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The slope gap distribution of a translation surface is a measure of how random the directions of the saddle connections on the surface are. It is known that Veech surfaces, a highly symmetric type of translation surface, have gap distributions that are piecewise real analytic. Beyond that, however, very little is currently known about the general behavior of the slope gap distribution, including the number of points of nonanalyticity or the tail.

We show that the limiting gap distribution of slopes of saddle connections on a Veech translation surface is always piecewise real analytic with *finitely* many points of nonanalyticity. We do so by taking an explicit parametrization of a Poincaré section to the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(X,\omega)$ associated to an arbitrary Veech surface (X,ω), and establishing a key finiteness result for the first return map under this flow. We use the finiteness result to show that the tail of the slope gap distribution of a Veech surface always has quadratic decay.

32G15, 37D40; 14H55

1 Introduction

We will study the slope gap distributions of Veech surfaces, a highly symmetric type of translation surface. *Translation surfaces* can be defined geometrically as finite collections of polygons with sides identified in parallel opposite pairs. If we place these polygons in the complex plane $\mathbb{C}$, the surface inherits a Riemann surface structure from $\mathbb{C}$, and the one-form dz gives rise to a well-defined holomorphic one-form on the surface. This leads to a second equivalent definition of a translation surfaces as a pair (X,ω) where X is a Riemann surface and ω is a holomorphic one-form on the surface. Every translation surface locally has the structure of $(\mathbb{C}, dz)$, except for at finitely many points that have total angle around them $2\pi k$ for some integer $k \geq 2$. These points are called *cone points* and correspond to the zeros of the one-form ω. A zero of order k gives rise to a cone point of angle $2\pi(k+1)$.

A translation surface inherits a flat metric from $\mathbb{C}$. *Saddle connections* are then straight-line geodesics connecting two cone points with no cone points in the interior. The *holonomy vector* of a saddle connection γ is then the vector describing how far and in what direction the saddle connection travels:

$$\boldsymbol{v}_\gamma = \int_\gamma \omega.$$

We will be interested in the distribution of directions of these vectors for various translation surfaces.

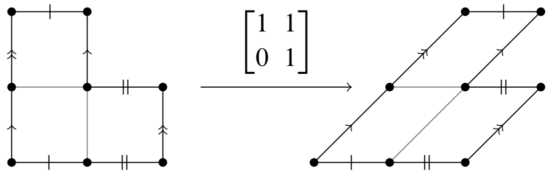

Figure 1: A matrix in $\mathrm{SL}(2,\mathbb{R})$ acting on a translation surface.

There is a natural $\mathrm{SL}(2,\mathbb{R})$ action on translation surfaces coming from the linear action of matrices on $\mathbb{R}^2$, as can be seen in Figure 1.

Sometimes this action produces a symmetry of the surface (X,ω). That is, after acting on the surface by the matrix, it is possible to cut and paste the new surface so that it looks like the original surface again. The collection of these symmetries is the stabilizer under the $\mathrm{SL}(2,\mathbb{R})$ action and is called the *Veech group* of the surface. It will be denoted by $\mathrm{SL}(X,\omega)$ and is a subgroup of $\mathrm{SL}(2,\mathbb{R})$. When the Veech group $\mathrm{SL}(X,\omega)$ of a translation surface has finite covolume in $\mathrm{SL}(2,\mathbb{R})$, the surface (X,ω) is called a *Veech surface*. Sometimes such surfaces are also called lattice surfaces since $\mathrm{SL}(X,\omega)$ is a lattice in $\mathrm{SL}(X,\mathbb{R})$. Veech surfaces have many nice properties, such as satisfying the *Veech dichotomy*: in any direction, every infinite trajectory on the surface is periodic or every infinite trajectory is equidistributed. For more information about translation and Veech surfaces see Hubert and Schmidt [6] and Zorich [13].

From work of Vorobets [12], it is known that, for almost every translation surface (X,ω) with respect to the Masur–Veech volume on any strata of translation surfaces (for details about Masur–Veech volume and strata, please see [13]), the angles of the saddle connections equidistribute in S^1. That is, if we let

$$\Lambda(X,\omega) := \{\text{holonomy vectors of saddle connections of } (X,\omega)\}$$

and normalize the circle to have total length 1, then for any interval $I \subset S^1$, as we let $R\to\infty$, the proportion of vectors in $\Lambda(X,\omega)$ of length $\le R$ that have direction in the interval I converges to the length of I.

A finer measure of the randomness of the saddle connection directions of a surface is its *gap distribution*, which we will now define. The idea of the gap distribution is that it records the limiting distribution of the spacings between the set of angles (or in our case, slopes) of the saddle connection directions of length up to a certain length R. We will be working with slope gap distributions rather than angle gap distributions because the slope gap distribution has deep ties to the horocycle flow on strata of translation surfaces. Thus, dynamical tools relating to the horocycle flow can be more easily applied to analyze the slope gap distribution.

Let us restrict our attention to the first quadrant and to slopes of at most 1, and define

$$\mathbb{S}(X,\omega) := \{\mathrm{slope}(\boldsymbol{v}) \mid \boldsymbol{v}\in\Lambda(X,\omega),\ 0<\mathrm{Re}(\boldsymbol{v}) \text{ and } 0\le \mathrm{Im}(\boldsymbol{v})\le \mathrm{Re}(\boldsymbol{v})\}.$$

We also allow ourselves to restrict to slopes of saddle connections of at most some length R in the ℓ_∞ metric, and define

$$\mathbb{S}_R(X,\omega) := \{\mathrm{slope}(\boldsymbol{v}) \mid \boldsymbol{v}\in\Lambda(X,\omega),\ 0<\mathrm{Re}(\boldsymbol{v}) \text{ and } 0\le \mathrm{Im}(\boldsymbol{v})\le \mathrm{Re}(\boldsymbol{v})\le R\}.$$

We let $N(R)$ denote the number of unique slopes $N(R) := |\mathbb{S}_R(X,\omega)|$. By results of Masur [7; 8], the growth of the number of saddle connections of length at most R in any translation surface is quadratic in R. We can order the slopes:

$$\mathbb{S}_R(X,\omega) = \{0 \le s_0^R < s_1^R < \cdots < s_{N(R)-1}^R\}.$$

Since $N(R)$ grows quadratically in R, we now define the *renormalized slope gaps* of (X,ω) to be

$$\mathbb{G}_R(X,\omega) := \{R^2(s_i^R - s_{i-1}^R) \mid 1 \le i \le N(R)-1 \text{ and } s_i \in \mathbb{S}_R(X,\omega)\}.$$

If there exists a limiting probability distribution function $f : [0,\infty) \to [0,\infty)$ for the renormalized slope gaps

$$\lim_{R\to\infty} \frac{|\mathbb{G}_R(X,\omega) \cap (a,b)|}{N(R)} = \int_a^b f(x)\,dx,$$

then f is called the *slope gap distribution* of the translation surface (X,ω). If the sequence of slopes of holonomy vectors of increasing length of a translation surface were independent and identically distributed uniform [0, 1] random variables, then a probability computation shows that the gap distribution would be a Poisson process of intensity 1. In all computed examples of slope gap distributions, however, this is not the case.

We give a brief overview of the literature on gap distributions of translation surfaces. In [2], Athreya and Chaika analyzed the gap distributions for typical surfaces and showed that, for almost every translation surface (with respect to the Masur–Veech volume), the gap distribution exists. They also showed that a translation surface is a Veech surface if and only if it has *no small gaps*, that is, if $\liminf_{R\to\infty}(\min(\mathbb{G}_R(X,\omega)) > 0$. In a later work [3], Athreya, Chaika and Lelièvre explicitly computed the gap distribution of the golden L, and in [1] Athreya gives an overview of results and techniques about gap distributions. Another relevant work is a paper by Taha [10] studying cross sections to the horocycle and geodesic flows on quotients of $\mathrm{SL}(2,\mathbb{R})$ by Hecke triangle groups. The computation of slope gap distributions involved understanding the first return map of the horocycle flow to a particular transversal of a quotient of $\mathrm{SL}(2,\mathbb{R})$.

In [11], Uyanik and Work computed the gap distribution of the octagon, and also showed that the gap distribution of any Veech surface exists and is piecewise real analytic. In [9], Sanchez went on to study the gap distributions of doubled slit tori. Up until then, all known slope gap distributions were for Veech surfaces. The above articles focus on gap distributions of *specific* translation surfaces and their $\mathrm{SL}(2,\mathbb{R})$ orbits. This work applies to any Veech surface and gives insight to the general behavior of the graph of the slope gap distribution of Veech surfaces. In fact, outside of [2], where it is shown that there are no small gaps, there is no other work in this direction with this level of generality.

Uyanik and Work gave an algorithm to compute the gap distribution of any Veech surface and showed that the gap distribution was piecewise analytic. However, their algorithm does not necessarily terminate in finite time and can make it seem like the gap distribution can have infinitely many points of nonanalyticity, as we will see in Section 2.3. We improve upon their algorithm to guarantee termination in finite time and show as a result that every Veech translation surface has a gap distribution with finitely many points of nonanalyticity. Uyanik and Work's algorithm starts by taking a transversal to the horocycle flow which a priori may break up into infinitely many components under the return map. Our key observation is that, by carefully choosing this transversal using the geometry of our surface, it will only break up into finitely many pieces, which will give the following theorem:

Theorem 1 *The slope gap distribution of any Veech surface has finitely many points of nonanalyticity.*

In addition, we show that the tail of the gap distribution of any Veech surface has a quadratic decay. Let $f(t) \sim g(t)$ mean that the ratio is bounded above and below by two positive constants as $t \to \infty$.

Theorem 2 *The slope gap distribution of any Veech surface has quadratic tail decay. That is, if f denotes the density function of the slope gap distribution, then*

$$\int_t^\infty f(x)\,dx \sim t^{-2}.$$

Thus, our results and the "no small gaps" result of [2] give a good understanding of the graph of the slope gap distribution of Veech surfaces: for some time the graph is identically zero before becoming positive. Afterward the graph has finitely many pieces where it is real analytic and may fluctuate up and down before it begins permanently decaying quadratically.

Organization In Section 2.1 we will go over background information on slope gap distributions, including how to relate the gap distribution to return times to a Poincaré section of the horocycle flow. In Section 2.2, we will outline the algorithm of Uyanik and Work, and observe some possible modifications. In Section 2.3, we will see how a couple steps of Uyanik and Work's algorithm apply to a specific Veech surface. A priori, the first return map to the Poincaré section breaks the section into infinitely many pieces, but after making some modifications to the parametrization we will see that there are in fact finitely many pieces. In Section 3 we will give a proof of Theorem 1. The strategy of the proof is to apply a compactness argument to show finiteness under our modified parametrization of the Poincaré section. We will show that, on a compact set that includes the Poincaré section, every point has a neighborhood that can contribute at most finitely many points of nonanalyticity to the gap distribution. This will give us that the slope gap distribution has finitely many points of nonanalyticity overall. In Section 4, as an application of Theorem 1, we prove quadratic decay of the slope gap distribution of Veech surfaces. Finally, in Section 5 we discuss a few further questions regarding slope gap distributions of translation surfaces.

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2 Background

2.1 A Poincaré section for the horocycle flow

In this section, we review a general strategy for computing the gap distribution of a translation surface by relating slope gap distributions to the horocycle flow. For more background and proofs of the statements given here, see [4] or [3].

Suppose that we wish to compute the slope gap distribution of a translation surface (X,ω). We let $\Lambda(X,\omega)$, sometimes shortened to just Λ, be the set of holonomy vectors of the surface. We may start by considering all of the holonomy vectors of (X,ω) in the first quadrant, with ℓ_∞ norm $\le R$. If we act on (X,ω) by the matrix

$$g_{-2\log(R)} = \begin{bmatrix} 1/R & 0 \\ 0 & R \end{bmatrix},$$

the slopes of the holonomy vectors of $g_{-2\log(R)}(X,\omega)$ in $[0,1]\times[0,R^2]$ are the same as R^2 times the slopes of the holonomy vectors of (X,ω) in $[0,R]\times[0,R]$, as we can see in Figure 2.

Another important observation is that the horocycle flow

$$h_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$$

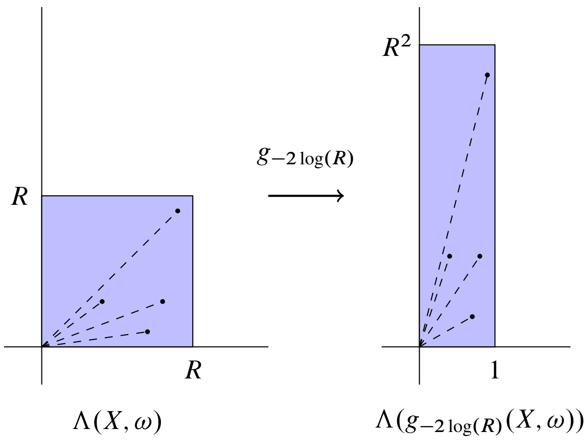

Figure 2: Upon renormalizing a surface (X,ω) by applying $g_{-2\log(R)}$, the slopes of the saddle connections of (X,ω) scale by R^2.

changes slopes of holonomy vectors by s. That is,

$$\mathrm{slope}(h_s(\boldsymbol{z})) = \mathrm{slope}(\boldsymbol{z}) - s$$

for $\boldsymbol{z} \in \mathbb{R}^2$. As a result, slope differences are preserved by the flow h_s.

Now we let the Veech group of the surface be $\mathrm{SL}(X,\omega)$ and we define a *Poincaré section* or transversal for horocycle flow on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(X,\omega)$. By transversal we mean a subset such that almost every orbit under horocycle flow intersects that subset in a nonempty countable discrete set of times. Two key related notions are given by the *return time* of a point in the transversal, which records how long it takes to return, and the *return map*, which outputs what the point has returned to in the transversal after flowing by the return time. Each of these are explicit in our situation and will be described below.

We consider the transversal given by the surfaces in the $\mathrm{SL}(2,\mathbb{R})$ orbit of (X,ω) with a short horizontal saddle connection of length ≤ 1. That is,

$$\Omega(X,\omega) = \{g\,\mathrm{SL}(X,\omega) \mid g\Lambda \cap ((0,1] \times \{0\}) \neq \varnothing\}.$$

By [1, Lemma 2.1], $\Omega(X,\omega)$ indeed is transversal for horocycle flow.

Then the slope gaps of (X,ω) for holonomy vectors of ℓ_∞ length $\le R$ are exactly $1/R^2$ times the set of $N(R)-1$ first return times to $\Omega(X,\omega)$ of the surface $g_{-2\log(R)}(X,\omega)$ under the horocycle flow h_s for $s \in [0, R^2]$. Here we are thinking of return times as the amount of time between each two successive times that the horocycle flow returns to the Poincaré section. In this way, the slope gaps of (X,ω) are related to the return times of the horocycle flow to the Poincaré section. Summarizing, since $\mathbb{G}_R(X,\omega)$ is the set of slope gaps renormalized by R^2, we have that

$$\mathbb{G}_R(X,\omega) = \{\text{first } N(R)-1 \text{ return times of } g_{-2\log(R)}(X,\omega) \text{ to } \Omega(X,\omega) \text{ under } h_s\}.$$

For a point z in the Poincaré section $\Omega(X,\omega)$, we denote by $R_h(z)$ the return time of z to $\Omega(X,\omega)$ under the horocycle flow. Then as one lets $R \to \infty$, this renormalization procedure gives us that

$$\lim_{R\to\infty} \frac{|\mathbb{G}_R(X,\omega) \cap (a,b)|}{N(R)} = \mu\{z \in \Omega(X,\omega) \mid R_h(z) \in (a,b)\},$$

where μ is the unique ergodic probability measure on $\Omega(X,\omega)$ for which the first return map under h_s is not supported on a periodic orbit. Computing the slope gap distribution then reduces to finding a convenient parametrization of the Poincaré section for the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(X,\omega)$, a suitable measure on this parametrization, and the first return time function to this the Poincaré section.

We note that this last point also makes it clear that every surface in the $\mathrm{SL}(2,\mathbb{R})$ orbit of a Veech surface has the same slope gap distribution. We also note that scaling the surface by c scales the gap distribution from $f(x)$ to $(1/c^4)f(x/c^2)$; see [11] for a proof of this latter fact.

2.2 Computing gap distributions for Veech surfaces

In [11], Uyanik and Work developed a general algorithm for computing the slope gap distribution for Veech surfaces. In particular, their algorithm finds a parametrization for the Poincaré section of any Veech surface and calculates the gap distribution by examining the first return time of the horocycle flow to this Poincaré section. In this section, we'll go over the basics of this algorithm. For more details about this algorithm as well as a proof of why it works, please see Uyanik and Work's original paper.

We start by supposing that (X,ω) is a Veech surface with $n<\infty$ cusps. Then we let $\Gamma_1,\dots,\Gamma_n$ be representatives of the conjugacy classes of maximal parabolic subgroups of $\mathrm{SL}(X,\omega)$. We are going to find a piece of the Poincaré section for each parabolic subgroup Γ_i. The idea here is that the set of shortest holonomy vectors of (X,ω) in each direction breaks up into $\bigcup_{i=1}^{n}\mathrm{SL}(X,\omega)\boldsymbol{v}_i$, where the $\boldsymbol{v}_i$ vectors are in the eigendirections of the generators of each Γ_i.

The Poincaré section is given by those elements $g\in\mathrm{SL}(X,\mathbb{R})/\mathrm{SL}(X,\omega)$ such that $g(X,\omega)$ has a short (length ≤ 1) horizontal holonomy vector:

$$\Omega(X,\omega)=\{g\,\mathrm{SL}(X,\omega)\mid g\Lambda\cap((0,1]\times\{0\})\neq\varnothing\}.$$

Here Λ is the set of holonomy vectors of (X,ω). Up to the action of $\mathrm{SL}(X,\omega)$, these short horizontal holonomy vectors are then just $g\boldsymbol{v}_i$ for a unique $\boldsymbol{v}_i$.

So $\Omega(X,\omega)$ then breaks up into a piece for each Γ_i, which we can parametrize as follows, depending on whether $-I\in\mathrm{SL}(X,\omega)$.

Case 1 $(-I\in\mathrm{SL}(X,\omega))$ In this case, $\Gamma_i\cong\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ and we can choose a generator P_i for the infinite cyclic factor of Γ_i that has eigenvalue 1. Up to possibly replacing P_i with its inverse, there exists a $C_i\in\mathrm{SL}(2,\mathbb{R})$ such that

$$S_i=C_iP_iC_i^{-1}=\begin{bmatrix}1&\alpha_i\\0&1\end{bmatrix}$$

for some $\alpha_i>0$ and that $C_i(X,\omega)$ has a shortest horizontal holonomy vector of $(1,0)$. The piece of the Poincaré section associated to Γ_i is then parametrized by all matrices $\boldsymbol{M}_{a,b}$ that take the saddle connection of $C_i(X,\omega)$ with holonomy vector $(1,0)$ to a short horizontal with holonomy vector $(|a|,0)$ of $\boldsymbol{M}_{a,b}C_i(X,\omega)$ with $-1\le a<0$ or $0<a\le 1$. With some linear algebra, we can see that this is given by matrices

$$\boldsymbol{M}_{a,b}=\begin{bmatrix}a&b\\0&1/a\end{bmatrix}$$

with $-1\le a<0$ or $0<a\le 1$.

Since S_i and $-I$ are in the Veech group of $C_i(X,\omega)$, this set of $\boldsymbol{M}_{a,b}$ has some redundancies. Quotienting out by $-I$ gives us the set of $\boldsymbol{M}_{a,b}$ with $0<a\le 1$ and arbitrary b. If we further quotient out by S_i, we

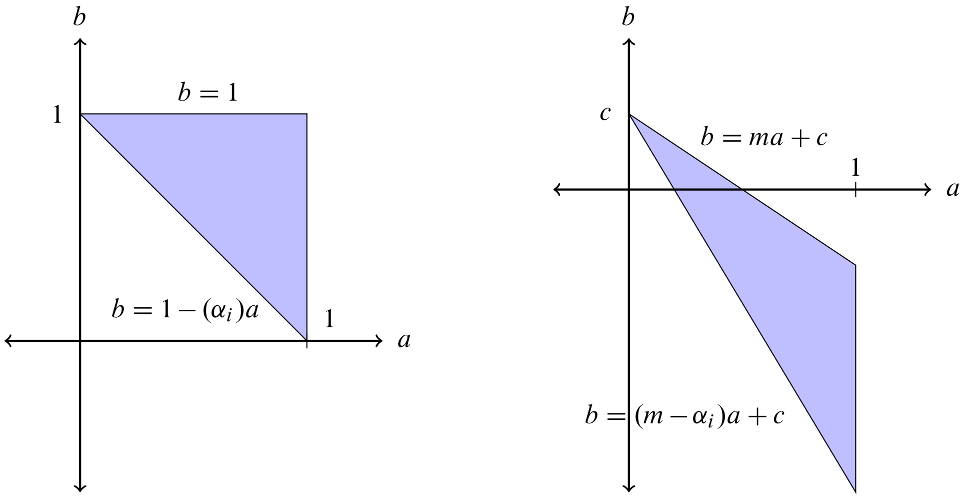

Figure 3: Two possible Poincaré section pieces Ω_i.

see that $\boldsymbol{M}_{a,b}$ is identified with $\boldsymbol{M}_{a,b+n(\alpha_i)a}$ for every $n \in \mathbb{Z}$. The result is that a Poincaré section piece associated to Γ_i can be parametrized by

$$\Omega_i = \{(a,b) \in \mathbb{R}^2 \mid 0 < a \leq 1 \text{ and } 1-(\alpha_i)a < b \leq 1\},$$

where each $(a,b) \in \Omega_i$ corresponds to $g\,\mathrm{SL}(X,\omega)$ for $g = \boldsymbol{M}_{a,b}C_i$.

Remark 3 While Ω_i is defined in this specific way in Uyanik and Work's paper, there is actually a lot more freedom in defining Ω_i. We just need to choose a fundamental domain for the $\boldsymbol{M}_{a,b}$ matrices under the action of $\langle S_i, -I\rangle$. To do this, we again let $0 < a \leq 1$, but for each a we choose a set of b values of length $(\alpha_i)a$ to account for the quotienting out by S_i. For any $m, c \in \mathbb{R}$, another such fundamental domain is

$$\Omega_i = \{(a,b) \in \mathbb{R}^2 \mid 0 < a \leq 1 \text{ and } ma + c - (\alpha_i)a < b \leq ma + c\}.$$

That is, instead of choosing Ω_i to be a triangle whose top line is $b = 1$ for $0 < a \leq 1$, we choose Ω_i to be a triangle whose top line is $b = ma + c$ for some slope m and b–intercept c. We see the distinction between these two Poincaré section pieces in Figure 3.

Furthermore, we can make similar modifications to Ω_i in Case 2 below. In this case, there will be another triangle with $a < 0$, and we have the freedom to choose the top line of the triangles with $a > 0$ and $a < 0$ independently. These modifications will be integral in our finiteness proofs.

Case 2 $(-I \notin \mathrm{SL}(X,\omega))$ This case breaks up into two subcases, depending on whether the generator P_i of $\Gamma_i \cong \mathbb{Z}$ has eigenvalue 1 or -1.

If P_i has eigenvalue 1, then we again can find

$$S_i = C_i P_i C_i^{-1} = \begin{bmatrix} 1 & \alpha_i \\ 0 & 1 \end{bmatrix}$$

for some $\alpha_i > 0$ and that $C_i(X,\omega)$ has a shortest horizontal holonomy vector of $(1,0)$. We again have that the matrices $\boldsymbol{M}_{a,b}$ parametrize the Poincaré section piece, but now we only can quotient out by the subgroup generated by S_i. The result is that the Poincaré section piece associated to Γ_i can be parametrized by

$$\Omega_i = \{(a,b) \in \mathbb{R}^2 \mid 0 < a \le 1 \text{ and } 1-(\alpha_i)a < b \le 1\} \cup \{(a,b) \in \mathbb{R}^2 \mid -1 \le a < 0 \text{ and } 1+(\alpha_i)a < b \le 1\},$$

where each $(a,b) \in \Omega_i$ corresponds to $g\,\mathrm{SL}(X,\omega)$ for $g = \boldsymbol{M}_{a,b}C_i$.

When P_i has eigenvalue -1, we can only find $C_i \in \mathrm{SL}(2,\mathbb{R})$ such that

$$S_i = C_i P_i C_i^{-1} = \begin{bmatrix} -1 & \alpha_i \\ 0 & -1 \end{bmatrix},$$

where $\alpha_i > 0$ and $C_i(X,\omega)$ has a shortest horizontal holonomy vector of $(1,0)$. We again quotient out our set of $\boldsymbol{M}_{a,b}$ matrices by the subgroup generated by S_i. The resulting Poincaré section piece associated to Γ_i can be parametrized by

$$\Omega_i = \{(a,b) \in \mathbb{R}^2 \mid 0 < a \le 1 \text{ and } 1-(2\alpha_i)a < b \le 1\},$$

where each $(a,b) \in \Omega_i$ corresponds to $g\,\mathrm{SL}(X,\omega)$ for $g = \boldsymbol{M}_{a,b}C_i$.

Having established what each piece of the Poincaré section associated to each Γ_i looks like, we also need to find the measure on the whole Poincaré section. The measure on the Poincaré section is the unique ergodic measure μ on $\Omega(X,\omega)$, which is a scaled copy of the Lebesgue measure on each of these pieces Ω_i of $\mathbb{R}^2$. The scaling factor is the total area of all the pieces of the transversal.

Upon finding the Poincaré section pieces, the return time function of the horocycle flow at a point $\boldsymbol{M}_{a,b}C_i(X,\omega)$ is the smallest positive slope of a holonomy vector of $\boldsymbol{M}_{a,b}C_i(X,\omega)$ which has short horizontal component. This is because of the way the horocycle flow acts on slopes. More precisely, if $\boldsymbol{v} = (x,y)$ is the holonomy vector of $C_i(X,\omega)$ such that $\boldsymbol{M}_{a,b}(x,y)$ is the holonomy vector on $\boldsymbol{M}_{a,b}(x,y)$ with the smallest positive slope among all holonomy vectors with a horizontal component of length ≤ 1, then the return time function at that point $(a,b) \in \Omega_i$ in the Poincaré section is given by the slope of $\boldsymbol{M}_{a,b}(x,y)$, which is

$$\frac{y}{a(ax+by)}.$$

We call such a vector $\boldsymbol{v} = (x,y)$ a *winner* or *winning saddle connection*. We note that while technically $\boldsymbol{v}$ is the holonomy vector of a saddle connection, we will often use the terms holonomy vector and saddle connection interchangeably. Our proof that the slope gap distribution of a Veech surface has finitely many points of nonanalyticity will rely on us showing that each piece Ω_i of the Poincaré section has finitely many winners.

Each such $\boldsymbol{v}$ would then be a winner on a convex polygonal piece of Ω_i, an example of which is given in Figure 6. Furthermore, the cumulative distribution function of the slope gap distribution would then be given by areas between the hyperbolic return time function level curves (see Figure 16 for an example

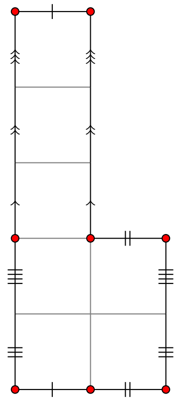

Figure 4: The surface $\mathscr{L}$ with cone point in red.

picture) and the sides of these polygons, and would therefore be piecewise real analytic with finitely many points of nonanalyticity. For more details about this process and a worked example of a computation of a slope gap distribution, please see [3].

2.3 Examples and difficulties

In this section, we will give an example of difficulties that arise from the choice of parametrization of the Poincaré section. In particular, it is possible for there to be infinitely many winning saddle connections under certain parametrizations, but only finitely many different winners under a different parametrization. For full computations of a gap distribution we refer to [3; 11].

We will take the surface $\mathscr{L}$ in Figure 4 and analyze the winning saddle connection on the component Ω_1 of the Poincaré section corresponding to the parabolic subgroup of $\mathrm{SL}(\mathscr{L})$ generated by $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$. $\mathscr{L}$ is a 7–square square-tiled surface with a single cone point.

Since $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ is in the Veech group and $\mathscr{L}$ has a length-1 horizontal saddle connection, the corresponding piece of the Poincaré section Ω_1 can be parametrized by matrices

$$\boldsymbol{M}_{a,b} = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$$

with $0 < a \le 1$ and $1 - a < b \le 1$. Notice that $\mathscr{L}$ has all saddle connections with coordinates $(n, 2)$ and $(n, 3)$ for $n \in \mathbb{Z}$, and no saddle connection with y–coordinate 1.

Proposition 4 *In a neighborhood of the point* $(0, 1)$ *on* Ω_1, *the winning saddle connection always has* y*–coordinate* 2.

Proof Take a saddle connection $\boldsymbol{v} = (n, k)$ with $k > 0$ such that $\boldsymbol{M}_{a,b}\boldsymbol{v}$ has horizontal component ≤ 1. We will show that if $k > 2$ and $a < \frac{1}{3}$, there is a saddle connection $\boldsymbol{w} = (m, 2)$ such that the slope of $\boldsymbol{M}_{a,b}\boldsymbol{w}$ is less than the slope of $\boldsymbol{M}_{a,b}\boldsymbol{v}$, and $\boldsymbol{M}_{a,b}\boldsymbol{w}$ has short horizontal component. Since there are no saddle connections with $k = 1$, this implies that the winning saddle connection must have y–coordinate 2.

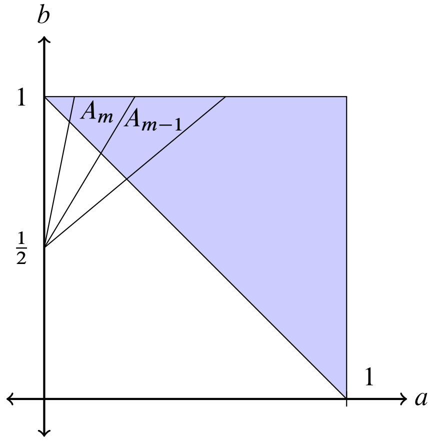

Figure 5: Regions A_m where $(-m, 2)$ is a winner.

The x–coordinate of $\boldsymbol{M}_{a,b}\boldsymbol{w}$ is $ma + 2b$. Since $\mathscr{L}$ has all $(m, 2)$ saddle connections, we may choose an m so that $1 - a < ma + 2b \leq 1$. The condition that $\text{slope}(\boldsymbol{M}_{a,b}\boldsymbol{w}) < \text{slope}(\boldsymbol{M}_{a,b}\boldsymbol{v})$ rearranges to

$$na + kb < \tfrac{1}{2}k(ma + 2b).$$

If $k > 2$, then since $\mathscr{L}$ is square-tiled $k \geq 3$, and when $a < \frac{1}{3}$ we have that $ma + 2b > 1 - a \geq \frac{2}{3}$; thus $\frac{1}{2}k(ma + 2b) > 1$. Since $\boldsymbol{M}_{a,b}\boldsymbol{v}$ has a short horizontal component $na + kb \leq 1$, so the above inequality is always true. □

Let A_m be the region where the saddle connection $(-m, 2)$ is the winning saddle connection. By Proposition 4, in the top left corner of Ω_1, A_m is the region where $\boldsymbol{M}_{a,b}(-m, 2) = (2b - ma, 2a^{-1})$ has smallest slope among all saddle connections with y–coordinate 2 and short horizontal component. The slope is $2a^{-1}/(2b - ma)$, so minimizing the slope is equivalent to maximizing $2b - ma$ with the constraint that $2b - ma \leq 1$, or in other words, $-m = \lfloor (1 - 2b)/a \rfloor$. But as $a \to 0$ inside the region Ω_1, $b \to 1$, so $-m \sim -1/a \to -\infty$. This implies that there infinitely many saddle connections that occur as winners in the top left corner of the Poincaré section.

By Remark 3 in Section 2.2, we can change the parametrization of the Poincaré section. One problem in our previous parametrization was that there were infinitely many winners in the upper left-hand corner $(0, 1)$ of our Poincaré section. To fix this, we will change our parametrization so that the upper left corner is at $\left(0, \frac{1}{2}\right)$ and the slope of the top line of our Poincaré section triangle is nicely compatible with the $(1, 2)$ holonomy vector. This will ensure that there are finitely many winners in the top left corner, and will result in finitely many winners across the entire Poincaré section. We will prove that we can always do this for arbitrary Veech surfaces in Section 3.

We will use the parametrization $0 < a \leq 1$ and $\frac{1}{2} - \frac{3}{2}a < b \leq \frac{1}{2} - \frac{1}{2}a$. This parametrization is chosen to ensure that the saddle connection $(1, 2)$ of $\mathscr{L}$ wins in a neighborhood of the top line segment, which prevents the problem that arises in the previous parametrization.

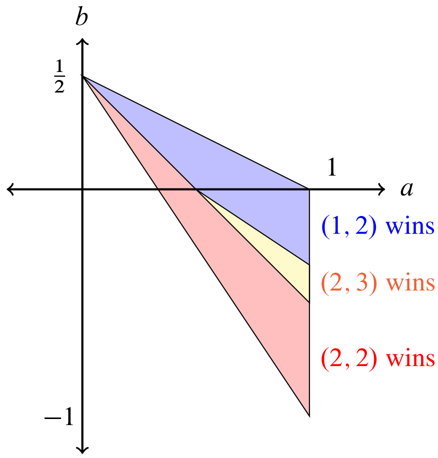

Figure 6: The new Poincaré section breaks up into three pieces, with saddle connection (1,2) winning in the blue region, (2,3) in the yellow region and (2,2) in the red region.

In this case, the only winners are the $(1,2)$, $(2,3)$ and $(2,2)$ saddle connections on $\mathscr{L}$:

(1) $(1,2)$ wins in the region
$$\left\{(a,b) \mid 0<a\le 1,\ \tfrac{1}{2}-a<b\le \tfrac{1}{2}-\tfrac{1}{2}a \text{ and } \tfrac{1}{3}-\tfrac{2}{3}a<b\right\}.$$

(2) $(2,3)$ wins in the region
$$\left\{(a,b) \mid \tfrac{1}{2}<a\le 1 \text{ and } \tfrac{1}{2}-a<b\le \tfrac{1}{3}-\tfrac{2}{3}a\right\}.$$

(3) $(2,2)$ wins in the region
$$\left\{(a,b) \mid 0<a\le 1 \text{ and } \tfrac{1}{2}-\tfrac{3}{2}a<b\le \tfrac{1}{2}-a\right\}.$$

To see this, notice that the saddle connection (x,y) is the winner at (a,b) if $\boldsymbol{M}_{a,b}(x,y)$ has smallest positive slope amongst all saddle connections with short horizontal component. $\boldsymbol{M}_{a,b}(x,y)$ has short horizontal component in the region with $0<a\le 1$ and $(-x/y)a<b\le 1/y-(x/y)a$. Minimizing the slope at (a,b) is equivalent to maximizing x/y over all saddle connections with a short horizontal component.

Working out the exact winners then comes down to casework. In this case, $\boldsymbol{M}_{a,b}(m,2)$ never has a short horizontal component for $m>2$ and (a,b) in the Poincaré section, and simple casework shows where $(1,2)$ and $(2,2)$ are the winners. For saddle connections with y–coordinate greater than 2, we need to understand those with $x/y>\frac{1}{2}$ which can potentially win against $(1,2)$ or $(2,2)$. In the yellow region $(2,3)$ wins, as $(2,2)$ does not have a short horizontal component for (a,b) in that region. All other saddle connections with $y=3$ and $x\ge 3$ do not have short horizontal component in the Poincaré section. For $y\ge 4$, a similar analysis shows that none of the saddle connections can appear as winners, giving the result.

3 Main theorem

In Section 2.3, we examined the 7–square square-tiled surface $\mathscr{L}$ and saw that, in one parametrization, it looked like the Poincaré section would admit infinitely many winning saddle connections and therefore give the possibility of infinitely many points of nonanalyticity in the slope gap distribution. However, when we strategically chose a different parametrization of this piece of the Poincaré section, there were only finitely many winners. Thus this piece of the Poincaré section could only contribute finitely many points of nonanalyticity to the slope gap distribution.

It is interesting to note that this implies that many of the potential points of nonanalyticity arising from the Uyanik–Work parametrization must cancel each other out and not result in points of nonanalyticity in the slope gap distribution. Choosing a strategic parametrization of the Poincaré section is one of the key ideas of the main theorem of this paper:

Theorem 1 *The slope gap distribution of any Veech surface has finitely many points of nonanalyticity.*

This section is devoted to the proof of this theorem. We will begin by giving an outline of the proof, and then will dive into the details of each step.

3.1 Outline

The idea is that after choosing strategic parametrizations of each piece of the Poincaré section of a Veech translation surface (X,ω), we will use compactness arguments to show that there are finitely many winners on each piece.

(1) We begin with a Veech translation surface (X,ω) and focus on a piece of its Poincaré section corresponding to one maximal parabolic subgroup in $\mathrm{SL}(X,\omega)$. Up to multiplication by an element of $\mathrm{GL}(2,\mathbb{R})$, we will assume that the generator of the parabolic subgroup has a horizontal eigenvector and (X,ω) has a horizontal saddle connection of length 1. Based on properties of the saddle connection set of (X,ω), we strategically choose a parametrization T_X of this Poincaré section piece. T_X will be some triangle in the plane.

(2) For any saddle connection $\boldsymbol{v}$ of (X,ω), we will define a strip $S_\Omega(\boldsymbol{v})$ that gives a set of points $(a,b)\in\mathbb{R}_{>0}\times\mathbb{R}$ where $\boldsymbol{v}$ is a potential winning saddle connection on the surface $\boldsymbol{M}_{a,b}(X,\omega)\in T_X$. We will start by showing various properties of these strips that we will make use of later on in the proof.

(3) We will then show that every point $(a,b)\in T_X$ in the interior of a strip has an open neighborhood with finitely many winning saddle connections.

(4) We show that every point (a,b) on the top edge of T_X has an open neighborhood with finitely many winning saddle connections.

(5) We then move on to show that points $(a,b)\in T_X$ that are either in the interior of T_X or on the bottom edge not including the right vertex with $a=1$ have an open neighborhood with finitely many winning saddle connections.

(6) Next we show that on the boundary $a = 1$ of T_X there are finitely many winning saddle connections.

(7) Using the finiteness on the right boundary, we show that any point $(a, b) \in T_X$ with $a = 1$ has an open neighborhood with finitely many winning saddle connections.

(8) By compactness of T_X, there is a finite cover of T_X with the open neighborhoods of points $(a, b) \in T_X$ that we found in our previous steps. Since each of the these open neighborhoods had finitely many winners, we find that there are finitely many winning saddle connections across all of T_X.

(9) Finally, we show that finitely many winners on each piece of the Poincaré section implies finitely many points of nonanalyticity of the slope gap distribution.

3.2 Proof

Using the method of [11] outlined in Section 2.2, it will suffice to show that every piece of the Poincaré section can be chosen so that there are only finitely many winning saddle connections. For most of the arguments in this section we will fix a piece of the Poincaré section and will work exclusively with it.

We recall that there is a piece of the Poincaré section for each conjugacy class of a maximal parabolic subgroup in $\mathrm{SL}(X, \omega)$. We will now fix such a maximal parabolic subgroup Γ_i and work with the corresponding component of the Poincaré section. Without loss of generality we may assume that (X, ω) has a horizontal saddle connection with x–component 1 and that Γ_i is generated by

$$P_i = \begin{bmatrix} 1 & \alpha_i \\ 0 & 1 \end{bmatrix}.$$

Using the notation of Section 2.2, this is essentially replacing (X, ω) with $C_i(X, \omega)$.

Since (X, ω) is a Veech surface with a horizontal saddle connection it has a horizontal cylinder decomposition [6], and therefore, for all $a \in \mathbb{R}$, there are only finitely many heights $0 \le h \le a$ such that (X, ω) has a saddle connection with y–component h. Let $y_0 > 0$ be the shortest vertical component of a saddle connection on (X, ω), and let $x_0 > 0$ be the shortest horizontal component of a saddle connection at height y_0. Our first step is to use this saddle connection to give a parametrization of the Poincaré section that is adapted to the geometry of (X, ω).

By Remark 3, we can choose the following parametrization of this piece of the Poincaré section, as pictured in Figure 7:

$$T_X = \left\{ (a, b) \;\middle|\; 0 < a \le 1 \text{ and } \frac{1 - x_0 a}{y_0} - na \le b \le \frac{1 - x_0 a}{y_0} \right\}.$$

Here n is either α_i or $2\alpha_i$ depending on which one is needed to fully parametrize this piece of the Poincaré section, as described in Section 2.2. In the case where $-I \notin \mathrm{SL}(X, \omega)$ and P_1 had eigenvalue 1, the Poincaré section has an additional triangle with $a < 0$. In particular, we can choose this triangle so that it consists of points $(-a, -b)$ for $(a, b) \in T_X$. But if $\boldsymbol{v}$ were the winning saddle connection for $\boldsymbol{M}_{a,b}(X, \omega)$, then $-\boldsymbol{v}$ would be the winning saddle connection for $\boldsymbol{M}_{-a,-b}(X, \omega)$, and hence when proving that there are only finitely many winners, it suffices to consider only the portion with $a > 0$.

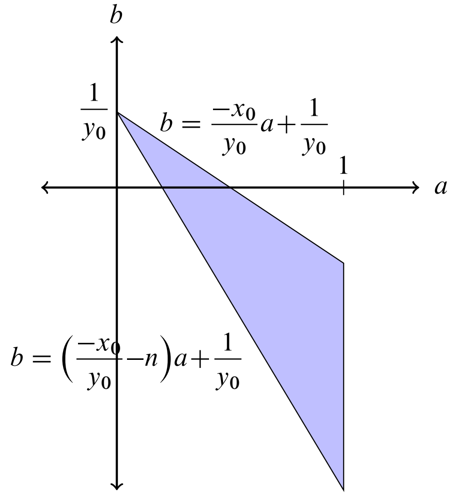

Figure 7: A Poincaré section piece for (X,ω) with $y_0 > 0$ the shortest vertical component and $x_0 > 0$ the shortest corresponding horizontal component of a saddle connection on (X,ω) with vertical component y_0.

Our goal now is to prove that the return time function is piecewise real analytic with finitely many pieces. We will do so by proving that there are finitely many winning saddle connections $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \Lambda(X,\omega)$ such that each point $(a,b) \in T_X$ has a winner $\boldsymbol{M}_{a,b}\boldsymbol{v}_i$ for some $1 \le i \le n$. We will repeat this for every T_X corresponding to each maximal parabolic subgroup.

To achieve this goal, we will first define an auxiliary set that will help us understand for what points $(a,b) \in T_X$ a particular $\boldsymbol{v} \in \Lambda(X,\omega)$ is a candidate winner. By a candidate winner, we mean that $\boldsymbol{M}_{a,b}\boldsymbol{v}$ has a positive x–coordinate at most 1 and a positive y–coordinate. If $\boldsymbol{v} = (x,y)$, the x–coordinate condition is the condition that $0 < ax + by \le 1$. We also note that for $\boldsymbol{M}_{a,b}(x,y)$ to be a winner, we need that $a^{-1}y > 0$. Since $a > 0$ on T_X, this condition reduces to saying that $y > 0$.

Definition 5 Given a saddle connection $\boldsymbol{v} = (x,y)$ with $y > 0$, we define $\mathcal{S}_\Omega(\boldsymbol{v})$ as the strip of points $(a,b) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $0 < ax + by \le 1$. This corresponds to the set of surfaces $\boldsymbol{M}_{a,b}(X,\omega)$ for which $\boldsymbol{M}_{a,b}\boldsymbol{v}$ is a potential winning saddle connection.

Let us note some properties of these strips $\mathcal{S}_\Omega(\boldsymbol{v})$ that we will use repeatedly in our proofs. We recall that we are assuming without loss of generality that (X,ω) has a short horizontal saddle connection of length 1. Considering the particular piece T_X of the Poincaré section, we recall that T_X is parametrized by matrices

$$\boldsymbol{M}_{a,b} = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$$

so that $\boldsymbol{M}_{a,b}(X,\omega)$ has a horizontal saddle connection of length ≤ 1.

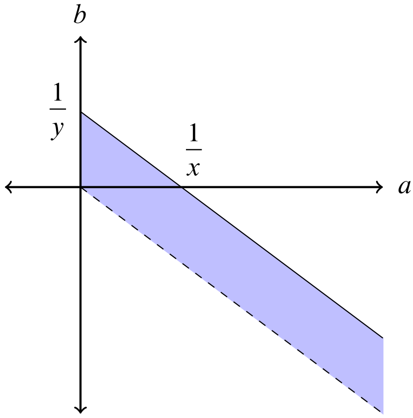

Figure 8: A strip $\mathcal{S}_\Omega(\boldsymbol{v})$ for $\boldsymbol{v}=(x,y)$. Here $y>0$. The slope of the upper and lower lines of the strip is $-x/y$.

Then, because (X,ω) is a Veech surface, it breaks up into horizontal cylinders, and therefore there exists a $y_0>0$ such that there is a saddle connection with height y_0 and furthermore that every saddle connection with positive height has height $\geq y_0$.

With these assumptions in place, we note the following useful properties of the strips $\mathcal{S}_\Omega(\boldsymbol{v})$ that are used implicitly throughout the proof:

(1) The strip $\mathcal{S}_\Omega(\boldsymbol{v})$ for $\boldsymbol{v}=(x,y)$ is sandwiched between a solid line that intersects the b–axis at $1/y$ and a dotted line that intersects the b–axis at 0. Both lines have slope $-x/y$. We also know that $y \geq y_0$, so $1/y \leq 1/y_0$.

(2) Fixing any $c>0$, there are only finitely many y–coordinates of saddle connections $\boldsymbol{v}$ such that $S_\Omega(\boldsymbol{v})$ intersects the y–axis at any point $\geq c$.

This is because (X,ω) being a Veech surface and having a horizontal saddle connection implies that the surface breaks up into finitely many horizontal cylinders of heights $h_1,\dots,h_n$ and every saddle connection with positive y–component must have a y–component that is a nonnegative linear combination of these h_i. Since there are finitely many such y values $\leq 1/c$, there are finitely many strips that intersect the y–axis at points $\geq c$.

(3) At a particular point $(a,b)\in T_X$, the winner is the saddle connection $\boldsymbol{v}=(x,y)\in\Lambda(X,\omega)$ such that $\boldsymbol{M}_{a,b}\boldsymbol{v}=(ax+by,a^{-1}y)$ has the least slope among those saddle connections satisfying $0<ax+by\leq 1$ and $a^{-1}y>0$. Since $a>0$ for any point in Ω_i and the reciprocal of the slope of $\boldsymbol{M}_{a,b}\boldsymbol{v}$ is $a^2x/y+ab$, this corresponds to the saddle connection with the greatest reciprocal slope, which corresponds to having the greatest x/y with $y>0$.

In terms of our strips, we're fixing the point (a,b) and looking for the strip $\mathcal{S}_\Omega(\boldsymbol{v})$ that contains (a,b) and has the least slope, since each strip has slope $-x/y$. We further note that, due to our choice of Poincaré section, no saddle connection $\boldsymbol{v} = (x, y)$ with $x < 0$ can ever be the winner at a point $(a, b) \in T_X$, since either (a, b) will not be in the strip $\mathcal{S}_\Omega(\boldsymbol{v})$ or the saddle connection (x_0, y_0) that defined T_X would win over (x, y) at (a, b). Because of this, from now on we will only consider saddle connections with $x \geq 0$ (and $y > 0$) when looking for potential winners.

(4) For any given $y > 0$, there are only finitely many saddle connection vectors $\boldsymbol{v} = (x, y)$ of (X, ω) with $x \geq 0$ such that $S_\Omega(\boldsymbol{v})$ intersects T_X.

This is because $S_\Omega(\boldsymbol{v})$ does not intersect T_X for x/y larger than some constant C that depends on T_X and y. Specifically, we can let $C = x_0/y_0 + n$, the negative of the slope of the bottom line that defines the triangle T_X. Since the saddle connection set is discrete, there are finitely many $x \geq 0$ for a given y such that $x/y \leq C$.

With these facts established, let us first prove a lemma that shows that winning saddle connections exist and that will be useful in proving Lemma 12.

Lemma 6 *Let $(a, b) \in T_X$ be such that $(a, 0)$ is a short horizontal saddle connection of $\boldsymbol{M}_{a,b}(X, \omega)$. Then $\boldsymbol{M}_{a,b}(X, \omega)$ either has a saddle connection $\boldsymbol{v} = (x, y)$ with $0 < x < a$ and $y > 0$, or there exist two saddle connections $\boldsymbol{v}_1 = (a, y)$ and $\boldsymbol{v}_2 = (0, y)$ with $y > 0$. This implies that every point in T_X has a winning saddle connection, or equivalently that every point in T_X is in some strip $S_\Omega(\boldsymbol{v})$ for some $\boldsymbol{v} = (x, y)$ with $y > 0$.*

Proof Let us take a horizontal saddle connection on our surface $\boldsymbol{M}_{a,b}(X, \omega)$ with holonomy vector $(a, 0)$, connecting two (possibly identical) cone points p and q. Then we will consider developing a width-a vertical strip on our surface extending upward with the open horizontal segment from p to q as its base. Since our surface is of finite area, this vertical strip must eventually hit a cone point r or come back to overlap our original open segment from p to q. Now we're going to define our vectors $\boldsymbol{v}$, or $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, in each case.

In the former case when the top edge of our vertical strip hits a cone point r in the interior of the edge, the straight segment from p to r cutting through our vertical strip gives us $\boldsymbol{v}$.

The latter case when the top edge of our vertical strip comes back to overlap our original open segment breaks up into two cases. If we have an incomplete overlap, then the top edge contains the cone point

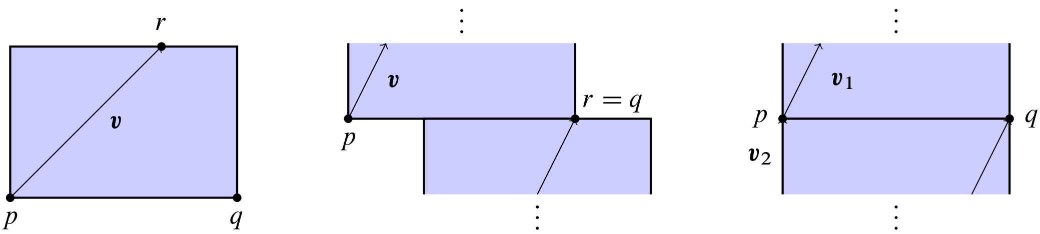

Figure 9: The vectors $\boldsymbol{v}$ or $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in the three different cases of vertical strip.

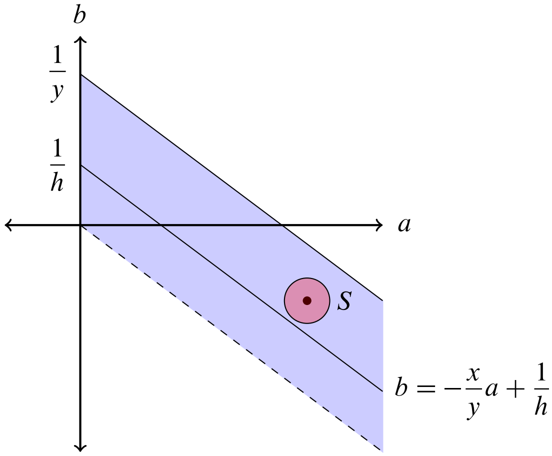

Figure 10: A choice of $1/h$ for a particular S.

$r = p$ or $r = q$, and the saddle connection from p on the bottom edge to r on the top edge gives us $\boldsymbol{v}$. If we have a complete overlap, then the saddle connection from p on the bottom edge to q on the top edge gives us our vector $\boldsymbol{v}_1$ and the saddle connection from p on the bottom edge to p on the top edge gives us our vector $\boldsymbol{v}_2$.

In any of these cases, letting $\boldsymbol{v}' = \boldsymbol{M}_{a,b}^{-1}(\boldsymbol{v})$ or $\boldsymbol{M}_{a,b}^{-1}(\boldsymbol{v}_1)$ gives us that $S_\Omega(\boldsymbol{v}')$ contains our initial point (a,b) and $\boldsymbol{v}'$ is a possible winning saddle connection. □

The following lemma will help us show that there are finitely many winning saddle connections on certain sets in T_X:

Lemma 7 *Let S be a closed set that is a subset of $S_\Omega(\boldsymbol{v})$ for $\boldsymbol{v} = (x,y)$ with $y > 0$. Then there are finitely many winning saddle connections on S.*

Proof Let S be a closed set contained in $S_\Omega(\boldsymbol{v})$ for a saddle connection $\boldsymbol{v} = (x,y)$ of (X,ω) with $y > 0$. By definition, $\boldsymbol{v}$ is a potential winning saddle connection on all of S. That is, for any point $(a,b) \in S$, $\boldsymbol{M}_{a,b}\boldsymbol{v}$ has positive y component and positive and short (≤ 1) x component.

We recall that for a point $(a',b') \subset S$ to have winner $\boldsymbol{v}' = (x',y') \neq \boldsymbol{v} = (x,y)$, we need that $\boldsymbol{v}'$ is a saddle connection of (X,ω), $x'/y' > x/y$, and that $(a',b') \subset S_\Omega(\boldsymbol{v}')$.

This corresponds to the strip $S_\Omega(\boldsymbol{v}')$ having a smaller slope than $S_\Omega(\boldsymbol{v})$ and still intersecting S. Given that S is closed and the bottom boundary of $S_\Omega(\boldsymbol{v})$ is open, there exists an $h > 0$ such that the line S is completely on or above the line $b = -(x/y)a + 1/h$. Furthermore, since the left boundary of $S_\Omega(\boldsymbol{v})$ is open, S is a positive distance away from the y–axis.

Then for $S_\Omega(\boldsymbol{v}')$ to intersect S and for $x'/y' > x/y$, we need that $y' < h$, since otherwise the strip $S_\Omega(\boldsymbol{v}')$ would have y–intercepts $1/y' \leq 1/h$ and 0 and would have smaller slope than that of $S_\Omega(\boldsymbol{v})$ and would therefore not intersect S.

But since (X,ω) is a Veech surface with a horizontal saddle connection, it decomposes into finitely many horizontal cylinders. Therefore, the set of possible vertical components y' of saddle connections are a discrete subset of $\mathbb{R}$, and thus there are finitely many vertical components of saddle connections that satisfy $y' < h$. Since there are finitely many saddle connections in the vertical strip $(0,1] \times (0,\infty)$ with vertical component less than h, there are finitely many possible winning saddle connections on S. □

We recall that our goal is to show that every point $(a,b) \in T_X$ has a neighborhood on which there are finitely many winners. This will allow us to use a compactness argument to prove that there are finitely many winners on all of T_X. Building off of the previous lemma, we show in the next lemma that certain points $(a,b) \in T_X$ have an open neighborhood on which there are finitely many winners:

Lemma 8 *Let (a,b) be in the interior of some strip $S_\Omega(\boldsymbol{v})$. Then there exists a neighborhood of (a,b) with finitely many winning saddle connections.*

Proof Let (a,b) be in the interior of the strip $S_\Omega(\boldsymbol{v})$ for $\boldsymbol{v}=(x,y)$ with $y>0$ and $x \geq 0$. Then we can find an $\epsilon > 0$ such that the closed ball of radius ϵ around (a,b) remains in the interior of the strip. That is, we choose an $\epsilon > 0$ such that

$$\overline{B_\epsilon((a,b))} \subset S_\Omega(\boldsymbol{v}).$$

We can then use Lemma 7 to conclude that there are finitely many winning saddle connections on $\overline{B_\epsilon((a,b))}$, and therefore on $B_\epsilon((a,b))$. □

We now look at points $(a,b) \in T_X$ that lie on the top edge of T_X and show that these points have a neighborhood with finitely many winners.

Lemma 9 *For any (a,b) that lies on the top edge of T_X, including the point $(0,1/y_0)$, there exists a neighborhood $B_\epsilon((a,b))$ such that there are finitely many winning saddle connections on $B_\epsilon((a,b)) \cap T_X$.*

Proof We recall that T_X is a triangle bounded by the lines $b=(-x_0/y_0)a+1/y_0$ on top, the line $a=1$ on the right and the line $b=(-x_0/y_0-n)a+1/y_0$ on the bottom.

We break up the proof of this lemma into cases, depending on the location of $(a,b) \in T_X \cup \{(0,1/y_0)\}$:

(1) $\boldsymbol{b=(-x_0/y_0)a+1/y_0}$ These points are on the top line of T_X. We recall that y_0 was chosen to be the least $y>0$ for which X has a saddle connection (x,y_0). Then x_0 was the least $x>0$ for which (x,y_0) was a saddle connection of X.

Let (a,b) be any point on the top line of T_X and let $\boldsymbol{v}=(x_0,y_0)$. Then (a,b) is on the top line of the strip $S_\Omega((x_0,y_0))$. We can find an $\epsilon>0$ such that $\overline{B_\epsilon((a,b))} \cap S_\Omega((x_0,y_0))$ is a closed subset of $S_\Omega((x_0,y_0))$. By Lemma 7, there are then finitely many winners on $B_\epsilon((a,b)) \cap S_\Omega((x_0,y_0))$.

(2) $\boldsymbol{(a,b)=(0,1/y_0)}$ This point is not in T_X but is the top left corner of the triangle that makes up T_X.

We can find a $y_1>y_0$ such that every saddle connection (x,y) of X with $y>y_0$ must satisfy that $y \geq y_1$. Thus, we can choose an $\epsilon>0$ such that $B_\epsilon((0,1/y_0)) \cap T_X \subset S_\Omega((x_0,y_0))$ and no strip $S_\Omega((x,y))$, for

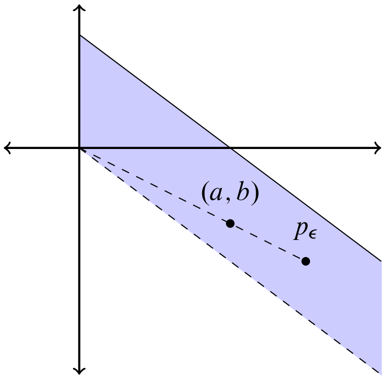

Figure 11: The strip $\mathcal{S}_\Omega(\boldsymbol{w}_\epsilon)$.

a saddle connection with $y > y_0$ and $x \geq 0$, intersects $B_\epsilon((0, 1/y_0))$. This would imply that the only possible winning saddle connections on $B_\epsilon((0, 1/y_0))$ are of the form (x, y_0) for $x \geq x_0$.

But if we fix $y = y_0$, since the set of saddle connections (x, y_0) is discrete and T_X is bounded below by the line $b = (-x_0/y_0 - n)a + 1/y_0$, there are only finitely many saddle connections $\boldsymbol{v} = (x, y_0)$ of (X, ω) whose strip $S_\Omega(\boldsymbol{v})$ intersects $B_\epsilon((0, 1/y_0))$ — exactly those x such that $x_0 \leq x \leq x_0 + ny_0$. We have shown then that only finitely many strips $S_\Omega(\boldsymbol{v})$, for holonomy vectors $\boldsymbol{v}$ that could win over (x_0, y_0), intersect $B_\epsilon((0, 1/y_0))$, and therefore there are only finitely many winners on this neighborhood. □

Having established that points $(a, b) \in T_X$ on the top edge of T_X have neighborhoods with finitely many winners, we now turn to points $(a, b) \in T_X$ that lie in the interior of T_X or on the bottom edge of T_X.

Lemma 10 *For any (a, b) that lies in the interior of T_X or on the bottom edge of T_X, excluding the vertex with $a = 1$, there exists a neighborhood $B_\epsilon((a, b))$ such that there are finitely many winning saddle connections on $B_\epsilon((a, b)) \cap T_X$.*

Proof By Lemma 8, it suffices to show that (a, b) lies on the interior of a strip $S_\Omega(\boldsymbol{v})$ for some saddle connection $\boldsymbol{v}$.

Because (a, b) is in T_X, it must lie in some strip $S_\Omega(\boldsymbol{v})$. If (a, b) is in the interior of $S_\Omega(\boldsymbol{v})$, then we are done. Otherwise, if (a, b) is on the boundary of $S_\Omega(\boldsymbol{v})$, we consider the points $p_\epsilon = ((1+\epsilon)a, (1+\epsilon)b)$, with winner $\boldsymbol{w}_\epsilon$. Since (a, b) lies in the interior of T_X or on the bottom edge of T_X, for $\epsilon > 0$ sufficiently small p_ϵ also lies in T_X. Moreover, notice that p_ϵ and (a, b) lie on the same line through the origin. This immediately implies that (a, b) lies in the interior of $S_\Omega(\boldsymbol{w}_\epsilon)$, as seen in Figure 11.

Indeed, by the definition of $S_\Omega(\boldsymbol{v})$ for any holonomy vector $\boldsymbol{v}$ as a half-open strip with the open bottom boundary passing through the origin, for all points $p \in S_\Omega(\boldsymbol{v})$ the points tp for $0 < t < 1$ lie in the interior, which gives the desired result. □

The combination of our previous lemmas shows that, for all $(a, b) \in T_X$ away from the right vertical boundary, there are only finitely many winners in a neighborhood of (a, b). We also want to show that, for each $(1, b)$ on the right vertical boundary, there are only finitely many winners in a neighborhood. We will do this in two steps. First we will show that there are finitely many winning saddle connections along the right boundary of T_X. We will then use this result to prove that every point $(1, b)$ on the right boundary of T_X has a neighborhood with finitely many winning saddle connections.

For our first result, we will need the following definition:

Definition 11 Given $(a, b) \in \mathbb{R}^2$, define the set $\mathcal{S}_\Lambda(a, b)$ as the strip of vectors $\boldsymbol{v} = (x, y) \in \mathbb{R}^2$ such that $0 < ax + by \le 1$ and $y > 0$. This corresponds to the set of vectors that are potential winners on the surface $\boldsymbol{M}_{a,b}(X, \omega)$.

We think of this definition as a sort of dual to Definition 5, where instead of thinking of the surfaces corresponding to a particular winning saddle connection, we think about the set of possible coordinates of winning saddle connections for a particular surface.

Lemma 12 *There are only finitely many winning saddle connections along the right vertical boundary $a = 1$ of T_X.*

Proof By Lemma 6, we know that every point $(1, b)$ on the right boundary of T_X has a winning saddle connection. The set of $b \in \mathbb{R}$ such that $(1, b) \in T_X$ is some interval $[c, d]$. We note that since $\left[\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right]$ is in the Veech group of our surface for some $\alpha > 0$, it suffices to show that there are finitely many winners for $b \in [c + n\alpha, d + n\alpha]$ for any $n \in \mathbb{Z}$. This is because (x, y) is the winner for b' if and only if $(x - n\alpha y, y)$ is the winner for $b' + n\alpha$. For convenience, we will prove that there are finitely many winners for $b \in [M, N] = [c + n\alpha, d + n\alpha]$ for an n such that $M, N > 0$.

For each such b, we let $\boldsymbol{v}_b$ be its corresponding winning saddle connection. We wish to show that the set of vectors $\boldsymbol{v}_b$ is finite. We suppose that $\{\boldsymbol{v}_b\}$ is infinite. Then, since $b \in [M, N]$, we must be able to find a convergent subsequence of $b_i \in \mathbb{R}$ with corresponding winning saddle connections (x_i, y_i) such that $b_i \to b'$ and $b', b_i \in [M, N]$ for all i. In particular, $b' > 0$.

We claim now that $S_\Lambda(1, b')$ cannot have a winning saddle connection, which would contradict Lemma 6. This corresponds to a saddle connection (x, y) in the strip $S_\Lambda(1, b')$ that maximizes x/y. The strip $S_\Lambda(1, b')$ satisfies that $y > 0$ and $0 < x + b'y \le 1$, or alternatively that $-(1/b')x < y \le -(1/b')x + 1/b'$. We recall that $b' > 0$. Figure 12 shows a depiction of this strip.

We suppose that the winning saddle connection (x', y') for b' lies in the interior of $S_\Lambda(1, b')$. If $x'/y' > x_i/y_i$ and $(x', y') \in S_\Lambda(1, b_i)$, then (x_i, y_i) could not be the winner for $(1, b_i)$ because (x', y') beats it and is still in the strip $S_\Lambda(1, b_i)$.

We let $\mathcal{C}_{b'}$ be the cone given by the intersection of $y < (y'/x')x$ and $y > (y'/(x'-1))x - y'/(x'-1)$. We notice that if $(x_i, y_i) \in C_{b'}$, then it follows that $(x', y') \in S_\Lambda(1, b_i)$. One can see this algebraically or visually by noting that if (x_i, y_i) is in the cone $\mathcal{C}_{b'}$ as depicted in Figure 13, then $S_\Lambda(1, b_i)$ contains (x_i, y_i) and is bounded by two lines with x–intercepts 0 and 1 and therefore must contain the point (x', y'). Furthermore, the first inequality defining the cone gives us that $x/y > x'/y'$.

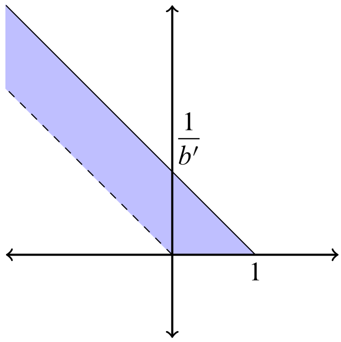

Figure 12: The strip $\mathcal{S}_\Lambda(1, b')$.

Therefore, if (x_i, y_i) is a winning saddle connection for some $(1, b_i)$, it cannot be in the open cone $\mathcal{C}_{b'}$ as defined above. Since $b_i \to b'$, this implies that for any $\epsilon > 0$ we can find an n large enough that, for all $i \geq n$, the strips $S_\Lambda(1, b_i)$ all lie in a region $\mathcal{S}_\epsilon$ that is the region where $(-1/b' + \epsilon)x \leq y \leq (-1/b' - \epsilon)x + (1/b' + \epsilon)$ and $y > 0$. Specifically, we will choose an ϵ such that the slopes of the two bounding lines of $\mathcal{S}_\epsilon$ are wedged between the slopes of the bounding lines of $\mathcal{C}_{b'}$. That is, we will choose $\epsilon > 0$ such that $(-1/b' - \epsilon) > y'/x'$ and $(-1/b' + \epsilon) < y'/(x' - 1)$. We call this latter region $\mathcal{S}_\epsilon$. Figure 13 illustrates these regions.

Given these conditions, we notice that $\mathcal{S}_\epsilon \backslash \mathcal{C}_{b'}$ is a compact set. With the possible exception of one point that equals (x', y'), the winning saddle connections (x_i, y_i) for $i \geq n$ must all be in this region. But the set of holonomy vectors of saddle connections of (X, ω), of which $\{(x_i, y_i)\}$ is a subset, is a discrete subset of $\mathbb{R}^2$ with no accumulation points, and so there are only finitely many $(x_i, y_i) \in \mathcal{S}_\epsilon \backslash \mathcal{C}_{b'}$. This is a

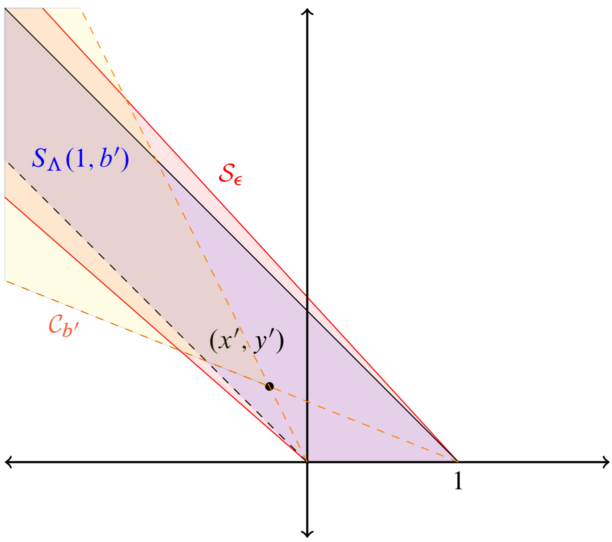

Figure 13: The strip $\mathcal{S}_\Lambda(1, b')$ with its winner (x', y') and cone $\mathcal{C}_{b'}$, along with the region $\mathcal{S}_\epsilon$ containing the winners (x_i, y_i) for $i \geq n$.

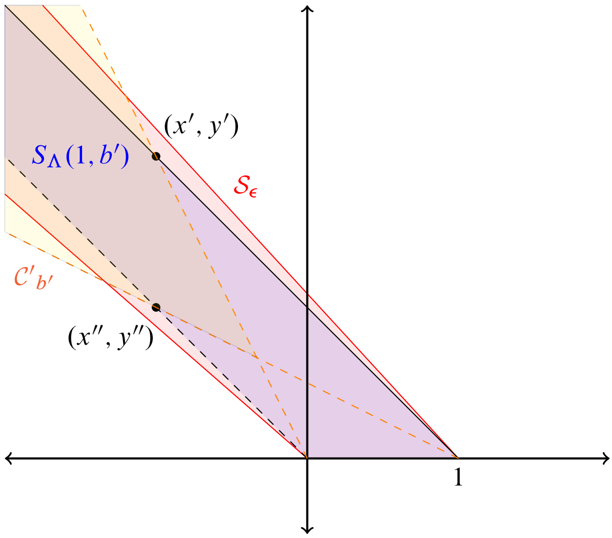

Figure 14: The strip $\mathcal{S}_\Lambda(1, b')$ with its winner (x', y'), the vector (x'', y'') on its open boundary and its cone $\mathcal{C}'_{b'}$, along with the region $\mathcal{S}_\epsilon$ containing the winners (x_i, y_i) for $i \geq n$.

contradiction, since the set $\{(x_i, y_i)\}$ is infinite. Hence, if $S_\Lambda((1, b'))$ contained a point (x', y'), it could not be in the interior of the strip.

We also consider the case when (x', y') is in on the boundary of $S_\Lambda(1, b')$. That is, we suppose that (x', y') is on the line $y = -(1/b')x + 1/b'$. If there exists a saddle connection in the interior of $S_\Lambda(1, b')$, we can appeal to the reasoning in the previous case to find a contradiction. Else, after potentially applying a shear to our surface, Lemma 6 guarantees that there is also a holonomy vector (x'', y'') on the open boundary $y = -(1/b')x$ of $S_\Lambda(1, b')$.

We now consider the cone $\mathcal{C}'_{b'}$ given by the intersection of the regions

$$y < \frac{y'}{x'}x \quad \text{and} \quad y > \frac{y''}{x''-1}x - \frac{y''}{x''-1}.$$

Similar to the previous case, we can find n large enough that the strips $S_\Lambda(1, b_i)$ all lie in a region $\mathcal{S}_\epsilon$ that is defined by $(-1/b' + \epsilon)x \leq y \leq (-1/b' - \epsilon)x + (1/b' + \epsilon)$ and $y > 0$. Here we again choose $\epsilon > 0$ such that the slopes of the two bounding lines of $\mathcal{S}_\epsilon$ are wedged between the slopes of the bounding lines of $\mathcal{C}'_{b'}$. That is, we will choose $\epsilon > 0$ such that $(-1/b' - \epsilon) > y'/x'$ and $(-1/b' + \epsilon) < y''/(x'' - 1)$. We call this latter region $\mathcal{S}_\epsilon$. Figure 14 illustrates these regions.

Since the set $\{(x_i, y_i)\}$ has no accumulation points and $\mathcal{S}_\epsilon \backslash \mathcal{C}'_{b'}$ is compact, all but finitely many of the (x_i, y_i) for $i \geq n$ must lie in the cone $\mathcal{C}'_{b'}$ and not be equal to (x', y') or (x'', y''). Let us consider one of these (x_i, y_i). The corresponding strip $\mathcal{S}_\Lambda(1, b_i)$ is the region between two parallel lines that intersect the x–axis at 1 and 0, including the line through 1 but not including the line through 0. Therefore $\mathcal{S}_\Lambda(1, b_i)$ must either contain (x', y') or (x'', y''), depending on if $b_i \leq b'$ or $b_i > b'$, respectively. If it contains (x', y'), then by similar reasoning as in the previous case (x', y') beats (x_i, y_i), and so (x_i, y_i) could not have been the winner for $(1, b_i)$. If it contains (x'', y''), then either (x'', y'') beats (x_i, y_i), which means that (x_i, y_i) was not the winner, or (x_i, y_i) was in the interior or $\mathcal{S}_\Lambda((1, b'))$, which contradicts that the interior of $S_\Lambda(1, b')$ did not contain any saddle connections. In either case, we have a contradiction.

Since we found a contradiction in both the cases when there was saddle connection in the interior and on the boundary of $S_\Lambda(1, b')$, we see that there must have been only finitely many winners on the right vertical boundary of T_X. □

We can now use the previous lemma to show that points on the right boundary of T_X have a neighborhood with finitely many winners.

Lemma 13 *Given any point $(a, b) \in T_X$ with $a = 1$, there exists a neighborhood $B_\epsilon((a, b))$ such that there are finitely many winning saddle connections on $B_\epsilon((a, b)) \cap T_X$.*

Proof Suppose that we have a point $(a, b) \in T_X$ with $a = 1$ and $b = b'$. Then Lemma 6 guarantees that $(1, b')$ is in some strip $S_\Omega(\boldsymbol{v})$. If $(1, b')$ is in the interior of $S_\Omega(\boldsymbol{v})$, then Lemma 8 shows that there is a neighborhood of $(1, b')$ in T_X with finitely many potential winners.

We now consider the case where $(1, b')$ is not in the interior of any strip. This means that $(1, b')$ is on the top boundary of some strip $S_\Omega(\boldsymbol{v})$. We will first deal with the case where $(1, b')$ is not on the top boundary of T_X: Every point $(1, b' + c)$ for $c > 0$ small enough must also be in some winning strip. Since Lemma 12 tells us that there are finitely many winning saddle connections on the right boundary of T_X where $a = 1$, this then implies that $(1, b')$ is on the bottom boundary of some other strip $S_\Omega(\boldsymbol{w})$, where $\boldsymbol{w}$ is the winning saddle connection for all $(1, b' + c)$ for $c > 0$ small enough.

Because there are finitely many winning saddle connections on the $a = 1$ line of T_X by Lemma 12, we can now choose an $\epsilon > 0$ small enough that $\boldsymbol{w}$ is the winning saddle connection for $(1, b' + c)$ and $\boldsymbol{v}$ is the winning saddle connection for $(1, b' - c)$ for any $0 < c \le \epsilon$.

We claim now that there are finitely many winning saddle connections on $B_\epsilon((1, b'))$. We recall that for a point $(a, b) \in B_\epsilon((1, b')) \cap T_X$ to have a winning saddle connection other than $\boldsymbol{v}$ or $\boldsymbol{w}$, there must be a strip $S_\Omega(\boldsymbol{u})$ for a saddle connection $\boldsymbol{u}$ that is steeper (has more negative slope) than $S_\Omega(\boldsymbol{v})$ or $S_\Omega(\boldsymbol{w})$ (whichever is the winner at (a, b)) and that contains (a, b).

Shrinking ϵ if necessary, $B_\epsilon((1, b'))$ lies above the line $b = -(x/y)a + 1/h$ for some $h > 0$ and $(x, y) = \boldsymbol{v}$. Then, as in the proof of Lemma 7, we can show that there are finitely many strips of saddle connections $\boldsymbol{u}$ of (X, ω) with strips $S_\Omega(\boldsymbol{u})$ intersecting $B_\epsilon((1, b'))$ and that are at least as steep as $S_\Omega(\boldsymbol{v})$.

If $S_\Omega(\boldsymbol{u})$ is at most as steep as $S_\Omega(\boldsymbol{w})$, then it cannot win for any point in $B_\epsilon((1, b')) \cap T_X$ since $\boldsymbol{w}$ or $\boldsymbol{v}$ would win instead.

If $S_\Omega(\boldsymbol{u})$ has steepness strictly between that of $\boldsymbol{w}$ and $\boldsymbol{v}$, then for $\boldsymbol{u}$ to be a winner for some point $(a, b) \in B_\epsilon((1, b')) \cap T_X$ we must have that $(a, b) \in S_\Omega(\boldsymbol{u}) \cap (S_\Omega(\boldsymbol{w}) \backslash S_\Omega(\boldsymbol{v}))$. But then, by slope considerations, $S_\Omega(\boldsymbol{u})$ must also intersect the $a = 1$ boundary of T_X in $B_\epsilon((1, b'))$ above the point $(1, b')$. But this contradicts that $\boldsymbol{w}$ and $\boldsymbol{v}$ were the only winners on the right boundary of T_X in $B_\epsilon((1, b'))$.

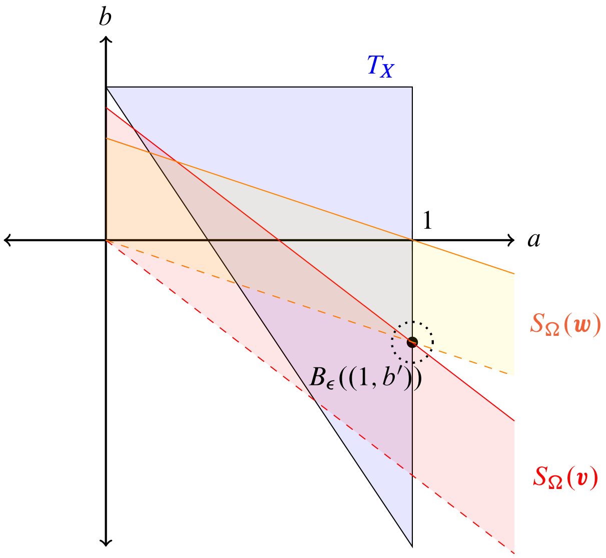

Figure 15: The winning strips $S_\Omega(\boldsymbol{v})$ and $S_\Omega(\boldsymbol{w})$ near $(1,b')$ on the right boundary of T_X.

Hence, only the finitely many saddle connections $\boldsymbol{u}$ with strips that intersect $B_\epsilon((1,b'))$ and have slope steeper than that of $S_\Omega(\boldsymbol{v})$ can be winners on $B_\epsilon((1,b')) \cap T_X$. □

Combining these lemmas shows that for all points in T_X there are finitely many winners in a neighborhood, and hence by compactness there are finitely many winners on T_X.

Proof of Theorem 1 We will consider $\overline{T}_X = T_X \cup \{(0, 1/y_0)\}$. This is a compact set. We showed in Lemmas 9, 10 and 13 that, for any point $(a,b) \in \overline{T}_X$, we can find a neighborhood $B_\epsilon((a,b))$ such that there are finitely many possible winning saddle connections on $B_\epsilon((a,b)) \cap T_X$. Since $\overline{T}_X$ is compact, it is covered by finitely many of these neighborhoods. Since a finite union of finite sets is finite, the set of possible winners on T_X is finite.

Each winning saddle connection $\boldsymbol{v}_i$ would then be a winner on a convex (see the remark below) polygonal piece of T_X. The cumulative distribution function of the slope gap distribution would then be given by the sums of areas between the level curves of the hyperbolic return time functions $y/(a(ax+by))$, as described in Section 2.2, and the sides of these polygons. Since there are finitely many polygonal pieces, the cumulative distribution function and therefore also the slope gap distribution would be piecewise real analytic with finitely many points of nonanalyticity. □

Remark 14 While it is not necessary for the proof of Theorem 1, we can see that each $\boldsymbol{v}$ is a winner on a convex polygonal piece of Ω_i. The convexity arises because the region where $\boldsymbol{v}$ wins is the intersection of finitely many convex regions: the strip $S_\Omega(\boldsymbol{v})$, the triangular region T_X, and finitely many half-planes that are the upper piece of the complement of $S_\Omega(\boldsymbol{v}')$ for other vectors $\boldsymbol{v}'$ that win on some region of $S_\Omega(\boldsymbol{v}) \cap T_X$.

4 Quadratic tail decay

As an application of the finiteness result (Theorem 1), we prove:

Theorem 2 *The slope gap distribution of any Veech surface has quadratic tail decay. That is, if f denotes the density function of the slope gap distribution, then*

$$\int_t^\infty f(x)\,dx \sim t^{-2}.$$

Proof We find the decay of the tail on a piece of the Poincaré section given by the triangle T_X. Doing this for all the pieces gives the decay of the tail.

The proof of Theorem 1 shows that there exists a minimal finite set of saddle connections $F \subset \Lambda(X,\omega)$ such that, for any point in the triangle T_X, there is some $\boldsymbol{v} \in F$ with $\boldsymbol{M}_{a,b}\boldsymbol{v}$ being the winning saddle connection. Let $S_\Omega(\boldsymbol{v}) \subset T_X$ denote the strip where $\boldsymbol{M}_{a,b}\boldsymbol{v}$ could win and $W_\Omega(\boldsymbol{v}) \subset S_\Omega(\boldsymbol{v})$ denote where $\boldsymbol{M}_{a,b}\boldsymbol{v}$ does win.

Fix $\boldsymbol{v} = (x,y) \in F$. Then the tail on the piece $W_\Omega(\boldsymbol{v})$ is proportional to the area of the set of points (a,b) in $W_\Omega(\boldsymbol{v})$ with

$$\operatorname{slope}(\boldsymbol{M}_{a,b}\boldsymbol{v}) = \frac{y}{a^2x+aby} > t \iff \frac{1}{at} - \frac{x}{y}a > b.$$

Let $m = x/y$. By adding the contribution that $W_\Omega(\boldsymbol{v})$ gives on the tail for each $\boldsymbol{v} \in F$, we get the full contribution to the tail. In what follows we work on one such winning saddle connection $\boldsymbol{v}$. Hence, it suffices to understand the portion of $W_\Omega(\boldsymbol{v})$ below the hyperbola $b = 1/(at) - ma$. Notice that this hyperbola approaches the line $b = -ma$ from above. Moreover, notice that the line $b = -ma$ is the bottom boundary of the strip $S_\Omega(\boldsymbol{v})$.

We have three situations, depending on how the line $b = -ma$ intersects T_X, as shown in Figure 16.

(1) Suppose $b = -ma$ doesn't intersect T_X. This means that the line $b = -ma$ avoids the bottom edge of T_X for $a \in [0,1]$. In this case we will only find contribution to the tail when the vertical of $\boldsymbol{v}$ is y_0, since otherwise we can choose large enough t so that the hyperbola misses $W_\Omega(\boldsymbol{v})$.

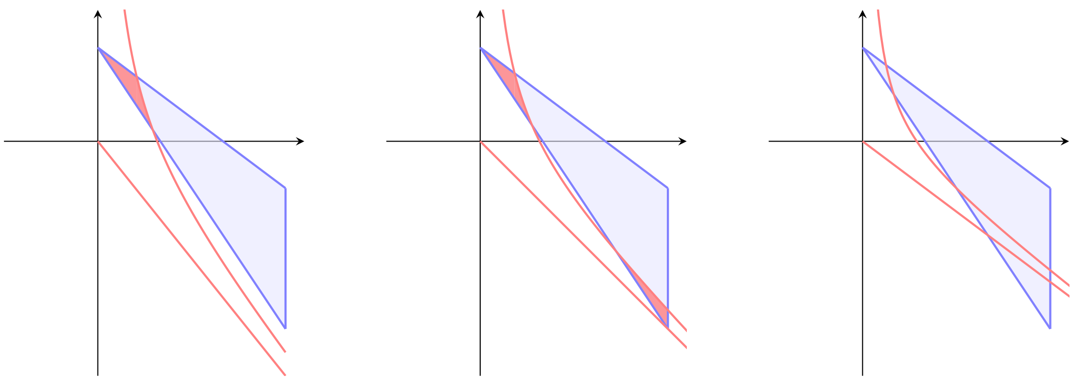

Figure 16: An illustration of cases (1)–(3) in the proof of Theorem 2 (from left to right).

An upper bound for the contribution of $W_\Omega(\boldsymbol{v})$ is just the part underneath the hyperbola and inside T_X. For t large, the hyperbola $b = 1/(at) - ma$ intersects the triangle twice. First it intersects at the top through the boundary line $b = 1/y_0 - (x_0/y_0)a$ at the point

$$a_{\mathrm{top}}^{+} = \frac{-1 + \sqrt{1 + 4y_0(my_0 - x_0)/t}}{2(my_0 - x_0)},$$

and then leaves through the bottom boundary line $b = 1/y_0 - (x_0/y_0 + n)a$ at the point

$$a_{\mathrm{bot}}^{+} = \frac{-1 + \sqrt{1 + 4y_0(my_0 - (x_0 + ny_0))/t}}{2(my_0 - (x_0 + ny_0))}.$$

Thus, the contribution is given by

$$\int_{a=0}^{a_{\mathrm{top}}^{+}} \int_{b=1/y_0-(x_0/y_0+n)a}^{1/y_0-(x_0/y_0)a} 1\, db\, da + \int_{a=a_{\mathrm{top}}^{+}}^{a_{\mathrm{bot}}^{+}} \int_{b=1/y_0-(x_0/y_0+n)a}^{1/(at)-ma} 1\, db\, da.$$

The first integral evaluates to $\frac{1}{2}n(a_{\mathrm{top}}^{+})^2$ and, by using a Taylor series on the square root, can be shown to decay like t^{-2}.

The second integral evaluates to

$$\left(\frac{1}{t}\log(a) + \frac{1}{2}\left(\frac{x_0}{y_0} + n - m\right)a^2 - \frac{1}{y_0}a\right)\Bigg|_{a=a_{\mathrm{top}}^{+}}^{a_{\mathrm{bot}}^{+}}.$$

By performing a Taylor series approximation on a_{top}^{+} and a_{bot}^{+}, we get that the second integral decays like t^{-3}.

Thus, the total decay on the integral is like t^{-2}.

(2) Now consider the case when $b = -ma$ intersects T_X at the bottom vertex of T_X. In this case $m = x_0/y_0 + n - 1/y_0$. If the vertical of y is the same as y_0, then we get a contribution to the tail at the top of T_X as in case (1). In fact, this is the only way we can get contribution at the top of T_X.

Now we find the contribution on the bottom of T_X. Thus, we are interested in the intersection of the hyperbola $b = 1/(at) - ma$ with the bottom boundary line of T_X given by $1/y_0 - (x_0/y_0 + n)a$. This is the point

$$a_{\mathrm{bot}}^{-} = \frac{-1 - \sqrt{1 + 4y_0(my_0 - (x_0 + ny_0))/t}}{2(my_0 - (x_0 + ny_0))}.$$

In fact, using that the line $b = -ma$ intersects the bottom of T_X, we get that $m = x_0/y_0 + n - 1/y_0$ and so we can see

$$a_{\mathrm{bot}}^{-} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4y_0}{t}}\right).$$

The contribution is then given by

$$\int_{a=a_{\mathrm{bot}}^{-}}^{1} \int_{b=1/y_0-(x_0/y_0+n)a}^{1/(at)-ma} 1\, db\, da.$$

This integral evaluates to

$$\frac{1}{2}\left(\frac{x_0}{y_0}+n-m\right)-\frac{1}{y_0}-\frac{1}{t}\log(a_{\mathrm{bot}}^-)-\frac{1}{2}\left(\frac{x_0}{y_0}+n-m\right)(a_{\mathrm{bot}}^-)^2+\frac{a_{\mathrm{bot}}^-}{y_0}.$$

By doing a Taylor series approximation on a_{bot}^- we can show that the decay is like t^{-2}.

(3) Now suppose that the line $b=-ma$ does intersect T_X and this intersection is above the bottom vertex of T_X, ie above $b=1/y_0-(x_0/y_0+n)$. We have two regions to consider: the top of the triangle and the region above between the hyperbola $b=1/(at)-ma$ and $b=-ma$. The behavior at the top of the triangle is identical to cases (1) and (2), and only occurs when the vertical y is the same as y_0. Thus we have quadratic decay there. We now focus on the second region and observe that each point on the bottom edge of $S_\Omega(\boldsymbol{v})$ must be in some other winning strip $S_\Omega(\boldsymbol{v}')$. There are finitely many such $\boldsymbol{v}'$, and we number them $\boldsymbol{v}_1,\dots,\boldsymbol{v}_n$. Thus $W_\Omega(\boldsymbol{v})\subset\bigl(S_\Omega(\boldsymbol{v})-\bigcup_{i=1}^n S_\Omega(\boldsymbol{v}_i)\bigr)$, which is some polygonal region whose closure is completely above the bottom boundary of $S_\Omega(\boldsymbol{v})$, $b=-ma$. Since the hyperbola $b=1/(at)-ma$ approaches $b=-ma$ as $t\to\infty$, for all t large enough the hyperbola is completely below $W_\Omega(\boldsymbol{v})$ and therefore $W_\Omega(\boldsymbol{v})$ has no contribution to the tail.

Adding up the contribution of every $\boldsymbol{v}\in F$, we see that there is a constant $C>0$ such that

$$\int_t^\infty f(x)\,dx\le\frac{C}{t^2}.$$

Now we compute a lower bound. Let $\boldsymbol{v}_0=(x_0,y_0)$ be the saddle connection used to define T_X, $S_\Omega(\boldsymbol{v}_0)$ denote the associated strip, and $b=1/(at)-(x_0/y_0)a$ be the associated hyperbola. We will use this specific saddle connection to find a lower bound to $\int_t^\infty f(x)\,dx$, essentially by using the argument from case (1) of the upper bound. That is, by analyzing the behavior at the top of the triangle. Either $\boldsymbol{v}_0$ is the winning saddle connection for every point on $S_\Omega(\boldsymbol{v}_0)$ or there is some other saddle connection $\boldsymbol{v}$ for which it is the winning saddle connection on $S_\Omega(\boldsymbol{v}_0)\cap S_\Omega(\boldsymbol{v})$. We deal with both cases.

(i) If $\boldsymbol{v}_0$ is the winning saddle connection for every point on $S_\Omega(\boldsymbol{v}_0)$, then a lower bound to $\int_t^\infty f(x)\,dx$ comes from the part underneath the hyperbola $b=1/(at)-(x_0/y_0)a$ and inside $S_\Omega(\boldsymbol{v}_0)$. We can choose t large enough that the hyperbola intersects $S_\Omega(\boldsymbol{v}_0)$ only once, at the point

$$a_{\mathrm{top}}^+=\frac{-1+\sqrt{1+4y_0(my_0-x_0)/t}}{2(my_0-x_0)}$$

with contribution given by

$$\int_{a=0}^{a_{\mathrm{top}}^+}\int_{b=1/y_0-((x_0/y_0)+n)a}^{1/y_0-x_0/y_0a}1\,db\,da.$$

Earlier we showed this decays like t^{-2}.

(ii) In the case that there is some other saddle connection $\boldsymbol{v}$ that is the winning saddle connection on $S_\Omega(\boldsymbol{v}_0)\cap S_\Omega(\boldsymbol{v})$ we have two subcases, depending on whether $\boldsymbol{v}$ has the same vertical as $\boldsymbol{v}_0$ or not. In the latter case we can choose t large enough that the contribution is the same as case (1). We now focus on when the vertical of $\boldsymbol{v}$ and $\boldsymbol{v}_0$ is the same. Furthermore, since we are looking for any lower bound, it suffices to assume that $\boldsymbol{v}$ has the least negative slope among all vectors that win in the intersection of $S_\Omega(\boldsymbol{v}_0) \cap S_\Omega(\boldsymbol{v})$. The contribution is given by

$$\int_{a=0}^{a_{\mathrm{top}}^+} \int_{b=1/y_0-(x/y)a}^{1/y_0-(x_0/y_0)a} 1\, db\, da.$$

The integral evaluates to $\frac{1}{2}(x/y - x_0/y_0)(a_{\mathrm{top}}^+)^2$ and, by using a Taylor series on the square root, can be shown to decay like t^{-2}. □

5 Further questions

We end with a few questions for further exploration:

(1) Are there bounds on the number of points of nonanalyticity of the slope gap distribution of a Veech surface?

In [5], linear upper and lower bounds in terms of n on the number of points were found for the translation surface given by gluing opposite sides of the $2n$–gon. These surfaces each have two cusps and have genus that grows linearly in n. This shows that bounds on the number of points of nonanalyticity based on the number of cusps is impossible. However, we can ask if there are bounds based on the genus of the surface.

(2) What can be said about the gap distributions of non-Veech surfaces?

In [2] it was shown that the limiting slope gap distribution exists for almost every translation surface, and in [9] the slope gap distributions for a special family of non-Veech surfaces were shown to be piecewise real analytic. We can ask if the limiting slope gap distributions are always piecewise real analytic, and if so, are there always finitely many points of nonanalyticity?

(3) Where do the points of nonanalyticity lie?

Beyond just understanding the number of points of nonanalyticity, we can ask about number-theoretic properties of the points themselves. In every example known to the authors of a limiting slope gap distribution, after rescaling, the points of nonanalyticity lie in the trace field of the Veech group. Given that the gap distribution is computed by integrating areas between hyperbolas in regions related to the geometry of the surface, it is natural to conjecture that points of nonanalyticity lie in quadratic extensions of the trace field.

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