$ by hypothesis. Hence, by Theorem 3.20, it suffices to show (7.3) when $i = 1$.

We write $D = \alpha_1 x_1 + \cdots + \alpha_s x_s$ with $x_k \neq x_j$. If $\deg D = 0$, then we are done by Lemma 5.2. Let us now assume that some $\alpha_i = 1$, say, for simplicity, $\alpha_1 = 1$. Then we can write $D = \tilde{D} + x_1$ and argue by induction on $\deg D$ as follows. By Lemma 3.11, we obtain an exact sequence

$$0 \to \overline{G}_{\tilde{D}} \otimes \Lambda_M^{-1} \to \overline{G}_D \otimes \Lambda_M^{-1} \to \overline{G}_{\tilde{D}} \to \overline{G}_{\tilde{D}}|_{M_0} \to 0,$$

where $M_0 = M_0(\Lambda(-2x_1))$. By the induction hypothesis, the first term in each sequence is Γ-acyclic. By Corollary 7.5, the last two terms in each sequence have vanishing higher cohomology and $H^0 = \mathbb{C}$ with a global section that does not vanish along $M_0(\Lambda(-2x_1))$. Thus

$$R\Gamma_{M_1(d)}(\overline{G}_D \otimes \Lambda_M^{-1}) = 0$$

by the hypercohomology spectral sequence $E_1^{p,q} = H^q(X, \mathcal{F}^p)$ and semicontinuity. So we can assume that $\alpha_k > 1$ for all k. Again, we write $D = \tilde{D} + x_1$ and get

$$0 \to \overline{G}_{\tilde{D}} \otimes \Lambda_M^{-1} \to F_{x_1} \otimes \overline{G}_{\tilde{D}} \otimes \Lambda_M^{-1} \to \overline{G}_{\tilde{D}} \to \overline{G}_{\tilde{D}}|_{M_0} \to 0. \tag{7.5}$$

The last two terms in (7.5) still have $R\Gamma = \mathbb{C}$, but now the global section vanishes along $M_0(\Lambda(-2x_1))$. Therefore, applying the same hypercohomology spectral sequence, we conclude that $F_{x_1} \otimes \overline{G}_{\tilde{D}} \otimes \Lambda_M^{-1}$ has the following cohomology: $h^p = 0$ for $p \ge 2$ and $h^0 = h^1 = 1$. By Remark 2.14, its stable deformation $\overline{G}_D \otimes \Lambda_M^{-1}$ must have $h^p = 0$ for $p \ge 2$ and $h^0 = h^1$. Hence, it suffices to show that $H^0(M_1(d), \overline{G}_D \otimes \Lambda_M^{-1}) = 0$, as claimed. □

In what follows, we focus on proving (7.4), under the assumptions of Proposition 7.6, and with $D = \alpha_1 x_1 + \cdots + \alpha_s x_s$, $\alpha_k > 1$. We recall the construction of $\overline{G}_D$ from the proof of Corollary 2.9 adapted to our case when D is not necessarily a fat point. Let $M = M_1(d)$.

Let $B_\alpha = \frac{\mathbb{C}[t_1,\ldots,t_\alpha]}{(\sigma_1,\ldots,\sigma_\alpha)}$, the covariant algebra, and let $\mathbb{B}_\alpha = \operatorname{Spec} B_\alpha$. Write the indexing set $\{1,\ldots,\alpha\}$ as a disjoint union of sets A_k of cardinality α_k for $k = 1,\ldots,s$, and denote $B = B_{\alpha_1} \otimes \cdots \otimes B_{\alpha_s}$. For every $j \in A_k$, we have a diagram of morphisms as in (2.3),

$$\begin{array}{ccccc} \mathbb{B}_{\alpha_1} \times \cdots \times \mathbb{B}_{\alpha_s} \times M & \xrightarrow{\pi_j} & \mathbb{D}_{\alpha_k} \times M & \xrightarrow{q_k} & C \times M \\ & \searrow^{\tau} & \iota \uparrow \downarrow \rho & \swarrow & \\ & & M & & \end{array} \tag{7.6}$$

We let $\mathcal{F}_k = q_k^* F$, where F is the universal bundle, and therefore $\overline{G}_D = \tau_*^{S_{\alpha_1} \times \cdots \times S_{\alpha_s}}(\bigotimes \pi_j^* \mathcal{F}_k \otimes \mathrm{sgn})$. Here τ_* does not change local sections of sheaves, but just forgets the B-algebra structure. Thus (7.4) is equivalent to the following: $\Lambda_M^{-1} \otimes \bigotimes \pi_j^* \mathcal{F}_k$ does not have skew-invariant global sections (with respect to each factor of $S_{\alpha_1} \times \cdots \times S_{\alpha_s}$).

The restriction of $\Lambda_M^{-1} \otimes \bigotimes \pi_j^* \mathcal{F}_k$ to the special fiber M is $\Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\otimes \alpha_k}$. While the group $S_{\alpha_1} \times \cdots \times S_{\alpha_s}$ acts trivially on the special fiber, the action on the vector bundle is still nontrivial (the action permutes tensor factors within each block).

LEMMA 7.8

Suppose that $s = 1$; that is, $D = \alpha x$ is a fat point. Write $\mathcal{F} = q_1^* F$, and let ρ be as in (7.6). Then $\operatorname{End} \rho_* \mathcal{F} = \mathbb{D}_\alpha$. *In particular, $\rho_* \mathcal{F}$ is indecomposable.*

Proof

We see that $\rho_* \mathcal{F} = \Phi_F(\mathcal{O}_{\alpha x})$, where Φ_F is the Fourier–Mukai functor with kernel F. The result follows from full faithfulness of Φ_F, which is given by Theorem 5.1. □

LEMMA 7.9

As a representation of $S_{\alpha_1} \times \cdots \times S_{\alpha_s}$, the space $H^0(M, \Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\otimes \alpha_k})$ is isomorphic to the direct sum $V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_s}$ of irreducible representations, where each V_{α_k} is the standard $(\alpha_k - 1)$-dimensional irreducible representation of S_{α_k} and the other factors S_{α_l}, $l \neq k$, act on V_{α_k} trivially. If we realize the representation V_{α_k} as $\{\sum a_j e_j \mid \sum a_j = 0\} \subset \mathbb{C}^{\alpha_k}$, then the vector $e_{j'} - e_{j''} \in V_{\alpha_k}$ corresponds to the global section $s_{j'j''}$ of $\Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\otimes \alpha_k}$ that can be written as a tensor product of the universal sections s_l of F_{x_l} with $l \neq k$, the universal sections s_k of F_{x_k} in positions $j \neq j', j''$, and the section of $\Lambda_M^{-1} \otimes F_{x_k} \otimes F_{x_k}$ (in positions j', j'') given by wedging (recall that Λ_M is the determinant of F_{x_k}).

Proof

The sections $s_{j'j''}$ satisfy the same linear relations as the difference vectors $e_{j'} - e_{j''}$, namely, that $s_{j_1 j_2} + s_{j_2 j_3} + \cdots + s_{j_{r-1} j_r} + s_{j_r j_1} = 0$ for $j_1, \ldots, j_r \in A_k$. Indeed, choose a basis $\{f_1, f_2\}$ in a fiber of the rank-2 bundle F_{x_k} so that the universal section is equal to f_2 and the determinant is given by $f_1 \wedge f_2$. After reordering of $j_1, \dots, j_r$, and ignoring factors of $s_{jj'}$ given by the universal sections s_l of F_{x_l} with $l \neq k$, we have

$$\begin{aligned} s_{12} + s_{23} + \dots + s_{r1} &= (f_1 \otimes f_2) \otimes f_2 \otimes \dots \otimes f_2 - (f_2 \otimes f_1) \otimes f_2 \otimes \dots \otimes f_2 \\ &\quad + f_2 \otimes (f_1 \otimes f_2) \otimes \dots \otimes f_2 - f_2 \otimes (f_2 \otimes f_1) \otimes \dots \otimes f_2 \\ &\quad + \dots = 0. \end{aligned}$$

Let $j_k = \min(A_k)$ for $k = 1, \dots, s$. It suffices to prove that the sections $s_{j_k j}$ for $k = 1, \dots, s$ and $j \in A_k \setminus \{j_k\}$ form a basis of $H^0(M, \Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\otimes \alpha_k})$. We prove this by induction on α. This is true if $\alpha = 0$ by Lemma 5.2 and if $\alpha = 1$ by Lemma 5.3. Let $\tilde{F} = F_{x_1}^{\otimes \alpha_1} \otimes \dots \otimes F_{x_s}^{\otimes(\alpha_s - 1)}$. We have the usual exact sequence obtained from Lemma 3.11:

$$0 \to \Lambda_M^{-1} \otimes \tilde{F} \to \Lambda_M^{-1} \otimes \tilde{F} \otimes F_{x_s} \to \tilde{F} \to \tilde{F}|_{M_0} \to 0, \tag{7.7}$$

where $M_0 = M_0(\Lambda(-2x_s))$. By Corollary 7.5, the last two terms have vanishing higher cohomology and $H^0 = \mathbb{C}$. If $\alpha_s = 1$ or, equivalently, $A_s = \{\alpha\}$, then the global section of $\tilde{F}$ does not vanish along M_0 and therefore $H^0(\Lambda_M^{-1} \otimes \tilde{F}) = H^0(\Lambda_M^{-1} \otimes \tilde{F} \otimes F_{x_s})$ by the corresponding hypercohomology spectral sequence, and the basis stays the same. On the other hand, if $\alpha \neq j_s$, then the global section of $\tilde{F}$ (the tensor product of universal sections) vanishes along M_0 inducing the zero map $H^0(\tilde{F}) \to H^0(\tilde{F}|_{M_0})$. Moreover, the section $s_{j_s \alpha} \in H^0(\Lambda_M^{-1} \otimes \tilde{F} \otimes F_{x_s})$ maps onto the global section of $\tilde{F}$. Thus the claim also follows from the hypercohomology spectral sequence. ☐

The sheaf $\bigotimes \pi_j^* \mathcal{F}_k$ carries a filtration by $B_{\geq d}(\bigotimes \pi_j^* \mathcal{F}_k)$, where $B_{\geq d}$ is the ideal of monomials of degree at least d. The associated graded object is $\mathrm{gr}(\bigotimes \pi_j^* \mathcal{F}_k) := \bigotimes_k F_{x_k}^{\otimes \alpha_k} \otimes_{\mathcal{O}_M} B$. If $\Lambda_M^{-1} \otimes \bigotimes \pi_j^* \mathcal{F}_k$ has a skew-invariant global section, an associated graded section will be a skew-invariant global section of $\Lambda_M^{-1} \otimes \mathrm{gr}(\bigotimes \pi_j^* \mathcal{F}_k)$.

By Frobenius reciprocity, the space of skew-invariants in $(V_{\alpha_1} \boxtimes \mathrm{Id} \boxtimes \dots \boxtimes \mathrm{Id}) \otimes B \subset H^0(M, \Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\alpha_k}) \otimes B$ has dimension $\alpha_1 - 1$ and basis

$$\sum_{i<j} \Big(\frac{\partial^r \Delta_1}{\partial t_i^r} - \frac{\partial^r \Delta_1}{\partial t_j^r} \Big) s_{ij} \boxtimes \Delta_2 \boxtimes \dots \boxtimes \Delta_s, \tag{7.8}$$

$r = 1, \dots, \alpha - 1$, where $\Delta_i \in \mathbb{C}[t_1, \dots, t_{\alpha_i}]$ is the Vandermonde determinant. Global sections of $H^0(M, \Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\alpha_k}) \otimes B$ coming from V_{α_k}, $k > 1$ are analogous. We will show that these global sections of $\Lambda_M^{-1} \otimes \mathrm{gr}(\bigotimes \pi_j^* \mathcal{F}_k)$ do not lift to sections of $\Lambda_M^{-1} \otimes \bigotimes \pi_j^* \mathcal{F}_k$.

LEMMA 7.10
It suffices to prove (7.4) for $s = 1$ and $\alpha = \alpha_1$.

Proof
We argue by induction on s. Let $\tilde{D} = \alpha_2 x_2 + \cdots + \alpha_s x_s$, and suppose that $H^0(\Lambda_M^{-1} \otimes \overline{G}_{\tilde{D}}) = 0$. Arguing as in the proof of Lemma 7.9, using the usual spectral sequences, we get $H^0(\Lambda_M^{-1} \otimes F_{x_1}^{\alpha_1} \otimes \overline{G}_{\tilde{D}}) = V_{\alpha_1}$, with a basis given by (7.8). Note that $\Delta_i \in B_{\alpha_i}$ is the element of top degree. Therefore, lifting basis elements to sections of $\overline{G}_D$ is equivalent to lifting them to $\overline{G}_{\alpha_1 x_1}$. □

From now on, we let $\alpha = \alpha_1$, $x = x_1$, and $\mathcal{F} = \mathcal{F}_1$. The space of skew-invariants in $H^0(\Lambda_M^{-1} \otimes F_x^{\otimes \alpha}) \otimes B_\alpha$ has basis $I_r = \sum_{i<j} (\frac{\partial^r \Delta}{\partial t_i^r} - \frac{\partial^r \Delta}{\partial t_j^r}) s_{ij}$, $r = 1, \ldots, \alpha - 1$. Writing, formally, $s_{ij} = e_i - e_j$, we also have $I_r = \sum_i \frac{\partial^r \Delta}{\partial t_i^r} e_i$. We claim that no I_r lifts to a global skew-invariant section $\widetilde{I_r}$ of $\Lambda_M^{-1} \otimes \bigotimes \pi_j^* \mathcal{F}$. We argue by induction on α.

LEMMA 7.11
Let $D = \alpha x$, $D' = (\alpha - 1)x$. Assuming that (7.4) holds for D', we have

$$H^0\Big(\mathbb{B}_\alpha \times M, \Lambda_M^{-1} \otimes \bigotimes \pi_j^* \mathcal{F} \otimes \mathrm{sgn}\Big)^{S_{\alpha-1}} = \mathbb{C}^{\alpha-1},$$

where $S_{\alpha-1} \subset S_\alpha$ is the subgroup fixing the last index.

Proof
We start with the Koszul complex on $C \times M$

$$0 \to \det F^\vee \to F^\vee \to \mathcal{O}_{C \times M} \to \mathcal{O}_{\mathcal{D}'} \to 0, \tag{7.9}$$

where $\mathcal{D}' \subset C \times M$ is the vanishing locus of the universal section. Recall that $\mathcal{D}'$ is smooth over C with fibers $M(\Lambda(-2x)) \subset M$ of codimension 2 over $x \in C$. In particular, $\mathcal{D}'$ is flat over C, and so the local generator $t \in \mathfrak{m}_x$ for $x \in C$ is not a zero divisor in $\mathcal{O}_{\mathcal{D}'}$. It follows that the pullback of (7.9) to $\mathbb{D}_\alpha \times M$ is also exact:

$$0 \to \Lambda_M^{-1} \to \mathcal{F}^\vee \to \mathcal{O}_{\mathbb{D}_\alpha \times M} \to \mathcal{O}_{\mathbb{D}_\alpha \times M(\Lambda(-2x))} \to 0.$$

We pull back to $\mathbb{B}_\alpha \times M$ and tensor with the locally free sheaf $\bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F}$ to obtain

$$\begin{aligned} 0 &\to \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F} \to \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \\ &\to \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F} \to \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F} \Big|_{\mathbb{B}_\alpha \times M(\Lambda(-2x))} \to 0. \end{aligned} \tag{7.10}$$

Next we compute $S_{\alpha-1}$-skew-invariant cohomology of the first, third, and fourth terms of (7.10). For each of these terms U, we have $H^0(U \otimes \mathrm{sgn})^{S_{\alpha-1}} = \rho_* \pi_{\alpha,*}^{S_\alpha - 1}(U \otimes \mathrm{sgn})$, which by Lemma 2.8 is a deformation of α copies of $\rho^* \pi_{\alpha,*}^{S_\alpha - 1}(U \otimes \mathrm{sgn})$ over $\mathbb{A}^1$. For the first term $U = \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F}$ in (7.10), we have that $\rho^* \pi_{\alpha,*}^{S_\alpha - 1}(U \otimes \mathrm{sgn})$ is isomorphic to $\Lambda_M^{-1} \otimes \overline{G}_{D'}$ (see the proof of Proposition 2.12), which is Γ-acyclic by the induction assumption. For the last two terms, $\rho^* \pi_{\alpha,*}^{S_\alpha - 1}(U \otimes \mathrm{sgn})$ is isomorphic to $\overline{G}_{D'}$ and $\overline{G}_{D'}|_{M(\Lambda(-2x))}$, respectively, both of which have $R\Gamma = \mathbb{C}$ by Corollary 7.5.

From this, it follows that $H^0(\Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}} = 0$, while $H^0(\bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}} = H^0(\bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F}|_{\mathbb{B}_\alpha \times M(\Lambda(-2x))} \otimes \mathrm{sgn})^{S_{\alpha-1}} = \mathbb{C}^\alpha$ and their higher cohomology vanishes. Furthermore, the last two groups are isomorphic to $\mathbb{D}_\alpha$ as $\mathbb{D}_\alpha$-modules and generated by the universal section $(\bigotimes_{j=1}^{\alpha-1} \pi_j^* \Sigma) \otimes \Delta_{\alpha-1}$, which under the restriction map to $\mathbb{B}_\alpha \times M(\Lambda(-2x))$ goes to $(\bigotimes_{j=1}^{\alpha-1} t_\alpha \pi_j^* \Sigma) \otimes \Delta_{\alpha-1}$. Therefore, the first page of the spectral sequence $E_1^{p,q} = H^q(X, \mathcal{F}^p)$ associated with (7.10) has the following shape:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & H^2(\mathbb{B}_\alpha \times M, \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}} & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & H^1(\mathbb{B}_\alpha \times M, \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}} & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & H^0(\mathbb{B}_\alpha \times M, \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}} & \longrightarrow & \mathbb{D}_\alpha & \xrightarrow{\cdot t_\alpha^{\alpha-1}} & \mathbb{D}_\alpha .
\end{array}
$$

We conclude that $H^0(\mathbb{B}_\alpha \times M, \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}} = \mathbb{C}^{\alpha-1}$. □

Proof of Proposition 7.6

We need to show that none of the $S_{\alpha-1}$-skew-invariant global sections found in Lemma 7.11 is S_α-skew-invariant. We can explicitly write a basis of $H^0(\mathbb{B}_\alpha \times M, \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}} = \mathrm{Hom}(\pi_\alpha^* \mathcal{F}, \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathcal{F} \otimes \mathrm{sgn})^{S_{\alpha-1}}$. Namely, consider the surjection $\pi_\alpha^* \mathcal{F} \twoheadrightarrow F_x$, followed by an isomorphism $F_x \xrightarrow{\sim} t_1^{\alpha-1} \mathcal{F} \simeq F_x$. Then we tensor with $\bigotimes_{j=2}^{\alpha-1} \pi_j^* \Sigma$, multiply by $t_2^{\alpha-3} t_3^{\alpha-4} \cdots t_{\alpha-2}$, and skew-symmetrize over $\{1, 2, \ldots, \alpha-1\}$. This way we obtain a morphism

$$\mu \in \operatorname{Hom}\Big(\pi_\alpha^* \mathscr{F}, \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathscr{F} \otimes \operatorname{sgn}\Big)^{S_{\alpha-1}}$$

and, therefore, also morphisms

$$\mu, t_\alpha \mu, \ldots, t_\alpha^{\alpha-2}\mu \in \operatorname{Hom}\Big(\pi_\alpha^* \mathscr{F}, \bigotimes_{j=1}^{\alpha-1} \pi_j^* \mathscr{F} \otimes \operatorname{sgn}\Big)^{S_{\alpha-1}}. \tag{7.11}$$

We claim that $t_\alpha^{\alpha-2}\mu \neq 0$, and therefore (7.11) gives a basis of the space $H^0(\mathbb{B}_\alpha \times M, \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^* \mathscr{F} \otimes \operatorname{sgn})^{S_{\alpha-1}}$ over $\mathbb{C}$. Indeed, notice that $t_\alpha^{\alpha-2}(t_1^{\alpha-1} t_2^{\alpha-3} t_3^{\alpha-4} \cdots t_{\alpha-2})$ is equal (up to sign) to the Vandermonde determinant $\Delta_\alpha \in \mathbb{B}_\alpha$, and it is also equal (up to a multiple) to $t_1^{\alpha-1}\Delta_{\alpha-1}$, where $\Delta_{\alpha-1}$ is the Vandermonde determinant in $t_1, \ldots, t_{\alpha-1}$. We show that these two expressions are not equal to zero. Let $\mathbb{B}_\alpha^{\text{top}}$ be the degree $\binom{\alpha}{2}$ component of $\mathbb{B}_\alpha$. Being spanned by Δ_α, $\mathbb{B}_\alpha^{\text{top}}$ is isomorphic to sgn as an S_α-module. Consider a monomial $m = t_1^{d_1} \cdots t_\alpha^{d_\alpha} \in \mathbb{B}_\alpha^{\text{top}}$. If $d_j = d_k$, then m is fixed by $(j\ k) \in S_\alpha$, so it must vanish. This leaves only the orbit of $t_1^{\alpha-1} t_2^{\alpha-2} \cdots t_{\alpha-1}$ under S_α, which all must be nonzero with

$$\sigma(t_1^{\alpha-1} t_2^{\alpha-2} \cdots t_{\alpha-1}) = (\operatorname{sgn}\sigma) t_1^{\alpha-1} t_2^{\alpha-2} \cdots t_{\alpha-1} \tag{7.12}$$

for $\sigma \in S_\alpha$. Monomials in $t_1^{\alpha-1}\Delta_{\alpha-1}$ of the form (7.12) have $\sigma(1) = 1$ and $\sigma(\alpha) = \alpha$. Moreover, they appear with a relative factor of $\operatorname{sgn}\sigma$ by antisymmetry of $\Delta_{\alpha-1}$, so they do not cancel in $\mathbb{B}_\alpha$, as claimed.

Therefore, $t_\alpha^{\alpha-2}\mu$ can be described as follows: it is the surjection $\pi_\alpha^* \mathscr{F} \twoheadrightarrow F_x$ followed by an isomorphism $F_x \xrightarrow{\sim} t_1^{\alpha-1} \mathscr{F} \simeq F_x$, twisted by $\bigotimes_{j=2}^{\alpha-1} \pi_j^* \Sigma$, multiplied by $\Delta_{\alpha-1}$ and then skew-symmetrized over $\{1, 2, \ldots, \alpha-1\}$. So the associated graded section of $t_\alpha^{\alpha-2}\mu$ is $\sum_{j=1}^{\alpha-1} s_{j\alpha} \cdot \Delta_\alpha \neq 0$ (cf. Lemma 7.9).

Finally, we check that no linear combination of (7.11) is S_α-skew-invariant. In fact, if $\alpha > 2$, the associated graded section does not involve s_{jk} for $j, k < \alpha$, while if $\alpha = 2$, the section is $s_{12}(f_1 - f_2)$, which is symmetric, not skew-symmetric. This completes the proof. □

8. Computation of $R\operatorname{Hom}(G_D, G_D)$

Now we will compute some of the Ext groups between G_D and $G_{D'}$, which will be needed in the proof of our semiorthogonal decomposition.

PROPOSITION 8.1

Let $d \leq 2g+1$ and $1 \leq i \leq v$. Suppose that D, D' are effective divisors, and let t be an integer satisfying

$$\deg D - i - 1 < t < d + g - 1 - 2i - \deg D'.$$

Then

$$H^p\big(M_i(d), G_D^\vee \otimes G_{D'} \otimes \Lambda_M^t\big) = 0$$

for every $p > \deg D - t$.

Proof
Let $\alpha = \deg D$, $\beta = \deg D'$. We first do the case $\alpha = \beta = 0$, for which we need to show vanishing of $H^p(M_i(d), \Lambda_M^t)$ for $p > -t$. If $t = 0$, this is trivial. If $t < 0$, observe that $i \geq -t$, so Theorem 6.1 gives $R\Gamma_{M_i}(\Lambda_M^t) = 0$. If $t > 0$, we notice that Λ_M^t has weight $-t$, with $1 + 2j - d - g < -t < j$ for every $1 < j \leq i$, so by Theorem 3.20 we must have $R\Gamma_{M_i}(\Lambda_M^t) = R\Gamma_{M_1}(\Lambda_M^t)$. But the latter is zero by Lemma 4.3, since $t \leq d + g - 4$.

Now we prove the result for $\beta = 0$ and $\alpha \geq 1$ by induction on α. Write $D = \tilde{D} + x$, and twist (3.3) by $G_{\tilde{D}}^\vee \otimes \Lambda_M^t$ to get an exact sequence

$$\begin{aligned} 0 &\to G_{\tilde{D}}^\vee \otimes \Lambda_M^{t-1} \to F_x^\vee \otimes G_{\tilde{D}}^\vee \otimes \Lambda_M^t \\ &\to G_{\tilde{D}}^\vee \otimes \Lambda_M^t \to G_{\tilde{D}}^\vee \otimes \Lambda_M^t|_{M_{i-1}(d-2)} \to 0. \end{aligned} \tag{8.1}$$

By induction, the first term has $H^p(M_i(d), G_{\tilde{D}}^\vee \otimes \Lambda_M^{t-1}) = 0$ for $p > \alpha - t$, and the third term has $H^p(M_i(d), G_{\tilde{D}}^\vee \otimes \Lambda_M^t) = 0$ for $p > \alpha - t - 1$. We see that on the last term we also have $H^p(M_{i-1}(d-2), G_{\tilde{D}}^\vee \otimes \Lambda_M^t) = 0$ for $p > \alpha - t - 1$. Indeed, if $i > 1$, this follows by induction, while if $i = 1$, we have $t \geq \alpha - 1$ and the restriction of $G_{\tilde{D}}^\vee \otimes \Lambda_M^t$ to $M_0(d-2) = \mathbb{P}^{d+g-4}$ is a deformation of a sum of line bundles $\bigoplus \mathcal{O}_{\mathbb{P}^{d+g-4}}(s_j)$ with $-(d+g-4) \leq -t \leq s_j \leq \alpha - t - 1 \leq 0$ (see Corollary 2.9, Remark 3.7). If $\alpha - t - 1 < 0$, this sum of line bundles is Γ-acyclic, and if $\alpha - t - 1 = 0$, this has vanishing cohomology H^p for $p > 0 = \alpha - t - 1$. In either case, we conclude that the last term has vanishing H^p for $p > \alpha - t - 1$ by semicontinuity. Taking the hypercohomology spectral sequence $E_1^{p,q} = H^q(X, \mathcal{F}^p)$ of (8.1), we conclude that $H^p(M_i(d), F_x^\vee \otimes G_{\tilde{D}}^\vee \otimes \Lambda_M^t) = 0$ for $p > \alpha - t$. Since $G_D^\vee \otimes \Lambda_M^t$ is a stable deformation over $\mathbb{A}^1$ of $F_x^\vee \otimes G_{\tilde{D}}^\vee \otimes \Lambda_M^t$ by Proposition 2.12, then by semicontinuity we also have $H^p(M_i(d), G_D^\vee \otimes \Lambda_M^t)$ for $p > t - \alpha$.

Finally, we do induction on $\beta \geq 1$. Similarly, write $D' = \tilde{D}' + y$, and twist (3.4) by $G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^t$ to get an exact sequence

$$\begin{aligned} 0 &\to G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^t \to G_D^\vee \otimes G_{\tilde{D}'} \otimes F_y \otimes \Lambda_M^t \\ &\to G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^{t+1} \to G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^{t+1}|_{M_{i-1}(d-2)} \to 0. \end{aligned}$$

By induction, the first term has $H^p = 0$ for $p > \alpha - t$ and the third one has $H^p = 0$ for $p > \alpha - t - 1$. The last term has vanishing pth cohomology for $p > \alpha - t -$

1, which follows by induction when $i > 1$. It remains to check the case $i = 1$. In this case, the restriction $G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^{t+1}|_{M_{i-1}(d-2)}$ is a deformation of a sum $\bigoplus \mathcal{O}_{\mathbb{P}^{d+g-4}}(s_j)$, with $-(d+g-4) \leq -t-\beta \leq \alpha - t - 1 \leq 0$. As before, we see that this has vanishing H^p for $p > \alpha - t - 1$ and the same is true for $G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^{t+1}|_{M_{i-1}(d-2)}$ by semicontinuity. The result then follows from taking the spectral sequence $E_1^{p,q} = H^q(X, \mathcal{F}^p)$ and semicontinuity. □

COROLLARY 8.2

Let $d \leq 2g+1$ and $0 \leq i \leq v$. If $\deg D \leq i$ and $\deg D' < d + g - 1 - 2i$, then we have

$$H^p\big(M_i(d), G_D^\vee \otimes G_{D'}\big) = 0$$

for every $p > \deg D$.

Proof

If $i = 0$, then D must be zero and the result follows from Corollary 7.5. For $i \geq 1$, this follows from taking $t = 0$ in Proposition 8.1. □

Using the previous results we can show that $G_D^\vee \otimes G_D$ has exactly one nontrivial global section, up to scalar multiplication. We need a lemma first.

LEMMA 8.3

Let $d \leq 2g+1$, and let D, D' be two effective divisors on C of $\deg D = \alpha \leq i$, $\deg D' < d + g - 2i - 1$. Write $D = x_1 + \cdots + x_\alpha$, in arbitrary order and possibly with repetitions. Then for every $k \leq \alpha$ we have $h^0(M_i(d), (\bigotimes_{j=1}^k F_{x_j}^\vee) \otimes \overline{G}_{D'}) \leq 1$.

Proof

If $i = 0$, then $\alpha = k = 0$ and this is given by Corollary 7.5. Let $i \geq 1$, so $d > 2$. We do induction on k. If $k = 0$, this still follows from Corollary 7.5. Otherwise, we use Lemma 3.11 to get an exact sequence

$$0 \to \bigotimes_{j=1}^{k-1} F_{x_j}^\vee \otimes \overline{G}_{D'} \otimes \Lambda_M^{-1} \to \bigotimes_{j=1}^{k} F_{x_j}^\vee \otimes \overline{G}_{D'}$$
$$\to \bigotimes_{j=1}^{k-1} F_{x_j}^\vee \otimes \overline{G}_{D'} \to \bigotimes_{j=1}^{k-1} F_{x_j}^\vee \otimes \overline{G}_{D'}\Big|_{M_{i-1}} \to 0,$$

where $M_{i-1} = M_{i-1}(\Lambda(-2x_k))$. The first term can be seen to be Γ-acyclic using Theorem 7.1. Indeed, here $t = -1 \notin [0,\ k-1]$ and the inequalities $(k-1) - i - 1 < -1 < d + g - 2i - 1 - \deg D'$ are satisfied since $k \leq \alpha \leq i$ and $\deg D' < d + g -$

$2i$. On the other hand, $h^0(M_i(d), (\bigotimes_{j=1}^{k-1} F_{x_j}^{\vee}) \otimes \overline{G}_{D'}) \leq 1$ by induction. Therefore, taking the hypercohomology spectral sequence $E_1^{p,q} = H^q(X, \mathcal{F}^p)$ of the Γ-acyclic complex above, we conclude that $h^0(M_i(d), (\bigotimes_{j=1}^{k} F_{x_j}^{\vee}) \otimes \overline{G}_{D'}) \leq 1$ as well. □

COROLLARY 8.4
Suppose that $d \leq 2g+1$, *and let* $0 \leq i \leq v$. *If* $\deg D \leq i$, *then*

$$\operatorname{Hom}_{M_i(d)}(G_D, G_D) = \operatorname{Hom}_{M_i(d)}(\overline{G}_D, \overline{G}_D) = \mathbb{C}.$$

Proof
We have $\operatorname{Hom}_{M_i(d)}(G_D, G_D) = H^0(M_i(d), G_D^{\vee} \otimes G_D)$. By Corollary 3.14, $G_D^{\vee} \otimes G_D \simeq \overline{G}_D^{\vee} \otimes \overline{G}_D$, so $\operatorname{Hom}_{M_i(d)}(G_D, G_D) = \operatorname{Hom}_{M_i(d)}(\overline{G}_D, \overline{G}_D)$ has dimension $h^0(G_D^{\vee} \otimes G_D) = h^0(\overline{G}_D^{\vee} \otimes \overline{G}_D)$, which by Corollary 2.9 and semicontinuity, is at most $h^0(M_i(d), (\bigotimes_{j=1}^{\deg D} F_{x_j}^{\vee}) \otimes \overline{G}_D)$. But by Lemma 8.3, this dimension is at most 1, since by hypothesis $\deg D \leq i < d + g - 2i - 1$. On the other hand, the identity provides a nontrivial map $G_D \to G_D$, so $\dim \operatorname{Hom}_{M_i(d)}(G_D, G_D)$ must be exactly 1. □

9. Full faithfulness

In this section, we construct fully faithful embeddings from $D^b(\operatorname{Sym}^{\alpha} C)$ to $D^b(M_i(\Lambda))$, for $0 \leq \alpha \leq i$, where $1 \leq i \leq v$ and $d \leq 2g-1$.

Definition 9.1
For $0 \leq \alpha \leq i$, let $\Phi_{\alpha}^i : D^b(\operatorname{Sym}^{\alpha} C) \to D^b(M_i(\Lambda))$ be the Fourier–Mukai functor determined by $F^{\boxtimes \alpha} \in D^b(\operatorname{Sym}^{\alpha} C \times M_i(\Lambda))$, where F is the universal bundle on $C \times M_i(\Lambda)$. Similarly, let $\overline{\Phi}_{\alpha}^i : D^b(\operatorname{Sym}^{\alpha} C) \to D^b(M_i(\Lambda))$ be the Fourier–Mukai functor given by $\overline{F}^{\boxtimes \alpha} \in D^b(\operatorname{Sym}^{\alpha} C \times M_i(\Lambda))$ (see Definition 2.3 for $F^{\boxtimes \alpha}$ and $\overline{F}^{\boxtimes \alpha}$).

Remark 9.2
For $\alpha = 0$, the functor $\Phi_0^i = \overline{\Phi}_0^i$ is simply the (derived) pullback of the map from $M_i(\Lambda)$ to a point.

We have already proved in Theorem 5.1 that $\Phi_1^1 = \Phi_F$ is fully faithful. The main result of the present section is a generalization of that result.

THEOREM 9.3
Suppose that $2 < d \leq 2g-1$. *For* $1 \leq i \leq v$, $0 \leq \alpha \leq i$, *both* Φ_{α}^i *and* $\overline{\Phi}_{\alpha}^i$ *are fully faithful functors.*

We will use induction to prove Theorem 9.3. First we need to investigate $R\,\mathrm{Hom}(G_D, G_{D'})$ between different divisors. We want to obtain Γ-acyclicity of $G_D^\vee \otimes G_{D'}$, for which we need some preliminary computations.

LEMMA 9.4

Let $d > 0$ and $0 \le i \le v$. Let D, D' be effective divisors on C with $D = \alpha x$ and $x \notin D'$. If $\alpha + \deg D' < d + g - 2i - 1$, then

$$R\Gamma_{M_i}(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha) = \mathbb{C}.$$

Moreover, if $i \ge 1$, then the unique (up to a scalar) global section of $G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha$ vanishes precisely along the union of codimension 2 loci $M_{i-1}(\Lambda(-2x))$ and $M_{i-1}(\Lambda(-2y))$ for $y \in \mathrm{supp}(D')$.

Proof

We use the fact that $G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha$ is a deformation over $\mathbb{A}^1$ of $(F_x^\vee)^{\otimes\alpha} \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k} \otimes \Lambda_M^\alpha \simeq F_x^{\otimes\alpha} \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k}$, where $D' = \sum y_k$. By Corollary 7.5, we see that $R\Gamma_{M_i}(F_x^{\otimes\alpha} \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k}) = \mathbb{C}$, so by semicontinuity and equality of the Euler characteristic, we must have $R\Gamma_{M_i}(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha) = \mathbb{C}$ as well. Furthermore, the global section of $G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha$ is a deformation of the global section of $F_x^{\otimes\alpha} \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k}$ over $\mathbb{A}^1$, which does not vanish outside of the union of loci $M_{i-1}(\Lambda(-2x))$ and $M_{i-1}(\Lambda(-2y_k))$. On the other hand, the tautological section of this bundle vanishes precisely along these loci. □

LEMMA 9.5

Suppose that $2 < d \le 2g + 1$ and $1 \le i \le v$. Let D, D' be effective divisors with $D = \alpha x$ and $D' = \beta x + \tilde{D}'$, $x \notin \tilde{D}'$. Suppose that $\alpha = \deg D \le i$ and $\deg D' < d + g - 2i - 1$. Then the map $R\Gamma_{M_i(d)}(\overline{G}_{\alpha x}^\vee \otimes \overline{G}_{\beta x}) \to R\Gamma_{M_i(d)}(\overline{G}_{\alpha x}^\vee \otimes \overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'})$ given by tensoring with the unique (up to scalar) section of $\overline{G}_{\tilde{D}'}$ (cf. Corollary 7.5) is an isomorphism.

Proof

We argue by induction on α. If $\alpha = 0$, this is clear, as the map $R\Gamma_{M_i(d)}(\overline{G}_{\beta x}) \to R\Gamma_{M_i(d)}(\overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'})$ is $\mathbb{C} \xrightarrow{\sim} \mathbb{C}$ (cf. Corollary 7.5).

For the inductive step, we argue as in the proof of Proposition 7.6, specifically as in Lemma 7.11: $\overline{G}_{\alpha x}^\vee \simeq \Lambda_M^{-\alpha} \otimes G_{\alpha x} = \Lambda_M^{-\alpha} \otimes \tau_*^{S_\alpha}(\bigotimes_{j=1}^\alpha \pi_j^* \mathcal{F})$, which is a direct summand in

$$\tau_*^{S_{\alpha-1}}\Big(\bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F}\Big). \tag{9.1}$$

Here $\mathcal{F} = q^*F = q_1^*F$ from (7.6). So it suffices to prove our claim for the bundle (9.1). As in the proof of Lemma 7.11, we have an exact sequence

$$0 \to \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha-1} \pi_j^*\mathcal{F} \to \Lambda_M^{-1} \otimes \bigotimes_{j=1}^{\alpha} \pi_j^*\mathcal{F}$$
$$\to \bigotimes_{j=1}^{\alpha-1} \pi_j^*\mathcal{F} \to \bigotimes_{j=1}^{\alpha-1} \pi_j^*\mathcal{F}\Big|_{\mathbb{B}_\alpha \times M_{i-1}(\Lambda(-2x))} \to 0, \tag{9.2}$$

to which we apply $\tau_*^{S_{\alpha-1}}$, then tensor with $\Lambda_M^{1-\alpha} \otimes \overline{G}_{\beta x}$ (resp., with $\Lambda_M^{1-\alpha} \otimes \overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'}$) and then compute $R\Gamma$. The resulting left term is a deformation of α copies of $\Lambda_M^{-1} \otimes \overline{G}^\vee_{(\alpha-1)x} \otimes \overline{G}_{\beta x}$ (resp., $\Lambda_M^{-1} \otimes \overline{G}^\vee_{(\alpha-1)x} \otimes \overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'}$), both of which are Γ-acyclic by Theorem 7.1.

Therefore, we have two exact triangles related by a commutative diagram:

$$\begin{array}{ccc}
R\Gamma(\Lambda_M^{-\alpha} \otimes U \otimes \pi_\alpha^*\mathcal{F} \otimes \overline{G}_{\beta x})^{S_{\alpha-1}} & \longrightarrow & R\Gamma(\Lambda_M^{-\alpha} \otimes U \otimes \pi_\alpha^*\mathcal{F} \otimes \overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'})^{S_{\alpha-1}} \\
\downarrow & & \downarrow \\
R\Gamma(\Lambda_M^{1-\alpha} \otimes U \otimes \overline{G}_{\beta x})^{S_{\alpha-1}} & \longrightarrow & R\Gamma(\Lambda_M^{1-\alpha} \otimes U \otimes \overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'})^{S_{\alpha-1}} \\
\downarrow & & \downarrow \\
R\Gamma(\Lambda_M^{1-\alpha} \otimes U \otimes \overline{G}_{\beta x}|_{\mathbb{B}_\alpha \times M'})^{S_{\alpha-1}} & \longrightarrow & R\Gamma(\Lambda_M^{1-\alpha} \otimes U \otimes \overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'}|_{\mathbb{B}_\alpha \times M'})^{S_{\alpha-1}} \\
\downarrow & & \downarrow
\end{array} \tag{9.3}$$

where $U = \bigotimes_{j=1}^{\alpha-1} \pi_j^*\mathcal{F}$, $M' = M_{i-1}(\Lambda(-2x))$, and the horizontal maps are multiplication by the universal section of $\overline{G}_{\tilde{D}'}$. The middle row of (9.3) is a deformation of α copies of the map $R\Gamma_{M_i(d)}(\overline{G}^\vee_{(\alpha-1)x} \otimes \overline{G}_{\beta x}) \to R\Gamma_{M_i(d)}(\overline{G}^\vee_{(\alpha-1)x} \otimes \overline{G}_{\beta x} \otimes \overline{G}_{\tilde{D}'})$, which is an isomorphism by the induction assumption. The same is true for the third row, on the moduli space $M_{i-1}(\Lambda(-2x))$. We conclude that the first row of (9.3) must also be an isomorphism, which completes the proof. □

LEMMA 9.6

Suppose that $2 < d \le 2g+1$ *and* $1 \le i \le v$. *Let* D, D' *be effective divisors with* $D = \alpha x$ *and* $\operatorname{mult}_x(D') \le \alpha - 1$. *Suppose that* $\alpha = \deg D \le i$ *and* $\deg D' < d + g - 2i - 1$. *If we assume that* $\overline{\Phi}^i_{\alpha'}$ *and* $\overline{\Phi}^{i-1}_{\alpha'}$ *are fully faithful for every* $\alpha' < \alpha$, *then* $R\Gamma_{M_i(d)}(\overline{G}^\vee_D \otimes \overline{G}_{D'}) = 0$.

Proof

By Lemma 9.5, it suffices to consider the case $D' = \beta x$, where $\beta < \alpha$. Moreover, arguing as in Lemma 9.5, we can assume that $\alpha = \beta + 1$, so it suffices to show that $R\Gamma_{M_i(d)}(\overline{G}_{\alpha x}^{\vee}, \overline{G}_{(\alpha-1)x}) = 0$ under the assumptions $\alpha \le i$, $\alpha < d + g - 2i$. Note that if $i = 1$, then $\alpha = 1$ and this follows (unconditionally) from Lemma 5.3. So we can assume that $i > 1$ and use full faithfulness of $\overline{\Phi}_{\alpha'}^{i}$ and $\overline{\Phi}_{\alpha'}^{i-1}$ as in the hypothesis. As in Lemma 9.5, we consider the exact sequence (9.2), twist it by $\Lambda_M^{1-\alpha} \otimes \overline{G}_{(\alpha-1)x}$, and take $S_{\alpha-1}$-invariant global sections. The resulting term on the left vanishes by semicontinuity and Theorem 7.1. It suffices to show that the second term vanishes, because it contains $R\Gamma_{M_i(d)}(\overline{G}_{\alpha x}^{\vee} \otimes \overline{G}_{(\alpha-1)x})$ as a direct summand. But the last two terms are deformations over $\mathbb{A}^1$ of α copies of the map $R\operatorname{Hom}_{M_i(d)}(\overline{G}_{(\alpha-1)x}, \overline{G}_{(\alpha-1)x}) \to R\operatorname{Hom}_{M_{i-1}(d-2)}(\overline{G}_{(\alpha-1)x}, \overline{G}_{(\alpha-1)x})$, which is an isomorphism by our assumption that $\overline{\Phi}_{\alpha-1}^{i}$ and $\overline{\Phi}_{\alpha-1}^{i-1}$ are fully faithful. This completes the proof. □

THEOREM 9.7

Suppose $2 < d \le 2g + 1$ *and* $1 \le i \le v$. *Let* D, D' *be effective divisors on* C, *with* $D \not\le D'$ *and satisfying* $\deg D \le i$ *and* $\deg D' < d + g - 2i - 1$. *If we assume that* $\overline{\Phi}_{\alpha'}^{i}$ *is fully faithful for every* $\alpha' < \deg D$, *then* $R\Gamma_{M_i(d)}(\overline{G}_D^{\vee} \otimes \overline{G}_{D'}) = 0$.

Proof

We do induction on $\deg D$. If $\deg D = 1$, then we have $D = x$ and $\operatorname{mult}_x(D') = 0$, so the result follows from Lemma 9.6 with $\alpha = 1$.

Let $\deg D > 1$, and so $i > 1$ as well. Since $D \not\le D'$, there is a point $x \in D$ with $\operatorname{mult}_x(D) = \alpha$, $\operatorname{mult}_x(D') \le \alpha - 1$. If $\operatorname{supp}(D) = \{x\}$, then $D = \alpha x$ is a fat point and the result follows from Lemma 9.6. Otherwise, we can find a point $y \ne x$ such that $\tilde{D} = D - y$ is effective. From (3.3), we get an exact sequence

$$
\begin{aligned}
0 &\to \overline{G}_{\tilde{D}}^{\vee} \otimes \overline{G}_{D'} \otimes \Lambda_M^{-1} \to F_y^{\vee} \otimes \overline{G}_{\tilde{D}}^{\vee} \otimes \overline{G}_{D'} \\
&\to \overline{G}_{\tilde{D}}^{\vee} \otimes \overline{G}_{D'} \to \overline{G}_{\tilde{D}}^{\vee} \otimes \overline{G}_{D'}|_{M_{i-1}(d-2)} \to 0.
\end{aligned}
$$

By induction, $R\Gamma_{M_i(d)}(\overline{G}_{\tilde{D}}^{\vee} \otimes \overline{G}_{D'}) = R\Gamma_{M_{i-1}(d-2)}(\overline{G}_{\tilde{D}}^{\vee} \otimes \overline{G}_{D'}) = 0$. On the other hand, the term $\overline{G}_{\tilde{D}}^{\vee} \otimes \overline{G}_{D'} \otimes \Lambda_M^{-1}$ satisfies the inequalities (7.1) with $t = -1 \notin [0, \deg \tilde{D}]$, so by Theorem 7.1 it is Γ-acyclic. As usual, the result follows from the hypercohomology spectral sequence and semicontinuity. □

Now we can prove the main result of this section.

Proof of Theorem 9.3

By Bondal–Orlov's criterion in [7], we only need to consider the images of skyscraper sheaves, $\Phi^i_\alpha(\mathcal{O}_{\{D\}}) = G_D$ and $\overline{\Phi}^i_\alpha(\mathcal{O}_{\{D\}}) = \overline{G}_D$. Namely, we need to show that for two divisors $D,\ D' \in \operatorname{Sym}^\alpha C$ we have

$$R^p\Gamma_{M_i(\Lambda)}(\overline{G}^\vee_D \otimes \overline{G}_{D'}) = \begin{cases} 0 & \text{if } D \neq D' \text{ or } p < 0 \text{ or } p > \alpha, \\ \mathbb{C} & \text{if } p = 0 \text{ and } D = D' \end{cases} \tag{9.4}$$

and similarly for $R\Gamma_{M_i(\Lambda)}(G^\vee_D \otimes G_{D'})$. Observe that since $R\Gamma_{M_i(\Lambda)}(\overline{G}^\vee_D \otimes \overline{G}_{D'}) = R\Gamma_{M_i(\Lambda)}(G^\vee_{D'} \otimes G_D)$ (cf. Corollary 3.14), full faithfulness of Φ^i_α is equivalent to that of $\overline{\Phi}^i_\alpha$, and it suffices to prove (9.4). We prove it by induction on α, where the case $\alpha = 0$ follows from the fact that $\mathcal{O}_{M_i(\Lambda)}$ is an exceptional object, since $M_i(\Lambda)$ is rational. So we assume that (9.4) holds for $\alpha' < \alpha$. If $D = D'$, then (9.4) follows directly from Corollaries 8.2 and 8.4. Now let $D \neq D'$ be different divisors of degree $\alpha \leq i$. Notice that $i \leq (d-1)/2 \leq g-1$, so the inequality $\alpha \leq d + g - 2i - 2$ holds. Therefore, in this case (9.4) follows from Theorem 9.7 by our induction hypothesis. We conclude that Φ^i_α and $\overline{\Phi}^i_\alpha$ are fully faithful functors. □

10. Proof of the semiorthogonal decomposition

Throughout this section we fix $d = \deg \Lambda = 2g-1$ so that $v = (d-1)/2 = g-1$. We are interested in the moduli spaces $M_i = M_i(\Lambda)$, where i will always be assumed to satisfy $1 \leq i \leq g-1$. Note that when $d = 2g-1$, the canonical bundle is $\omega_{M_i} = \mathcal{O}_i(-3, 3-3g) = \Lambda^{-1}_M \otimes \zeta^{-1} \otimes \theta^{-1}$ (see [39, 6.1] and Definition 3.10).

By abuse of notation, we will denote the essential image $\Phi^i_\alpha(\operatorname{Sym}^\alpha C)$ simply by Φ^i_α, and the image $\overline{\Phi}^i_\alpha(\operatorname{Sym}^\alpha C)$ by $\overline{\Phi}^i_\alpha$, which by Theorem 9.3 are admissible subcategories of $D^b(M_i)$ equivalent to $D^b(\operatorname{Sym}^\alpha C)$. In particular, Φ^i_0 is the full triangulated subcategory generated by $\mathcal{O}_{M_i}$, which is equivalent to $D^b(\mathrm{pt})$.

Definition 10.1

We define the following full triangulated subcategories of $D^b(M_i)$:

$$\mathcal{A}_{2k} := \Phi^i_{2k} \otimes \Lambda^{-k}_M \otimes \theta^{-1}, \quad 0 \leq 2k \leq i,$$

$$\mathcal{B}_{2k} := \Phi^i_{2k} \otimes \Lambda^{-k}_M, \quad 0 \leq 2k \leq i,$$

$$\mathcal{C}_{2k+1} := \overline{\Phi}^i_{2k+1} \otimes \Lambda^{-k}_M \otimes \zeta \otimes \theta^{-1}, \quad 0 \leq 2k+1 \leq i,$$

$$\mathcal{D}_{2k+1} := \overline{\Phi}^i_{2k+1} \otimes \Lambda^{-k}_M \otimes \zeta, \quad 0 \leq 2k+1 \leq i.$$

Each of these subcategories is equivalent to some $D^b(\operatorname{Sym}^\alpha C)$ with either $\alpha = 2k$ or $\alpha = 2k+1$. These four families of subcategories constitute the building blocks of our semiorthogonal decomposition on $D^b(M_i)$. We will see that different subcategories of the form $\mathcal{A}_{2k}$ are semiorthogonal to each other, and the same is true for subcategories within the other three blocks. We need the following lemma.

LEMMA 10.2
Let $\mathcal{D}_1$, $\mathcal{D}_2$ be admissible subcategories of a triangulated category $\mathcal{D}$, and let Ω_1, Ω_2 be spanning classes (see [20, Section 3.2]) of $\mathcal{D}_1$, $\mathcal{D}_2$. If we have that $\mathrm{Hom}_{\mathcal{D}}(A, B[k]) = 0$ *for every $A \in \Omega_1$, $B \in \Omega_2$, and $k \in \mathbb{Z}$, then* $\mathrm{Hom}_{\mathcal{D}}(F, G) = 0$ *for every $F \in \mathcal{D}_1$, $G \in \mathcal{D}_2$.*

Proof
We need to show that $\mathcal{D}_1 \subset {}^{\perp}\mathcal{D}_2$ or, equivalently, $\mathcal{D}_2 \subset \mathcal{D}_1^{\perp}$.

First we see that $\Omega_1 \subset {}^{\perp}\mathcal{D}_2$. Let $A \in \Omega_1$. Since $\mathcal{D} = \langle \mathcal{D}_2, {}^{\perp}\mathcal{D}_2 \rangle$, we can fit A in an exact triangle $D \to A \to D' \to D[1]$, where $D \in {}^{\perp}\mathcal{D}_2$ and $D' \in \mathcal{D}_2$. Applying $\mathrm{Hom}(\cdot, B)$ for $B \in \Omega_2$, we get a long exact sequence where $\mathrm{Hom}(D, B[k]) = 0$ by definition and $\mathrm{Hom}(A, B[k]) = 0$ by hypothesis. Therefore, $\mathrm{Hom}(D', B[k]) = 0$ for every k and every $B \in \Omega_2$, so $D' \simeq 0$ since Ω_2 is a spanning class of $\mathcal{D}_2$. As a consequence, $A \simeq D \in {}^{\perp}\mathcal{D}_2$.

Now let $G \in \mathcal{D}_2$. Similarly, there is an exact triangle $D \to G \to D' \to D[1]$ with $D \in \mathcal{D}_1$, $D' \in \mathcal{D}_1^{\perp}$. Applying $\mathrm{Hom}(A, \cdot)$ with $A \in \Omega_1$, we now see that $\mathrm{Hom}(A, D[k]) = \mathrm{Hom}(A, G[k]) = 0$ by the previous discussion and therefore $D' \simeq 0$. This implies that $G \simeq D \in \mathcal{D}_1^{\perp}$, as desired. □

PROPOSITION 10.3
Let $k > l$ and $0 \le 2l < 2k \le i$. Then

$$\mathrm{Hom}_{D^b(M_i)}(\mathcal{A}_{2k}, \mathcal{A}_{2l}) = 0, \qquad \mathrm{Hom}_{D^b(M_i)}(\mathcal{B}_{2k}, \mathcal{B}_{2l}) = 0.$$

Similarly, if $k < l$ and $0 \le 2k + 1 < 2l + 1 \le i$, then we have

$$\mathrm{Hom}_{D^b(M_i)}(\mathcal{C}_{2k+1}, \mathcal{C}_{2l+1}) = 0, \qquad \mathrm{Hom}_{D^b(M_i)}(\mathcal{D}_{2k+1}, \mathcal{D}_{2l+1}) = 0.$$

Proof
Let us first show semiorthogonality between subcategories of the form $\mathcal{A}_{2k}$, $\mathcal{A}_{2l}$, $k > l$, as well as semiorthogonality between those of the form $\mathcal{B}_{2k}$, $\mathcal{B}_{2l}$, $k > l$. Since skyscraper sheaves $\mathcal{O}_{\{D\}}$ of closed points $D \in \mathrm{Sym}^{\alpha} C$ are a spanning class of $D^b(\mathrm{Sym}^{\alpha} C)$ (see [20, Proposition 3.17]), Lemma 10.2 says that semiorthogonality can be checked on closed points. That is, it suffices to show that for $D \in \mathrm{Sym}^{2k} C$, $D' \in \mathrm{Sym}^{2l} C$, with $0 \le 2l < 2k \le i \le g-1$, we have $R\Gamma_{M_i}(G_D^{\vee} \otimes G_{D'} \otimes \Lambda_M^{k-l}) = 0$. But this follows from Theorem 7.1 (and Remark 7.2). Indeed, the inequalities

$$2k - i - 1 < k - l < d + g - 2i - 1 - 2l$$

are equivalent to $k+l<i+1$ and $k+l+2i<d+g-1$, which are guaranteed by the fact that $k+l<i\leq(d-1)/2<g$ in this case. Also, since $k>l$, we have $2k\notin[k-l,\ k+l]$. This proves the first two semiorthogonality statements.

Similarly, in order to prove semiorthogonality between subcategories $\mathcal{C}_{2k+1}$, $\mathcal{C}_{2l+1}$, $k<l$, as well as between $\mathcal{D}_{2k+1}$, $\mathcal{D}_{2l+1}$, $k<l$, we need to prove that for $D\in\operatorname{Sym}^{2k+1}C$, $D'\in\operatorname{Sym}^{2l+1}C$, with $0\leq 2k+1<2l+1\leq i\leq g-1$, we must have

$$R\Gamma_{M_i}(\overline{G}_D^\vee\otimes\overline{G}_{D'}\otimes\Lambda_M^{k-l})=0.$$

Again, this can be proved using Theorem 7.1: the inequalities

$$2k+1-i-1<k-l<d+g-1-2i-(2l+1)$$

are equivalent to $k+l<i$ and $k+l+2i<d+g-2$, both of which follow from the fact that $k+l+1<i\leq(d-1)/2<g$ in this case. Similarly, $k<l$ implies $k-l\notin[0,\ 2k+1]$. This proves the required vanishing. □

THEOREM 10.4

Let $d=2g-1$ and $1\leq i\leq g-1$. On $D^b(M_i)$, we have a semiorthogonal list of admissible subcategories arranged in four blocks

$$\mathcal{A},\mathcal{C},\mathcal{B},\mathcal{D}, \tag{10.1}$$

where

$$\mathcal{A}=\langle\mathcal{A}_{2k}\rangle_{0\leq 2k\leq i},\qquad \mathcal{C}=\langle\mathcal{C}_{2k+1}\rangle_{1\leq 2k+1\leq i},$$
$$\mathcal{B}=\langle\mathcal{B}_{2k}\rangle_{0\leq 2k\leq\min(i,\ g-2)},\qquad \mathcal{D}=\langle\mathcal{D}_{2k+1}\rangle_{1\leq 2k+1\leq\min(i,\ g-2)}$$

as given in Definition 10.1. Within the blocks $\mathcal{A}$ and $\mathcal{B}$, the subcategories are arranged in increasing order of k. Within the blocks $\mathcal{C}$ and $\mathcal{D}$, the subcategories are arranged in decreasing order of k.

Proof

All of these are admissible subcategories of $D^b(M_i)$ by Theorem 9.3, and we have already shown in Proposition 10.3 that, within each of the four blocks in (10.1), the corresponding subcategories are semiorthogonal in the given order. It remains to prove semiorthogonality between different blocks.

Step 1

Between $\mathcal{A}$ and $\mathcal{C}$: we show that $\operatorname{Hom}_{D^b(M_i)}(\mathcal{C}_{2k+1},\mathcal{A}_{2l})=0$. By Lemma 10.2, this amounts to showing that

$$R\Gamma_{M_i}(\overline{G}_D^\vee \otimes G_{D'} \otimes \Lambda_M^{k-l} \otimes \zeta^{-1}) = 0$$

for $D \in \mathrm{Sym}^{2k+1} C$, $D' \in \mathrm{Sym}^{2l} C$, with $0 \le 2k+1, 2l \le i \le (d-1)/2 = g-1$. We can apply Theorem 4.1 (and Remark 4.2) since the inequalities

$$2k + 1 - g < k - l < d - 2l - i - 1$$

are equivalent to $k + l < g - 1$ and $k + l + i < d - 1$, which hold in this case as $k + l < i \le (d-1)/2 = g - 1$. This gives the corresponding semiorthogonality.

Step 2

Between $\mathcal{A}$ and $\mathcal{B}$: let us show that $\mathrm{Hom}_{D^b(M_i)}(\mathcal{B}_{2k}, \mathcal{A}_{2l}) = 0$. Again by Lemma 10.2, we need to show that $R\Gamma_{M_i}(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{k-l} \otimes \theta^{-1}) = 0$ when $D \in \mathrm{Sym}^{2k} C$, $D' \in \mathrm{Sym}^{2l} C$, $0 \le 2k, 2l \le i \le (d-1)/2 = g-1$ and $2k \le g-2$. By Serre duality, given that $\omega_{M_i} = \Lambda_M^{-1} \otimes \zeta^{-1} \otimes \theta^{-1}$, this is equivalent to showing that $G_{D'}^\vee \otimes G_D \otimes \Lambda_M^{l-k-1} \otimes \zeta^{-1}$ is Γ-acyclic on M_i under the conditions above. This is given by Theorem 4.1 because

$$2l - g < l - k - 1 < d - 2k - i - 1$$

is equivalent to $l + k < g - 1$ and $l + k + i < d$, and these inequalities hold since $l + k + i \le 2i \le d - 1$ and $2l + 2k \le g - 1 + g - 2$ in this case.

Step 3

Between $\mathcal{A}$ and $\mathcal{D}$: for $\mathrm{Hom}_{D^b(M_i)}(\mathcal{D}_{2k+1}, \mathcal{A}_{2l})$, we need to show that $R\Gamma_{M_i}(\overline{G}_D^\vee \otimes G_{D'} \otimes \Lambda_M^{k-l} \otimes \zeta^{-1} \otimes \theta^{-1}) = 0$ whenever $D \in \mathrm{Sym}^{2k+1} C$, $D' \in \mathrm{Sym}^{2l} C$, $0 \le 2l, 2k+1 \le i \le (d-1)/2 = g-1$. Again by Serre duality, this is equivalent to Γ-acyclicity of $G_{D'}^\vee \otimes \overline{G}_D \otimes \Lambda_M^{l-k-1}$.

If $l \le k$, we check that this is given by Theorem 7.1. Indeed, the corresponding inequalities

$$2l - i - 1 < l - k - 1 < d + g - 2i - 1 - (2k+1)$$

are equivalent to $k + l < i$ and $l + k + 2i < d + g - 1$. The former follows from $2l, 2k+1 \le i$ and the latter follows from $l + k < i < g$ and $2i \le d - 1$. Also, the fact that $k \ge l$ implies $l - k - 1 \notin [0, 2l]$.

On the other hand, if $l > k$, we rewrite $G_{D'}^\vee \otimes \overline{G}_D \otimes \Lambda_M^{l-k-1} \simeq G_D^\vee \otimes \overline{G}_{D'} \otimes \Lambda_M^{k-l}$ using Corollary 3.14. Again, we can use Theorem 7.1. Indeed, we see that the inequalities

$$(2k+1) - i - 1 < k - l < d + g - 2i - 1 - 2l$$

are equivalent to the ones above and hence are satisfied, while now $l > k$ guarantees $k - l \notin [0, 2k+1]$. Thus, Theorem 7.1 gives the required Γ-acyclicity.

Step 4
Next we show semiorthogonality between $\mathcal{C}$ and $\mathcal{B}$. This amounts to Γ-acyclicity of $G_D^\vee \otimes \overline{G}_{D'} \otimes \Lambda_M^{k-l} \otimes \zeta \otimes \theta^{-1} = G_D^\vee \otimes \overline{G}_{D'} \otimes \Lambda_M^{k-l-1} \otimes \zeta^{-1}$ (cf. Definition 3.10) for $D \in \mathrm{Sym}^{2k} C$, $D' \in \mathrm{Sym}^{2l+1} C$, where $0 \le 2k, 2l+1 \le i \le (d-1)/2 = g-1$. We check that Theorem 4.1 can be applied in this case:

$$2k - g < k - l - 1 < d - (2l+1) - i - 1$$

is equivalent to $k + l < g - 1$ and $k + l + i < d - 1$, both of which hold in our case. This proves that $\mathrm{Hom}_{D^b(M_i)}(\mathcal{B}_{2k}, \mathcal{C}_{2l+1}) = 0$.

Step 5
To show that $\mathrm{Hom}_{D^b(M_i)}(\mathcal{D}_{2k+1}, \mathcal{C}_{2l+1}) = 0$, we need to check that $\overline{G}_D^\vee \otimes \overline{G}_{D'} \otimes \Lambda_M^{k-l} \otimes \theta^{-1}$ is Γ-acyclic on M_i, where $D \in \mathrm{Sym}^{2k+1} C$, $D' \in \mathrm{Sym}^{2l+1} C$, $1 \le 2k+1, 2l+1 \le i \le (d-1)/2 = g-1$ and $2k+1 \le g-2$. By Serre duality, this is equivalent to Γ-acyclicity of $\overline{G}_{D'}^\vee \otimes \overline{G}_D \otimes \Lambda_M^{l-k-1} \otimes \zeta^{-1}$ and this follows from Theorem 4.1 since

$$2l + 1 - g < l - k - 1 < d - (2k+1) - i - 1$$

is equivalent to $l + k + 1 < g - 1$ and $l + k + i < d - 1$, both of which hold given the conditions above.

Step 6
Finally, we show semiorthogonality between blocks from $\mathcal{B}$ and $\mathcal{D}$. We need to show that if $D \in \mathrm{Sym}^{2k+1} C$, $D' \in \mathrm{Sym}^{2l} C$, $0 \le 2k+1, 2l \le i \le (d-1)/2 = g-1$, we have $R\Gamma_{M_i}(\overline{G}_D^\vee \otimes G_{D'} \otimes \Lambda_M^{k-l} \otimes \zeta^{-1}) = 0$. We can use Theorem 4.1 since

$$2k + 1 - g < k - l < d - 2l - i - 1$$

is equivalent to the inequalities $k + l < g - 1$ and $k + l + i < d - 1$, again both of which hold in our situation. We conclude that $\mathrm{Hom}_{D^b(M_i)}(\mathcal{D}_{2k+1}, \mathcal{B}_{2l}) = 0$.

This completes the proof of the theorem. □

Remark 10.5
On $D^b(M_{g-1})$, this defines a semiorthogonal list of admissible subcategories $\mathcal{A}_0, \mathcal{A}_2, \ldots\ldots\ldots, \mathcal{C}_3, \mathcal{C}_1, \mathcal{B}_0, \mathcal{B}_2, \ldots\ldots\ldots, \mathcal{D}_3, \mathcal{D}_1$ where we have two copies of $D^b(\mathrm{Sym}^\alpha C)$ for $0 \le \alpha \le g-2$ and one copy of $D^b(\mathrm{Sym}^{g-1} C)$. We have chosen $D^b(\mathrm{Sym}^{g-1} C)$ to appear in the block $\mathcal{A}$ when $g-1$ is even and in $\mathcal{C}$ when $g-1$ is odd, but in fact any other choice of even and odd blocks would be valid too.

Indeed, a similar computation in the proof of Theorem 10.4 still gives the required semiorthogonalities.

Now let $i = g-1$, and call $\xi : M_{g-1} \to N$ the last map in (3.1), where $N = M_C(2,\Lambda)$ is the space of stable rank-2 vector bundles with determinant Λ. The Picard group of N is generated by an ample line bundle θ_N, such that $\xi^*\theta_N = \theta$ (see [39, 5.8, 5.9], [32, Proposition 2.1]). Then we have the following corollary.

COROLLARY 10.6
Let $\mathcal{E}$ be the Poincaré bundle of the moduli space $N = M_C(2,\Lambda)$ over a curve of genus at least 3, normalized so that $\det \pi_! \mathcal{E} = \mathcal{O}_N$ and $\det \mathcal{E}_x = \theta_N$, and where Λ is a line bundle on C of arbitrary odd degree. For $i = 0, \ldots, g-1$, let $\mathcal{G}_i \subset D^b(N)$ (resp., $\overline{\mathcal{G}}_i$) be the essential image of the Fourier–Mukai functor with kernel $\mathcal{E}^{\boxtimes i}$ (resp., $\overline{\mathcal{E}}^{\boxtimes i}$). Then

$$\begin{array}{ccccc} \theta_N^* \otimes \mathcal{G}_0, & (\theta_N^*)^2 \otimes \mathcal{G}_2, & (\theta_N^*)^3 \otimes \mathcal{G}_4, & (\theta_N^*)^4 \otimes \mathcal{G}_6, & \ldots, \\ \ldots, & (\theta_N^*)^4 \otimes \overline{\mathcal{G}}_7, & (\theta_N^*)^3 \otimes \overline{\mathcal{G}}_5, & (\theta_N^*)^2 \otimes \overline{\mathcal{G}}_3, & \theta_N^* \otimes \overline{\mathcal{G}}_1 \\ \mathcal{G}_0, & \theta_N^* \otimes \mathcal{G}_2, & (\theta_N^*)^2 \otimes \mathcal{G}_4, & (\theta_N^*)^3 \otimes \mathcal{G}_6, & \ldots, \\ \ldots, & (\theta_N^*)^3 \otimes \overline{\mathcal{G}}_7, & (\theta_N^*)^2 \otimes \overline{\mathcal{G}}_5, & \theta_N^* \otimes \overline{\mathcal{G}}_3, & \overline{\mathcal{G}}_1 \end{array} \tag{10.2}$$

is a semiorthogonal sequence of admissible subcategories of $D^b(N)$. There are two blocks isomorphic to $D^b(\mathrm{Sym}^i C)$ for each $i = 0, \ldots, g-2$ and one block isomorphic to $D^b(\mathrm{Sym}^{g-1} C)$.

Proof
If Λ, Λ' are two line bundles of odd degree, then it is easy to see that $M_C(2,\Lambda) \simeq M_C(2,\Lambda')$, so we can assume that $d = \deg \Lambda = 2g-1$, as before. Observe that ξ^* is fully faithful. Indeed, ξ is a projective birational morphism of nonsingular varieties, so we have $R\xi_*(\mathcal{O}_{M_{g-1}}) = \mathcal{O}_N$ by [39, 5.12] and [19, (2), pp. 144–145]. Then by adjointness,

$$\mathrm{Hom}_{D^b(M_{g-1})}(\xi^* A, \xi^* B) = \mathrm{Hom}_{D^b(N)}(A, R\xi_* \xi^* B) = \mathrm{Hom}_{D^b(N)}(A, B).$$

The pullback $\xi^*(\mathcal{E})$ is a vector bundle on $C \times M_{g-1}$ whose restriction to each $C \times \{(E,\phi)\} \subset C \times M_{g-1}$ is exactly $\mathcal{E}$. Thus, it has to coincide with the universal bundle F up to twist by a line bundle on M_{g-1}, so that $\xi^*\mathcal{E} = F \otimes L$. Then $\xi^* \det \mathcal{E}_x = \Lambda_M \otimes L^2$, which by the normalization chosen must be $\xi^*\theta_N = \theta$, so $L = \zeta$. Thus $\xi^*(\mathcal{E}) = F \otimes \zeta$ and the result follows from Theorem 10.4, together with the fact that $\zeta^{2k} \otimes \theta^{-k} \simeq \Lambda_M^{-k}$ under our assumption $d = 2g-1$. $\square$

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