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AN AVOIDANCE PRINCIPLE AND MARGULIS FUNCTIONS FOR EXPANDING TRANSLATES OF UNIPOTENT ORBITS

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ABSTRACT. We prove an avoidance principle for expanding translates of unipotent orbits for some quotients of semisimple Lie groups. In addition, we prove a quantitative isolation result of closed orbits and give an upper bound on the number of closed orbits of bounded volume. The proofs of our results rely on the construction of a Margulis function and the theory of finite dimensional representations of semisimple Lie groups.

1. INTRODUCTION

Avoidance principles—quantifying how much time trajectories avoid certain subsets of the ambient space—have been fruitful in the study of dynamical systems. An important example is the non-divergence of unipotent flows which goes back to Margulis [26]. A quantitative version of non-divergence appears in Dani [6] and was key in Ratner's seminal theorems on unipotent flows [28, 29, 30, 31].

Two successful strategies to prove such avoidance principles are the construction of *Margulis functions* which originated in the influential work of Eskin–Margulis–Mozes [13] and the linearization technique of Dani–Margulis [8].

The flexibility offered by the construction of Margulis functions makes them applicable to settings where unipotent dynamics are not available or poorly understood. For example, they appear in the important work of Benoist–Quint [3, 4, 5] and the recent generalizations of Eskin–Lindenstrauss [10, 11] on stationary measures of homogeneous spaces. Additionally, Margulis functions are utilized in Eskin–Mirzakhani–Mohammadi [15] to prove an avoidance principle that was crucially used to show an analog of Ratner's orbit closure theorem.

We highlight some other examples to indicate the breadth of Margulis functions, but we recommend the wonderful survey of Eskin–Mozes [16] for a more complete overview of the literature. Margulis functions appear: in the setting of Teichmüller dynamics by Eskin–Masur [14] and Athreya [1], in the space

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of lattices by Kadyrov–Kleinbock–Lindenstrauss–Margulis [21] and Kleinbock–Mirzadeh [23], for finite homogeneous spaces by Guan–Shi [20] and Rodriguez-Hertz–Wang [32], for infinite homogeneous spaces by Mohammadi–Oh [27], and in the space of closed subgroups of a semisimple Lie group equipped with the Chabauty topology in the work of Gelander–Levit–Margulis [19] and Fraczyk–Gelander [17].

We use Margulis functions and the theory of finite dimensional representations of semisimple Lie groups to prove an avoidance principle. Broadly speaking, our results rely on the hyperbolicity of diagonal actions and the fact that the perturbation by a foliation often places one in a general position where one expects expansion by the diagonal direction.

Throughout this paper, G will be a semisimple algebraic Lie group without compact factors and H will be a semisimple subgroup of G without compact factors such that $C_G(H)$ is finite. We let $X := G/\Gamma$ where Γ is a lattice.

We equip $\mathrm{Lie}(G)$ with an inner product that induces a right-invariant Riemannian metric on G. The notions of distance and volumes make sense with respect to this Riemanninan metric. Denote by by $\mathrm{inj}(x)$ the injectivity radius at point x. See the next section for formal descriptions of these notions.

DEFINITION 1. For a pair of positive real numbers (V,d), we say that a point $x \in X$ is (V,d)-*Diophantine with respect to* H if the following holds: for any intermediate subgroup $H \subseteq S \subsetneq G$ and any closed S-orbit $Y = Sx'$ with $\mathrm{vol}(Y) \leq V$, the distance between x and Y is at least d; namely, $\mathrm{dist}(x,Y) \geq d$.

For $r > 0$, if a point $x \in X$ is (V,d)-Diophantine with respect to H and $\mathrm{inj}(x) \geq r$, then we say that x is (V,d,r)-*Diophantine with respect to* H.

We fix a one parameter subgroup of diagonalizable elements $\{a_t\} \subseteq H$ and let U be the part of the unstable horospherical subgroup with respect to $\{a_t\}$ that is also in H:

$$\{u \in H : a_t u a_{-t} \to e \ as \ t \ \to -\infty\}.$$

We work with the operators

$$(A_{r,t} f)(x) = \frac{1}{m_U(B_r^U)} \int_{B_r^U} f(a_t u x)\, dm_U(u),$$

where B_r^U is the ball of radius r in U and m_U is the Haar measure on the Lie subgroup U normalized so that B_1^U has measure 1. Here the implicit metric on U comes from the identification of $\mathrm{Lie}(U)$ with a Euclidean space. See the next section for details. When considering $A_{1,t}$, we use the notation A_t.

We use the operators $A_{r,t}$ to prove a result on the behavior of points of the form $a_t u x$ for $u \in B_1^U$ and large $t > 0$. The following is our main theorem.

THEOREM 2 (Avoidance Principle)**.** *Let G be a semisimple group without compact factors and H be a semisimple subgroup without compact factors such that $C_G(H)$ is finite. Let $X = G/\Gamma$ where Γ is a lattice. There exist absolute constants $D = D(\dim(G)) > 0$, $A = A(G/\Gamma, H) > 0$, and $C = C(G/\Gamma, H) > 0$ such that the following dichotomy holds: for any $x \in X$, there exists $T_x > 0$ such that for any pair of $T > T_x$ and $R > 2$, either:*

(1) *x is not $(R, 1/T)$-Diophantine with respect to H or*
(2) *for all $t \geq A \log T$,*

$$m_U\left(\left\{u \in B_1^U : a_t ux \text{ is not } (R, R^{-D}, R^{-D})\text{-Diophantine w.r.t. } H\right\}\right) < CR^{-1}.$$

Moreover, if $K \subseteq X$ is compact, then T_x can be chosen to be uniform over all $x \in K$.

REMARK 3. We note a generalization of our main result for solvable epimorphic subgroups. Recall, a subgroup G' of a real algebraic group G is called epimorphic in G if any G'-fixed vector is also G-fixed for any finite dimensional algebraic representation of G. Proposition 2.2. of Shah and Weiss [34] gives an analogous result to our Linear Algebra Lemma (Lemma 8) for solvable epimorphic groups. Hence, it is plausible that our result can be further generalized so that B_1^U in condition (2) is replaced by B_1^N where $N \subseteq U$ is an algebraic unipotent subgroup normalized by $\{a_t\}$, such that the subgroup generated by $\{a_t\}$ and N is solvable and epimorphic in G.

A version of Theorem 2 was used in Lindenstrauss–Mohammadi–Wang [25] for $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ to obtain an absolute Diophantine estimate. Additionally, Lindenstrauss–Margulis–Mohammadi–Shah [24] prove a similar avoidance principle for unipotent flows, but work in a more general setting. It is also similar to the work of Bénard–de Saxcé[2].

To prove our main result, we need results on the quantitative isolation of closed orbits which are interesting in their own right. The following theorem is analogous to Lemma 10.3.1 of Einsiedler–Margulis–Venkatesh [9].

THEOREM 4 (Quantitative Isolation of closed orbits)**.** *There exists a global constant $D = D(\dim(G)) > 0$ such that the following holds: for all intermediate subgroup $H \subseteq S \subsetneq G$ and closed S-orbits $Y = Sy$ and $Z = Sz$ of finite volume,*

$$\mathrm{dist}(Y \cap K, Z) \gg_K \mathrm{vol}(Y)^{-D} \mathrm{vol}(Z)^{-D},$$

where K is a compact subset of X.

We note that the proof of [9, Lemma 10.3.1], relies on uniform spectral gap for periodic S-orbits ($H \subseteq S \subsetneq G$) in congruence quotients. Our proof is arguably softer. In particular, it does not require Γ to be arithmetic. The main idea of the proof is to estimate the size of the additive constant of a Margulis function, and goes back to Margulis' unpublished notes (see also [27, Theorem 1.1]).

Using Theorem 4 above, an upper bound can be obtained on the number of closed orbits of bounded volume. A qualitative version of this theorem is originally due to Dani–Margulis [8]. The theorem below is analogous to Corollary 10.7 of Mohammadi–Oh [27] for geometrically finite quotients of $\mathbb{H}^3$.

THEOREM 5 (Upper bound for the number of closed orbits of bounded volume)**.** *There exists a global constant $D = D(\dim(G)) \gg 1$ such that for any intermediate subgroup $H \subseteq S \subsetneq G$,*

$$\#\{Y : Y = Sy \text{ is a closed } S\text{-orbit and } \mathrm{vol}(Y) \leq R\} \ll R^D.$$

2. PRELIMINARIES

In this section we fix notation.

Equip Lie(G) with the Killing form. This induces

1. a norm $\|\cdot\|$ on Lie(G),
2. a right-invariant Riemannian metric on G that induces a right-invariant metric on G denoted as dist_G,
3. a metric on $X = G/\Gamma$ denoted as dist so that the canonical projection $G \to X$ is a local isometry, and
4. a volume for a closed orbit H-orbit on X induced from the Riemannian structure on G which we denote with vol.

With respect to the norm $\|\cdot\|$ on Lie(G), we can define the unit ball in Lie(G) which we denote as $B_1^{\mathrm{Lie}(G)}$.

We choose a inner product on Lie(U) that comes from the identification of Lie(U) to $\mathbb{R}^{d_U}$ where d_U denotes the dimension of Lie(U). For any $\eta > 0$, we can use the inner product on $Lie(U)$ to define a norm (resp. metric) on Lie(U) (resp. U). This allows us to make sense of the unit ball in Lie(U) which we denote as $B_1^{\mathrm{Lie}(U)}$ (resp. in U which we denote as B_1^U).

For each $x \in X$, we denote by inj(x) the injectivity radius at point x; the supremum of all $\eta > 0$ for which the projection map $g \to gx$ from G to $X = G/\Gamma$ is injective on B_η^G. In Section 6, we shall choose a specific $\varepsilon_X > 0$ and denote $X_{\varepsilon_X} := \{x \in X : \mathrm{inj}(x) \geq \varepsilon_X\}$ as the compact part of X. Since the exponential map Lie(G) $\to G$ defines a local diffeomorphism, there exists an absolute constant $\sigma_0 > 1$ such that for all $w \in$ Lie(G) with $\|w\| \leq \epsilon_X$ and $x \in X_{\varepsilon_X}$,

$$\sigma_0^{-1}\|w\| \leq \mathrm{dist}(x, \exp(w)x) \leq \sigma_0\|w\|.$$

By noting that the canonical projection $G \to X$ is a local isometry, we have a way of locally measuring distances in X with the norm on Lie(G).

For any intermediate subgroup $H \subseteq S \subseteq G$, we denote the dimension of Lie(S) by dim(S) or simply, d_S.

We will denote the Haar measure on G by m_G. For the horospherical subgroup U of G, we denote the Haar measure on U by m_U.

Let T denote a maximal Cartan subgroup containing $(a_t)_{t\in\mathbb{R}}$. Let $\rho : G \to GL(V)$ be a finite dimensional representation. Let Φ denote the root system of Lie(G) and decompose the vector space into weight spaces $V = \oplus_{\beta\in\Phi} V_\beta$ where

$$V_\beta = \{v \in V : \rho(\tau)v = \exp(\beta(\log(\tau)))v, \forall \tau \in T\}$$

is the weight space with weight $\beta \in \Phi$. Choose a basis $(v_{\beta,i})_{i=1}^{\dim V_\beta}$ so that every $v \in V$ can be written in the form

$$v = \sum_{\beta\in\Phi} \sum_{i=1}^{\dim V_\beta} c_{\beta,i} v_{\beta,i}$$

for some scalars $c_{\beta,i}$.

Let S be an intermediate subgroup with $H \subseteq S \subsetneq G$ and consider the decomposition of Lie(G) given by Lie(G) = Lie(S) $\oplus V_S$ where V_S is Ad(Lie(S))-invariant, but not necessarily irreducible. If we decompose V_S into Ad(Lie(S))-invariant subspaces, then each subspace will be non-zero since $C_G(H)$ is finite. This will be an important fact that we use throughout the paper when working with the adjoint representation.

We end the section by introducing two results from Einsiedler–Margulis–Venkatesh [9] on intermediate subgroups $H \subset G$.

LEMMA 6 ([9, Lemma 3.4.1]). *Suppose $H \subseteq G$ are semisimple Lie groups without compact factors such that $C_G(H)$ is finite. Then there are only finitely many intermediate subgroups $H \subseteq S \subsetneq G$. Each such S is semisimple and without compact factors.*

LEMMA 7 ([9, Appendix A]). *If G is a semisimple Lie group without compact factors, then there exists a finite collection of semisimple subgroups $\mathscr{H}$ such that the following holds: for any semisimple Lie subgroup $H \subseteq G$ with no compact factors and $C_G(H)$ finite, there exist $H' \in \mathscr{H}$ and $g \in G$ such that $H = gH'g^{-1}$.*

3. LINEAR ALGEBRA LEMMA

In this section we state some key technical lemmas related to the action of horospherical subgroups and diagonal subgroups from [22] and [33] and prove extensions of these results. The main result of this section applies to representations that are not necessarily irreducible.

LEMMA 8 (Linear algebra lemma). *Suppose $\rho : G \to GL(V)$ is a faithful finite dimensional representation of a semisimple Lie group G. Suppose V decomposes into non-trivial and irreducible subspaces $V = \oplus_i V_i$. There exists an absolute constant $0 < \delta_0 = \delta_0(\dim(G), V) \ll 1$ such that for all $0 < \delta < \delta_0$ and $0 < c < 1$, there exists $t_{\delta,c} = t_{\delta,c}(G,H) > 0$ with*

$$\frac{1}{m_U\big(B_2^U\big)} \int_{B_2^U} \frac{1}{\|\rho(a_t u)v\|^\delta}\, dm_U(u) < \frac{c}{\|v\|^\delta}$$

for every $v \in V$, $t \geq t_{\delta,c}$.

We postpone the proof until the next page after collecting some lemmas. In brief, these lemmas show that the action of the diagonal and horspherical subgroups on a vector space expand the norm. While we will follow the exposition of Katz [22], we would like to draw the reader's attention to Shah [33], specifically Section 5.

The following lemmas are essentially Lemma 3.1 and Lemma 3.2 of [22].

LEMMA 9 ([22, Lemma 3.1]). *Let $u = \exp(\underline{u}) \in B_r^U$. There exists polynomials $f_{\beta,j} : B_r^{\mathrm{Lie}(U)} \to \mathbb{R}$ with*

$$\rho(u)v = \sum_{\beta \in \Phi} \sum_{j=1}^{\dim V_\beta} f_{\beta,j}(\underline{u})\, v_{\beta,j}$$

for any $v \in V$.

Proof. The proof of Lemma 3.1 in [22] works for any $u \in B_r^U$ where $r > 0$. □

LEMMA 10 (Anchor Lemma, [22, Lemma 3.2])**.** *Let $\rho : G \to GL(V)$ be a finite dimensional irreducible representation of a semisimple Lie group G. Then for any $r > 0$ and non-zero $v \in V$, there is a positive root $\beta \in \Phi^+$ and $1 \le j \le \dim V_\beta$ such that*

$$\sup_{u \in B_r^U} \left| f_{\beta,j}(u) \right| > 0.$$

Proof. The proof of Lemma 3.2 in [22] works for any open ball B_r^U with $r > 0$. □

By the Anchor lemma, the projection of the action of U in the expanding direction is nonzero. Thus, the norm under the action of a_t grows. By noting that Lemmas 3.1 and 3.2 of Katz [22] hold for any open ball B_r^U, we have the following minor generalization of Lemma 2.3 of [22].

LEMMA 11. *Suppose $\rho : G \to GL(V)$ is an irreducible finite dimensional representation of a semisimple Lie group G. There exists $0 < \delta_0 = \delta_0(\dim(G)) \ll 1$ such that for all $0 < \delta < \delta_0$ and $0 < c < 1$, there exists $t_c > 0$ with*

$$\frac{1}{m_U\big(B_2^U\big)} \int_{B_2^U} \frac{1}{\|\rho(a_t u)v\|^\delta}\, dm_U(u) < \frac{c}{\|v\|^\delta}$$

for every $v \in V$ and $t \ge t_c$.

We now give a proof of the lemma stated at the beginning of the section.

Proof of Lemma 8. This essentially follows from the irreducible version of [22]. We equip $V = \oplus_i V_i$ with the max norm. That is, for $v = (v_i)$, $\|v\| = \max_i \|v_i\|$. We also note that the inequality we aim to prove is independent of the choice of norm.

Let $0 < c < 1$ and $0 < \delta < \delta_0$. We will choose δ_0 in the course of the proof.

Given $v = (v_i)$, let i_0 be the index with $\|v\| = \|v_{i_0}\|$. Then,

$$\|\rho(a_t u)v\| = \max_i \|\rho(a_t u)v_i\| \ge \|\rho(a_t u)v_{i_0}\|.$$

By the irreducible case (Lemma 11), we have the existence of $\delta_i \in (0,1)$ such that for every $\delta \in (0,\delta_i)$ contraction occurs for every $v_i \in V_i$ and t sufficiently large. To finish the proof, take $\delta_0 := \min_i \delta_i$ and we have

$$\begin{aligned}\frac{1}{m_U\big(B_2^U\big)} \int_{B_2^U} \frac{1}{\|\rho(a_t u)v\|^\delta}\, dm_U(u) &\le \frac{1}{m_U\big(B_2^U\big)} \int_{B_2^U} \frac{1}{\|\rho(a_t u)v_{i_0}\|^\delta}\, dm_U(u)\\ &< \frac{c}{\|v_{i_0}\|^\delta} = \frac{c}{\|v\|^\delta}\end{aligned}$$

for every $v \in V$ and t sufficiently large. □

Now we apply the above Linear algebra lemma to a specific representation which will be used to control the height function. First we give a definition. See also the end of Section 2 of Eskin–Margulis [12].

DEFINITION 12 (Maximal Parabolic Subgroups)**.** When a lattice Γ is non-uniform, we define a finite collection Δ of maximal parabolic subgroups of G as follows. A parabolic subgroup P of G is called Γ-rational if $\Gamma \cap R_u(P)$ is a lattice in $R_u(P)$, where $R_u(P)$ is the unipotent radical of P. If G is of real rank 1, then we let $\Delta = \{P_0\}$ where P_0 is a Γ-rational minimal parabolic subgroup of G. The existence of P_0 follows from Garland–Raghunathan [18]. If the real rank of G is not less than 2, then by the Margulis Arithmiticity theorem, Γ is arithmetic. Hence we let $\Delta = \{P_1, P_2, ..., P_r\}$ where P_k are standard parabolic subgroups of G, with respect to its maximal $\mathbb{Q}$-split torus A_0. For every $1 \le k \le r$, there exists a finite-dimensional irreducible representation $\rho_k : G \to GL(W_k)$ and vectors $w_k \in W_k$ such that the stabilizer of $\mathbb{R}w_k$ is P_k.

REMARK 13 (Upper bound on the dimension of W_k)**.** For later computational purposes (see Lemma 19 and Lemma 33), we take W_k to have dimension no greater than $\dim(G)^2$. We can do so by choosing

$$W_k := \wedge^{\dim(R_u(P_k))}\mathrm{Lie}(R_u(P_k)) \subseteq \wedge^{\dim(R_u(P_k))}\mathrm{Lie}(G)$$

and w_k to be a normalized diagonal element of W_k.

The following result is an analogue of Condition A of Eskin–Margulis [12]. We can deduce it by applying Lemma 8 to the representation above.

LEMMA 14 (Linear algebra lemma for height functions)**.** *Let Γ be a non-uniform lattice of G and consider the representation $\rho : G \to GL(W_k)$. Then there exists $0 < \delta_1 = \delta_1(\dim(G)) \ll 1$ such that for all $0 < \delta < \delta_1$ and $0 < c < 1$, there exists $t_{\delta,c} = t_{\delta,c}(G, H) > 0$ such that for every $v \in Gw_k$ and $t \ge t_{\delta,c}$,*

$$\frac{1}{m_U(B_2^U)} \int_{B_2^{\mathrm{Lie}(U)}} \frac{1}{\|\rho_k(a_t u)v\|^\delta}\, dm_U(u) < \frac{c}{\|v\|^\delta}.$$

4. ABSTRACT MARGULIS INEQUALITY

In this section we prove an abstract result that yields exponential decay for Margulis functions.

THEOREM 15. *Suppose $F : X \to (0,\infty)$ satisfies the following properties*

- *(Log Continuity) For any compact subset $K \subset H$, there exists $\sigma = \sigma_F(K) > 1$ such that for all $g \in K$ and $x \in X$,*
$$\sigma^{-1}F(x) \le F(gx) \le \sigma F(x)$$
- *(Margulis Inequality for F) There exist constants $0 < c < 1$, $t = t_c \gg 1$, and $b > 0$ such that for any $x \in X$,*
$$(A_{2,t}F)(x) := \frac{1}{m_U(B_2^U)} \int_{B_2^U} F(a_t ux)\, dm_U(u) < cF(x) + b.$$

Then, there exist absolute constants $C > 0$ and $B > 0$ such that for any $t \ge t_c$,

$$(A_t F) \le C \cdot c^{t/t_c} F + B.$$

Proof. **Step 1:** We find an upper bound on $A_{nt}F$.

Recall,

$$(A_{2,t}f)(x) := \frac{1}{m_U(B_2^U)} \int_{B_2^U} F(a_t ux)\, dm_U(u).$$

By iterating our operator, we have

$$\left(A_{2,t}^n F\right)(x) < c^n F(x) + B',$$

where $B' = O(b) = b\sum_{j=0}^n c^j$. On the other hand,

$$\begin{aligned}&\left(A_{2,t}^n F\right)(x)\\ &= \frac{1}{m_U^n\left(B_2^U\right)} \int_{\left(B_2^U\right)^{n-1}} \left(\frac{1}{m_U\left(B_2^U\right)} \int_{B_2^U} F\left(a_{nt}\phi_n(\vec{u})u_1 x\right) dm_U(u_1) \right) d(m_U)^{n-1}(\vec{u}),\end{aligned}$$

where $\phi : (B_2^U)^{n-1} \to U$ is given by

$$\vec{u} = (u_n, \dots, u_2) \mapsto \phi_n(\vec{u}) = \left(a_{-(n-1)t} u_n a_{(n-1)t}\right) \cdots (a_{-2t} u_3 a_{2t})(a_{-t} u_2 a_t).$$

Hence, there must exist some $\vec{u} \in (B_2^U)^{n-1}$ so that

$$\frac{1}{m_U\left(B_2^U\right)} \int_{B_2^U} F\left(a_{nt}\phi_n(\vec{u})u_1 x\right) dm_U(u_1) < c^n F(x) + B'.$$

Now note that since t is large, $\phi((B_2^U)^{n-1}) \subseteq B_1^U$ and so $\phi(\vec{u})^{-1} B_1^U \subseteq B_2^U$. Hence,

$$\begin{aligned}&\frac{1}{m_U\left(B_2^U\right)} \int_{\phi(\vec{u})^{-1} B_1^U} F\left(a_{nt}\phi_n(\vec{u})u_1 x\right) dm_U(u_1)\\ &\leq \frac{1}{m_U\left(B_2^U\right)} \int_{B_2^U} F\left(a_{nt}\phi_n(\vec{u})u_1 x\right) dm_U(u_1).\end{aligned}$$

Make the substitution $v = \phi(\vec{u})^{-1} u$ and note that m_U is translation invariant to obtain

$$\frac{1}{m_U\left(B_2^U\right)} \int_{\phi(\vec{u})^{-1} B_1^U} F\left(a_{nt}\phi_n(\vec{u})u_1 x\right) dm_U(u_1) = \frac{1}{m_U\left(B_2^U\right)} \int_{B_1^U} F\left(a_{nt} v x\right) dm_U(v).$$

Putting everything together, we obtain

$$(A_{nt}F)(x) = \int_{B_1^U} F(a_{nt} v x)\, dm_U(v) < m_U\left(B_2^U\right) c^n F(x) + B,$$

where $B = B' m_U(B_2^U)$.

Step 2: Now we use the previous step to show for arbitrarily large $t \geq t_c$ we have a bound on $A_t F$.

Let $K_1 = \{a_t : 0 \leq t \leq t_c\}$ be a fixed compact set and let $\sigma_1 = \sigma_F(K_1)$ be the constant from the log continuity property of F. Suppose that $t \geq t_c$, and let $\lfloor t/t_c \rfloor = n$. Since $0 \leq t - nt_c < t_c$, by log-continuity of F,

$$\begin{aligned}\int_{B_1^U} F(a_t u x)\, dm_U(u) &= \int_{B_1^U} F\left(a_{t-nt_c} \cdot a_{nt_c} u x\right) dm_U(u)\\ &\leq \sigma_1 \int_{B_1^U} F(a_{nt_c} u x)\, dm_U(u).\end{aligned}$$

By Step 1,

$$\int_{B_1^U} F(a_{nt_c} u x)\, dm_U(u) = (A_{nt}F)(x) \leq m_U\left(B_2^U\right) c^n F(x) + B$$

and thus,

$$\int_{B_1^U} F(a_t ux)\,dm_U(u) \le \sigma_1\left(c^n m_U\big(B_2^U\big)F(x)+B\right) = \sigma_1 m_U\big(B_2^U\big)c^{\lfloor t/t_c\rfloor}F(x)+\sigma_1 B$$
$$\le \sigma_1 m_U\big(B_2^U\big)c^{t/t_c-1}F(x)+\sigma_1 B.$$

Letting $C=\sigma_1 m_U(B_2^U)c^{-1}$ and relabeling $\sigma_1 B$ as B finishes the proof. □

5. HEIGHT FUNCTIONS AND MARGULIS INEQUALITY

For this section, we make the following assumption: Γ will be taken to be a non-uniform lattice of G.

Thus, the space $X=G/\Gamma$ has a cuspidal part and we construct a height function h on X that measures how high a point $x\in X$ is in the cusp. We prove that h satisfies a Margulis inequality.

This is why we assume Γ is non-uniform; the height function only makes sense in this setting. In Section 6.1, we will make a small modification so that our results hold in the compact case as well.

The height function in use is essentially the same as in Eskin–Margulis [12]. However, instead of taking the average over some random walk on X, we average over the expanding translates of B_1^U, the unit ball in the horospherical subgroup.

THEOREM 16. *For any $0<\delta<\delta_1$ (δ_1 as in Lemma 14), there exists a height function $h=h_\delta : X\to(0,\infty)$ such that the following holds: for any $0<c<1$, there exists $t_c>0$ such that for any $t\ge t_c$, there exists an absolute constant $B_t>0$ such that*

$$A_{2,t}h < ch + B_t.$$

We will use Lemma 14 in the construction of the height function of Theorem 16. As such, we only consider $0<\delta<\delta_1$. Also note that Theorem 16 directly implies the (Margulis inequality) hypothesis in Theorem 15 for h. In Lemma 19 we shall see that h is log-continuous also, and thus we get the following exponential decay property for h.

COROLLARY 17. *For any $0<\delta<\delta_1$, height function $h=h_\delta$ (the same height function as in Theorem 16) satisfies the following: there exist $t_h>0$, $C_h>0$ and $B_h>0$ such that for all $t\ge t_h$,*

$$A_t h \le \frac{C_h}{2^{t/t_h}}\cdot h + B_h.$$

Proof. The result directly follows from Theorem 15, Theorem 16, and Lemma 19. We note that t_h is equal to $t_{1/2}$, defined as in Theorem 16. □

Construction of the height function h. The following construction is from Section 3.2 of Eskin-Margulis [12]. The construction of individual functions d_k, which are height functions with respect to parabolic subgroups P_k from Definition 12, appears in Dani–Margulis [7] too. See also Guan–Shi [20] where they used this height function to compute Hausdorff dimensions of divergent trajectories.

Let P_0 denote a minimal Γ-rational parabolic subgroup of G. Then we have the Langlands decomposition $P_0 = M_0 A_0 N_0$ where M_0 is semisimple, A_0 is abelian, and N_0 is the unipotent radical of P_0. If G has real rank greater than 2, then we let A_0 to be the fixed maximal $\mathbb{Q}$-split torus of G in Definition 12. Let $\mathfrak{a}$ denote the Lie algebra of A_0. We shall identify $\mathfrak{a}$ with its dual via the Killing form. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ denote the roots which we view as elements of the dual of $\mathfrak{a}$. A *Siegel set* is a set $\mathfrak{S} = K\mathscr{M}\mathscr{A}\mathscr{N}$ where K is the maximal compact subgroup of G, $\mathscr{M} \subseteq M_0$ and $\mathscr{N} \subseteq N_0$ are compact, and $\mathscr{A} = \mathscr{P}\{a \in A_0 : \alpha_k(\log a) < C$ for all $1 \leq k \leq r\}$ for some positive constant C.

Note that for appropriate choices of $\mathscr{M}$, $\mathscr{N}$, and C, there exists a finite set $J \subseteq G$ such that for every $g \in G$, the intersection $\mathfrak{S} \cap g\Gamma J$ is not empty. See Dani–Margulis [8] for details.

For $1 \leq k \leq r$, define $d_k(g) := \|\rho_k(g) w_k\|$ where $\rho_k : G \to GL(W_k)$ and $w_k \in W_k$ are defined as in Definition 12, with respect to parabolic subgroups P_k. By structure theory, there exists absolute constants C_0 and $c_1, c_2, \ldots, c_r$ such that for each $1 \leq k \leq r$,

$$d_k(g) = d_k(a) \ \text{ and } \ |\log(d_k(a)) - c_k \omega_k(\log a)| < C_0$$

for all $g \in G$ where $g = kman$ is the Langlands decomposition of g with respect to P_k and ω_k is the co-root corresponding to α_k; i.e., $\omega_k(\alpha_k) = 1$ and $\omega_k(\alpha_j) = 0$ for all $j \neq k$. Let

$$\beta_k(g) = \max_{\gamma \in \Gamma} \frac{1}{d_k(g\gamma)^{1/c_k}}.$$

Also, for $x = g\Gamma \in X$, we shall define $\beta_k(x) := \beta_k(g)$.

REMARK 18. There exists an absolute constant $C = C(G/\Gamma) > 1$ such that for any $g \in G$ and $g_1 \in \mathfrak{S} \cap g\Gamma J$,

$$C^{-1}\beta_k(g_1) < \beta_k(g) < C\beta_k(g_1).$$

Lastly, we choose a sequence of positive real numbers $\{q_k\}_{k=1}^r$ so that $\sum_k q_k \omega_k$ belongs to the postive Weyl chamber of $\mathfrak{a}$ and let

$$h_k(g) = \beta_k(g)^{1/q_k}.$$

For later use (see Proposition 22), we shall take $\{q_k\}_{k=1}^r$ to be normalized so that

$$\min_{1 \leq k \leq r} \{c_k q_k\} = 1.$$

Equipped with this construction from Eskin-Margulis [12], we define the height function that we use.

Our height function h will be defined to be

$$h := C_* \sum_{k=1}^r h_k^{\delta_*}$$

for some δ_* and C_*. The condition $\delta_* \ll 1$ will be later verified in the proof of Proposition 22 so that h satisfies the Margulis inequality, and the constant $C_* \gg 1$ will be later chosen in Remark 24 so that h is bounded away from 1.

LEMMA 19 (Log continuity of the height function). *For any compact subset $K \subset G$, there exists $\sigma_h = \sigma_h(K) \geq 1$ such that for all $g \in K$ and $x \in X$,*

$$\sigma_h^{-1} \cdot h(x) \leq h(gx) \leq \sigma_h \cdot h(x).$$

Moreover, σ_h can be chosen to be a constant only depending on a compact set K and $\dim(G)$, *each of which will be independent of lattice* Γ.

Proof. For each $1 \leq k \leq r$, the map $d_k = \|\rho_k(\cdot)w_k\|$ is log continuous. Thus, there exists $\sigma_k = \sigma_k(K)$ such that for all $g' \in K$ and $g \in G$,

$$\sigma_k^{-1} \cdot d_k(g) \leq d_k(g'g) \leq \sigma_k \cdot d_k(g).$$

Note that σ_k only depends on compact set K and $\dim(W_k)$. By Remark 13, $\dim(W_k). \leq \dim(G)^2)$ so that $\sigma := \max_{1\leq k\leq r}\{\sigma_k\}$ is a constant only depending on K and $\dim(G)$.

Now for each $1 \leq k \leq r$, let $\sigma'_k := \sigma^{1/c_k}$, where c_k are the constants used to define β_k. Then, for all $g' \in K$ and $g \in G$,

$$(\sigma'_k)^{-1} \cdot \beta_k(g) \leq \beta_k(g'g) \leq \sigma'_k \cdot \beta_k(g).$$

If the maximum in $\beta_k(g'g)$ is achieved by the same $\gamma \in \Gamma$ as in $\beta_k(g)$, then the result directly follows from log continuity of d_k and definition of σ'_k. Suppose that the maximum is achieved by different choice of γ;

$$\beta_k(g) = \frac{1}{d_k(g\gamma_0)^{1/c_k}} \quad \text{and} \quad \beta_k(g'g) = \frac{1}{d_k(g'g\gamma_1)^{1/c_k}}$$

for some $\gamma_0 \neq \gamma_1 \in \Gamma$. Then,

$$\beta_k(g'g) > \frac{1}{d_k(g'g\gamma_0)^{1/c_k}} = d_k(g'g\gamma_0)^{-1/c_k} \geq (\sigma \cdot d_k(g\gamma_0))^{-1/c_k} = (\sigma'_k)^{-1} \cdot \beta_k(g)$$

and

$$\beta_k(g'g) = d_k(g'g\gamma_1)^{-1/c_k} \leq (\sigma^{-1} \cdot d_k(g\gamma_1))^{-1/c_k} < \sigma'_k \cdot d_k(g\gamma_0))^{-1/c_k} = \sigma'_k \cdot \beta_k(g).$$

Lastly, by definition

$$h := C_* \sum_{k=1}^{r} h_k^{\delta_*} = C_* \sum_{k=1}^{r} \beta_k^{\delta_*/q_k},$$

so it follows that the height function h has log continuity with constant

$$\sigma_h := \max_{1\leq k\leq r} \left\{(\sigma'_k)^{\delta_*/q_k}\right\} = \sigma^{\max_{1\leq k\leq r}\{\delta_*/c_k q_k\}} = \sigma^{\delta_*/\min_{1\leq k\leq r}\{c_k q_k\}} = \sigma^{\delta_*}.$$

(Here, we are using the fact that the $\{q_k\}_{k=1}^r$ are normalized to satisfy the condition that $\min_{1\leq k\leq r}\{c_k q_k\} = 1$.) Since $\delta_* = \frac{1}{2}\delta_1$ is a constant only dependent on $\dim(G)$ (see Lemma 14), σ_h is only dependent on the compact set K and $\dim(G)$. □

The following lemma is a direct result of equation (33) from Eskin–Margulis [12].

LEMMA 20. *For any constant $C > 1$, there exists an absolute constant $D_C = D_C(G/\Gamma) > 1$ such that the following holds: if for some $1 \le k \le r$ and $g \in G$ there exist $g_1 \in \mathfrak{S} \cap g\Gamma J$ and $g_2 (\neq g_1) \in g\Gamma J$ such that $d_k(g_2) < C d_k(g_1)$, then*

$$h_k(g) \le D_C \prod_{j \neq k} h_j^{\lambda_{j,k}}(g)$$

where $\lambda_{j,k} = \frac{q_j |\langle \alpha_j, \alpha_k \rangle|}{q_k \langle \alpha_k, \alpha_k \rangle}$.

In view of Remark 18, we can rewrite the above Lemma 20 as follows.

LEMMA 21 (Upper bound for h_k)**.** *For any constant $C > 1$, there exists an absolute constant $D'_C = D'_C(G/\Gamma) > 1$ such that the following holds. If $\beta_k(g) = \frac{1}{d_k(g\gamma_0)^{1/c_k}}$ for some $\gamma_0 \in \Gamma$ and there exists $\gamma_1 (\neq \gamma_0) \in \Gamma$ such that $d_k(g\gamma_1) < C d_k(g\gamma_0)$, then*

$$h_k(g) \le D'_C \prod_{j \neq k} h_j^{\lambda_{j,k}}(g).$$

Now, we replace Condition A of Eskin–Margulis [12] with Lemma 14 and prove an Margulis inequality for our averaging operator A_t.

PROPOSITION 22 (Upper bound for $A_t h_k^\delta$)**.** *For any $0 < \delta < \delta_1$ and $0 < c < 1$, for all $t \ge t_{\delta,c}$ (where $t_{\delta,c}$ is as in Lemma 14), there exists an absolute constant $D_t > 0$ (depending only on t) such that for any $1 \le k \le r$,*

$$\left(A_{2,t} h_k^\delta\right)(g) := \frac{1}{m_U\left(B_2^U\right)} \int_{B_2^U} h_k^\delta(a_t u g)\, dm_U(u) \le c h_k^\delta(g) + D_t \prod_{j \neq k} h_j^{\delta \lambda_{j,k}}(g)$$

for any $g \in G$.

Proof. Let $1 \le k \le r$ be fixed. If for every $a_t u g$ (varying u over B_2^U) the maximum in β_k is achieved by the exact same $\gamma \in \Gamma$ as in $\beta_k(g)$, then we get $(A_{2,t} h_k^\delta)(g) \le c h_k^\delta(g)$ directly from Lemma 14, since

$$h_k(g) = \frac{1}{d_k(g)^{(\delta/c_k q_k)}} = \frac{1}{\|\rho_k(g) w_k\|^{(\delta/c_k q_k)}}$$

and $\delta/c_k q_k < \delta_1/c_k q_k \le \delta_1$. (Here we are using the fact that the $\{q_k\}_{k=1}^r$ are normalized to satisfy $\min_{1 \le k \le r}\{c_k q_k\} = 1$.)

Suppose that the maximum in β_k is achieved by a different γ for some $u \in B_2^U$;

$$\beta_k(g) = \frac{1}{d_k(g\gamma_0)^{1/c_k}} \quad \text{and} \quad \beta_k(g'g) = \frac{1}{d_k(g'g\gamma_1)^{1/c_k}}$$

for some $g' \in a_t B_2^U$ and $\gamma_0 \neq \gamma_1 \in \Gamma$. By definition of β_k, compactness of $a_t B_2^U$, and log continuity of d_k, we have

$$d_k(g\gamma_1) < C d_k(g'g\gamma_1) < C d_k(g'g\gamma_0) < C^2 d_k(g\gamma_0),$$

where $C = C_{t,k} = \sigma_{d_k}(a_t B_2^U)$. By Lemma 21 and by log continuity of h_k,

$$\frac{1}{m_U(B_2^U)} \int_{B_2^U} h_k^\delta(a_t u g)\, dm_U(u) \le \frac{1}{m_U(B_2^U)} \int_{B_2^U} C' h_k^\delta(g)\, dm_U(u) = C' h_k^\delta(g)$$
$$\le C' D'_{C^2} \prod_{j\neq k} h_j^{\delta\lambda_{j,k}}(g),$$

where $C' = C'_{t,k} = \sigma_{h_k}(a_t B_1^U)$, and D'_{C^2} is defined as in Lemma 21, for our choice of $C = C_{t,k}$. We note that $C = C_{t,k}$, $C' = C'_{t,k}$, and $D'_{C^2} = D'_{C^2_{t,k}}$ are constants only depending on t and k. Thus, by taking $D_t := \max_{1\le k\le r}\{C'_{t,k} D'_{C^2_{t,k}}\}$, we get the desired result. □

THEOREM 23. *For any triple of* $0 < \delta < \delta_1$, $0 < c < 1$, *and* $t \ge t_{\delta,c/2}$ *(*$t_{\delta,c/2}$ *as in Lemma 14), there exists* $0 < \varepsilon = \varepsilon_{t,c} \ll 1$ *such that the height function* $h = h_{t,c} := \sum_{k=1}^r (\varepsilon h_k)^\delta$ *satisfies*

$$A_{2,t} h \le c h + 1.$$

Proof. By Proposition 22, we have

$$A_{2,t} h_k^\delta \le \frac{c}{2}\cdot h_k^\delta + D_t \cdot \prod_{j\neq k} h_j^{\delta\lambda_{j,k}}.$$

Taking the sum over all $1 \le k \le r$ and multiplying ε^δ to both sides of the above equation yields,

$$A_{2,t}\left(\sum_{k=1}^r (\varepsilon h_k)^\delta\right) \le \frac{c}{2}\cdot \sum_{k=1}^r (\varepsilon h_k)^\delta + D_t \cdot \sum_{k=1}^r \varepsilon_k \prod_{j\neq k} (\varepsilon h_j)^{\delta\lambda_{j,k}},$$

where $\varepsilon_k = \varepsilon^{\delta\cdot(1-\sum_{j\neq k}\lambda_{j,k})}$ for each $1 \le k \le r$.

Since $\sum_k q_k \omega_k$ belongs to the positive Weyl chamber, we have that

$$\sum_{j\neq k} \lambda_{j,k} = \sum_{j\neq k} \frac{q_j |\langle \alpha_j, \alpha_k\rangle|}{q_k \langle \alpha_k, \alpha_k\rangle} < 1$$

for each $1 \le k \le r$. See also Equation (35) of [12]. Hence, by Jensen's inequality

$$\prod_{j\neq k} (\varepsilon h_j)^{\delta\lambda_{j,k}} = \exp\left(\sum_{j\neq k} \lambda_{j,k}\cdot \left(\log(\varepsilon h_j)^\delta\right) + \left(1 - \sum_{j\neq k}\lambda_{j,k}\right)\cdot 0\right)$$
$$\le \sum_{j\neq k} \lambda_{j,k}\cdot \exp\left(\log(\varepsilon h_j)^\delta\right) + \left(1 - \sum_{j\neq k}\lambda_{j,k}\right)\cdot \exp(0)$$
$$= \sum_{j\neq k} \lambda_{j,k} (\varepsilon h_j)^\delta + \left(1 - \sum_{j\neq k}\lambda_{j,k}\right)\cdot 1 \le \sum_{j\neq k} (\varepsilon h_j)^\delta + 1$$

and thus,

$$A_{2,t}\left(\sum_{k=1}^{r}(\varepsilon h_k)^{\delta}\right) \le \frac{c}{2}\cdot\sum_{k=1}^{r}(\varepsilon h_k)^{\delta} + \left(D_t\sum_{k=1}^{r}\varepsilon_k\right)\cdot\left(\sum_{k=1}^{r}(\varepsilon h_k)^{\delta}+1\right)$$
$$= \left(\frac{c}{2}+D_t\sum_{k=1}^{r}\varepsilon_k\right)\cdot\sum_{k=1}^{r}(\varepsilon h_k)^{\delta} + \left(D_t\sum_{k=1}^{r}\varepsilon_k\right).$$

Since $\varepsilon_k = \varepsilon^{\delta\cdot(1-\sum_{j\neq k}\lambda_{j,k})}$ and $\sum_{j\neq k}\lambda_{j,k} < 1$ for each $1 \le k \le r$, we can choose $\varepsilon = \varepsilon_{\delta,c} \ll 1$ small enough so that $\left(D_t\sum_{k=1}^{r}\varepsilon_k\right) < \frac{c}{2} < 1$. □

Proof of Theorem 16. For fixed $0 < \delta < \delta_1$, the class of height functions $\{h_{t,c}\}$ with $0 < c < 1$ and $t \ge t_{\delta,c/2}$ defined as in Theorem 23 are linear. That is, if we let $h = h_\delta = \sum_{k=1}^{r} h_k^{\delta}$, then

$$h_{t,c} = \sum_{k=1}^{r}(\varepsilon_{t,c}h_k)^{\delta} = \varepsilon_{t,c}^{\delta}\cdot\sum_{k=1}^{r}h_k^{\delta} = \varepsilon_{t,c}^{\delta}\cdot h.$$

By Theorem 23, for any pair of $0 < c < 1$ and $t \ge t_{\delta,c/2}$, we have

$$(A_{2,t}h_{t,c})(x) \le ch_{t,c}(x) + 1.$$

Multiply $B_t := 1/\varepsilon_{t,c}^{\delta}$ to both sides of the inequality and we get

$$(A_{2,t}h)(x) \le ch(x) + B_t.$$

Lastly, take $t_c = t_{\delta,c/2}$ and we are done. □

REMARK 24 (Lower bound for h)**.** For computational reasons that will become apparent later, we want our h to be large and bounded away from 1. Since $\{d_k(g) : g \in \mathfrak{S}\}$ ($\mathfrak{S}$ is the Siegel set) is bounded away from zero, by Remark 18 we have that for each $1 \le k \le r$, h_k is bounded away from zero and therefore, $h_\delta = \sum_{k=1}^{r} h_k^{\delta}$ is bounded away from zero. Thus, by multiplying some large $C \gg 1$, we can make our height function $h := Ch_\delta$ to be no less than 2. Note that multiplication by a constant does not effect Margulis inequalities (see proof of Theorem 16) except that it only makes the additive constant B_t bigger; our newly defined $h = Ch_\delta$ also satisfies Theorem 16 and Corollary 17.

For the remainder of the paper, if Γ is a non-uniform lattice, then we fix $\delta_* := \frac{1}{2}\delta_1$ and set our height function to be $h = C_* h_{\delta_*} \ge 2$. However, note that the exact value of δ_* is not important and all arguments in the later sections apply for any choice of $0 < \delta_* < \delta_1$.

6. RETURN LEMMA AND NUMBER OF NEARBY SHEETS

Let $H \subseteq S \subsetneq G$ be an intermediate orbit. For a closed S-orbit $Y = Sy$ and point $x \in X$, we shall define a window set $I_Y(x)$ which collects all the sheets of Y that are nearby x. Roughly, the idea is to collect sheets of Y within the injective ball $B^G_{\mathrm{inj}(x)}(x)$, but the exact size of our windows will be much smaller, and will be given in terms of the height function h. A formal definition of $I_Y(x)$ will be given in Definition 31.

The aim of this section is to show that $\#I_Y(X)$, the number of nearby sheets, is bounded in terms of volume of Y.

PROPOSITION 25 (Number of nearby sheets). *There exists a global constant*

$$C_1 = C_1(\dim(G)) > 0$$

such that for any intermediate subgroup $H \subseteq S \subsetneq G$, closed S-orbit $Y = Sy$, and $x \in X$, we have

$$\#I_Y(x) < C_1 \mathrm{vol}(Y).$$

6.1. **Height function and radius of injectivity.** First we compare $h(x)$, the value of the our height function at $x \in X$, with $\mathrm{inj}(x)$, the injectivity radius at point x. We note that the following Proposition is as an analog of Lemma 6.3 of Benoist-Quint [3].

PROPOSITION 26. *If Γ is a non-uniform lattice of G, then there is absolute constants $C_2 > 0$ and $m > 0$ such that for all $x \in X$,*

$$\mathrm{inj}(x)^{-1} \le C_2 h(x)^m.$$

Proof. Let dist_G denote the left invariant Riemannian metric on G. Suppose that for some $g_1 \neq g_2 \in B_\epsilon^G(e)$ and $x = g\Gamma \in X$, $g_1 x = g_2 x \in X$. Then, for any $\gamma \in \Gamma$, $(\gamma g)^{-1}(g_1^{-1} g_2)(g\gamma)$ is in Γ. Moreover, for each point in G, there is a neighborhood on which the metric dist_G is Lipschitz equivalent to the metric derived from matrix norm. Thus,

$$2\epsilon \ge \mathrm{dist}_G\left(e, g_1^{-1} g_2\right) \ge \mathrm{dist}_G\left(e, (\gamma g)^{-1}\left(g_1^{-1} g_2\right) g\gamma\right) \left\|\mathrm{Ad}(g\gamma)^{-1}\right\|^{-1}.$$

Since Γ is a lattice, $\inf_{e \neq \gamma \in \Gamma} \mathrm{dist}_G(e, \gamma) > 0$ and thus,

$$\mathrm{inj}(x) \gg \min_{\gamma \in \Gamma} \left\|\mathrm{Ad}(g\gamma)^{-1}\right\|^{-1}.$$

Since $\mathfrak{S} \cap g\Gamma J$ is non-empty, we choose $g' \in \mathfrak{S} \cap g\Gamma J$ and take its Langlands decomposition $g' = k'a'n'$ with respect to P_0; we get $\|\mathrm{Ad}(g')\| \asymp \|\mathrm{Ad}(a')\|$ and

$$\|\mathrm{Ad}(a')\| \ll \left(\frac{1}{\min_{1 \le k \le r} \exp(\omega_k(\log(a')))}\right)^r.$$

By Remark 18, $\min_{\gamma \in \Gamma} \|\mathrm{Ad}(g\gamma)^{-1}\|^{-1}$ is comparable with $\|\mathrm{Ad}(g')^{-1}\|^{-1}$ and $h(x)$ is comparable with $h(g')$. Therefore,

$$\mathrm{inj}(x)^{-1} \ll h(x)^m$$

where

$$m := \frac{r}{\min_{1 \le k \le r}\{\delta_* / q_k\}} = \frac{r}{\delta_*} \cdot \max_{1 \le k \le r}\{q_k\}.$$

(Here, δ_* is from the definition of height function h, as in Remark 24, and $\{q_k\}_{1 \le k \le r}$ are the positive real numbers used to define $h_k = \beta_k^{1/q_k}$.) □

Later in the proof of Proposition 25 and also in the construction of Margulis function F_Y (see Theorem 38), we shall see that Proposition 26 plays a key role, together with Theorem 16 and Corollary 17. That is, the key property of the height function h is that it is a Margulis function that is comparable with the injectivity radius.

We now generalize our definition of height function $h : X \to [2, \infty)$ to the case when Γ is cocompact, so that Theorem 16, Corollary 17, and Proposition 26 all holds true also when Γ is cocompact.

DEFINITION 27. If Γ is a cocompact lattice of G, then we define $h : X \to (0, \infty)$ to be the constant function $h \equiv 2$.

Proof of Theorem 16 and Corollary 17 (for cocompact Γ). All constant functions are Margulis functions. That is, for any $c > 0$, $t > 0$, and $x \in X$, we have

$$\begin{aligned}\left(A_{2,t} h\right)(x) &:= \frac{1}{m_U\left(B_2^U\right)} \int_{B_2^U} h(a_t u x)\, d m_U(u) \\ &= \frac{1}{m_U\left(B_2^U\right)} \int_{B_2^U} 2\, d m_U(u) = 2 < c h(x) + 2.\end{aligned}$$

Corollary 17 can be proved in a similar way. Simply take $t_h = 1$; for any $t \geq 2 t_h$, we have

$$(A_t h)(x) = 2 < \frac{1}{2^{t/t_h}} h(x) + 2. \qquad \square$$

Proof of Proposition 26 (for cocompact Γ). If Γ is a cocompact lattice of G, then $X = G/\Gamma$ is compact and $\varepsilon_0 := \inf_{x \in X} \operatorname{inj}(x) > 0$. Therefore, for any $x \in X$,

$$\operatorname{inj}(x)^{-1} \leq \frac{1}{\varepsilon_0} = \frac{1}{2\varepsilon_0} h(x). \qquad \square$$

For the remainder of the paper, the height function h will refer to either of the two; if Γ is non-uniform, then h will refer to the function defined in Section 4 and if Γ is cocompact, then h will refer to the constant function in Definition 27 above.

6.2. **Return lemma and number of sheets.** In Lemma 32, we shall first give a weaker bound on $\#I_Y(x)$, depending on both $\operatorname{vol}(Y)$ and $h(x)$. This shows that $\#I_Y(x)$ is uniformly bounded in terms of $\operatorname{vol}(Y)$ in the compact part of X. In the case where Γ is non-uniform, we shall make use of height function h to analyze the cuspidal part of X. We shall show that if the sheets of Y are very dense near a point x in the cuspidal part of X, then for some $a_t u x$ which lies in the compact part of X the sheets of Y must be very dense near $a_t u x$ too. This gives us the desired result; even when x is high up in the cuspidal part of X, there is a uniform bound for $\#I_Y(x)$, only in terms of $\operatorname{vol}(Y)$. Results in this section are analogous to Section 8 of Mohammadi–Oh [MO20] and Section 8 of Eskin–Mirzakhani–Mohammadi [EMM15].

LEMMA 28 (Return lemma)**.** *There exists a global constant $C_3 = C_3(\dim(G)) > 0$ and $Q = Q(G/\Gamma, H)$ such that the following holds: for every $x \in X$, there exists $u \in B_1^U$ so that $a_{t_x} u x \in X_{\mathrm{cpt}} := \{x \in X : h(x) \leq Q\}$ where $t_x = C_3 \log(h(x))$.*

Proof. By Lemma 7, there exists $H' \in \mathscr{H}$ and $g \in G$ such that $H = gH'g^{-1}$. For each $H' \in \mathscr{H}$, we fix a one parameter subgroup $\{a'_t\}$ of diagonalizable elements in H', and let U' be the unstable horospherical subgroup of G with respect to $\{a'_t\}$. Then, for H we let

$$\{a_t\} = g\{a'_t\}g^{-1} \text{ and } B_1^U = gB_1^{U'}g^{-1}.$$

Let t'_h, C'_h, and B'_h be the constants from Corollary 17, with respect to $H' \subset G$. That is, t'_h, C'_h, and B'_h are constants so that for all $t \geq t'_h$ and $x \in X$,

$$A'_t h(x) := \int_{B_1^{U'}} h(a'_t u' x)\, dm_{U'}(u') \leq \frac{C'_h}{2^{t/t'_h}} \cdot h(x) + B'_h.$$

Take $C_3 := \frac{1}{\log 2} \max_{H' \in \mathscr{H}}(t'_h)$. Also, let $\sigma_g > 1$ be the absolute constant following from the log continuity of h so that for any $x \in X$, $\sigma_g^{-1} h(x) \leq h(g^{\pm 1}x) \leq \sigma_g h(x)$. (Here, g is an element of G so that $H = gH'g^{-1}$, as defined above. Note that g is unbounded, dependent on H and so is σ_g. However, this will result in only Q being dependent of H, not C_3.) Then, for every $x \in X$,

$$\begin{aligned}
\left(A'_{t_x} h\right)\left(g^{-1}x\right) &= \int_{B_1^{U'}} h\left(a'_{t_x} u'\left(g^{-1}x\right)\right) dm_{U'}(u') \leq \frac{C'_h}{2^{t_x/t'_h}} \cdot h\left(g^{-1}x\right) + B'_h \\
&\leq \frac{C'_h}{2^{\frac{1}{\log 2} t'_h \log(h(x))/t'_h}} \cdot \sigma_g h(x) + B'_h \leq C'_h \sigma_g + B'_h
\end{aligned}$$

and thus, there exists $u' \in B_1^{U'}$ such that $h(a'_{t_x} u' g^{-1} x) \leq C'_h \sigma_g + B'_h$.

Now, if we take $u := gu'g^{-1} (\in B_1^U)$, then we have that

$$\begin{aligned}
h(a_{t_x} ux) &= h\left(ga'_{t_x} g^{-1} \cdot gu'g^{-1}x\right) = h\left(ga'_{t_x} u' g^{-1} x\right) \\
&\leq \sigma_g h\left(a'_{t_x} u' g^{-1} x\right) = \sigma_g (C'_h \sigma_g + B'_h).
\end{aligned}$$

Take $Q := \sigma_g(C'_h \sigma_g + B'_h)$. All that is left is to show that C_3 is a constant that only depends on $\dim(G)$ (and hence is independent of H and lattice Γ).

Recall that t'_h (see the proof of Theorem 23 and also the proof of Corollary 17) is the constant $t'_{\delta_*,1/4} = t'_{\delta_*,1/4}(G, H')$ as in Lemma 14. From Lemma 14, we have that $\delta_* = \frac{1}{2}\delta_1$ depends only on $\dim(G)$ and therefore, $t'_{\delta_*,1/4}$ is a constant that only depends on G and H' (see Lemma 14).

Since $\mathscr{H}$ is finite data that only depend on $\dim(G)$ (Lemma 7),

$$C_3 = \frac{1}{\log 2} \max_{H' \in \mathscr{H}} (t'_h)$$

is a constant that only depends on $\dim(G)$. The fact that C_3 only depends on $\dim(G)$ will be later used to show that constant D in Theorem 2 only depends on $\dim(G)$. □

REMARK 29. Let $\sigma_U > 1$ be a global constant such that for all $v \in \mathrm{Lie}(G)$ and $u \in B_1^U$,

$$\sigma_U^{-1} \|v\| \leq \|u.v\| \leq \sigma_U \|v\|,$$

where $u.v$ denotes the adjoint representation of $u \in G$ on $v \in \mathrm{Lie}(G)$. For later computational purposes (see the proof of Proposition 25), we shall assume that $Q > \sigma_U$. Even if we replace $Q = \sigma_g(C'_h\sigma_g + B'_h)$ by $Q = \max\{\sigma_g(C'_h\sigma_g + B'_h), \sigma_U\}$, Lemma 28 still holds.

We shall also assume that $C_3 \geq 1$ (this will also be used in the proof of Proposition 25). Even if we replace $C_3 = \frac{1}{\log 2}\max_{H'\in\mathscr{H}}(t'_h)$ with

$$C_3 = \max\left\{\frac{1}{\log 2}\max_{H'\in\mathscr{H}}\left(t'_h\right), 1\right\},$$

Lemma 28 still holds.

REMARK 30. Note that by Proposition 26, $X_{\mathrm{cpt}} \subseteq X_{\epsilon_X} = \{x \in X : \mathrm{inj}(x) \geq \epsilon_X\}$, where $\epsilon_X = C_2{}^{-1}Q^{-m}$ (where C_2 and m as in Proposition 26):

$$\mathrm{inj}(a_{t_x}ux) \geq C_2{}^{-1}h(x)^{-m} \geq C_2{}^{-1}Q^{-m} = \epsilon_X.$$

Recall, from the preliminaries that there exists an absolute constants $\sigma_0 > 1$ such that for all $w \in \mathrm{Lie}(G)$ with $\|w\| \leq \epsilon_X$,

$$\sigma_0^{-1}\|w\| \leq \mathrm{dist}(x, \exp(w)x) \leq \sigma_0\|w\|.$$

DEFINITION 31. Let S be a subgroup with $H \subseteq S \subsetneq G$ and consider the decomposition of $\mathrm{Lie}(G)$ given by $\mathrm{Lie}(G) = \mathrm{Lie}(S) \oplus V_S$ where V_S is $\mathrm{Ad}(\mathrm{Lie}(S))$-invariant, but not necessarily irreducible. For each closed S-orbit $Y = Sy$ and $x \in X$, we define the set

$$I_Y(x) = \left\{v \in V_S \smallsetminus \{0\} : \|v\| \leq \varepsilon_h \cdot h(x)^{-\kappa},\ \exp(v)x \in Y\right\}$$

where global constants $\kappa \gg 1$ and $\varepsilon_h \ll 1$ are chosen to be; $\kappa := \max\{m, 3C_3\}$ and $\varepsilon_h = \min\{2^\kappa\epsilon_X, \frac{1}{2}\sigma_0^{-1}C_2^{-1}\}$ (m and C_2 as in Proposition 26, C_3 as in Lemma 28, and ϵ_X as in Remark 30).

LEMMA 32. *Let $Y = Sy$ be a closed S-orbit where $H \subseteq S \subsetneq G$. For all $x \in X$, we have that*

$$\#I_Y(x) < C_4 h(x)^{d_S m}\mathrm{vol}(Y)$$

where d_S is the dimension of $\mathrm{Lie}(S)$ *and* $C_4 = (4C_2)^{d_S}$.

Proof. For any $x \in X$ and $v \in I_Y(x)$,

$$\mathrm{dist}(x, \exp(v)x) \leq \sigma_0\|v\| \leq \sigma_0\varepsilon_h h(x)^{-\kappa} \leq \frac{1}{2}C_2{}^{-1}h(x)^{-m} \leq \frac{1}{2}\mathrm{inj}(x).$$

(Since $h \geq 2$ and $\varepsilon_h \leq 2^\kappa\epsilon_X$, $\|v\| \leq \varepsilon_h h(x)^{-\kappa} \leq \epsilon_X$ and thus $\mathrm{dist}(x, \exp(v)x) \leq \sigma_0\|v\|$. Also, since $h \geq 1$, $\kappa \leq m$, and $\varepsilon_h \leq \frac{1}{2}\sigma_0^{-1}C_2^{-1}$, we have $\sigma_0\varepsilon_h h(x)^{-\kappa} \leq \frac{1}{2}C_2{}^{-1}h(x)^{-m}$.)

It follows that for each $v \in I_Y(x)$, $\mathrm{inj}(\exp(v)x) \geq \frac{1}{4}\mathrm{inj}(x)$, which means that the balls

$$\left(B_Y(\exp(v)x, \mathrm{inj}(x)/4)\right)_{v\in I_Y(x)}$$

are disjoint from each other. Hence,

$$\begin{aligned}\#I_Y(x)\cdot\mathrm{vol}\left(B_S(e, \mathrm{inj}(x)/4\right) &= \mathrm{vol}\left\{\cup(B_Y(\exp(v)x, \mathrm{inj}(x)/4)) : v \in I_Y(x)\right\} \\ &\leq \mathrm{vol}\left(B_Y(x, \mathrm{inj}(x)\right) \leq \mathrm{vol}(Y).\end{aligned}$$

Therefore,

$$\begin{aligned}\#I_Y(x) &\le \mathrm{vol}\big(B_S(e,\mathrm{inj}(x)/4)\big)^{-1}\cdot \mathrm{vol}(Y)\\ &< 4^{d_S}\mathrm{inj}(x)^{-d_S}\cdot \mathrm{vol}(Y) \le 4^{d_S}C_2{}^{d_S}h(x)^{d_S m}\mathrm{vol}(Y).\end{aligned}$$

□

Proof of Proposition 25. If $x\in X$ is so that $h(x)\le Q^2$, then by Lemma 32,

$$\#I_Y(x) < (4C_2)^{d_S}Q^{2d_S m}\mathrm{vol}(Y).$$

Suppose that $h(x) > Q^2$. By Lemma 28, there exist $u\in B_1^U$ and $t_x = C_3\log(h(x))$ such that $h(a_{t_x}ux)\le Q$.

We claim that if $v\in I_Y(x)$, then $a_{t_x}u.v\in I_Y(a_{t_x}ux)$ and moreover, the map $a_{t_x}u : I_Y(x)\to I_Y(a_{t_x}ux)$ which sends $v\mapsto a_{t_x}u.v$ is injective. If $v\in I_Y(x)$, then

$$\begin{aligned}\|a_{t_x}u.v\| &\le e^{t_x}\sigma_U\|v\| = \exp(C_3\log(h(x)))\sigma_U\|v\| = \sigma_U h(x)^{C_3}\|v\|\\ &\le \sigma_U h(x)^{C_3}\varepsilon_h h(x)^{-\kappa} \le \sigma_U\varepsilon_h h(x)^{-2\kappa/3} \le \sigma_U\varepsilon_h Q^{2\cdot(-2\kappa/3)}\\ &= \sigma_U\varepsilon_h Q^{-4\kappa/3} \le \varepsilon_h Q^{-\kappa}\end{aligned}$$

since $\kappa\ge 3C_3\ge 3$, $h(x) > Q^2$, and $Q\ge\sigma_U$. On the other hand, $h(a_{t_x}ux)^{-\kappa}\ge Q^{-\kappa}$ and thus,

$$\big\|a_{t_x}u.v\big\| < \varepsilon_h h\big(a_{t_x}ux\big)^{-\kappa}.$$

Moreover,

$$a_{t_x}\exp(v)x = \exp(a_{t_x}u.v)a_{t_x}ux\in Y$$

since Y is S-invariant. Therefore, $a_{t_x}u.v\in I_Y(a_{t_x}u_s x)$ and the map $v\mapsto a_{t_x}u.v$ from $I_Y(x)$ to $I_Y(a_{t_x}ux)$ is injective. Consequently,

$$\#I_Y(x)\le \#I_Y\big(a_{t_x}ux\big) < (4C_2)^{d_S}Q^{2d_S m}\mathrm{vol}(Y).$$

We complete the proposition by taking $C_1 = (4C_2)^{d_G}Q^{2d_G m}$ where d_G denotes the dimension of $\mathrm{Lie}(G)$. □

We end this section with the following lemma on κ, which will be later used to show that constant D in Theorem 2 is an absolute constant that only depends on $\dim(G)$.

LEMMA 33. *Constant κ in Definition 31 has an upper bound as a function of* $\dim(G)$.

Proof. Since $\kappa := \max\{m, 3C_3\}$ and C_3 is a constant that only depends on $\dim(G)$ (see Lemma 28), it is enough to show that m from Proposition 26 has an upper-bound as a function of $\dim(G)$.

Recall that (see the proof of Proposition 26)

$$m = \frac{r}{\delta_*}\cdot\max_{1\le k\le r}\{q_k\} = \frac{r}{\frac{1}{2}\delta_1}\cdot\max_{1\le k\le r}\{q_k\},$$

where δ_1 is as in Lemma 14 and q_k are as in definition of h_k. By Lemma 14, δ_1 is a constant that only depends on $\dim(G)$. From reduction theory, we have that $r\le\dim(G)$. Lastly, since the root system $\{\omega_k\}$ is a datum that only depends on G, the value of $\max_{1\le k\le r}\{q_k\}$ also has an upper bound in terms of $\dim(G)$. □

7. MARGULIS FUNCTION: CONSTRUCTION AND ESTIMATES

In this section we construct Margulis functions that we associate to the orbit of an intermediate subgroup. The main result of this section is a Margulis inequality for the functions we consider (Theorem 38).

DEFINITION 34 (Margulis function)**.** For an intermediate subgroup $H \subseteq S \subsetneq G$ and closed S-orbit $Y = Sy$, define $f_Y := X \to (0,\infty)$ by

$$f_Y(x) := \begin{cases} \sum_{v\in I_Y(x)} \|v\|^{-\delta_F}, & \text{if } I_Y(x) \neq \emptyset \\ h(x), & \text{otherwise} \end{cases}.$$

where $\delta_F := \min\{\delta_0/2, 1/\kappa\}$, δ_0 as in Lemma 8 applied to the adjoint representation of G on the Lie algebra $\mathrm{Lie}(G)$ and κ as in Definition 31.

For $\lambda \geq 1$, define $F_{\lambda,Y} : X \to (0,\infty)$ by

$$F_{\lambda,Y}(x) = f_Y(x) + \lambda \mathrm{vol}(Y) h(x).$$

We will later fix an explicit λ in Theorem 38 so that $F_{\lambda,Y}$ satisfies a Margulis inequality.

REMARK 35. Note that by Lemma 8 and Lemma 33, δ_F can be thought of as a constant that only depends on $\dim(G)$. Later, this will imply that constant D in Theorem 2 is only dependent on $\dim(G)$.

PROPOSITION 36 (Log continuity of $F_{\lambda,Y}$)**.** *Let K be a compact subset of S. Then there exists an absolute constant $\sigma = \sigma(K)$ (only depending on K and independent on the choice of Y and λ) such that for all Margulis function $F_{\lambda,Y}$, point $x \in X$, and $g \in K$,*

$$\sigma^{-1} F_{\lambda,Y}(x) \leq F_{\lambda,Y}(gx) \leq \sigma F_{\lambda,Y}(x).$$

Proof. Recall that we denote the adjoint representation of elements $g \in G$ and $v \in \mathrm{Lie}(G)$ as $g.v$. Since K is compact, there exists $R_K \geq 1$ so that

$$R_K^{-1}\|v\| \leq \|g.v\| \leq R_K\|v\|$$

for every $g \in K$ and $v \in \mathrm{Lie}(G)$. Also, by the log continuity of h, there exists $\sigma_h = \sigma_h(K) \geq 1$ so that

$$\sigma_h^{-1} \cdot h(x) \leq h(gx) \leq \sigma_h \cdot h(x)$$

for every $g \in K$ and $x \in X$.

If $I_Y(gx)$ is empty, then

$$f_Y(gx) = h(gx) \leq \sigma_h \cdot h(x)$$

Now suppose that $I_Y(gx)$ is not empty. Set $\varepsilon = R_K^{-1}\varepsilon_h h(x)^{-\kappa}$. Note that we can write

$$f_Y(gx) = \sum_{\substack{v\in I_Y(gx)\\ \|v\|<\varepsilon}} \|v\|^{-\delta_F} + \sum_{\substack{v\in I_Y(gx)\\ \|v\|\geq\varepsilon}} \|v\|^{-\delta_F}.$$

By Proposition 25 since $\#I_Y(gx) \le C_1\mathrm{vol}(Y)$, then we have the following bound for the second term above,

$$\sum_{\substack{v\in I_Y(gx)\\ \|v\|\ge\varepsilon}} \|v\|^{-\delta_F} \le C_1\mathrm{vol}(Y)\varepsilon^{-\delta_F} = \left(C_1\left(R_K\varepsilon_h^{-1}\right)^{\delta_F}\mathrm{vol}(Y)\right)\cdot h(x)^{\delta_F\kappa} \le C_5\mathrm{vol}(Y)h(x),$$

where $C_5 := C_1(R_K\varepsilon_h^{-1})^{\delta_F}$. (Here we are using that $h \ge 1$ and $\delta_F \le 1/\kappa$.) If there is no $v \in I_Y(gx)$ with $\|v\| < \varepsilon$, then this proves the claim. If there is $v \in I_Y(gx)$ with $\|v\| < \varepsilon$, then

$$\left\|g^{-1}.v\right\| < R_K\varepsilon = \varepsilon_h h(x)^{-\kappa}.$$

Thus, $g^{-1}.v \in I_Y(x)$. Setting $v' = g^{-1}v$ yields that

$$\sum_{\substack{v\in I_Y(gx)\\ \|v\|<\varepsilon}} \|v\|^{-\delta_F} \le \sum_{v'\in I_Y(x)} \|g.v'\|^{-\delta_F} \le R_K^{\delta_F}\sum_{v'\in I_Y(x)} \|v'\|^{-\delta_F} = R_K^{\delta_F} f_Y(x).$$

In total, we have a bound of the form

$$f_Y(gx) \le R_K^{\delta_F} f_Y(x) + (C_5\mathrm{vol}(Y)+\sigma_h)h(x).$$

Thus,

$$\begin{aligned} F_{\lambda,Y}(gx) &= f_Y(gx) + \lambda\mathrm{vol}(Y)h(gx) \\ &\le R_K^{\delta_F} f_Y(x) + (C_5\mathrm{vol}(Y)+\sigma_h)h(x) + \lambda\mathrm{vol}(Y)\cdot\sigma_h h(x). \end{aligned}$$

Note that $\mathrm{vol}(Y)$ is bounded away from zero. That is, there exists an absolute constant $\tau > 0$ such that for any intermediate subgroup $H \subseteq S \subsetneq G$ and closed S-orbit $Y = Sy$, $\mathrm{vol}(Y) > \tau$. (See Lemma 37 below.) Since $\lambda \ge 1$, we have

$$\begin{aligned} F_{\lambda,Y}(gx) &\le R_K^{\delta_F} f_Y(x) + \left(C_5 + \sigma_h\tau^{-1} + \sigma_h\lambda\right)\mathrm{vol}(Y)h(x) \\ &\le R_K^{\delta_F} f_Y(x) + \left(C_5 + \sigma_h\tau^{-1} + \sigma_h\right)\lambda\mathrm{vol}(Y)h(x). \end{aligned}$$

Put $\sigma := \max\{R_K^{\delta_F}, C_5 + \sigma_h(\tau^{-1}+1)\}$. We note that σ is a constant only dependent on the compact set K, independent of Y and λ.

The remaining inequality

$$\sigma^{-1}F_{\lambda,Y}(x) \le F_{\lambda,Y}(gx)$$

is proved in a similar fashion. □

LEMMA 37. *There exists an absolute constant $\tau > 0$ such that for any intermediate subgroup $H \subseteq S \subsetneq G$ and closed S-orbit $Y = Sy$,* $\mathrm{vol}(Y) > \tau$.

Proof. This follows from the quantitative non-divergence of Dani–Margulis [7]. Let U' be a 1-parameter unipotent subgroup of H. By the quantitative non-divergence of the action of U' on X, there exists some $\rho > 0$ such that $m_Y(X \smallsetminus X_\rho) < 0.01$ for every closed S-orbit $Y = Sy$ ($H \subseteq S \subsetneq G$ and $y \in Y$), where m_Y is the probability Haar measure on Y.

Note that we have the Lie algebra decomposition $\mathrm{Lie}(G) = \mathrm{Lie}(S) \oplus V_S$. Let $\eta \asymp \rho$ be so that the map $g \mapsto gx$ is injective for all $x \in X_\rho$ and all

$$g \in \mathrm{Box}(\eta) := \exp\left(B_\eta^{\mathrm{Lie}(S)}\right)\exp\left(B_\eta^{V_S}\right).$$

For any connected component C of $Y \cap \mathrm{Box}(\eta)z$ with $z \in X_\rho$, there exists some $v \in V_S$ such that

$$C = C_v := \exp\left(B_\eta^{\mathrm{Lie}(S)}\right)\exp(v)z.$$

Since $m_Y(X \smallsetminus X_\rho) < 0.01$, $Y \cap X_\rho \neq \emptyset$ and thus,

$$\mathrm{vol}(Y) \geq \eta^{d_S} \geq \eta^{d_H}. \qquad \square$$

THEOREM 38 (Margulis Inequality for F_Y)**.** *Let $H \subseteq S \subsetneq G$ be an intermediate subgroup and $Y = Sy$ a closed S-orbit. Let $F_{\lambda,Y}$ $(\lambda \geq 1)$ denote the Margulis functions associated to Y. For any $0 < c < 1$, there exists $t = t_{F,c} > 0$ such that there exists global constants $\lambda_1 = \lambda_1(G/\Gamma, H) \geq 1$ and $E_1 = E_1(G/\Gamma, H) > 0$ such that the following holds for any closed orbit $Y = Sx$ $(H \subseteq S \subsetneq G)$ and its corresponding Margulis function $F_Y := F_{\lambda_1,Y}$:*

$$A_{2,t}F_Y \leq cF_Y + E_1\mathrm{vol}(Y).$$

Proof. We have that $\mathrm{Lie}(G) = \mathrm{Lie}(S) \oplus V_S$ where V_S is $\mathrm{Ad}(H)$-invariant, but has no $\mathrm{Ad}(H)$-invariant vectors. Since $\delta_F \leq \delta_0/2 < \delta_0$, by the Linear Algebra Lemma (Lemma 8) there exists $t'_c > 0$ so that

$$\frac{1}{m_U(B_2^U)}\int_{B_2^U}\frac{1}{\|a_t u.v\|^{\delta_F}}\,dm_U(u) < \frac{c}{\|v\|^{\delta_F}}$$

for every $v \in V_S$ and $t \geq t'_c$.

On the other hand, by Theorem 16 there exists $t''_c > 0$ such that for all $t \geq t''_c$, there exists absolute constant B_t (depending on t) so that

$$A_{2,t}h \leq \frac{c}{2}h + B_t. \tag{1}$$

Let t'_c and t''_c be as above and take $t = t_{F,c} = \max\{t'_c, t''_c\}$.

We first find a bound on $A_t f_Y$. Then we pick a particular value for λ and combine the first bound with the bound from Theorem 16 to reach the desired bound on F_Y.

Recall that (a_t) and U are subsets of H. Fix the compact set $K_t = a_t B_1^U \subseteq H$ and let $R_{K_t} \geq 1$ be a constant so that

$$R_{K_t}^{-1}\|v\| \leq \|g.v\| \leq R_{K_t}\|v\|$$

for every $g \in K_t$ and $v \in V_S$.

If $I_Y(gx)$ is empty for every $g \in K_t$, then F_Y is just a constant multiple of the height function h and thus by (1),

$$\begin{aligned}A_{2,t}F_Y = A_{2,t}(\lambda\mathrm{vol}(Y)h) &\leq \lambda\mathrm{vol}(Y)\left(\frac{c}{2}h + B_t\right)\\ &\leq c(\lambda\mathrm{vol}(Y)h) + \lambda\mathrm{vol}(Y)B_t \leq cF_Y + \lambda B_t\mathrm{vol}(Y).\end{aligned}$$

Suppose that $I_Y(gx)$ is not empty. Set $\varepsilon = R_{K_t}^{-1}\varepsilon_h h(x)^{-\kappa}$. Note that we can write

$$f_Y(gx) = \sum_{\substack{v\in I_Y(gx)\\ \|v\|<\varepsilon}}\|v\|^{-\delta_F} + \sum_{\substack{v\in I_Y(gx)\\ \|v\|\geq\varepsilon}}\|v\|^{-\delta_F},$$

and by the calculation from Proposition 36, we obtain a bound of the form,

$$f_Y(gx) \le \sum_{v' \in I_Y(x)} \|g.v'\|^{-\delta_F} + (C_5 \mathrm{vol}(Y) + \sigma_h) h(x)$$

where $C_5 = C_1(R_{K_t}\varepsilon_h^{-1})^{\delta_F}$ and $\sigma_h = \sigma_h(K_t)$ is the constant from the log continuity property of h. Integrating over B_2^U yields

$$\begin{aligned} &\frac{1}{m_U(B_2^U)} \int_{B_2^U} f_Y(a_t ux)\, dm_U(u) \\ &\le \sum_{v' \in I_Y(x)} \frac{1}{m_U(B_2^U)} \int_{B_2^U} \|a_t u.v'\|^{-\delta_F}\, dm_U(u) + (C_5 \mathrm{vol}(Y) + \sigma_h) h(x) \end{aligned}$$

and since $t \ge t_c'$ by Lemma 8,

$$\sum_{v' \in I_Y(x)} \frac{1}{m_U(B_2^U)} \int_{B_2^U} \|a_t u.v'\|^{-\delta_F}\, dm_U(u) < c \sum_{v' \in I_Y(x)} \|v\|^{-\delta_F}.$$

To summarize, we get the bound

$$\frac{1}{m_U(B_2^U)} \int_{B_2^U} f_Y(a_t ux)\, dm_U(u) \le c f_Y(x) + (C_5 + \sigma_h \tau^{-1}) \mathrm{vol}(Y) h(x) \tag{2}$$

where τ is as in Lemma 37.

Combining equation (1) and equation (2) together we have

$$\begin{aligned} A_{2,t} F_Y &\le c \cdot f_Y + \left(C_5 + \sigma_h \tau^{-1}\right) \mathrm{vol}(Y) \cdot h + \lambda \mathrm{vol}(Y) \left(\frac{c}{2} \cdot h + B_t\right) \\ &= c \cdot f_Y + \left(C_5 + \sigma_h \tau^{-1} + \frac{\lambda c}{2}\right) \mathrm{vol}(Y) \cdot h + \lambda B_t \mathrm{vol}(Y). \end{aligned}$$

Now choose $\lambda_1 := 2(C_5 + \sigma_h \tau^{-1})/c$ so that $\left(C_5 + \sigma_h \tau^{-1} + \frac{\lambda_1 c}{2}\right) = \lambda_1 c$ and we get the desired result,

$$A_{2,t} F_Y \le c F_Y + E_1 \mathrm{vol}(Y)$$

where $E_1 = \lambda_1 B_t = 2(C_5 + \sigma_h \tau^{-1}) B_t / c$. Note that $\lambda_1 \ge 1$, since C_5 and σ_h are constants larger than 1 and τ and c are constants smaller than 1. We remark that both λ_1 and E_1 depend on G/Γ and H, but are independent of Y. □

For the remainder of the paper, F_Y will refer to the Margulis function $F_{\lambda_1, Y}$ with fixed $\lambda_1 = 2(C_5 + \sigma_h \tau^{-1})/c$ as in Theorem 38.

COROLLARY 39 (Exponential decay). *There exists global constants $C_F > 0$ and $E_2 > 0$ such that for any closed orbit Y and for any $t \ge t_F$ (where $t_F := t_{F,1/2}$ defined as in Theorem 38),*

$$(A_t F_Y)(x) \le \frac{C_F}{2^{t/t_F}} F_Y(x) + E_2 \mathrm{vol}(Y)$$

Proof. The result follows from Theorem 15, Proposition 36, and Theorem 38. Especially, the fact that C_F and E_2 are global constants, independent of Y follows from the fact that the log continuity constants for F_Y depends only on the compact set and is independent on the choice of Y (see Proposition 36). □

The following is following result is standard, see [27, Lemma 7.3] or [15, Lemma 11.1].

PROPOSITION 40 (Margulis function on average)**.** *Let $H \subseteq S \subsetneq G$ denote an intermediate subgroup and $Y = Sy$ be a closed S-orbit. Let F_Y denote the associated Margulis function from Theorem 38. Let μ be an A-ergodic U-invariant measure with $\mu(Y) = 0$. Then*

$$F_Y \in L^1(\mu).$$

Proof. In this proof we will drop the subscript Y in F_Y for simplicity. For $k \in \mathbb{N}$, let $F_k := \min(F, k)$. Take t to be t_F, the constant obtained from Corollary 39.

By Moore's ergodicity theorem, we have that the action of $A = \{a_t : t \in \mathbb{R}\}$ is ergodic on X. Then, by the Birkhoff ergodic theorem, for μ-a.e. $x \in X$ and $k \in \mathbb{N}$,

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} F_k(a_{nt} x) = \int F_k \, d\mu.$$

There exists some $x_0 \in X$ such that for m_U-a.e. $u \in B_1^U$,

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} F_k(a_{nt} u x_0) = \int F_k \, d\mu.$$

Thus, by Egoroff's theorem, for each $k \in \mathbb{N}$ there exists a subset $E_k \subseteq B_1^U$ with $m_U(E_k) > \frac{1}{2}$ and $N_k \in \mathbb{N}$ such that for every $N > N_k$ and $u \in E_k$,

$$\frac{1}{N} \sum_{n=1}^{N} F_k(a_{nt} u x_0) > \frac{1}{2} \int F_k \, d\mu.$$

Integrate this inequality over B_1^U to obtain

$$\frac{1}{N} \sum_{n=1}^{N} \int_{B_1^U} F_k(a_{nt} u x_0) \, dm_U(u) > \frac{1}{2} \int F_k \, d\mu.$$

By Corollary 39, for all $n \in \mathbb{N}$,

$$\int_{B_1^U} F_k(a_{nt} u x_0) \, dm_U(u) \le \int_{B_1^U} F(a_{nt} u x_0) \, dm_U(u) < \frac{C}{2^n} F(x_0) + b,$$

where $C = C_F$ and $b = E_2 \mathrm{vol}(Y)$.

Choose n_0 so that $\frac{1}{2^{n_0}} F(x_0) \le 1$. Then for each $n \ge n_0$ and $N > \max(N_k, k n_0)$,

$$\begin{aligned}
\frac{1}{2} \int F_k \, d\mu &< \frac{1}{N} \sum_{n=1}^{N} \int_{B_1^U} F_k(a_{nt} u x_0) \, dm_U(u) \\
&= \frac{1}{N} \sum_{n=1}^{n_0} \int_{B_1^U} F_k(a_{nt} u x_0) \, dm_U(u) + \frac{1}{N} \sum_{n=n_0+1}^{N} \int_{B_1^U} F_k(a_{nt} u x_0) \, dm_U(u) \\
&\le \frac{n_0 k}{N} + \frac{1}{N} \sum_{n=n_0+1}^{N} \left(\frac{C}{2^n} F(x_0) + b \right) \\
&\le 1 + \frac{1}{N} \sum_{n=n_0+1}^{N} (C + b) = C + b + 1.
\end{aligned}$$

Thus,

$$\int F_k \, d\mu \le 2(C+b+1).$$

Taking $k \to \infty$ and using the monotone convergence theorem, we have $F \in L^1(\mu)$. □

8. ISOLATION OF CLOSED ORBITS

In this section, we prove Theorem 4 and Theorem 5. Results in this section are analogous to Section 10 of [27].

Proof of Theorem 4. We shall prove the following: for any two distinct closed S-orbits $Y = Sy$ and $Z = Sz$ ($H \subseteq S \subsetneq G$) of finite volume,

$$\mathrm{dist}(Y \cap K, Z) \gg_K \mathrm{vol}(Y)^{-1/\delta_F} \mathrm{vol}(Z)^{-1/\delta_F},$$

where K is a compact subset of X and δ_F is as in Definition 34. Recall that δ_F is a global constant only depending on G and H (and thus, independent of the choice of Γ, see Remark 35).

Let m_Y denote the Haar probability measure on Y. Since m_Y is an A-ergodic S-invariant probability measure, $m_Y(A_t F_Z) = m_Y(F_Z)$. Thus, by integrating the Margulis inequality

$$A_t(F_Z) < cF_Z + E_2 \mathrm{vol}(Z)$$

(c is some positive constant smaller than 1) from Corollary 39 over Y, we get

$$m_Y(F_Z) \le \frac{E_2}{1-c} \mathrm{vol}(Z).$$

Since K is compact, $\epsilon = \epsilon_K := \min_{x \in K} \mathrm{inj}(x) > 0$ and so for log continuity of F_Z applied to B^S_ϵ, there exists $\sigma = \sigma_{F_Z}(B^S_\epsilon) > 1$ such that for any $x \in X$ and $g \in B^S_\epsilon$,

$$F_Z(x) \le \sigma F_Z(gx).$$

Recall that the log continuity coefficients for F_Z is independent of closed orbit Z that only depends on the set that g belongs to (Proposition 36). Thus, σ is a global constant that only depends on K.

For any point $y \in Y \cap K$,

$$\begin{aligned} f_Z(y) \le F_Z(y) &\le \frac{1}{m_Y\left(B^S_\epsilon(y)\right)} \int_{g \in B^S_\epsilon(e)} \sigma F_Z(gy) \, dm_Y(gy) \\ &\le \frac{1}{m_Y\left(B^S_\epsilon(y)\right)} \sigma m_Y(F_Z) \le \frac{\sigma}{\epsilon^{d_S}} \cdot \frac{E_2}{1-c} \mathrm{vol}(Y)\mathrm{vol}(Z). \end{aligned}$$

Lastly, we observe that $\mathrm{dist}(y,Z)^{-\delta_F} \ll f_Z(y)$. If $I_Z(y)$ is non-empty, then

$$f_Z(y) = \sum_{v \in I_Z(y)} \|v\|^{-\delta_F} \ge \mathrm{dist}(y,Z)^{-\delta_F}.$$

If $I_Z(y)$ is empty, then $\mathrm{dist}(y,Z) > \varepsilon_h h(y)^{-\kappa}$ and so

$$f_Z(y) = h(y) \ge h(y)^{\kappa \delta_F} \ge \varepsilon_h^{\delta_F} \mathrm{dist}(y,Z)^{-\delta_F}.$$

(Here, we are using that $h \ge 1$ and $\delta_F := \min\{\delta_0/2, 1/\kappa\} \le 1/\kappa$.) □

Proof of Theorem 5. We shall prove the following: there exists a global constant $C_6 > 0$ such that for any intermediate subgroup $H \subseteq S \subsetneq G$,

$$\#\{Y : Y = Sy \text{ is a closed } S\text{-orbit and } \mathrm{vol}(Y) \le R\} < C_6 R^{d_G/\delta_F},$$

where d_G is the dimension of $\mathrm{Lie}(G)$ and δ_F is as in Definition 34.

We define constants $\rho > 0$ and $\eta > 0$ as in Lemma 37. Let $\rho > 0$ be a constant such that $m_Y(X \setminus X_\rho) < 0.01$ for every closed S-orbit $Y = Sy$ ($H \subseteq S \subsetneq G$ and $y \in Y$) and let $\eta \asymp \rho$ be a constant so that the map $g \mapsto gx$ is injective for all $x \in X_\tau$ and all

$$g \in \mathrm{Box}(\eta) := \exp\left(B_\eta^{\mathrm{Lie}(S)}\right)\exp\left(B_\eta^{V_S}\right)$$

(here, $\mathrm{Lie}(G) = \mathrm{Lie}(S) \oplus V_S$ is the Lie algebra decomposition). Then, for any connected component C of $Y \cap \mathrm{Box}(\eta)z$ with $z \in X_\rho$, there exists some $v \in V_S$ such that

$$C = C_v := \exp\left(B_\eta^{\mathrm{Lie}(S)}\right)\exp(v)z.$$

For $R > 0$, let

$$\mathscr{Y}(R) = \{Y : Y = Sy \text{ is closed } S\text{-orbit and } R/2 < \mathrm{Vol}(Sy) \le R\}.$$

By Theorem 4, for any distinct connected components C_v and $C_{v'}$ in

$$\left(\bigcup_{Y \in \mathscr{Y}(2^k)} Y\right) \cap \mathrm{Box}(\eta)z,$$

we have that

$$\|v - v'\| \gg_\rho 2^{-2k/\delta_F}.$$

The cardinality of any $2^{-2k/\delta_F}$-separated set in $B_\eta^{V_S}$ is, up to multiplicative constant,

$$\left(2^{2k/\delta_F}\right)^{(d_G - d_S)}.$$

Since $\mathrm{vol}(\mathrm{Box}(\eta)) = \eta^{d_G}$, we can cover X_ρ by $M = O(\eta^{-d_G})$ many sets of the form $\mathrm{Box}(\eta)z$. Choose such a cover $\{\mathrm{Box}(\eta)z_j : j = 1, \ldots, M\}$.

Then,

$$\begin{aligned}\#\mathscr{Y}\left(2^k\right) &\le 2^{-k+1} \sum_{Y \in \mathscr{Y}(2^k)} \mathrm{vol}(Y)\\ &\ll 2^{-k+1} \sum_{j=1}^{M} \sum_{C_v \in \mathrm{Box}(\eta)z_j} \mathrm{vol}(C_v).\end{aligned}$$

Since (1) $\mathrm{vol}(C_v) = \eta^{d_S} \ll 1$ for each C_v, (2) $\#\{C_v \in \mathrm{Box}(\eta)z_j\} \ll (2^{2k/\delta_F})^{(d_G - d_S)}$ for each $\mathrm{Box}(\eta)z_j$, and (3) $M = O(1)$, we have

$$2^{-k+1} \sum_{j=1}^{M} \sum_{C_v \in \mathrm{Box}(\eta)z_j} \mathrm{vol}(C_v) \ll 2^{2k(d_G - d_S)/\delta_F - k + 1}.$$

Recall that $\operatorname{vol}(Sy) \geq \eta^{d_S}$ since the volume of the orbit needs to contain at least one connected component C_v in some $\operatorname{Box}(\eta) z_j$. Let $n_0 = \lfloor d_S \log_2(\eta) \rfloor$ and $n_R = \lceil \log_2(R) \rceil$. Since

$$\{Sy : Sy \text{ is closed and } \operatorname{vol}(Sy) \leq R\} \subseteq \bigcup_{k=n_0}^{n_R} \mathscr{Y}\big(2^k\big),$$

we get

$$\#\{Y : Y = Sy \text{ is a closed } S\text{-orbit and } \operatorname{vol}(Y) \leq R\} \leq \sum_{k=n_0}^{n_R} \#\mathscr{Y}\big(2^k\big) \ll R^{2d_G/\delta_F}. \quad \square$$

9. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 2.

Proof of Theorem 2. For each point $x \in X$, our choice of T_x will be $T_x := h(x)^{1/\delta_F}$. Note that $h(x)$ is bounded in the compact part of X and thus, T_x can be chosen uniformly within a compact subset of X.

We fix a point $x \in X$. Let (T, R) be a pair of real numbers such that $T > T_x$ and $R > 2$ and suppose that x is $(R, 1/T)$-Diophantine with respect to H. Our final goal will be to show that there exist absolute constants $D = D(\dim(G))$, $A = A(G/\Gamma, H)$, and $C = C(G/\Gamma, H)$ independent of x, R, and T, such that condition (2) of Theorem 2 holds: for all $t \geq A \log T$,

$$m_U\big(\big\{u \in B_1^U : a_t u x \text{ is not } (R, R^{-D}, R^{-D})\text{-Diophantine}\big\}\big) < CR^{-1}.$$

Recall that $x' \in X$ is (R, R^{-D}, R^{-D})-Diophantine with respect to H if and only if

(1) $\operatorname{inj}(x) \geq R^{-D}$ and
(2) for all intermediate subgroups $H \subseteq S \subsetneq G$ and all closed S-orbit $Y = Sx'$ with $\operatorname{vol}(Y) \leq R$, we have $\operatorname{dist}(x, Y) \geq R^{-D}$.

Step 1: Recurrence to the compact part. First, we show that there exists $D_1 = D_1(\dim(G)) > 0$ and $A_1 = A_1(G/\Gamma, H) > 0$ such that for all $D \geq D_1$ and $A \geq A_1$, the following is true: for all $t \geq A \log T$,

$$m_U\big(\big\{u \in B_1^U : \operatorname{inj}(a_t u x) < R^{-D}\big\}\big) \ll R^{-1}.$$

Take $A_1 = \delta_F t_h / \log 2$. Then, for any $t \geq A \log T$,

$$t \geq A \log T \geq A_1 \log T_x = A_1 \log(h(x)^{1/\delta_F}) \geq A_1/\delta_F \cdot \log 2 = t_h$$

and so by Corollary 17,

$$\begin{aligned}(A_t h)(x) &= \int_{B_1^U} h(a_t u x)\, dm_U(u) \leq \frac{C_h}{2^{t/t_h}} h(x) + B_h \\ &\leq \frac{C_h}{2^{A \log T/t_h}} h(x) + B_h \leq \frac{C_h}{2^{A_1 \log T_x/t_h}} h(x) + B_h \\ &= \frac{C_h}{2^{(A_1/\delta_F t_h)\cdot \log(h(x))}} h(x) + B_h = C_h + B_h.\end{aligned}$$

For $x' \in X$, if $\text{inj}(x') \le R^{-D}$, then by Proposition 26,

$$h(x') \ge C_2^{-1/m} \cdot \text{inj}(x)^{-1/m} > C_2^{-1/m} \cdot R^{D/m}.$$

Applying this observation to points $a_t ux$ and using Chebyshev's theorem we obtain,

$$\begin{aligned} m_U\left(\left\{u \in B_1^U : \text{inj}(a_t ux) \le R^{-D}\right\}\right) &\le m_U\left(\left\{u \in B_1^U : h(a_t ux) > C_2^{-1/m} \cdot R^{D/m}\right\}\right) \\ &< C_2^{1/m} R^{-D/m} \cdot (A_t h)(x) \\ &\le C_7 R^{-D/m}, \end{aligned}$$

where $C_7 := C_2^{1/m}(C_h + B_h)$ is a global constant. Recall that m is a constant that depends only on $\dim(G)$ (see proof of Lemma 35). Thus, by taking $D_1 = m$, we get the desired result.

Step 2: Avoidance principle. Let $H \subseteq S \subsetneq G$ be an intermediate orbit. First we shall fix a single closed S-orbit $Y = Sy$ with volume less that R, and show that there exists $D_2 = D_2(\dim(G)) > 0$ and $A_2 = A_2(G/\Gamma, H) > 0$ such that for all $D \ge D_2$ and $A \ge A_2$, the following is true: for all $t \ge A \log T$,

$$m_U\left(\left\{u \in B_1^U : \text{dist}(a_t ux, Y) < R^{-D}\right\}\right) \ll R^{-1}.$$

Then, we will use Lemma 6 and Corollary 5 to piece together the results for different choices of Y: we show that there exists $D_3 = D_3(\dim(G)) > D_2$ such that for all $D \ge D_3$ and $A \ge A_2$, for all $t \ge A \log T$,

$$m_U\left(\left\{u \in B_1^U : \text{dist}(a_t ux, Y) < R^{-D} \text{ for some } Y \in O_R\right\}\right) \ll R^{-1}$$

(here, $O_R = \{Y = Sy : H \subseteq S \subsetneq G, Y \text{ is closed}, \text{vol}(Y) < R\}$ is the set of all closed orbits of volume less than R).

Step 2.1: Avoiding a single closed orbit. Construct Margulis functions f_Y and F_Y with respect to Y as in Section 6. If $I_Y(x)$ is empty, then

$$f_Y(x) = h(x) \le T_x^{\delta_F} \le T^{\delta_F}.$$

Otherwise, we have

$$f_Y(x) = \sum_{v \in I_Y(x)} \|v\|^{-\delta_F} \le \text{dist}(x, Y)^{-\delta_F} \# I_Y(x).$$

Since x is $(R, 1/T)$–Diophantine with respect to H, x is $(R, 1/T)$–Diophantine with respect to Y and thus, $\text{dist}(x, Y) \ge 1/T$. Combining with Proposition 25, we have

$$\text{dist}(x, Y)^{-\delta_F} \# I_Y(x) \le \text{dist}(x, Y)^{-\delta_F} \cdot C_1 \text{vol}(Y) \le C_1 T^{\delta_F} R.$$

Thus, using that $h(x) \le T^{\delta_F}$, we conclude

$$F_Y(x) = f_Y(x) + \lambda_1 \text{vol}(Y) h(x) \le C_1 T^{\delta_F} R + \lambda_1 T^{\delta_F} R = C_8 T^{\delta_F} R,$$

where $C_8 := C_1 + \lambda_1$ is a global constant independent of Y.

Now take $A_2 = \delta_F t_F / \log 2$ (here, t_F is as in Theorem 38). If $t \ge A \log T$, then

$$t \ge A \log T \ge A_2 \log T_x \ge A_2 \log h(x)^{1/\delta_F} \ge A_2/\delta_F \cdot \log 2 = t_F$$

and so by Corollary 39 we have

$$\begin{aligned}
A_t(F_Y(x)) &\le \frac{C_F}{2^{t/t_F}} F_Y(x) + E_2 \mathrm{vol}(Y) \\
&\le \frac{C_F}{2^{A \log T/t_F}} \cdot C_8 T^{\delta_F} R + E_2 R \\
&\le \left(\frac{1}{2^{A_2/t_F}}\right)^{\log T} \cdot C_F C_8 T^{\delta_F} R + E_2 R \\
&= (C_F C_8 + E_2) R.
\end{aligned}$$

If $\mathrm{dist}(x', Y) < R^{-D}$ for some $x' \in X$, then either there exists $v \in I_Y(x')$ with $\|v\| < R^{-D}$ or $\varepsilon_h h(x')^{-\kappa} < R^{-D}$. In either case, we have that

$$F_Y(x') = f_Y(x') + \lambda_1 \mathrm{vol}(Y) h(x') > \min\left\{R^{D\delta_F}, \lambda_1 \tau \varepsilon_h^{1/\kappa} R^{D/\kappa}\right\} \ge C_9 \cdot R^{D\delta_F},$$

where $C_9 := \lambda_1 \tau \varepsilon_h^{1/\kappa}$ is a global constant independent of Y. (Here, we are again using that $\delta_F := \min\{\delta_0/2, 1/\kappa\}$ and thus $\delta_F \le 1/\kappa$.)

Applying this observation to points $a_t u x$ and using Chebyshev's theorem we obtain,

$$\begin{aligned}
m_U\left(\left\{u \in B_1^U : \mathrm{dist}(a_t u x, Y) < R^{-D}\right\}\right) &\le m_U\left(\left\{u \in B_1^U : F_Y(a_t u x) \ge C_9 \cdot R^{D\delta_F}\right\}\right) \\
&< C_9^{-1} R^{-D\delta_F} \cdot A_t(F_Y(x)) \\
&\le C_{10} R^{-(D\delta_F - 1)},
\end{aligned}$$

where $C_{10} := (C_F C_8 + D_2)/C_9$ is another global constant independent of Y. Recall that δ_F is a global constant that depends only on G and H (see Remark 35). Take $D_2 = 2/\delta_F$.

Step 2.2: Avoiding all closed orbits of small volume. By Lemma 6, the number of intermediate subgroups $H \subseteq S \subsetneq G$ is finite; we shall denote this number as $N(G,H)$. By Theorem 5, for each fixed S, the number of closed S-orbits Y with $\mathrm{vol}(Y) < R$ is bounded by $C_6 R^{d_G/\delta_F}$. Therefore, the cardinality of the set

$$O_R = \left\{Y = Sy : H \subseteq S \subsetneq G, Y \text{ is closed}, \mathrm{vol}(Y) < R\right\}$$

is bounded above by $N(G,H) \cdot C_6 R^{d_G/\delta_F}$ and so

$$\begin{aligned}
&m_U\left(\left\{u \in B_1^U : \mathrm{dist}(a_t u x, Y) < R^{-D} \text{ for some } Y \in O_R\right\}\right) \\
&\quad \le \sum_{Y \in O_R} m_U\left(\left\{u \in B_1^U : \mathrm{dist}(a_t u x, Y) < R^{-D}\right\}\right) \\
&\quad \le \sum_{Y \in O_R} C_9 R^{-(D\delta_F - 1)} \\
&\quad < N(G,H) \cdot C_6 R^{d_G/\delta_F} \cdot C_{10} R^{-(D\delta_F - 1)} \\
&\quad = N(G,H) C_6 C_{10} \cdot R^{-D\delta_F + d_G/\delta_F + 1}.
\end{aligned}$$

Take $D_3 = D_3(\dim(G)) := (d_G/\delta_F + 2)/\delta_F$ (so that $-D\delta_F + d_G/\delta_F + 1 \ge -1$ for any $D \ge D_3$) and we get the desired avoidance principle: for all $D \ge D_3$ and

$A \geq A_2$, for all $t \geq A \log T$,

$$m_U\left(\{u \in B_1^U : \mathrm{dist}(a_t ux, Y) < R^{-D} \text{ for some } Y \in O_R\}\right) < C_{11} R^{-1}$$

(here, $C_{11} := N(G,H)C_6C_{10}$ is another global constant).

Now combine the results of Step 1 and Step 2. Take

$$D = D(\dim(G)) := \max\{D_1, D_3\},$$

$A = A(G/\Gamma, H) := \max\{A_1, A_2\}$, and $C = C(G/\Gamma, H) := C_7 + C_{11}$, and we get

$$\begin{aligned} &m_U\left(\{u \in B_1^U : a_t ux \text{ is not } (R, R^{-D}, R^{-D})\text{-Diophantine with respect to } H\}\right) \\ &\leq m_U\left(\{u \in B_1^U : \mathrm{inj}(a_t ux) < R^{-D}\}\right) \\ &\quad + m_U\left(\{u \in B_1^U : \mathrm{dist}(a_t ux, Y) < R^{-D} \text{ for some } Y \in O_R\}\right) \\ &< C_7 R^{-1} + C_{11} R^{-1} = CR^{-1}. \end{aligned}$$

□

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