Pacific Journal of Mathematics

ULTRAPRODUCT METHODS FOR MIXED q-GAUSSIAN ALGEBRAS

MARIUS JUNGE AND QIANG ZENG

Volume 331 No. 1 July 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 331, No. 1, 2024
https://doi.org/10.2140/pjm.2024.331.99

ULTRAPRODUCT METHODS FOR MIXED q-GAUSSIAN ALGEBRAS

MARIUS JUNGE AND QIANG ZENG

We provide a unified ultraproduct approach for constructing Wick words in mixed q-Gaussian algebras which are generated by $s_j = a_j + a_j^*$ for $j = 1, \dots, N$, where $a_i a_j^* - q_{ij} a_j^* a_i = \delta_{ij}$. Here we also allow equality in $-1 \le q_{ij} = q_{ji} \le 1$. Using the ultraproduct method, we construct an approximate comultiplication of the mixed q-Gaussian algebras. Based on this we prove that these algebras are weakly amenable and strongly solid in the sense of Ozawa and Popa. We also encode Speicher's central limit theorem in the unified ultraproduct method, and show that the Ornstein–Uhlenbeck semigroup is hypercontractive, the Riesz transform associated to the number operator is bounded, and the number operator satisfies the L_p Poincaré inequalities with constants $C\sqrt{p}$.

1. Introduction

Group measure space constructions go back to the original work of Murray and von Neumann [1936]. In the last decades Popa and his collaborators have solved many open problems about fundamental groups and uniqueness of Cartan subalgebras; see, e.g., [Ozawa and Popa 2010a; 2010b; Popa and Vaes 2010; 2014; Houdayer and Shlyakhtenko 2011]. In parallel, von Neumann algebras generated by q-commutation relations (motivated by physics and number theory) were introduced by Bożejko and Speicher [1991], and further investigated by Bożejko, Kümmerer, and Speicher [Bożejko et al. 1997], Shlyakhtenko [2004], Nou [2004], Śniady [2004], Ricard [2005], Kennedy and Nica [2011], and Avsec [2011], among others. More recently, Dabrowski [2014] and Guionnet and Shlyakhtenko [2014] have shown that for small q, the q-Gaussian algebras are isomorphic to free group factors. All these results on factoriality, embeddability in R^ω, and approximation properties face a similar problem: how to derive properties of von Neumann algebras from combinatorial structures given by the original q-commutation relations.

MSC2010: primary 46L36, 46L53; secondary 46N50, 81S05.
Keywords: q-Gaussian algebras, Wick product, hypercontractivity, Riesz transform, Poincaré inequality, approximation property, strong solidity.

In this paper we study generalized q-commutation relations: Given a symmetric matrix $Q=(q_{ij})_{i,j=1}^N$, $q_{ij}\in[-1,1]$, Speicher [1993] considered variables satisfying

$$a_i a_j^* - q_{ij} a_j^* a_i = \delta_{ij}. \tag{1-1}$$

The mixed q-Gaussian algebra Γ_Q is generated by the self-adjoint variables $s_j = a_j^* + a_j$ and admits a normal faithful tracial state (see Section 3 for more details). Bożejko and Speicher [1994] systematically constructed the Fock space representation of the so-called braid relations, which is more general than (1-1). Then various properties were studied in, e.g., [Nou 2004; Krȯlak 2000; 2005]. As for (1-1), Lust-Piquard [1999] showed the L_p boundedness of the Riesz transforms associated to the number operator of the system when $q_{ii}<1$. Other generalized Gaussian systems related to our investigation have also been studied; see, e.g., [Guţă and Maassen 2002; Guţă 2003].

It is very tempting to believe that mixed q-Gaussian algebras behave in any respect the same way as the q-Gaussian algebras with constant q. Indeed, the L_2 space of such an algebra admits a decomposition

$$L_2(\Gamma_Q)=\bigoplus_{k=0}^{\infty} H_Q^k$$

into finite-dimensional subspaces H_Q^k of dimension N^k, which are eigenspaces of the number operator. For fixed $q_{ij}=q$ the number operator can be defined in a functorial way following Voiculescu's lead [Voiculescu et al. 1992] for $q=0$. Indeed, for every real Hilbert H one finds the q-Gaussian von Neumann algebra $\Gamma_q(H)$ and a group homomorphism $\alpha: O(H)\to \mathrm{Aut}(\Gamma_q(H))$ such that for $o\in O(H)$ and $h\in H$

$$\alpha(o)(s^q(h))=s^q(o(h)).$$

Here, $s^q(e_j)=s_j$ and (e_j) is an orthonormal basis for the N-dimensional Hilbert space H. Then

$$T_t = E\alpha(o_t)\pi,$$

where $\pi:\Gamma_q(H)\to\Gamma_q(H\oplus H)$ is the natural embedding with conditional expectation E, and

$$o_t=\begin{pmatrix} e^{-t}\,\mathrm{id} & -\sqrt{1-e^{-2t}}\,\mathrm{id}\\ \sqrt{1-e^{-2t}}\,\mathrm{id} & e^{-t}\,\mathrm{id}\end{pmatrix}.$$

For nonconstant $Q=(q_{ij})$ we can no longer refer to functoriality directly. One of the first results in this paper is to provide a unified approach to the Ornstein–Uhlenbeck semigroup for $|q_{ij}|\le 1$ including the classical cases $q=1$ for bosons and $q=-1$ for fermions. The fact that the dimension of the eigenspace H_Q^k is not more than N^k uniformly for all Q is based on thorough analysis of different forms of Wick words and probabilistic estimates (see Section 3).

Another new feature of these generalized relations comes from studying the operators

$$\Phi(x) = E(s_{N+1} x s_{N+1}),$$

where x is generated by $s_1, \ldots, s_N$. For constant $q_{ij} = q$ we find $\Phi(x) = q^{l(x)}x$ can be easily computed in terms of the length function $l(x) = k$ if $x \in H_Q^k$. The formula for general Q is vastly more complicated. However, such expressions are crucial building blocks in proving strong solidity.

Let us recall some notions in operator algebras. We always assume the von Neumann algebras to be finite in this paper. Recall that a von Neumann algebra $\mathcal{M}$ has the weak* completely bounded approximation property (w*CBAP) if there exists a net of normal, completely bounded, finite-rank maps $\phi_\alpha : \mathcal{M} \to \mathcal{M}$ such that $\|\phi_\alpha\|_{\mathrm{cb}} \leq C$ for all α, and $\phi_\alpha \to \mathrm{id}$ in the point weak* topology. Here, $\|\cdot\|_{\mathrm{cb}}$ denotes the completely bounded norm. The infimum of such constants C is called the Cowling–Haagerup constant and is denoted by $\Lambda_{\mathrm{cb}}(\mathcal{M})$. Cowling and Haagerup [1989] showed that a discrete group G is weakly amenable if and only if its group von Neumann algebra LG has w*CBAP. Thus, a von Neumann algebra with w*CBAP is also said to be weakly amenable. If $\Lambda_{\mathrm{cb}}(\mathcal{M}) = 1$, $\mathcal{M}$ is said to have the weak* completely contractive approximation property (w*CCAP). See, e.g., [Brown and Ozawa 2008] for more details of the approximation properties. Following Ozawa and Popa [2010a], a von Neumann algebra $\mathcal{M}$ is called strongly solid if the normalizer $\mathcal{N}_{\mathcal{M}}(P) := \{u \in \mathcal{U}(\mathcal{M}) : uPu^* = P\}$ of any diffuse amenable subalgebra $P \subset \mathcal{M}$ generates an amenable von Neumann algebra. Here, $\mathcal{U}(\mathcal{M})$ is the set of unitary operators in $\mathcal{M}$.

Theorem 1.1. *Γ_Q has w*CCAP and is strongly solid provided* $\max_{1\leq i,j\leq N} |q_{ij}| < 1$.

These properties extend similar results due to Avsec [2011] for q-Gaussian von Neumann algebras. The w*CCAP for Γ_Q is proved using a transference method based on Avsec's w*CCAP result for the q-Gaussian algebras. Then we show a weak containment result of certain bimodules. These results, together with a modification of Popa's s-malleable deformation estimate, leads to strong solidity using a, by now, standard argument. The method used here follows that of [Houdayer and Shlyakhtenko 2011; Avsec 2011]. However, the techniques are more difficult than the case of q-Gaussian algebras. We have to use some nontrivial tricks to achieve certain results similar to those in [Avsec 2011].

The ultraproduct method plays an essential role in many aspects of this paper. It is well known that CCAP is a stepping stone for proving strong solidity. The transference method mentioned above relies entirely on an embedding of Γ_Q into an ultraproduct of von Neumann algebras which preserves the Wick words. This allows us to transfer the CCAP result of the constant q case of Avsec to the current mixed q

case. The argument can be illustrated using the following commutative diagram:

$$\begin{array}{ccc} \Gamma_Q & \xrightarrow{\pi_{\mathcal{U}}} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m) \,\overline{\otimes}\, \Gamma_{\widetilde{Q}\otimes \mathbb{1}_m} \\ \downarrow{\scriptstyle \psi_\alpha} & & \downarrow{\scriptstyle \varphi_\alpha(A)\otimes \mathrm{id}} \\ \Gamma_Q & \xrightarrow{\pi_{\mathcal{U}}} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m) \,\overline{\otimes}\, \Gamma_{\widetilde{Q}\otimes \mathbb{1}_m} \end{array}$$

The notation will be explained in the proof of Theorem 5.5. The map $\pi_{\mathcal{U}}$ can be understood as an approximate comultiplication. Without the help of the ultraproduct method above, we will have to extend directly the argument for the constant q case to the mixed q case, which may be very hard due to the involved combinatorial structure.

We also prove some analytic properties for Γ_Q following the unified ultraproduct approach. The cornerstone is a Wick word decomposition result, whose proof involves some complicated combinatorial and probabilistic arguments. In this context, the ultraproduct construction provides a natural framework to encode Speicher's central limit theorem; see [Speicher 1992; 1993; Junge 2006]. Furthermore, the Wick words are identified as some special sequences in the ultraproduct of spin matrix models. Once we have the Wick word decomposition, it follows immediately that the Ornstein–Uhlenbeck semigroup $(T_t)_{t\geq 0}$ associated to Γ_Q is hypercontractive: For $1 \leq p, r < \infty$,

$$\|T_t\|_{L_p \to L_r} = 1 \quad \text{if and only if} \quad e^{-2t} \leq \frac{p-1}{r-1}.$$

Here, $L_p = L_p(\Gamma_Q, \tau_Q)$ is the noncommutative L_p space associated to the canonical tracial state τ_Q on Γ_Q. This result is a vast generalization of the work of Biane [1997] and Junge et al. [2015]. Indeed, we obtain hypercontractivity results for free products of q-Gaussian algebras and, in particular, free products of Clifford algebras. More exotic choices may be obtained for general q_{ij}. We also recover and extend the result of Lust-Piquard [1999] on the boundedness of Riesz transforms. Let A be the number operator of Γ_Q, which is also the generator of T_t. Define the gradient form (Meyer's "carré du champ") associated to A as

$$\Gamma(f, g) = \tfrac{1}{2}(A(f^*)g + f^*Ag - A(f^*g))$$

for f, g in the domain of A. We show that

(a) for $p \geq 2$,

$$c_p^{-1}\|A^{1/2}f\|_p \leq \max\left\{\|\Gamma(f, f)^{1/2}\|_p, \|\Gamma(f^*, f^*)^{1/2}\|_p\right\} \leq K_p\|A^{1/2}f\|_p$$

with $c_p = O(p^2)$ and $K_p = O(p^{3/2})$;

(b) for $1 < p < 2$,

$$K_{p'}^{-1}\|A^{1/2}f\|_p \le \inf_{\substack{\delta(f)=g+h\\ g\in G_p^c, h\in G_p^r}} \left\{\|E(g^*g)^{1/2}\|_p + \|E(hh^*)^{1/2}\|_p\right\} \le C_p\|A^{1/2}f\|,$$

with $K_{p'} = O(1/(p-1)^{3/2})$ and $C_p = O(1/(p-1)^2)$, where δ is a derivation related to the Riesz transforms and G_p^c and G_p^r are two Gaussian spaces (all will be defined below).

Moreover, we obtain the L_p Poincaré inequalities:

$$\|f - \tau_Q(f)\|_p \le C\sqrt{p}\max\left\{\|\Gamma(f,f)^{1/2}\|_p, \Gamma(f^*,f^*)^{1/2}\|_p\right\} \quad \text{for } p \ge 2.$$

This is an extension of similar results for the Walsh and Fermionic system in [Efraim and Lust-Piquard 2008]. It is known that the constant $C\sqrt{p}$ in such inequalities is crucial for proving concentration and transportation inequalities; see, e.g., [Zeng 2014].

The paper is organized as follows. Some preliminaries and notation are presented in Section 2. We construct the mixed q-Gaussian algebras and the Ornstein–Uhlenbeck semigroup in Section 3, where the Wick word decomposition result is also proved with a lengthy argument. The analytic properties are proved in Section 4, and the strong solidity is proved in Section 5.

2. Preliminaries and notation

2A. ***Notation.*** We write $[N] = \{1, 2, \dots, N\}$ for $N \in \mathbb{N}$. The set of nonnegative integers is denoted by $\mathbb{Z}_+$. For $n \in \mathbb{N}$, we denote by M_n the algebra of $n \times n$ matrices. We will use some notation to analyze combinatorial structures following [Speicher 1992; Junge et al. 2015]. Denote by $P(d)$ the set of all partitions of $[d] = \{1, \dots, d\}$. For $\sigma, \pi \in P(d)$, we write $\sigma \le \pi$ or $\pi \ge \sigma$ if σ is a refinement of π. We denote the integer valued vectors by $\underline{i}$, $\underline{j}$, etc. Given $\underline{i} = (i_1, \dots, i_d) \in [N]^d$, we associate a partition $\sigma(\underline{i})$ to $\underline{i}$ by requiring $k, l \in [d]$ belonging to the same block of $\sigma(\underline{i})$ if and only if $i_k = i_l$.

We denote by $|S|$ or $\#S$ the cardinality of a (finite) set S. If d is an even integer, we define $P_2(d)$ to be the set of pair partitions of $[d]$, i.e., $P_2(d)$ consists of $\pi = \{V_1, \dots, V_{d/2}\}$ such that $|V_k| = 2$ for every block V_k. Write $V_k = \{e_k, z_k\}$ with $e_k < z_k$ and $e_1 < e_2 < \cdots < e_{d/2}$. Given $\pi \in P_2(d)$, the set of crossings of π is denoted by

$$I(\pi) = \left\{\{k,l\} \mid 1 \le k, l \le d/2 \text{ and } e_k < e_l < z_k < z_l\right\}. \tag{2-1}$$

For $d \in \mathbb{N}$, we denote by $P_1(d)$ the one element set of singleton partition of $[d]$, i.e., $P_1(d) = \{\sigma_0\}$ and $\sigma_0 = \{\{1\}, \{2\}, \dots, \{d\}\}$. Let $P_{1,2}(d)$ denote the set of partitions

consisting of only singletons and pair blocks, and $P_r(d) = P(d) \setminus P_{1,2}(d)$. Let $\sigma \in P_{1,2}(d)$ be given by

$$\sigma = \{V_1, \ldots, V_{s+u}\},$$

where the V_j's are singletons ($V_j = \{e_j = z_j\}$) or pair blocks ($V_j = \{e_j, z_j\}$). Assume there are s singleton blocks and u pair blocks in σ. Let σ_p be a subpartition consisting of the u pair blocks of $\sigma \in P_{1,2}(d)$. Denote by $I_p(\sigma) := I(\sigma_p)$ the set of pair crossings of σ given in (2-1) and define

$$I_{\mathrm{sp}}(\sigma) = \big\{\{r, t\} : e_r < e_t = z_t < z_r\big\}$$

to be the set of crossings between pairs and singletons.

Given a discrete group G, the left regular representation is $\lambda : G \to \ell_2(G)$, $\lambda(g)\delta_h = \delta_{gh}$ for $g, h \in G$, and $(\delta_g)_{g\in G}$ is a canonical basis of $\ell_2(G)$. The group von Neumann algebra of G is denoted by LG and the canonical trace by τ_G. The Kronecker delta function is denoted by $\delta_{i,j}$. The use of two δ's will not appear in the same place. It should be clear from the context which one we are using. We let $\mathbb{1}_n$ denote the $n \times n$ matrix with all entries equal to 1.

2B. ***Spin matrix model.*** We consider a general spin matrix model, following [Lust-Piquard 1998; Junge et al. 2015]. Fix a finite integer N. Let $J_{N,m} = [N] \times [m]$ and $J_N = [N] \times \mathbb{N}$. We usually do not specify the dependence on N and simply write $J_m = J_{N,m}$ and $J = J_N$ if there is no ambiguity. We equip J_m with the lexicographical order. Let $\varepsilon : J \times J \to \{-1, 1\}$ be a map satisfying $\varepsilon(x, y) = \varepsilon(y, x)$ and $\varepsilon(x, x) = -1$ for all $x, y \in J$. Consider the complex unital algebra $\mathcal{A}_m = \mathcal{A}_m(N, \varepsilon)$ generated by $(x_i(k))_{(i,k)\in J_m}$, where the $x_i(k)$'s satisfy $x_i(k)^* = x_i(k)$ and

$$x_i(k)x_j(l) - \varepsilon((i,k),(j,l))x_j(l)x_i(k) = 2\delta_{(i,k),(j,l)}$$

for $(i, k), (j, l) \in J_m$. It is well known that the $x_i(k)$'s can be represented as tensor products of Pauli matrices. Thus $\mathcal{A}_m$ can be represented as a matrix subalgebra of $M_{2^{Nm}}$. A generic element of $\mathcal{A}_m$ can be written as a linear combination of words of the form

$$x_B = x_{i_1}(k_1) \cdots x_{i_d}(k_d),$$

where $B = \{(i_1, k_1), \ldots, (i_d, k_d)\} \subset J_m$. We say x_B is a reduced word if the $x_{i_r}(k_r)$'s in x_B are pairwise different for $r = 1, \ldots, d$. Using the commutation relation, every word ξ can be written in the reduced form, denoted by $\widetilde{\xi}$. There is a canonical normalized trace τ_m on $\mathcal{A}_m$ such that $\tau_m(x_B) = \delta_{B,\varnothing}$ for a reduced word x_B.

2C. ***Pisier's method for multi-index summations.*** Let $\sigma \in P(d)$ be a partition. In the following we need to estimate the L_p norm of

$$\sum_{\underline{k}\in[m]^d : \sigma(\underline{k}) \geq \sigma} x_1(k_1) \cdots x_d(k_d),$$

where $x_i(k_i) \in \bigcap_{p<\infty} L_p(\tau)$, and $L_p(\tau)$ is a noncommutative L_p space associated to a trace τ. To this end, we follow Pisier's method [2000]. As illustrated in the proof of [Pisier 2000, Sublemma 3.3], one can find $\xi_1(k_1), \ldots, \xi_d(k_d) \in \mathrm{LG}$ such that $\tau_G(\xi_1(k_1)\cdots\xi_d(k_d)) = 1$ if and only if $\sigma(\underline{k}) \geq \sigma$ and $\tau_G(\xi_1(k_1)\cdots\xi_d(k_d)) = 0$ otherwise. Here, G is a suitable product of free groups, LG is the von Neumann algebra of G, and τ_G is the canonical trace on LG.

Let us explain this in more detail using an example. We denote by $\mathbb{F}_m$ the free group with free generators $(g_i)_{i\in[m]}$. Suppose $d = 6$ and $\sigma = \{\{1, 3, 5\}, \{2, 6\}, \{4\}\}$. In this case, $G = \mathbb{F}_m \times \mathbb{F}_m \times \mathbb{F}_m$ and for $i \in [m]$,

$$\begin{aligned} \xi_1(i) &= \lambda(g_i)^* \otimes 1 \otimes 1, & \xi_2(i) &= 1 \otimes \lambda(g_i)^* \otimes 1, \\ \xi_3(i) &= \lambda(g_i) \otimes 1 \otimes \lambda(g_i)^*, & \xi_4(i) &= 1 \otimes 1 \otimes 1, \\ \xi_5(i) &= 1 \otimes 1 \otimes \lambda(g_i), & \xi_6(i) &= 1 \otimes \lambda(g_i) \otimes 1. \end{aligned}$$

Then $\tau_G(\xi_1(k_1)\cdots\xi_6(k_6)) = 1$ if and only if $k_1 = k_3 = k_5$ and $k_2 = k_6$.

Returning to the general setting, consider the algebraic tensor product $\mathrm{LG} \otimes L_p(\tau)$. Since $\tau_G \otimes \mathrm{id}$ extends to contractions on L_p, using Hölder's inequality, we have

$$\begin{aligned} \Big\| \sum_{\underline{k}\in[m]^d : \sigma(\underline{k})\geq\sigma} x_1(k_1)\cdots x_d(k_d) \Big\|_p &= \Big\| \sum_{\underline{k}\in[m]^d} \tau_G(\xi_1(k_1)\cdots\xi_d(k_d)) x_1(k_1)\cdots x_d(k_d) \Big\|_p \\ &\leq \Big\| \sum_{\underline{k}\in[m]^d} \xi_1(k_1)\otimes x_1(k_1)\cdots\xi_d(k_d)\otimes x_d(k_d) \Big\|_p \\ &\leq \prod_{i=1}^{d} \Big\| \sum_{k_i=1}^{m} \xi_i(k_i)\otimes x_i(k_i) \Big\|_{pd}. \end{aligned}$$

If i belongs to a singleton block of σ, then $\xi_i(k_i) = 1$ and

$$\Big\| \sum_{k_i=1}^{m} \xi_i(k_i)\otimes x_i(k_i) \Big\|_{pd} = \Big\| \sum_{k_i=1}^{m} x_i(k_i) \Big\|_{pd}.$$

If i does not belong to any singleton of σ, then it is well known that

$$\Big\| \sum_{k_i=1}^{m} \xi_i(k_i)\otimes x_i(k_i) \Big\|_{pd} = \Big\| \sum_{k_i=1}^{m} \lambda(g_i)\otimes x_i(k_i) \Big\|_{pd}.$$

By [Pisier 2000, Lemma 3.4], we have, for any even integer $p \geq 2$,

$$\begin{aligned} &\Big\| \sum_{k_i=1}^{m} \lambda(g_i)\otimes x_i(k_i) \Big\|_p \\ &\qquad \leq \frac{3\pi}{4} \max\left\{ \Big\| \Big(\sum_{k_i} x_i(k_i)^* x_i(k_i) \Big)^{\frac{1}{2}} \Big\|_p, \ \Big\| \Big(\sum_{k_i} x_i(k_i) x_i(k_i)^* \Big)^{\frac{1}{2}} \Big\|_p \right\}. \end{aligned}$$

We record this result as follows. Denote by σ_{sing} and σ_{ns} the union of singletons and the union of nonsingleton blocks of σ respectively. Thus we have $\#\sigma_{\mathrm{sing}} + \#\sigma_{ns} = d$.

Proposition 2.1. *Let $\sigma \in P(d)$ be a partition and $x_i(k_i) \in L_p(\tau)$ for $\underline{k} \in [m]^d$ and $i \in [d]$. Then, for any even integer $p \geq 2$,*

$$\Big\| \sum_{\underline{k}:\sigma(\underline{k})\geq\sigma} x_1(k_1)\cdots x_d(k_d) \Big\|_p \leq \Big(\frac{3\pi}{4}\Big)^{\#\sigma_{ns}} \prod_{i\in\sigma_{\mathrm{sing}}} \Big\| \sum_{k_i=1}^m x_i(k_i) \Big\|_{pd} \prod_{i\in\sigma_{ns}} \max\Big\{ \Big\| \Big(\sum_{k_i=1}^m x_i(k_i)^* x_i(k_i) \Big)^{\frac12} \Big\|_{pd}, \Big\| \Big(\sum_{k_i=1}^m x_i(k_i) x_i(k_i)^* \Big)^{\frac12} \Big\|_{pd} \Big\}.$$

This result will be used in a slightly more general setting. We may have other fixed operators, y_j's, inside the product $x_1(k_1)\cdots x_d(k_d)$. In this case, we may simply attach the y_j's to their adjacent $x_i(k_i)$'s and then invoke Proposition 2.1.

3. Construction and Wick word decomposition

The algebra we study here can be constructed using purely operator algebraic techniques if $\max_{1\leq i,j\leq N} |q_{ij}| < 1$ as shown in [Bożejko and Speicher 1994]. However, we use the probabilistic approach due to Speicher [1992; 1993]. This is convenient for studying the analytic properties following Biane's original idea [1997]. The main result of this section is Theorem 3.8. Although the proof is unexpectedly lengthy, the analytic properties are easy consequences of this result. As a byproduct, we also provide an alternative construction of the Fock space representation.

3A. ***Speicher's CLT and von Neumann algebra ultraproducts.*** Let $Q = (q_{ij})_{i,j=1}^N$ be a symmetric matrix where $q_{ij} = q(i,j) \in [-1,1]$. Note that we do not specify the values on the diagonal. Following the notation of Section 2B, we consider a probability space $(\Omega, \mathbb{P})$ and a family of independent random variables $\varepsilon((i,k),(j,l)) : \Omega \to \{-1,1\}$ for $(i,k) < (j,l)$ with distribution

$$\begin{aligned} \mathbb{P}\big(\varepsilon((i,k),(j,l)) = -1\big) &= \tfrac12(1 - q(i,j)), \\ \mathbb{P}\big(\varepsilon((i,k),(j,l)) = 1\big) &= \tfrac12(1 + q(i,j)), \end{aligned} \tag{3-1}$$

so that $\mathbb{E}[\varepsilon((i,k),(j,l))] = q(i,j)$. Here, $(i,k),(j,l) \in [N]\times\mathbb{N}$. Given $\omega \in \Omega$, the commutation/anticommutation relation is fixed. We understand all generators $x_i(k)(\omega)$ to depend on ω. Restricting $k \in [m]$ we get random $\mathcal{A}_m$. Because the dependence on ω should be clear from the context, we will not write ω in the following to simplify notation. Let $\widetilde{x}_i(m) = \frac{1}{\sqrt{m}} \sum_{k=1}^m x_i(k)$.

The following central limit theorem result was due to Speicher [1993] and is a generalization of [Speicher 1992]. We streamline Speicher's proof in the appendix for the reader's convenience. The same strategy will be used repeatedly when we prove Theorem 3.8.

Theorem 3.1. *Let* $\underline{i} \in [N]^s$. *Then*

$$\lim_{m\to\infty} \tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m)) = \delta_{s\in 2\mathbb{Z}} \sum_{\substack{\sigma\in P_2(s)\\ \sigma\le\sigma(\underline{i})}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)) \quad a.s.$$

Here and in what follows, we understand $\prod_{\{i,j\}\in\varnothing} q(i,j)$ *to be equal to* 1.

By Theorem 3.1, we can find a full probability set $\Omega_0 \subset \Omega$ such that the convergence holds for all $\omega \in \Omega_0$. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By the well-known ultraproduct construction of von Neumann algebras (see, e.g., [Brown and Ozawa 2008, Appendix A]), we have a finite von Neumann algebra $\mathcal{A}_\mathcal{U} := \prod_{m,\mathcal{U}} \mathcal{A}_m$ with normal faithful tracial state $\tau_\mathcal{U} = \lim_{m,\mathcal{U}} \tau_m$. Put $\mathcal{A}_\mathcal{U}^\infty = \bigcap_{p<\infty} L_p(\mathcal{A}_\mathcal{U})$. For each $\omega \in \Omega_0$,

$$(\widetilde{x}_i(m)(\omega))^\bullet \in \mathcal{A}_\mathcal{U}^\infty.$$

Here and in what follows, we write $(\widetilde{x}_i(m)(\omega))^\bullet$ for the element represented by $(\widetilde{x}_i(m)(\omega))_{m\in\mathbb{N}}$ in the ultraproduct. We have the moment formula

$$\tau_\mathcal{U}\big((\widetilde{x}_{i_1}(m)(\omega))^\bullet\cdots(\widetilde{x}_{i_s}(m)(\omega))^\bullet\big) = \delta_{s\in 2\mathbb{Z}} \sum_{\substack{\sigma\in P_2(s)\\ \sigma\le\sigma(\underline{i})}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)). \tag{3-2}$$

It follows that

$$\tau_\mathcal{U}(|(\widetilde{x}_i(m)(\omega))^\bullet|^p) \le Cp^p.$$

By the uniqueness argument in [Junge 2006, Section 6], the von Neumann algebras generated by the spectral projections of the $(\widetilde{x}_i(m)(\omega))^\bullet$, where $i = 1,\dots,N$, for different $\omega \in \Omega_0$ are isomorphic. We denote by Γ_Q any von Neumann algebra in the isomorphic class with generators $(\widetilde{x}_i(m)(\omega))^\bullet$, where $i = 1,\dots,N$. This algebra was introduced by Speicher [1993] and studied in [Bożejko and Speicher 1994; Lust-Piquard 1999]. Note that $(\widetilde{x}_i(m)(\omega))^\bullet$ may be an unbounded operator, therefore may not be in Γ_Q. But, by our construction, it belongs to $\Gamma_Q^\infty := \bigcap_{p<\infty} L_p(\Gamma_Q, \tau_\mathcal{U})$. In the following, whenever we say that the $(\widetilde{x}_i(m)(\omega))^\bullet$, where $i = 1,\dots,N$, are generators of Γ_Q, we always mean $(\widetilde{x}_i(m)(\omega))^\bullet \in \Gamma_Q^\infty$ and Γ_Q is generated by the spectral projections of the $(\widetilde{x}_i(m)(\omega))^\bullet$'s. We call Γ_Q *the* mixed q-Gaussian algebra, and Q the structure matrix of Γ_Q. Sometimes we also write $\tau_Q = \tau_\mathcal{U}|_{\Gamma_Q}$.

There is another way of constructing Γ_Q. All the $x_i(k)$'s are in fact in $L_\infty(\Omega;\mathcal{A}_m)$ and thus $\widetilde{x}_i(m) \in L_\infty(\Omega;\mathcal{A}_m)$. Here, the trace on $L_\infty(\Omega;\mathcal{A}_m)$ is given by $\mathbb{E}\otimes\tau_m$. By the same CLT argument as for Theorem 3.1, we find the moment formula (A-4) in the limit, which is the same as (3-2). Therefore, as before, the $(\widetilde{x}_i(m))^\bullet$ for $i = 1,\dots,N$

generate a von Neumann algebra, denoted by Γ_Q^a. We call it the average model. Using the uniqueness results in [Junge 2006, Section 6], we have that Γ_Q^a is isomorphic to Γ_Q. When we write $(\widetilde{x}_i(m))^\bullet$, it can mean either an element in $\bigcap_{p<\infty} L_p\big(\prod_{m,\mathcal{U}} L_\infty(\Omega;\mathcal{A}_m)\big)$ or simply $(\widetilde{x}_i(m)(\omega))^\bullet$ for some $\omega\in\Omega_0$. It should be clear from the context which one we are using. In fact, we may simply write $x_1,\dots,x_N$ for the generators of Γ_Q if we are not concerned with the construction.

By considering different structure matrix Q, we can construct various examples as special cases of Γ_Q. The same philosophy was used before by Lust-Piquard [1999].

Example 3.2. $\Gamma_q(H)$, *where* $q\in[-1,1]$ *is fixed.* If $q(i,j)=q$ all $1\le i,j\le N$, then we recover the classical q-Gaussian algebra $\Gamma_q(H)$, where H is a real Hilbert space with $\dim H=N$.

Example 3.3. $*_{i=1}^n \Gamma_{q_i}(H_i)$, *where* $q_i\in[-1,1]$ *is fixed for* $i=1,\dots,n$. Here, the H_i's are real Hilbert spaces with $\dim H_i=d_i$. Let $N=d_1+\cdots+d_n$. Define Q as follows. For $k=0,\dots,n-1$ and $1\le\alpha,\beta\le d_{k+1}$, put

$$q\bigg(\sum_{j=1}^k d_j+\alpha,\sum_{j=1}^k d_j+\beta\bigg)=q_{k+1},$$

and $q(\alpha,\beta)=0$ otherwise. Then by the moment formula (3-2), we recover $*_{i=1}^n\Gamma_{q_i}(H_i)$. The case $q_i=-1$ for all $i=1,\dots,n$ was considered in [Junge et al. 2015].

Example 3.4. $\overline{\bigotimes}_{i=1}^n(\Gamma_{q_i}(H_i)*\Gamma_{p_i}(K_i))$, *where* $q_i,\,p_i\in[-1,1]$ *are fixed.* Here, the H_i's and K_i's are real Hilbert spaces with $\dim H_i=d_i$ and $\dim K_i=d_i'$. Let $N=\sum_{i=1}^n d_i+d_i'$. For $k=0,\dots,n-1$, define

$$\widetilde{q}\bigg(\sum_{j=1}^i(d_j+d_j')+\alpha,\sum_{j=1}^i(d_j+d_j')+\beta\bigg)=\begin{cases}q_i & \text{if } 1\le\alpha,\beta\le d_{i+1},\\ p_i & \text{if } d_{i+1}+1\le\alpha,\beta\le d_{i+1}+d_{i+1}',\\ 0 & \text{if } 1\le\alpha\le d_{i+1}<\beta\le d_{i+1}+d_{i+1}',\end{cases}$$

and $\widetilde{q}(\alpha,\beta)=1$ otherwise. Let $Q=(\widetilde{q}_{\alpha,\beta})_{1\le\alpha,\beta\le N}$. By the moment formula (3-2), this model gives mixed products of q-Gaussian algebras. For example, consider the von Neumann algebra of the integer lattice $L(\mathbb{Z}^n)$. We may identify $L(\mathbb{Z}^n)$ with $\overline{\bigotimes}_{i=1}^n\Gamma_0(\mathbb{R})$ via $\lambda(g_k)\mapsto x_k$, where the g_k's are the generators of $\mathbb{Z}^n$ and the x_k's are generators of $\overline{\bigotimes}_{i=1}^n\Gamma_0(\mathbb{R})$. Alternatively, by extending $\lambda(g_k)\mapsto x_{2k-1}x_{2k}$, we may embed $L(\mathbb{Z}^n)$ into $\overline{\bigotimes}_{i=1}^n\Gamma_{-1}(\mathbb{R})*\Gamma_{-1}(\mathbb{R})$.

3B. ***Wick word decomposition.*** For our later development, we need an analogue of Wick word decomposition, i.e., rewriting $(\widetilde{x}_{i_1}(m))^\bullet\cdots(\widetilde{x}_{i_d}(m))^\bullet$ as a linear combination of Wick words (to be defined) so that we can analyze the Ornstein–Uhlenbeck semigroup easily. This procedure is conceptually clear with the help of Fock

space representation because $(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet$ belongs to $L_2(\Gamma_Q)$ and $L_2(\Gamma_Q)$ should coincide with the Fock space, which is spanned by Wick products; see [Bożejko et al. 1997; Bożejko and Speicher 1994]. However, we do not know the explicit formula for the decomposition of $(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet$ in terms of matrix models. Moreover, the known Fock space construction usually requires $\max_{i,j}|q_{ij}| < 1$.

Our approach is again probabilistic. We refer the readers to Section 2A for the notation used in the following. By definition

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet = \left(\frac{1}{m^{d/2}} \sum_{\underline{k}\in[m]^d} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\right)^\bullet.$$

Note that

$$\sum_{\underline{k}\in[m]^d} x_{i_1}(k_1)\cdots x_{i_d}(k_d) = \sum_{\sigma\in P_{1,2}(d)} \sum_{\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d) + \sum_{\sigma\in P_r(d)} \sum_{\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d).$$

We first record a simple algorithm which we will refer to later on.

Proposition 3.5. *Let $\underline{i}\in[N]^d$, $\underline{k}\in[m]^d$, $\sigma(\underline{k})\le\sigma(\underline{i})$ and $\sigma(\underline{k})\in P_{1,2}(d)$. Then there is a specific algorithm to interchange $x_{i_\alpha}(k_\alpha)$'s in $x_{i_1}(k_1)\cdots x_{i_d}(k_d)$ such that*

(1) *$x_{i_1}(k_1)\cdots x_{i_d}(k_d) = \varepsilon(\underline{i},\underline{k})x_{j_1}(l'_1)\cdots x_{j_s}(l'_s)\cdots x_{j_d}(l'_d)$, where $\varepsilon(\underline{i},\underline{k})$ is a random sign resulting from interchanging $x_{i_\alpha}(k_\alpha)$'s which is given by*

$$\varepsilon(\underline{i},\underline{k}) = \prod_{\{r,t\}\in I_{\mathrm{sp}}(\sigma(\underline{k}))} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]) \times \prod_{\{r,t\}\in I_p(\sigma(\underline{k}))} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]);$$

(2) *$(l'_1,\dots,l'_s)$ are pairwise different and maintain their relative positions in $\underline{k}$, i.e., $(l'_1,\dots,l'_s)$ is obtained from $\underline{k}$ by removing the k_α's which correspond to pair blocks*;

(3) *$l'_{s+1}=l'_{s+2},\dots,l'_{d-1}=l'_d$.*

Proof. Since $\sigma(\underline{k})\in P_{1,2}(d)$, for each k_α in $\underline{k}$, there is at most one k_β in $\underline{k}$ equal to k_α. We can find the first k_α corresponding to a singleton in $\sigma(\underline{k})$, and move $x_{i_\alpha}(k_\alpha)$ to the beginning of the word by interchanging it with the $x_{i_\beta}(k_\beta)$'s which are to the left of $x_{i_\alpha}(k_\alpha)$. Rename this $x_{i_\alpha}(k_\alpha)$ to be $x_{j_1}(l'_1)$. This process produces a product of random signs of the form $\varepsilon((i_\alpha,k_\alpha),(i_\beta,k_\beta))$, where k_α corresponds to a singleton and k_β corresponds to a pair block in $\sigma(\underline{k})$. Then we repeat this procedure for the second k_α corresponding to a singleton in $\sigma(\underline{k})$, and rename it $x_{j_2}(l'_2)$. We continue until all the $x_{i_\alpha}(k_\alpha)$ corresponding to singletons in $\sigma(\underline{k})$

are in front of the remaining $x_{i_\beta}(k_\beta)$'s corresponding to pair blocks in $\sigma(\underline{k})$. In this way, we get $x_{j_1}(l'_1)\cdots x_{j_s}(l'_s)$ and a product of random signs. Afterwards, we rename the variable $x_{i_\alpha}(k_\alpha)$ right-adjacent to $x_{j_s}(l'_s)$ to be $x_{j_{s+1}}(l'_{s+1})$. Then move the other term with the same k_α to the right of $x_{j_{s+1}}(l'_{s+1})$, and call it $x_{j_{s+2}}(l'_{s+2})$. This produces a product of $\varepsilon((i_\alpha,k_\alpha),(i_\beta,k_\beta))$, where k_β and k_α correspond to different pair blocks. Repeat this procedure for the next pair of k_α's. After finitely many steps, the algorithm will stop and we obtain $\varepsilon(\underline{i},\underline{k})x_{j_1}(l'_1)\cdots x_{j_s}(l'_s)\cdots x_{j_d}(l'_d)$ with the desired three properties. □

We write

$$(l'_1,\dots,l'_d)=(k_{\pi(1)},\dots,k_{\pi(d)}), \tag{3-3}$$

where π is a permutation determined by the algorithm. Similarly, $(j_1,\dots,j_d)=(i_{\pi(1)},\dots,i_{\pi(d)})$. Let

$$l_1=l'_1,\quad \dots,\quad l_s=l'_s,\quad l_{s+1}=l'_{s+1}=l'_{s+2},\quad \dots,\quad l_{s+u}=l'_{d-1}=l'_d. \tag{3-4}$$

Here, s and u are the number of singletons and pair blocks of $\sigma(\underline{k})$, respectively.

Lemma 3.6. *Let $\sigma\in P_{1,2}(d)$. Then, for all $2<p<\infty$ and fixed $\omega\in\Omega$,*

$$\lim_{m\to\infty}\Big\|\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma}x_{i_1}(k_1)\cdots x_{i_d}(k_d) - \frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma}E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\Big\|_{L_p(\mathcal{A}_m,\tau_m)}=0.$$

Here, $\mathcal{N}_s(\underline{k})$ denotes the von Neumann algebra generated by all the $x_{i_\alpha}(k_\alpha)$'s, where the k_α's correspond to singleton blocks in $\sigma(\underline{k})$.

Proof. Let s and u denote the number of singletons and pair blocks of σ, respectively. Clearly, $s+2u=d$ and there are

$$m_{(s+u)}:=m(m-1)\cdots(m-s-u+1)$$

vectors $\underline{k}\in[m]^d$ with $\sigma(\underline{k})=\sigma$. Let $\underline{l}$ be given in (3-4). $\underline{l}$ is a vector of length $s+u$. Let $\delta_1,\dots,\delta_m$ be i.i.d. random selectors uniformly distributed on $\{1,2,\dots,s+u\}$ which are independent from $L_\infty(\Omega;\mathcal{A}_m)$. If all the l_α's are pairwise different, then by independence,

$$\mathbb{E}_\delta(1_{[\delta_{l_1}=1]}1_{[\delta_{l_2}=2]}\cdots 1_{[\delta_{l_{s+u}}=s+u]})=(s+u)^{-s-u},$$

where $\mathbb{E}_\delta$ is the expectation with respect to the δ_{l_q}'s. Define random sets B_q for $q=1,\dots,s+u$ by

$$B_q=\{l_q\in[m]:\delta_{l_q}=q\}.$$

Then, for each instance of the δ_{l_q}'s, the B_q's are pairwise disjoint and their union is $[m]$. By (3-3), there is a 1-to-1 correspondence between $\underline{k}$ and $\underline{l}$. We may rewrite

$$\begin{aligned}
&\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\\
&\quad= \sum_{\underline{l}\in[m]^{s+u}:\sigma(\underline{l})\in P_1(s+u)} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\\
&\quad= (s+u)^{s+u} \sum_{\underline{l}:\sigma(\underline{l})\in P_1(s+u)} \mathbb{E}_\delta[1_{[\delta_{l_1}=1]}1_{[\delta_{l_2}=2]}\cdots 1_{[\delta_{l_{s+u}}=s+u]}x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\\
&\quad= (s+u)^{s+u}\mathbb{E}_\delta\Bigg(\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\Bigg),
\end{aligned}$$

where $\sigma(\underline{l})\in P_1(s+u)$ amounts to saying that all the l_q's are pairwise different. For $q=s, s+1,\ldots,s+u$, let $\mathcal{N}_q(\underline{k})$ be the von Neumann algebra generated by

$$\{x_{j_\alpha}(l'_\alpha) : \alpha\le s+2(q-s)\}.$$

Recall that $l_{s+u}=k_{\pi(d-1)}=k_{\pi(d)}$. Let

$$w_{\underline{i},\underline{l}}(l_{s+u}) = \sum_{l_{s+u-1}\in B_{s+u-1}}\cdots\sum_{l_1\in B_1} x_{i_1}(k_1)\cdots x_{i_d}(k_d).$$

Here we only fix l_{s+u} and sum over all the other indices. It is straightforward to check that

$$\{w_{\underline{i},\underline{l}}(l_{s+u}) - E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\}_{l_{s+u}\in B_{s+u}}$$

is a sequence of martingale differences. Using the noncommutative Burkholder–Gundy inequality [Pisier and Xu 1997], we have

$$\begin{aligned}
&\Bigg\|\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1}\big(x_{i_1}(k_1)\cdots x_{i_d}(k_d) - E_{\mathcal{N}_{s+u-1}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Bigg\|_p\\
&\quad= \Bigg\|\sum_{l_{s+u}\in B_{s+u}}\big(w_{\underline{i},\underline{l}}(l_{s+u}) - E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\big)\Bigg\|_p\\
&\quad\le C_p\Bigg\|\Bigg(\sum_{l_{s+u}\in B_{s+u}}|w_{\underline{i},\underline{l}}(l_{s+u}) - E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))|^2\\
&\qquad\qquad + \big|\big(w_{\underline{i},\underline{l}}(l_{s+u}) - E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\big)^*\big|^2\Bigg)^{\frac{1}{2}}\Bigg\|_p\\
&\quad=: \Psi.
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}\Psi &\le C_p\sqrt{|B_{s+u}|}\sup_{l_{s+u}\in B_{s+u}}\|w_{\underline{i},\underline{l}}(l_{s+u})-E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\|_p\\ &\le 2C_p\sqrt{|B_{s+u}|}\sup_{l_{s+u}\in B_{s+u}}\|w_{\underline{i},\underline{l}}(l_{s+u})\|_p.\end{aligned}$$

Recall that $k_\alpha = l_{\pi^{-1}(\alpha)}$ if α is a singleton of σ. In this case, $l_\beta \in B_\beta$ if and only if $k_\alpha \in B_{\pi^{-1}(\alpha)}$, where $\pi(\beta)=\alpha$. Replacing p by a larger even integer if necessary, arguing as for Proposition 2.1, or simply adding zeros to apply Proposition 2.1, we find

$$\|w_{\underline{i},\underline{l}}(l_{s+u})\|_p \le \Big(\frac{3\pi}{4}\Big)^{2u}\prod_{\alpha\in\widetilde{\sigma}_{\mathrm{sing}}}\Big\|\sum_{k_\alpha\in B_{\pi^{-1}(\alpha)}}x_{i_\alpha}(k_\alpha)\Big\|_{pd}\prod_{\alpha\in\widetilde{\sigma}_{ns}}m^{1/2}.$$

Here, $\widetilde{\sigma}$ is obtained from σ by erasing one pair block containing $\pi(d)$ so that $\#\widetilde{\sigma}_{ns}=2(u-1)$. We mention one subtlety here in applying Proposition 2.1. Since l_{s+u} is fixed, the term $x_{i_\alpha}(l_{s+u})$ is regarded to "attach" to its adjacent term. For instance, $x_{i_{j'}}(k_{j'})x_{i_\alpha}(l_{s+u})x_{i_j}(k_j)$ is regarded as a product of two terms, i.e., $[x_{i_{j'}}(k_{j'})x_{i_\alpha}(l_{s+u})]x_{i_j}(k_j)$ or $x_{i_{j'}}(k_{j'})[x_{i_\alpha}(l_{s+u})x_{i_j}(k_j)]$. Using the noncommutative Khintchine inequality [Lust-Piquard 1986; Lust-Piquard and Pisier 1991] or the Burkholder–Gundy inequality [Pisier and Xu 1997], we have, for $\alpha\in\widetilde{\sigma}_{\mathrm{sing}}$,

$$\begin{aligned}&\Big\|\sum_{k_\alpha\in B_{\pi^{-1}(\alpha)}}x_{i_\alpha}(k_\alpha)\Big\|_{pd}\\ &\qquad\le C_{pd}\max\Big\{\Big\|\sum_{k_\alpha}x_{i_\alpha}(k_\alpha)^*x_{i_\alpha}(k_\alpha)\Big\|_{pd/2}^{1/2},\Big\|\sum_{k_\alpha}x_{i_\alpha}(k_\alpha)x_{i_\alpha}(k_\alpha)^*\Big\|_{pd/2}^{1/2}\Big\}\\ &\qquad\le C_{pd}m^{1/2}.\end{aligned}$$

It follows that $\|w_{\underline{i},\underline{l}}(l_{s+u})\|_p \le C_{p,\sigma}m^{s/2+u-1}$ and thus $\Psi\le C_{p,\sigma}m^{s/2+u-1/2}$. We have shown that

$$\begin{aligned}(3\text{-}5)\quad &\frac{1}{m^{d/2}}\Big\|\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1}\big(x_{i_1}(k_1)\cdots x_{i_d}(k_d)\\ &\qquad\qquad -E_{\mathcal{N}_{s+u-1}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Big\|_p\le\frac{C_{p,\sigma}}{m^{1/2}}.\end{aligned}$$

Repeating the argument $u-1$ times by replacing u with $u-1,u-2,\dots,1$, we find

$$\begin{aligned}&\frac{1}{m^{q-s/2}}\Big\|\sum_{l_q\in B_q}\cdots\sum_{l_1\in B_1}\big(E_{\mathcal{N}_q(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]-E_{\mathcal{N}_{q-1}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Big\|_p\\ &\qquad\qquad\le\frac{C_{p,\sigma}}{m^{1/2}},\end{aligned}$$

for $q = s+u-1, \dots, s+1$. In this iteration argument, we use the same "attaching" procedure as described above in order to apply Proposition 2.1. By the triangle inequality, we have

$$\frac{1}{m^{d/2}} \Big\| \sum_{l_{s+u} \in B_{s+u}} \cdots \sum_{l_1 \in B_1} \big(x_{i_1}(k_1) \cdots x_{i_d}(k_d) - E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \big) \Big\|_p \le \frac{C_{p,\sigma}}{m^{1/2}}.$$

Hence, by Jensen's inequality,

$$\begin{aligned} &\Big\| \frac{1}{m^{d/2}} \sum_{\underline{k}:\sigma(\underline{k})=\sigma} x_{i_1}(k_1) \cdots x_{i_d}(k_d) - \frac{1}{m^{d/2}} \sum_{\underline{k}:\sigma(\underline{k})=\sigma} E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \Big\|_{L_p(\mathcal{A}_m, \tau_m)} \\ &\le \frac{(s+u)^{s+u}}{m^{d/2}} \mathbb{E}_\delta \Big\| \sum_{l_{s+u} \in B_{s+u}} \cdots \sum_{l_1 \in B_1} \big(x_{i_1}(k_1) \cdots x_{i_d}(k_d) - E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \big) \Big\|_p \\ &\le \frac{C_{p,\sigma}}{m^{1/2}}. \end{aligned}$$

In the last inequality, the upper bound holds for every instance of δ and thus holds for the average. The proof is complete by sending $m \to \infty$. □

Lemma 3.7. *Let $\sigma \in P_r(d)$. Then, for all $p < \infty$ and fixed $\omega \in \Omega$,*

$$\lim_{m \to \infty} \Big\| \frac{1}{m^{d/2}} \sum_{\underline{k}:\sigma(\underline{k})=\sigma} x_{i_1}(k_1) \cdots x_{i_d}(k_d) \Big\|_{L_p(\mathcal{A}_m, \tau_m)} = 0.$$

Proof. We follow the same argument as for Lemma 3.6 and only indicate the differences. For $\sigma \in P_r(d)$, there is at lease one block with more than two elements. Without loss of generality, assume there is only one block in σ with more than two elements. Suppose this block has, say, three elements. We list the running indices $\underline{k}$ in the sum as $\{l_1, \dots, l_s, l_{s+1}, \dots, l_{s+u}, l_{s+u+1}\}$, where there are s singletons, u pairs and one block with three elements in σ. Using the random selectors, it suffices to show that

$$\frac{1}{m^{d/2}} \Big\| \sum_{l_{s+u+1} \in B_{s+u+1}} \cdots \sum_{l_1 \in B_1} x_{i_1}(k_1) \cdots x_{i_d}(k_d) \Big\|_p \to 0$$

as $m \to \infty$, where $B_1, \dots, B_{s+u+1}$ are disjoint random sets with union $[m]$. Denote by $\mathcal{N}_{s+u}(\underline{k})$ the von Neumann algebra generated by $x_{i_{\pi(\alpha)}}(l'_\alpha)$ for all $\alpha \le s+2u$, where $\underline{l}'$ is a permutation of $\underline{k}$ so that $l_1 = l'_1, \dots, l_s = l'_s$, $l_{s+1} = l'_{s+1} = l'_{s+2}$, etc. Then using the noncommutative Burkholder–Gundy inequality, we can show that

$$\begin{aligned} &\frac{1}{m^{d/2}} \Big\| \sum_{l_{s+u+1} \in B_{s+u+1}} \cdots \sum_{l_1 \in B_1} \big(x_{i_1}(k_1) \cdots x_{i_d}(k_d) - E_{\mathcal{N}_{s+u}(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \big) \Big\|_p \\ &\qquad \le \frac{C_{p,\sigma} m^{s/2+u+1/2}}{m^{s/2+u+3/2}} \to 0 \end{aligned}$$

as $m \to \infty$. It remains to show

$$\frac{1}{m^{d/2}} \left\| \left(\sum_{l_{s+u+1} \in B_{s+u+1}} \cdots \sum_{l_1 \in B_1} E_{\mathcal{N}_{s+u}(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \right) \right\|_p \to 0.$$

Note that

$$\frac{1}{m^{d/2}} \left\| \left(\sum_{l_{s+u+1} \in B_{s+u+1}} \cdots \sum_{l_1 \in B_1} E_{\mathcal{N}_{s+u}(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \right) \right\|_p$$
$$\leq \frac{1}{m} \sum_{l_{s+u+1} \in B_{s+u+1}} \frac{1}{m^{(d-2)/2}} \left\| \sum_{l_{s+u} \in B_{s+u}} \cdots \sum_{l_1 \in B_1} x_{i_1}(k_1) \cdots x_{i_d}(k_d) \right\|_p.$$

Now apply Proposition 2.1 with the same "attaching" procedure as above, yielding

$$\left\| \sum_{l_{s+u} \in B_{s+u}} \cdots \sum_{l_1 \in B_1} x_{i_1}(k_1) \cdots x_{i_d}(k_d) \right\|_p \leq C_{p,\sigma} m^{s/2+u},$$

which gives a decay factor and completes the proof. □

Theorem 3.8. *Let* $(\widetilde{x}_j(m))^\bullet \in \bigcap_{p<\infty} L_p\big(\prod_{m,\mathcal{U}} L_\infty(\Omega; \mathcal{A}_m)\big)$ *for* $j = 1, \ldots, d$. *Then*

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet = \sum_{\substack{\sigma \in P_{1,2}(d) \\ \sigma \leq \sigma(\underline{i})}} w_\sigma(\underline{i}),$$

where the equality holds for all $\omega \in \Omega$ *and*

$$w_\sigma(\underline{i}) = \left(\frac{1}{m^{d/2}} \sum_{\underline{k} \in [m]^d : \sigma(\underline{k}) = \sigma} E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \right)^\bullet. \tag{3-6}$$

Proof. By Lemmas 3.6 and 3.7, we have

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet = \sum_{\sigma \in P_{1,2}(d)} \left(\frac{1}{m^{d/2}} \sum_{\underline{k} \in [m]^d : \sigma(\underline{k}) = \sigma} E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] \right)^\bullet.$$

By Proposition 3.5, we write

$$(j_1, \ldots, j_d) = (i_{\pi(1)}, \ldots, i_{\pi(d)}) \quad \text{and} \quad (l'_1, \ldots, l'_d) = (k_{\pi(1)}, \ldots, k_{\pi(d)}).$$

It follows that

$$E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)] = \varepsilon(\underline{i}, \underline{k}) x_{j_1}(l'_1) \cdots x_{j_s}(l'_s) \delta_{j_{s+1}, j_{s+2}} \cdots \delta_{j_{d-1}, j_d}.$$

Note that $E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1) \cdots x_{i_d}(k_d)]$ is nonzero only if $j_{s+1} = j_{s+2}, \ldots, j_{d-1} = j_d$. Since $\sigma(\underline{k}) = \sigma$, we have $\sigma \leq \sigma(\underline{i})$. □

If $\sigma \leq \sigma(\underline{i})$, we call the $w_\sigma(\underline{i})$ defined in (3-6) the arbitrary Wick words. By Theorem 3.8,

$$L_2(\Gamma_Q) \subset L_2\text{-span}\{w_\sigma(\underline{i}) : \underline{i} \in [N]^d, \sigma \in P_{1,2}(d), d \in \mathbb{N}\}.$$

Here and in what follows, L_p- span W means the $L_p(\tau_{\mathcal{U}})$ closure of linear combinations of elements in W. We want to identify $L_2(\Gamma_Q)$ with the span of fewer Wick words. Let $\underline{i} \in [N]^s$ for $s \in \mathbb{N}$. We define the special Wick words

$$w(\underline{i}) = \left(\frac{1}{m^{s/2}} \sum_{\underline{k} \in [m]^s : \sigma(\underline{k}) \in P_1(s)} x_{i_1}(k_1) \cdots x_{i_s}(k_s) \right)^{\bullet}. \tag{3-7}$$

Let $\underline{i}' \in [N]^{s'}$. In order to understand the inner product of $w(\underline{i})$ and $w(\underline{i}')$, we first introduce some notions. Let $\{2 \cdot 1, 2 \cdot 2, \ldots, 2 \cdot s\}$ be a multiset, each element with multiplicity 2. One can regard it as a set of cardinality $2s$ given by $[2s] = \{1, 2, \ldots, s, \widetilde{1}, \widetilde{2}, \ldots, \widetilde{s}\}$. Let σ^b be a partition of the set $[2s]$. We call it a bipartite pair partition of $[2s]$ if

$$\sigma^b = \left\{ \{e_k, z_k\} : e_k = 1, 2, \ldots, s, z_k = \widetilde{1}, \widetilde{2}, \ldots, \widetilde{s} \right\}.$$

Let $P_2^b(2s)$ denote the set of all bipartite pair partitions. Let $\underline{i}, \underline{i}' \in [m]^s$, where $\underline{i}'$ is understood as a map $\underline{i}' : \{\widetilde{1}, \widetilde{2}, \ldots, \widetilde{s}\} \to [m]$. Define the concatenation operation by

$$\underline{i} \sqcup \underline{i}' = (i_1, \ldots, i_s, i'_{\widetilde{1}}, \ldots, i'_{\widetilde{s}}). \tag{3-8}$$

We denote by $\sigma(\underline{i} \sqcup \underline{i}')$ the partition induced by $\underline{i}$ and $\underline{i}'$ on the multiset $[2s]$. For example, $\{k, l, \widetilde{k}\}$ are in the same block of $\sigma(\underline{i} \sqcup \underline{i}')$ if $i_k = i_l = i'_{\widetilde{k}}$. Given $\sigma^b \in P_2^b(2s)$, define the set of bipartite crossings by

$$I^b(\sigma^b) = \left\{ \{k, l\} : 1 \le k, l \le s, e_k < e_l, z_l > z_k \right\}.$$

Recall that $\langle w(\underline{i}), w(\underline{i}') \rangle = \tau_{\mathcal{U}}[w(\underline{i}')^* w(\underline{i})]$.

Proposition 3.9. *Let $w(\underline{i})$ and $w(\underline{i}')$ be special Wick words. Then there exists a full probability set $\Omega_0 \subset \Omega$ such that for all $\omega \in \Omega_0$,*

$$\langle w(\underline{i}), w(\underline{i}') \rangle = \begin{cases} \displaystyle\sum_{\substack{\sigma^b \in P_2^b(2s) \\ \sigma^b \le \sigma(\underline{i} \sqcup \underline{i}')}} \prod_{\{r,t\} \in I^b(\sigma^b)} q(i(e_r), i(e_t)) & \text{if } \{i_1, \ldots, i_s\} = \{i'_1, \ldots, i'_{s'}\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\{i_1, \ldots, i_s\} = \{i'_1, \ldots, i'_{s'}\}$ means that $\underline{i}$ and $\underline{i}'$ are equal as multisets, i.e., both the elements and their multiplicities are the same.

Proof. We follow the same argument as for Theorem 3.1. By definition,

$$\langle w(\underline{i}), w(\underline{i}') \rangle = \lim_{m, \mathcal{U}} \frac{1}{m^{(s+s')/2}} \sum_{\substack{\underline{k}, \underline{k}' : \sigma(\underline{k}) \in P_1(s) \\ \sigma(\underline{k}') \in P_1(s')}} \tau_m[x_{i'_{s'}}(k'_{s'}) \cdots x_{i'_1}(k'_1) x_{i_1}(k_1) \cdots x_{i_s}(k_s)].$$

Since all the k_α's are pairwise different, $\tau_m[x_{i'_{s'}}(k'_{s'}) \cdots x_{i'_1}(k'_1) x_{i_1}(k_1) \cdots x_{i_s}(k_s)] = 0$ unless $s = s'$. Moreover, every $x_{i_\alpha}(k_\alpha)$ has to be the same as exactly one $x_{i'_\beta}(k'_\beta)$

to contribute to the sum. This implies $\underline{i}$ and $\underline{i}'$ are equal as multisets. We rewrite

$$\sum_{\substack{\underline{k},\underline{k}':\sigma(\underline{k})\in P_1(s)\\ \sigma(\underline{k}')\in P_1(s')}} \tau_m[x_{i'_{s'}}(k'_{s'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$= \sum_{\sigma(\underline{k}),\sigma(\underline{k}')\in P_1(s)} \tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$= \sum_{\substack{\sigma^b\in P_2^b(2s)\\ \sigma^b\le\sigma(\underline{i}\sqcup\underline{i}')}} \sum_{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b} \tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)].$$

If $\{r,t\}\in I^b(\sigma^b)$, then we have to switch $x_{i(e_r)}(k(e_r))$ and $x_{i(e_t)}(k(e_t))$ to cancel the corresponding $x_{i(z_r)}(k(z_r))$ and $x_{i(z_t)}(k(z_t))$ terms. It follows that

$$\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)] = \prod_{\{r,t\}\in I^b(\sigma^b)} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]).$$

Since $\underline{k}\in P_1(s)$, by independence, we have

$$\frac{1}{m^s}\sum_{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b} \mathbb{E}\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$= \frac{m(m-1)\cdots(m-s+1)}{m^s}\prod_{\{r,t\}\in I^b(\sigma^b)} q(i(e_r),i(e_t)).$$

Hence, if $\underline{i}=\underline{i}'$ as multisets, then

$$\mathbb{E}\langle w(\underline{i}),w(\underline{i}')\rangle = \sum_{\substack{\sigma^b\in P_2^b(2s)\\ \sigma^b\le\sigma(\underline{i}\sqcup\underline{i}')}} \prod_{\{r,t\}\in I^b(\sigma^b)} q(i(e_r),i(e_t)).$$

To show almost sure convergence, let

$$X_m = \frac{1}{m^s}\sum_{\substack{\sigma^b\in P_2^b(2s)\\ \sigma^b\le\sigma(\underline{i}\sqcup\underline{i}')}} \sum_{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b} \tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)].$$

Since $\mathbb{P}(\omega:|X_m-\mathbb{E}X_m|>\eta)\le \mathrm{Var}(X_m)/\eta^2$, by the Borel–Cantelli lemma, it suffices to show that $\sum_{m=1}^\infty \mathrm{Var}(X_m)<\infty$. But

$$\mathrm{Var}(X_m) = \frac{1}{m^{2s}}\sum_{\sigma^b,\pi^b\in P_2^b(2s)} \sum_{\substack{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b\\ \sigma(\underline{\ell}\sqcup\underline{\ell}')=\pi^b}} V_{\underline{k},\underline{\ell}},$$

where

$$V_{\underline{k},\underline{\ell}} = \mathbb{E}(\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$\times \tau_m[x_{i'_s}(\ell'_s)\cdots x_{i'_1}(\ell'_1)x_{i_1}(\ell_1)\cdots x_{i_s}(\ell_s)])$$
$$- \mathbb{E}(\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)])$$
$$\times \mathbb{E}(\tau_m[x_{i'_s}(\ell'_s)\cdots x_{i'_1}(\ell'_1)x_{i_1}(\ell_1)\cdots x_{i_s}(\ell_s)])$$

$$= \mathbb{E}\Bigg[\prod_{\{r,t\}\in I^b(\sigma^b)} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]) \times \prod_{\{r',t'\}\in I^b(\pi^b)} \varepsilon([i(e_{r'}),\ell(e_{r'})],[i(e_{t'}),\ell(e_{t'})])\Bigg] - \prod_{\{r,t\}\in I^b(\sigma^b)} q(i(e_r),i(e_t)) \prod_{\{r',t'\}\in I^b(\pi^b)} q(i(e_{r'}),i(e_{t'})).$$

By independence, $V_{\underline{k},\underline{\ell}}$ is nonzero only if there are two pairs $\{r,t\} \in I^b(\sigma^b)$ and $\{r',t'\} \in I^b(\pi^b)$ such that $\{k(e_r),k(e_t)\} = \{\ell(e_{r'}),\ell(e_{t'})\}$. In this case,

$$\#\big\{\underline{k},\underline{k}',\underline{\ell},\underline{\ell}' : \sigma(\underline{k}\sqcup\underline{k}') = \sigma^b \text{ and } \sigma(\underline{\ell}\sqcup\underline{\ell}') = \pi^b\big\} \le m^s m^{s-2} = m^{2s-2}.$$

Since $V_{\underline{k},\underline{\ell}}$ is uniformly bounded and $C(s) := [\#P_2^b(2s)]^2$ is independent from m,

$$\sum_{m=1}^{\infty} \mathrm{Var}(X_m) \le \sum_{m=1}^{\infty} \frac{C(s)}{m^2} < \infty. \qquad \square$$

Recall the notation $I_p(\sigma)$ and $I_{\mathrm{sp}}(\sigma)$ from Section 2A. For $\underline{i} \in [N]^d$ and $\sigma \in P_{1,2}(d)$ with $\sigma \le \sigma(\underline{i})$, put

$$f_\sigma(\underline{i}) = \prod_{\{r,t\}\in I_p(\sigma)} q(i(e_r),i(e_t)) \prod_{\{r,t\}\in I_{\mathrm{sp}}(\sigma)} q(i(e_r),i(e_t)), \tag{3-9}$$

with the convention that the product over an empty index set is 1.

Proposition 3.10. *Let $\sigma \in P_{1,2}(d)$ and $\sigma' \in P_{1,2}(d')$ be partitions. Let $w_\sigma(\underline{i})$ and $w_{\sigma'}(\underline{i}')$ be arbitrary Wick words as defined in (3-6). Then, for almost all $\omega \in \Omega$,*

$$\langle w_\sigma(\underline{i}), w_{\sigma'}(\underline{i}')\rangle = \begin{cases} \langle f_\sigma(\underline{i}) w(\underline{i}_{\mathrm{np}}),\, f_{\sigma'}(\underline{i}') w(\underline{i}'_{\mathrm{np}})\rangle & \text{if } \sigma \le \sigma(\underline{i}) \text{ and } \sigma' \le \sigma(\underline{i}'), \\ 0 & \text{otherwise.} \end{cases} \tag{3-10}$$

Here, $\underline{i}_{\mathrm{np}}$ is the vector obtained by removing coordinates in $\underline{i}$ which correspond to the pair blocks of σ.

Example 3.11. Suppose that $\underline{i} = (2,4,7,4,7)$ and $\sigma = \{\{1\},\{2\},\{4\},\{3,5\}\}$. Then $\underline{i}_{\mathrm{np}} = (2,4,4)$.

Proof of Proposition 3.10. By definition,

$$\langle w_\sigma(\underline{i}), w_{\sigma'}(\underline{i}')\rangle = \lim_{m,\mathcal{U}} \frac{1}{m^{(d+d')/2}} \sum_{\substack{\underline{k},\underline{k}' : \sigma(\underline{k})=\sigma \\ \sigma(\underline{k}')=\sigma'}} \tau_m[x_{i'_{d'}}(k'_{d'}) \cdots x_{i'_1}(k'_1) x_{i_1}(k_1) \cdots x_{i_d}(k_d)].$$

Note that $w_\sigma(\underline{i})$ is nonzero only if $\sigma \le \sigma(\underline{i})$. Then $k_\alpha = k_\beta$ implies $i_\alpha = i_\beta$. By (A-2), we may assume that, in $x_{i'_{d'}}(k'_{d'}) \cdots x_{i'_1}(k'_1) x_{i_1}(k_1) \cdots x_{i_d}(k_d)$, if $k_\alpha = k_\beta$ for $\alpha \ne \beta$, then $k'_\gamma \ne k_\alpha$ for all $\gamma \in [d']$. In other words, $k_\alpha \ne k'_\gamma$ for all $\alpha \in [d]$ and $\gamma \in [d']$ if

both of them belong to pair blocks. Applying Proposition 3.5 to $x_{i_1}(k_1)\cdots x_{i_d}(k_d)$ and $x_{i'_1}(k'_1)\cdots x_{i'_{d'}}(k'_{d'})$, we find

$$\tau_m[x_{i'_{d'}}(k'_{d'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_d}(k_d)] = \varepsilon(\underline{i},\underline{k})\varepsilon(\underline{i}',\underline{k}')\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)],$$

where $\underline{\ell}\in P_1(s)$, $\underline{\ell}'\in P_1(s')$, $\underline{j}=\underline{i}_{\mathrm{np}}$, $\underline{j}'=\underline{i}'_{\mathrm{np}}$, and $\varepsilon(\underline{i},\underline{k})\varepsilon(\underline{i}',\underline{k}')$ is given in Proposition 3.5. By independence, we have

$$\begin{aligned}\mathbb{E}\tau_m[x_{i'_{d'}}(k'_{d'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_d}(k_d)] &= \prod_{\{r,t\}\in I_p(\sigma)} q(i(e_r),i(e_t)) \prod_{\{r,t\}\in I_{\mathrm{sp}}(\sigma)} q(i(e_r),i(e_t)) \prod_{\{r',t'\}\in I_p(\sigma')} q(i(e_{r'}),i(e_{t'}))\\ &\quad\times \prod_{\{r',t'\}\in I_{\mathrm{sp}}(\sigma')} q(i(e_{r'}),i(e_{t'}))\mathbb{E}\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)].\end{aligned}$$

As shown in Proposition 3.9, $\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)]$ is zero if $\underline{\ell}$ and $\underline{\ell}'$ are not equal as multisets, and there is nothing more to prove. Assume $\underline{\ell}$ and $\underline{\ell}'$ are equal. Let u and u' be the number of pair blocks in σ and σ'. By Proposition 3.9, we find

$$\begin{aligned}\mathbb{E}\langle w_\sigma(\underline{i}), w_{\sigma'}(\underline{i}')\rangle &= \lim_{m,\mathcal{U}} \frac{m\cdots(m-s+1)}{m^s}\cdot\frac{(m-s)\cdots(m-s-u-u'+1)}{m^{u+u'}}\\ &\quad\times f_\sigma(\underline{i})f_{\sigma'}(\underline{i}')\mathbb{E}\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)]\\ &= f_\sigma(\underline{i})f_{\sigma'}(\underline{i}')\mathbb{E}\langle w(\underline{i}_{\mathrm{np}}), w(\underline{i}'_{\mathrm{np}})\rangle.\end{aligned}$$

The almost sure convergence follows from the same argument as for Proposition 3.9 using the Borel–Cantelli lemma and independence. □

In the two proofs above, the Borel–Cantelli lemma may be avoided if we use the average model Γ_Q^a; see Section 3A. Note that for $\underline{i}\in[m]^s$, $w_\sigma(\underline{i})=w(\underline{i})$ for any $\sigma\in P_1(s)$.

Corollary 3.12. *Let $\sigma\le\sigma(\underline{i})$. We have $w_\sigma(\underline{i})=f_\sigma(\underline{i})w(\underline{i}_{\mathrm{np}})$ for almost all $\omega\in\Omega$.*

Proof. Since $\tau_{\mathcal{U}}$ is faithful on Γ_Q, it suffices to show

$$\tau_{\mathcal{U}}\big((w_\sigma(\underline{i})-f_\sigma(\underline{i})w(\underline{i}_{\mathrm{np}}))^*(w_\sigma(\underline{i})-f_\sigma(\underline{i})w(\underline{i}_{\mathrm{np}}))\big)=0.$$

But, by Proposition 3.10, we have

$$\langle w(\underline{i}_{\mathrm{np}}), w_\sigma(\underline{i})\rangle = f_\sigma(\underline{i})\langle w(\underline{i}_{\mathrm{np}}), w(\underline{i}_{\mathrm{np}})\rangle.$$

From here the claim follows by linearity. □

This result yields the identification

$$L_2\text{-}\operatorname{span}\{w_\sigma(\underline{i}):\underline{i}\in[N]^d,\sigma\in P_{1,2}(d),d\in\mathbb{Z}_+\}=L_2\text{-}\operatorname{span}\{w(\underline{i}):\underline{i}\in[N]^s,s\in\mathbb{Z}_+\},$$

with the inner product relation given by (3-10).

Proposition 3.13. $L_2(\Gamma_Q) = L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [N]^s, s \in \mathbb{Z}_+\}$.

Proof. Write $H_w := L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [N]^s, s \in \mathbb{Z}_+\}$. By Theorem 3.8 and Corollary 3.12, $L_2(\Gamma_Q) \subset H_w$. It remains to show that $H_w \subset L_2(\Gamma_Q)$. We proceed by induction on the length s of special Wick words $w(\underline{i})$. First observe that if $\sigma(\underline{i}) \in P_1(s)$, then the only partition $\sigma \leq \sigma(\underline{i})$ is $\sigma(\underline{i})$ itself. In this case, by Theorem 3.8, we have

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_s}(m))^\bullet = w_{\sigma(\underline{i})}(\underline{i}) = w(\underline{i}) \in L_2(\Gamma_Q), \tag{3-11}$$

since every $(\widetilde{x}_{i_1}(m))^\bullet$ is in $\bigcap_{p<\infty} L_p(\Gamma_Q)$. If $s = 1$,

$$w(\underline{i}) = \left(\frac{1}{\sqrt{m}} \sum_{k_1=1}^{m} x_{i_1}(k_1)\right)^\bullet \in L_2(\Gamma_Q)$$

by definition. If $s = 2$ and $i_1 \neq i_2$, then $w(\underline{i}) \in L_2(\Gamma_Q)$ by (3-11). If $i_1 = i_2$, using Theorem 3.8, we find

$$(\widetilde{x}_{i_1}(m))^\bullet (\widetilde{x}_{i_2}(m))^\bullet = w_{\sigma(\underline{i})}(\underline{i}) + w_{\sigma_0}(\underline{i}) = 1 + w_{\sigma_0}(\underline{i}),$$

where $\sigma_0 \in P_1(2)$. It follows that $w(\underline{i}) = w_{\sigma_0}(\underline{i}) \in L_2(\Gamma_Q)$. Suppose $w(\underline{i}) \in L_2(\Gamma_Q)$ for all $\underline{i}$ with $|\underline{i}| < s$. Consider $\underline{i} \in [N]^s$. We know $w(\underline{i}) \in L_2(\Gamma_Q)$ if $\sigma(\underline{i}) \in P_1(s)$. If $\sigma(\underline{i}) \notin P_1(s)$, by Theorem 3.8, we have

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_s}(m))^\bullet = w_{\sigma_0}(\underline{i}) + \sum_{\substack{\sigma \in P_{1,2}(s) \\ \sigma \leq \sigma(\underline{i}), \sigma \notin P_1(s)}} w_\sigma(\underline{i}), \tag{3-12}$$

where $\sigma_0 \in P_1(s)$. By Corollary 3.12, we have $w_\sigma(\underline{i}) = f_\sigma(\underline{i}) w(\underline{i}_{\mathrm{np}})$, and $\underline{i}_{\mathrm{np}}$ is a vector of dimension at most $s-2$. By the induction hypothesis, $w_\sigma(\underline{i}) \in L_2(\Gamma_Q)$ for $\sigma \notin P_1(s)$. We deduce from (3-12) that $w(\underline{i}) = w_{\sigma_0}(\underline{i}) \in L_2(\Gamma_Q)$. □

3C. ***Fock spaces and mixed q-commutation relations.*** From the work in the previous section, we can describe the Fock space and creation/annihilation operators associated to Γ_Q. Given a vector $\underline{i}$, we denote by $|\underline{i}|$ the number of nonzero coordinates in $\underline{i}$. Let $H_Q^s = \operatorname{span}\{w(\underline{i}) : |\underline{i}| = s\}$. We define the mixed Fock space by

$$\mathcal{F}_Q = \bigoplus_{s=0}^{\infty} H_Q^s. \tag{3-13}$$

Clearly, $\mathcal{F}_Q = L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [N]^s, s \in \mathbb{Z}_+\}$, which can be further identified with $L_2(\Gamma_Q)$ by Proposition 3.13.

Proposition 3.14. *Let* $x_j = \left(\frac{1}{\sqrt{m}} \sum_{k=1}^m x_j(k)\right)^\bullet \in \Gamma_Q^\infty$ *for* $j = 1, \ldots, N$ *be generators of* Γ_Q *and* $w(\underline{i}) \in H_Q^s$. *Then*

$$x_j w(\underline{i}) = w(j \sqcup \underline{i}) + \sum_{l=1}^{s} \delta_{j,i_l} w(\underline{i} - i_l) \prod_{r=1}^{l-1} q(i_r, i_l).$$

Here, $j \sqcup \underline{i} = (j, i_1, \ldots, i_s) \in [N]^{s+1}$ is the concatenation operation defined in (3-8), $\underline{i} - i_l = (i_1, \ldots, i_{l-1}, i_{l+1}, \ldots, i_s)$ with $|\underline{i} - i_l| = s - 1$, and we understand the product over empty index set to be 1. Therefore,

$$x_j = \sum_{s=0}^{\infty} P_{s+1} x_j P_s + \sum_{s=1}^{\infty} P_{s-1} x_j P_s,$$

where $P_s : \mathcal{F}_Q \to H_Q^s$ is the orthogonal projection.

Proof. By definition,

$$\begin{aligned} x_j w(\underline{i}) &= \left(\frac{1}{\sqrt{m}} \sum_{k_0=1}^{m} x_j(k_0) \frac{1}{m^{s/2}} \sum_{\underline{k} \in [m]^s : \sigma(\underline{k}) \in P_1(s)} x_{i_1}(k_1) \cdots x_{i_s}(k_s) \right)^{\bullet} \\ &= \left(\frac{1}{m^{(s+1)/2}} \sum_{k_0 \sqcup \underline{k} \in [m]^{s+1} : \sigma(k_0 \sqcup \underline{k}) \in P_1(s+1)} x_j(k_0) x_{i_1}(k_1) \cdots x_{i_s}(k_s) \right)^{\bullet} \\ &\quad + \sum_{l=1}^{s} \left(\frac{1}{m^{(s+1)/2}} \sum_{\substack{k_0 \sqcup \underline{k} \in [m]^{s+1} : k_0 = k_l \\ \sigma(\underline{k}) \in P_1(s)}} x_j(k_0) x_{i_1}(k_1) \cdots x_{i_s}(k_s) \right)^{\bullet}. \end{aligned}$$

The first term in the above equation is clearly the special Wick word $w(j \sqcup \underline{i})$. To understand the second one, we define $\sigma_l \in P_{1,2}(s+1)$ by $\sigma_l = \sigma(k_0 \sqcup \underline{k})$ for $k_0 = k_l$ and $\underline{k} \in P_1(s)$, i.e.,

$$\sigma_l = \big\{ \{1, l+1\}, \{2\}, \ldots, \{l\}, \{l+2\}, \ldots, \{s+1\} \big\}.$$

Using (the proof of) Lemma 3.6, we deduce that the arbitrary Wick word satisfies

$$\left(\frac{1}{m^{(s+1)/2}} \sum_{\substack{k_0 \sqcup \underline{k} \in [m]^{s+1} : k_0 = k_l \\ \sigma(\underline{k}) \in P_1(s)}} x_j(k_0) x_{i_1}(k_1) \cdots x_{i_s}(k_s) \right)^{\bullet} = w_{\sigma_l}(j \sqcup \underline{i}).$$

Note that $w_{\sigma_l}(j \sqcup \underline{i})$ is nonzero only if $\sigma_l \le \sigma(j \sqcup \underline{i})$ or equivalently $j = i_l$. Using (3-9) and Corollary 3.12, we find

$$w_{\sigma_l}(j \sqcup \underline{i}) = \delta_{j, i_l} \prod_{r=1}^{l-1} q(i_r, i_l) w(\underline{i} - i_l). \qquad \square$$

Define operators c_j and a_j acting on $\mathcal{F}_Q$ by

$$c_j w(\underline{i}) = w(j \sqcup \underline{i}), \quad a_j w(\underline{i}) = \sum_{l=1}^{s} \delta_{j, i_l} w(\underline{i} - i_l) \prod_{r=1}^{l-1} q(i_r, i_l). \tag{3-14}$$

Clearly $x_j = c_j + a_j$, $c_j = \sum_{s=0}^{\infty} P_{s+1} x_j P_s$ and $a_j = \sum_{s=1}^{\infty} P_{s-1} x_j P_s$. Since $x_j = x_j^*$, we have $c_j^* = a_j$. We call c_j and a_j the creation and annihilation operators respectively for $j = 1, \ldots, N$. The following result is simply a recapitulation; see, e.g., [Brown and Ozawa 2008] for more about QWEP C*-algebras.

Corollary 3.15. *Let $\widetilde{\Gamma}_Q$ be the von Neumann algebra generated by (the spectral projections of) $c_j + a_j$ for $j = 1, \ldots, N$. Then $\Gamma_Q = \widetilde{\Gamma}_Q$. In particular, $\widetilde{\Gamma}_Q$ is QWEP.*

Proposition 3.16. *For $j, k = 1, \ldots, N$, c_j and c_j^* satisfy the mixed q-commutation relation*

$$c_k^* c_j - q(j,k) c_j c_k^* = \delta_{j,k} 1. \tag{3-15}$$

Proof. Let $w(\underline{i}) \in H_Q^s$. Then

$$c_j^* c_j w(\underline{i}) = c_j^* w(j \sqcup \underline{i}) = w(\underline{i}) + \sum_{l=1}^{s} \delta_{j,i_l} w(j \sqcup (\underline{i} - i_l)) q(j, i_l) \prod_{r=1}^{l-1} q(i_r, i_l).$$

But

$$c_j c_j^* w(\underline{i}) = \sum_{l=1}^{s} \delta_{j,i_l} w(j \sqcup (\underline{i} - i_l)) \prod_{r=1}^{l-1} q(i_r, i_l).$$

Hence $c_j^* c_j w(\underline{i}) - q(j,j) c_j c_j^* w(\underline{i}) = w(\underline{i})$. If $j \neq k$, then

$$c_k^* c_j w(\underline{i}) = c_k^* w(j \sqcup \underline{i}) = \sum_{l=1}^{s} \delta_{k,i_l} w(j \sqcup (\underline{i} - i_l)) q(j, i_l) \prod_{r=1}^{l-1} q(i_r, i_l),$$

and

$$c_j c_k^* w(\underline{i}) = \sum_{l=1}^{s} \delta_{k,i_l} w(j \sqcup (\underline{i} - i_l)) \prod_{r=1}^{l-1} q(i_r, i_l).$$

Hence $c_k^* c_j w(\underline{i}) - q(j,k) c_j c_k^* w(\underline{i}) = 0$. □

Remark 3.17. The Fock space representation was studied in more general setting by Bożejko and Speicher [1994]. Let (e_i) be an orthonormal basis (o.n.b.) of a Hilbert space H. One can construct the Fock space $\mathcal{F}_Q(H)$ following [Bożejko and Speicher 1994; Lust-Piquard 1999]. Let Ω be the vacuum state and W be the Wick product, i.e.,

$$W(e_{i_1} \otimes \cdots \otimes e_{i_s})\Omega = e_{i_1} \otimes \cdots \otimes e_{i_s}.$$

The Wick product was studied in detail in [Krolak 2000]. Suppose $\underline{i} \in [N]^s$ and $\underline{j} \in [N]^{s'}$. We have

$$\langle w(\underline{i}), w(\underline{j}) \rangle = \langle e_{i_1} \otimes \cdots \otimes e_{i_s}, e_{j_1} \otimes \cdots \otimes e_{j_{s'}} \rangle,$$

where the left side is given by Proposition 3.9 and the right side is understood as the inner product in $\mathcal{F}_Q(H)$; see [Bożejko and Speicher 1994; Lust-Piquard 1999]. Our argument shows that one can alternatively implement (3-15) and construct the Fock space using the probabilistic approach (Speicher's CLT) and the von Neumann algebra ultraproduct. If $\sup_{ij} q_{ij} < 1$ and $\underline{i} \in [N]^s$, we can identify our special Wick

words $w(\underline{i})$ with $W(e_{i_1} \otimes \cdots \otimes e_{i_s})$. Thus we can also identify H_Q^s with $H^{\otimes s}$. This identification will play an important role when we study the operator algebraic properties of Γ_Q in later parts of this paper.

3D. ***The Ornstein–Uhlenbeck semigroup on*** $\boldsymbol{\Gamma_Q}$**.** Let T_t^m denote the Ornstein–Uhlenbeck semigroup acting on $\mathcal{A}_m$; see [Biane 1997, Section 2.1]. T_t^m is given by

$$T_t^m x_{i_1}(k_1) \cdots x_{i_d}(k_d) = e^{-td} x_{i_1}(k_1) \cdots x_{i_d}(k_d)$$

if $\sigma(\underline{k}) \in P_1(d)$. Let us first recall an elementary fact.

Lemma 3.18. *Let $(\mathcal{N}, \tau)$ be noncommutative W^* probability space, where $\mathcal{N}$ is a von Neumann algebra and τ is a normal faithful tracial state. Let $T : \mathcal{N} \to \mathcal{N}$ be a $*$-preserving linear normal map with pre-adjoint map $T_* : \mathcal{N}_* \to \mathcal{N}_*$. Suppose T is self-adjoint on $L_2(\mathcal{N}, \tau)$. Then $T = T_*|_{\mathcal{N}}$.*

Proof. Let $x, y \in \mathcal{N}$. Denote the dual pairing between $x_* \in \mathcal{N}_*$ and x by (x_*, x), which can be implemented by $(x_*, x) = \tau(x_* x)$. Since $\mathcal{N} \subset \mathcal{N}_* = L_1(\mathcal{N}, \tau)$,

$$(Tx, y) = \tau((Tx)y) = \langle Tx, y^* \rangle_{L_2(\mathcal{N},\tau)} = \tau(x(Ty)) = (x, Ty) = (T_* x, y). \qquad \square$$

Let $(T_t^m)_* : L_1(\mathcal{A}_m) \to L_1(\mathcal{A}_m)$ be the pre-adjoint map of T_t. By Lemma 3.18, it coincides with T_t on $\mathcal{A}_m$. Let $\prod_{m,\mathcal{U}} L_1(\mathcal{A}_m)$ be the ultraproduct of Banach spaces $L_1(\mathcal{A}_m)$. Recall that $\mathcal{A}_\mathcal{U} = \prod_{m,\mathcal{U}} \mathcal{A}_m$ is the von Neumann algebra ultraproduct in Section 3A. Note we have the canonical inclusion $L_1(\mathcal{A}_\mathcal{U}, \tau_\mathcal{U}) \subset \prod_{m,\mathcal{U}} L_1(\mathcal{A}_m, \tau_m)$. Let $((T_t^m)_*)^\bullet$ be the usual ultraproduct of $(T_t^m)_*$. If $(x_m)^\bullet \in \mathcal{A}_\mathcal{U}$, then

$$((T_t^m)_*)^\bullet (x_m)^\bullet = (T_t^m x_m)^\bullet \in \mathcal{A}_\mathcal{U}$$

because $\sup_m \|T_t^m x_m\| \le \sup_m \|x_m\| < \infty$. Hence, $((T_t^m)_*)^\bullet$ leaves $\mathcal{A}_\mathcal{U}$ invariant. We have checked the commutative diagram

$$\begin{array}{ccccc}
\mathcal{A}_\mathcal{U} & \hookrightarrow & L_1(\mathcal{A}_\mathcal{U}, \tau_\mathcal{U}) & \hookrightarrow & \prod_{m,\mathcal{U}} L_1(\mathcal{A}_m, \tau_m) \\
\downarrow{\scriptstyle ((T_t^m)_*)^\bullet|_{\mathcal{A}_\mathcal{U}}} & & \downarrow{\scriptstyle ((T_t^m)_*)^\bullet|_{L_1(\mathcal{A}_\mathcal{U},\tau_\mathcal{U})}} & & \downarrow{\scriptstyle ((T_t^m)_*)^\bullet} \\
\mathcal{A}_\mathcal{U} & \hookrightarrow & L_1(\mathcal{A}_\mathcal{U}, \tau_\mathcal{U}) & \hookrightarrow & \prod_{m,\mathcal{U}} L_1(\mathcal{A}_m, \tau_m)
\end{array}$$

We define $T_t = \big(((T_t^m)_*)^\bullet|_{L_1(\mathcal{A}_\mathcal{U},\tau_\mathcal{U})}\big)^*$. Then, by construction, $T_t : \mathcal{A}_\mathcal{U} \to \mathcal{A}_\mathcal{U}$ is a normal unital completely positive map which is self-adjoint on $L_2(\mathcal{A}_\mathcal{U}, \tau_\mathcal{U})$. By Lemma 3.18 again, T_t coincides with $((T_t^m)_*)^\bullet$ on $\mathcal{A}_\mathcal{U}$ and thus on $L_2(\mathcal{A}_\mathcal{U}, \tau_\mathcal{U})$. Since $\Gamma_Q \subset \mathcal{A}_\mathcal{U}$ is a von Neumann subalgebra, $L_2(\Gamma_Q) \subset L_2(\mathcal{A}_\mathcal{U}) \subset L_1(\mathcal{A}_\mathcal{U})$. Therefore, for $\underline{i} \in [N]^s$ and $w(\underline{i}) \in L_2(\Gamma_Q)$,

$$T_t w(\underline{i}) = \left(\frac{1}{m^{s/2}} \sum_{\underline{k}:\sigma(\underline{k})\in P_1(s)} e^{-ts} x_{i_1}(k_1) \cdots x_{i_s}(k_s) \right)^\bullet = e^{-ts} w(\underline{i}) \in L_2(\Gamma_Q).$$

Since $L_2(\Gamma_Q) \cap \mathcal{A}_\mathcal{U} = \Gamma_Q$, T_t leaves Γ_Q invariant. Also, we see that $(T_t)_{t\geq 0}$ is a strongly continuous semigroup in $L_2(\Gamma_Q)$. Note that in general $(T_t)_{t\geq 0}$ may not be a point σ-weakly continuous semigroup in t, hence may not extend to a strongly continuous semigroup on $L_2(\mathcal{A}_\mathcal{U})$. By Theorem 3.8,

$$T_t\big((\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet\big) = \sum_{\substack{\sigma\in P_{1,2}(d)\\ \sigma\leq\sigma(\underline{i})}} e^{-t|\sigma_{\mathrm{sing}}|} w_\sigma(\underline{i}) = \sum_{\substack{\sigma\in P_{1,2}(d)\\ \sigma\leq\sigma(\underline{i})}} e^{-t|\underline{i}_{\mathrm{np}}|} f_\sigma(\underline{i}) w(\underline{i}_{\mathrm{np}}),$$

where $|\sigma_{\mathrm{sing}}|$ is the number of singletons in σ, and $|\underline{i}_{\mathrm{np}}|$ is the dimension of $\underline{i}_{\mathrm{np}}$. $f_\sigma(\underline{i})$ and $\underline{i}_{\mathrm{np}}$ are defined in (3-9) and Proposition 3.10. The generator of T_t is the number operator, denoted by A.

4. Analytic properties

Our goal of this section is to prove some analytic properties for Γ_Q. This will be done via a limit procedure, as was used in [Biane 1997; Junge et al. 2015] for proving hypercontractivity.

4A. ***Hypercontractivity.*** Biane [1997, Theorem 5] proved the Ornstein–Uhlenbeck semigroup acting on $\mathcal{A}_m = \mathcal{A}_m(N, \varepsilon)$ is hypercontractive.

Theorem 4.1. *Let $1 \leq p, r < \infty$. Then, for every $\omega \in \Omega$,*

$$\|T_t^m\|_{L_p\to L_r} = 1 \quad \textit{if and only if} \quad e^{-2t} \leq \frac{p-1}{r-1}.$$

With the hard work done in the previous section, it is very easy to prove the following result.

Theorem 4.2. *Let T_t be the Ornstein–Uhlenbeck semigroup on Γ_Q for an arbitrary $N \times N$ symmetric matrix Q with entries in $[-1, 1]$. Then, for $1 \leq p, r < \infty$,*

$$\|T_t\|_{L_p\to L_r} = 1 \quad \textit{if and only if} \quad e^{-2t} \leq \frac{p-1}{r-1}.$$

Proof. The "only if" part follows verbatim Biane's argument [1997, p. 461]. For the converse, since the special Wick words span $L_p(\Gamma_Q)$, it suffices to prove that if $e^{-2t} \leq (p-1)/(r-1)$ then

$$\left\| T_t\left(\sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i}) \right) \right\|_r \leq \left\| \sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i}) \right\|_p,$$

where $\sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i})$ is a finite linear combination of special Wick words.

But, by Theorem 4.1,

$$\left\| T_t^m\left(\sum_{\underline{i}} \frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}} \sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))} x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})}) \right) \right\|_r$$
$$\le \left\| \sum_{\underline{i}} \frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}} \sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))} x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})}) \right\|_p.$$

Since there is a canonical inclusion $L_p(\Gamma_Q)\subset \prod_{m,\mathcal{U}} L_p(\mathcal{A},\tau_m)$, we have

$$\left\| \sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i}) \right\|_p = \lim_{m,\mathcal{U}} \left\| \sum_{\underline{i}} \frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}} \sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))} x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})}) \right\|_p.$$

Similarly,

$$\left\| T_t\left(\sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i}) \right) \right\|_r$$
$$= \left\| \sum_{\underline{i}} \alpha_{\underline{i}} e^{-t|\underline{i}|} w(\underline{i}) \right\|_r$$
$$= \lim_{m,\mathcal{U}} \left\| \sum_{\underline{i}} \frac{\alpha_{\underline{i}} e^{-t|\underline{i}|}}{m^{d(\underline{i})/2}} \sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))} x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})}) \right\|_r$$
$$= \lim_{m,\mathcal{U}} \left\| T_t^m\left(\sum_{\underline{i}} \frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}} \sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))} x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})}) \right) \right\|_r.$$

The assertion follows immediately. □

This result in particular implies the hypercontractivity results for $\Gamma_q(H)$ due to Biane [1997] and for the free product of $\Gamma_{-1}(\mathbb{R}^n)$ obtained in [Junge et al. 2015]. See also [Królak 2005] for another generalization with the braid relation. Using the standard argument [Biane 1997], the log-Sobolev inequality follows from optimal hypercontractivity bounds. Recall that A is the number operator associated to Γ_Q.

Corollary 4.3 (log-Sobolev inequality). *For any finite linear combination of special Wick words* $f=\sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i})$,

$$\tau_Q(|f|^2 \ln |f|^2) - \|f\|_2^2 \ln \|f\|_2^2 \le 2\tau_Q(fAf^*).$$

4B. ***Derivations.*** Given the $N\times N$ matrix $Q=(q_{ij})$, we define a $2N\times 2N$ matrix Q' by

$$Q' = Q\otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}.$$

Recall that $\mathcal{A}_m(N,\varepsilon)$ is the spin matrix system with Nm generators as in Section 2B and $J_{N,m} = [N]\times[m]$. We can extend the function ε to $J_{2N,m}\times J_{2N,m}$ as follows:

$$\varepsilon'((i,k),(j,l)) = \begin{cases} \varepsilon((i,k),(j,l)) & \text{if } (i,k),(j,l)\in J_{N,m},\\ \varepsilon((i-N,k),(j,l)) & \text{if } 1\le j<N+1\le i\le 2N,\\ \varepsilon((i,k),(j-N,l)) & \text{if } 1\le i<N+1\le j\le 2N,\\ \varepsilon((i-N,k),(j-N,l)) & \text{if } N+1\le i,j\le 2N. \end{cases}$$

In other words, $\varepsilon' = \varepsilon\otimes\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. We may write ε for ε' without causing any ambiguity. Now we define a linear map

$$\begin{aligned}(4\text{-}1)\quad \delta : \mathcal{A}_m(N,\varepsilon)&\to \mathcal{A}_m(2N,\varepsilon'),\\ x_{i_1}(k_1)x_{i_2}(k_2)\cdots x_{i_n}(k_n)&\\ &\mapsto \sum_{\alpha=1}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n),\end{aligned}$$

where $x_{i_1}(k_1)x_{i_2}(k_2)\cdots x_{i_n}(k_n)$ is assumed to be in the reduced form. See also [Lust-Piquard 1999]. It is easy to see that δ is $*$-preserving.

Lemma 4.4. *δ is a derivation, i.e., $\delta(\xi\eta) = \delta(\xi)\eta + \xi\delta(\eta)$ for two words ξ and η.*

Proof. The assertion follows from the fact that δ is the derivative of certain one parameter group of automorphisms; see [Lust-Piquard 1998; 1999; Efraim and Lust-Piquard 2008]. We provide a direct elementary proof here. Note that if ξ and η are two reduced words with no common generators, the derivation property follows easily from (4-1). It remains to verify the derivation property when ξ and η have common generators. Let $\xi = x_{i_1}(k_1)\cdots x_{i_n}(k_n)\in\mathcal{A}_m(N,\varepsilon)$ be a reduced word, and let a be an arbitrary generator. Assume $a = x_{i(\alpha_0)}(k(\alpha_0))$ and write the reduced form of $a\xi$ as $\widetilde{a\xi}$. Then

$$\begin{aligned}\widetilde{a\xi} = \varepsilon((i_1,k_1),(i_{\alpha_0},k_{\alpha_0}))\cdots\varepsilon((i_{\alpha_0-1},k_{\alpha_0-1}),(i_{\alpha_0},k_{\alpha_0}))x_{i_1}(k_1)&\\ \cdots\check{x}_{i_{\alpha_0}}(k_{\alpha_0})\cdots x_{i_n}(k_n),&\end{aligned}$$

where $\check{x}$ means the generator x is omitted in the expression. We have

$$\begin{aligned}\delta(\widetilde{a\xi}) = \varepsilon((i_1,k_1),(i_{\alpha_0},k_{\alpha_0}))\cdots\varepsilon((i_{\alpha_0-1},k_{\alpha_0-1}),(i_{\alpha_0},k_{\alpha_0}))&\\ \times\sum_{\alpha=1,\alpha\neq\alpha_0}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n).&\end{aligned}$$

Here we understand that if $\alpha=\alpha_0-1$ then $i_{\alpha+1}$ is actually $i_{\alpha+2}$ because $x_{i_{\alpha_0}}(k_{\alpha_0})$ is omitted. Similar remark applies when $\alpha=\alpha_0+1$ and we will follow this convention

to ease notation in this proof. On the other hand,

$$\begin{aligned}\delta(a)\xi + a\delta(\xi) &= x_{i_{\alpha_0}+N}(k_{\alpha_0})x_{i_1}(k_1)\cdots x_{i_n}(k_n) + a x_{i_1}(k_1)\cdots x_{i_{\alpha_0}+N}(k_{\alpha_0})\cdots x_{i_n} \\ &\quad + \prod_{j=1}^{\alpha_0-1} \varepsilon((i_j,k_j),(i_{\alpha_0},k_{\alpha_0})) \\ &\quad \times \sum_{\alpha=1,\alpha\neq\alpha_0}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n) \\ &= \prod_{j=1}^{\alpha_0-1} \varepsilon((i_j,k_j),(i_{\alpha_0},k_{\alpha_0})) \\ &\quad \times \sum_{\alpha=1,\alpha\neq\alpha_0}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n).\end{aligned}$$

Here we used the commutation relation given by ε in both equalities. Hence,

$$\delta(\widetilde{a\xi}) = \delta(a)\xi + a\delta(\xi). \tag{4-2}$$

Now assume $\delta(\widetilde{\eta\xi}) = \delta(\eta)\xi + \eta\delta(\xi)$, where both ξ and η are reduced words and the generators of η are all in ξ, i.e., η is a subword of ξ. We want to show that $\delta(\widetilde{a\eta\xi}) = \delta(\widetilde{a\eta})\xi + a\eta\delta(\xi)$, where a is a generator. Note that $\widetilde{a\eta\xi} = a\widetilde{\eta\xi}$. By (4-2) and the induction hypothesis,

$$\begin{aligned}\delta(\widetilde{a\eta\xi}) &= \delta(a)\eta\xi + a\delta(\widetilde{\eta\xi}) \\ &= \delta(a)\eta\xi + a\delta(\eta)\xi + a\eta\delta(\xi) = \delta(\widetilde{a\eta})\xi + a\eta\delta(\xi).\end{aligned}$$

The derivation property is verified when η is a subword of ξ. For arbitrary reduced words ξ and η, using the commutation relation we can write $\eta = \eta_1\eta_2$ so that η_1 and ξ have no common generators and the generators of η_2 are in ξ. Then

$$\begin{aligned}\delta(\widetilde{\eta\xi}) &= \delta(\eta_1\widetilde{\eta_2\xi}) = \delta(\eta_1)\eta_2\xi + \eta_1\delta(\widetilde{\eta_2\xi}) = \delta(\eta_1)\eta_2\xi + \eta_1\delta(\eta_2)\xi + \eta_1\eta_2\delta(\xi) \\ &= \delta(\eta_1\eta_2)\xi + \eta_1\eta_2\delta(\xi) = \delta(\eta)\xi + \eta\delta(\xi).\end{aligned}$$

□

This lemma implies in particular that $\delta(\xi)$ can be defined by (4-1) and is equal to $\delta(\widetilde{\xi})$ even if ξ is not a reduced word. We will simply write $\delta(\xi)$ for any word ξ in the following. If we denote by A^m the number operator associated to the spin system $\mathcal{A}_m(N,\varepsilon)$, the gradient form is defined as

$$\Gamma^m(f,g) = \tfrac{1}{2}(A^m(f^*)g + f^*A^m(g) - A^m(f^*g))$$

for $f, g \in \mathcal{A}_m(N,\varepsilon)$. The superscript m is used to distinguish the operators from their counterparts defined for the limiting algebra Γ_Q. We may simply omit this superscript if there is no ambiguity.

Lemma 4.5. *Let $f, g \in \mathcal{A}_m(N, \varepsilon)$. Then*

$$\Gamma(f, g) = E(\delta(f)^*\delta(g)),$$

where $E : \mathcal{A}_m(2N, \varepsilon) \to \mathcal{A}_m(N, \varepsilon)$ is the conditional expectation satisfying

$$E(x_B) = \delta_{B\cap(\{N+1,\dots,2N\}\times[m]),\varnothing} x_B$$

for a reduced word x_B.

Proof. By linearity, it suffices to check $\Gamma(f, g) = E(\delta(f)^*\delta(g))$ if f and g are reduced words in $\mathcal{A}_m(N, \varepsilon)$. Let $X_B = x_{i_1}(k_1)\cdots x_{i_n}(k_n)$ and $X_C = x_{j_1}(l_1)\cdots x_{j_s}(l_s)$ be two reduced words, where $B, C \subset [N]\times[m]$ consist of (i_α, k_α) and (j_β, l_β) respectively. By the derivation property (4-1),

$$E(\delta(X_B)^*\delta(X_C)) = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{s} E(x_{i_n}(k_n)\cdots x_{i_\alpha+N}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta+N}(l_\beta)\cdots x_{j_s}(l_s)).$$

We claim that the only nonzero terms in the above sum are those with $(i_\alpha, k_\alpha) = (j_\beta, l_\beta)$. Indeed, the conditional expectation simply computes the trace of generators with subscript greater than N in the reduced form of

$$x_{i_n}(k_n)\cdots x_{i_\alpha+N}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta+N}(l_\beta)\cdots x_{j_s}(l_s).$$

Thus $x_{i_\alpha+N}(k_\alpha)$ and $x_{j_\beta+N}(l_\beta)$ have to be the same to cancel out in order to contribute to the sum. It follows that

$$\begin{aligned} E(\delta(X_B)^*\delta(X_C)) &= \sum_{\alpha,\beta:(i_\alpha,k_\alpha)=(j_\beta,l_\beta)} E\big(x_{i_n}(k_n)\cdots x_{i_\alpha+N}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1) \\ &\qquad \cdots x_{j_\beta+N}(l_\beta)\cdots x_{j_s}(l_s)\big) \\ &= \sum_{\alpha,\beta:(i_\alpha,k_\alpha)=(j_\beta,l_\beta)} x_{i_n}(k_n)\cdots x_{i_\alpha}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta}(l_\beta)\cdots x_{j_s}(l_s). \end{aligned} \tag{4-3}$$

Here we used the extended commutation relation on $\mathcal{A}(2N, \varepsilon)$ given by ε in the last equality. Since X_B and X_C are reduced, given $(i_\alpha, k_\alpha) \in B$ there is at most one $(j_\beta, l_\beta) \in C$ such that they are equal, and vice versa. We see that there are $|B \cap C|$ terms in the sum of (4-3). Hence, we find

$$E(\delta(X_B)^*\delta(X_C)) = |B \cap C| X_B^* X_C.$$

On the other hand,

$$\begin{aligned} \Gamma(X_B, X_C) &= \tfrac{1}{2}(A(X_B^*)X_C + X_B^* A(X_C) - A(X_B^* X_C)) \\ &= \tfrac{1}{2}(|B| + |C| - |B\triangle C|) \\ &\qquad \times x_{i_n}(k_n)\cdots x_{i_\alpha}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta}(l_\beta)\cdots x_{j_s}(l_s). \end{aligned}$$

Note that we have the same word here as the summand of (4-3). Since $2|B\cap C| = |B|+|C|-|B\triangle C|$, we must have

$$\Gamma(X_B, X_C) = |B\cap C| X_B^* X_C = E(\delta(X_B)^*\delta(X_C)). \qquad \square$$

Let $w(\underline{i}) \in H_Q^s$ be a special Wick word with length $s \in \mathbb{Z}_+$. We define a linear map $\delta : H_Q^s \to H_{Q'}^s$

$$\delta(w(\underline{i})) = \left(\frac{1}{m^{s/2}} \sum_{\underline{k}:\sigma(\underline{k})\in P_1(s)} \delta^m[x_{i_1}(k_1)\cdots x_{i_s}(k_s)]\right)^{\bullet}, \tag{4-4}$$

where δ^m is the derivation defined in (4-1). Here we used Remark A.1 implicitly. Note that δ^m is bounded when acting on words with fixed length s although it is not uniformly (in m) bounded on $\mathcal{A}_m$. Hence $\delta = (\delta^m)^{\bullet}$ is well-defined on H_Q^s. Since $L_2(\Gamma_Q) = \bigoplus_{s=0}^{\infty} H_Q^s$, we can define δ on each H_Q^s by (4-4). By definition, δ is densely defined on $\mathcal{F}_Q = L_2(\Gamma_Q)$ and $\mathrm{Dom}(\delta) = \mathrm{Dom}(A)$ can be identified with the linear span of special Wick words with finite length, where A is the number operator on $L_2(\Gamma_Q)$. Since each $w(\underline{i})$ is actually in Γ_Q^{∞}, $\delta(w(\underline{i}))$ is in $\Gamma_{Q'}^{\infty}$.

Proposition 4.6. *$\delta : L_2(\Gamma_Q) \to L_2(\Gamma_{Q'})$ is a closed derivation.*

Proof. Let $P_s : L_2(\Gamma_Q) \to H_Q^s$ and $P_s' : L_2(\Gamma_{Q'}) \to H_{Q'}^s$ be the orthogonal projections. Suppose $x_n \in \mathrm{Dom}(\delta)$, $\lim_{n\to\infty} \|x_n\|_2 = 0$ and $\lim_{n\to\infty} \|\delta(x_n) - y\|_2 = 0$. Then $P_s x_n \to 0$ for each $s \in \mathbb{Z}_+$. It follows that

$$P_s'\delta(x_n) = \delta(P_s(x_n)) \to 0 \quad \text{as } n \to \infty.$$

But $P_s'\delta(x_n) \to P_s' y$, we find $P_s' y = 0$ and thus $y = 0$. Hence δ is closed. The derivation property follows from the definition (4-4), (4-1) and Remark A.1. $\square$

Denote by $\mathcal{A}_{\mathcal{U}}(N)$ the von Neumann algebra ultraproduct of $\mathcal{A}_m(N)$. Then $E = (E^m)^{\bullet} : \mathcal{A}_{\mathcal{U}}(2N) \to \mathcal{A}_{\mathcal{U}}(N)$ is the canonical conditional expectation, where $E^m : \mathcal{A}_m(2N) \to \mathcal{A}_m(N)$ is given in Lemma 4.5. Since $\Gamma_Q \subset \mathcal{A}_{\mathcal{U}}(N)$ as a von Neumann subalgebra, there is a trace-preserving conditional expectation $E : \Gamma_{Q'} \to \Gamma_Q$ which extends to contractions on L_p for $1 \le p < \infty$. Recall that $\Gamma(\cdot,\cdot)$ is the gradient form associated with the number operator A on Γ_Q.

Proposition 4.7. *Let $f, g \in \mathrm{Dom}(\delta)$. Then*

$$\Gamma(f, g) = E(\delta(f)^*\delta(g)).$$

Proof. By linearity, it suffices to check the claim for $f = w(\underline{i})$ and $g = w(\underline{i}')$. By the construction of conditional expectation, the proof of Lemma 4.5 and the

construction of the Ornstein–Uhlenbeck semigroup on $L_2(\Gamma_Q)$ in Section 3D,

$$\begin{aligned}
&E[\delta(w(\underline{i}))^*\delta(w(\underline{i}'))]\\
&= \left(\frac{1}{m^{(d+d')/2}} E^m\left(\delta^m\left(\sum_{\underline{k}:\sigma(\underline{k})\in P_1(d)} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\right)^*\right.\right.\\
&\qquad\left.\left.\times\, \delta^m\left(\sum_{\underline{k}':\sigma(\underline{k}')\in P_1(d')} x_{i'_1}(k'_1)\cdots x_{i'_{d'}}(k'_{d'})\right)\right)\right)^{\bullet}\\
&= \tfrac{1}{2}[A(w(\underline{i}^*)w(\underline{i}') + w(\underline{i})^*A(w(\underline{i}')) - A(w(\underline{i})^*w(\underline{i}'))] = \Gamma(w(\underline{i}), w(\underline{i}')),
\end{aligned}$$

where A is the number operator on Γ_Q. □

4C. ***Riesz transforms.*** Lust-Piquard [1998] showed the boundedness of Riesz transforms for the general spin system. Let $T \in \mathcal{A}_m(N,\varepsilon)$ with $\tau_m(T) = 0$. Recall that the Riesz transforms satisfy $R_j(T) = D_j(A^m)^{-1/2}(T)$, where D_j is the annihilation operator and $A^m = \sum_{j=1}^{Nm} D_j^* D_j$ is the number operator for the spin system $\mathcal{A}_m(N,\varepsilon)$. By [Lust-Piquard 1998, Lemma 3.2 and Proposition 1.3], we have

$$\widetilde{K}_{p'}^{-1}\|T\|_p \le \left\|\sum_{j=1}^{Nm} P_j R_j(T)\right\|_p \le \widetilde{K}_p\|T\|_p \quad \text{for } 1<p<\infty, \tag{4-5}$$

where $\widetilde{K}_p = O(p^3/(p-1)^{3/2})$, $1/p+1/p'=1$, and P_j is a certain tensor of Pauli matrices in the general spin system; see [Lust-Piquard 1998, Definition 2.1]. It is known that $\|\sum_{j=1}^{Nm} P_j R_j(T)\|_p = \|\delta^m(A^m)^{-1/2}(T)\|_p$ (see [Lust-Piquard 1999, p. 547]), where δ^m is the derivation defined in (4-1). By considering $T = (A^m)^{1/2} f$, (4-5) can be rewritten as

$$\widetilde{K}_{p'}^{-1}\|(A^m)^{1/2} f\|_p \le \|\delta^m(f)\|_p \le \widetilde{K}_p\|(A^m)^{1/2} f\|_p. \tag{4-6}$$

Now it is easy to recover the main result in [Lust-Piquard 1999]. Recall that A is the number operator on Γ_Q.

Theorem 4.8 (Lust-Piquard). *Let $1<p<\infty$ and $1/p+1/p'=1$. Let δ be defined by (4-4). Then, for any $f \in \mathrm{Dom}(\delta)$,*

$$\widetilde{K}_{p'}^{-1}\|A^{1/2} f\|_p \le \|\delta(f)\|_p \le \widetilde{K}_p\|A^{1/2} f\|_p,$$

where $\widetilde{K}_p = O(p^3/(p-1)^{3/2})$.

Proof. We may assume without loss of generality that $f = \sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i})$ is a finite linear combination of special Wick words. Write $w(\underline{i}) = (X(\underline{i}, m))^{\bullet}$. Then

$$\|\delta(f)\|_p = \left\|\sum_{\underline{i}} \alpha_{\underline{i}}\delta(w(\underline{i}))\right\|_p = \lim_{m,\mathcal{U}} \left\|\sum_{\underline{i}} \alpha_{\underline{i}}\delta^m(X(\underline{i}, m))\right\|_p.$$

Similarly,

$$\|A^{1/2}f\|_p = \lim_{m,\mathcal{U}} \Big\| \sum_{\underline{i}} \alpha_{\underline{i}} \sqrt{|\underline{i}|}\, X(\underline{i}, m) \Big\|_p = \lim_{m,\mathcal{U}} \|(A^m)^{1/2} f\|_p.$$

The assertion follows from (4-6) with a limiting procedure. □

In fact, we can give more precise estimates using the gradient form. Let

$$G_p = L_p\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [2N]^s, s \in \mathbb{N}, 1 \le i_k \le N \text{ for all but at most one } k\}.$$

Since $L_p(\Gamma_Q) \subset G_p \subset L_p(\Gamma_{Q'})$, we have $E : G_p \to L_p(\Gamma_Q)$ given by the restriction of the conditional expectation $E : \Gamma_{Q'} \to \Gamma_Q$. If $f \in \Gamma_{Q'}$, we define $\|f\|_{L_p^c(E)} = \|E(f^*f)^{1/2}\|_p$ and $\|f\|_{L_p^r(E)} = \|f^*\|_{L_p^c(E)}$. The conditional $L_p(\Gamma_{Q'})$ space is

$$L_p^{rc}(E) = \begin{cases} L_p^r(E) + L_p^c(E) & \text{if } 1 \le p \le 2, \\ L_p^r(E) \cap L_p^c(E) & \text{if } 2 \le p \le \infty. \end{cases}$$

Define G_p^r (resp. G_p^c) as the space of G_p with the norm inherited from $L_p^r(E)$ (resp. $L_p^c(E)$). Now we follow [Junge et al. 2014] to derive a Khintchine-type inequality. First, since $E : \Gamma_{Q'} \to \Gamma_Q$ extends to contractions on L_p for $1 \le p < \infty$, we have, for $f \in L_p(\Gamma_{Q'})$ and $2 \le p < \infty$,

$$\max\big\{\|E(f^*f)^{1/2}\|_p, \|E(ff^*)^{1/2}\|_p\big\} \le \|f\|_{L_p}. \tag{4-7}$$

This means that $L_p(\Gamma_{Q'}) \subset L_p^{rc}(E)$ contractively for $2 \le p < \infty$.

Lemma 4.9. *Let $E : G_p \to L_p(\Gamma_Q)$ be as above. Then, for $2 \le p < \infty$,*

$$\|f\|_{G_p} \le C\sqrt{p}\max\big\{\|E(f^*f)^{1/2}\|_p, \|E(ff^*)^{1/2}\|_p\big\} \le C\sqrt{p}\|f\|_{G_p},$$

and for $1 < p \le 2$,

$$\|f\|_{G_p} \le \inf_{\substack{f=g+h \\ g\in G_p^c, h\in G_p^r}} \big\{\|E(g^*g)^{1/2}\|_p + \|E(hh^*)^{1/2}\|_p\big\} \le C\sqrt{\frac{p}{p-1}}\|f\|_{G_p}.$$

Proof. Let $2 \le p < \infty$. The right inequality is a special case of (4-7). For the left inequality, let $n \in \mathbb{N}$ and $\underline{i} \in [2N]^s$. For $j = 1, \dots, n$, define

$$\phi_j : [2N]^s \to [2N]^{ns}, \quad \phi_j(\underline{i}) = \underline{0} \sqcup \cdots \underline{0} \sqcup \underline{i} \sqcup \underline{0} \cdots \sqcup \underline{0},$$

where $\underline{i}$ occurs in the j-th position. Put $\widetilde{\pi}_j(w(\underline{i})) = w(\phi_j(\underline{i}))$, where $w(\phi_j(\underline{i}))$ is the special Wick word associated to $\phi_j(\underline{i})$. Define

$$\pi_n : \Gamma_{Q'} \to \Gamma_{Q'\otimes \mathbb{1}_n}, \quad \pi_n(w(\underline{i})) = \frac{1}{\sqrt{n}} \sum_{j=1}^n w(\phi_j(\underline{i})).$$

Here, $\mathbb{1}_n$ is the $n \times n$ matrix with all entries equal to 1. The map π_n extends to a trace-preserving $*$-homomorphism. Alternatively, one may define π_n via the second

quantization functor as in [Lust-Piquard 1999]. It is crucial to observe that the $\widetilde{\pi}_j(w(\underline{i}))$, where $j=1,\dots,n$, are fully independent over $\Gamma_{Q\otimes\mathbb{1}_n}$ (see [Junge and Zeng 2013]) if $w(\underline{i})\in G_p$, as can be checked from the definition of $E_{\Gamma_Q}:\Gamma_{Q'}\to\Gamma_Q$. We may assume f is a finite linear combination of special Wick words in G_p. By the noncommutative Rosenthal inequality [Junge and Xu 2008; Junge and Zeng 2013],

$$\begin{aligned}&\|\pi_n(f)\|_p\\ &\le \frac{Cp}{\sqrt{n}}\Big(\sum_{j=1}^n \|\widetilde{\pi}_j(f)\|_p^p\Big)^{\frac{1}{p}}\\ &\quad+\frac{C\sqrt{p}}{\sqrt{n}}\max\left\{\left\|\Big(\sum_{j=1}^n E[\widetilde{\pi}_j(f)^*\widetilde{\pi}_j(f)]\Big)^{\frac12}\right\|_p,\left\|\Big(\sum_{j=1}^n E[\widetilde{\pi}_j(f)\widetilde{\pi}_j(f)^*]\Big)^{\frac12}\right\|_p\right\}.\end{aligned}$$

We have extended the conditional expectation $E:\Gamma_{Q'}\to\Gamma_Q$ to $E:\Gamma_{Q'\otimes\mathbb{1}_n}\to\Gamma_{Q\otimes\mathbb{1}_n}$. Note that $E[\widetilde{\pi}_j(f)^*\widetilde{\pi}_j(f)]=\widetilde{\pi}_j[E(f^*f)]$ and $\|\widetilde{\pi}_j(f)\|_p=\|f\|_p$. Sending $n\to\infty$, for $2<p<\infty$, we have

$$\|f\|_{G_p}\le C\sqrt{p}\max\big\{\|E(f^*f)^{1/2}\|_p,\|E(ff^*)^{1/2}\|_p\big\}. \tag{4-8}$$

For the case $1<p<2$, we argue by duality. Define the orthogonal projection $P:L_2^{rc}(E)\to G_2\cap L_2^{rc}(E)$. By orthogonality, for $g\in\Gamma_{Q'}$,

$$E(g^*g)=E(Pg^*Pg)+E(P^\perp g^*P^\perp g)\ge E(Pg^*Pg).$$

Similarly, $E(gg^*)\ge E(PgPg^*)$. Since

$$\max\big\{\|E(Pg^*Pg)^{1/2}\|_p,\|E(PgPg^*)^{1/2}\|_p\big\}\le\max\big\{\|E(g^*g)^{1/2}\|_p,\|E(gg^*)^{1/2}\|_p\big\},$$

we deduce from (4-8) that P extends to a bounded projection with norm

$$\|P:L_p^{rc}(E)\to L_p(\Gamma_{Q'})\|\le C\sqrt{p}$$

for $2\le p<\infty$. For $1<p\le 2$ and $f\in G_2$, since $P^*=P$, we have by duality

$$\|f\|_{L_p^{rc}(E)}=\|Pf\|_{L_p^{rc}(E)}\le C\sqrt{p'}\|f\|_{L_p(\Gamma_{Q'})},$$

where $1/p+1/p'=1$. By density, this inequality extends to $f\in G_p$. It suffices to consider the decomposition of $f\in G_p$ in $G_p^c+G_p^r$ when we compute $\|f\|_{L_p^{rc}(E)}$. This gives the right inequality. The left inequality follows from duality and (4-7). □

Remark 4.10. In fact, the above argument also shows that G_p is complemented in $L_p(\Gamma_{Q'})$. Morally speaking, G_p is a Γ_Q–Γ_Q bimodule corresponding to differential forms of order one.

Corollary 4.11. (a) *Let $2\le p<\infty$. Then, for every $f\in\mathrm{Dom}(A)$,*

$$c_p^{-1}\|A^{1/2}f\|_p\le\max\big\{\|\Gamma(f,f)^{1/2}\|_p,\|\Gamma(f^*,f^*)^{1/2}\|_p\big\}\le K_p\|A^{1/2}f\|_p,$$

where $c_p=O(p^2)$ and $K_p=O(p^{3/2})$.

(b) *Let* $1 < p \le 2$. *Then, for every* $f \in \mathrm{Dom}(A)$,

$$K_{p'}^{-1}\|A^{1/2}f\|_p \le \inf_{\substack{\delta(f)=g+h\\ g\in G_p^c, h\in G_p^r}} \big\{\|E(g^*g)^{1/2}\|_p + \|E(hh^*)^{1/2}\|_p\big\} \le C_p\|A^{1/2}f\|,$$

where $K_{p'} = O(1/(p-1)^{3/2})$ *and* $C_p = O(1/(p-1)^2)$.

Proof. Note that $\delta(f) \in G_p$ if $f \in \mathrm{Dom}(A)$. Since $E(\delta(f)^*\delta(f)) = \Gamma(f, f)$, using Lemma 4.9 for $2 \le p < \infty$, we have

$$\|\delta(f)\|_p \le C\sqrt{p}\max\big\{\|\Gamma(f, f)^{1/2}\|_p, \|\Gamma(f^*, f^*)^{1/2}\|_p\big\} \le C\sqrt{p}\|\delta(f)\|_p.$$

Now apply Theorem 4.8 to conclude (a). For the constants, $K_p = O(p^{3/2})$ is trivial. Since $\widetilde{K}_{p'} = O(p'^3/(p'-1)^{3/2}) = O(p^{3/2})$, we have $c_p \le O(p^2)$. Assertion (b) follows similarly using Lemma 4.9 and Theorem 4.8. □

Compared with Theorem 4.8, which was proved in [Lust-Piquard 1999], this result is closer to Lust-Piquard's original formulation of the Riesz transforms on the Walsh system and the fermions given in [Lust-Piquard 1998]. In particular, we get the exact order of constants as in [Lust-Piquard 1998].

4D. ***L_p Poincaré inequalities.*** Efraim and Lust-Piquard [2008] proved that the L_p Poincaré inequalities ($2 \le p < \infty$),

$$\|f - \tau_m(f)\|_p \le C\sqrt{p}\max\big\{\|\Gamma^m(f, f)^{1/2}\|_p, \Gamma^m(f^*, f^*)^{1/2}\|_p\big\}, \tag{4-9}$$

hold for Walsh systems and CAR algebras. In fact, the same proof also works for the general spin matrix system $\mathcal{A}_m$ with some technical variants as shown in [Lust-Piquard 1998]. Indeed, Lemmas 6.2–6.5 in [Efraim and Lust-Piquard 2008] hold for the general spin systems, from which (4-9) follows. Recall that we denote by A the number operator on Γ_Q.

Theorem 4.12. *Let* $2 \le p < \infty$. *Then, for every* $f \in \mathrm{Dom}(A)$,

$$\|f - \tau_Q(f)\|_p \le C\sqrt{p}\max\big\{\|\Gamma(f, f)^{1/2}\|_p, \Gamma(f^*, f^*)^{1/2}\|_p\big\}.$$

Proof. Assume without loss of generality that $f = \sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i}) = (f^m)^\bullet$ is a finite linear combination of special Wick words. Note that $E(\delta(f)^*\delta(f)) = (E^m[\delta^m(f^m)\delta^m(f^m)])^\bullet$. Then the assertion follows from (4-9) and a limiting procedure as for Theorems 4.2 and 4.8 with the help of Lemma 4.5 and Proposition 4.7. □

5. Strong solidity

5A. ***CCAP.*** Let $\Gamma_q(H)$ be the q-Gaussian von Neumann algebra associated to a real Hilbert space H with $\dim H \ge 2$; see, e.g., [Bożejko et al. 1997] for more information on $\Gamma_q(H)$. Avsec showed that $\Gamma_q(H)$ for $-1 < q < 1$ has the weak* completely contractive approximation property (w*CCAP) in [Avsec 2011]. In

particular, $\Gamma_q(H)$ is weakly amenable. Our goal here is to prove that Γ_Q also has w*CCAP if $\max_{1\le i,j\le N}|q_{ij}|<1$. Our argument is based on Avsec's result.

Assume that $\max_{i,j}|q_{ij}|<1$. We may find q such that $\max_{i,j}|q_{ij}|<q<1$. Let $Q=q\widetilde{Q}$, where $\widetilde{Q}=(\widetilde{q}_{ij})$ satisfies $\max_{i,j}|\widetilde{q}_{ij}|<1$. For $h\in H$, let $c^q(h)$ and $(c^q)^*(h)$ be the creation and annihilation operators, respectively, acting on the q-Fock space $\mathcal{F}_q(H)$, where $\dim H=N$. We write the q-Gaussian variables as $s^q(h)=c^q(h)+(c^q)^*(h)$. In particular, for an orthonormal basis (o.n.b.) (e_j) of H, we write $s_j^q=s^q(e_j)$. Similarly, we write $s^Q(h)=c^Q(h)+(c^Q)^*(h)$ for the mixed q-Gaussian variables of Γ_Q; see [Lust-Piquard 1999]. In particular, $s_j^Q=s^Q(e_j)$. We write $x_{i,j}=s^{\widetilde{Q}\otimes\mathbb{1}_n}(f_i\otimes e_j)$, where (f_i) is an o.n.b. of ℓ_2^N, and (e_j) is an o.n.b. of ℓ_2^n. Clearly, the $x_{i,j}$'s generate $\Gamma_{\widetilde{Q}\otimes\mathbb{1}_n}$. We first construct an "approximate comultiplication" for Γ_Q.

Proposition 5.1. *Let $\pi_{\mathcal{U}}:\Gamma_Q\to\prod_{m,\mathcal{U}}\Gamma_q(\ell_2^m)\,\overline{\otimes}\,\Gamma_{\widetilde{Q}\otimes\mathbb{1}_m}$ be a $*$-homomorphism given by*

$$\pi_{\mathcal{U}}(s_i^Q)=\left(\frac{1}{\sqrt{m}}\sum_{k=1}^m s_k^q\otimes x_{i,k}\right)^{\bullet}.$$

Then $\pi_{\mathcal{U}}$ is trace-preserving. Therefore, Γ_Q is isomorphic to the von Neumann algebra generated by $\pi_{\mathcal{U}}(s_i^Q)$.

Proof. Let d be an even integer. By the moment formula (3-2),

$$\begin{aligned}\sum_{\underline{k}\in[m]^d}\tau_q\otimes\tau_{\widetilde{Q}\otimes\mathbb{1}_m}[(s_{k_1}^q\cdots s_{k_d}^q)\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})]&\\ =\sum_{\sigma\in P_2(d),\sigma\le\sigma(\underline{i})}\ \sum_{\sigma(\underline{k})=\sigma}q^{\#I(\sigma)}\prod_{\{r,t\}\in I(\sigma)}\widetilde{q}(i(e_r),i(e_t))&\\ =\sum_{\sigma\in P_2(d),\sigma\le\sigma(\underline{i})}\ \sum_{\sigma(\underline{k})=\sigma}\prod_{\{r,t\}\in I(\sigma)}q(i(e_r),i(e_t)),&\end{aligned}$$

where $I(\sigma)$ is the set of inversions for the partition σ. Counting the number of $\underline{k}$ with $\sigma(\underline{k})=\sigma$, we have

$$\tau_{\mathcal{U}}\left(\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d}(s_{k_1}^q\cdots s_{k_d}^q)\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})\right)^{\bullet}=\sum_{\substack{\sigma\in P_2(d)\\ \sigma\le\sigma(\underline{i})}}\ \prod_{\{r,t\}\in I(\sigma)}q(i(e_r),i(e_t)).$$

This coincides with $\tau_Q(s_{i_1}^Q\cdots s_{i_d}^Q)$ given by (3-2). □

Now we want to understand the image of Wick words of Γ_Q under $\pi_{\mathcal{U}}$. We need a Wick word decomposition result similar to Theorem 3.8. For $\underline{i}\in[N]^d$, we define

$$w^s(\underline{i})=\left(\frac{1}{m^{d/2}}\sum_{\underline{k}:\sigma(\underline{k})\in P_1(d)}(s_{k_1}^q\cdots s_{k_d}^q)\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})\right)^{\bullet}. \tag{5-1}$$

Proposition 5.2. *Following the notation of Proposition 5.1, we have*

$$\pi_{\mathcal{U}}(s_{i_1}^{Q}\cdots s_{i_d}^{Q}) = \sum_{\substack{\sigma\in P_{1,2}(d)\\ \sigma\le\sigma(\underline{i})}} w_\sigma^s(\underline{i}).$$

Here, $w_\sigma^s(\underline{i}) = f_\sigma(\underline{i})w^s(\underline{i}_{\mathrm{np}})$, $f_\sigma(\underline{i})$ *and* $\underline{i}_{\mathrm{np}}$ *are the same as those in Proposition 3.10.*

Proof. Following verbatim the argument for Theorem 3.8, we have

$$\pi_{\mathcal{U}}(s_{i_1}^{Q}\cdots s_{i_d}^{Q}) = \sum_{\sigma\in P_{1,2}(d)} w_\sigma^s(\underline{i}).$$

Here we have

$$w_\sigma^s(\underline{i}) = \left(\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma} E_{\mathcal{N}_s(\underline{k})}[(s_{k_1}^q\cdots s_{k_d}^q)\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})]\right)^{\bullet},$$

and $\mathcal{N}_s(\underline{k})$ is the von Neumann algebra generated by all the $s_{k_\alpha}^q\otimes x_{i_\alpha,k_\alpha}$'s, where the k_α's correspond to singleton blocks in $\underline{k}$. To simplify the conditional expectation in the ultraproduct, we denote by $\mathcal{N}_s^1(\underline{k})$ and $\mathcal{N}_s^2(\underline{k})$ the von Neumann algebras generated by the $s_{k_\alpha}^q\otimes\Gamma_{\widetilde{Q}\otimes\mathbb{1}_m}$'s and $\Gamma_q(\ell_2^m)\otimes x_{i_\alpha,k_\alpha}$'s, respectively, where the k_α's correspond to singleton blocks in $\underline{k}$. Clearly, $\mathcal{N}_s(\underline{k})\subset\mathcal{N}_s^1(\underline{k})\cap\mathcal{N}_s^2(\underline{k})$. We claim

$$E_{\mathcal{N}_s^2(\underline{k})}(1\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})) = \begin{cases} f_{\sigma,\widetilde{Q}}(\underline{i})1\otimes(x_{j_1,l_1}\cdots x_{j_s,l_s}) & \text{if } \sigma=\sigma(\underline{k})\le\sigma(\underline{i}),\\ 0 & \text{otherwise,}\end{cases} \tag{5-2}$$

where $(l_1,\dots,l_s)$ is obtained by deleting pair blocks in $\underline{k}$, which also gives the corresponding $(j_1,\dots,j_s)$, and

$$f_{\sigma,\widetilde{Q}}(\underline{i}) = \prod_{\{r,t\}\in I_p(\sigma)}\widetilde{q}(i(e_r),i(e_t))\prod_{\{r,t\}\in I_{\mathrm{sp}}(\sigma)}\widetilde{q}(i(e_r),i(e_t)).$$

Unlike in the matrix models, the x_{i_α,k_α}'s do not have commutation relations. We check (5-2) by calculating the inner product of $E_{\mathcal{N}_s^2(\underline{k})}(1\otimes x_{i_1,k_1}\cdots x_{i_d,k_d})$ and monomials generated by the $1\otimes x_{i_\alpha,k_\alpha}$'s in $\mathcal{N}_s^2(\underline{k})$. Let $1\otimes x_{i'_1,k'_1}\cdots x_{i'_n,k'_n}\in\mathcal{N}_s^2(\underline{k})$ be a monomial. Since $E_{\mathcal{N}_s^2(\underline{k})}$ is trace-preserving, by the moment formula (3-2) for mixed q-Gaussian algebras,

$$\tau_{\widetilde{Q}\otimes\mathbb{1}_m}[x_{i'_n,k'_n}\cdots x_{i'_1,k'_1}E_{\mathcal{N}_s^2(\underline{k})}(x_{i_1,k_1}\cdots x_{i_d,k_d})] = \begin{cases} f_{\sigma,\widetilde{Q}}(\underline{i})\tau_{\widetilde{Q}\otimes\mathbb{1}_m}(x_{i'_n,k'_n}\cdots x_{i'_1,k'_1}x_{j_1,l_1}\cdots x_{j_s,l_s}) & \text{if } \sigma=\sigma(\underline{k})\le\sigma(\underline{i}),\\ 0 & \text{otherwise.}\end{cases}$$

Hence (5-2) is verified. Similarly, it can be checked that

$$E_{\mathcal{N}_s^1(\underline{k})}((s_{k_1}^q\cdots s_{k_d}^q)\otimes 1) = q^{\#I_p(\sigma)+\#I_{\mathrm{sp}}(\sigma)}s_{l_1}^q\cdots s_{l_s}^q.$$

Note that $f_\sigma(\underline{i}) = q^{\#I_p(\sigma)+\#I_{\mathrm{sp}}(\sigma)}f_{\sigma,\widetilde{Q}}(\underline{i})$. The assertion follows from the fact that $E_{\mathcal{N}_s(\underline{k})} = E_{\mathcal{N}_s(\underline{k})}E_{\mathcal{N}_s^1(\underline{k})}E_{\mathcal{N}_s^2(\underline{k})}$. □

Proposition 5.3. *$\pi_{\mathcal{U}}$ extends to an isomorphism*

$$L_2(\Gamma_Q) \cong L_2\text{-}\operatorname{span}\{w^s(\underline{i}) : \underline{i} \in [N]^d, d \in \mathbb{Z}_+\}.$$

Proof. Put $H_W = L_2\text{-}\operatorname{span}\{w^s(\underline{i}) : \underline{i} \in [N]^d, d \in \mathbb{Z}_+\}$. By Proposition 5.2, we know that $\pi_{\mathcal{U}}(L_2(\Gamma_Q)) \subset H_W$. The converse containment follows from the same induction argument as for Proposition 3.13. □

Remark 5.4. In fact, one can prove that $\pi_{\mathcal{U}}(w(\underline{i})) = w^s(\underline{i})$ using the Fock space representation. Since we do not need this fact, we leave it to the reader.

Now we are ready for the first main result of this section.

Theorem 5.5. *Γ_Q has the weak* completely contractive approximation property for all Q with $\max_{1\le i,j\le N}|q_{ij}| < 1$.*

Proof. Let H be a real Hilbert space and $-1 < q < 1$. In [Avsec 2011], Avsec proved that there exists a net of finite-rank maps $\varphi_\alpha(A)$ which converges to the identity map on $\Gamma_q(H)$ in the point-weak* topology and such that $\|\varphi_\alpha(A)\|_{\mathrm{cb}} \le 1+\varepsilon$ for some prescribed ε. Here, $\|\cdot\|_{\mathrm{cb}}$ is the completely bounded norm and $\varphi_\alpha(A)$ only depends on the number operator A on $\Gamma_q(H)$. Let $Q = q\widetilde{Q}$ as above. Consider the diagram

$$\begin{array}{ccc} \Gamma_Q & \xrightarrow{\pi_{\mathcal{U}}} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m) \,\overline{\otimes}\, \Gamma_{\widetilde{Q}\otimes \mathbb{1}_m} \\ \Big\downarrow{\scriptstyle \psi_\alpha} & & \Big\downarrow{\scriptstyle \varphi_\alpha(A)\otimes \mathrm{id}} \\ \Gamma_Q & \xrightarrow{\pi_{\mathcal{U}}} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m) \,\overline{\otimes}\, \Gamma_{\widetilde{Q}\otimes \mathbb{1}_m} \end{array}$$

where we define $\psi_\alpha = \pi_{\mathcal{U}}^{-1} \circ (\varphi_\alpha(A)\otimes \mathrm{id}) \circ \pi_{\mathcal{U}}$. Here, $\varphi_\alpha(A)\otimes \mathrm{id}$ is well-defined on the ultraproduct of von Neumann algebras because it is uniformly bounded in each $\Gamma_q(\ell_2^m) \,\overline{\otimes}\, \Gamma_{\widetilde{Q}\otimes \mathbb{1}_m}$. By an argument similar to that in Section 3D, $\varphi_\alpha(A)\otimes \mathrm{id}$ is a normal map. Note that ψ_α is well-defined because $\pi_{\mathcal{U}}$ is injective and $\varphi_\alpha(A)\otimes \mathrm{id}$ acts as a multiplier. We claim that ψ_α is the desirable completely contractive approximation of identity. By construction, the only nontrivial thing to check is that ψ_α is of finite rank. To this end, it suffices to show that $\varphi(A)_\alpha \otimes \mathrm{id}$ restricted to

$$\pi_{\mathcal{U}}(L_2(\Gamma_Q)) = L_2\text{-}\operatorname{span}\{w^s(\underline{i}) : \underline{i} \in [N]^d, d \in \mathbb{N}\}$$

is of finite rank thanks to Proposition 5.1 and 5.3. Since $\varphi_\alpha(A)$ is of finite rank, suppose its range is $\operatorname{span}\{s_{k_1}^q \cdots s_{k_n}^q : \sigma(\underline{k}) \in \bigcup_{n\in\mathbb{N}} P_1(n), \underline{k} \in B\}$ for some finite set B. Then the range of $\varphi_\alpha(A)\otimes \mathrm{id}\,|_{\pi_{\mathcal{U}}(L_2(\Gamma_Q))}$ is

$$\operatorname{span}\left\{ s_{k_1}^q \cdots s_{k_n}^q \otimes x_{i_1,k_1} \cdots x_{i_n,k_n} : \sigma(\underline{k}) \in \bigcup_{n\in\mathbb{N}} P_1(n), \underline{k} \in B \right\}.$$

Therefore $\varphi_\alpha(A)\otimes \mathrm{id}\,|_{\pi_{\mathcal{U}}(L_2(\Gamma_Q))}$ is a finite-rank map. □

5B. ***Strong solidity.*** We follow closely the argument in [Avsec 2011; Houdayer and Shlyakhtenko 2011]. The strategy is to first prove a weak containment result of bimodules and then use it to prove strong solidity of Γ_Q. See, e.g., [Brown and Ozawa 2008; Avsec 2011] for more details on bimodules and weak containment. For simplicity, we write $Q' = Q \otimes \mathbb{1}_2 = Q \otimes \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ as in Section 4B. Here we assume

$$\max_{1 \le i,j \le N} |q_{ij}| < q^2 < q < 1. \tag{5-3}$$

Recall that $L_2^0(\Gamma_{Q'})$ denotes the subspace of $L_2(\Gamma_{Q'})$ which consists of mean zero elements. Define the following subspaces of $L_2^0(\Gamma_{Q'})$:

$$F_m = L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [2N]^s, s \in \mathbb{N}, s \ge m, \exists i_1, \dots, i_m \in \{N+1, \dots, 2N\}\},$$
$$E_m = \bigoplus_{k=0}^{m} F_k.$$

Clearly, $E_m^\perp$ is a Γ_Q–Γ_Q-subbimodule of $L_2^0(\Gamma_{Q'})$. We want to show that $E_m^\perp$ is weakly contained in the coarse bimodule $L_2(\Gamma_Q) \otimes L_2(\Gamma_Q)$ for m large enough. By Proposition 3.13, we may identify $L_2(\Gamma_{Q'})$ with the Fock space $\mathcal{F}_{Q'}$. For $\xi, \eta \in L_2^0(\Gamma_{Q'})$, define $\Phi_{\xi,\eta} : L_2(\Gamma_Q) \to L_2(\Gamma_Q)$ by

$$\Phi_{\xi,\eta}(x) = E_{\Gamma_Q}(\xi x \eta).$$

To distinguish the left action and the right action of $\Gamma_{Q'}$ on $L_2(\Gamma_{Q'})$, we write $l(h_i)$ (resp. $r(h_i)$) as the left (resp. right) creation operator associated to h_i acting on the Fock space $\mathcal{F}_{Q'}$, i.e.,

$$l(h_i)(h_{j_1} \otimes \cdots \otimes h_{j_n}) = h_i \otimes h_{j_1} \otimes \cdots \otimes h_{j_n},$$
$$r(h_i)(h_{j_1} \otimes \cdots \otimes h_{j_n}) = h_{j_1} \otimes \cdots \otimes h_{j_n} \otimes h_i.$$

Here, the h_i's are elements in $\mathbb{C}^{2N} = \mathbb{C}^N \oplus \mathbb{C}^N$. We write $l(h_i)^*$ (resp. $r(h_i)^*$) as the left (resp. right) annihilation operator acting on the Fock space $\mathcal{F}_{Q'}$. See more details for these operations in [Bożejko and Speicher 1994; Lust-Piquard 1999]. One can also define them following Section 3C after choosing an o.n.b. Write

$$H'^s = \operatorname{span}\{w(\underline{i}) \in L_2(\Gamma_{Q'}) : \underline{i} \in [2N]^s\} \quad \text{and} \quad H^s = \operatorname{span}\{w(\underline{i}) \in L_2(\Gamma_Q) : \underline{i} \in [N]^s\}.$$

Lemma 5.6. *Assume (5-3). Let $(e_i)_{i=1}^{2N}$ be an o.n.b. of $\mathbb{C}^{2N}$. Suppose $\underline{i} \in [2N]^{n_1}$ and $\underline{j} \in [2N]^{n_2}$. If $\{N+1, \dots, 2N\}$ contains exactly n elements of $\{i_{r+1}, \dots, i_{n_1}\}$, then*

$$\left\| E_{\Gamma_Q}\left[l(e_{i_1}) \cdots l(e_{i_r}) l(e_{i_{r+1}})^* \cdots l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_s}) r(e_{j_{s+1}})^* \cdots r(e_{j_{n_2}})^*(x) \right] \right\|_2$$
$$\le C_{q,n_1,n_2} q^{(\alpha-(n_2-s)-(n_1-r-n))n} \|x\|_2$$

for all $x \in H^\alpha$.

Proof. Note that among all possible configurations, the assertion is nontrivial only if

$$i_1, \dots, i_r, j_{s+1}, \dots, j_{n_2} \le N.$$

By [Bożejko and Speicher 1994, Theorem 3.1],

$$\|r(e_{j_s})^*\| \le \frac{1}{\sqrt{1-q}}, \quad \|l(e_{i_r})\| \le \frac{1}{\sqrt{1-q}}. \tag{5-4}$$

We may assume without loss of generality that $r = 0$ and $s = n_2$ and estimate the norm of $l(e_{i_1})^* \cdots l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}})$. The idea is that all the e_{i_r}'s with $i_r > N$ have to pair with the e_{j_s}'s to cancel out, and moving across the element x will yield a power of q. Let us assume $i_{n_1} > N$ to illustrate the argument. Note that by Remark 3.17, H^α can be identified with $(\mathbb{C}^N \oplus 0)^{\otimes \alpha}$ via

$$w(\underline{i}) \mapsto W(e_{i_1} \otimes \cdots \otimes e_{i_\alpha}) \mapsto e_{i_1} \otimes \cdots \otimes e_{i_\alpha}.$$

First assume $x = e_{k_1} \otimes \cdots \otimes e_{k_\alpha}$. Using (3-14) (or the formula on p. 109 of [Bożejko and Speicher 1994]), we find

$$\begin{aligned} &l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}}) x \\ &\quad = \sum_{m=1}^{n_2} \delta_{i_{n_1}, j_m} \prod_{s=1}^{\alpha} q_{i_{n_1}, k_s} \prod_{r=m+1}^{n_2} q_{i_{n_1}, j_r} x \otimes e_{j_{n_2}} \otimes \cdots \otimes \check{e}_{j_m} \otimes \cdots \otimes e_{j_1} \\ &\quad = \sum_{m=1}^{n_2} \delta_{i_{n_1}, j_m} \prod_{s=1}^{\alpha} q_{i_{n_1}, k_s} \prod_{r=m+1}^{n_2} q_{i_{n_1}, j_r} r(e_{j_1}) \cdots \check{r}(e_{j_m}) \cdots r(e_{j_{n_2}}) x, \end{aligned}$$

where $\check{e}_{j_m}$ and $\check{r}(e_{j_m})$ mean that e_{j_m} and $r(e_{j_m})$ are omitted in the expression. The difficulty is that the coefficient in front of x depends on x. In order to extend the above equation to arbitrary $x \in H^\alpha$, we will find a linear operator for any fixed m via deformation and enlargement of the algebra. Define $\widetilde{q}_{ij} = q_{ij}/q$ for $1 \le i, j \le N$, $\widetilde{Q} = (\widetilde{q}_{ij})$, and

$$P = \begin{pmatrix} Q \otimes \mathbb{1}_2 & \widetilde{Q} \otimes \mathbb{1}_2 \\ \widetilde{Q} \otimes \mathbb{1}_2 & Q \otimes \mathbb{1}_2 \end{pmatrix}.$$

Note that (5-3) implies that $\max_{ij} |p_{ij}| < q$. We can construct new von Neumann algebras Γ_P and $\Gamma_{P \otimes \mathbb{1}_{n_2+1}}$. Clearly, we have the relation

$$\Gamma_Q \hookrightarrow \Gamma_{Q'} \hookrightarrow \Gamma_P \hookrightarrow \Gamma_{P \otimes \mathbb{1}_{n_2+1}}.$$

We continue to denote by $E_{\Gamma_Q} : \Gamma_{P \otimes \mathbb{1}_{n_2+1}} \to \Gamma_Q$ the conditional expectation. Let $\hat{i}_{n_1} = i_{n_1} + 2N$, and let

$$\hat{j}_r = \begin{cases} j_r + 2N & \text{if } j_r > N, \\ j_r & \text{otherwise.} \end{cases}$$

For fixed m, let

$$\widetilde{i}_{n_1} = \hat{i}_{n_1} + 4mN = i_{n_1} + 2N + 4mN, \quad \widetilde{j}_m = j_m + 4mN,$$

and let $\widetilde{j}_r = \hat{j}_r$ for $r \neq m$. In $L_2(\Gamma_{P\otimes \mathbb{1}_{n_2+1}})$, observing the repetition pattern in the matrix P, we have

$$\begin{aligned} &r(e_{\widetilde{j}_1}) \cdots r(e_{\widetilde{j}_{m-1}})[l(e_{\widetilde{i}_{n_1}})^* r(e_{\widetilde{j}_m})] r(e_{\widetilde{j}_{m+1}}) \cdots r(e_{\widetilde{j}_{n_2}})x \\ &\quad = \sum_{u=m}^{n_2} \delta_{\widetilde{i}_{n_1},\widetilde{j}_u} \prod_{s=1}^{\alpha} \widetilde{q}_{i_{n_1},k_s} \prod_{v=u+1}^{n_2} p_{\hat{i}_{n_1},\hat{j}_v} r(e_{\widetilde{j}_1}) \cdots r(e_{\widetilde{j}_{m-1}}) \cdots \check{r}(e_{\widetilde{j}_u}) \cdots r(e_{\widetilde{j}_{n_2}})x \\ &\quad = \prod_{s=1}^{\alpha} \widetilde{q}_{i_{n_1},k_s} \prod_{v=m+1}^{n_2} p_{\hat{i}_{n_1},\hat{j}_v} r(e_{\widetilde{j}_1}) \cdots \check{r}(e_{\widetilde{j}_m}) \cdots r(e_{\widetilde{j}_{n_2}})x, \end{aligned}$$

where $p_{\hat{i}_{n_1},\hat{j}_v} = q_{i_{n_1},j_v}$ if $j_v \in \{N+1,\dots,2N\}$ and $p_{\hat{i}_{n_1},\hat{j}_v} = \widetilde{q}_{i_{n_1},j_v}$ otherwise. Note that the term $4mN$ is used to guarantee that $l(e_{\widetilde{i}_{n_1}})^*$ only annihilates $e_{\widetilde{j}_m}$. Let

$$I(m) = \big\{ j_v : v \in \{m+1,\dots,n_2\},\ j_v \le N \big\}.$$

Then

$$\begin{aligned} (5\text{-}5)\quad &E_{\Gamma_Q}[l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}})x] \\ &= q^\alpha \sum_{m=1}^{n_2} \delta_{i_{n_1},j_m} \prod_{s=1}^{\alpha} \widetilde{q}_{i_{n_1},k_s} \prod_{r=m+1}^{n_2} q_{i_{n_1},j_r} E_{\Gamma_Q}[r(e_{j_1}) \cdots \check{r}(e_{j_m}) \cdots r(e_{j_{n_2}})x] \\ &= q^\alpha \sum_{m=1}^{n_2} \delta_{i_{n_1},j_m} q^{\#I(m)} \\ &\qquad \times \prod_{s=1}^{\alpha} \widetilde{q}_{i_{n_1},k_s} \prod_{v=m+1}^{n_2} p_{\hat{i}_{n_1},\hat{j}_v} E_{\Gamma_Q}[r(e_{\widetilde{j}_1}) \cdots \check{r}(e_{\widetilde{j}_m}) \cdots r(e_{\widetilde{j}_{n_2}})x] \\ &= q^\alpha \sum_{m=1}^{n_2} \delta_{i_{n_1},j_m} q^{\#I(m)} \\ &\qquad \times E_{\Gamma_Q}\big[r(e_{\widetilde{j}_1}) \cdots r(e_{\widetilde{j}_{m-1}})[l(e_{\widetilde{i}_{n_1}})^* r(e_{\widetilde{j}_m})] r(e_{\widetilde{j}_{m+1}}) \cdots r(e_{\widetilde{j}_{n_2}})x\big]. \end{aligned}$$

Here, the conditional expectation is used in the second equality so that the change in $\underline{i}$ and $\underline{j}$ will not affect the resultant value in $L_2(\Gamma_Q)$. Note that the summand in (5-5) does not depend on x for each fixed m. By linearity, (5-5) holds for any $x \in H^\alpha$. We deduce from (5-4) and the triangle inequality that

$$\big\| E_{\Gamma_Q}[l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}})x] \big\|_2 \le C_{q,n_2} q^\alpha \|x\|_2$$

for all $x \in H^\alpha$. Since $l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}})x$ is a linear combination of words with fixed length, the above argument can be easily extended to handle more than one annihilator. To get a norm estimate on $E_{\Gamma_Q}[l(e_{i_1})^* \cdots l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}})(x)]$, it

suffices to consider the configuration yielding the minimal power of q. This occurs if

$$i_1, \ldots, i_n, j_{n_2-n+1}, \ldots, j_{n_2} \in \{N+1, \ldots, 2N\}.$$

In this situation, $l(e_{i_1})^*, \ldots, l(e_{i_n})^*$ need to cross at least $\alpha - (n_1 - n)$ terms to cancel with the e_{j_s}'s. This gives $q^{[\alpha-(n_1-n)]n}$. Using (5-4) to estimate the norm of $l(e_{i_{n+1}})^* \cdots l(e_{i_{n_1}})^*$ gives a constant C_{q,n_1}. Proceeding like so finishes the proof. □

We will use the normal form theorem of Wick products [Bożejko et al. 1997; Królak 2000] to estimate the norm of $\Phi_{\xi,\eta}$. We achieve this via the following result.

Lemma 5.7. *Assume (5-3). Let $\xi \in H'^{n_1} \cap F_n$ and $\eta \in H'^{n_2} \cap F_n$. Then, for $\alpha > 2(n_1+n_2)$ and $x \in H^\alpha$, we have*

$$\|\Phi_{\xi,\eta}(x)\|_2 \le C_{q,\xi,\eta} q^{n\alpha/2} \|x\|_2.$$

Moreover, $\Phi_{\xi,\eta}(x) \in \bigoplus_{\beta=\alpha-n_1-n_2+2n}^{\alpha+n_1+n_2-2n} H^\beta$.

Proof. First we assume $\xi = w(\underline{i})$, $\eta = w(\underline{j})$ and identify x as a vector in $(\mathbb{C}^N \oplus 0)^\alpha$. By the normal form theorem of Wick products [Bożejko et al. 1997] and [Królak 2000, Theorem 1], we have

$$\begin{aligned} w(\underline{i}) &= W(e_{i_1} \otimes \cdots \otimes e_{i_{n_1}}) \\ &= \sum_{r=0}^{n_1} \sum_{\sigma \in S_{n_1}/(S_r \times S_{n_1-r})} K(Q,\sigma) l(e_{\sigma(i_1)}) \cdots l(e_{\sigma(i_r)}) l(e_{\sigma(i_{r+1})})^* \cdots l(e_{\sigma(i_{n_1})})^*, \end{aligned}$$

where $\sigma(i_r) = i_{\sigma^{-1}(r)}$, and $K(Q,\sigma)$ is a product of certain entries of Q and only depending on Q and σ. The precise value of $K(Q,\sigma)$ is irrelevant here. We only need the fact that $|K(Q,\sigma)| \le C_{q,n_1}$ for some constant C_{q,n_1} depending on q and n_1. We have a similar formula for $w(\underline{j})$. It follows that

$$\begin{aligned} \Phi_{\xi,\eta}(x) = \sum_{r=0}^{n_1} \sum_{s=0}^{n_2} \sum_{\substack{\sigma \in S_{n_1}/(S_r \times S_{n_1-r}) \\ \pi \in S_{n_2}/(S_s \times S_{n_2-s})}} & K(Q,\sigma) K(Q,\pi) E_{\Gamma_Q}\big[l(e_{\sigma(i_1)}) \cdots l(e_{\sigma(i_r)}) \\ & \cdot l(e_{\sigma(i_{r+1})})^* \cdots l(e_{\sigma(i_{n_1})})^* r(e_{\pi(j_1)}) \cdots r(e_{\pi(j_s)}) r(e_{\pi(j_{s+1})})^* \cdots r(e_{\pi(j_{n_2})})^*(x) \big]. \end{aligned} \tag{5-6}$$

By Lemma 5.6,

$$\begin{aligned} \big\| E_{\Gamma_Q}\big[l(e_{\sigma(i_1)}) \cdots l(e_{\sigma(i_r)}) l(e_{\sigma(i_{r+1})})^* \cdots l(e_{\sigma(i_{n_1})})^* r(e_{\pi(j_1)}) \cdots r(e_{\pi(j_s)}) \\ r(e_{\pi(j_{s+1})})^* \cdots r(e_{\pi(j_{n_2})})^*(x) \big] \big\|_2 \\ \le C_{q,n_1,n_2} q^{(\alpha-(n_2-s)-(n_1-r-n))n} \|x\|_2. \end{aligned}$$

Since $\alpha - n_1 - n_2 + s + r + n \ge \frac{1}{2}\alpha$, it follows from the triangle inequality that

$$\|\Phi_{\xi,\eta}(x)\|_2 \le C_{q,n_1,n_2} q^{n\alpha/2} \|x\|_2.$$

Now suppose ξ, η are linear combinations of special Wick words. Using the triangle inequality again, we have proved the first assertion. As for the range of $\Phi_{\xi,\eta}$, a moment of thought shows that the summand in (5-6) is of length $\alpha - n_1 - n_2 + 2s + 2r$ and that $0 \le r \le n_1 - n, n \le s \le n_2$ because we must have $\sigma(i_1), \dots, \sigma(i_r), \pi(j_{s+1}), \dots, \pi(j_{n_2}) \le N$ so that the right-hand side of (5-6) is nonzero. This gives the "moreover" part of the lemma. □

Lemma 5.8. *Let $K = \bigoplus_{n=0}^{\infty} K_n$ and $T : K \to K$ be an operator such that*

(i) $\dim(K_n) \le d^n$;

(ii) $\|T|_{K_n}\| \le C\alpha^n$ *for* $n \ge n_0$;

(iii) $\alpha^2 d < 1$.

Then T is Hilbert–Schmidt.

Proof. Let $P_n : K \to K_n$ be the orthogonal projection. Then

$$\operatorname{tr}(T^*T) = \sum_n \operatorname{tr}((TP_n)^*TP_n) \le \sum_n \|TP_n\|^2 d^n \le C \sum_n \alpha^{2n} d^n.$$

Since the series is absolutely convergent the assertion follows immediately. □

Lemma 5.9. *Let $\xi, \eta \in F_n$ and $n > -\ln N / \ln q$. Then $\Phi_{\xi,\eta} : L_2(\Gamma_Q) \to L_2(\Gamma_Q)$ is Hilbert–Schmidt.*

Proof. Write $L_2(\Gamma_Q) = \bigoplus_{s=0}^{\infty} H^s$. Then $\dim(H^\alpha) \le N^\alpha$ and $q^n N < 1$. By Lemma 5.7, we have

$$\left\| \Phi_{\xi,\eta}|_{H^\alpha} \right\| \le C_{q,\xi,\eta} (q^{n/2})^\alpha.$$

The assertion follows from Lemma 5.8. □

Proposition 5.10. *Let $n > -\ln N / \ln q$. Then $E_{n-1}^{\perp}$ is weakly contained in the coarse bimodule $L_2(\Gamma_Q) \otimes L_2(\Gamma_Q)$.*

Proof. The proof is given in [Avsec 2011, Proposition 4.1] using Lemma 5.9. □

Let $R_t : \mathbb{R}^N \oplus \mathbb{R}^N \to \mathbb{R}^N \oplus \mathbb{R}^N$ be the orthogonal transform

$$R_t = \begin{pmatrix} e^{-t}\,\mathrm{id} & -\sqrt{1-e^{-2t}}\,\mathrm{id} \\ \sqrt{1-e^{-2t}}\,\mathrm{id} & e^{-t}\,\mathrm{id} \end{pmatrix},$$

where $\mathrm{id} : \mathbb{R}^N \to \mathbb{R}^N$ is the identity operator and we understand the canonical o.n.b. in $0 \oplus \mathbb{R}^N$ to be $\{e_{N+1}, \dots, e_{2N}\}$. Recall from [Lust-Piquard 1999, Lemma 3.1] that there is a second quantization functor Γ_Q which sends the category of Hilbert spaces to the category of mixed q-Gaussian algebras. Let $\alpha_t = \Gamma_Q(R_t)$. Then α_t is a trace-preserving $*$-automorphism on $\Gamma_{Q'}$ and extends to an isometry on $L_2(\Gamma_{Q'})$. It is easy to check that $T_t = E_{\Gamma_Q} \circ \alpha_t$ coincides with the Ornstein–Uhlenbeck semigroup on Γ_Q defined in Section 3D. The following is a modification of Popa's s-malleable

deformation estimate [Popa 2008, Lemma 2.1]. The proof modifies slightly that of [Avsec 2011, Proposition 5.1]. We provide the difference here for the reader's convenience. Recall that $L_2^0(\Gamma_{Q'}) = \bigoplus_{m=1}^{\infty} H'^m$.

Proposition 5.11. *Let $P_k : L_2^0(\Gamma_{Q'}) \to E_k^{\perp}$ be the orthogonal projection. Then, for $k \geq 1$, we have*

$$\|(\alpha_{t^k} - \mathrm{id})(x)\|_2 \leq C_k \|P_{k-1}\alpha_t(x)\|_2$$

for $x \in \bigoplus_{n=k}^{\infty} H^n \subset L_2(\Gamma_Q)$ and $t < 2^{-k}$.

Proof. Note $\alpha_{t^k} - \mathrm{id}$ and $P_{k-1}\alpha_t$ preserve the length of n-tensors for $n \geq k$ and $t > 0$. It suffices to prove the assertion for $x \in H^n$ with $n \geq k$. Identify H^n with $(\mathbb{C}^N \oplus 0)^{\otimes n}$. Let $x = e_{i_1} \otimes \cdots \otimes e_{i_n}$ and $y = e_{j_1} \otimes \cdots \otimes e_{j_n}$. Then

$$\langle P_{k-1}\alpha_t(x),\, P_{k-1}\alpha_t(y)\rangle = \sum_{m=k}^{n} \langle P_{F_m}\alpha_t(x),\, P_{F_m}\alpha_t(y)\rangle,$$

where the inner product is given by Proposition 3.9, and $P_{F_m} : L_2^0(\Gamma_{Q'}) \to F_m$ is the orthogonal projection. By the second quantization [Lust-Piquard 1999, Lemma 3.1],

$$\alpha_t(e_{i_1} \otimes \cdots \otimes e_{i_n}) = (e^{-t}e_{i_1} + \sqrt{1 - e^{-2t}}\, e_{N+i_1}) \otimes \cdots \otimes (e^{-t}e_{i_n} + \sqrt{1 - e^{-2t}}\, e_{N+i_n}).$$

It follows that

$$P_{F_m}\alpha_t(x) = \sum_{B \subset \{1,\ldots,n\}, |B|=m} (1 - e^{-2t})^{m/2} e^{-t(n-m)} e_{\pi_B(i_1)} \otimes \cdots \otimes e_{\pi_B(i_n)},$$

where $\pi_B(i_k) = N + i_k$ for $k \in B$ and $\pi_B(i_k) = i_k$ otherwise. Similarly, we get

$$P_{F_m}\alpha_t(y) = \sum_{C \subset \{1,\ldots,n\}, |C|=m} (1 - e^{-2t})^{m/2} e^{-t(n-m)} e_{\pi_C(j_1)} \otimes \cdots \otimes e_{\pi_C(j_n)},$$

where

$$\pi_C(j_k) = N + j_k \quad \text{for } k \in C \qquad \text{and} \qquad \pi_C(j_k) = j_k \quad \text{otherwise.}$$

By Proposition 3.9, $\langle P_{F_m}\alpha_t(x),\, P_{F_m}\alpha_t(y)\rangle$ is nonzero only if $\{\pi_B(i_1), \ldots, \pi_B(i_n)\}$ and $\{\pi_C(j_1), \ldots, \pi_C(j_n)\}$ are equal as multisets. Hence, the indices in B have to be paired with the indices in C when we compute $\langle e_{\pi_B(i_1)} \otimes \cdots \otimes e_{\pi_B(i_n)},\, e_{\pi_C(j_1)} \otimes \cdots \otimes e_{\pi_C(j_n)}\rangle$ using Proposition 3.9. For every fixed B, pairing all the possible C with B and the corresponding C^c with B^c gives all the bipartite partitions of $\underline{i} \sqcup \underline{j}$. Using Proposition 3.9 again, we see that

$$\langle P_{F_m}\alpha_t(x),\, P_{F_m}\alpha_t(y)\rangle = (1 - e^{-2t})^m e^{-2t(n-m)} \sum_{B \subset \{1,\ldots,n\}, |B|=m} \langle x, y\rangle.$$

By linearity, this identity holds for arbitrary $x, y \in H^n$. Hence,

$$\langle P_{k-1}\alpha_t(x),\, P_{k-1}\alpha_t(y)\rangle = \sum_{m=k}^{n} (1 - e^{-2t})^m e^{-2t(n-m)} \binom{n}{m} \langle x, y\rangle.$$

Since the Ornstein–Uhlenbeck semigroup T_t is self-adjoint on $L_2(\Gamma_Q)$ and α_t is trace-preserving, we have, for $x, y \in (\mathbb{C}^N \oplus 0)^{\otimes n}$,

$$\big\langle (\alpha_{t^k} - \mathrm{id})(x), (\alpha_{t^k} - \mathrm{id})(y) \big\rangle = 2\big(\langle x, y\rangle - \langle x, T_{t^k}(y)\rangle\big) = 2(1 - e^{-nt^k})\langle x, y\rangle.$$

The rest of the proof is just numerical estimate, which is provided in the proof of [Avsec 2011, Proposition 5.1]. □

The following is the main result of this section.

Theorem 5.12. *Let Q be a real symmetric $N \times N$ matrix with $\max_{1\le i,j\le N}|q_{ij}| < 1$ and $N < \infty$. Then Γ_Q is strongly solid.*

Proof. The proof is the same as that of [Avsec 2011, Theorem B], with the help of Theorem 5.5 and Propositions 5.10 and 5.11. The argument in [Avsec 2011] follows literally the same strategy as that of [Houdayer and Shlyakhtenko 2011, Theorem 3.5], which in turn is a suitable modification of [Ozawa and Popa 2010a; 2010b]. □

Appendix: Speicher's central limit theorem

Proof of Theorem 3.1. The proof is rephrased from [Speicher 1993] and also follows [Junge et al. 2015]. We first show that the convergence holds on average, and then prove almost sure convergence using the Borel–Cantelli lemma. We write

$$\begin{aligned}\tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m)) &= \frac{1}{m^{s/2}}\sum_{\underline{k}\in[m]^s}\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s)) \\ &= \frac{1}{m^{s/2}}\sum_{\sigma\in P(s)}\sum_{\substack{\underline{k}\in[m]^s\\ \sigma(\underline{k})=\sigma}}\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s)) \\ &=: \frac{1}{m^{s/2}}\sum_{\sigma\in P(s)}\Delta_\sigma.\end{aligned} \tag{A-1}$$

By the commutation/anticommutation relation, $\Delta_\sigma = 0$ if σ contains a singleton. Note $|\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))| \le 1$. If σ has r blocks, then $\Delta_\sigma \le m(m-1)\cdots(m-r+1)$. Hence,

$$\lim_{m\to\infty}\frac{1}{m^{s/2}}\Delta_\sigma = 0 \tag{A-2}$$

for $r < s/2$ and thus for $\sigma \in P(s)\setminus P_2(s)$ since the singleton case is automatically true. Our argument so far is independent of $\omega \in \Omega$, so that (A-2) holds for all $\omega \in \Omega$. The theorem follows immediately from (A-2) if s is odd. Thus, we only need to consider $\sigma \in P_2(s)$ in (A-1). To this end, we write $\sigma = \{\{e_1, z_1\}, \ldots, \{e_{s/2}, z_{s/2}\}\}$. Since $\sigma(\underline{k}) = \sigma$ is a pair partition, if $k_j = k_l$, then $i_j = i_l$ in order for $x_{i_j}(k_j)$ and $x_{i_l}(k_l)$ to cancel out. Hence we may assume $\sigma \le \sigma(\underline{i})$. In this case, if $\{r, t\} \in I(\sigma)$,

then we have to switch $x_{i(e_r)}(k(e_r))$ and $x_{i(e_t)}(k(e_t))$ to cancel the corresponding $x_{i(z_r)}(k(z_r))$ and $x_{i(z_t)}(k(z_t))$ terms, which yields

$$\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s)) = \prod_{\{r,t\}\in I(\sigma)} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]). \tag{A-3}$$

By independence and counting the elements in $\{\underline{k}\in[m]^s \mid \sigma(\underline{k})=\sigma\}$, we find

$$\begin{aligned}\mathbb{E}(\Delta_\sigma) &= \sum_{\substack{\underline{k}\in[m]^s\\ \sigma(\underline{k})=\sigma}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t))\\ &= m(m-1)\cdots(m-s/2+1)\prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t)).\end{aligned}$$

Combining these, we have

$$\lim_{m\to\infty}\mathbb{E}\big(\tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m))\big) = \sum_{\substack{\sigma\in P_2(s)\\ \sigma\le\sigma(\underline{i})}}\prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t)). \tag{A-4}$$

It remains to prove the almost sure convergence. Put $X_m=\tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m))$ and $E_m(\alpha)=\{\omega:|X_m-\mathbb{E}X_m|\ge\alpha\}$. Then we only need to show $\mathbb{P}(\limsup_m E_m(\alpha))=0$. By the Borel–Cantelli lemma and Chebyshev's inequality, it suffices to show that

$$\sum_{m=1}^{\infty}\mathbb{P}(E_m(\alpha)) \le \frac{1}{\alpha^2}\sum_{m=1}^{\infty}\operatorname{Var}(X_m) < \infty \quad \text{for any } \alpha>0,$$

where $\operatorname{Var}(X_m)$ is the variance of X_m. Decompose X_m as $X_m=Y_m+Z_m$, where Y_m corresponds to sum over all pair partitions in (A-1) and $Z_m=X_m-Y_m$. Since (A-2) holds for $\sigma\in P(s)\setminus P_2(s)$, we have $\lim_{m\to\infty}X_m-Y_m=0$ for all $\omega\in\Omega$. But Z_m is uniformly bounded, so $\lim_{m\to\infty}\operatorname{Var}(Z_m)=0$. Therefore, we only need to show that $\sum_{m=1}^{\infty}\operatorname{Var}(Y_m)<\infty$. Write

$$\operatorname{Var}(Y_m)=\frac{1}{m^s}\sum_{\sigma,\pi\in P_2(s)}\sum_{\substack{\underline{k}:\sigma(\underline{k})=\sigma\\ \underline{l}:\sigma(\underline{l})=\pi}} V_{\underline{k},\underline{l}},$$

where

$$\begin{aligned}V_{\underline{k},\underline{l}} &= \mathbb{E}[\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))\tau_m(x_{i_1}(l_1)\cdots x_{i_s}(l_s))]\\ &\quad -\mathbb{E}[\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))]\mathbb{E}[\tau_m(x_{i_1}(l_1)\cdots x_{i_s}(l_s))]\\ &= \mathbb{E}\Bigg(\prod_{\{r,t\}\in I(\sigma)}\varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)])\\ &\qquad\times\prod_{\{r',t'\}\in I(\pi)}\varepsilon([i(e_{r'}),l(e_{r'})],[i(e_{t'}),l(e_{t'})])\Bigg)\\ &\qquad -\prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t))\prod_{\{r',t'\}\in I(\pi)} q(i(e_{r'}),i(e_{t'})).\end{aligned} \tag{A-5}$$

Let us analyze the product in the third and fourth lines of (A-5). If $\{k(e_r), k(e_t)\} \neq \{l(e_{r'}), l(e_{t'})\}$ for all $\{r,t\} \in I(\sigma)$ and $\{r',t'\} \in I(\pi)$, then $V_{\underline{k},\underline{l}} = 0$. In order to contribute for $\mathrm{Var}(Y_m)$, there exists at least one pair $\{r,t\} \in I(\sigma)$ and one pair $\{r',t'\} \in I(\pi)$ such that $\{k(e_r), k(e_t)\} = \{l(e_{r'}), l(e_{t'})\}$. In this case, we have

$$\#\{\underline{k}, \underline{l} : \sigma(\underline{k}) = \sigma, \sigma(\underline{l}) = \pi\} \leq m^{s/2} m^{s/2-2} = m^{s-2}.$$

Note that $|V_{\underline{k},\underline{l}}| \leq 1$ and $C(s) := [\#P_2(s)]^2$ does not depend on m. It follows that

$$\sum_{m=1}^{\infty} \mathrm{Var}(Y_m) \leq \sum_{m=1}^{\infty} \frac{C(s)}{m^2} < \infty. \qquad \square$$

Remark A.1. In the above argument, we assumed that the $\varepsilon((i,k),(j,l))$'s are independent for different indices. However, the independence assumption can be weakened if the structure matrix is of the form $Q \otimes \mathbb{1}_n$, where Q is an $N \times N$ symmetric matrix with entries in $[-1,1]$. In this case we require that the $\varepsilon((i,k),(j,l))$'s be independent (up to symmetric assumption) with (3-1) for $(i,k),(j,l) \in [N] \times \mathbb{N}$ and then

$$\varepsilon((i+\alpha N, k),(j+\beta N, l)) = \varepsilon((i,k),(j,l)) \tag{A-6}$$

for $\alpha, \beta = 1, \ldots, n-1$. In other words, $\varepsilon = \varepsilon|_{1 \leq i,j \leq N} \otimes \mathbb{1}_n$. To verify the claim, we only need to show the dependence introduced in (A-6) will not destroy the proof of Theorem 3.1. Indeed, by (A-2) it suffices to consider pair partitions. Suppose $i_\beta = i_\alpha + N$. Then α and β are not in the same pair block of $\sigma(\underline{i})$. It follows that $k_\alpha \neq k_\beta$ since $\sigma(\underline{k}) \leq \sigma(\underline{i})$. (If $k_\alpha = k_\beta$, then $i_\alpha = i_\beta$ in order for $x_{i_\alpha}(k_\alpha)$ and $x_{i_\beta}(k_\beta)$ to cancel.) Hence, the random signs in (A-3) are pairwise different. Note that unlike the case in the proof of Theorem 3.1, now we may have

$$\varepsilon((i_\alpha, k_\alpha),(i_\gamma, k_\gamma)) \quad \text{and} \quad \varepsilon((i_\beta, k_\beta),(i_\gamma, k_\gamma)) = \varepsilon((i_\alpha, k_\beta),(i_\gamma, k_\gamma))$$

in (A-3), but the two random signs are not equal because $k_\alpha \neq k_\beta$. In other words, the second coordinates (k_α, k_γ) in $\varepsilon((i_\alpha, k_\alpha),(i_\gamma, k_\gamma))$ are never the same for random signs in (A-3) even under the weaker condition (A-6) so that (i_α, i_γ) may be the same for different random signs. The point is that the independence structure in the proof of Theorem 3.1 is given via the second coordinates' k_α's. The rest of the argument is the same as for Theorem 3.1. We invite the interested reader to consider the simplest case $Q = q\mathbb{1}_N$. In this case we can take $\varepsilon((i,k),(j,l)) = \varepsilon((i,k),(i,l))$ and require $\varepsilon((i,k),(i,l))$ to be independent for different k and l up to symmetry.

By this remark, the moment formula (3-2) remains valid with the weaker condition (A-6). The same discussion applies in other parts of the paper when the CLT argument is invoked with (A-6). This subtlety is crucial for our limiting argument in Section 4.

Acknowledgements

Junge was partially supported by NSF grants DMS-1201886 and DMS-1501103. A large part of this work was done when Zeng was a graduate student at the University of Illinois. He is thankful for financial support from the graduate college dissertation fellowship. He also thanks the Center of Mathematical Sciences and Applications at Harvard University for financial support.

References

[Avsec 2011] S. Avsec, "Strong solidity of the q-Gaussian algebras for all $-1 < q < 1$", 2011. arXiv 1110.4918

[Biane 1997] P. Biane, "Free hypercontractivity", *Comm. Math. Phys.* **184**:2 (1997), 457–474. MR Zbl

[Bożejko and Speicher 1991] M. Bożejko and R. Speicher, "An example of a generalized Brownian motion", *Comm. Math. Phys.* **137**:3 (1991), 519–531. Zbl

[Bożejko and Speicher 1994] M. Bożejko and R. Speicher, "Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces", *Math. Ann.* **300**:1 (1994), 97–120. MR Zbl

[Bożejko et al. 1997] M. Bożejko, B. Kümmerer, and R. Speicher, "q-Gaussian processes: non-commutative and classical aspects", *Comm. Math. Phys.* **185**:1 (1997), 129–154. MR Zbl

[Brown and Ozawa 2008] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Grad. Stud. Math. **88**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Cowling and Haagerup 1989] M. Cowling and U. Haagerup, "Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one", *Invent. Math.* **96**:3 (1989), 507–549. MR Zbl

[Dabrowski 2014] Y. Dabrowski, "A free stochastic partial differential equation", *Ann. Inst. Henri Poincaré Probab. Stat.* **50**:4 (2014), 1404–1455. MR Zbl

[Efraim and Lust-Piquard 2008] L. B. Efraim and F. Lust-Piquard, "Poincaré type inequalities on the discrete cube and in the CAR algebra", *Probab. Theory Related Fields* **141**:3-4 (2008), 569–602. MR Zbl

[Guionnet and Shlyakhtenko 2014] A. Guionnet and D. Shlyakhtenko, "Free monotone transport", *Invent. Math.* **197**:3 (2014), 613–661. MR Zbl

[Guţă 2003] M. Guţă, "The q-product of generalised Brownian motions", pp. 121–134 in *Quantum probability and infinite dimensional analysis* (Burg, 2001), Quantum Probab. White Noise Anal. **15**, World Sci. Publ., River Edge, NJ, 2003. MR Zbl

[Guţă and Maassen 2002] M. Guţă and H. Maassen, "Generalised Brownian motion and second quantisation", *J. Funct. Anal.* **191**:2 (2002), 241–275. MR Zbl

[Houdayer and Shlyakhtenko 2011] C. Houdayer and D. Shlyakhtenko, "Strongly solid II_1 factors with an exotic MASA", *Int. Math. Res. Not.* **2011**:6 (2011), 1352–1380. MR Zbl

[Junge 2006] M. Junge, "Operator spaces and Araki–Woods factors: a quantum probabilistic approach", *Int. Math. Res. Pap.* (2006), art. id. 76978. MR Zbl

[Junge and Xu 2008] M. Junge and Q. Xu, "Noncommutative Burkholder/Rosenthal inequalities, II: Applications", *Israel J. Math.* **167** (2008), 227–282. MR Zbl

[Junge and Zeng 2013] M. Junge and Q. Zeng, "Noncommutative Bennett and Rosenthal inequalities", *Ann. Probab.* **41**:6 (2013), 4287–4316. MR Zbl

[Junge et al. 2014] M. Junge, T. Mei, and J. Parcet, "Noncommutative Riesz transforms: dimension free bounds and Fourier multipliers", 2014. arXiv 1407.2475

[Junge et al. 2015] M. Junge, C. Palazuelos, J. Parcet, M. Perrin, and E. Ricard, "Hypercontractivity for free products", *Ann. Sci. Éc. Norm. Supér.* (4) **48**:4 (2015), 861–889. MR Zbl

[Kennedy and Nica 2011] M. Kennedy and A. Nica, "Exactness of the Fock space representation of the q-commutation relations", *Comm. Math. Phys.* **308**:1 (2011), 115–132. MR Zbl

[Krȯlak 2000] I. Krȯlak, "Wick product for commutation relations connected with Yang–Baxter operators and new constructions of factors", *Comm. Math. Phys.* **210**:3 (2000), 685–701. MR Zbl

[Królak 2005] I. Królak, "Contractivity properties of Ornstein–Uhlenbeck semigroup for general commutation relations", *Math. Z.* **250**:4 (2005), 915–937. MR Zbl

[Lust-Piquard 1986] F. Lust-Piquard, "Inégalités de Khintchine dans C_p $(1 < p < \infty)$", *C. R. Acad. Sci. Paris Sér. I Math.* **303**:7 (1986), 289–292. MR Zbl

[Lust-Piquard 1998] F. Lust-Piquard, "Riesz transforms associated with the number operator on the Walsh system and the fermions", *J. Funct. Anal.* **155**:1 (1998), 263–285. MR Zbl

[Lust-Piquard 1999] F. Lust-Piquard, "Riesz transforms on deformed Fock spaces", *Comm. Math. Phys.* **205**:3 (1999), 519–549. MR Zbl

[Lust-Piquard and Pisier 1991] F. Lust-Piquard and G. Pisier, "Noncommutative Khintchine and Paley inequalities", *Ark. Mat.* **29**:2 (1991), 241–260. MR Zbl

[Murray and Von Neumann 1936] F. J. Murray and J. Von Neumann, "On rings of operators", *Ann. of Math.* (2) **37**:1 (1936), 116–229. MR Zbl

[Nou 2004] A. Nou, "Non injectivity of the q-deformed von Neumann algebra", *Math. Ann.* **330**:1 (2004), 17–38. MR Zbl

[Ozawa and Popa 2010a] N. Ozawa and S. Popa, "On a class of II_1 factors with at most one Cartan subalgebra", *Ann. of Math.* (2) **172**:1 (2010), 713–749. MR Zbl

[Ozawa and Popa 2010b] N. Ozawa and S. Popa, "On a class of II_1 factors with at most one Cartan subalgebra, II", *Amer. J. Math.* **132**:3 (2010), 841–866. MR Zbl

[Pisier 2000] G. Pisier, "An inequality for p-orthogonal sums in non-commutative L_p", *Illinois J. Math.* **44**:4 (2000), 901–923. MR Zbl

[Pisier and Xu 1997] G. Pisier and Q. Xu, "Non-commutative martingale inequalities", *Comm. Math. Phys.* **189**:3 (1997), 667–698. MR Zbl

[Popa 2008] S. Popa, "On the superrigidity of malleable actions with spectral gap", *J. Amer. Math. Soc.* **21**:4 (2008), 981–1000. MR Zbl

[Popa and Vaes 2010] S. Popa and S. Vaes, "Group measure space decomposition of II_1 factors and W^*-superrigidity", *Invent. Math.* **182**:2 (2010), 371–417. MR Zbl

[Popa and Vaes 2014] S. Popa and S. Vaes, "Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups", *Acta Math.* **212**:1 (2014), 141–198. MR Zbl

[Ricard 2005] E. Ricard, "Factoriality of q-Gaussian von Neumann algebras", *Comm. Math. Phys.* **257**:3 (2005), 659–665. MR Zbl

[Shlyakhtenko 2004] D. Shlyakhtenko, "Some estimates for non-microstates free entropy dimension with applications to q-semicircular families", *Int. Math. Res. Not.* **2004**:51 (2004), 2757–2772. MR Zbl

[Śniady 2004] P. Śniady, "Factoriality of Bożejko–Speicher von Neumann algebras", *Comm. Math. Phys.* **246**:3 (2004), 561–567. MR Zbl

[Speicher 1992] R. Speicher, "A noncommutative central limit theorem", *Math. Z.* **209**:1 (1992), 55–66. MR Zbl

[Speicher 1993] R. Speicher, "Generalized statistics of macroscopic fields", *Lett. Math. Phys.* **27**:2 (1993), 97–104. MR Zbl

[Voiculescu et al. 1992] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series **1**, Amer. Math. Soc., Providence, RI, 1992. MR Zbl

[Zeng 2014] Q. Zeng, "Poincaré type inequalities for group measure spaces and related transportation cost inequalities", *J. Funct. Anal.* **266**:5 (2014), 3236–3264. MR Zbl

Received October 2, 2016.

MARIUS JUNGE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, IL
UNITED STATES
mjunge@illinois.edu

QIANG ZENG
MATHEMATICS DEPARTMENT
NORTHWESTERN UNIVERSITY
EVANSTON, IL
UNITED STATES
Current address:
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
BEIJING
CHINA
qzeng.math@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 331 No. 1 July 2024