

The strong Künneth theorem for topological periodic cyclic homology

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Abstract

Topological periodic cyclic homology (i.e., \mathbb{T} -Tate fixed points of THH) has the structure of a strong symmetric monoidal functor of smooth and proper dg categories over a perfect field of finite characteristic.

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Introduction

Trace methods have produced powerful tools for computing algebraic K -theory. In these methods, one obtains information about the K -theory spectrum by mapping it to more computable theories like topological Hochschild homology (THH) and topological cyclic homology (TC). As the name suggests, THH is a “topological” analogue of Hochschild homology, where tensor product is replaced with smash product and we work over the sphere spectrum instead of the integers, resulting in a theory which is the same rationally but richer at finite primes. Despite the name, TC is not the topological analogue of cyclic homology, but is more closely related to negative cyclic homology. In contrast to the algebraic setting, this construction does not provide evident topological analogues of positive or periodic cyclic homology. Since TC has been so successful for calculations in algebraic K -theory, topologists have focused comparatively little attention on such analogues until recently.

In [14], Hesselholt studied a topological analogue for periodic cyclic homology, motivated by the Deninger program [9] and the recent progress on it by Connes-Consani [8]. Deninger proposed an approach to the classical Riemann hypothesis in analogy with Deligne’s proof of Weil’s Riemann hypothesis. A basic ingredient in this approach is a suitable infinite-dimensional cohomology theory and some kind of endomorphism playing the role of the Frobenius automorphism; there is then a conjectural expression for the zeta function in terms of regularized determinants. Using periodic cyclic homology, Connes-Consani [8, 1.1] produced a cohomology theory and expression for the product of Serre’s archimedean local factors of the Hasse-Weil zeta function of a smooth projective variety over a number field in terms of a regularized determinant for an endomorphism coming from the action of λ -operations on this cohomology theory. Hesselholt [14] defined a theory TP as the Tate \mathbb{T} -fixed points of THH . We refer to this theory as *topological periodic cyclic homology* as it is the topological analogue of periodic cyclic homology, though we note that it is not always itself periodic. Using TP in place of the Connes-Consani theory, Hesselholt establishes a non-archimedean version of their results, producing a regularized determinant expression for the Hasse-Weil zeta function of a smooth projective variety over a finite field.

The purpose of this paper is to prove an important structural property of TP . It is well-known to experts that TP has the natural structure of a lax symmetric monoidal functor from dg categories (or spectral categories) over a commutative ring R (or commutative \mathbb{S} -algebra R) to the derived category of $TP(R)$ -modules. In other words, TP satisfies a lax Künneth formula. We establish a strong Künneth formula for TP when restricted to smooth and proper dg categories over a perfect field of characteristic $p > 0$.

THEOREM A (Strong Künneth Formula). *Let k be a perfect field of characteristic $p > 0$ and let \mathcal{X} and \mathcal{Y} be k -linear dg categories. The natural lax symmetric monoidal transformation*

$$TP(\mathcal{X}) \wedge_{TP(k)}^{\mathbf{L}} TP(\mathcal{Y}) \longrightarrow TP(\mathcal{X} \otimes_k \mathcal{Y})$$

is a weak equivalence when \mathcal{X} and \mathcal{Y} are smooth and proper over k .

Here $\wedge^{\mathbf{L}}$ denotes the derived smash product: This theorem is a derived category statement rather than a homotopy groups statement. We get a homotopy groups

statement from the Künneth spectral sequence [10, IV.4.7], which has E^2 -term

$$E_{*,*}^2 = \mathrm{Tor}_{*,*}^{TP_*(k)}(TP_*(\mathcal{X}), TP_*(\mathcal{Y}))$$

and converges strongly to $\pi_*(TP(\mathcal{X}) \wedge_{TP(k)}^{\mathbf{L}} TP(\mathcal{Y}))$. When k is a perfect field of characteristic p , $\pi_*TP(k) \cong \mathbb{W}k[v, v^{-1}]$, where $\mathbb{W}k$ denotes the p -typical Witt vectors, and v is an element in degree 2. In particular, $\mathrm{Tor}_{s,*}^{TP_*(k)}$ vanishes for $s > 1$ and the spectral sequence degenerates to a short exact sequence. We state this as the following corollary.

COROLLARY A-1. *Let k be a perfect field of characteristic $p > 0$ and let \mathcal{X} and \mathcal{Y} be smooth and proper k -linear dg categories. Then there is a natural short exact sequence*

$$0 \rightarrow TP_*(\mathcal{X}) \otimes_{TP_*(k)} TP_*(\mathcal{Y}) \rightarrow TP_*(\mathcal{X} \otimes_k \mathcal{Y}) \rightarrow \mathrm{Tor}_{*-1,*}^{TP_*(k)}(TP_*(\mathcal{X}), TP_*(\mathcal{Y})) \rightarrow 0.$$

This exact sequence splits, but not naturally.

We also get a strong Künneth theorem on homotopy groups after inverting p ; indeed, for the Hesselholt work on TP , the statements of the main results only involve $TP_*[1/p]$. Since inverting p commutes with the smash product and $TP_*(k)[1/p]$ is a “graded field” in the sense that all graded modules over it are free (in particular, $\mathrm{Tor}_{s,*} = 0$ for $s > 0$), we have the following immediate corollary.

COROLLARY A-2. *Let k be a perfect field of characteristic $p > 0$ and let \mathcal{X} and \mathcal{Y} be k -linear dg categories. The natural map on homotopy groups with p inverted*

$$TP_*(\mathcal{X})[1/p] \otimes_{TP_*(k)[1/p]} TP_*(\mathcal{Y})[1/p] \longrightarrow TP_*(\mathcal{X} \otimes_k \mathcal{Y})[1/p]$$

is an isomorphism when \mathcal{X} and \mathcal{Y} are smooth and proper over k .

Along the way to proving Theorem A, we prove the following finiteness result for TP . This result is surprising in light of the relationship between TP and the étale cohomology of the de Rham-Witt sheaves [14, 6.8] and the extreme non-finiteness of this cohomology for supersingular $K3$ surfaces [17, §II.7.2]. (We thank Lars Hesselholt for calling our attention to this example.)

THEOREM B. *Let k be a perfect field of characteristic $p > 0$ and let \mathcal{X} be a smooth and proper k -linear dg category. Then $TP(\mathcal{X})$ is a small $TP(k)$ -module; in particular $TP_*(\mathcal{X})$ is finitely generated over $TP_*(k)$.*

We also prove analogues of Theorems A and B for C_p -Tate THH . For THH , as we review in Section 15, the analogue of Theorem A is well-known to hold in much wider generality (without the smooth and proper hypotheses, or the field hypothesis), but Theorem B appears to be new even in this case. Since the other cases take this case as input, we include the statement here in the generality we prove it in Section 16. We emphasize that the statement is non-equivariant.

THEOREM C. *Let R be a commutative ring orthogonal spectrum and \mathcal{X} a smooth and proper R -spectral category. Then $THH(\mathcal{X})$ is a small $THH(R)$ -module.*

Theorem C is easy to deduce from the strong Künneth theorem for THH and the interpretation of smooth and proper in terms of duality [33, 5.4.2], [2, 3.7], but we provide a simplicial argument in Section 16.

An interesting application for Theorem A comes from the theory of noncommutative motives. Periodic cyclic homology plays an important role in Kontsevich's ideas on noncommutative motives over a field of characteristic 0, as developed by Tabuada [32], serving as a noncommutative replacement for de Rham cohomology. The fact that $TP_*[1/p]$ satisfies a Künneth formula allows TP to take the place of periodic cyclic homology in the setting of noncommutative motives over a perfect field of characteristic p . Notably, Tabuada [30] (see also [31]) has recently used our Theorem A to show that the numerical Grothendieck group of a smooth and proper variety (the generalization of the group of algebraic cycles up to numerical equivalence) is finitely generated. Tabuada also applies Theorem A to establish a conjecture of Kontsevich [19] that the category of numerical noncommutative motives is abelian semi-simple, generalizing work of Janssen [18] for numerical motives.

Finally, this paper also achieves some technical results that may be of interest to homotopy theorists. Some highlights include:

- Construction of an explicit A_∞ coalgebra structure, parametrized by the little 1-cubes operad, for cellular approximations of the diagonal for the cell structure on the geometric realization of a simplicial set and the analogue for a simplicial space. (See Section 10.)
- Construction of models for the co-family universal space \widetilde{EG} with action by arbitrary A_∞ or E_∞ operads (see Section 2) and a filtration equivalent to the standard filtration (as the cofiber of $EG \rightarrow S^0$) that respects the operad structure (see Section 9).
- Construction of a point-set lax symmetric monoidal model of the Tate construction (see Section 2) and a point-set lax monoidal filtered version of the Tate construction (see Section 11).
- Construction of the balanced smash product for right and left modules over A_∞ ring orthogonal spectra, and homotopical comparison for some common A_∞ operads as the operad varies. (See Section 18.)
- Study of the T-analogue of the Hesselholt-Madsen construction of the Tate spectral sequence, which (unlike the case of a finite group) differs at E^2 from the Greenlees Tate spectral sequence, before becoming isomorphic at E^3 . (See Sections 3 and 12–13.)

The reader interested in homotopy theory might wonder about generalizing Theorems A and B for commutative ring spectra more general than Hk and to other closed subgroups of \mathbb{T} beyond $\{1\}$, C_p , and \mathbb{T} . Besides connectivity of Hk , the proof of Theorem A depends on Theorem B and the observation in Proposition 8.1 that the canonical map $THH(Hk) \rightarrow Hk$ makes Hk a small $THH(Hk)$ -module in the Borel category of equivariant $THH(Hk)$ -modules. Theorem B depends on the fact that the commutative ring $THH_*(Hk)$ is Noetherian and the commutative rings $TP_*(k)$ and $\pi_*(THH(k))^{tC_p}$ are a graded PID and a graded field, respectively; for the theorem, it is enough that they have finite global dimension. For other subgroups $C_{p^n} < \mathbb{T}$ with $n > 1$, $\pi_*(THH(k))^{tC_p}$ has infinite global dimension.

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1. Orthogonal G -spectra and the Tate fixed points

This section sets out some conventions for the rest of the paper and reviews certain aspects of the homotopy theory of equivariant orthogonal spectra and the construction of the Tate fixed point spectra. Throughout the paper, G denotes a compact Lie group, but starting in Section 3, we specialize to the case when G is a finite group or $G = \mathbb{T}$, the circle group of unit complex numbers.

Let \mathcal{S} denote the category of non-equivariant orthogonal spectra and \mathcal{S}^G the category of orthogonal G -spectra, which we understand to be indexed on all finite-dimensional orthogonal G -representations (at least up to isomorphism). We understand *weak equivalence* (when used as an unmodified specific term) to refer to the weak equivalences in the stable model structure of [23, III.4.2], or equivalently, the positive stable model structure [23, III.5.3]; the weak equivalences are the maps that induce isomorphisms on homotopy groups π_*^H for all $H < G$, defined by

$$\pi_q^H X = \operatorname{colim}_{V < U} \operatorname{colim}_{n \geq \max\{0, -q\}} \pi_{q+n}((\Omega^V(X(\mathbb{R}^n \oplus V)))^H).$$

In this notation, $H < G$ means H is a closed subgroup of G (not necessarily proper), U denotes a fixed infinite dimensional G -inner product space containing a representative of each finite dimensional G -representation, and $V < U$ means that V is a finite dimensional G -stable vector subspace of U . The homotopy category obtained by inverting these weak equivalences is called the *stable category* (or G -stable category or G -equivariant stable category) and denoted here as $\operatorname{Ho}(\mathcal{S}^G)$. We call the objects of $\operatorname{Ho}(\mathcal{S}^G)$ *spectra* (or G -spectra or G -equivariant spectra).

The Tate construction is most naturally viewed as a functor not from the stable category but from a localization called the “Borel stable category”, which is formed by inverting the “Borel equivalences”.

DEFINITION 1.1. A map of orthogonal G -spectra is a *Borel equivalence* if it induces an isomorphism on $\pi_* = \pi_*^{\{e\}}$ (where $\{e\}$ denotes the trivial subgroup of G). The *Borel colocal model structure* on orthogonal G -spectra is the \mathcal{F} -model structure of [23, IV.6.5ff] for $\mathcal{F} = \{\{e\}\}$, the *Borel local model structure* is the Bousfield \mathcal{F} -module structure of [23, IV.6.3ff] for $\mathcal{F} = \{\{e\}\}$, and the *Borel category* $\operatorname{Ho}^B(\mathcal{S}^G)$ is the homotopy category obtained by inverting the Borel equivalences.

For orthogonal G -spectra X and Y , the relationship between maps in the stable category and maps in the Borel category is given by the formula [23, IV.6.11]

$$\begin{aligned} \operatorname{Ho}^B(\mathcal{S}^G)(X, Y) &\cong \operatorname{Ho}(\mathcal{S}^G)(X \wedge EG_+, Y \wedge EG_+) \\ &\cong \operatorname{Ho}(\mathcal{S}^G)(X \wedge EG_+, Y) \cong \operatorname{Ho}(\mathcal{S}^G)(X, Y^{EG}). \end{aligned}$$

As always, EG denotes the universal space, i.e., a free G -CW space whose underlying non-equivariant space is contractible. The notation Y^{EG} denotes the derived G -spectrum of unbased non-equivariant maps from EG to Y . The last two isomorphisms indicate that the (co)localization $\operatorname{Ho}(\mathcal{S}^G) \rightarrow \operatorname{Ho}^B(\mathcal{S}^G)$ has both a left and a right adjoint: The left is modeled by the functor $(-) \wedge EG_+$ and the right is modeled by applying fibrant replacement in the stable or positive stable model structure followed by the point-set mapping G -spectrum functor $(-)^{EG}$.

The previous formula indicates that the Borel category may be viewed as the full subcategory of the G -stable category consisting of the free G -spectra. The Borel category can also be constructed as the homotopy category of G -objects in orthogonal spectra, giving rise to the Lewis-May slogan “free G -spectra live in the trivial universe”. To be precise, let $\mathcal{S}[G]$ denote the category where an object is an orthogonal spectrum X together with an associative and unital structure map $G_+ \wedge X \rightarrow X$ and a morphism is a map of orthogonal spectra preserving the action maps. The category $\mathcal{S}[G]$ then has a model structure with weak equivalences and fibrations the underlying weak equivalences and fibrations of orthogonal spectra; the cofibrations are retracts of cell complexes built using free G -cells on spheres. Let $\kappa: \mathcal{S}[G] \rightarrow \mathcal{S}^G$ be the functor that fills in the non-trivial representations by the formula

$$\kappa X(V) := \mathcal{I}(\mathbb{R}^{\dim V}, V) \wedge_{O(\dim V)} X(\mathbb{R}^{\dim V})$$

where $\mathcal{I}(\mathbb{R}^{\dim V}, V)$ denotes the G -space of non-equivariant linear isometries from $\mathbb{R}^{\dim V}$ to V . As observed in [23, V.1.5], κ is an equivalence of categories with inverse the functor that forgets the non-trivial representation indexes. Since the formula for the homotopy groups of an orthogonal G -spectrum in the case $H = \{e\}$ simplifies to

$$\pi_q X = \operatorname{colim}_{n \geq \max\{0, -q\}} \pi_{q+n} X(\mathbb{R}^n)$$

(as the map $\operatorname{colim}_n \pi_{q+n} X(\mathbb{R}^n) \rightarrow \operatorname{colim}_n \pi_{q+n} \Omega^V X(\mathbb{R}^n \oplus V)$ is an isomorphism for all $V < U$), κ sends weak equivalences to Borel equivalences and κ^{-1} sends Borel equivalences to weak equivalences. Thus, κ and κ^{-1} also induce equivalences on homotopy categories $\operatorname{Ho}(\mathcal{S}[G]) \simeq \operatorname{Ho}^B(\mathcal{S}^G)$ (with κ a Quillen left adjoint in both model structures in Definition 1.1).

If A is an associative ring orthogonal G -spectrum, we also have Borel local and colocal model structures on the category $\mathcal{M}\text{od}_A$ of equivariant A -modules. We call the homotopy category the *Borel derived category*, denoted $\operatorname{Ho}^B(\mathcal{M}\text{od}_A)$. Just as in the base case of $A = \mathbb{S}$, we have

$$\begin{aligned} \operatorname{Ho}^B(\mathcal{M}\text{od}_A)(X, Y) &\cong \operatorname{Ho}(\mathcal{M}\text{od}_A)(X \wedge EG_+, Y \wedge EG_+) \\ &\cong \operatorname{Ho}(\mathcal{M}\text{od}_A)(X \wedge EG_+, Y) \cong \operatorname{Ho}(\mathcal{M}\text{od}_A)(X, Y^{EG}). \end{aligned}$$

As usual, we can interpret $\operatorname{Ho}^B(\mathcal{M}\text{od}_A)(X, Y)$ as the π_0 of a derived mapping spectrum $\mathbf{R}^B F_A^G(X, Y)$. We can construct $\mathbf{R}^B F_A(X, Y)$ as the derived G -fixed points of the derived mapping G -spectrum

$$\mathbf{R} F_A(X \wedge EG_+, Y \wedge EG_+) \simeq \mathbf{R} F_A(X \wedge EG_+, Y) \simeq \mathbf{R} F_A(X, Y^{EG}),$$

or equivalently, as the homotopy G -fixed points of the derived mapping G -spectrum $\mathbf{R} F_A(X, Y)$.

We now define the Tate fixed point functor, following [13]. In the definition, \widetilde{EG} denotes a based G -space of the homotopy type of the cofiber of the map $EG_+ \rightarrow S^0$; we will become more picky about models in later sections.

DEFINITION 1.2. Define the *Tate fixed points* to be the functor

$$(-)^{tG}: \operatorname{Ho}^B(\mathcal{S}^G) \longrightarrow \operatorname{Ho}(\mathcal{S})$$

from the Borel category to the stable category constructed as the composite of derived functors

$$X^{tG} = (X^{EG} \wedge \widetilde{EG})^G$$

where $(-)^{EG}$ is the right derived inclusion functor $\mathrm{Ho}^B(\mathcal{S}^G) \rightarrow \mathrm{Ho}(\mathcal{S}^G)$. We write $\pi_*^{tG} X$ for $\pi_*(X^{tG})$.

We can construct a point-set model for the Tate fixed point functor using a fibrant replacement functor R_G in the (positive) stable model structure in orthogonal G -spectra; we will be more specific about R_G in later sections.

CONSTRUCTION 1.3. For a chosen fibrant replacement functor R_G in the stable or positive stable model structure on orthogonal G -spectra and a chosen model for $S^0 \rightarrow \widetilde{EG}$, let $T_G: \mathcal{S}^G \rightarrow \mathcal{S}$ be the composite point-set functor

$$T_G(X) = (R_G((R_G X)^{EG} \wedge \widetilde{EG}))^G.$$

As constructed, T_G sends Borel equivalences of orthogonal G -spectra to weak equivalences of orthogonal spectra and its derived functor is canonically naturally isomorphic to $(-)^{tG}$.

REMARK 1.4. Construction 1.3 gives us a lot of latitude in the choices, which we take advantage of in later sections. We pause to note that any choices give naturally weakly equivalent functors by an essentially unique natural transformation (where “essentially unique” means a contractible space of choices): The choice of fibrant replacement functor is unique up to essentially unique natural weak equivalence, the space of equivariant self-maps of the G -space EG is contractible, and the space of equivariant based self-maps of the based G -space \widetilde{EG} is based homotopy equivalent to S^0 .

2. A lax Künneth theorem for Tate fixed points

In this section, we prove a lax version of the Künneth theorem for the Tate fixed points (for an arbitrary compact Lie group G). These results are regarded as well-known, but the exposition of this section sets up some of the constructions and notation we need in later sections. We deduce our stable category results from stronger point-set results that we expect to be useful in future work.

Construction 1.3, our point-set model T_G for the Tate fixed point functor, left the choice of the fibrant replacement functor R_G and the model for \widetilde{EG} unspecified. Our strategy in this section is to make choices so that T_G becomes a lax symmetric monoidal functor. The first step is to choose R_G to be lax symmetric monoidal.

LEMMA 2.1. *Let G be a compact Lie group. The positive stable model structure on orthogonal G -spectra admits a topologically enriched lax symmetric monoidal fibrant replacement functor.*

As the details of the construction are unimportant, we give the proof in Section 20.

We next choose a model for \widetilde{EG} . McClure [26, §2] showed that any model of \widetilde{EG} has the structure of an equivariant algebra over some non-equivariant E_∞ operad (an N_∞ operad in the terminology of [4]). Turning this around, given a non-equivariant E_∞ or A_∞ operad \mathcal{O} , we would like a tractable model of \widetilde{EG} which is an \mathcal{O} -algebra in based G -spaces (using \wedge as the symmetric monoidal product). We need the flexibility to consider varying \mathcal{O} : in this section, using Boardman’s linear isometries operad \mathcal{L} lets us take advantage of the symmetric monoidal structures considered in [5], and in future sections we need to use the Boardman-Vogt little 1-cubes operad \mathcal{C}_1 (and certain variants).

In the following, we take \mathcal{O} to be an E_∞ or A_∞ operad (in unbased non-equivariant spaces), and for convenience of exposition, we require \mathcal{O} to satisfy $\mathcal{O}(0) = *$. (If not, the discussion below changes in that I and S^0 get replaced with $\mathcal{O}(0) \times I$ and $\mathcal{O}(0)_+$, respectively.) For a based space X , we denote by $\mathbb{O}X$ the free \mathcal{O} -algebra on X in based spaces, $\mathbb{O}X = \bigvee \mathcal{O}(n)_+ \wedge_{\Sigma_n} X^{(n)}$, where (n) denotes the n th smash power.

CONSTRUCTION 2.2. Let $\widetilde{E_{\mathcal{O}}G}$ be the \mathcal{O} -algebra formed as the pushout in \mathcal{O} -algebras

$$S^0 \longleftarrow \mathbb{O}(G \times \partial I)_+ \longrightarrow \mathbb{O}(G \times I)_+$$

where the rightward map is induced by the inclusion and the leftward map is induced by the map $G \times \partial I \rightarrow S^0$ sending $G \times \{1\}$ to the basepoint and $G \times \{0\}$ to the non-basepoint.

It is clear that $\widetilde{E_{\mathcal{O}}G}$ is non-equivariantly contractible because the basepoint lies in the unit component. Neglecting the \mathcal{O} -algebra structure, the underlying G -space has a filtration by G -equivariant cofibrations induced by homogeneous degree in the elements of I ,

$$(2.3) \quad S^0 = E_0 \subset E_1 \subset \cdots \subset \widetilde{E_{\mathcal{O}}G},$$

with quotients

$$E_n/E_{n-1} \cong \mathcal{O}(n)_+ \wedge_{\Sigma_n} (G_+^n \wedge I^n / \partial I^n).$$

Indeed, E_n is the pushout in unbased spaces of the evident map

$$\mathcal{O}(n) \times_{\Sigma_n} (G^n \times \partial(I^n)) \longrightarrow E_{n-1}$$

and the inclusion

$$\mathcal{O}(n) \times_{\Sigma_n} (G^n \times \partial I^n) \longrightarrow \mathcal{O}(n) \times_{\Sigma_n} (G^n \times I^n).$$

Because $\mathcal{O}(n)$ is Σ_n -free, it follows that for any nontrivial $H < G$, the fixed point subspace E^H is exactly the subspace S^0 . This shows that $\widetilde{E_{\mathcal{O}}G}$ is a model of \widetilde{EG} .

The result of this construction is that when we use a lax symmetric monoidal fibrant replacement functor to define T_G , we now have a natural map

$$(2.4) \quad \mathcal{O}(n)_+ \wedge T_G X_1 \wedge \cdots \wedge T_G X_n \longrightarrow T_G(X_1 \wedge \cdots \wedge X_n),$$

compatible with the operadic multiplication. This map is also natural in the operad \mathcal{O} .

Using $\widetilde{E_{\mathcal{L}}G}$ for Boardman's linear isometries operad \mathcal{L} allows us to construct a point-set lax symmetric monoidal model of $(-) \wedge \widetilde{EG}$ as follows. We use the model of spectra given by the symmetric monoidal category of S -modules inside the weak symmetric monoidal category of $\mathcal{L}(1)$ -spectra in orthogonal spectra constructed by Blumberg-Hill in [5] following the ideas of EKMM. Specifically, let $\mathcal{S}^G[\mathcal{L}(1)]$ denote the category with objects orthogonal G -spectra X equipped with an associative and unital action map $\mathcal{L}(1)_+ \wedge X \rightarrow X$, and morphisms the maps $X \rightarrow Y$ compatible with the action. Recall that this is a *weak* symmetric monoidal category, that is, a category equipped with a product which satisfies all the axioms of a symmetric monoidal category except that the unit map need not be an isomorphism. As in EKMM, there is a full symmetric monoidal subcategory of the unital objects in $\mathcal{S}^G[\mathcal{L}(1)]$, which we write as $\mathcal{S}_{\text{BH}}^G$ (denoted in [5] as $G\mathcal{S}_{\mathbb{R}^\infty}$), the analogue of EKMM S -modules. Furthermore, [5, 4.8] constructs a strong symmetric monoidal functor $J: \mathcal{S}^G[\mathcal{L}(1)] \rightarrow \mathcal{S}_{\text{BH}}^G$ and a natural weak equivalence $J \rightarrow \text{Id}$.

Both $\mathcal{S}^G[\mathcal{L}(1)]$ and $\mathcal{S}_{\text{BH}}^G$ admit model structures with weak equivalences determined by the underlying equivalences in \mathcal{S}^G ; although [5] works with the stable equivalences, the arguments clearly hold for the Borel equivalences as well. In $\mathcal{S}^G[\mathcal{L}(1)]$, the fibrations are also determined by the forgetful functor to \mathcal{S}^G . Note that the lax symmetric monoidal fibrant replacement functor R_G on \mathcal{S}^G induces a lax symmetric monoidal fibrant replacement functor on $\mathcal{S}^G[\mathcal{L}(1)]$: as the fibrations in $\mathcal{S}^G[\mathcal{L}]$ are the maps whose underlying maps in \mathcal{S}^G are fibrations, $R_G X$ is clearly fibrant, and R_G inherits a lax symmetric monoidal structure on $\mathcal{S}^G[\mathcal{L}(1)]$.

The point of introducing this setup is that for the model $\widetilde{E_{\mathcal{L}}G}$, $\Sigma^\infty \widetilde{E_{\mathcal{L}}G}$ is a commutative monoid in the weak symmetric monoidal structure on $\mathcal{L}(1)$ -spectra and therefore

$$J((-) \wedge \widetilde{E_{\mathcal{L}}G}) \cong J((-) \wedge \Sigma^\infty \widetilde{E_{\mathcal{L}}G})$$

is a strong symmetric monoidal functor from orthogonal G -spectra to the category of $J(\Sigma^\infty \widetilde{E_{\mathcal{L}}G})$ -modules in $\mathcal{S}_{\text{BH}}^G$. The following theorem is an immediate consequence of the construction.

THEOREM 2.5. *Taking R_G to be a lax symmetric monoidal fibrant replacement functor and the model $\widetilde{E_{\mathcal{L}}G}$ for $\widetilde{E}G$, the functor T_G of Construction 1.3 lifts to a lax symmetric monoidal functor $\mathcal{S}^G \rightarrow \mathcal{S}^G[\mathcal{L}]$ and $J \circ T_G$ has the compatible structure of a lax symmetric monoidal functor $\mathcal{S}^G \rightarrow \mathcal{S}_{\text{BH}}^G$.*

NOTATION 2.6. For any A_∞ or E_∞ operad \mathcal{O} in unbased spaces with $\mathcal{O}(0) = *$, write $T_G^\mathcal{O}$ for the functor T_G of Construction 1.3 using a lax symmetric monoidal fibrant replacement functor and the model $\widetilde{E_{\mathcal{O}}G}$ for $\widetilde{E}G$. Write $JT_G^\mathcal{L}: \mathcal{S}^G \rightarrow \mathcal{S}_{\text{BH}}^G$ for the composite functor $J \circ T_G^\mathcal{L}$ in the previous theorem.

Note that the functor J preserves all weak equivalences and so the functor $JT_G^\mathcal{L}$ induces a lax symmetric monoidal functor from the Borel stable category to the stable category; in these terms, the previous theorem specializes to the following homotopy category statement.

COROLLARY 2.7. *The Tate fixed point functor $(-)^{tG}: \text{Ho}^B(\mathcal{S}^G) \rightarrow \text{Ho}(\mathcal{S})$ has a lax symmetric monoidal structure.*

More generally, if A is a commutative ring orthogonal G -spectrum (or even an \mathcal{L} -algebra in orthogonal G -spectra), the previous theorem specializes to show that Tate fixed point functor restricts to a lax symmetric monoidal functor from the Borel derived category of A -modules to the derived category of $JT_G^\mathcal{L}A$ -modules.

3. The Tate spectral sequences

In this section, we review the construction and properties of certain ‘‘Tate’’ spectral sequences computing π_*^{tG} . We discuss two different spectral sequences, one based on the Greenlees Tate filtration of [11, §1] and another introduced by Hesselholt-Madsen in [15, §4.3]. In the case of finite group G , these spectral sequences agree from E^2 on and in the case when $G = \mathbb{T}$, they agree from E^3 on. The former spectral sequence is aesthetically superior, but we can prove sharper multiplicativity properties for the latter, and we need these sharper properties for the work in Section 5. We begin with the case where G is a finite group as that is conceptually simpler.

For G a finite group and P_* a projective $\mathbb{Z}[G]$ -resolution of \mathbb{Z} in which each P_n is finitely generated, Tate cohomology is constructed by putting together P_* and the dual (contragradient) $\mathbb{Z}[G]$ -complex $Q^* = \text{Hom}(P_*, \mathbb{Z})$. If X is a $\mathbb{Z}[G]$ -module, the Tate cohomology $\hat{H}_G^{-*}(X)$ is the homology of the G -fixed point chain complex of the chain complex

$$(3.1) \quad \cdots \leftarrow Q^n \otimes X \leftarrow \cdots \leftarrow Q^0 \otimes X \leftarrow P_0 \otimes X \leftarrow \cdots \leftarrow P_n \otimes X \leftarrow \cdots$$

using the augmentation $P_0 \rightarrow \mathbb{Z}$ and its dual $\mathbb{Z} \rightarrow Q^0$. (Here $Q^0 \otimes X$ sits in degree 0.) Alternatively, let \tilde{P}_* denote the augmented resolution: $\tilde{P}_0 = \mathbb{Z}$, $\tilde{P}_n = P_{n+1}$ for $n > 0$. The Tate cohomology can also be calculated as the homology of the G -fixed point chain complex of

$$(3.2) \quad \text{Tot}^\oplus(\text{Hom}(P_*, X) \otimes \tilde{P}_*) \cong Q^* \otimes \tilde{P}_* \otimes X$$

(total complex formed with \oplus). The construction (3.1) has a simpler form, particularly when G is a cyclic group and we take P_* to be the minimal resolution; for the construction (3.2), it is easy to construct the cup product in Tate cohomology, using the (unique up to chain homotopy) chain maps $P_* \rightarrow P_* \otimes P_*$ and $\tilde{P}_* \otimes \tilde{P}_* \rightarrow \tilde{P}_*$ consistent with the maps $\mathbb{Z} \rightarrow \mathbb{Z} \otimes \mathbb{Z}$ and $\mathbb{Z} \otimes \mathbb{Z} \rightarrow \mathbb{Z}$, respectively.

The Greenlees Tate filtration and Hesselholt-Madsen Tate filtration do the analogous constructions in topology. For any G -CW model of EG , the cellular chain complex of EG is a $\mathbb{Z}[G]$ -resolution P_* of \mathbb{Z} . For an orthogonal G -spectrum X , we consider the tower of orthogonal G -spectra

$$\begin{aligned} \cdots &\longrightarrow F(EG/EG_n, X) \longrightarrow F(EG/EG_{n-1}, X) \longrightarrow \cdots \\ &\cdots \longrightarrow F(EG/EG_0, X) \longrightarrow F(EG/EG_{-1}, X) = X^{EG}, \end{aligned}$$

where $F(-, X)$ denotes the orthogonal G -spectrum of (non-equivariant) maps and where we understand EG_{-1} as the empty set. When X is fibrant, this is a tower of fibrations whose inverse limit $F(EG/EG, X)$ is trivial. Taking the model of \widetilde{EG} given as the homotopy cofiber of the map $EG_+ \rightarrow S^0$,

$$\widetilde{EG} = (EG \times I) \cup_{(EG \times \partial I)} S^0$$

we get a filtration with \widetilde{EG}_n ($n \geq 0$) the homotopy cofiber of $(EG_{n-1})_+ \rightarrow S^0$ and $\widetilde{EG}_0 = S^0$; the cellular chain complex of this filtration is \tilde{P}_* in the notation above.

We get the Greenlees Tate filtration [11, §1] (cf. [15, 4.3.6]) by continuing the tower above using the filtration on \widetilde{EG} :

$$X^{EG} = X^{EG} \wedge \widetilde{EG}_0 \longrightarrow X^{EG} \wedge \widetilde{EG}_1 \longrightarrow \cdots \longrightarrow X^{EG} \wedge \widetilde{EG}_n \longrightarrow \cdots.$$

In the spectral sequence associated to this \mathbb{Z} -indexed sequence, the E^1 -term is canonically naturally isomorphic to

$$\cdots \leftarrow Q^n \otimes \pi_* X \leftarrow \cdots \leftarrow Q^0 \otimes \pi_* X \leftarrow P_0 \otimes \pi_* X \leftarrow \cdots \leftarrow P_n \otimes \pi_* X \leftarrow \cdots,$$

precisely the complex of (3.1). Because each homotopy cofiber is G -free and has G -free homotopy groups, the homotopy groups of the G -fixed point spectra are the G -fixed points of the homotopy groups. The spectral sequence associated to the induced \mathbb{Z} -indexed sequence on G -fixed points therefore has E^1 -term the complex

$$\cdots \leftarrow (Q^0 \otimes \pi_* X)^G \leftarrow (P_0 \otimes \pi_* X)^G \leftarrow \cdots$$

of G -fixed points, and so has E^2 -term given by $E_{i,j}^2 = \hat{H}_G^{-i}(\pi_j X)$. Because $\text{holim}_n F(EG/EG_{n-1}, X)^G \simeq *$, this spectral sequence is conditionally convergent (in the sense of [7, 5.10]) to the colimit $\pi_*^{tG} X$.

DEFINITION 3.3. Let G be a finite group and let X be an orthogonal G -spectrum. The *Greenlees Tate filtration* on X^{tG} is the \mathbb{Z} -indexed sequence

$$\cdots \longrightarrow X_{-1}^{tG} \longrightarrow X_0^{tG} \longrightarrow X_1^{tG} \longrightarrow \cdots,$$

where X_n^{tG} is the composite of derived functors

$$X_n^{tG} = \begin{cases} F(EG/EG_{-n-1}, X)^G & \text{if } n < 0 \\ (X^{EG} \wedge \widetilde{EG}_n)^G & \text{if } n \geq 0 \end{cases}$$

and the maps are induced by the maps $EG/EG_{n-1} \rightarrow EG/EG_n$ and $\widetilde{EG}_n \rightarrow \widetilde{EG}_{n+1}$. The *Greenlees Tate spectral* sequence is the associated conditionally convergent spectral sequence

$$E_{i,j}^2 = \hat{H}_G^{-i}(\pi_j X) \Longrightarrow \pi_{i+j}^{tG}(X).$$

The Hesselholt-Madsen Tate filtration [15, §4.3] follows the pattern of (3.2). Because we use a version of this filtration for the arguments in the remainder of the paper, it is convenient to set up a point-set model for it rather than just a construction in the stable category. For an orthogonal G -spectrum X , using the chosen lax symmetric monoidal fibrant replacement functor R_G , let

$$(3.4) \quad T_G X_{i,j} = (R_G(F(EG/EG_{j-1}, R_G X) \wedge \widetilde{EG}_i))^G$$

for any $i, j \geq 0$. If the filtration on EG and \widetilde{EG} comes from a G -CW structure as above, then $T_G X_{0,j}$ is a point-set model for X_{-j}^{tG} and $T_G X_{i,0}$ is a point-set model for X_i^{tG} in the Greenlees Tate filtration. We have canonical maps $T_G X_{i,j} \rightarrow T_G X_{i',j'}$ for any $i \leq i'$, $j \geq j'$, making $T_G X_{-, -}$ a functor from $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ to orthogonal G -spectra. Using the functor minus, $- : (\mathbb{N}, \leq) \times (\mathbb{N}, \geq) \rightarrow (\mathbb{Z}, \leq)$, for any $n \in \mathbb{Z}$, define the point-set functor

$$(3.5) \quad \bar{T}_n X = \text{hocolim}_{i-j \leq n} T_G X_{i,j}$$

from (\mathbb{Z}, \leq) to orthogonal spectra using the categorical bar construction as the point-set model for hocolim. Then for $n < n'$, we have an induced map $\bar{T}_n X \rightarrow \bar{T}_{n'} X$. The consistent system of maps $T_G X_{i,j} \rightarrow T_G X$ induce a consistent system of maps $\bar{T}_n X \rightarrow T_G X$ and a weak equivalence $\text{hocolim} \bar{T}_n X \rightarrow T_G X$. Hesselholt-Madsen [15, 4.3.4] proves that the E^1 -term of the spectral sequence associated to this \mathbb{Z} -indexed sequence is canonically naturally isomorphic as a chain complex to

$$(\text{Tot}^\oplus(\text{Hom}(P_*, \pi_* X) \otimes \tilde{P}_*))^G$$

(specifically, $E_{i,j}^1$ is the degree i part of the total complex for $\pi_j X$), and

$$E_{i,j}^2 \cong \hat{H}_G^{-i}(\pi_j X).$$

Moreover, Hesselholt-Madsen [15, 4.3.6] constructs a zigzag of maps consistent with the abutment to $\pi_*^{tG} X$ and inducing an isomorphism on E^2 between this spectral sequence and the Greenlees Tate spectral sequence, so this spectral sequence also conditionally converges to the colimit $\pi_*^{tG} X$.

DEFINITION 3.6. Let G be a finite group and let X be an orthogonal G -spectrum. The *Hesselholt-Madsen Tate filtration* on X^{tG} is the \mathbb{Z} -indexed sequence

$$\cdots \longrightarrow \bar{T}_{-1}X \longrightarrow \bar{T}_0X \longrightarrow \bar{T}_1X \longrightarrow \cdots,$$

constructed in the previous paragraph. The *Hesselholt-Madsen Tate spectral sequence* is the associated conditionally convergent spectral sequence whose E^1 -term is

$$E_{i,j}^1 = (\text{Tot}^\oplus(\text{Hom}(P_*, \pi_j X) \otimes \tilde{P}_*))_i^G$$

(for P_* the cellular chain complex of EG , a locally finite free $\mathbb{Z}[G]$ -resolution of \mathbb{Z}); it is isomorphic to the Greenlees Tate spectral sequence from E^2 onward.

As constructed, the Hesselholt-Madsen Tate filtration is a filtration in the traditional sense of $\text{colim}_n \bar{T}_n X \xrightarrow{\sim} X^{tG}$. Recall that a map $f: A \rightarrow B$ in any topologically enriched category is called an *h-cofibration* when it satisfies the *homotopy extension property*: Given a map $g: B \rightarrow C$, a homotopy of $g \circ f: A \rightarrow C$ may be extended to a homotopy of $g: B \rightarrow C$. For orthogonal spectra, this is equivalent to the map

$$B \cup_{A \wedge \{0\}_+} A \wedge I_+ \longrightarrow B \wedge I_+$$

admitting a retraction, and so *h-cofibrations* are preserved by many point-set constructions including smash product and gluing. They also have important homotopy colimit properties including:

- If $A \rightarrow B$ is an *h-cofibration* of orthogonal spectra and $A \rightarrow B$ is any map of orthogonal spectra, then the pushout $B \cup_A C$ represents the homotopy pushout (left derived functor of pushout). In particular, the quotient B/A represents the left derived quotient, the homotopy cofiber in the stable category.
- If $A_0 \rightarrow A_1 \rightarrow \cdots$ is a system of *h-cofibrations* of orthogonal spectra, then the colimit represents the homotopy colimit.

It is well-known and straightforward to prove that the categorical bar construction model of the homotopy colimit has the property that the inclusion of a subcategory induces an *h-cofibration* on homotopy colimits. In the particular case of the Hesselholt-Madsen Tate filtration, this is the following observation.

PROPOSITION 3.7. *The maps $\bar{T}_n X \rightarrow \bar{T}_{n+1} X$ in the point-set model of the Hesselholt-Madsen Tate filtration above are *h-cofibrations*.*

Technically, both the filtrations and the E^1 -terms of the spectral sequences in Definitions 3.3 and 3.6 depend on the choice of G -CW structure on EG . While it is tempting to use one that gives the minimal resolution, we will use the G -CW structure on EG coming from the standard two-sided bar construction model: We use the model of EG constructed as the geometric realization of the simplicial G -space

$$B_n(G, G, *) = G \times (G^n) \times *$$

with the usual face and degeneracy maps (induced by multiplication/projection and by inclusion of the unit, respectively) and the G -action on the lefthand factor. The geometric realization filtration is evidently the cellular filtration of a G -CW structure. The following two lemmas are well-known but in particular follow from the more delicate study of the structure we perform in Sections 9 and 10.

LEMMA 3.8. *For the standard bar construction model of EG , the diagonal map $EG \rightarrow EG \times EG$ admits an equivariant cellular approximation that is homotopy coassociative and homotopy counital through equivariant cellular maps.*

LEMMA 3.9. *For the standard bar construction model of EG , the space \widetilde{EG} admits a pairing $\widetilde{EG} \wedge \widetilde{EG} \rightarrow \widetilde{EG}$ that is equivariant and filtered and is homotopy unital and associative through equivariant filtered maps.*

Since the space of equivariant maps $\widetilde{EG} \wedge \widetilde{EG} \rightarrow \widetilde{EG}$ is homotopy discrete with components corresponding to the trivial map and a weak equivalence, the multiplication on \widetilde{EG} in Lemma 3.9 is compatible with the multiplication constructed on the models in Section 2. In particular, the maps in the previous two lemmas induce the same pairing in the stable category $T_G(X) \wedge^L T_G(Y) \rightarrow T_G(X \wedge^L Y)$ as Corollary 2.7.

We can use the preceding two lemmas to give the Hesselholt-Madsen Tate spectral sequence a natural lax monoidal structure starting at the E^1 -term. The equivariant cellular model of the diagonal map $EG \rightarrow EG \times EG$ induces equivariant maps

$$EG_{n-1} \longrightarrow (EG_{i-1} \times EG) \cup (EG \times EG_{j-1}) \subset EG \times EG$$

for $i + j = n$, which in turn induce equivariant maps

$$\begin{aligned} EG/EG_{i+j-1} &\longrightarrow (EG \times EG)/((EG_{i-1} \times EG) \cup (EG \times EG_{j-1})) \\ &\cong (EG/EG_{i-1}) \wedge (EG/EG_{j-1}) \end{aligned}$$

and equivariant maps

$$\begin{aligned} F(EG/EG_{i-1}, R_G X) \wedge F(EG/EG_{j-1}, R_G Y) \\ \longrightarrow F(EG/EG_{i-1} \wedge EG/EG_{j-1}, R_G X \wedge R_G Y) \\ \longrightarrow F(EG/EG_{i+j-1}, R_G(X \wedge Y)). \end{aligned}$$

Likewise the equivariant cellular model for the multiplication $\widetilde{EG} \wedge \widetilde{EG} \rightarrow \widetilde{EG}$ induces

$$(X \wedge \widetilde{EG}_i) \wedge (Y \wedge \widetilde{EG}_j) \longrightarrow (X \wedge Y) \wedge \widetilde{EG}_{i+j}.$$

Returning to T_G , we then obtain natural maps

$$T_G X_{i,j} \wedge T_G Y_{i',j'} \longrightarrow T_G(X \wedge Y)_{i+i',j+j'}.$$

These are functorial in i, i', j, j' and define maps on the Hesselholt-Madsen Tate filtration

$$\bar{T}X_m \wedge \bar{T}Y_n \longrightarrow \bar{T}(X \wedge Y)_{m+n},$$

respecting the $(\mathbb{N}, \leq) \times (\mathbb{N}, \leq)$ -structure on both sides. This in turn induces a pairing

$$(3.10) \quad E_{*,*}^1(X) \otimes E_{*,*}^1(Y) \longrightarrow E_{*,*}^1(X \wedge^L Y)$$

converging to the standard pairing on π_*^{tG} . The homotopy coassociativity of EG and homotopy associativity of \widetilde{EG} then give us filtered homotopies between the two maps

$$\bar{T}W_\ell \wedge \bar{T}X_m \wedge \bar{T}Y_n \longrightarrow \bar{T}(W \wedge X \wedge Y)_{\ell+m+n}$$

and so both associations induce the same map

$$E_{*,*}^1(W) \otimes E_{*,*}^1(X) \otimes E_{*,*}^1(Y) \longrightarrow E_{*,*}^1(W \wedge^L X \wedge^L Y).$$

Similar observations apply to the unit using the canonical map $\mathbb{S} \rightarrow T_G \mathbb{S}_{0,0}$. Hesselholt-Madsen [15, 4.3.5] shows that such a pairing induces the usual pairing on Tate cohomology on the E^2 -term.

We can describe this pairing algebraically as follows. Lemma 3.8 endows the resolution P_* with an equivariant coassociative and counital comultiplication $P_* \rightarrow P_* \otimes P_*$ and Lemma 3.9 endows \tilde{P} with an equivariant differential graded algebra structure. In fact, for our construction in Section 10, the coalgebra structure on P_* is induced by the Alexander-Whitney map. The resulting differential graded algebra plays a central role in our formulas, and we use the following notation.

NOTATION 3.11. Let HM_* be the differential graded algebra

$$(\text{Tot}^\oplus(P_*, \mathbb{Z}) \otimes \tilde{P})^G,$$

where P_* denotes the cellular chain complex of the standard bar construction model of EG and where the multiplication is induced by Lemmas 3.8 and 3.9 and the unit by the augmentation $P_0 \rightarrow \mathbb{Z}$ and isomorphism $\mathbb{Z} \rightarrow \tilde{P}_0$. Let $HM_{*,*}$ be the bigraded ring which is HM_* concentrated in degree 0 in the second grading (the *internal* grading).

The pairing on E^1 -terms (3.10) makes the E^1 -term for \mathbb{S} into a differential graded algebra, which acts on both sides on the E^1 -term for any orthogonal G -spectrum. By the formula for the E^1 -term in Definition 3.6, we see that as a bigraded ring, the E^1 -term for \mathbb{S} is $HM_* \otimes \pi_* \mathbb{S}$. In particular, the E^1 -term for every orthogonal G -spectrum is naturally a bimodule over $HM_{*,*}$.

Using the fact that $P_0 = \mathbb{Z}[G]$ and $\tilde{P}_0 = \mathbb{Z}$, we get a natural map

$$\pi_* X \longrightarrow (\text{Hom}(P_0, \pi_* X) \otimes \tilde{P}_0)^G \longrightarrow E_{0,*}^1(X)$$

where $x \in \pi_j X$ goes to the unique equivariant homomorphism $P_0 \rightarrow \pi_j X$ sending 1 to x . It is easy to see that this is a monoidal natural transformation. The following proposition is then clear from inspection of the multiplication.

PROPOSITION 3.12. *For any orthogonal G -spectrum X , the E^1 -term of the Hesselholt-Madsen Tate spectral sequence is a bimodule over $HM_{*,*}$. The map of left modules from $HM_{*,*} \otimes \pi_* X$ to the E^1 -term for X is a map of $HM_{*,*}$ -bimodules and an isomorphism.*

Regarding $HM_{*,*} \otimes \pi_*(-)$ as a functor from G -spectra to $HM_{*,*}$ -bimodules, the unit of $HM_{*,*} \otimes \pi_* \mathbb{S}$ and the pairing

$$(HM_{*,*} \otimes \pi_*(X)) \otimes (HM_{*,*} \otimes \pi_*(Y)) \longrightarrow HM_{*,*} \otimes \pi_*(X \wedge Y)$$

induced by the multiplication of $HM_{*,*}$ and the usual pairing on π_* makes $HM_{*,*} \otimes \pi_*(-)$ into a lax monoidal functor. The isomorphism of the previous proposition is then a monoidal natural transformation from this functor to the E^1 -term of the Hesselholt-Madsen spectral sequence.

We now turn to the case of the circle group \mathbb{T} . We write $\mathbb{C}(1)$ for \mathbb{C} with the standard action of \mathbb{T} as the group of unit complex numbers, $S(\mathbb{C}(1)^n)$ for the unit sphere in $\mathbb{C}(1)^n$, and $S^{\mathbb{C}(1)^n}$ for the one-point compactification of $\mathbb{C}(1)^n$. When working in the \mathbb{T} -equivariant stable category, we write $S^{n\mathbb{C}(1)}$ for the suspension spectrum of $S^{\mathbb{C}(1)^n}$ for $n \in \mathbb{N}$, and we extend this notation to representation spheres $S^{n\mathbb{C}(1)}$ for all $n \in \mathbb{Z}$. We have the standard bar construction model for $E\mathbb{T}$, which comes with a filtration from geometric realization, but to better match the

numbering in the finite group case, we define $E\mathbb{T}_{2n+1} = E\mathbb{T}_{2n}$ and let $E\mathbb{T}_{2n}$ be the geometric realization n -skeleton; we use the corresponding numbering for the filtration on $\widetilde{E\mathbb{T}}$. In this numbering, there are well-known \mathbb{T} -equivariant homotopy equivalences

$$E\mathbb{T}_{2n} \simeq S(\mathbb{C}(1)^n), \quad \widetilde{E\mathbb{T}}_{2n} \simeq S^{\mathbb{C}(1)^n}$$

as well as \mathbb{T} -equivariant homotopy equivalences

$$E\mathbb{T}/E\mathbb{T}_{2n-1} = E\mathbb{T}/E\mathbb{T}_{2n-2} \simeq E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}.$$

These \mathbb{T} -equivariant homotopy equivalences give us natural isomorphisms in the \mathbb{T} -equivariant stable category

$$X^{E\mathbb{T}/E\mathbb{T}_{2n-1}} \simeq X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{-n\mathbb{C}(1)} \quad \text{and} \quad X^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}}_{2n} \simeq X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{n\mathbb{C}(1)}.$$

The *Greenlees Tate filtration* is then

$$(3.13) \quad \cdots \longrightarrow (X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{-n\mathbb{C}(1)})^{\mathbb{T}} \longrightarrow \cdots \longrightarrow (X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{-\mathbb{C}(1)})^{\mathbb{T}} \longrightarrow (X^{E\mathbb{T}})^{\mathbb{T}} \\ \longrightarrow (X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{\mathbb{C}(1)})^{\mathbb{T}} \longrightarrow \cdots \longrightarrow (X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{n\mathbb{C}(1)})^{\mathbb{T}} \longrightarrow \cdots$$

(even indices only displayed, odd indices equal to previous even index, with $(X^{E\mathbb{T}})^{\mathbb{T}}$ in index 0). Using the identification of the quotient $S^{\mathbb{C}(1)}/S^0$ as $\mathbb{T}_+ \wedge S^1$, the associated graded spectra are

$$(\mathbb{T}_+ \wedge X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{(n-1)\mathbb{C}(1)} \wedge S^1)^{\mathbb{T}} \simeq (\mathbb{T}_+ \wedge \Sigma^{2n-1} X)^{\mathbb{T}} \simeq \Sigma^{2n} X,$$

where the last weak equivalence is the Adams isomorphism. The *Greenlees Tate spectral sequence* is the spectral sequence associated to the \mathbb{Z} -indexed Greenlees Tate filtration (3.13). It has

$$(3.14) \quad E_{i,j}^1 = E_{i,j}^2 = \pi_j X \implies \pi_{i+j}^{t\mathbb{T}} X,$$

conditionally converging to the colimit $\pi_{i+j}^{t\mathbb{T}} X$. (In general $E^{2r-1} = E^{2r}$ for all r since the filtration is concentrated in even indices.) Moreover, because the associative and commutative pairing (in the equivariant stable homotopy category)

$$S^{m\mathbb{C}(1)} \wedge^{\mathbf{L}} S^{n\mathbb{C}(1)} \xrightarrow{\sim} S^{(m+n)\mathbb{C}(1)}$$

is compatible with the maps in the Greenlees Tate filtration, the Greenlees Tate spectral sequence has a natural associative and unital pairing on E^1 compatible with the pairing on $\pi_*^{t\mathbb{T}}$. In terms of the description of the E^1 -term in (3.14), the pairing on E^1 is induced by $\pi_* X \otimes \pi_* Y \rightarrow \pi_*(X \wedge^{\mathbf{L}} Y)$.

Although the Greenlees Tate filtration and Greenlees Tate spectral sequence are ideal for many purposes, we have not succeeded in making the pairing coherent enough for the argument in Section 5. For that argument we work with the Hesselholt-Madsen filtration, which does not have as a clean a spectral sequence for $G = \mathbb{T}$; in particular it does not agree at E^2 with the Greenlees Tate spectral sequence (see 3.19–3.20 below). We construct a point-set model for the Hesselholt-Madsen filtration exactly as in Definition 3.6, except using the doubled indexing for the filtration on $E\mathbb{T}$ and $\widetilde{E\mathbb{T}}$ as in the preceding paragraph.

DEFINITION 3.15. For $G = \mathbb{T}$, the *Hesselholt-Madsen Tate filtration* is the filtration by h -cofibrations

$$\cdots \longrightarrow \bar{T}_{-1} X \longrightarrow \bar{T}_0 X \longrightarrow \bar{T}_1 X \longrightarrow \cdots$$

where $\bar{T}_n X = \text{hocolim}_{i-j \leq n} T_{\mathbb{T}} X_{i,j}$ for

$$T_{\mathbb{T}} X_{i,j} = (R_{\mathbb{T}}(F(E\mathbb{T}/E\mathbb{T}_{j-1}, R_{\mathbb{T}} X) \wedge \widetilde{E\mathbb{T}_i}))^{\mathbb{T}}.$$

The *Hesselholt-Madsen Tate spectral sequence* is the spectral sequence associated to this filtration.

In the finite group case, conditional convergence of the Hesselholt-Madsen spectral sequence followed from conditional convergence of the Greenlees Tate spectral sequence, where it was clear from construction. For the case $G = \mathbb{T}$, we prove it separately. As the details are not needed in the main argument, we defer the proof of the following lemma to Section 12.

LEMMA 3.16. *For $G = \mathbb{T}$, the Hesselholt-Madsen Tate spectral sequence converges conditionally to the colimit $\pi_*^{t\mathbb{T}} X$.*

Lemmas 3.8 and 3.9 hold for $G = \mathbb{T}$ using these filtrations. We therefore again obtain a monoidal structure on the spectral sequence as in (3.10). The E^1 -term for \mathbb{S} is a bigraded ring that acts on both sides on the E^1 -term for an arbitrary X . We make the following computation in Section 12, where we specify the generators precisely.

THEOREM 3.17. *For $G = \mathbb{T}$, the E^1 -term for the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence for $X = \mathbb{S}$ is as a bigraded ring the free graded commutative $\pi_* \mathbb{S}$ -algebra on a generator x in bidegree $(2, 0)$, a generator y in bidegree $(2, -1)$, and a generator z in bidegree $(-2, 0)$ subject to the relation $y^2 = 0$.*

In the previous statement, for the $\pi_* \mathbb{S}$ action, we regard elements of $\pi_* \mathbb{S}$ as concentrated in degree 0 for the first grading, i.e., elements of $\pi_n \mathbb{S}$ are in bidegree $(0, n)$. In analogy with Notation 3.11, we use the following notation in the case $G = \mathbb{T}$, so that the E^1 -term of the Hesselholt-Madsen Tate spectral sequence for \mathbb{S} is isomorphic as a bigraded ring to $HM_{*,*} \otimes \pi_* \mathbb{S}$.

NOTATION 3.18. Let $HM_{*,*} = \mathbb{Z}[x, y, z]/y^2$ where x is in bidegree $(2, 0)$ and y is in bidegree $(2, -1)$, and z is in bidegree $(-2, 0)$.

For an arbitrary orthogonal \mathbb{T} -spectrum X , using the canonical isomorphism

$$HM_{*,*} \otimes \pi_* X \cong (HM_{*,*} \otimes \pi_* \mathbb{S}) \otimes_{\pi_* \mathbb{S}} \pi_* X,$$

the E^1 -term of the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence for X naturally becomes a bimodule over $HM_{*,*}$. We have a canonical map $\pi_* X \rightarrow E_{0,*}^1$ induced by the identification of

$$i^* X \simeq (R_{\mathbb{T}} F(E\mathbb{T}_0, R_{\mathbb{T}} X))^{\mathbb{T}}$$

as the homotopy cofiber of $T_{\mathbb{T}} X_{0,1} \rightarrow T_{\mathbb{T}} X_{0,0}$, where i^* denotes the underlying non-equivariant spectrum. We then get an induced a map of left $HM_{*,*}$ -modules from $HM_{*,*} \otimes \pi_* X$ to the E^1 -term for X . We prove in Section 12 the following theorem.

THEOREM 3.19. *The map from $HM_{*,*} \otimes \pi_* X$ to the E^1 -term of the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence for X is an isomorphism of $HM_{*,*}$ -bimodules and a monoidal transformation.*

Non-multiplicatively, the previous theorem identifies the E^1 -term of the Hesselholt-Madsen Tate spectral sequence for an orthogonal \mathbb{T} -spectrum X as having $E_{2i,j}^1$ a countable direct sum of copies of $\pi_j X \oplus \pi_{j+1} X$ and $E_{2i-1,j}^1$ zero.

Although the work above is all we need for the main results of the paper, we can say more about the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence and its relationship to the Greenlees \mathbb{T} -Tate spectral sequence. We prove the following theorems in Section 13.

THEOREM 3.20. *In the notation of Definition 3.15, the inclusion of $T_{\mathbb{T}}X_{0,j}$ in $\bar{T}_{-j}X$ and $T_{\mathbb{T}}X_{i,0}$ in \bar{T}_iX induce a (non-monoidal) filtered map from the Greenlees filtration to the Hesselholt-Madsen filtration for $G = \mathbb{T}$. The induced map on spectral sequences is split injective on $E^1 = E^2$ and an isomorphism from E^3 onward.*

THEOREM 3.21. *In the Hesselholt-Madsen Tate spectral sequence for $G = \mathbb{T}$, the d_1 differential is 0. The d_2 differential satisfies the Leibniz rule, is given on generators of $HM_{*,*} \otimes \pi_*(\mathbb{S})$ by*

$$\begin{aligned} d_2(x \otimes 1) &= 1 \otimes \eta \\ d_2(y \otimes 1) &= xz \otimes 1 + yz \otimes \eta - 1 \otimes 1 \\ d_2(z \otimes 1) &= z^2 \otimes \eta \end{aligned}$$

and is given on elements v of π_*X by $d_2(1 \otimes v) = -z \otimes \zeta v$, where ζ denotes the action of the fundamental class of $\pi_1\mathbb{T}$ on π_*X .

The $E^1 = E^2$ term of the Greenlees \mathbb{T} -Tate spectral sequence for X is (multiplicatively) isomorphic to $\mathbb{Z}[t, t^{-1}] \otimes \pi_*X$, where t is an element of $E^1_{-2,0}$ for $X = \mathbb{S}$. The element t has $d_2(t \otimes 1) = t^2 \otimes \eta$ and is usually chosen so that for $v \in \pi_*X$, $d_2(1 \otimes v) = t \otimes \zeta v$. With this choice of t (which is now uniquely specified), the map in Theorem 3.20 then takes $t^n \otimes v$ to

$$\begin{cases} (-z)^n & n \geq 0 \\ (-x)^{|n|} \otimes v - (-x)^{|n|-1}y \otimes \zeta v & n < 0. \end{cases}$$

From Theorem 3.21, we see that the ring map $HM_{*,*} \rightarrow \mathbb{Z}[t, t^{-1}]$ sending x to $-t^{-1}$, y to 0, and z to $-t$ induces a multiplicative map of spectral sequences from the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence to the Greenlees \mathbb{T} -Tate spectral sequence that splits the (non-multiplicative) map in Theorem 3.20.

4. Topological periodic cyclic homology

Having reviewed the definition and basic properties of the Tate fixed point functor in the previous three sections, we now review the definition and basic properties of topological periodic cyclic homology (TP). We begin with a very rapid review of relevant aspects of the theory of THH of spectral categories and $\mathcal{M}od_R$ -categories for R a commutative ring orthogonal spectrum. We define TP of spectral categories and, as discussed below, we rely on the equivalence of k -linear dg categories and $\mathcal{M}od_{Hk}$ -categories to define TP of dg categories.

Let $\mathcal{C}at^{\mathcal{S}}$ denote the category of small spectral categories and spectral functors: An object of $\mathcal{C}at^{\mathcal{S}}$ is a small category enriched in orthogonal spectra and a morphism in $\mathcal{C}at^{\mathcal{S}}$ is an enriched functor. As explained in [6, §3], the topological Hochschild-Mitchell complex N^{cy} yields a functor from orthogonal spectra to orthogonal \mathbb{T} -spectra. Since we are working with Borel equivalences, the left derived functor provides an adequate model for THH . (In other contexts, the construction is more subtle; see [1, §5].)

To make sense of the left derived functor THH , we say that a small spectral category is *pointwise cofibrant* if each mapping spectrum $\mathcal{C}(x, y)$ is a cofibrant orthogonal spectrum (in the standard model structure). A spectral functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *pointwise equivalence* if the induced function on object sets is the identity and each map of spectra $\mathcal{C}(x, y) \rightarrow \mathcal{D}(x, y)$ is a weak equivalence. The argument of [6, 2.7] shows that there is an endofunctor Q of $\text{Cat}^{\mathcal{S}}$ that lands in pointwise cofibrant small spectral categories and a natural transformation $Q \rightarrow \text{Id}$ through pointwise equivalences. The topological Hochschild-Mitchell complex N^{cy} composed with Q then takes pointwise equivalences of spectral categories to Borel equivalences of orthogonal \mathbb{T} -spectra.

Although pointwise cofibrant replacement is technically convenient, we are typically interested in less rigid notions of equivalence on $\text{Cat}^{\mathcal{S}}$. We say that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *Dwyer-Kan equivalence* (or *DK-equivalence*) if the induced functor $\pi_0(\mathcal{C}) \rightarrow \pi_0(\mathcal{D})$ is essentially surjective and each map of orthogonal spectra $\mathcal{C}(x, y) \rightarrow \mathcal{D}(Fx, Fy)$ is a weak equivalence [6, 5.1]. We say that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a Morita equivalence if the induced functor on “triangulated closures” is a DK-equivalence (see [6, §5]). Here the triangulated closure of a small spectral category \mathcal{C} is a spectral category \mathcal{C}' such that $\text{Ho}(\mathcal{C}')$ is the pre-triangulated closure of $\text{Ho}(\mathcal{C})$ [6, 5.5].

The composite $N^{cy} \circ Q$ sends Morita equivalences of spectral categories to Borel equivalences of orthogonal \mathbb{T} -spectra [6, 5.12]. We define topological Hochschild homology (THH) to be the resulting left derived functor

$$THH: \text{Ho}^M(\text{Cat}^{\mathcal{S}}) \longrightarrow \text{Ho}^B(\mathcal{S}^{\mathbb{T}}),$$

where $\text{Ho}^M(\text{Cat}^{\mathcal{S}})$ denotes the homotopy category of spectral categories obtained by formally inverting the Morita equivalences. The work of Shipley [29, 4.2.8–9] and Patchkoria-Sagave [27, 3.8] (cf. [6, 3.5]) shows that $THH(\mathcal{C})$ coincides with the classical Bökstedt construction of topological Hochschild homology as functors to $\text{Ho}^B(\mathcal{S}^{\mathbb{T}})$.

DEFINITION 4.1. *Topological periodic cyclic homology*

$$TP: \text{Ho}^M(\text{Cat}^{\mathcal{S}}) \longrightarrow \text{Ho}(\mathcal{S})$$

is the composite derived functor $TP(\mathcal{C}) = THH(\mathcal{C})^{t\mathbb{T}}$.

The smash product of spectral categories endows $\text{Cat}^{\mathcal{S}}$ with a symmetric monoidal structure. This smash product can be left derived using the pointwise cofibrant replacement functor Q [3, 4.1]. The standard Milnor product argument makes N^{cy} a strong symmetric monoidal functor, which passes to the homotopy category (inverting Morita equivalences) to make THH a strong symmetric monoidal functor [3, 6.8, 6.10]. As an immediate consequence of this discussion and Corollary 2.7, we have:

PROPOSITION 4.2. *TP has the canonical structure of a lax symmetric monoidal functor $\text{Ho}^M(\text{Cat}^{\mathcal{S}}) \rightarrow \text{Ho}(\mathcal{S})$.*

For R a commutative ring orthogonal spectrum, the definition of $\mathcal{M}\text{od}_R$ -category is completely analogous to the definition of spectral category, using the category of R -modules in place of the category of orthogonal spectra. We denote the category of $\mathcal{M}\text{od}_R$ -categories as $\text{Cat}_R^{\mathcal{S}}$ and the homotopy category obtained by formally inverting the Morita equivalences as $\text{Ho}^M(\text{Cat}_R^{\mathcal{S}})$. When R is cofibrant

as a commutative ring orthogonal spectrum (which we can assume without loss of generality), N^{cy} preserves Morita equivalences between pointwise cofibrant $\mathcal{M}od_R$ -categories (where we understand cofibrant in the sense of the standard model structure on the category of R -modules). We then have the following proposition for TP of $\mathcal{M}od_R$ -categories, using the model $JT_{\mathbb{T}}^{\mathcal{L}} N^{cy}(R)$ (in Notation 2.6; see also Theorem 2.5).

PROPOSITION 4.3. *Let R be a cofibrant commutative orthogonal ring spectrum. Then TP has the canonical structure of a lax symmetric monoidal functor $\text{Ho}^M(\mathcal{C}at_R^{\mathcal{S}}) \rightarrow \text{Ho}(\mathcal{M}od_{TP(R)})$.*

Next, let $\mathcal{C}at_k^{\text{dg}}$ denote the category of small k -linear dg categories for a commutative ring k : An object consists of a small category enriched in chain complexes of k -modules and a morphism is an enriched functor. Choosing a cofibrant commutative ring orthogonal spectrum Hk representing the Eilenberg-Mac Lane ring spectrum, we have a symmetric monoidal equivalence of homotopy categories $\text{Ho}(\mathcal{C}at_k^{\text{dg}}) \simeq \text{Ho}(\mathcal{C}at_{Hk}^{\mathcal{S}})$ and $\text{Ho}^M(\mathcal{C}at_k^{\text{dg}}) \simeq \text{Ho}^M(\mathcal{C}at_{Hk}^{\mathcal{S}})$, q.v. [6, §2]. Using this equivalence, we obtain TP as a lax symmetric monoidal functor

$$\text{Ho}^M(\mathcal{C}at_k^{\text{dg}}) \longrightarrow \text{Ho}(\mathcal{M}od_{TP(k)})$$

where we write $TP(k)$ for $JT_{\mathbb{T}}^{\mathcal{L}}(N^{cy}(Hk))$.

5. A filtration argument (Proof of Theorem A)

The purpose of this section is to give an outline of the proof of the main theorem, assuming the existence of a point-set model of TP with a filtration that satisfies certain properties. Since construction of a model satisfying these properties is technically involved, it is useful to abstract out the key principles of the argument here, leaving the construction and verification of properties to Sections 9–11. The same basic outline also proves an analogous theorem for C_p -Tate of THH in the same context (k -linear dg categories for k a perfect field of characteristic $p > 0$). Given the properties of the point-set model described below, we prove the main theorem using a comparison of spectral sequences argument.

For the purposes of this section, let G be a finite group or the circle group \mathbb{T} of unit complex numbers, and we consider a point-set functor

$$T^M: \mathcal{S}^G \setminus \mathbb{S} \longrightarrow \mathcal{S} \setminus \mathbb{S}$$

where $\mathcal{S}^G \setminus \mathbb{S}$ denotes the category of orthogonal G -spectra with a structure map from \mathbb{S} and likewise $\mathcal{S} \setminus \mathbb{S}$ denotes the category of orthogonal spectra with a structure map from \mathbb{S} . We intend to apply this functor to the THH of Hk -categories, where any choice of object of \mathcal{X} induces a map of orthogonal \mathbb{T} -spectra $\mathbb{S} \rightarrow N^{cy}(\mathcal{X})$. We ask for T^M to come with the following additional structure:

- (i) T^M is a lax monoidal functor.
- (ii) T^M comes with a natural filtration by h -cofibrations

$$\cdots \longrightarrow T_1^M \longrightarrow T_0^M \longrightarrow T_1^M \longrightarrow \cdots$$

with $\text{holim } T_{-n}^M \simeq *$ and $\text{colim } T_n^M = T^M$.

- (iii) The filtration is compatible with the monoidal structure in the sense that the structure map $\mathbb{S} \rightarrow T^M$ has a natural lift to T_0^M and we have natural maps

$$T_m^M(X) \wedge T_n^M(Y) \longrightarrow T_{m+n}^M(X \wedge Y)$$

for all $m, n \in \mathbb{Z}$, which induce in the colimit the lax monoidal transformation $T^M(X) \wedge T^M(Y) \rightarrow T^M(X \wedge Y)$.

Moreover, we require that as a functor to the stable category T^M is naturally isomorphic to $(-)^{tG}$ by an isomorphism taking the filtration to the Hesselholt-Madsen Tate filtration, Definition 3.15 (case $G = \mathbb{T}$) or 3.6 (case G finite), and we fix such an isomorphism.

The Hesselholt-Madsen Tate filtration comes with a map $\pi_* X$ into $E_{0,*}^1$; it derives from a natural transformation from the underlying non-equivariant orthogonal spectrum $i^* X$ to the 0th filtration quotient. We will need the following additional technical hypothesis about the filtration $T_*^M X$ and this map.

HYPOTHESIS 5.1. The given natural transformation in the stable category $i^* X \rightarrow T_0^M(X)/T_{-1}^M(X)$ comes from a zigzag of point-set monoidal natural transformations.

Specifically, we construct in Section 11 a zigzag of point-set monoidal functors of the form

$$i^* X \xrightarrow{\simeq} RR(X) \xleftarrow{\simeq} TT(X) \longrightarrow T_0^M X / T_{-1}^M X$$

where RR and TT are specific point-set monoidal functors constructed there.

Assuming the existence of the functor T^M with properties above, the remainder of the section discusses and outlines a proof of the following theorem.

THEOREM 5.2. *Let $G = \mathbb{T}$ or C_p for a prime p . Let k be a perfect field of characteristic p , and let X and Y be G -equivariant $N^{\text{cy}}(Hk)$ -modules under $N^{\text{cy}}(Hk)$ with the property that $\pi_*(X)$ and $\pi_*(Y)$ are finitely generated as graded modules over $THH_*(k) \cong \pi_*(N^{\text{cy}}(Hk))$. Then the induced map*

$$T^M(X) \wedge_{T^M(N^{\text{cy}}(Hk))}^{\mathbf{L}} T^M(Y) \longrightarrow T^M(X \wedge_{N^{\text{cy}}(Hk)}^{\mathbf{L}} Y)$$

is a weak equivalence.

This implies Theorem A and its analogue for $G = C_p$ as follows. Let \mathcal{X}' and \mathcal{Y}' be pointwise cofibrant Hk -spectral categories modeling smooth and proper k -linear dg categories \mathcal{X} and \mathcal{Y} , and take X and Y in the statement of Theorem 5.2 to be $N^{\text{cy}}(\mathcal{X}')$ and $N^{\text{cy}}(\mathcal{Y}')$, using any object of \mathcal{X} and \mathcal{Y} to obtain the structure maps $N^{\text{cy}}(Hk) \rightarrow N^{\text{cy}}(\mathcal{X}')$ and $N^{\text{cy}}(Hk) \rightarrow N^{\text{cy}}(\mathcal{Y}')$. Although the map in Theorem A implicitly uses the lax symmetric monoidal structure on T_G constructed in Section 2, we show in Section 18 that the monoidal structure on the model T^M induces the same map in the stable category. Since $THH_*(k) \cong k[t]$ (with t in degree 2) is a Noetherian ring, it follows from Theorem C, which is proved independently in Section 16, that the THH_* of a smooth and proper k -linear dg category is finitely generated over $THH_*(k)$. The Künneth theorem for THH ,

$$N^{\text{cy}}(\mathcal{X}') \wedge_{N^{\text{cy}}(Hk)} N^{\text{cy}}(\mathcal{Y}') \simeq N^{\text{cy}}(\mathcal{X}' \wedge_{Hk} \mathcal{Y}'),$$

q.v. Theorem 15.1, gives us a weak equivalence

$$T^M(X \wedge_{N^{\text{cy}}(Hk)}^{\mathbf{L}} Y) \simeq T^M(N^{\text{cy}}(\mathcal{X}' \wedge_{Hk} \mathcal{Y}')) \simeq TP(\mathcal{X} \otimes \mathcal{Y}).$$

Putting this all together reduces Theorem A to the statement of Theorem 5.2.

We now outline the proof of Theorem 5.2. For convenience, write $A = N^{\text{cy}}(Hk)$. Without loss of generality, we can take X and Y in the statement to be cofibrant in the category of G -equivariant A -modules under A ; then $X \wedge_A Y$ represents the derived smash product $X \wedge_{N^{\text{cy}}(Hk)}^{\mathbf{L}} Y$. The filtration on $T^M(X \wedge_A Y)$ induces a

spectral sequence, the Hesselholt-Madsen Tate spectral sequence for $X \wedge_A Y$. In our work below, we call this the *righthand spectral sequence* as it is the codomain in a map of spectral sequences.

We construct the *lefthand spectral sequence* as follows. As we review in Section 6, the compatibility of the filtration with the lax monoidal structure on T^M allow us to construct $T^M X \wedge_{T^M A} T^M Y$ as a filtered object. Actually, we are more interested in the derived smash product $T^M X \wedge_{T^M A}^L T^M Y$, and we argue in Section 6 that we can model the derived smash product by a filtered object where the comparison map to $T^M X \wedge_{T^M A} T^M Y$ is filtered; we choose such a model and denote it as $T^M X \wedge_{T^M A}^L T^M Y$. The filtration on $T^M X \wedge_{T^M A}^L T^M Y$ then gives the lefthand spectral sequence. The compatibility of the filtration with the lax monoidal structure of T^M implies that the map

$$(5.3) \quad T^M X \wedge_{T^M A}^L T^M Y \longrightarrow T^M(X \wedge_A Y)$$

is filtered, and that gives us a map of spectral sequences from the lefthand spectral sequence to the righthand spectral sequence. The main step in the proof of Theorem 5.2 is the following result, which we prove in Section 7.

THEOREM 5.4. *The map of spectral sequences from the lefthand spectral sequence to the righthand spectral sequence is an isomorphism on E^1 -terms.*

To apply standard spectral sequence comparison techniques, we need the following two results on the convergence of the spectral sequences.

PROPOSITION 5.5. *The righthand spectral sequence is a half-plane spectral sequence with entering differentials, conditionally converging to $\pi_*(T^M(X \wedge_A Y))$.*

PROOF. We have $\pi_* A \cong THH_*(k) \cong k[t]$ (with t in degree 2) is connective, and by hypothesis in Theorem 5.2, $\pi_* X$ and $\pi_* Y$ are finitely generated over $\pi_* A$; they are therefore bounded below. It follows that $X \wedge_A Y$ is bounded below and so the explicit formula for the E^1 -term in Theorem 3.19 (in the case $G = \mathbb{T}$) or Proposition 3.12 (in the case $G = C_p$) shows that $E_{i,*}^1$ is bounded below with bound independent of i . Conditional convergence is Lemma 3.16 in the case $G = \mathbb{T}$ and [15, 4.3.6] in the case $G = C_p$. \square

In Section 8, we prove the following lemma.

LEMMA 5.6. *The lefthand spectral sequence is conditionally convergent, converging to $\pi_*(T^M X \wedge_{T^M A}^L T^M Y)$.*

Theorem 5.4, Proposition 5.5, and Lemma 5.6 together imply that the map (5.3) is a weak equivalence (by [7, 7.2]), which proves Theorem 5.2.

6. Filtered modules over filtered ring orthogonal spectra

This section begins the argument for Theorem 5.4 and Lemma 5.6. Theorem 5.4 compares a spectral sequence constructed from a smash product of Tate filtrations to the spectral sequence of the Tate filtration of the smash product. To develop the tools to do this, we study the homotopy theory of filtered modules over filtered ring orthogonal spectra and the derived smash product. We use the following terminology in this section and the next two.

DEFINITION 6.1. A *filtered spectrum* X_* consists of a sequence of maps of orthogonal spectra

$$\cdots \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots,$$

or equivalently a functor from the poset (\mathbb{Z}, \leq) to the category of orthogonal spectra. The category of filtered spectra has objects the filtered spectra and maps the natural transformations. We make filtered spectra a symmetric monoidal category using the Day convolution [24, §21] for the symmetric monoidal product $+$ on (\mathbb{Z}, \leq) . Let \mathbb{S}_* denote the filtered spectrum satisfying $\mathbb{S}_n = *$ for $n < 0$ and $\mathbb{S}_n = \mathbb{S}$ and maps the identity maps for $n \geq 0$.

The Day convolution formulation of the smash product is a shortcut to produce the strong symmetric monoidal structure on the category, but in this case the construction is easy to describe. Given filtered spectra X_* and Y_* , the smash product $Z_* = X_* \wedge Y_*$ is given by the formula

$$Z_n = \bigcup_{i+j=n} X_i \wedge Y_j$$

or more formally,

$$\begin{aligned} Z_n = \underset{s \rightarrow -\infty, t \rightarrow \infty}{\text{colim}} \quad & X_s \wedge Y_{n-s} \cup_{X_s \wedge Y_{n-s-1}} X_{s+1} \wedge Y_{n-s-1} \cup_{X_{s+1} \wedge Y_{n-s-2}} \cdots \\ & \cdots \cup_{X_{t-2} \wedge Y_{n-t+1}} X_{t-1} \wedge Y_{n-t+1} \cup_{X_{t-1} \wedge Y_{n-t}} X_t \wedge Y_{n-t}. \end{aligned}$$

The smash product Z_* comes with canonical maps $X_i \wedge Y_j \rightarrow Z_{i+j}$ and is characterized by the property that maps of filtered spectra $Z_* \rightarrow W_*$ are in natural bijective correspondence with systems of maps $X_i \wedge Y_j \rightarrow W_{i+j}$ that make the evident diagrams in i and j commute. The spectrum \mathbb{S}_* is the unit for the smash product.

DEFINITION 6.2. A *filtered associative ring spectrum* consists of a filtered spectrum A_* , a map $\eta: \mathbb{S}_* \rightarrow A_*$, and a map $\mu: A_* \wedge A_* \rightarrow A_*$ satisfying the usual monoid relations (unit and associativity diagrams). A *filtered left A_* -module* (resp., *filtered right A_* -module*) consists of a filtered spectrum M_* and a map $\xi: A_* \wedge M_* \rightarrow M_*$ (resp., $\xi: M_* \wedge A_* \rightarrow M_*$) satisfying the usual action relations (unit and associativity diagrams).

Using the characterization of the smash product, we obtain an external formulation of filtered associative ring spectra: a filtered associative ring spectrum consists of a filtered spectrum A_* together with a map $\eta_0: \mathbb{S} \rightarrow A_0$ and maps

$$\mu_{i,j}: A_i \wedge A_j \longrightarrow A_{i+j}$$

such that the following unit diagrams

$$\begin{array}{ccc} \mathbb{S} \wedge A_n & \xrightarrow{\eta_0 \wedge \text{id}} & A_0 \wedge A_n \\ & \searrow \cong & \downarrow \mu_{0,n} \\ & & A_n \end{array} \quad \begin{array}{ccc} A_n \wedge A_0 & \xleftarrow{\text{id} \wedge \eta_0} & A_n \wedge \mathbb{S} \\ \downarrow \mu_{n,0} & \swarrow \cong & \\ A_n & & \end{array}$$

and associativity and structure map diagrams

$$\begin{array}{ccc}
 A_i \wedge A_j \wedge A_k & \xrightarrow{\mu_{i,j} \wedge \text{id}} & A_{i+j} \wedge A_k \\
 \text{id} \wedge \mu_{j,k} \downarrow & & \downarrow \mu_{i+j,k} \\
 A_i \wedge A_{j+k} & \xrightarrow{\mu_{i,j+k}} & A_{i+j+k}
 \end{array}
 \quad
 \begin{array}{ccc}
 A_i \wedge A_j & \xrightarrow{a_{i,m} \wedge a_{j,n}} & A_m \wedge A_n \\
 \mu_{i,j} \downarrow & & \downarrow \mu_{m,n} \\
 A_{i+j} & \xrightarrow{a_{i+j,m+n}} & A_{m+n}
 \end{array}$$

commute, where we have written $a_{k,l}: A_k \rightarrow A_l$ for the structure maps of A_* . Filtered left and right A_* -modules admit a similar external formulation.

EXAMPLE 6.3. Let T_*^M be a functor satisfying the properties laid out in Section 5. For A an associative ring orthogonal \mathbb{T} -spectrum, $T_*^M A$ has the canonical structure of a filtered associative ring spectrum. If X and Y are right and left A -modules, then $T_*^M X$ and $T_*^M Y$ have the canonical structure of right and left filtered $T_*^M A$ -modules.

The argument in Section 5 uses a filtered version of the balanced smash product, which is constructed as follows.

DEFINITION 6.4. Let A_* be a filtered associative ring spectrum, let M_* be a filtered right A_* -module and let N_* be a filtered left A_* -module. Define the balanced smash product $M_* \wedge_{A_*} N_*$ to be the coequalizer

$$M_* \wedge A_* \wedge N_* \rightrightarrows M_* \wedge N_* \longrightarrow M_* \wedge_{A_*} N_*$$

where one map is induced by the right A_* -action on M_* and the other is induced by the left A_* -action on N_* . The coequalizer is a filtered spectrum (typically with no extra structure).

Since smash products and coequalizers commute with sequential colimits, we see that

$$\text{colim}(M_* \wedge_{A_*} N_*) \cong (\text{colim } M_*) \wedge_{(\text{colim } A_*)} (\text{colim } N_*).$$

In other words, the balanced smash product above gives a filtration on the balanced smash product of the underlying unfiltered modules.

A filtered spectrum naturally gives a spectral sequence on homotopy groups. To avoid the ambiguity of writing $E_{i,j}^1(X)$ for the E^1 -term, we introduce the following notation.

DEFINITION 6.5. For a filtered spectrum X_* , let $\pi_{i,j}^{\text{Gr}} X_* = \pi_{i+j} C(X_i, X_{i-1})$, where $C(X_i, X_{i-1})$ denotes the homotopy cofiber of the structure map $X_{i-1} \rightarrow X_i$.

It is also useful to work directly with the orthogonal spectra $C(X_n, X_{n-1})$ whose homotopy groups represent $\pi_{n,*}^{\text{Gr}}$, the associated graded spectra of the filtration. We follow the traditional route of defining a point-set functor Gr with good structural properties, and regard $C(X_n, X_{n-1})$ as a model for the left derived functor. Before defining Gr , we define the target category.

DEFINITION 6.6. The category of graded spectra is the category of functors from the discrete category \mathbb{Z} to orthogonal spectra: A *graded spectrum* X_* consists of a sequence of orthogonal spectra X_n for each $n \in \mathbb{Z}$; a map of graded spectra $X_* \rightarrow Y_*$ consists of a sequence of maps of orthogonal spectra $X_n \rightarrow Y_n$ for $n \in \mathbb{Z}$.

The category of graded spectra becomes a symmetric monoidal category using the Day convolution for the symmetric monoidal product $+$ on \mathbb{Z} :

$$(X_* \wedge Y_*)_n = \bigvee_{i+j=n} X_i \wedge Y_j.$$

DEFINITION 6.7. The *associated graded* functor Gr from filtered spectra to graded spectra is defined by

$$(\text{Gr } X_*)_n := \text{Gr}_n(X_*) := X_n/X_{n-1}.$$

We emphasize that in the definition, X_n/X_{n-1} denotes the point-set quotient and not the homotopy cofiber. The functor Gr has a right adjoint Z that makes a graded spectrum into a filtered spectrum using the trivial map for structure maps. In particular Gr preserves colimits. It also clearly preserves smash products:

PROPOSITION 6.8. *Gr is a strong symmetric monoidal functor.*

We define a *graded associative ring spectrum* and *graded left and right modules* over a graded associative ring spectrum in terms of the smash product with unit maps, multiplication, and action maps in the usual way for graded spectra just as we did for filtered spectra in Definition 6.2. The previous proposition implies that Gr takes filtered associative ring spectra to graded associative ring spectra and filtered left and right modules over a filtered associative ring spectrum A_* to graded left and right modules over the graded associative ring spectrum $\text{Gr } A_*$. The previous proposition and the fact that Gr preserves colimits then implies that Gr preserves the balanced smash product.

PROPOSITION 6.9. *Let A_* be a filtered associative ring spectrum, let M_* be a filtered right A_* -module and let N_* be a filtered left A_* -module. Then there is a canonical natural isomorphism*

$$\text{Gr}(M_* \wedge_{A_*} N_*) \cong (\text{Gr } M_*) \wedge_{(\text{Gr } A_*)} (\text{Gr } N_*).$$

We now turn to the homotopy theory of filtered and graded spectra. To avoid confusion we do not use the unmodified phrase “weak equivalence” in the context of filtered and graded spectra. We always write *objectwise weak equivalence* for a map $X_* \rightarrow Y_*$ that is a weak equivalence $X_n \rightarrow Y_n$ for all n . We call a map of filtered spectra $X_* \rightarrow Y_*$ a *total weak equivalence* when it induces a weak equivalence

$$\text{hocolim}_{n \rightarrow \infty} X_n \longrightarrow \text{hocolim}_{n \rightarrow \infty} Y_n.$$

Objectwise weak equivalences are total weak equivalences, but not vice versa. However, the following result provides a useful converse under an additional hypothesis.

PROPOSITION 6.10. *Let $f: X_* \rightarrow Y_*$ be a map of filtered spectra that induces an isomorphism on $\pi_{*,*}^{\text{Gr}}$. Then f is a total weak equivalence if and only if it is an objectwise weak equivalence.*

PROOF. We only need to show the direction that assumes f is a total weak equivalence; the converse is clear. Write X_∞ for $\text{hocolim } X_n$ and Y_∞ for $\text{hocolim } Y_n$. The hypothesis is then that the map $C(X_{n+1}, X_n) \rightarrow C(Y_{n+1}, Y_n)$ is a weak equivalence for all n ; by induction, we see that $C(X_{n+i}, X_n) \rightarrow C(Y_{n+i}, Y_n)$ is a weak equivalence for all n, i , and passing to the homotopy colimit (in i), we see that $C(X_\infty, X_n) \rightarrow C(Y_\infty, Y_n)$ is a weak equivalence for all n . When f is a total weak equivalence, then $X_\infty \rightarrow Y_\infty$ is a weak equivalence, and it follows that $X_n \rightarrow Y_n$ is a weak equivalence for all n . \square

Standard results [16, 11.6.1], [28, 4.1] provide a closed model structure on filtered spectra for the objectwise weak equivalences.

PROPOSITION 6.11. *The category of filtered spectra has a compactly generated topological model structure in the sense of [24, 5.9, 5.12] with weak equivalences and fibrations defined objectwise.*

PROPOSITION 6.12. *The categories of filtered associative ring spectra and filtered left and right modules over a filtered associative ring spectrum have compactly generated topological closed model category structures with weak equivalences and fibrations defined objectwise.*

We call the homotopy categories of the preceding model structures the *filtered derived category* of left and right A_* -modules. We have a derived smash product of an A_* -module and a filtered spectrum, denoted \wedge^L , which may be constructed by cofibrant approximation of either object. Using this, we have an evident notion of homotopical module in the derived category. For us, the case that is most useful is that of homotopical left B_* -modules in the filtered derived category of right A_* -modules, so we make this definition explicitly; other types of homotopical modules are defined analogously.

DEFINITION 6.13. Let A_* and B_* be a filtered associative ring spectra. A *homotopical left B_* -module* in the filtered derived category of right A_* -modules consists of a right A_* -module N_* and a map

$$\xi: B_* \wedge^L N_* \longrightarrow N_*$$

in the filtered derived category of right A_* -modules satisfying the usual unit and associativity conditions.

The next proposition, a special case of [21, 8.2], studies the homotopy theory of the balanced smash product. It implies in particular that the filtered derived smash product may be computed by deriving either variable.

PROPOSITION 6.14. *Let A_* be a filtered associative ring spectrum. The left derived bifunctor $\text{Tor}^{A_*}(-, -)$ of the balanced product $(-) \wedge_{A_*} (-)$ exists and can be constructed by cofibrant replacement of either variable. In particular, for each right module M_* and left module N_* , $\text{Tor}^{A_*}(-, N_*)$ is the left derived functor of $(-) \wedge_{A_*} N_*$ and $\text{Tor}^{A_*}(M_*, -)$ is the left derived functor of $M_* \wedge (-)$.*

Given this proposition, there is no source of confusion for which derived functor \wedge^L denotes; we now switch to writing $M_* \wedge_{A_*}^L N_*$ in place of $\text{Tor}^{A_*}(M_*, N_*)$.

As a consequence of the proposition, if M_* has the structure of a homotopical left A_* -module in the category of filtered right A_* -modules, then $M_* \wedge_{A_*}^L (-)$ has the natural structure of a homotopical left A_* -module in filtered spectra.

For Propositions 6.11–6.14, analogous statements hold in context of graded spectra with easier proofs. As discussed above, the functor Gr from filtered spectra to graded spectra has a right adjoint, and from the description above, it is clear that the right adjoint preserves objectwise fibrations and objectwise weak equivalences. Thus, Gr is a Quillen left adjoint; its left derived functor LGr exists and may be constructed by applying Gr to a cofibrant replacement. Alternatively, as per the motivation for introducing Gr , we can use a homotopy cofiber construction. For work below it is useful to have a wide class of objects where the point-set functor Gr models the derived functor. We introduce the following terminology.

DEFINITION 6.15. A filtered spectrum X_* is *reasonably filtered* when the structure maps $X_n \rightarrow X_{n+1}$ are all h -cofibrations.

In what follows, “reasonably filtered” will always refer to the underlying filtered spectrum. We elide “reasonably filtered filtered...” to “reasonably filtered...” for filtered spectra, filtered associative ring spectra, and filtered left and right modules.

PROPOSITION 6.16. Gr preserves objectwise weak equivalences between reasonably filtered spectra.

Cofibrant objects in the model category of filtered spectra are reasonably filtered, and so Gr computes the derived functor LGr on all reasonably filtered spectra. When A_* is a reasonably filtered associative ring spectrum, cofibrant left and right filtered A_* -modules are reasonably filtered, and so the functors Gr from filtered left and right A_* -modules to graded left and right $\text{Gr } A_*$ -modules have left derived functors, calculated by applying Gr to reasonably filtered replacements. We also have the following observation about the balanced smash product.

PROPOSITION 6.17. Let A_* be a reasonably filtered associative ring spectrum, let M_* be a cofibrant right A_* -module and let N_* be a cofibrant left A_* -module. Then the balanced smash product $M_* \wedge_{A_*} N_*$ is a reasonably filtered spectrum.

Boardman [7, 5.10] defines conditional convergence of spectral sequences in terms of \lim and \lim^1 . In the current context of filtered spectra, the spectral sequence associated to the filtration on X_* is conditionally convergent if and only if $\text{holim}_n X_{-n} \simeq *$. With this in mind, we make the following definition.

DEFINITION 6.18. A filtered spectrum X_* is *conditionally convergent* when $\text{holim}_n X_{-n} \simeq *$

A first easy observation about conditional convergence of filtered spectra is that it is invariant under objectwise weak equivalences, and so conditional convergence may be studied in the homotopy category of filtered spectra.

Because homotopy limits commute with cofiber sequences and with other homotopy limits, the following propositions are clear.

PROPOSITION 6.19. If X_* and Y_* are conditionally convergent, then for any map $X_* \rightarrow Y_*$, the homotopy cofiber is conditionally convergent.

PROPOSITION 6.20. If $d \mapsto X_*(d)$ is a small diagram and each $X_*(d)$ is conditionally convergent, then $\text{holim}_d X_*(d)$ is conditionally convergent.

For a filtered spectrum X_* , the suspension $\Sigma^m X_*$ is defined objectwise, $(\Sigma^m X_*)_n = \Sigma^m X_n$, where we understand $\Sigma^m X = X \wedge F_{-m} S^0$ when $m < 0$. (Here $F_{-m} S^0$ is a particular cofibrant model of S^m ; see [24, 1.3]). The shift $X_*[t]$ is defined by $(X_*[t])_n = X_{n-t}$. The following proposition is also clear.

PROPOSITION 6.21. If X_* is conditionally convergent, then so is any suspension and shift.

For a filtered associative ring spectrum A_* , a finite cell filtered right A_* -module is a filtered right A_* -module that can be built in finitely many stages using cofiber sequences involving suspensions and shifts.

PROPOSITION 6.22. If N_* is a conditionally convergent left A_* -module and M_* is a finite cell filtered right A_* -module, then $M_* \wedge_{A_*}^L N_*$ is conditionally convergent.

PROOF. $M_* \wedge_{A_*}^L N_*$ is built in finitely many stages using cofiber sequences involving suspensions and shifts of N_* . \square

7. Comparison of the lefthand and righthand spectral sequences (Proof of Theorem 5.4)

This section is devoted to the proof of Theorem 5.4. We prove a slightly more general result: We assume that that A is a commutative ring orthogonal T -spectrum, and X and Y are cofibrant A -modules under A . (We can also take G to be any closed subgroup of \mathbb{T} .)

Let TY'_* be a filtered $T_*^M A$ -module cofibrant replacement for $T_*^M Y$. In particular $\text{Gr } TY'_* \rightarrow \text{Gr } T_*^M Y$ is a weak equivalence. Theorem 5.4 asserts that the filtered map

$$(7.1) \quad T_*^M X \wedge_{T_*^M A} TY'_* \longrightarrow T_*^M (X \wedge_A Y)$$

induces a weak equivalence on $\pi_{*,*}^{\text{Gr}}$. Note that this is a map of homotopical left $T_*^M A$ -modules and hence the induced map on $\pi_{*,*}^{\text{Gr}}$ is a map of left $\pi_{*,*}^{\text{Gr}} T_*^M A$ -modules. Theorem 3.19 (in the case $G = \mathbb{T}$) or Proposition 3.12 (in the case G is finite) gives an isomorphism of bigraded rings

$$\pi_{*,*}^{\text{Gr}} T_*^M A \cong HM_{*,*} \otimes \pi_* A.$$

In particular (7.1) is a map of left $HM_{*,*}$ -modules.

By Hypothesis 5.1, we have a zigzag of monoidal functors, which as indicated takes the form

$$i^* \xrightarrow{\sim} RR \xleftarrow{\sim} TT \longrightarrow \text{Gr}_0 T_*^M,$$

where i^* denotes the forgetful functor to non-equivariant orthogonal spectra. Choosing cofibrant replacements $TTY' \rightarrow TT(Y)$ and $RRY' \rightarrow RR(Y)$ in the categories of left $TT(A)$ -modules and left $RR(A)$ -modules, respectively, we can then choose lifts $TTY' \rightarrow \text{Gr}_0 TY'$ and $TTY' \rightarrow RRY'$ (in left $TT(A)$ -modules) and $i^* Y \rightarrow RRY'$ (in left $i^* A$ -modules). Since the natural transformations are monoidal, we get a commutative diagram of graded spectra

$$\begin{array}{ccc} i^* X \wedge_{i^* A} i^* Y & \xrightarrow{=} & i^* X \wedge_{i^* A} i^* Y \\ \simeq \downarrow & & \downarrow \simeq \\ RR(X) \wedge_{RR(A)} RRY' & \longrightarrow & RR(X \wedge_A Y) \\ \simeq \uparrow & & \uparrow \simeq \\ TT(X) \wedge_{TT(A)} TTY' & \longrightarrow & TT(X \wedge_A Y) \\ \downarrow & & \downarrow \\ \text{Gr } T_*^M X \wedge_{\text{Gr } T_*^M A} \text{Gr } TY' & \longrightarrow & \text{Gr}(T_*^M (X \wedge_A Y)) \end{array}$$

where we view the ungraded spectra as concentrated in degree 0. On the righthand side, since

$$\pi_{*,*}^{\text{Gr}}(T_*^M (X \wedge_A Y)) \cong \pi_{*,*} \text{Gr}(T_*^M (X \wedge_A Y))$$

is a left $HM_{*,*}$ -module, we get an induced map of $HM_{*,*}$ -modules

$$HM_{*,*} \otimes \pi_* (i^* X \wedge_{i^* A} i^* Y) \longrightarrow \pi_{*,*}^{\text{Gr}}(T_*^M (X \wedge_A Y))$$

which by construction (and Hypothesis 5.1) is the usual isomorphism. We are now reduced to proving the following lemma.

LEMMA 7.2. *The induced map of bigraded abelian groups*

$$HM_{*,*} \otimes \pi_*(i^*X \wedge_{i^*A} i^*Y) \longrightarrow \pi_{*,*}^{\text{Gr}}(T_*^M X \wedge_{T_*^M A} TY')$$

is an isomorphism.

PROOF. We have trigraded Eilenberg-Moore spectral sequences to compute both sides. On the left we take the tensor of the torsion free bigraded abelian group $HM_{*,*}$ with the Eilenberg-Moore spectral sequence for the smash product $i^*X \wedge_{i^*A} i^*Y$; this has E^2 -term

$$E_{i,j,*}^2 = HM_{j,*} \otimes \text{Tor}_{i,*}^{\pi_* A}(\pi_* X, \pi_* Y).$$

On the right the usual Eilenberg-Moore spectral sequence for the balanced smash product of associated graded modules is naturally trigraded with E^2 -term

$$E_{i,*,*}^2 = \text{Tor}_i^{\pi_{*,*}^{\text{Gr}} A}(\pi_{*,*}^{\text{Gr}} X, \pi_{*,*}^{\text{Gr}} Y) \cong \text{Tor}_{i,*,*}^{HM_{*,*} \otimes \pi_* A}(HM_{*,*} \otimes \pi_* X, HM_{*,*} \otimes \pi_* Y).$$

Since the map is induced by maps of rings and modules, we get a homomorphism of spectral sequences. Since the isomorphism $HM_{*,*} \otimes \pi_* Z \cong \pi_{*,*}^{\text{Gr}} Z$ of Theorem 3.19 or Proposition 3.12 is induced by the same map $\pi_* Z \rightarrow \pi_{*,*}^{\text{Gr}} T^M Z$ as in Hypothesis 5.1, the induced map on E^2 -terms

$$HM_{*,*} \otimes \text{Tor}_{*,*}^{\pi_* A}(\pi_* X, \pi_* Y) \longrightarrow \text{Tor}_{*,*,*}^{HM_{*,*} \otimes \pi_* A}(HM_{*,*} \otimes \pi_* X, HM_{*,*} \otimes \pi_* Y)$$

is the evident isomorphism. \square

8. Conditional convergence of the lefthand spectral sequence (Proof of Lemma 5.6)

We employ the terminology of Section 6. In this terminology, Lemma 5.6 is precisely the assertion that the filtered spectrum $T_*^M X \wedge_{T_*^M A} TY'$ is conditionally convergent. The proof relies properties specific to perfect fields of finite characteristic.

As indicated in Section 6, our homotopical work is in the Borel equivariant stable category, where we can specify an object as an orthogonal spectrum indexed on $\{\mathbb{R}^n\}$ with a (point-set) \mathbb{T} -action. We denote by $i_* Hk$ the Eilenberg-Mac Lane spectrum Hk (indexed on $\{\mathbb{R}^n\}$) with the trivial action. For any fixed model of Hk as a commutative ring orthogonal spectrum, the augmentation map $N^{\text{cy}}(Hk) \rightarrow i_* Hk$ is a point-set map of commutative ring orthogonal spectra with \mathbb{T} -action, in particular making $i_* Hk$ an $N^{\text{cy}}(Hk)$ -module. On the point-set level the map is induced by the multiplication map $Hk \wedge \cdots \wedge Hk \rightarrow Hk$, or viewing $N^{\text{cy}}(Hk)$ as the tensor $i_* Hk \otimes \mathbb{T}$ in the point-set category of commutative ring orthogonal spectra with \mathbb{T} -action, it is induced by the map of \mathbb{T} -spaces $\mathbb{T} \rightarrow \mathbb{T}/\mathbb{T}$. The following proposition is specific to the case of perfect fields of finite characteristic.

PROPOSITION 8.1. *Let k be a perfect field of finite characteristic. In the Borel derived category of left $N^{\text{cy}}(Hk)$ -modules, $i_* Hk$ is finite.*

PROOF. We have $THH_*(k) \cong k[t]$ (with t in degree 2), so it suffices to show that there exists a map in the Borel stable category $\Sigma^2 \mathbb{S} \rightarrow THH(k)$ sending the fundamental class to t , or equivalently, that t is in the image of the map $\pi_*^{h\mathbb{T}} THH(k) \rightarrow \pi_* THH(k)$. The homotopy fixed point spectral sequence is conditionally convergent in this case and concentrated in even degrees, and so strongly convergent with $E_2 = E_\infty$. In particular, the map $\pi_*^{h\mathbb{T}} THH(k) \rightarrow \pi_* THH(k)$ is surjective. \square

Throughout this section, when we refer to i_*Hk as an equivariant $N^{\text{cy}}(Hk)$ -module, we always mean the structure in Proposition 8.1. In the argument for Lemma 5.6, we will use Postnikov towers built from i_*Hk .

PROPOSITION 8.2. *Let $A = N^{\text{cy}}(Hk)$ and let X be an equivariant A -module. If $\pi_n X = 0$ for $n < N$ then there exists a tower of equivariant A -modules*

$$\cdots \longrightarrow X_{m+1} \longrightarrow X_m \longrightarrow \cdots \longrightarrow X_N \longrightarrow X_{N-1} = *$$

and a map of equivariant A -modules from X to the system such that:

- (i) *The map $X \rightarrow \text{holim } X_m$ is a Borel equivalence.*
- (ii) *Each homotopy fiber $\text{Fib}(X_{m+1} \rightarrow X_m)$ is Borel equivalent as an equivariant A -module to a wedge of copies of $\Sigma^{m+1} i_* Hk$.*

PROOF. By way of notation, recall from Section 1 that $\mathbf{R}F_A(-, -)$ denotes the equivariant derived mapping spectrum for the derived category of left A -modules, an object of the \mathbb{T} -equivariant stable category. In contrast, $\mathbf{R}^B F_A^{\mathbb{T}}(-, -)$ denotes the derived mapping spectrum in the Borel derived category of left A -modules, an object of the (non-equivariant) stable category, and the relationship is that $\mathbf{R}^B F_A^{\mathbb{T}}(-, -)$ is the homotopy fixed point spectrum of $\mathbf{R}F_A(-, -)$. We then have isomorphisms

$$\begin{aligned} \text{Ho}^B(\mathcal{M}\text{od}_A)(X, \Sigma^n i_* Hk) &\cong \pi_{-n} \mathbf{R}^B F_A^{\mathbb{T}}(X, i_* Hk) \cong \pi_{-n}^{h\mathbb{T}} \mathbf{R}F_A(X, i_* Hk) \\ &\cong \pi_{-n}^{h\mathbb{T}} \mathbf{R}F_{i_* Hk}(i_* Hk \wedge_A^L X, i_* Hk), \end{aligned}$$

the last isomorphism induced by change of scalars. By the Hurewicz theorem for (non-equivariant) A -modules and the hypothesis that $\pi_n X = 0$ for $n < N$, we have that the map $\pi_N X \rightarrow \pi_N(i_* Hk \wedge_A^L X)$ is an isomorphism and $\pi_n(i_* Hk \wedge_A^L X) = 0$ for $n < N$. We see that $\pi_n \mathbf{R}F_{i_* Hk}(i_* Hk \wedge_A^L X, i_* Hk) = 0$ for $n > -N$, and so from the homotopy fixed point spectral sequence, we deduce that the map

$$\begin{aligned} \pi_{-N}^{h\mathbb{T}} \mathbf{R}F_{i_* Hk}(i_* Hk \wedge_A^L X, i_* Hk) &\longrightarrow \pi_{-N} \mathbf{R}F_{i_* Hk}(i_* Hk \wedge_A^L X, i_* Hk) \\ &\cong \pi_{-N} \mathbf{R}F_{Hk}(Hk \wedge_{i^* A}^L i^* X, Hk) \end{aligned}$$

is an isomorphism. This constructs enough maps $X \rightarrow \Sigma^N i_* Hk$ to produce a map $X \rightarrow \bigvee \Sigma^N i_* Hk$ that induces an isomorphism on π_N . The remainder of the construction is parallel to the usual construction of the Postnikov tower, using the techniques above to construct the maps. \square

We are ready to prove Lemma 5.6.

PROOF OF LEMMA 5.6. We use the Postnikov tower construction of Proposition 8.2 applied to Y . By construction, each Y_m is built in finitely many fiber sequences from a wedge of copies of $\Sigma^n i_* Hk$. These are finite wedges since $\pi_n X$ is finitely generated over k . We also know from Proposition 8.1 that $i_* Hk$ is finite in the Borel derived category of A -modules. It follows that $T_*^M Y_m$ is a finite filtered $T_*^M A$ -module, and by Proposition 6.22, $T_*^M X \wedge_{T_*^M A}^L T_*^M Y_m$ is conditionally convergent.

Proposition 6.14 allows us to choose a cofibrant filtered $T_*^M A$ -module replacement $TX'_* \rightarrow T_*^M X$ so that $TX'_* \wedge_{T_*^M A} (-)$ models $T_*^M X \wedge_{T_*^M A}^L (-)$. Proposition 6.20 now implies that

$$\text{holim}_m(TX'_* \wedge_{T_*^M A} T_*^M Y_m)$$

is conditionally convergent. We have a map of filtered spectra

$$TX'_* \wedge_{T_*^M A} T^M Y_* \longrightarrow \text{holim}_m(TX'_* \wedge_{T_*^M A} T_*^M Y_m)$$

that induces an isomorphism on $\pi_{*,*}^{\text{Gr}}$ since $\pi_* X$ is bounded below (for example, see Lemma 7.2). The proof is completed by showing that this map is an objectwise weak equivalence. We do this by applying Proposition 6.10 and showing that the map is a total weak equivalence. To see that the map is a total weak equivalence, it suffices to observe that TX' is small as a $T^M A$ -module. This is the content of Theorem B in the case of main interest when $X = N^{\text{cy}}(\mathcal{X}')$ and $G = \mathbb{T}$, and follows in the current generality by its generalization, Theorem 17.4 (together with Proposition 17.3), proved in Section 17. \square

9. Constructing the filtered model: The positive filtration

In this section, we start the construction of the filtered functor T^M outlined in Section 5. As in the construction of the Hesselholt-Madsen Tate filtration in Section 3, we construct the integrally graded filtration from a positive filtration, arising from a filtration on \widetilde{EG} , and a negative filtration, arising from a filtration on EG . In both cases, the technical work is to construct the multiplicative structure we require. This section handles the work for the positive filtration and the next one the work for the negative filtration.

The positive filtration on the Tate fixed points arises from the G -cellular filtration on \widetilde{EG} . It should have S^0 in filtration level zero and free G -cells in every positive degree in the pattern specified by the cell structure of the standard model discussed in Section 3. Recall that when $G = \mathbb{T}$, we double the natural filtration degrees as explained there. However, for the purposes of this section, we will work with the natural filtration degrees in order to give a uniform treatment, introducing a different notation to avoid confusion.

To construct a multiplicative version of this filtration, we use a model $\widetilde{E}_{\mathcal{O}}\mathbb{T}$ for an A_∞ operad \mathcal{O} (with $\mathcal{O}(0) = *$) and construct a new filtration, the “pseudocellular filtration”, which is coarser than the homogeneous filtration (2.3). Here we understand an A_∞ operad to have spaces the homotopy type of CW complexes, the identity element to be a non-degenerate basepoint, and to come with a weak equivalence of operads $\mathcal{O} \rightarrow \mathcal{A}\text{ss}$, where $\mathcal{A}\text{ss}$ denotes the associative operad $\mathcal{A}\text{ss}(n) = \Sigma_n$. We have a corresponding non- Σ operad $\overline{\mathcal{O}}$ where $\overline{\mathcal{O}}(n)$ is the component of $\mathcal{O}(n)$ lying over the identity permutation in Σ_n ; then \mathcal{O} is canonically isomorphic to the induced operad $(\mathcal{O}(n) \cong \overline{\mathcal{O}}(n) \times \mathcal{A}\text{ss}(n))$. In this notation, $\widetilde{E}_{\mathcal{O}}G$ is formed by starting with S^0 and iteratively gluing on the cells $\overline{\mathcal{O}}(n) \times G^n \times I^n$.

DEFINITION 9.1. Let \mathcal{O} be an A_∞ operad with $\mathcal{O}(0) = *$. We define the *pseudocellular filtration*, an increasing filtration on $\widetilde{E}_{\mathcal{O}}G$, as follows. We put S^0 in filtration level 0. For an element $(g_1, \dots, g_n) \in G^n$, let

$$q(g_1, \dots, g_n) = n - (\delta(g_1, g_2) + \dots + \delta(g_{n-1}, g_n)),$$

where $\delta(g, h) = 1$ if $g = h$ and 0 otherwise. An element of $\widetilde{E}_{\mathcal{O}}G$ in the image of

$$a, (g_1, \dots, g_n), (t_1, \dots, t_n) \in \overline{\mathcal{O}}(n) \times G^n \times I^n$$

is placed in filtration level $q(g_1, \dots, g_n)$. For the purposes of this section, we write $\widetilde{E}_{\mathcal{O}}G_n^{pc}$ for the subspace in pseudocellular filtration level n . For G , finite, we define $\widetilde{E}_{\mathcal{O}}G_n = \widetilde{E}_{\mathcal{O}}G_n^{pc}$, and for $G = \mathbb{T}$, we define $\widetilde{E}_{\mathcal{O}}G_{2n+1} = \widetilde{E}_{\mathcal{O}}G_{2n} = \widetilde{E}_{\mathcal{O}}G_n^{pc}$.

Although different representatives of the same point may have different q -values, the filtration is well-defined: The point lies in all filtration levels of all representatives and all higher levels. The function q subtracts from the homogeneous degree the number of consecutive repeats; an element in homogeneous filtration level n is also in pseudocellular filtration level n (and possibly lower). Since G acts diagonally on G^n , the filtration is G -equivariant. The filtration is multiplicative in the following sense.

PROPOSITION 9.2. *The operad action maps $\overline{\mathcal{O}}(m)_+ \wedge \widetilde{E_{\mathcal{O}}G}^{(m)} \rightarrow \widetilde{E_{\mathcal{O}}G}$ preserve filtration using the pseudocellular filtration on $\widetilde{E_{\mathcal{O}}G}$ and the smash power of the pseudocellular filtration on $\widetilde{E_{\mathcal{O}}G}^{(m)}$.*

PROOF. Representing an element x of $\widetilde{E_{\mathcal{O}}G}^{(m)}$ as

$$a^i, (g_1^i, \dots, g_{n_i}^i), (t_1, \dots, t_{n_i})$$

for $i = 1, \dots, m$, and given an element $a \in \overline{\mathcal{O}}(m)$, the composition takes a, x to an element y of $\widetilde{E_{\mathcal{O}}G}$ that is represented by

$$b, (h_1, \dots, h_n), (u_1, \dots, u_n)$$

where $b = a \circ (a_1, \dots, a_i)$, $n = n_1 + \dots + n_m$, and the h_k 's and u_k 's are the lists obtained by flattening the arrays of g_j^i 's and t_j^i 's (respectively) lexicographically with the lower index first. The element x is in filtration level

$$\sum_{i=1}^m q(g_1, \dots, g_{n_i}) = \sum_{i=1}^m \left(n_i - \sum_{j=1}^{n_i-1} \delta(g_j^i, g_{j+1}^i) \right)$$

which is at least as big as

$$\sum_{i=1}^m \left(n_i - \sum_{j=1}^{n_i-1} \delta(g_j^i, g_{j+1}^i) \right) - \sum_{i=1}^{m-1} \delta(g_{n_i}^i, g_1^{i+1}) = n - \sum_{k=1}^{n-1} \delta(h_k, h_{k+1}),$$

which filtration level contains y . \square

To show that this filtration is equivalent to the standard one, consider the standard (simplicial) filtration on EG coming from the bar construction $EG = B(G, G, *)$ where

$$B_n(G, G, *) = G \times (G^n).$$

We have the same filtration when we look at the W -construction, the geometric realization of the simplicial set

$$WG_n = G^{n+1},$$

where the faces are induced by projection maps

$$d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$$

and the degeneracies by diagonal maps

$$s_i(g_0, \dots, g_n) = (g_0, \dots, g_{i-1}, g_i, g_i, g_{i+1}, \dots, g_n).$$

This has the diagonal g -action, and the map $WG_{\bullet} \rightarrow B_{\bullet}(G, G, *)$ defined by

$$g_0, \dots, g_n \mapsto g_0, (g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n)$$

is a G -equivariant simplicial isomorphism. We get another model closer to the operadic construction by regarding WG_{\bullet} as a Δ -set, forgetting the degeneracies.

Let WG^Δ denote the Δ -set geometric realization, where we iteratively glue $WG_n \times \Delta[n]$ along $WG_n \times \partial\Delta[n]$ without collapsing the degeneracies. The Δ -set geometric realization gives a filtration but not one that corresponds to the filtration on WG : The map $WG^\Delta \rightarrow WG$ is filtered and an equivariant weak equivalence, but the filtration pieces do not map by weak equivalences. We can fix this by enlarging the filtration making use of the forgotten degeneracies: We define the pseudocellular filtration on WG^Δ by taking WG_n^Δ to consist of all elements in the image of $WG_m \times \Delta[m]^\Delta$ for $m \leq n$, where $\Delta[m]^\Delta$ denotes the Δ -set geometric realization of $\Delta[m]$. Using this filtration, the map $WG^\Delta \rightarrow WG$ is an equivariant weak equivalence on each filtration level.

DEFINITION 9.3. Let \widetilde{WG}^Δ be the cofiber of the map $WG_+^\Delta \rightarrow S^0$ collapsing WG^Δ to the non-basepoint. Let \widetilde{WG}_n^Δ be the cofiber of $(WG_{n-1}^\Delta)_+ \rightarrow S^0$ for the pseudocellular filtration on WG^Δ , where we understand WG_{-1}^Δ to be the empty set.

The equivariant weak equivalence $WG^\Delta \rightarrow EG$ induces an equivariant weak equivalence $\widetilde{WG}^\Delta \rightarrow \widetilde{EG}$, and the following proposition is clear by construction.

PROPOSITION 9.4. *The canonical map $\widetilde{WG}^\Delta \rightarrow \widetilde{EG}$ is filtered for the standard filtration on \widetilde{EG} and an equivariant weak equivalence on each filtration level.*

Next we define a map $\widetilde{E_O G} \rightarrow \widetilde{WG}^\Delta$. Concretely, \widetilde{WG}^Δ is built from S^0 by iteratively attaching $G^n \times \Delta[n-1] \times I$ along $G^n \times \partial(\Delta[n-1] \times I)$ by the map that identifies an element

$$(g_1, \dots, g_n), x, t$$

of $G^n \times \partial(\Delta[n-1] \times I) \subset G^n \times \Delta[n-1] \times I$ with

$$(g_1, \dots, \hat{g}_i, \dots, g_n), y, t$$

when $x = d^{i-1}(y)$ for some $y \in \Delta[n-2]$, with the basepoint if $t = 1$, and with the non-basepoint element of $S^0 = \widetilde{WG}_0^\Delta$ when $t = 0$. We can identify the image of $G^m \times \Delta[m]^\Delta$ in WG^Δ in terms of repeated coordinates. Concisely, a point is in \widetilde{WG}_n^Δ exactly when it has a representative $(g_1, \dots, g_n), x, t$ with $q(g_1, \dots, g_n) \leq n$, where q is the function defined in Definition 9.1.

We define the map $\widetilde{E_O G} \rightarrow \widetilde{WG}^\Delta$ as follows. We send $S^0 = \widetilde{E_O G}_0$ by the identity into $S^0 = \widetilde{WG}_0^\Delta$, and we send the element of $\widetilde{E_O G}$ represented by

$$a, (g_1, \dots, g_n), (t_1, \dots, t_n),$$

to the basepoint if $\sum t_i \geq 1$, to the non-basepoint element of S^0 if $t_i = 0$ for all i , and otherwise to the element represented by

$$(g_1, \dots, g_n), x, t$$

where $t = \sum t_i$ and x has barycentric coordinates

$$t_1/t, t_2/t, \dots, t_n/t.$$

It is clear from the gluing relations on $\widetilde{E_O G}$ that this is well-defined, continuous, and equivariant. Moreover, it is clear from the construction of the filtration that it is filtered for the pseudocellular filtration. The following proposition completes the work we need on the pseudocellular filtration.

PROPOSITION 9.5. *The map $\widetilde{E}_{\mathcal{O}}G_n^{pc} \rightarrow \widetilde{WG}_n^{\Delta}$ is an equivariant weak equivalence for all n .*

PROOF. For every non-trivial $H < G$, the induced map on H -fixed points is the identity map $S^0 \rightarrow S^0$, so it suffices to show that it is a non-equivariant weak equivalence. For both $\widetilde{E}_{\mathcal{O}}G$ and \widetilde{WG}^{Δ} , the n th piece of the filtration is built from the $(n-1)$ st piece of the filtration by attaching cells of a certain form along boundaries. For $\widetilde{E}_{\mathcal{O}}G$, these cells are of the form

$$\overline{\mathcal{O}}(n) \times G^n \times I^{n+m}$$

with boundary

$$(\overline{\mathcal{O}}(n) \times sG^n \times I^{n+m}) \cup (\overline{\mathcal{O}}(n) \times G^n \times \partial I^{n+m}) \subset \overline{\mathcal{O}}(n) \times G^n \times I^{n+m}$$

for $m \geq 0$, where sG^n denotes the subspace where at least one coordinate is the same as the following coordinate. The cells for \widetilde{WG}^{Δ} are in one-to-one correspondence but of the form

$$G^n \times \Delta[n-1+m] \times I$$

with boundary

$$(sG^n \times \Delta[n-1+m] \times I) \cup (G^n \times \partial(\Delta[n-1+m] \times I)) \subset G^n \times \Delta[n-1+m] \times I.$$

The map $\widetilde{E}_{\mathcal{O}}G_n^{pc} \rightarrow \widetilde{WG}_n^{\Delta}$ sends each cell of $\widetilde{E}_{\mathcal{O}}G_n^{pc}$ to the corresponding cell of $\widetilde{WG}_n^{\Delta}$ by a homotopy equivalence, which is a homotopy equivalence on the boundary. \square

10. Constructing the filtered model: The negative filtration

This section continues the work on constructing the filtered functor T^M by constructing the negative part of the Hesselholt-Madsen multiplicative filtration. The negative filtration arises from the \mathbb{T} -cellular filtration of $E\mathbb{T}$ and the diagonal map $E\mathbb{T} \rightarrow E\mathbb{T} \times E\mathbb{T}$ (partly) induces the multiplication. Of course, the diagonal is not compatible with the filtration; we must modify the diagonal to make it cellular. Our approach is to parametrize different diagonal maps using a variant of the little 1-cubes operad, the “overlapping little 1-cubes” operad.

DEFINITION 10.1. The *overlapping little 1-cubes* \mathcal{O}_1^{Ξ} has n th space the subspace of elements $((x_1, y_1), \dots, (x_n, y_n))$ of $(I^2)^n$ satisfying $x_i < y_i$, with Σ_n acting in the usual way on $(-)^n$. Composition is just like in the little 1-cubes operad:

$$\begin{aligned} & ((x_1, y_1), \dots, (x_n, y_n)) \circ_i ((x'_1, y'_1), \dots, (x'_{n'}, y'_{n'})) \\ &= ((x_1, y_1), \dots, (x_{i-1}, y_{i-1}), (x_i + (y_i - x_i)x'_1, x_i + (y_i - x_i)y'_1), \\ & \quad \dots, (x_i + (y_i - x_i)x'_{n'}, x_i + (y_i - x_i)y'_{n'}), (x_{i+1}, y_{i+1}), \dots, (x_n, y_n)) \end{aligned}$$

Identifying I^2 with the increasing linear endomorphisms of I by $(x, y) \leftrightarrow f_{(x,y)}$ where $f_{(x,y)}(t) = x + (y - x)t$, the composition formula can be written more conceptually as

$$\begin{aligned} & (f_{(x_1, y_1)}, \dots, f_{(x_n, y_n)}) \circ_i (f_{(x'_1, y'_1)}, \dots, f_{(x'_{n'}, y'_{n'}))) \\ &= (f_{(x_1, y_1)}, \dots, f_{(x_{i-1}, y_{i-1})}, f_{(x_i, y_i)} \circ f_{(x'_1, y'_1)}, \\ & \quad \dots, f_{(x_i, y_i)} \circ f_{(x'_{n'}, y'_{n'})}, f_{(x_i, y_i)}, \dots, f_{(x_n, y_n)}). \end{aligned}$$

Note that the distinguished point $((0, 1), \dots, (0, 1))$ in $\mathcal{O}_1^{\Xi}(n)$ (corresponding to the identity map $I \rightarrow I$) induces a map of operads from the commutative operad into \mathcal{O}_1^{Ξ} ; this map is a spacewise equivariant homotopy equivalence. We have a canonical map from the little 1-cubes operad \mathcal{C}_1 to \mathcal{O}_1^{Ξ} as the spacewise inclusion of the subsets where the open intervals $(x_1, y_1), \dots, (x_n, y_n) \subset I$ do not overlap.

The purpose of the overlapping little 1-cubes operad \mathcal{O}_1^{Ξ} is that it has a natural coaction on the geometric realization of a simplicial space, generalizing the diagonal map. We recall that for a space X , the space of continuous maps $\text{End}_X^{\text{op}}(n) = \text{Map}(X, X^n)$ naturally forms an operad, and a *coaction* of an operad \mathcal{O} on the space X is a map of operads $\mathcal{O} \rightarrow \text{End}_X^{\text{op}}$. For example, the set of diagonal maps $X \rightarrow X^n$ give a coaction of Com on any space. The theorem is the following.

THEOREM 10.2. *The operad \mathcal{O}_1^{Ξ} has a natural coaction on the geometric realization of a simplicial space $|Z_{\bullet}|$ such that the composite coaction of Com is the set of diagonal maps and the composite coaction of \mathcal{C}_1 ,*

$$|Z_{\bullet}| \times \mathcal{C}_1(n) \longrightarrow |Z_{\bullet}|^n$$

is filtered for the simplicial filtration on $|Z_{\bullet}|$.

We emphasize that the target uses the cartesian product filtration for the filtration on $|Z_{\bullet}|^n$ rather than the simplicial filtration on $|Z_{\bullet}|$. We apply this to the standard model of EG formed from the bar construction. In the case when G is finite, the simplicial filtration is just the cellular filtration. In the case when $G = \mathbb{T}$, it is a renumbering of the cellular filtration which has one free \mathbb{T} -cell in each even dimension. (The cartesian product filtration on $E\mathbb{T}^n$ for $n > 1$ is not closely related to a cellular filtration since \mathbb{T} is positive dimensional.) Recall that we define EG_n to be the n th geometric filtration level when G is finite and $EG_{2n+1} = EG_{2n}$ to be the n th geometric filtration level when $G = \mathbb{T}$. Then for any partition $n = n_1 + \dots + n_m$, the map

$$EG \times \mathcal{C}_1(m) \longrightarrow EG \times \dots \times EG$$

takes the subspace $EG_{n-1} \times \mathcal{C}_1(m)$ to the subspace

$$(EG^m)_{n-1} \subset (EG_{n_1-1} \times EG \times \dots \times EG) \cup \dots \cup (EG \times \dots \times EG \times EG_{n_m-1}).$$

We therefore get a map of based G -spaces

$$EG/EG_{n-1} \wedge \mathcal{C}_1(m)_+ \longrightarrow EG/EG_{n_1-1} \wedge \dots \wedge EG/EG_{n_m-1}.$$

Then for any orthogonal G -spectra X_1, \dots, X_m , the cotensor adjunction induces a map of orthogonal G -spectra

$$\begin{aligned} \mathcal{C}_1(m)_+ \wedge F(EG/EG_{n_1-1}, X_1) \wedge \dots \wedge F(EG/EG_{n_m-1}, X_m) \\ \longrightarrow F(EG/EG_{n-1}, X_1 \wedge \dots \wedge X_m). \end{aligned}$$

In the next section, we convert this structure into a monoidal structure using the usual Moore trick to convert \mathcal{C}_1 structures to strictly associative structures.

We conclude the section with the proof of Theorem 10.2.

PROOF OF THEOREM 10.2. Recall the Milnor coordinates $u_1 \leq u_2 \leq \dots \leq u_n$ on the standard n -simplex $\Delta[n]$ give a homeomorphism of $\Delta[n]$ with a subspace of I^n and relate to the barycentric coordinates t_0, \dots, t_n by the formulas

$$u_j = \sum_{i=0}^{j-1} t_i \quad t_j = u_{j+1} - u_j$$

(where in the righthand formula $u_0 = 0$ and $u_{n+1} = 1$). Then any element of the geometric realization of a simplicial space Z_\bullet is specified (non-uniquely) by an element $z \in Z_n$ and Milnor coordinates $u_1 \leq \dots \leq u_n$, and an element of $\mathcal{O}_1^\Xi(m)$ is specified by a sequence of subintervals of I , $(x_1, y_1), \dots, (x_m, y_m)$. On this element, the coaction map has the form

$$\begin{aligned} (z, u_1 \leq \dots \leq u_n), ((x_1, y_1), \dots, (x_m, y_m)) &\in (Z_n \times \Delta[n]) \times \mathcal{O}_1^\Xi(m) \\ \mapsto (z, v_1^1 \leq \dots \leq v_n^1), \dots, (z, v_1^m \leq \dots \leq v_n^m) &\in (Z_n \times \Delta[n])^m. \end{aligned}$$

To specify the v_j^i 's, we note that the linear function $f_{(x,y)}$ has a unique weakly increasing left inverse function $g_{(x,y)}: I \rightarrow I$ given by the formula

$$g_{(x,y)}(t) = \begin{cases} 0 & t < x \\ (t-x)/(y-x) & x \leq t \leq y \\ 1 & t > y. \end{cases}$$

We let $v_j^i = g_{(x_i, y_i)}(u_j)$.

This formula describes a continuous map

$$(Z_n \times \Delta[n]) \times \mathcal{O}_1^\Xi(m) \longrightarrow (Z_n \times \Delta[n])^m \longrightarrow |Z_\bullet|^m.$$

This map is compatible with the simplicial gluing instructions and so constructs a map

$$|Z_\bullet| \otimes \mathcal{O}_1^\Xi(m) \longrightarrow |Z_\bullet|^m$$

or equivalently a map $\mathcal{C}(m) \rightarrow \text{End}_{|Z_\bullet|}^{\text{op}}(m)$. A straightforward check of the explicit formulas shows that this is a map of operads.

For the composite coaction of \mathcal{C} , we are looking at the case

$$(x_1, y_1), \dots, (x_m, y_m) = (0, 1), \dots, (0, 1)$$

and $g_{(0,1)}$ is the identity endomorphism of I and so $v_j^i = u_j$. The coaction is therefore the set of diagonal maps.

For the composite coaction of \mathcal{C}_1 , we are looking at the case when the open intervals $(x_1, y_1), \dots, (x_m, y_m)$ are non-overlapping. In this case, for each fixed j , at most one v_j^i can lie in $(0, 1)$ (i.e., be different from 0 or 1). It follows that each $v_1^i \leq \dots \leq v_n^i$ lies in a n_i -face of $\Delta[n]$ for some $0 \leq n_i \leq n$ that can be chosen so that $n_1 + \dots + n_m \leq n$. Thus, the image of $(Z_n \times \Delta[n]) \times \mathcal{C}_1(m)$ in $(Z_n \times \Delta[n])^m$ lands in the n th filtration level. \square

11. Constructing the filtered model and verifying the hypotheses of Section 5

In this section, we construct the functor T^M postulated in Section 5 and establish the properties stated there. Specifically, we construct T^M as a lax monoidal functor (property (i)), filtered (property (ii)) so that the filtration on T is lax monoidal (property (iii)). Moreover, we show that T is naturally isomorphic to $(-)^{tG}$ (as functors to the stable category) via an isomorphism that takes the filtration on T^M to the Hesselholt-Madsen Tate filtration and satisfies Hypothesis 5.1.

As in Section 3, we construct the integer graded filtration out of a functor from the poset $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ to orthogonal spectra.

CONSTRUCTION 11.1. For $i, j \in \mathbb{N} \times \mathbb{N}$, let

$$\bar{T}_{i,j}X = (R_G(F(EG/EG_{j-1}, R_G(X)) \wedge \widetilde{E_{C_1}G_i}))^G$$

where R_G is a lax symmetric monoidal fibrant approximation functor, EG is the standard bar construction model, and \mathcal{C}_1 is the Boardman-Vogt little 1-cubes operad with $\widetilde{E_{C_1}G}$ as in Construction 2.2. The filtration on EG is the standard simplicial filtration when G is finite and twice that when $G = \mathbb{T}$, just as we used in Sections 3 and 10, and the filtration on $\widetilde{E_{C_1}G}$ is the renumbered pseudocellular filtration as defined in Definition 9.1.

Using the work of the previous two sections, $\bar{T}_{*,*}X$ comes with canonical maps

$$(11.2) \quad \bar{\mathcal{C}}_1(n)_+ \wedge (\bar{T}_{i_1,j_1}X_1 \wedge \cdots \wedge \bar{T}_{i_n,j_n}X_n) \longrightarrow \bar{T}_{i,j}(X_1 \wedge \cdots \wedge X_n)$$

for $i_k, j_k \in \mathbb{N} \times \mathbb{N}$ and $i = \sum i_k$, $j = \sum j_k$, using the maps

$$\begin{aligned} EG/EG_{j-1} \wedge \bar{\mathcal{C}}_1(n)_+ &\longrightarrow EG/EG_{j_1-1} \wedge \cdots \wedge EG/EG_{j_n-1}, \quad \text{and} \\ \bar{\mathcal{C}}_1(n)_+ \wedge (\widetilde{E_{C_1}G}_{i_1} \wedge \cdots \wedge \widetilde{E_{C_1}G}_{i_n}) &\longrightarrow \widetilde{E_{C_1}G}_i \end{aligned}$$

of the previous two sections. Moreover, these maps are consistent with the operadic multiplication in the obvious way. For the construction postulated in Section 5, we need to rectify the $\bar{\mathcal{C}}_1$ in the formulas above to $\overline{\mathcal{A}\text{ss}}$; the standard trick for doing this is the Moore construction.

CONSTRUCTION 11.3. Let $\bar{T}_{*,*}^{\bar{M}}X = (\bar{T}_{*,*}X) \wedge \mathbb{R}_+^{>0}$ where $\mathbb{R}_+^{>0}$ denotes the set of positive real numbers. Let $\mu_n: (\mathbb{R}^{>0})^n \rightarrow \bar{\mathcal{C}}_1(n)$ be the map that takes $\ell_1, \dots, \ell_n \in (\mathbb{R}^{>0})^n$ to the element of $\bar{\mathcal{C}}_1(n)$ consisting of the subintervals

$$[0, \frac{\ell_1}{\ell}], [\frac{\ell_1}{\ell}, \frac{\ell_1 + \ell_2}{\ell}], \dots, [\frac{\ell_1 + \cdots + \ell_{n-1}}{\ell}, 1]$$

where $\ell = \ell_1 + \cdots + \ell_n$. Let

$$\mu: \bar{T}_{i,j}^{\bar{M}}X \wedge \bar{T}_{i',j'}^{\bar{M}}Y \longrightarrow \bar{T}_{i+i',j+j'}^{\bar{M}}(X \wedge Y)$$

be the map

$$\bar{T}_{i,j}X \wedge \mathbb{R}_+^{>0} \wedge \bar{T}_{i',j'}Y \wedge \mathbb{R}_+^{>0} \longrightarrow \bar{T}_{i+i',j+j'}(X \wedge Y) \wedge \mathbb{R}_+^{>0}$$

obtained using the canonical map $\bar{T}_{i,j}X \wedge \bar{T}_{i',j'}Y \rightarrow \bar{T}_{i+i',j+j'}(X \wedge Y)$ above for $\mu_2: \mathbb{R}^{>0} \times \mathbb{R}^{>0} \rightarrow \bar{\mathcal{C}}_1(2)$ and the map $+: \mathbb{R}^{>0} \times \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$.

For example, for $\ell_1, \ell_2 \in \mathbb{R}^{>0}$, the restriction of the map above to

$$\bar{T}_{i,j}X \wedge \{\ell_1\}_+ \wedge \bar{T}_{i',j'}Y \wedge \{\ell_2\}_+$$

uses $\mu_2(\ell_1, \ell_2)$ for the map

$$\bar{T}_{i,j}X \wedge \bar{T}_{i',j'}Y \longrightarrow \bar{T}_{i+i',j+j'}(X \wedge Y)$$

and restricts to

$$\bar{T}_{i,j}X \wedge \{\ell_1\}_+ \wedge \bar{T}_{i',j'}Y \wedge \{\ell_2\}_+ \longrightarrow \bar{T}_{i+i',j+j'}(X \wedge Y) \wedge \{\ell_1 + \ell_2\}_+.$$

Since

$$\mu_2(\ell_1, \mu_2(\ell_2, \ell_3)) = \mu_3(\ell_1, \ell_2, \ell_3) = \mu_2(\mu_2(\ell_1, \ell_2), \ell_3)$$

and $+: \mathbb{R}^{>0} \times \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$ is associative, the diagram

$$\begin{array}{ccc} \bar{T}_{i,j}^{\bar{M}} X \wedge \bar{T}_{i',j'}^{\bar{M}} Y \wedge \bar{T}_{i'',j''}^{\bar{M}} Z & \xrightarrow{\text{id} \wedge \mu} & \bar{T}_{i,j}^{\bar{M}} X \wedge (\bar{T}_{i'+i'',j'+j''}^{\bar{M}} (Y \wedge Z)) \\ \mu \wedge \text{id} \downarrow & & \downarrow \mu \\ \bar{T}_{i+i',j+j'}^{\bar{M}} (X \wedge Y) \wedge \bar{T}_{i'',j''}^{\bar{M}} Z & \xrightarrow{\mu} & \bar{T}_{i+i'+i'',j+j'+j''}^{\bar{M}} (X \wedge Y \wedge Z) \end{array}$$

commutes for all orthogonal G -spectra X, Y, Z , and all $i, i', i'', j, j', j'' \geq 0$.

CONSTRUCTION 11.4. Define the filtered spectrum $T_*^{\bar{M}} X$ as

$$T_n^{\bar{M}} X = \text{hocolim}_{i-j \leq n} T_{i,j}^{\bar{M}} X$$

using the bar construction model for the homotopy colimit as in Section 3. Let $T^{\bar{M}} X = \text{colim} T_n^{\bar{M}} X$.

As in Proposition 3.7, $T_*^{\bar{M}} X$ is reasonably filtered. The maps

$$\bar{T}_{i,j}^{\bar{M}} X \longrightarrow (R_G(F(EG_+, R_G(X)) \wedge \widetilde{E_{C_1}G}))^G = T_G^{C_1} X$$

therefore induce a weak equivalence $T^{\bar{M}} X \rightarrow T_G X$. (See Construction 1.3 and Notation 2.6 for the definition of $T_G X$ and $T_G^{C_1}$, respectively.)

By construction $T_*^{\bar{M}}$ inherits an associative pairing

$$\mu: T_*^{\bar{M}} X \wedge T_*^{\bar{M}} Y \longrightarrow T_*^{\bar{M}} (X \wedge Y)$$

in the category of filtered spectra. To make this pairing unital as well, we need a slight modification that takes as input a unital spectrum.

CONSTRUCTION 11.5. For X an orthogonal G -spectrum under \mathbb{S} , we have a canonical map $\mathbb{S} \rightarrow \bar{T}_{0,0} X$, whence a map $\mathbb{S} \rightarrow \bar{T}_{i,0} X$ for $i \geq 0$. Let

$$\bar{T}_{i,j}^M X = \bar{T}_{i,j}^{\bar{M}} X = \bar{T}_{i,j} X \wedge \mathbb{R}_+^{>0}$$

for $j > 0$ and define $\bar{T}_{i,0}^M X$ as the pushout

$$\begin{array}{ccc} \mathbb{S} \wedge \mathbb{R}_+^{>0} & \longrightarrow & \bar{T}_{i,0} X \wedge \mathbb{R}_+^{>0} \\ \downarrow & & \downarrow \\ \mathbb{S} \wedge \mathbb{R}_+^{\geq 0} & \longrightarrow & \bar{T}_{i,0}^M X \end{array}$$

where $\mathbb{R}^{\geq 0}$ denotes the space of non-negative real numbers. Define $T_*^M X$ by $T_n^M X = \text{hocolim}_{i-j \leq n} \bar{T}_{i,j}^M X$ and let $T^M X = \text{colim} T_n^M X$.

The lax associative pairing on $\bar{T}_{*,*}^{\bar{M}}$ extends to $\bar{T}_{*,*}^M$ where it is now lax monoidal using the canonical map

$$\mathbb{S} \cong \mathbb{S} \wedge \{0\}_+ \longrightarrow \bar{T}_{0,0}^{\bar{M}} (\mathbb{S})$$

as unit. We then get a lax symmetric monoidal structure on the functor T_*^M . This gives the first part of the following theorem, which shows that T_*^M satisfies the hypotheses required for the argument of Section 5.

THEOREM 11.6. T_*^M is a lax monoidal functor from orthogonal G -spectra under \mathbb{S} to reasonably filtered spectra, is naturally isomorphic to the Tate functor in the homotopy category of filtered spectra, and satisfies Hypothesis 5.1.

PROOF. We constructed the lax symmetric monoidal structure above. For the comparison we note that the inclusion of \mathbb{S} in $T_{0,0}X$ and the collapse map $\mathbb{R} \rightarrow *$ induce a natural transformation

$$\bar{T}_{i,j}^M X \longrightarrow \bar{T}_{i,j} X$$

of functors from orthogonal G -spectra to $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ -diagrams in orthogonal spectra. This map is a weak equivalence for all X ; see [25, 6.2] (or Proposition 19.2 below).

To verify Hypothesis 5.1, we let

$$TT(X) = \bar{T}_{0,0}^M X \cup_{\bar{T}_{0,1}^M X} (\bar{T}_{0,1}^M X \wedge I),$$

the standard model of the homotopy cofiber (where we use 1 as the basepoint of I). This functor has a monoidal structure coming from the unit of $T_{0,0}^M$ and the pairing on $T_{*,*}^M$ and the pairing max on I . This is canonically isomorphic as a monoidal functor (reversing the direction of the interval) to the quotient of the categorical bar construction homotopy colimit for the diagram

$$\bar{T}_{0,1}^M X \longrightarrow \bar{T}_{0,0}^M X$$

by the inclusion of $T_{0,1}^M X$. We then have a natural transformation of monoidal functors $TT \rightarrow (T_0^M / T_{-1}^M)$ induced by the inclusion in the diagram of $\bar{T}_{0,0}^M$ and $\bar{T}_{0,1}^M$.

We define

$$RR(X) = (R_G(F((EG_0)_+, R_G X)))^G.$$

We then have the monoidal natural transformation

$$i^* X \longrightarrow i^* R_G R_G X \longrightarrow (R_G(F((EG_0)_+, R_G X)))^G = RR(X).$$

To construct the map $TT(X) \rightarrow RR(X)$, we use the map

$$\begin{aligned} \bar{T}_{0,0}^M X \longrightarrow \bar{T}_{0,0} X &= (R_G(F(EG_+, R_G X) \wedge \mathbb{S}))^G \\ &\longrightarrow (R_G(F((EG_0)_+, R_G X) \wedge \mathbb{S}))^G \cong RR(X) \end{aligned}$$

Since

$$F(EG/EG_0, R_G X) \longrightarrow F(EG_+, R_G X) \longrightarrow F(EG_{0+}, R_G X)$$

is a point-set fiber sequence, we see that the map $TT(X) \rightarrow RR(X)$ is a weak equivalence. The composite map $\bar{T}_{0,1}^M X \rightarrow RR(X)$ factors through

$$(R_G(F(EG/EG_0, R_G X) \wedge \mathbb{S}))^G,$$

and therefore is the point-set trivial map. As a consequence, the natural transformation $TT \rightarrow RR$ factors through the monoidal natural transformation $TT \rightarrow \bar{T}_{0,0}^M / \bar{T}_{0,1}^M$. Thus, it remains to show that the transformation $T_{0,0}^M \rightarrow RR$ is monoidal, but this is clear because the coaction of \mathcal{O}_1^{Ξ} on the zeroth filtration level of EG is the diagonal. \square

12. The E^1 -term of the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence

In this section we prove Lemma 3.16 and Theorems 3.17 and 3.19. All three results require a careful study of the filtration, and this is where we begin. In this section, $E_{*,*}^1(X)$ will always refer to the E^1 -term of the Hesselholt-Madsen Tate spectral sequence for an orthogonal \mathbb{T} -spectrum X .

Our calculation of the E^1 -term will use a filtration on $X^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}}$ corresponding to the Hesselholt-Madsen Tate filtration before taking \mathbb{T} -fixed points. Let

$$\bar{U}X_{i,j} = F(E\mathbb{T}/E\mathbb{T}_{j-1}, R_{\mathbb{T}}X) \wedge \widetilde{E\mathbb{T}}_i$$

and let

$$\bar{U}_n X = \operatorname{hocolim}_{i-j \leq n} UX_{i,j}, \quad \bar{\bar{U}}_n X = \operatorname{hocolim}_{i-j \leq n} R_{\mathbb{T}} UX_{i,j}.$$

Because the point-set fixed point functor commutes with the categorical bar construction of homotopy colimits and the derived fixed point functor commutes with homotopy colimits, we have the following proposition. (See (3.5) for the notation $\bar{T}_n X$.)

PROPOSITION 12.1. *The canonical map $(\bar{\bar{U}}_n X)^{\mathbb{T}} \rightarrow \bar{T}_n X$ is an isomorphism and the canonical map $(\bar{\bar{U}}_n X)^{\mathbb{T}} \rightarrow (R_{\mathbb{T}}(\bar{U}_n X))^{\mathbb{T}}$ is a weak equivalence.*

This proposition then gives a canonical identification of the E^1 -term of the Hesselholt-Madsen Tate spectral sequence for X as

$$E_{s,t}^1(X) = \pi_{s+t}^{\mathbb{T}}(\bar{U}_s X / \bar{U}_{s-1} X),$$

with pairing induced by the pairing on $\bar{U}_*(-)$ (itself induced by the pairing on $\bar{U}(-)_{*,*}$). We can now dispense with $\bar{T}_n X$, $T_{\mathbb{T}} X_{i,j}$, and $\bar{\bar{U}}_n X$ and work exclusively with the equivariant \mathbb{T} -spectra $\bar{U}_*(-)$ and $\bar{U}(-)_{*,*}$.

To study the filtration quotients $\bar{U}_i X / \bar{U}_{i-1} X$, we need another homotopy colimit. Let

$$\bar{U}X_{m,n}^{\leq, \geq} = \operatorname{hocolim}_{i \leq m, j \geq n} \bar{U}X_{i,j}.$$

Since (m, n) is the final object in the category in the homotopy colimit, the inclusion $\bar{U}X_{m,n} \rightarrow \bar{U}X_{m,n}^{\leq, \geq}$ is a homotopy equivalence of orthogonal \mathbb{T} -spectra. It is easy to see that for $n \geq 0$

$$\bar{U}_n X = (\bar{U}X_{n,0}^{\leq, \geq}) \cup_{(\bar{U}X_{n,1}^{\leq, \geq})} (\bar{U}X_{n+1,1}^{\leq, \geq}) \cup_{(\bar{U}X_{n+1,2}^{\leq, \geq})} (\bar{U}X_{n+2,2}^{\leq, \geq}) \cup \cdots$$

and for $n \leq 0$

$$\bar{U}_n X = (\bar{U}X_{0,-n}^{\leq, \geq}) \cup_{(\bar{U}X_{0,-n+1}^{\leq, \geq})} (\bar{U}X_{1,-n+1}^{\leq, \geq}) \cup_{(\bar{U}X_{1,-n+2}^{\leq, \geq})} (\bar{U}X_{2,-n+2}^{\leq, \geq}) \cup \cdots.$$

This gives us what we need to prove Lemma 3.16, conditional convergence of the Hesselholt-Madsen Tate spectral sequence.

PROOF OF LEMMA 3.16. Using the weak equivalence

$$\begin{aligned} \bar{U}X_{i,j} &= F(E\mathbb{T}/E\mathbb{T}_{j-1}, R_{\mathbb{T}}X) \wedge \widetilde{E\mathbb{T}}_i \\ &\simeq (X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{-j\mathbb{C}(1)} \wedge^{\mathbf{L}} S^{i\mathbb{C}(1)}) \simeq (X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{(i-j)\mathbb{C}(1)}), \end{aligned}$$

we see in particular that $\bar{U}_n X$ is weakly equivalent to the cofiber of a map

$$\bigvee_{\ell \geq 0} X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{(n-1)\mathbb{C}(1)} \longrightarrow \bigvee_{\ell \geq 0} X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{n\mathbb{C}(1)}$$

where the map on the ℓ th summand is the Greenlees Tate inclusion into the ℓ th summand minus the Greenlees Tate inclusion into the $(\ell + 1)$ st summand. Conditional convergence in the sense of [7, 5.10] is equivalent to showing that

$\text{holim}_n(R_{\mathbb{T}}\bar{U}_n X)^{\mathbb{T}} \simeq *$. Because of the identification of $\bar{U}_n X$ as a homotopy cofiber above, it suffices to show

$$\text{holim}_n \left(\bigvee (X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{-n\mathbb{C}(1)})^{\mathbb{T}} \right) \simeq *$$

or, equivalently,

$$\text{holim}_n \left(\bigvee ((X^{E\mathbb{T}})^{E\mathbb{T}/E\mathbb{T}_{2n-1}})^{\mathbb{T}} \right) \simeq *$$

This is clear when $\pi_* X^{E\mathbb{T}}$ is bounded above and follows in the general by considering the Postnikov tower for $X^{E\mathbb{T}}$. \square

Returning to the discussion of the quotients $\bar{U}_s X / \bar{U}_{s-1} X$, the work above allows us to identify these in terms of the double homotopy cofibers of maps for the $\bar{U} X_{i,j}$. Writing $C(B, A)$ for the homotopy cofiber of an understood map $A \rightarrow B$, we have that for $s > 0$, $\bar{U}_s X / \bar{U}_{s-1} X$ is weakly equivalent to the cofiber of the inclusion

$$\begin{aligned} & (\bar{U} X_{s-1,0}^{\leq, \geq}) \cup_{(\bar{U} X_{s-1,1}^{\leq, \geq})} (\bar{U} X_{s,1}^{\leq, \geq}) \cup_{(\bar{U} X_{s,2}^{\leq, \geq})} (\bar{U} X_{s+1,2}^{\leq, \geq}) \cup \cdots \\ & \longrightarrow \bar{U}_s X = (\bar{U} X_{s,0}^{\leq, \geq}) \cup_{(\bar{U} X_{s,1}^{\leq, \geq})} (\bar{U} X_{s+1,1}^{\leq, \geq}) \cup_{(\bar{U} X_{s+1,2}^{\leq, \geq})} (\bar{U} X_{s+2,2}^{\leq, \geq}) \cup \cdots \end{aligned}$$

which is easily seen to be the wedge of double homotopy cofibers

$$\bigvee_{n \geq 0} C(C(\bar{U} X_{s+n,n}, \bar{U} X_{s+n-1,n}), C(\bar{U} X_{s+n,n+1}, \bar{U} X_{s+n-1,n+1})).$$

Similarly, for $s \leq 0$, $\bar{U}_s X / \bar{U}_{s-1} X$ is weakly equivalent to the wedge

$$\begin{aligned} & C(\bar{U} X_{0,-s}, \bar{U} X_{0,-s+1}) \vee \\ & \bigvee_{n > 0} C(C(\bar{U} X_{n,n-s}, \bar{U} X_{n-1,n-s}), C(\bar{U} X_{n,n-s+1}, \bar{U} X_{n-1,n-s+1})). \end{aligned}$$

(The difference in formulas arises because for $s > 0$, $\bar{U} X_{s,0}$ admits maps from $\bar{U} X_{s-1,0}$ and $\bar{U} X_{s,1}$, whereas for $s \leq 0$, $\bar{U} X_{0,-s}$ only admits a map from $\bar{U} X_{0,-s+1}$.)

For convenience of later reference, we will use the following abbreviations

$$\begin{aligned} (12.2) \quad & C\bar{U} X_n = C(\bar{U} X_{0,n}, \bar{U} X_{0,n+1}) \\ & CC\bar{U} X_{m,n} = C(C(\bar{U} X_{m,n}, \bar{U} X_{m-1,n}), C(\bar{U} X_{m,n+1}, \bar{U} X_{m-1,n+1})). \end{aligned}$$

We note that since the pairing on $\bar{U}_*(-)$ is induced by the pairing on $\bar{U}(-)_{*,*}$, the pairings on the associated graded is the wedge of pairings on the pieces

$$\begin{aligned} (12.3) \quad & CC\bar{U} X_{m,n} \wedge CC\bar{U} Y_{m',n'} \longrightarrow CC\bar{U}(X \wedge Y)_{m+m',n+n'} \\ & C\bar{U} X_n \wedge CC\bar{U} Y_{m',n'} \longrightarrow CC\bar{U}(X \wedge Y)_{m',n+n'} \\ & CC\bar{U} X_{m,n} \wedge C\bar{U} Y_{n'} \longrightarrow CC\bar{U}(X \wedge Y)_{m,n+n'} \\ & C\bar{U} X_n \wedge C\bar{U} Y_{n'} \longrightarrow C\bar{U}(X \wedge Y)_{n+n'}. \end{aligned}$$

We proceed by studying these pairings; results translate back to E^1 -terms using the identification

$$(12.4) \quad \bigoplus_{s,t} E_{s,t}^1(X) = \bigoplus_{n \geq 0} \pi_q^{\mathbb{T}}(C\bar{U} X_n) \oplus \bigoplus_{m > 0, n \geq 0} \pi_q^{\mathbb{T}}(CC\bar{U} X_{m,n})$$

where the n, q summand on the right lies in the $s = n, t = q + n$ summand on the left and the m, n, q summand on the right lies in the $s = m - n, t = q - m + n$ summand on the left.

We have canonical identifications of the homotopy cofibers

$$\begin{aligned} C(\bar{U}X_{m,n}, \bar{U}X_{m-1,n}) &\simeq F(E\mathbb{T}/E\mathbb{T}_{\lfloor(n-1)/2\rfloor}, R_{\mathbb{T}}X) \wedge (S^{\mathbb{C}(1)^{\lfloor m/2\rfloor}}/S^{\mathbb{C}(1)^{\lfloor(m-1)/2\rfloor}}) \\ C(\bar{U}X_{m,n}, \bar{U}X_{m,n+1}) &\simeq F(E\mathbb{T}_{\lfloor n/2\rfloor}/E\mathbb{T}_{\lfloor(n-1)/2\rfloor}, R_{\mathbb{T}}X) \wedge S^{\mathbb{C}(1)^{\lfloor m/2\rfloor}}; \end{aligned}$$

the first cofiber is trivial if m is odd and the second is trivial if n is odd. It follows that the double homotopy cofibers are trivial if either m or n is odd, and when both are even, we have

$$\begin{aligned} CC\bar{U}X_{2m,2n} &= C(C(\bar{U}X_{2m,2n}, \bar{U}X_{2m-1,2n}), C(\bar{U}X_{2m,2n+1}, \bar{U}X_{2m-1,2n+1})) \\ &\simeq F(E\mathbb{T}_{2n}/E\mathbb{T}_{2n-1}, R_{\mathbb{T}}X) \wedge (S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}}), \end{aligned}$$

Since $E\mathbb{T}_{2n}/E\mathbb{T}_{2n-1}$ and $S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}}$ are all of the form $\mathbb{T}_+ \wedge Z$ for some based \mathbb{T} -spaces Z (depending on m or n), the associated graded pieces $\bar{U}_sX/\bar{U}_{s-1}X$ are all (non-canonically) of the form $\mathbb{T}_+ \wedge Z$ for some spectra Z . The next step is to review the relationship between the homotopy groups π_* and $\pi_*^{\mathbb{T}}$ of such spectra.

Writing $\pi_*^{\mathbb{S}}(\mathbb{T})$ for the stable homotopy groups of \mathbb{T}_+ , the inclusion of the unit $S^0 \rightarrow \mathbb{T}_+$ and the collapse map $\mathbb{T}_+ \rightarrow S^0$ induce a canonical splitting

$$\pi_*^{\mathbb{S}}(\mathbb{T}) \cong \pi_*\mathbb{S} \oplus \pi_*(\Sigma\mathbb{S}).$$

We denote the canonical generator in $\pi_0^{\mathbb{S}}(\mathbb{T})$ as 1 and the generator in $\pi_1^{\mathbb{S}}(\mathbb{T})$ (determined by the orientation on \mathbb{T} as the unit complex numbers) as ζ ; the multiplication on \mathbb{T} makes $\pi_*^{\mathbb{S}}(\mathbb{T})$ a graded ring, and 1 is the unit in this structure. It is well-known (and essentially the definition of the Hopf invariant) that the element ζ satisfies $\zeta^2 = \eta\zeta$ where η denotes the non-zero element of $\pi_1\mathbb{S} \cong \mathbb{Z}/2$. For any orthogonal \mathbb{T} -spectrum X , we then have a canonical action of $\pi_*^{\mathbb{S}}(\mathbb{T})$ on π_*X , or equivalently, a degree 1 operator ζ satisfying $\zeta^2x = \eta\zeta x$.

PROPOSITION 12.5. *Let X be an orthogonal \mathbb{T} -spectrum which is isomorphic in the equivariant stable category to $\mathbb{T}_+ \wedge Z$ for some orthogonal \mathbb{T} -spectrum Z . Then the map $\pi_*^{\mathbb{T}}X \rightarrow \pi_*X$ is injective and is surjective onto the kernel of multiplication by ζ , which is equal to the image of multiplication by $\zeta + \eta$.*

PROOF. The image of $\pi_*^{\mathbb{T}}X$ in π_*X is clearly contained in the kernel of multiplication by ζ . The kernel of multiplication by ζ clearly contains the image of multiplication by $\zeta + \eta$; an easy calculation shows that they are equal under the hypotheses. It suffices to consider the case when Z comes from a non-equivariant orthogonal spectrum (using the isomorphism between the diagonal action and the action on just \mathbb{T}). In the case $Z = \mathbb{S}$, the transfer gives a weak equivalence $\pi_*\Sigma\mathbb{S} \rightarrow (\mathbb{T}_+ \wedge \mathbb{S})^{\mathbb{T}}$ and the composite

$$\pi_*\Sigma\mathbb{S} \longrightarrow \pi_*(\mathbb{T}_+ \wedge \mathbb{S})^{\mathbb{T}} \longrightarrow \pi_*(\mathbb{T}_+ \wedge \mathbb{S}) \cong \pi_*^{\mathbb{S}}(\mathbb{T})$$

takes the fundamental class of $\pi_1\Sigma\mathbb{S}$ to the element $\zeta + \eta 1$; the theorem follows in this case. For arbitrary Z , naturality of the transfer map implies that

$$\pi_*\Sigma Z \longrightarrow \pi_*(\mathbb{T}_+ \wedge Z)^{\mathbb{T}} \longrightarrow \pi_*(\mathbb{T}_+ \wedge Z) \cong \pi_*^{\mathbb{S}}(\mathbb{T}) \otimes_{\pi_*\mathbb{S}} \pi_*Z$$

takes an element $x \in \pi_nZ \cong \pi_{n+1}\Sigma Z$ to $(\zeta + \eta 1) \otimes x = (\zeta + \eta)(1 \otimes x)$. \square

Now we are ready to start identifying elements in $\pi_*(C\bar{U}\mathbb{S}_n)$ and $\pi_*(CC\bar{U}\mathbb{S}_{m,n})$. We can then apply Proposition 12.5 (and (12.4)) to compute $E_{*,*}^1(\mathbb{S})$ and with some

more work, $E_{*,*}^1(X)$ for general X . Using the canonical orientation on $S^{\mathbb{C}(1)^m}$, we get canonical generators

$$a_m \in \pi_{2m-1}\Sigma^\infty(S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}}), \quad b_m \in \pi_{2m}\Sigma^\infty(S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}})$$

where b_m is the image of the fundamental class of $S^{\mathbb{C}(1)^m}$ and a_m maps to the fundamental class in $S^{\mathbb{C}(1)^{m-1}}$; $\pi_*\Sigma^\infty(S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}})$ is then the free $\pi_*\mathbb{S}$ -module generated by a_m and b_m . In terms of the action of $\pi_*^S(\mathbb{T})$, it is easy to see geometrically that $\zeta a_1 = b_1$. Using the cofiber sequence

$$\mathbb{T}_+ \longrightarrow S^0 \longrightarrow S^{\mathbb{C}(1)} \longrightarrow S^{\mathbb{C}(1)}/S^0 \cong \Sigma\mathbb{T}_+$$

we see that $\zeta b_1 = \eta b_1$; the identification

$$S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}} \cong (S^{\mathbb{C}(1)})^{(m-1)} \wedge (S^{\mathbb{C}(1)}/S^0)$$

shows that in general

$$(12.6) \quad \begin{aligned} \zeta a_m &= b_m + (m-1)\eta a_m \\ \zeta b_m &= m\eta b_m. \end{aligned}$$

To produce corresponding formulas for $F(E\mathbb{T}_{2n}/E\mathbb{T}_{2n-1}, R_{\mathbb{T}}\mathbb{S})$, we use the following lemma proved in the next section.

LEMMA 12.7. *There exists a unique system of isomorphisms in the Borel stable category $E\mathbb{T}/E\mathbb{T}_{2n-1} \xrightarrow{\simeq} E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}$ such that:*

(i) *The diagram*

$$\begin{array}{ccc} E\mathbb{T}/E\mathbb{T}_{2n-1} & \longrightarrow & E\mathbb{T}/E\mathbb{T}_{2i-1} \wedge E\mathbb{T}/E\mathbb{T}_{2j-1} \\ \simeq \downarrow & & \downarrow \simeq \\ E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n} & \longrightarrow & (E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^i}) \wedge (E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^j}) \end{array}$$

commutes for all $i + j = n$, where the top horizontal map is induced by the filtered approximation to the diagonal on $E\mathbb{T}$ constructed in Section 9 and the bottom horizontal map is induced by the diagonal map on $E\mathbb{T}$ and the homeomorphism

$$S^{\mathbb{C}(1)^n} = S^{\mathbb{C}(1)^{i+j}} \cong S^{\mathbb{C}(1)^i \oplus \mathbb{C}(1)^j} \cong S^{\mathbb{C}(1)^i} \wedge S^{\mathbb{C}(1)^j}.$$

(ii) *The diagram*

$$\begin{array}{ccc} E\mathbb{T}/E\mathbb{T}_{2m-1} & \longrightarrow & E\mathbb{T}/E\mathbb{T}_{2m} \\ \simeq \downarrow & & \downarrow \simeq \\ E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^m} & \longrightarrow & E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^{m+1}} \end{array}$$

commutes for $m = 0$, where the bottom map is induced by the inclusion of $\mathbb{C}(1)^m$ in $\mathbb{C}(1)^{m+1}$ as the first m coordinates.

Moreover, the diagram in (ii) commutes for all $m \geq 0$.

The lemma gives us a canonical weak equivalence

$$F(E\mathbb{T}/E\mathbb{T}_{2m-1}, R_{\mathbb{T}}X) \simeq X^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{-m\mathbb{C}(1)},$$

and we then define generators a'_m, b'_m for $\pi_*F(E\mathbb{T}_{2m}/E\mathbb{T}_{2m-1}, R_{\mathbb{T}}\mathbb{S})$ as above. Namely, b'_m is the generator of π_{-2m} in the image of the fundamental class of

$$\pi_{-2m}F(E\mathbb{T}/E\mathbb{T}_{2m-1}, R_{\mathbb{T}}\mathbb{S}) \cong \pi_{-2m}(\mathbb{S}^{E\mathbb{T}} \wedge^{\mathbf{L}} S^{-m\mathbb{C}(1)}) \cong \pi_{-2m}S^{-m\mathbb{C}(1)}$$

(for the standard orientation) and a'_m is the generator of π_{-2m-1} that maps to the fundamental class of

$$\pi_{-2m-2}F(E\mathbb{T}/E\mathbb{T}_{2m}R_{\mathbb{T}}\mathbb{S}) = \pi_{-2m-2}F(E\mathbb{T}/E\mathbb{T}_{2m+1}R_{\mathbb{T}}\mathbb{S}) \cong \pi_{-2m-2}S^{-(m+1)\mathbb{C}(1)}.$$

As above, we have

$$(12.8) \quad \begin{aligned} \zeta a'_m &= b'_m + (-m-1)\eta a'_m = b'_m + (m-1)\eta a'_m \\ \zeta b'_m &= -m\eta b'_m = m\eta b'_m. \end{aligned}$$

Using the elements introduced above, we can identify the homotopy groups of $CC\bar{U}X_{2m,2n}$ as

$$\pi_*(CC\bar{U}X_{2m,2n}) \cong \pi_*\mathbb{S}\langle a'_n, b'_n \rangle \otimes_{\pi_*\mathbb{S}} \pi_*\mathbb{S}\langle a_m, b_m \rangle \otimes_{\pi_*\mathbb{S}} \pi_*X$$

using the map

$$\begin{aligned} \pi_*(F(E\mathbb{T}_{2n}/E\mathbb{T}_{2n-1}, R_{\mathbb{T}}\mathbb{S})) \otimes_{\pi_*\mathbb{S}} \pi_*\Sigma^\infty(S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}}) \otimes_{\pi_*\mathbb{S}} \pi_*X \\ \longrightarrow \pi_*(F(E\mathbb{T}_{2n}/E\mathbb{T}_{2n-1}, R_{\mathbb{T}}\mathbb{S}) \wedge (S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}}) \wedge X) \\ \longrightarrow \pi_*(F(E\mathbb{T}_{2n}/E\mathbb{T}_{2n-1}, R_{\mathbb{T}}X) \wedge (S^{\mathbb{C}(1)^m}/S^{\mathbb{C}(1)^{m-1}})) = \pi_*(CC\bar{U}X_{2m,2n}), \end{aligned}$$

where $\pi_*\mathbb{S}\langle - \rangle$ denotes the free $\pi_*\mathbb{S}$ -module on the given generators (in their natural degrees). We likewise have

$$\pi_*(C\bar{U}X_{2n}) \cong \pi_*\mathbb{S}\langle a'_n, b'_n \rangle \otimes_{\pi_*\mathbb{S}} \pi_*X.$$

We next describe the pairing for $X = \mathbb{S}$. The induced pairing on homotopy groups of the pairing in (12.3) in this case takes the form

$$\begin{aligned} (\pi_*\mathbb{S}\langle a'_n, b'_n \rangle \otimes_{\pi_*\mathbb{S}} \pi_*\mathbb{S}\langle a_m, b_m \rangle) \otimes_{\pi_*\mathbb{S}} (\pi_*\mathbb{S}\langle a'_{n'}, b'_{n'} \rangle \otimes_{\pi_*\mathbb{S}} \pi_*\mathbb{S}\langle a_{m'}, b_{m'} \rangle) \\ \longrightarrow \pi_*\mathbb{S}\langle a'_{n+n'}, b'_{n+n'} \rangle \otimes_{\pi_*\mathbb{S}} \pi_*\mathbb{S}\langle a_{m+m'}, b_{m+m'} \rangle. \end{aligned}$$

the analogous formulas for the other three pairings, dropping either $\langle a_m, b_m \rangle$, $\langle a_{m'}, b_{m'} \rangle$, or both tensor factors also hold.

Looking at the pairing on $\bar{U}\mathbb{S}_{m,n}$, it is clear that the pairing on the $F(E\mathbb{T}/E\mathbb{T}_{2n-1}, R_{\mathbb{T}}\mathbb{S})$ -part and on the $S^{m\mathbb{C}(1)}$ -part are independent of each other; Lemma 12.7 describes what happens on the former, and on the latter it is given by the usual isomorphism

$$S^{m\mathbb{C}(1)} \wedge S^{m'\mathbb{C}(1)} \simeq S^{(m+m')\mathbb{C}(1)}.$$

This implies that for the pairings above we have $b'_n \otimes b'_{n'} \mapsto b'_{n+n'}$ and $b_m \otimes b_{m'} \mapsto b_{m+m'}$. In addition, we see that $a'_n \otimes a'_{n'} \mapsto 0$ and $a_m \otimes a_{m'} \mapsto 0$ for dimension reasons. Since the pairings on the spectrum level are \mathbb{T} -equivariant, the algebraic maps above commute with the action of ζ . Then

$\zeta(a_m \otimes a_{m'}) = (\zeta a_m) \otimes a_{m'} - a_m \otimes (\zeta a_{m'}) = b_m \otimes a_{m'} - a_m \otimes b_{m'} + (m-m')\eta a_m \otimes a_{m'}$ goes to 0 since $a_m \otimes a_{m'} \mapsto 0$, and we see that $b_m \otimes a_{m'}$ and $a_m \otimes b_{m'}$ go to the same element of $\pi_*\mathbb{S}\langle a_{m+m'}, b_{m+m'} \rangle$. For dimension reasons, these elements go to an integral multiple of $a_{m+m'}$; since

$$\begin{aligned} \zeta(b_m \otimes a_{m'}) &= (\zeta b_m) \otimes a_{m'} + b_m \otimes (\zeta a_{m'}) \\ &= m\eta b_m \otimes a_{m'} + b_m \otimes b_{m'} + (m'-1)\eta b_m \otimes a_{m'} \\ &= b_m \otimes b_{m'} + (m+m'-1)\eta b_m \otimes a_{m'}, \end{aligned}$$

we see $b_m \otimes a_{m'} \mapsto a_{m+m'}$. The same observations apply to $b'_n \otimes a'_{n'}$ and $a'_n \otimes b'_{n'}$ to show that these both map to $a'_{n+n'}$. We summarize this in the following proposition.

PROPOSITION 12.9. *The pairings of homotopy groups π_* from (12.3) are induced by the tensor products of the maps*

$$\begin{array}{ll} b'_n \otimes b'_{n'} \mapsto b'_{n+n'} & b_m \otimes b_{m'} \mapsto b_{m+m'} \\ b'_n \otimes a'_{n'} \mapsto a'_{n+n'} & b_m \otimes a_{m'} \mapsto a_{m+m'} \\ a'_n \otimes b'_{n'} \mapsto a'_{n+n'} & a_m \otimes b_{m'} \mapsto a_{m+m'} \\ a'_n \otimes a'_{n'} \mapsto 0 & a_m \otimes a_{m'} \mapsto 0. \end{array}$$

We see from the formulas above that $b_0 \in \pi_0(C\bar{U}\mathbb{S}_0)$ acts by the identity and satisfies $\zeta b_0 = 0$, and so gives the identity element of $E_{0,0}^1(\mathbb{S})$. We now identify some other key elements of $\pi_*(\bar{U}\mathbb{S}_s/\bar{U}\mathbb{S}_{s-1})$.

NOTATION 12.10. Let

$$\begin{array}{ll} \bar{x} = b'_0 \otimes b_1 + \eta b'_0 \otimes a_1, & \bar{x} \in \pi_2(CC\bar{U}\mathbb{S}_{2,0}) \\ \bar{y} = a'_0 \otimes b_1 - b'_0 \otimes a_1 & \bar{y} \in \pi_1(CC\bar{U}\mathbb{S}_{2,0}) \\ \bar{z} = b'_1 + \eta a'_1, & \bar{z} \in \pi_{-2}(C\bar{U}\mathbb{S}_2). \end{array}$$

Applying the formulas (12.6) and (12.8) for the action of ζ and using Proposition 12.5 to identify $\pi_*^{\mathbb{T}}$ as a subset of π_* for $CC\bar{U}\mathbb{S}_{2,0}$ and $C\bar{U}\mathbb{S}_2$, we define elements x, y, z in $E_{*,*}^1(\mathbb{S})$ as follows.

PROPOSITION 12.11. *The elements \bar{x} , \bar{y} , and \bar{z} lift to (unique) elements*

$$\begin{array}{l} x \in \pi_2^{\mathbb{T}}(CC\bar{U}\mathbb{S}_{2,0}) \subset E_{2,0}^1(\mathbb{S}) \\ y \in \pi_1^{\mathbb{T}}(CC\bar{U}\mathbb{S}_{2,0}) \subset E_{2,-1}^1(\mathbb{S}) \\ z \in \pi_{-2}^{\mathbb{T}}(C\bar{U}\mathbb{S}_2) \subset E_{-2,0}^1(\mathbb{S}). \end{array}$$

We can now prove Theorems 3.17 and 3.19

PROOF OF THEOREM 3.17. Eliding notation for the distinction between an element of $\pi_*^{\mathbb{T}}$ and its image in π_* , the multiplication formulas of Proposition 12.9 give us

$$\begin{array}{l} x^m = b'_0 \otimes b_m + m\eta b'_0 \otimes a_m \\ y^2 = 0 \\ z^n = b'_n + n\eta a'_n. \end{array}$$

From here we see

$$\begin{array}{l} x^m z^n = b'_n \otimes b_m + n\eta a'_n \otimes b_m + m\eta b'_n \otimes a_m + mn\eta^2 a'_n \otimes a_m \\ x^m y = a'_0 \otimes b_{m+1} - b'_0 \otimes a_{m+1} + m\eta a'_0 \otimes a_{m+1} \\ x^m y z^n = a'_n \otimes b_{m+1} - b'_n \otimes a_{m+1} + (n+m)\eta a'_n \otimes a_{m+1} \\ y z^n = a'_n \otimes b_1 - b'_n \otimes a_1 + n\eta a'_n \otimes a_1 \end{array}$$

for $m > 0, n \geq 0$. Since $\{a'_n \otimes a_m, a'_n \otimes b_m, b'_n \otimes a_m, b'_n \otimes b_m\}$ is a set of generators for the free $\pi_*\mathbb{S}$ -module $\pi_*(CC\bar{U}\mathbb{S}_{2m,2n})$ and $\{a'_n, b'_n\}$ is a set of generators for the free $\pi_*\mathbb{S}$ -module $\pi_*(C\bar{U}\mathbb{S})_{2n}$, we see from the formulas above that the map from

the free (tri)graded commutative $\pi_*\mathbb{S}$ -algebra on $\bar{x}, \bar{y}, \bar{z}$ (in the appropriate degrees; q.v. 12.10) modulo $\bar{y}^2 = 0$ injects into

$$\bigoplus_{n \geq 0} \pi_*(C\bar{U}\mathbb{S}_{2n}) \oplus \bigoplus_{m > 0, n \geq 0} \pi_*(CC\bar{U}\mathbb{S}_{2m, 2n})$$

and it follows that it injects into

$$\bigoplus_{n \geq 0} \pi_*^{\mathbb{T}}(C\bar{U}\mathbb{S}_{2n}) \oplus \bigoplus_{m > 0, n \geq 0} \pi_*^{\mathbb{T}}(CC\bar{U}\mathbb{S}_{2m, 2n}).$$

To see it is surjective, we note that by Proposition 12.9, the elements

$$\begin{aligned} (\zeta + \eta)(b'_n \otimes b_m) &= (m + n + 1)\eta b'_n \otimes b_m & (\zeta + \eta)b'_n &= (n + 1)\eta b'_n \\ (\zeta + \eta)(b'_n \otimes a_m) &= b'_n \otimes b_m + (n + m)\eta b'_n \otimes a_m & (\zeta + \eta)a'_n &= b'_n + n\eta a'_n \\ (\zeta + \eta)(a'_n \otimes b_m) &= b'_n \otimes b_m + (n + m)\eta a'_n \otimes b_m \\ (\zeta + \eta)(a'_n \otimes a_m) &= b'_n \otimes a_m - a'_n \otimes b_m + (n + m + 1)\eta a'_n \otimes a_m \end{aligned}$$

generate $\pi_*^{\mathbb{T}}(CC\bar{U}\mathbb{S}_{2m, 2n})$ (in the left column) and $\pi_*^{\mathbb{T}}(C\bar{U}\mathbb{S}_{2n})$ (in the right column) as $\pi_*\mathbb{S}$ -modules. From this we see that

- $x^m z^n, x^{m-1} y z^n$ generate $\pi_*^{\mathbb{T}}(CC\bar{U}\mathbb{S})_{2m, 2n}$ for $m \geq 1, n \geq 0$
- z^n generates $\pi_*^{\mathbb{T}}(C\bar{U}\mathbb{S}_{2n})$ for $n \geq 0$.

The theorem now follows from (12.4). \square

PROOF OF THEOREM 3.19. The cofiber $C\bar{U}X_0$ of $\bar{U}X_{0,1} \rightarrow \bar{U}X_{0,0}$ has a canonical weak equivalence to $R_{\mathbb{T}}F(E\mathbb{T}_0/ET_{-1}, X) = R_{\mathbb{T}}F(\mathbb{T}_+, R_{\mathbb{T}}X)$. The map

$$i^*X \longrightarrow R_{\mathbb{T}}F(\mathbb{T}_+, R_{\mathbb{T}}X)^{\mathbb{T}} \longrightarrow i^*R_{\mathbb{T}}F(\mathbb{T}_+, R_{\mathbb{T}}X) \simeq \mathbf{R}F(\mathbb{T}_+, i^*X)$$

in the non-equivariant stable category is adjoint to the \mathbb{T} -action map

$$\mathbb{T}_+ \wedge i^*X \longrightarrow i^*X,$$

so the map

$$\pi_*^{\mathbb{S}} \mathbb{T} \otimes_{\pi_*\mathbb{S}} \pi_*X \longrightarrow \pi_*^{\mathbb{S}} \mathbb{T} \otimes_{\pi_*\mathbb{S}} \pi_*(C\bar{U}X_0) \longrightarrow \pi_*X$$

sends $1 \otimes x$ to x and $\zeta \otimes x$ to ζx , where the second map is induced by the counit (evaluation)

$$\mathbb{T}_+ \wedge \mathbf{R}F(\mathbb{T}_+, i^*X) \longrightarrow i^*X.$$

We note that the evaluation map is \mathbb{T} -equivariant and so in the case of $X = \mathbb{S}$ satisfies

$$\begin{aligned} 1 \otimes b'_0 &\mapsto 1 & 1 \otimes a'_0 &\mapsto 0 \\ \zeta \otimes b'_0 &\mapsto 0 & \zeta \otimes a'_0 &\mapsto -1 \end{aligned}$$

(the first formula coming from Lemma 12.7, the remainder following for equivariance and dimension reasons). In terms of our identification

$$\pi_*(C\bar{U}X_0) \cong \pi_*\mathbb{S}\langle a'_0, b'_0 \rangle \otimes_{\pi_*\mathbb{S}} \pi_*X,$$

we therefore have that the map $\pi_*X \rightarrow \pi_*(C\bar{U}X_0)$ satisfies the formula

$$(12.12) \quad v \mapsto b'_0 \otimes v - a'_0 \otimes \zeta v.$$

By Proposition 12.9, we see that the map in the statement

$$HM_{*,*} \otimes \pi_*X \longrightarrow E_{*,*}^1(X)$$

satisfies the following formulas

$$\begin{aligned}
 (12.13) \quad & z^n \otimes v \mapsto (b_n + n\eta a'_n) \otimes v - a'_n \otimes \zeta v \\
 & x^m z^n \otimes v \mapsto (b'_n \otimes b_m + n\eta a'_n \otimes b_m + m\eta b'_n \otimes a_m + mn\eta^2 a'_n \otimes a_m) \otimes v \\
 & \quad - (a'_n \otimes b_m + m\eta a'_n \otimes a_m) \otimes \zeta v \\
 & x^m y z^n \otimes v \mapsto (a'_n \otimes b_{m+1} - b'_n \otimes a_{m+1} + (n+m)\eta a'_n \otimes a_{m+1}) \otimes v \\
 & \quad + a'_n \otimes a_{m+1} \otimes \zeta v \\
 & y z^n \otimes v \mapsto (a'_n \otimes b_1 - b'_n \otimes a_1 + n\eta a'_n \otimes a_1) \otimes v + a'_n \otimes a_1 \otimes \zeta v
 \end{aligned}$$

for all $m > 0, n \geq 0$, from which it is clear that the map is injective. Surjectivity follows from looking at the image of multiplication by $\zeta + \eta$ just as in the proof of Theorem 3.17 above.

The bimodule statement follows from the monoidality statement, which we now check. For $v \in \pi_* X$, $w \in \pi_* Y$, and writing $v \wedge w$ for the smash product element in $\pi_*(X \wedge Y)$, the map

$$(HM_* \otimes \pi_* X) \otimes (HM_* \otimes \pi_* Y) \longrightarrow E_{*,*}^1(X) \otimes E_{*,*}^1(Y) \longrightarrow E_{*,*}^1(X \wedge Y)$$

takes $(1 \otimes v) \otimes (1 \otimes w)$ to

$$\begin{aligned}
 & (b'_0 \otimes v - a'_0 \otimes \zeta v)(b'_0 \otimes w - a'_0 \otimes \zeta w) \\
 & = b'_0 \otimes (v \wedge w) - a'_0 \otimes ((\zeta v) \wedge w) - (-1)^{|v|} a'_0 \otimes (v \wedge (\zeta w)) \\
 & = b'_0 \otimes (v \wedge w) - a'_0 \otimes \zeta(v \wedge w).
 \end{aligned}$$

This agrees with the image of $v \otimes w$ under the map

$$(HM_{*,*} \otimes \pi_* X) \otimes (HM_{*,*} \otimes \pi_* Y) \longrightarrow HM_{*,*} \otimes \pi_*(X \wedge Y) \longrightarrow E_{*,*}^1(X \wedge Y)$$

induced by the multiplication on HM_* . More generally, for any $\alpha \in HM_*$, both maps send $(1 \otimes x) \otimes (\alpha \otimes y)$ to the same element of $E_{*,*}^1(X \wedge Y)$. The monoidality statement follows. \square

13. Comparison of the Hesselholt-Madsen and Greenlees \mathbb{T} -Tate Spectral Sequences

We continue the work of the previous section to prove Theorems 3.20 and 3.21, which are not needed in the rest of the memoir. As we use Theorem 3.21 to prove Theorem 3.20, we begin there.

PROOF OF THEOREM 3.21. Since the multiplication comes from a pairing of filtered objects, all differentials satisfy the Leibniz rule. By definition, the connecting map ∂ for the cofiber sequence $S^0 \rightarrow S^{\mathbb{C}(1)} \rightarrow S^{\mathbb{C}(1)}/S^0$ takes $a_1 \in \pi_1(S^{\mathbb{C}(1)}/S^0)$ to the generator $1 \in \pi_0(S^0)$ and takes the generator $b_1 \in \pi_2(S^{\mathbb{C}(1)}/S^0)$ to 0. Likewise, in the cofiber sequences for the maps $S^{-(m+1)\mathbb{C}(1)} \rightarrow S^{-m\mathbb{C}(1)}$ that arise from filtration on $E\mathbb{T}$, the connecting maps satisfy $\partial a'_m = b'_{m+1}$ and $\partial b'_m = 0$. We can then read off the formulas for $d_2 x$, $d_2 y$, and $d_2 z$ in the spectral sequence for $X = \mathbb{S}$ from the definition of these classes in Notation 12.10:

$$\begin{aligned}
 d_2 x &= \eta b'_0 = \eta \\
 d_2 y &= b'_1 \otimes b_1 - b'_0 = b'_1 \otimes b_1 - 1 \\
 d_2 z &= \eta b'_2 = z^2.
 \end{aligned}$$

To finish the identification of $d_2 y$, we note that

$$\begin{aligned} xz + \eta yz &= (b'_1 \otimes b_1 + \eta a'_1 \otimes b_1 + \eta b'_1 \otimes a_1 + \eta^2 a'_1 \otimes a_1) \\ &\quad + \eta(a'_1 \otimes b_1 - b'_1 \otimes a_1 + \eta a'_1 \otimes a_1) = b'_1 \otimes b_1. \end{aligned}$$

For $v \in \pi_* X$, the tensor product notation in the statement conflicts with the tensor product notation we have been using above. For clarity, we continue to write elements in $\pi_*(C\bar{U}X_*)$ using the symbol \otimes , and switch to writing elements of $HM_{*,*} \otimes X$ using the symbol \boxtimes . For $v \in \pi_* X$, by (12.12), the element $1 \boxtimes v$ in $HM_{*,*} \otimes \pi_* X$ is the element $b'_0 \otimes v - a'_0 \otimes \zeta v$ in $\pi_*(C\bar{U}X_*)$, giving the formula

$$d_2(1 \boxtimes v) = -b'_1 \otimes \zeta v \in \pi_*(C\bar{U}X_0).$$

Since $1 \boxtimes \zeta v \in HM_{*,*} \otimes \pi_* X$ is the element $b'_0 \otimes \zeta v - \eta a'_0 \otimes \zeta v \in \pi_*(C\bar{U}X_0)$, we see that $z \boxtimes \zeta v \in HM_{*,*} \otimes \pi_* X$ is the element

$$(b'_1 + \eta a'_1)(b'_0 \otimes \zeta v - \eta a'_0 \otimes \zeta v) = b'_1 \otimes \zeta v \in \pi_*(C\bar{U}X_2).$$

This then gives the formula $d_2(1 \boxtimes v) = -z \boxtimes \zeta v$ in $HM_{*,*} \otimes \pi_* X$. \square

PROOF OF THEOREM 3.20. It is clear from the description in the statement that the specified map is a map of filtered spectra and induces a map of spectral sequences. For the purposes of this proof, we write $GE_{*,*}^2$ for the $E^2 = E^1$ term of the Greenlees \mathbb{T} -Tate spectral sequence as a complex with its d_2 differential, and analogously $HME_{*,*}^2$ for the $E^2 = E^1$ term of the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence. The Greenlees spectral sequence arises from $\pi_*^{\mathbb{T}}$ of a filtered spectrum; write GC_s for the level n cofiber so that (by definition)

$$GE_{s,t}^2 = \pi_{s+t}^{\mathbb{T}} GC_s.$$

We have $GC_n \simeq *$ for n odd and

$$GC_{2n} \simeq C(S^{n\mathbb{C}(1)}, S^{(n-1)\mathbb{C}(1)}) \wedge X$$

for all $n \in \mathbb{Z}$. The filtered map from the Greenlees filtration to the Hesselholt-Madsen filtration then sends GC_n isomorphically (in the stable category) to $C\bar{U}X_{|n|}$ for $n \leq 0$ or to $C(\bar{U}X_{n,0}, \bar{U}_{n-1,0})$ for $n > 0$. Let

$$b''_n \in \pi_{2n} C(S^{n\mathbb{C}(1)}, S^{(n-1)\mathbb{C}(1)})$$

be the element that is the image of the fundamental class of $S^{n\mathbb{C}(1)}$ (for the standard orientation) and let

$$a''_n \in \pi_{2n-1} C(S^{n\mathbb{C}(1)}, S^{(n-1)\mathbb{C}(1)})$$

be the element that the connecting map in the cofiber sequence takes to the fundamental class of $S^{(n-1)\mathbb{C}(1)}$. Then

$$\pi_* GC_{2n} = \pi_* \mathbb{S} \langle a''_n, b''_n \rangle \otimes_{\pi_* \mathbb{S}} \pi_* X$$

and $GE_{*,*}^2$ is the kernel of the action of $\zeta \in \pi_1^{\mathbb{S}\mathbb{T}}$ (Proposition 12.5) on the above, which can be computed by the Leibniz rule from its action on each tensor factor.

In this notation, the map $GE_{*,*}^2 \rightarrow HME_{*,*}^2$ is induced by the map

$$b''_n \mapsto \begin{cases} b'_{|n|} & n \leq 0 \\ b'_0 \otimes b_n & n > 0 \end{cases}$$

$$a''_n \mapsto \begin{cases} a'_{|n|} & n \leq 0 \\ b'_0 \otimes a_n & n > 0 \end{cases}$$

in the notation of the previous section.

We introduce the notation HMC_* for the graded spectrum analogous to GC_* that we studied (in terms of its further decomposition) in the previous section,

$$HMC_n = \bar{U}_n X / \bar{U}_{n-1} X \simeq \begin{cases} C\bar{U}X_{|n|} \vee \bigvee_{m>0} CC\bar{U}X_{m,m+|n|} & n \leq 0 \\ \bigvee_{m \geq n} CC\bar{U}X_{m,m-n} & n > 0. \end{cases}$$

Since the filtration cofibers GC_* and HMC_* are trivial in odd degrees, the d_2 differential makes $\pi_* GC_*$ and $\pi_* HMC_*$ into differential graded $\pi_*^{\mathbb{S}\mathbb{T}}$ -modules. To prove the theorem, it suffices to show that the map $\pi_* GC_* \rightarrow \pi_* HMC_*$ is a chain homotopy equivalence of differential graded $\pi_*^{\mathbb{S}\mathbb{T}}$ -modules. Since the map for general X is $(-) \otimes_{\pi_* \mathbb{S}} \pi_* X$ applied to the map in the case $X = \mathbb{S}$, it suffices to treat the case $X = \mathbb{S}$.

Write i for the map $\pi_* GC_* \rightarrow \pi_* HMC_*$ and define the map $r: \pi_* HMC_* \rightarrow \pi_* GC_*$ by

$$\begin{aligned} b'_n \otimes b_m &\mapsto b''_{m-n} & b'_n &\mapsto b''_n \\ a'_n \otimes b_m &\mapsto a''_{m-n} & a'_n &\mapsto a''_n \\ b'_n \otimes a_m &\mapsto a''_{m-n} \\ a'_n \otimes a_m &\mapsto 0. \end{aligned}$$

As the differential takes a_n ($n > 1$), a'_n ($n \geq 0$), a''_n ($n \in \mathbb{Z}$) to b_{n-1} , b'_{n+1} , b''_{n-1} (respectively), takes b_n ($n > 0$), b'_n ($n \geq 0$), b''_n ($n \in \mathbb{Z}$) to 0, and in the exceptional cases satisfies

$$\begin{aligned} d(a'_n \otimes a_1) &= b'_{n+1} \otimes a_1 - a'_n \\ d(b'_n \otimes a_1) &= b'_n, \end{aligned}$$

we see that r preserves the differential. Using the formulas for the action of ζ in the previous section (and the corresponding formulas in $\pi_* GC_*$), we see that r is a map of $\pi_*^{\mathbb{S}\mathbb{T}}$ -modules. Moreover, the composite $r \circ i$ is the identity on $\pi_* GC_*$.

To complete the argument, we construct a chain homotopy B from $i \circ r$ to the identity on $\pi_* HMC_*$ that satisfies $\zeta B = -B\zeta$. Since $1 - i \circ r = 0$ on the image of i , we can regard $1 - i \circ r$ as a homomorphism $(\pi_* HMC_*)/i \rightarrow \pi_* HMC_*$ and construct B as $(1 - i \circ r) \circ A$ for A a nullhomotopy of $(\pi_* HMC_*)/i$ satisfying $\zeta A = -A\zeta$. As a $\pi_* \mathbb{S}$ -module, $(\pi_* HMC_*)/i$ is free on elements

$$\begin{aligned} a'_n \otimes a_m &\quad (m > 0, n \geq 0), & a'_n \otimes b_m &\quad (m > 0, n \geq 0), \\ b'_n \otimes a_m &\quad (m > 0, n > 0), & b'_n \otimes b_m &\quad (m > 0, n > 0). \end{aligned}$$

We extend the notation of these elements to all $m \in \mathbb{Z}, n \geq 0$ by understanding the elements as zero when m, n is outside the indicated ranges above. Then for all

$m \in \mathbb{Z}, n \geq 0$, the following formulas are valid for the differential on $(\pi_* HM_*)/i$:

$$\begin{aligned} d(a'_n \otimes a_m) &= b'_{n+1} \otimes a_m - a'_n \otimes b_{m-1} \\ d(a'_n \otimes b_m) &= b'_{n+1} \otimes b_m \\ d(b'_n \otimes a_m) &= b'_n \otimes b_m \\ d(b'_n \otimes b_m) &= 0, \end{aligned}$$

and the following formulas are valid for the action of ζ on $(\pi_* HM_*)/i$:

$$\begin{aligned} \zeta(a'_n \otimes a_m) &= b'_n \otimes a_m - a'_n \otimes b_m + (m+n)\eta a'_n \otimes a_m \\ \zeta(a'_n \otimes b_m) &= b'_n \otimes b_m + (m+n-1)\eta a'_n \otimes b_m \\ \zeta(b'_n \otimes a_m) &= b'_n \otimes b_m + (m+n-1)\eta b'_n \otimes a_m \\ \zeta(b'_n \otimes b_m) &= (m+n)\eta b'_n \otimes b_m. \end{aligned}$$

For $m \geq 1$, let $A(a'_0 \otimes a_m) = 0$, $A(a'_0 \otimes b_m) = 0$, and for $m \geq 1, n \geq 1$, let

$$\begin{aligned} A(a'_n \otimes a_m) &= 0 \\ A(a'_n \otimes b_m) &= a'_{n-1} \otimes a_m + \cdots + a'_0 \otimes a_{m-(n-1)} \\ A(b'_n \otimes a_m) &= a'_{n-1} \otimes a_m + \cdots + a'_0 \otimes a_{m-(n-1)} \\ A(b'_n \otimes b_m) &= a'_{n-1} \otimes b_m + \cdots + a'_0 \otimes b_{m-(n-1)} \\ &\quad - (b'_{n-1} \otimes a_m + \cdots + b'_0 \otimes a_{m-(n-1)}) \end{aligned}$$

We note that $A(a'_n \otimes b_m) = A(b'_n \otimes a_m)$ and

$$A(b'_n \otimes b_m) = -(\zeta + (m-n-1)\eta)A(b'_n \otimes a_m);$$

from this it follows that $\zeta A = -A\zeta$. For $m \geq 1, n = 0$, we have

$$\begin{aligned} (Ad + dA)(a'_0 \otimes a_m) &= A(b'_1 \otimes a_m - a'_0 \otimes b_{m-1}) = a'_0 \otimes a_m \\ (Ad + dA)(a'_0 \otimes b_m) &= A(b'_1 \otimes b_m) = a'_0 \otimes b_m - b'_0 \otimes a_m = a'_0 \otimes b_m \end{aligned}$$

and in the remaining cases $m \geq 1, n \geq 1$, we have

$$\begin{aligned} (Ad + dA)(a'_n \otimes a_m) &= A(b'_{n+1} \otimes a_m - a'_n \otimes b_{m-1}) \\ &= a'_n \otimes a_m + \cdots + a'_0 \otimes a_{m-n} - (a'_{n-1} \otimes a_{m-1} + \cdots + a'_0 \otimes a_{m-n}) = a'_n \otimes a_m \end{aligned}$$

$$\begin{aligned} (Ad + dA)(a'_n \otimes b_m) &= A(b'_{n+1} \otimes b_m) + d(a'_{n-1} \otimes a_m + \cdots + a'_0 \otimes a_{m-(n-1)}) \\ &= a'_n \otimes b_m + \cdots + a'_0 \otimes b_{m-n} - (b'_n \otimes a_m + \cdots + b'_0 \otimes a_{m-n}) \\ &\quad + b'_n \otimes a_m - a'_{n-1} \otimes b_{m-1} + \cdots + b'_1 \otimes a_{m-(n-1)} - a'_0 \otimes b_{m-n} \\ &= a'_n \otimes b_m - b'_0 \otimes a_{m-n} = a'_n \otimes b_m \end{aligned}$$

$$\begin{aligned} (Ad + dA)(b'_n \otimes a_m) &= A(b'_n \otimes b_{m-1}) + d(a'_{n-1} \otimes a_m + \cdots + a'_0 \otimes a_{m-(n-1)}) \\ &= a'_{n-1} \otimes b_{m-1} + \cdots + a'_0 \otimes b_{m-n} - (b'_{n-1} \otimes a_{m-1} + \cdots + b'_0 \otimes a_{m-n}) \\ &\quad + b'_n \otimes a_m - a'_{n-1} \otimes b_{m-1} + \cdots + b'_1 \otimes a_{m-(n-1)} - a'_0 \otimes b_{m-n} \\ &= b'_n \otimes a_m - b'_0 \otimes a_{m-n} = b'_n \otimes a_m \end{aligned}$$

$$\begin{aligned}
(Ad + dA)(b'_n \otimes b_m) &= d(a'_{n-1} \otimes b_m + \cdots + a'_0 \otimes b_{m-(n-1)}) \\
&\quad - (b'_{n-1} \otimes a_m + \cdots + b'_0 \otimes a_{m-(n-1)})) \\
&= b'_n \otimes b_m + \cdots + b'_1 \otimes b_{m-(n-1)} - (b'_{n-1} \otimes b_{m-1} + \cdots + b'_0 \otimes b_{m-n}) \\
&= b'_n \otimes b_m - b'_0 \otimes b_{m-n} = b'_n \otimes b_m.
\end{aligned}$$

This completes the proof. \square

The end of Section 3 specifies a formula for the map from the Greenlees \mathbb{T} -Tate spectral sequence to the Hesselholt-Madsen \mathbb{T} -Tate spectral sequence in terms of the usual identification of the E^2 -term of the former and our notation for the E^2 -term of the latter. Given that the inclusion of $\pi_* X$ in homological degree zero agree for these spectral sequences, it can be seen by naturality from the case $X = \Sigma^\infty \mathbb{T}_+$, that the formula there is the unique one consistent with the differentials of the spectral sequences. Alternatively, it can be calculated directly as in (12.12) that for an element $v \in \pi_* X$, under the usual identification of the E^2 -term of the Greenlees spectral sequence, up to sign, the element $t^n \otimes v \in \mathbb{Z}[t, t^{-1}] \otimes \pi_* X$ goes to the element

$$\begin{cases} b'_n \otimes v + n\eta a'_n \otimes v - a'_n \otimes \zeta v & n \geq 0 \\ b'_0 \otimes b_{|n|} \otimes v + n\eta b'_0 \otimes a_{|n|} \otimes v - b'_0 \otimes a_{|n|} \otimes \zeta v & n \leq 0. \end{cases}$$

The formulas at the end of Section 3 can then be checked directly and the sign determined to be $(-1)^n$.

14. Coherence of the equivalences $E\mathbb{T}/E\mathbb{T}_{2n-1} \simeq E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}$ (Proof of Lemma 12.7)

In this section we prove Lemma 12.7. We first observe uniqueness. We note that

$$\begin{aligned}
\text{Ho}^B(\mathcal{S}^{\mathbb{T}})(\Sigma^\infty E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}, \Sigma^\infty E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}) \\
\cong \text{Ho}(\mathcal{S}^{\mathbb{T}})(\Sigma^\infty E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}, \Sigma^\infty E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}) \\
\cong \text{Ho}(\mathcal{S}^{\mathbb{T}})(\Sigma^\infty E\mathbb{T}_+, \Sigma^\infty E\mathbb{T}_+) \cong \pi_0^{h\mathbb{T}}(\mathbb{S}),
\end{aligned}$$

which is isomorphic to \mathbb{Z} by [12, B^T]. Thus, for the moment assume there exists a Borel equivalence $\Sigma^\infty E\mathbb{T}/E\mathbb{T}_{2n-1} \simeq \Sigma^\infty E\mathbb{T}_+ \wedge S^{\mathbb{C}(1)^n}$ for all $n \geq 0$. Then there are exactly two choices for each n , which induce opposite sign isomorphisms on $H_{2n} \cong \mathbb{Z}$, and diagrams involving only spectra Borel equivalent to $\Sigma^\infty E\mathbb{T}/E\mathbb{T}_{2n-1}$ (for fixed n) commute exactly when they commute on H_{2n} . It follows that the diagram in (i) determines a unique choice in the case $n = 0$ and in the other cases a unique choice once the choice for $n = 1$ is specified. The diagram in (ii) determines a unique choice in the case $n = 1$.

The proof of existence is a construction, which we do on the space level. First, it is convenient to change models. Recall that $S(\mathbb{C}(1)^n)$ denotes the unit sphere in $\mathbb{C}(1)^n$, where $\mathbb{C}(1)$ denotes the complex numbers \mathbb{C} with the standard action of \mathbb{T} as the unit complex numbers. Let $S(\mathbb{C}(1)^\infty)$ be the union of $S(\mathbb{C}(1)^n)$, where we include $\mathbb{C}(1)^n$ in $\mathbb{C}(1)^{n+1}$ as the vectors with 0 in the last coordinate. Then $S(\mathbb{C}(1)^\infty)$ is a model of $E\mathbb{T}$ and so we have a unique \mathbb{T} -homotopy class of \mathbb{T} -homotopy equivalence from $S(\mathbb{C}(1)^\infty)$ to our standard model of $E\mathbb{T}$. Looking at

the induced map on \mathbb{T} -quotients, the restriction

$$\mathbb{C}P^n = S(\mathbb{C}(1)^n)/\mathbb{T} \longrightarrow E\mathbb{T}/\mathbb{T} \cong B\mathbb{T}$$

factors through $B\mathbb{T}_{2n-2}$ (with our doubled filtration index convention) by obstruction theory, and so the map $S(\mathbb{C}(1)^n) \rightarrow E\mathbb{T}$ factors up to \mathbb{T} -homotopy through $E\mathbb{T}_{2n-2}$ by principal bundle theory. This constructs a \mathbb{T} -homotopy commutative diagram of \mathbb{T} -spaces

$$\begin{array}{ccccccc} S(\mathbb{C}(1)) & \longrightarrow & S(\mathbb{C}(1)^2) & \longrightarrow & S(\mathbb{C}(1)^3) & \longrightarrow & \cdots \longrightarrow S(\mathbb{C}(1)^\infty) \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \downarrow \simeq \\ E\mathbb{T}_0 & \longrightarrow & E\mathbb{T}_2 & \longrightarrow & E\mathbb{T}_4 & \longrightarrow & \cdots \longrightarrow E\mathbb{T}. \end{array}$$

Choosing homotopies, it suffices to construct and study the maps using the model $S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^n)$ in place of $E\mathbb{T}/E\mathbb{T}_{2n-1} = E\mathbb{T}/E\mathbb{T}_{2n-2}$.

REMARK 14.1. A concrete construction of the maps $S(\mathbb{C}(1)^n) \rightarrow E\mathbb{T}_{2n-2}$ is as follows. For the model WT of Section 9, we can use the map that sends an element

$$(z_1, \dots, z_n) \in S(\mathbb{C}(1)^n) \subset \mathbb{C}(1)^n$$

to the element represented by

$$\left(\frac{z_{i_0}}{\|z_{i_0}\|}, \dots, \frac{z_{i_m}}{\|z_{i_m}\|} \right), (\|z_{i_0}\|^2, \dots, \|z_{i_m}\|^2) \in \mathbb{T}^{m+1} \times \Delta[m]$$

where z_{i_0}, \dots, z_{i_m} drops the coordinates z_j that are 0, and the element of $\Delta[m]$ is specified in barycentric coordinates. The corresponding map to the model $E\mathbb{T}$ sends the element to

$$\left(\frac{z_{i_0}}{\|z_{i_0}\|}, \frac{z_{i_0}^{-1}}{\|z_{i_0}\|} \frac{z_{i_1}}{\|z_{i_1}\|}, \dots, \frac{z_{i_{m-1}}^{-1}}{\|z_{i_{m-1}}\|} \frac{z_{i_m}}{\|z_{i_m}\|} \right), (\|z_{i_0}\|^2, \dots, \|z_{i_m}\|^2) \in \mathbb{T}^{m+1} \times \Delta[m].$$

In the case $n = 1$, this map is clearly the canonical isomorphism $\mathbb{T} \cong \mathbb{T} \times \Delta[0]$. More generally, the map is surjective and generically one-to-one on the top dimensional cell, and so is a homology isomorphism on the underlying non-equivariant homotopy spheres. Since the map is also \mathbb{T} -equivariant and both sides are free \mathbb{T} -cell complexes, the map is a \mathbb{T} -homotopy equivalence.

CONSTRUCTION 14.2. We construct a \mathbb{T} -homeomorphism

$$h_{n,q}: S(\mathbb{C}(1)^{n+q})/S(\mathbb{C}(1)^n) \longrightarrow S^{\mathbb{C}(1)^n} \wedge S(\mathbb{C}(1)^q)_+$$

by constructing a \mathbb{T} -equivariant map

$$\bar{h}_{n,q}: S(\mathbb{C}(1)^{n+q}) \longrightarrow S^{\mathbb{C}(1)^n} \wedge S(\mathbb{C}(1)^q)_+$$

that sends all of $S(\mathbb{C}(1)^n)$ to the basepoint. For $\vec{x} = (w_1, \dots, w_n, z_1, \dots, z_q) \in \mathbb{C}(1)^{n+q}$ with $\|\vec{x}\| = 1$, write $\vec{w} = (w_1, \dots, w_n) \in \mathbb{C}(1)^n$ and $\vec{z} = (z_1, \dots, z_q) \in \mathbb{C}(1)^q$. We have $\vec{w} \in S(\mathbb{C}(1)^n) \subset S(\mathbb{C}(1)^{n+q})$ exactly when $\vec{z} = 0$, in which case we define $\bar{h}_{n,q}(\vec{x})$ to be the basepoint as required; otherwise we define

$$\bar{h}_{n,q}(\vec{x}) = \frac{\vec{w}}{\|\vec{z}\|} \wedge \frac{\vec{z}}{\|\vec{z}\|} \in S^{\mathbb{C}(1)^n} \wedge S(\mathbb{C}(1)^q)_+.$$

This is continuous since $\vec{w}/\|\vec{z}\|$ goes to the basepoint as \vec{z} goes to 0. The factorization $h_{n,q}$ is a \mathbb{T} -equivariant continuous bijection of compact spaces and hence a \mathbb{T} -homeomorphism.

The maps $h_{n,q}$ are consistent for varying q to define a \mathbb{T} -homeomorphism

$$h_{n,\infty}: S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^n) \xrightarrow{\cong} S^{\mathbb{C}(1)^n} \wedge S(\mathbb{C}(1)^\infty)_+.$$

Collapsing $S(\mathbb{C}(1)^\infty)$ to a point, we then get a composite map

$$h_n: S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^n) \longrightarrow S^{\mathbb{C}(1)^n}$$

that is a Borel equivalence.

PROPOSITION 14.3. *The diagram*

$$\begin{array}{ccc} S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^m) & \longrightarrow & S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^{m+1}) \\ h_m \downarrow & & \downarrow h_{m+1} \\ S^{\mathbb{C}(1)^m} & \xrightarrow{\quad} & S^{\mathbb{C}(1)^{m+1}} \end{array}$$

commutes up to \mathbb{T} -homotopy where the top horizontal map is the quotient map and the bottom horizontal map is induced by the inclusion of $\mathbb{C}(1)^m$ into $\mathbb{C}(1)^{m+1}$.

PROOF. Let \bar{i}_m be the self-map of $S(\mathbb{C}(1)^\infty)$ induced by the map $\mathbb{C}(1)^\infty \rightarrow \mathbb{C}(1)^\infty$ that is the identity on $\mathbb{C}(1)^m$ but sends the standard basis vector e_n to e_{n+1} for $n > m$; let i_m be the induced self-map of $S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^m)$. Then \bar{i}_m is a \mathbb{T} -equivariantly homotopic to the identity through a homotopy that preserves $S(\mathbb{C}(1)^m)$ pointwise, and therefore i_m is also \mathbb{T} -equivariantly homotopic to the identity. Since the diagram

$$\begin{array}{ccccc} S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^m) & \xrightarrow{i_m} & S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^m) & \longrightarrow & S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^{m+1}) \\ & \searrow h_m & & & \downarrow h_{m+1} \\ & & S^{\mathbb{C}(1)^m} & \xrightarrow{\quad} & S^{\mathbb{C}(1)^{m+1}} \end{array}$$

commutes, the diagram in the statement commutes up to \mathbb{T} -homotopy. \square

If we choose $i, j \geq 0$, letting $n = i + j$, we define a \mathbb{T} -equivariant map

$$\delta_{i,j}: S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^n) \longrightarrow S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^i) \wedge S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^j)$$

as follows. Writing an element of $S(\mathbb{C}(1)^\infty)$ as

$$\vec{x} = (v_1, \dots, v_i, w_1, \dots, w_j, z_1, z_2, \dots),$$

taking $\vec{v} = (v_1, \dots, v_i, 0, 0, \dots) \in \mathbb{C}(1)^\infty$, $\vec{w} = (w_1, \dots, w_j, 0, 0, \dots) \in \mathbb{C}(1)^\infty$, and

$$\vec{z} = (\underbrace{0, \dots, 0}_n, z_1, z_2, \dots) \in \mathbb{C}(1)^\infty,$$

we define

$$\delta_{i,j}(\vec{x}) = \frac{\vec{v} + \vec{z}}{\|\vec{v} + \vec{z}\|} \wedge \frac{\vec{w} + \vec{z}}{\|\vec{w} + \vec{z}\|} \in S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^i) \wedge S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^j)$$

where we understand the point as the basepoint when $\vec{z} = 0$. We use this map in the following proposition.

PROPOSITION 14.4. *For any $i, j \geq 0$ and $n = i + j$, The diagram*

$$\begin{array}{ccc} S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^n) & \xrightarrow{\delta_{i,j}} & S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^i) \wedge S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^j) \\ h_n \downarrow & & \downarrow h_i \wedge h_j \\ S^{\mathbb{C}(1)^n} & \xrightarrow{\cong} & S^{\mathbb{C}(1)^i} \wedge S^{\mathbb{C}(1)^j} \end{array}$$

commutes up to \mathbb{T} -homotopy where the bottom horizontal map is the canonical isomorphism

$$S^{\mathbb{C}(1)^n} = S^{\mathbb{C}(1)^{i+j}} \cong S^{\mathbb{C}(1)^i \oplus \mathbb{C}(1)^j} \cong S^{\mathbb{C}(1)^i} \wedge S^{\mathbb{C}(1)^j}.$$

PROOF. In the $\vec{x}; \vec{v}, \vec{w}, \vec{z}$ notation above the down-then-right map sends \vec{x} to

$$\frac{\vec{v}}{\|\vec{z}\|} \wedge \frac{\vec{w}}{\|\vec{z}\|}$$

while the right-then-down map sends \vec{x} to

$$\frac{\vec{v}}{\|\vec{v} + \vec{z}\| \|\vec{z}\|} \wedge \frac{\vec{w}}{\|\vec{w} + \vec{z}\| \|\vec{z}\|}.$$

The homotopy

$$\frac{\vec{v}}{\|\vec{v} + \vec{z}\|^t \|\vec{z}\|} \wedge \frac{\vec{w}}{\|\vec{w} + \vec{z}\|^t \|\vec{z}\|}.$$

is \mathbb{T} -equivariant. \square

We are now ready to prove Lemma 12.7.

PROOF. As discussed above, uniqueness follows immediately from existence. We use the map in the Borel stable category given by the zigzag

$$ET/ET_{2n-1} \xleftarrow{\cong} S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^n) \xrightarrow{h_{n,\infty}} S(\mathbb{C}(1)^\infty)_+ \wedge S^{\mathbb{C}(1)^n} \xrightarrow{\cong} ET_+ \wedge S^{\mathbb{C}(1)^n}.$$

Proposition 14.4 shows that the diagram in (ii) commutes for all m . To deduce that the diagram in (i) commutes, it suffices to show that the diagram

$$\begin{array}{ccc} S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^n) & \xrightarrow{\delta_{i,j}} & S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^i) \wedge S(\mathbb{C}(1)^\infty)/S(\mathbb{C}(1)^j) \\ \downarrow & & \downarrow \\ ET/ET_{2n-1} & \longrightarrow & ET/ET_{2i-1} \wedge ET/ET_{2j-1} \end{array}$$

commutes after applying H_{2n} . This is a straightforward homology calculation, keeping track of signs. \square

15. The strong Künneth theorem for THH

The following well-known result does not appear to have a good reference in the literature.

THEOREM 15.1. *Let R be a commutative ring orthogonal spectrum. Let X and Y be R -algebras. The symmetric monoidal structure map*

$$THH(X) \wedge_{THH(R)}^L THH(Y) \longrightarrow THH(X \wedge_R^L Y)$$

is an isomorphism in the Borel derived category of $THH(R)$ -modules.

We can assume without loss of generality that R is cofibrant as a commutative ring orthogonal spectrum and X and Y are cofibrant as R -algebras. Then the cyclic bar constructions $N^{\text{cy}}(X)$, $N^{\text{cy}}(Y)$, and $N^{\text{cy}}(R)$ model $THH(X)$, $THH(Y)$, and $THH(R)$, respectively. The map in question is induced by the map

$$N^{\text{cy}}X \wedge N^{\text{cy}}Y \longrightarrow N^{\text{cy}}(X \wedge Y) \longrightarrow N^{\text{cy}}(X \wedge_R Y)$$

which coequalizes the maps

$$N^{\text{cy}}X \wedge N^{\text{cy}}R \wedge N^{\text{cy}}Y \rightrightarrows N^{\text{cy}}X \wedge N^{\text{cy}}Y$$

where the top arrow is the $N^{\text{cy}}R$ -action on $N^{\text{cy}}X$ and the bottom arrow is the $N^{\text{cy}}R$ -action on $N^{\text{cy}}Y$. It suffices to prove the following stronger point-set theorem.

THEOREM 15.2. *Let R be a commutative ring orthogonal spectrum. Let X and Y be R -algebras. The symmetric monoidal structure map*

$$N^{\text{cy}}(X) \wedge_{N^{\text{cy}}(R)} N^{\text{cy}}(Y) \longrightarrow N^{\text{cy}}(X \wedge_R Y)$$

is an isomorphism in the point-set category of equivariant $N^{\text{cy}}(R)$ -modules.

PROOF. Since coequalizers commute with geometric realization, looking at simplicial level n , it suffices to show that the map

$$(X \wedge \cdots \wedge X) \wedge_{R \wedge \cdots \wedge R} (Y \wedge \cdots \wedge Y) \longrightarrow (X \wedge_R Y) \wedge \cdots \wedge (X \wedge_R Y)$$

is an isomorphism and this is clear by inspection. \square

REMARK 15.3. Since the map in Theorem 15.1 is natural in the category $\text{Ho}^M(\mathcal{C}\text{at}_R^{\mathcal{S}})$ obtained from the category of R -spectral categories by formally inverting the Morita equivalences, the result generalizes to the case when X and Y are R -spectral categories.

REMARK 15.4. We have consistently worked in the Borel derived category throughout rather than the equivariant derived category. It is clear from Theorem 15.2 that the strong Künneth theorem holds for THH in any category where THH is modeled by a functor of N^{cy} on cofibrant objects of the given types. In particular, it applies to the norm model of THH constructed as $N_e^{\mathbb{T}} = I_{\mathbb{R}^\infty}^U N^{\text{cy}}$ as in [1].

16. THH of smooth and proper algebras (Proof of Theorem C)

The purpose of this section is to prove Theorem C of the introduction. Toën-Vaquié [34, 2.6] show that any smooth and proper k -linear dg category \mathcal{X} is Morita equivalent to a smooth and proper dg k -algebra, and so by Morita invariance of THH [6, 5.12], it suffices to prove the following theorem.

THEOREM 16.1. *Let R be a commutative ring orthogonal spectrum and A a R -algebra which is smooth and proper when viewed as an R -spectral category. Then $THH(A)$ is a small $THH(R)$ -module.*

We emphasize that the statement here and the work in this section is non-equivariant.

We recall that an R -module M is small means that maps in the derived category out of it $\mathcal{D}_R(M, -)$ commutes with arbitrary coproducts; this is equivalent to M being weakly equivalent to the homotopy retract of a finite cell R -module. For the hypothesis of the theorem, A is a proper as an R -spectral category precisely when

A is small as an R -module and A is a smooth as an R -spectral category precisely when A is small as an $A \wedge_R^L A^{\text{op}}$ -module (q.v. [2, §3.2]), or more precisely, when for some (hence any) cofibrant replacement $A' \rightarrow A$ in R -algebras, A is small as an $A' \wedge_R A'^{\text{op}}$ -module.

Without loss of generality, we model R as a cofibrant commutative ring orthogonal spectrum and we model A as a cofibrant R -algebra. These hypotheses will be enough to ensure that all smash products we use below in the proof of Theorem 16.1 represent derived smash products.

The proof of Theorem 16.1 involves a comparison of different models representing $THH(A)$ as a $THH(R)$ -module, some of which involve elaborate simplicial constructions that play off the smash product of orthogonal spectra against the smash products \wedge_R for R -modules. To avoid confusion, we will always write $\wedge_{\mathbb{S}}$ in this section for the smash product of orthogonal spectra to contrast with \wedge_R . Likewise, we will refer to orthogonal spectra with no extra structure as \mathbb{S} -modules to contrast with the extra structure of R -modules or A -modules.

The basic building blocks of the constructions of this section are the two-sided and cyclic bar constructions relative to \mathbb{S} and R . We begin by establishing conventions and notation for these constructions in this section. In this section only, we use $N_{\bullet}^{\mathbb{S}}$ and $N_{\bullet}^{\text{cy};\mathbb{S}}$ to denote the two-sided bar and cyclic bar constructions using $\wedge_{\mathbb{S}}$, and $B^{\mathbb{S}}$ and $B^{\text{cy};\mathbb{S}}$ for the geometric realizations. We also use corresponding notation replacing \mathbb{S} with R . To be specific about the face maps: in

$$N_q^{\mathbb{S}}(X, A, Y) = X \wedge_{\mathbb{S}} \underbrace{A \wedge_{\mathbb{S}} \cdots \wedge_{\mathbb{S}} A}_{q} \wedge_{\mathbb{S}} Y,$$

d_0 uses the left action of A on Y and d_q uses the right action of A on X . We call X the *leftside module* and Y the *rightside module*; the leftside module is a right module and the rightside module is a left module. If the leftside module is an A -bimodule, then the left action of A on X induces a left action of A on $N^{\mathbb{S}}(X, A, Y)$; if Y is also an A -bimodule, then $N^{\mathbb{S}}(X, A, Y)$ inherits an A -bimodule structure. We have similar observations using R in place of A or in place of \mathbb{S} .

Recall that an A -bimodule (in the category of \mathbb{S} -modules) consists of an \mathbb{S} -module M together with commuting left and right A -module action maps $A \wedge_{\mathbb{S}} M \rightarrow M$ and $M \wedge_{\mathbb{S}} A \rightarrow M$; this is equivalent data to the structure of either a left or right $A \wedge_{\mathbb{S}} A^{\text{op}}$ -module on M . Likewise, an A -bimodule (in the category of R -modules) consists of an R -module M and commuting left and right A -module action maps $A \wedge_R M \rightarrow M$ and $M \wedge_R A \rightarrow M$, with equivalent data a (left or right) $A \wedge_R A^{\text{op}}$ -module structure on M .

Given an A -bimodule M (in the category of \mathbb{S} -modules), when forming the cyclic bar construction

$$N_q^{\text{cy};\mathbb{S}}(A; M) = M \wedge_{\mathbb{S}} \underbrace{A \wedge_{\mathbb{S}} \cdots \wedge_{\mathbb{S}} A}_{q},$$

our convention is to have d_0 use the left action of A on M and d_q the right action of A on M . We then have a canonical isomorphism

$$N_{\bullet}^{\text{cy};\mathbb{S}}(A; M) \cong M \wedge_{A \wedge_{\mathbb{S}} A^{\text{op}}} N_{\bullet}^{\mathbb{S}}(A, A, A)$$

where we use the (previously unused) left A -module structure on the leftside A and right A -module structure on the rightside A . Writing things this way, it is

natural to regard M as a right $A \wedge_{\mathbb{S}} A^{\text{op}}$ -module and $N_{\bullet}^{\mathbb{S}}(A, A, A)$ as a simplicial left $A \wedge_{\mathbb{S}} A^{\text{op}}$ -module.

In the context of commutative ring orthogonal spectra, the two-sided bar construction $B^{\mathbb{S}}(R, R, R)$ and cyclic bar construction $B^{\text{cy}, \mathbb{S}}(R)$ are special cases of tensors. In the following, we write \otimes for the tensor of a commutative ring orthogonal spectrum with an unbased space (in the category of commutative ring orthogonal spectra). Then $B^{\mathbb{S}}(R, R, R)$ is canonically isomorphic to the tensor $R \otimes \Delta[1]$ and $B^{\text{cy}, \mathbb{S}}(R)$ is canonically isomorphic to the tensor $R \otimes (\Delta[1]/\partial\Delta[1])$.

We are now ready to begin the constructions of the various models of $THH(A)$. We use the following shorthand notation for some of the constructions described above:

NOTATION 16.2. Let RI denote $R \otimes \Delta[1]$. Let $R_0 = R \otimes \{0\}$ and $R_1 = R \otimes \{1\}$; then R_0 and R_1 are canonically isomorphic to R as commutative ring orthogonal spectra, but the subscripts keep track of the maps of commutative ring orthogonal spectra $R \rightarrow RI$. (In fact, we have analogous maps R_t for $t \in (0, 1)$, although we do not use these.)

NOTATION 16.3. Let $ARI = A \wedge_{R_0} RI$, the extension of scalars RI -algebra. We regard ARI as an R algebra via $R_1 \rightarrow RI$.

Recalling that d_0 on the tautological element of $\Delta[1]_1$ corresponds to the vertex 1 and d_1 to the vertex 0, we see that ARI is the geometric realization of the simplicial associative ring orthogonal spectrum

$$ARI_{\bullet} = N_{\bullet}^{\mathbb{S}}(A, R, R),$$

and its R -algebra structure comes from the rightside R .

The inclusion of A in ARI_{\bullet} is map of simplicial associative ring orthogonal spectra with A constant but not a map of simplicial R -algebras. The collapse map $ARI_{\bullet} \rightarrow A$ is a map of R -algebras and a simplicial homotopy equivalence of simplicial associative ring orthogonal spectra.

NOTATION 16.4. Let O_{\bullet} be the simplicial model of the circle obtained by gluing 2 copies of $\Delta[1]_{\bullet}$ along 0 and along 1. To be definite later, we label one copy (a) and the other (b). Let RO_{\bullet} be the simplicial commutative ring orthogonal spectrum $R \otimes O_{\bullet}$ ($\wedge_{\mathbb{S}}$ -power of R in each simplicial degree). We write RO for the geometric realization, which is canonically isomorphic to the commutative ring orthogonal spectrum $RI \wedge_{R_0 \wedge_{\mathbb{S}} R_1} RI$. We have canonical maps of commutative ring orthogonal spectra $R_0, R_1 \rightarrow RO$.

NOTATION 16.5. We let ARO be the extension of scalars RO -algebra $RO \wedge_{R_0} A$. When we regard this as an R -algebra it will be via $R_1 \rightarrow RO$.

We can also describe ARO as the geometric realization of a simplicial associative ring orthogonal spectrum (or simplicial R -algebra). We have 2 maps of R -algebras $ARI \rightarrow ARO$ corresponding to the 2 maps of R -algebras $RI \rightarrow RO$ (corresponding to the 2 maps of $\Delta[1]_{\bullet}$ into O_{\bullet}). We use the “(a)” map to define a left ARI -module structure on ARO and the “(b)” map to define a right ARI -module structure on ARO ; together these give a ARI -bimodule structure in R -modules, a $ARI \wedge_R ARI^{\text{op}}$ -module structure. This bimodule structure commutes with the RO -module structure.

NOTATION 16.6. Let $R\partial\Delta[2] = R \otimes \partial\Delta[2]$ and $R\partial\Delta[2]_{\bullet} = R \otimes \partial\Delta[2]_{\bullet}$.

We have a map of simplicial sets $\alpha: \partial\Delta[2]_{\bullet} \rightarrow O_{\bullet}$ sending the 1-simplex $\{0, 1\}$ to the point $\{0\}$ (i.e., collapsing $d^2\Delta[1]_{\bullet}$), sending $\{0, 2\}$ to (a) and $\{1, 2\}$ to (b). This induces a weak equivalence of commutative ring orthogonal spectra $R\partial\Delta[2] \rightarrow RO$. Indeed, this map is homotopic in the category of commutative ring orthogonal spectra to an isomorphism, just as the geometric realization of $\partial\Delta[2]_{\bullet} \rightarrow O_{\bullet}$ is homotopic in the category of spaces to a homeomorphism. For consistency with this map, when we refer to $R\partial\Delta[2]$ as an R -module it will be via $R \otimes \{2\}$. (The vertex 2 goes to the vertex 1 in the simplicial map $\partial\Delta[2]_{\bullet} \rightarrow O_{\bullet}$.)

The inclusion $d^2: \Delta[1]_{\bullet} \rightarrow \partial\Delta[2]_{\bullet}$ induces a map of commutative ring orthogonal spectra $RI \rightarrow R\partial\Delta[2]$.

NOTATION 16.7. Let D be the $R\partial\Delta[2]$ -module $R\partial\Delta[2] \wedge_{d^2, RI} B^{\mathbb{S}}(A, A, A)$.

The maps $d^0, d^1: \Delta[1]_{\bullet} \rightarrow \Delta[2]_{\bullet}$ induce a pair of maps $RI \rightarrow R\partial\Delta[2]$, giving D a pair of commuting RI -module structures – indeed an $RI \wedge_{R_1} RI$ -module structure; using the left and right action of A on $B^{\mathbb{S}}(A, A, A)$, these extend to a ARI -bimodule structure on D (with the left action of ARI corresponding to the 1-simplex $\{0, 2\}$ of $\Delta[2]_{\bullet}$ and the right action of ARI corresponding to the 1-simplex $\{1, 2\}$). In fact, the ARI -bimodule structure is a $ARI \wedge_R ARI^{\text{op}}$ -module structure that commutes with the $R\partial\Delta[2]$ -module structure.

The A -bimodule (in \mathbb{S} -modules) weak equivalence $B^{\mathbb{S}}(A, A, A) \rightarrow A$ induces a weak equivalence $D \rightarrow ARO$, which is a map of ARI -bimodules in R -modules and of $R\partial\Delta[2]$ -modules (using the map $R\partial\Delta[2] \rightarrow RO$ for the $R\partial\Delta[2]$ -module structure on ARO). Applying $B^{\text{cy}; R}(ARI; -)$, we get a weak equivalence of $R\partial\Delta[2]$ -modules

$$(16.8) \quad B^{\text{cy}; R}(ARI; D) \longrightarrow B^{\text{cy}; R}(ARI; ARO).$$

We will see below that these are models of $THH(A)$; the weak equivalence (16.8) is the first key ingredient to the proof of Theorem 16.1 below.

The last model arises by applying the Dennis-Waldhausen Morita trick [6, §6] to $B^{\text{cy}; R}(ARI; D)$. To do so, we need to identify $B^{\text{cy}; R}(ARI; D)$ as the geometric realization of the simplicial \mathbb{S} -module

$$(*) \quad N_{\bullet}^{\text{cy}; R}(ARI; N_{\bullet}^{\text{cy}; \mathbb{S}}(A; ARI \wedge_R ARI^{\text{op}})).$$

In (*), we use the above map of associative ring orthogonal spectra $A \rightarrow ARI$ to get a map of associative ring orthogonal spectra

$$A \wedge_{\mathbb{S}} A^{\text{op}} \longrightarrow ARI \wedge_{\mathbb{S}} ARI^{\text{op}} \longrightarrow ARI \wedge_R ARI^{\text{op}},$$

and this gives commuting right $A \wedge_{\mathbb{S}} A^{\text{op}}$ -module and left $ARI \wedge_R ARI^{\text{op}}$ -module structures on $ARI \wedge_R ARI^{\text{op}}$; the canonical isomorphism (symmetry isomorphism on smash factors)

$$(ARI \wedge_R ARI^{\text{op}})^{\text{op}} \cong ARI \wedge_R ARI^{\text{op}}$$

makes this commuting right $A \wedge_{\mathbb{S}} A^{\text{op}}$ and $ARI \wedge_R ARI^{\text{op}}$ -module structures, allowing us to do construction (*). To show that $B^{\text{cy}; R}(ARI; D)$ is the geometric realization of the simplicial \mathbb{S} -module (*), we just need to produce an isomorphism of left $ARI \wedge_R ARI^{\text{op}}$ -modules between D and $B^{\text{cy}; \mathbb{S}}(A; ARI \wedge_R ARI)$.

By construction D is isomorphic to the geometric realization of

$$R \otimes \partial\Delta[2]_{\bullet} \wedge_{d^2} N_{\bullet}^{\mathbb{S}}(A, A, A).$$

We can construct $\partial\Delta[2]_{\bullet}$ as the diagonal of a bisimplicial set obtained by gluing a copy of Λ_2^2 (the faces d^0 and d^1 of $\partial\Delta[2]_{\bullet}$) in a first simplicial direction along

vertices 0 and 1 to a copy of $\Delta[1]$ (the face d^2) in a second simplicial direction. Taking the geometric realization of the first simplicial direction, we see that D is isomorphic to the geometric realization of

$$R \otimes \Lambda_2^2 \wedge_{R_0 \wedge_{\mathbb{S}} R_1} N_{\bullet}^{\mathbb{S}}(A, A, A) \cong (R \otimes \Lambda_2^2 \wedge_{R_0 \wedge_{\mathbb{S}} R_1} (A \wedge_{\mathbb{S}} A^{\text{op}})) \wedge_{A \wedge_{\mathbb{S}} A^{\text{op}}} N_{\bullet}^{\mathbb{S}}(A, A, A)$$

(to be clear, using the R -action corresponding to the vertex 0 of $\Delta[2]_{\bullet}$ to attach A and corresponding to the vertex 1 to attach A^{op}). Breaking Λ_2^2 as $\Delta[1] \cup_{\{1\}} \Delta[1]$, this is easily recognized as

$$(ARI \wedge_R ARI^{\text{op}}) \wedge_{A \wedge_{\mathbb{S}} A^{\text{op}}} N_{\bullet}^{\mathbb{S}}(A, A, A) \cong N^{\text{cy};\mathbb{S}}(A; ARI \wedge_R ARI^{\text{op}}).$$

The isomorphism constructed preserves the ARI -bimodule (in R -module) structure, using the standard left action of ARI on ARI and right action on ARI^{op} .

The previous paragraph constructs an isomorphism from $B^{\text{cy};R}(ARI; D)$ to the geometric realization of the simplicial \mathbb{S} -module

$$N_{\bullet}^R(ARI, ARI, ARI) \wedge_{ARI \wedge_R ARI^{\text{op}}} (ARI \wedge_R ARI^{\text{op}}) \wedge_{A \wedge_{\mathbb{S}} A^{\text{op}}} N_{\bullet}^{\mathbb{S}}(A, A, A),$$

Using the symmetry isomorphism of $\wedge_{\mathbb{S}}$ to switch the sides we put $N_{\bullet}^{\mathbb{S}}(A, A, A)$ and $N_{\bullet}^R(ARI, ARI, ARI)$ on, we get an isomorphism with

$$N_{\bullet}^{\text{cy};\mathbb{S}}(A; N_{\bullet}^R(ARI, ARI, ARI))$$

and so an isomorphism of \mathbb{S} -modules

$$(16.9) \quad B^{\text{cy};R}(ARI; D) \cong B^{\text{cy};\mathbb{S}}(A; B^R(ARI, ARI, ARI)).$$

(This is the Dennis-Waldhausen Morita trick.) We transport the $R\partial\Delta[2]$ -module structure on $B^{\text{cy};R}(ARI; D)$ constructed above to $B^{\text{cy};\mathbb{S}}(A; B^R(ARI, ARI, ARI))$ along this isomorphism. We emphasize that this constructs a well-defined $R\partial\Delta[2]$ -module structure on $B^{\text{cy};\mathbb{S}}(A; B^R(ARI, ARI, ARI))$, but we can describe it more concretely as follows. We write $B^{\text{cy};\mathbb{S}}(A; B^R(ARI, ARI, ARI))$ as the geometric realization of the multisimplicial \mathbb{S} -module

$$N_i^{\text{cy};\mathbb{S}}(A; N_n^R(ARI_j, ARI_m, ARI_k)).$$

For fixed m, n , grouping together the i, j, k -terms, we are looking at

$$\begin{aligned} (**)\quad & A^{\wedge_{\mathbb{S}}(i)} \wedge_{\mathbb{S}} (ARI_j \wedge_R ARI_k \wedge_R ARI_m^{\wedge_R(n)}) \\ &= A^{\wedge_{\mathbb{S}}(i)} \wedge_{\mathbb{S}} \left((A \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(j)} \wedge_{\mathbb{S}} R) \wedge_R (A \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(k)} \wedge_{\mathbb{S}} R) \wedge_R ARI_m^{\wedge_R(n)} \right) \end{aligned}$$

where the R -module structure on ARI_{\bullet} for \wedge_R is the last factor of R in the expansion. Viewing $\partial\Delta[2]_{\bullet}$ as the diagonal of a trisimplicial set, $R\partial\Delta[2]$ is the geometric realization of the trisimplicial set

$$R_0 \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(i)} \wedge_{\mathbb{S}} R_1 \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(j)} \wedge_{\mathbb{S}} R_2 \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(k)},$$

where R_0, R_1 , and R_2 are all copies of R that we have marked by numbers to keep track of them in the formulas that follow. (The numbers correspond to the vertices of $\partial\Delta[2]$.) We can rewrite the above in terms of \wedge_{R_2} as

$$R_0 \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(i)} \wedge_{\mathbb{S}} \left((R_1 \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(j)} \wedge_{\mathbb{S}} R_2) \wedge_{R_2} (R^{\wedge_{\mathbb{S}}(k)} \wedge_{\mathbb{S}} R_2) \right),$$

and we can move R_0 inside the last factor

$$R^{\wedge_{\mathbb{S}}(i)} \wedge_{\mathbb{S}} \left((R_1 \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(j)} \wedge_{\mathbb{S}} R_2) \wedge_{R_2} (R_0 \wedge_{\mathbb{S}} R^{\wedge_{\mathbb{S}}(k)} \wedge_{\mathbb{S}} R_2) \right).$$

In terms of the formula (**), we have $R^{\wedge_{\mathbb{S}}(i)}$ acting on $A^{\wedge_{\mathbb{S}}(i)}$, we have $R^{\wedge_{\mathbb{S}}(j)}$ and $R^{\wedge_{\mathbb{S}}(k)}$ acting on the respective factors in (**), we have R_2 acting on the righthand R -factors (used in \wedge_R), and finally R_0 acts on the A in the k -factor while R_1 acts on the A in the j -factor.

The collapse maps

$$B^R(ARI, ARI, ARI) \longrightarrow ARI \longrightarrow A$$

induce a weak equivalence of \mathbb{S} -modules

$$(16.10) \quad B^{\text{cy};R}(ARI; D) \longrightarrow B^{\text{cy};\mathbb{S}}(A; B^R(ARI, ARI, ARI)) \longrightarrow B^{\text{cy};\mathbb{S}}(A).$$

This is a map of $R\partial\Delta[2]$ -modules when we give $B^{\text{cy};\mathbb{S}}(A)$ the module structure induced by the map

$$\beta: \partial\Delta[2]_{\bullet} \longrightarrow \partial\Delta[2]_{\bullet}/\Lambda_2^2 \cong \Delta[1]_{\bullet}/\partial\Delta[1]$$

and the usual $B^{\text{cy};\mathbb{S}}(R) \cong R \otimes \Delta[1]/\partial\Delta[1]$ module structure on $B^{\text{cy}}(A)$.

In the end, we have constructed in (16.8), (16.9), and (16.10) a zigzag of weak equivalences of $R\partial\Delta[2]$ -modules relating $B^{\text{cy};R}(ARI; ARO)$ and $B^{\text{cy}}(A)$. We are now ready to prove Theorem 16.1

PROOF OF THEOREM 16.1. The hypothesis that A is proper over R means that A is weakly equivalent to a homotopy retract of a finite cell R -module. Since the derived functor $RO \wedge_{R_0} (-)$ from R -modules to RO -modules preserves cofiber sequences, we see that ARO is weakly equivalent to a homotopy retract of a finite cell RO -module. Since the map $R \otimes \alpha: R\Delta[2] \rightarrow RO$ is a weak equivalence, we see that ARO is weakly equivalent to a homotopy retract of a finite cell $R\partial\Delta[2]$ -module. The hypothesis that A is smooth over R is that A is weakly equivalent to a homotopy retract of a finite cell $A \wedge_R A$ -module. Using the weak equivalence of R -algebras $ARI \rightarrow A$, we see that ARI and $B^R(ARI, ARI, ARI)$ are weakly equivalent to homotopy retracts of finite cell right $ARI \wedge_R ARI$ -modules. Since

$$B^{\text{cy};R}(ARI; ARO) \cong B^R(ARI, ARI, ARI) \wedge_{ARI \wedge_R ARI^{\text{op}}} ARO,$$

we see that $B^{\text{cy};R}(ARI; ARO)$ is weakly equivalent to the homotopy retract of a finite cell $R\partial\Delta[2]$ -module.

Since the map $R \otimes \beta: R \otimes \partial\Delta[2] \rightarrow B^{\text{cy};\mathbb{S}}(R)$ is a weak equivalence, the forgetful functor from $B^{\text{cy};\mathbb{S}}(R)$ -modules to $R \otimes \partial\Delta[2]$ -modules is the right adjoint of a Quillen equivalence. We conclude that $B^{\text{cy};\mathbb{S}}(A)$ is weakly equivalent to a homotopy retract of a finite cell $B^{\text{cy};\mathbb{S}}(R)$ -module. \square

REMARK 16.11. The zigzag of weak equivalences between $B^{\text{cy};R}(ARI; ARO)$ and $B^{\text{cy}}(A)$ can be interpreted as an isomorphism in the stable category

$$THH(A) \simeq THH^R(A; THH(R; A)).$$

We note that although $THH(R; A)$ is weakly equivalent as an A -module or A^{op} -module to $THH(R) \wedge_R^{\mathbf{L}} A$, it is generally not for the $A \wedge_R A^{\text{op}}$ structure in the weak equivalence: if it were, we would then have a weak equivalence $THH(A) \simeq THH^R(A) \wedge_R^{\mathbf{L}} THH(R)$. This does not hold in the example of $R = H\mathbb{Z}$ and $A = H\mathbb{Z}[i]$ as shown by the calculation in [22].

17. The finiteness theorem for TP (Proof of Theorem B)

The purpose of this section is to prove Theorem B and its analogue for $G = C_p$. For convenience of reference, we state the combined theorem here.

THEOREM 17.1. *Let k be a perfect field of characteristic $p > 0$ and let $G = \mathbb{T}$ or $G = C_p$. If \mathcal{X} is a smooth and proper k -linear dg category, then $T_G(THH(\mathcal{X}))$ is a finite $T_G(THH(k))$ -module.*

We have left off the second statement, because we use it to deduce the first: the main work of the section is to prove the following theorem.

THEOREM 17.2. *Let k be a perfect field of characteristic $p > 0$, and let G be a closed subgroup of \mathbb{T} . If B is an Hk -algebra with $THH_*(B)$ finitely generated over $THH_*(k)$, then $\pi_*^{tG} THH(B)$ is finitely generated over $\pi_*^{tG} THH(k)$.*

To deduce Theorem 17.1 from Theorem 17.2, we note that by Morita invariance of THH [6, 5.12], it suffices to consider dg k -algebras that are smooth and proper as k -linear dg categories in Theorem 17.1. Theorem C then implies that the Hk -algebras corresponding to smooth and proper dg k -algebras satisfy the hypotheses of Theorem 17.2. From here Theorem 17.1 is a consequence of the following observation.

PROPOSITION 17.3. *Let $G = \mathbb{T}$ or $G = C_p$ and let X be a $T_G(THH(k))$ -module. Then X is a finite $T_G(THH(k))$ -module if and only if $\pi_* X$ is finitely generated over $\pi_*(T_G(THH(k)))$.*

PROOF. When $G = \mathbb{T}$, $\pi_*(T_{\mathbb{T}}(THH(k))) \cong \mathbb{W}k[v, v^{-1}]$ where $\mathbb{W}k$ denotes the p -typical Witt vectors on k and v is an element of $TP_{-2}(k) = \pi_{-2}(T_{\mathbb{T}}(THH(k)))$, a particular choice of which Hesselholt constructed in [14, 4.2]. As a graded ring $\pi_*(T_{\mathbb{T}}(THH(k)))$ is a graded PID; specifically, all graded ideals are of the form (p^n) for some n . As a consequence every finitely generated $T_{\mathbb{T}}(THH(k))$ -module is a wedge of copies of suspensions $T_{\mathbb{T}}(THH(k))$ and suspensions of the cofiber of multiplication by p^n ; a module is finite over $T_{\mathbb{T}}(THH(k))$ if and only if its homotopy groups are finitely generated over $\pi_*(T_{\mathbb{T}}(THH(k)))$. In the case $G = C_p$, $\pi_*(T_{C_p}(THH(k))) \cong k[v, v^{-1}]$, a graded field in the sense that every graded module over it is free. Every $T_{C_p}(THH(k))$ -module is a wedge of suspensions of $T_{C_p}(THH(k))$ and a $T_{C_p}THH(k)$ -module is finite if and only if its homotopy groups are finitely generated over $\pi_* T_{C_p} THH(k)$. \square

For other closed subgroups $C_r < \mathbb{T}$, writing $r = p^n m$ with $(p, m) = 1$, we have a weak equivalence $T_{C_r} THH(A) \simeq T_{C_p^n} THH(A)$ induced by the transfer. For $n > 1$, $\pi^{tC_p^n} THH(k) \cong \mathbb{W}k_n[v, v^{-1}]$, where $\mathbb{W}k_n$ denotes the Witt vectors of length n . This graded ring has infinite global (projective) dimension and there exist $T_{C_p^n} THH(k)$ -modules whose homotopy groups are finitely generated but which are not themselves small. For example, $T_{C_p} THH(k)$ is a $T_{C_p^n} THH(k)$ -module that is not small for $n > 1$ but has homotopy groups $(\pi_* T_{C_p^n} THH(k))/p$.

For the proof of Theorem 17.2, we prove the following slightly more general theorem.

THEOREM 17.4. *Let k be a perfect field of characteristic $p > 0$, let G be a closed subgroup of \mathbb{T} , and let X be a G -equivariant $N^{\text{cy}}(Hk)$ -module with $\pi_* X$ finitely generated over $THH_*(k)$. Then $\pi_*^{tG} X$ is finitely generated over $\pi_*^{tG} THH(k)$.*

PROOF. We can assume without loss of generality that $G = \mathbb{T}$ or $G = C_{p^r}$ for some $r > 0$. In the case when $G = \mathbb{T}$, we let $r = \infty$ and we understand $p^\infty = 0$ and $\mathbb{W}k_\infty = \mathbb{W}k$.

First we note that the conditionally convergent Greenlees Tate spectral sequence is strongly convergent: The hypothesis that $\pi_*(X)$ is finitely generated over $THH_*(k)$ implies that $E_{i,j}^2 = 0$ for j small enough (below the minimum degree of a generator) and each $E_{i,j}^2$ is a finite dimensional vector space over k .

The Greenlees Tate spectral sequence for X is a module over the Greenlees Tate spectral sequence for $THH(k)$. In the latter, the E^2 -term is isomorphic as a graded ring to $k[\bar{v}, \bar{v}^{-1}, t]$ in the case $G = \mathbb{T}$ and $k[\bar{v}, \bar{v}^{-1}, t, b]/b^2$ in the case when G is finite, where \bar{v} is in bidegree $(-2, 0)$ (the image of $v \in \pi_{-2}^{tG} THH(k)$ in the spectral sequence), t is in bidegree $(0, 2)$ (the image of a generator of $\pi_2 THH(k)$), and b is in bidegree $(1, 0)$. The elements \bar{v} and t are infinite cycles while $d^{2r+1}b = \bar{v}^r t^r$, q.v. [14, 6.2]. When $r = 1$, t becomes zero on E^4 , but otherwise, we can choose t so that it represents pv^{-1} in the spectral sequence.

The E^2 -term for X is naturally isomorphic as a graded $k[\bar{v}, \bar{v}^{-1}, t]$ -module to $THH_*(X) \otimes_{THH_*(k)} k[\bar{v}, \bar{v}^{-1}, t]$ in the case when $G = \mathbb{T}$ and $THH_*(X) \otimes_{THH_*(k)} k[\bar{v}, \bar{v}^{-1}, t, b]/b^2$ in the case when G is finite. In either case, the hypothesis that $THH_*(X)$ is finitely generated over $THH_*(k)$ implies that the E^2 -term for X is finitely generated over $k[\bar{v}, \bar{v}^{-1}, t]$. Since $k[\bar{v}, \bar{v}^{-1}, t]$ is Noetherian, the E^∞ -term is also finitely generated over $k[\bar{v}, \bar{v}^{-1}, t]$. The theorem now follows from a standard spectral sequence comparison argument; we give the full proof in the current context.

Choose elements $\bar{x}_1, \dots, \bar{x}_n$ in $E_{*,*}^\infty$ that generate E^∞ as a $k[\bar{v}, \bar{v}^{-1}, t]$ -module. Choose representatives x_1, \dots, x_n ; we need to show that x_1, \dots, x_n generate $\pi_*^{tG} X$ over $\mathbb{W}k_r[v, v^{-1}]$. Let $y \neq 0$ be any element of $\pi_*^{tG} X$ and let \bar{y} denote the element in $E_{i,j}^\infty$ representing y , where y is in filtration level i (but not $i-1$) and total degree $i+j$. Define $i_\ell, j_\ell \in \mathbb{Z}$ by $\bar{x}_\ell \in E_{i_\ell, j_\ell}^\infty$. It will also be convenient to write $d = i+j$ for the total degree of y and $d_\ell = i_\ell + j_\ell$ for the total degree of x_ℓ . Since y is arbitrary, it suffices to show that y is in the submodule of $\pi_*^{tG} X$ generated by x_1, \dots, x_n over $\pi_*^{tG} THH(k)$.

Since $E_{*,*}^\infty$ is generated by $\bar{x}_1, \dots, \bar{x}_n$ over $k[\bar{v}, \bar{v}^{-1}, t]$, we can write

$$\bar{y} = \bar{a}_1^0 \bar{v}^{\frac{1}{2}(i_1-i)} t^{\frac{1}{2}(j-j_1)} \bar{x}_1 + \dots + \bar{a}_n^0 \bar{v}^{\frac{1}{2}(i_n-i)} t^{\frac{1}{2}(j-j_n)} \bar{x}_n$$

for some $\bar{a}_1^0, \dots, \bar{a}_n^0 \in k$, where we must have $\bar{a}_\ell^0 = 0$ if $i_\ell - i$ is odd, $j - j_\ell$ is odd, or $j - j_\ell < 0$. Let $a_\ell^0 = \omega(\bar{a}_\ell^0) \in \mathbb{W}k$ using the Teichmüller character and let

$$\begin{aligned} y_0 &= a_1^0 v^{\frac{1}{2}(i_1-i)} (pv^{-1})^{\frac{1}{2}(j-j_1)} x_1 + \dots + a_n^0 v^{\frac{1}{2}(i_n-i)} (pv^{-1})^{\frac{1}{2}(j-j_n)} x_n \\ &= a_n^0 p^{\frac{1}{2}(j-j_1)} v^{\frac{1}{2}(d_1-d)} x_1 + \dots + a_n^0 p^{\frac{1}{2}(j-j_n)} v^{\frac{1}{2}(d_n-d)} x_n. \end{aligned}$$

We again have $a_\ell^0 = 0$ when $d_\ell - d$ is odd, $j - j_\ell$ is odd, or $j - j_\ell < 0$, so this formula makes sense. Let $z_1 = y - y_0$. Since y_0 also represents \bar{y} in $E_{i,j}^\infty$, we must have that z_1 is in filtration degree $i-1$ or lower, so represents an element \bar{z}_1 in $E_{i-s_1, j+s_1}^\infty$ for some $s_1 > 0$. Writing \bar{z}_1 in terms of the generators $\bar{x}_1, \dots, \bar{x}_n$, we must have

$$\bar{z}_1 = \bar{a}_1^0 \bar{v}^{\frac{1}{2}(i_1-i+s_1)} t^{\frac{1}{2}(j-j_1+s_1)} \bar{x}_1 + \dots + \bar{a}_n^0 \bar{v}^{\frac{1}{2}(i_n-i+s_1)} t^{\frac{1}{2}(j-j_n+s_1)} \bar{x}_n$$

for some $\bar{a}_1^1, \dots, \bar{a}_n^1 \in k$. Let $a_\ell^1 = \omega(\bar{a}_\ell^1)$ and let

$$\begin{aligned} y_1 &= y_0 + a_1^1 p^{\frac{1}{2}(j-j_1+s_1)} v^{\frac{1}{2}(d_1-d)} x_1 + \dots + a_n^1 p^{\frac{1}{2}(j-j_n+s_1)} v^{\frac{1}{2}(d_n-d)} x_n \\ &= (a_1^0 p^{\frac{1}{2}(j-j_1)} + a_1^1 p^{\frac{1}{2}(j-j_1+s_1)}) v^{\frac{1}{2}(d_1-d)} x_1 + \dots \\ &\quad + (a_n^0 p^{\frac{1}{2}(j-j_n)} + a_n^1 p^{\frac{1}{2}(j-j_n+s_1)}) v^{\frac{1}{2}(d_n-d)} x_n. \end{aligned}$$

Let $z_2 = y - y_2$; then z_2 must be in filtration level $i - 1 - s_1$, so represents an element \bar{z}_2 in $E_{i-s_2,j+s_2}^\infty$ for some $s_2 > s_1$. Inductively construct as above y_m, z_m , with $\bar{z}_m \in E_{i-s_m,j+s_m}^\infty$ for a strictly increasing sequence of positive integers s_m and with y_m of the form

$$\begin{aligned} y_m &= (a_1^0 p^{\frac{1}{2}(j-j_1)} + \dots + a_1^m p^{\frac{1}{2}(j-j_1+s_m)}) v^{\frac{1}{2}(d_1-d)} x_1 + \dots \\ &\quad + (a_n^0 p^{\frac{1}{2}(j-j_n)} + \dots + a_n^m p^{\frac{1}{2}(j-j_n+s_m)}) v^{\frac{1}{2}(d_n-d)} x_n, \end{aligned}$$

where $a_\ell^m = 0$ if $j - j_\ell + s_m$ is odd or negative or if $d_\ell - d$ is odd. Because $s_m \rightarrow \infty$, the coefficients

$$a_\ell^0 p^{\frac{1}{2}(j-j_\ell)} + \dots + a_\ell^m p^{\frac{1}{2}(j-j_\ell+s_m)}$$

converge to define an element y_∞ in the submodule of $\pi_*^{tG} X$ generated by x_1, \dots, x_n over $\pi_*^{tG} THH(k)$. The difference $y - y_\infty$ is in filtration level s_m for all m , and so $y = y_\infty$ since the filtration on homotopy groups is complete. \square

18. Comparing monoidal models

In Section 11 we defined a lax monoidal model T^M for the Tate fixed point functor for any finite group G and in Section 2 we defined a lax symmetric monoidal model $JT_G^{\mathcal{L}}$. In this section we argue that these define the same lax monoidal structure on the Tate fixed points viewed as a functor to the stable category. This comparison is in itself is not enough to compare the map

$$T^M X \wedge_{T^M A}^{\mathbf{L}} T^M Y \longrightarrow T^M(X \wedge_A^{\mathbf{L}} Y)$$

we used in Section 5 to the map

$$JT_G^{\mathcal{L}} X \wedge_{JT_G^{\mathcal{L}} A}^{\mathbf{L}} JT_G^{\mathcal{L}} Y \longrightarrow JT_G^{\mathcal{L}}(X \wedge_A^{\mathbf{L}} Y)$$

implicitly in the statement of the main theorem; we also make that comparison here.

Both arguments use an elaboration of the operadic structure on $T_G^{\mathcal{O}}$ described in (2.4), which for $n = 2$ takes the form

$$\mathcal{O}(2)_+ \wedge T_G^{\mathcal{O}}(X) \wedge T_G^{\mathcal{O}}(Y) \longrightarrow T_G^{\mathcal{O}}(X \wedge Y).$$

This structure uses the diagonal map on EG . Any operad \mathcal{O} admits a canonical map $\mathcal{O} \rightarrow \text{Com} \rightarrow \mathcal{O}_1^{\Xi}$ and we view the structure of (2.4) as corresponding to this map. We can use any map of operads $\mathcal{O} \rightarrow \mathcal{O}_1^{\Xi}$ and then the structure in (2.4) generalizes to this context, where we use the coaction of \mathcal{O}_1^{Ξ} on EG in place of the diagonal map. This structure is natural in maps of $(E_\infty$ or A_∞) operads over \mathcal{O}_1^{Ξ} .

Now consider the maps of operads over \mathcal{O}_1^{Ξ}

$$\mathcal{C}_1 \longleftarrow \mathcal{L} \times \mathcal{C}_1 \longrightarrow \mathcal{L} \times \mathcal{O}_1^{\Xi} \longleftarrow \mathcal{L}$$

where the lefthand map is projection, the middle map is induced by the inclusion of \mathcal{C}_1 in \mathcal{O}_1^{Ξ} and the righthand map is induced by the identity on \mathcal{L} and the map

$\mathcal{L} \rightarrow \mathcal{C}\text{om} \rightarrow \mathcal{O}_1^{\Xi}$. The backward maps are weak equivalence, and looking at the structure above, we get a natural commuting diagram

$$\begin{array}{ccc}
 \bar{\mathcal{C}}_1(2)_+ \wedge T_G^{\mathcal{C}_1} X \wedge T_G^{\mathcal{C}_1} Y & \longrightarrow & T_G^{\mathcal{C}_1}(X \wedge Y) \\
 \simeq \uparrow & & \uparrow \simeq \\
 (\mathcal{L}(2) \times \bar{\mathcal{C}}_1(2))_+ \wedge T_G^{\mathcal{L} \times \mathcal{C}_1} X \wedge T_G^{\mathcal{L} \times \mathcal{C}_1} Y & \longrightarrow & T_G^{\mathcal{L} \times \mathcal{C}_1}(X \wedge Y) \\
 \simeq \downarrow & & \downarrow \simeq \\
 (\mathcal{L}(2) \times \mathcal{O}_1^{\Xi}(2))_+ \wedge T_G^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} X \wedge T_G^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} Y & \longrightarrow & T_G^{\mathcal{L} \times \mathcal{O}_1^{\Xi}}(X \wedge Y) \\
 \simeq \uparrow & & \uparrow \simeq \\
 \mathcal{L}(2)_+ \wedge T_G^{\mathcal{L}} X \wedge T_G^{\mathcal{L}} Y & \longrightarrow & T_G^{\mathcal{L}}(X \wedge Y)
 \end{array}$$

(where we have used the identity permutation subspace $\bar{\mathcal{C}}_1(2)$ of $\mathcal{C}_1(2)$ for the entries involving \mathcal{C}_1). Precomposing with the universal map from the derived smash product to the smash product and restricting to the case when X and Y are cofibrant, we get an analogous diagram with the smash product replaced by the derived smash product. Together with the canonical map $\mathbb{S} \rightarrow T_G^{\mathcal{O}} \mathbb{S}$, the horizontal maps give structure maps for lax monoidal structures on the Tate fixed point functor and the vertical maps imply that these structures coincide. We can now prove the following theorem.

THEOREM 18.1. *The canonical isomorphism in the stable category between JT_G and T^M preserves the lax monoidal structure.*

PROOF. The comparison of unit maps is clear. The canonical isomorphism in the stable category $JT_G^{\mathcal{L}} \rightarrow T_G^{\mathcal{L}}$ is symmetric monoidal and the canonical map in the stable category

$$\mathcal{L}(2)_+ \wedge T_G^{\mathcal{L}} X \wedge^{\mathbf{L}} T_G^{\mathcal{L}} Y \longrightarrow T_G^{\mathcal{L}} X \wedge^{\mathbf{L}} T_G^{\mathcal{L}} Y$$

is an isomorphism, so the work above reduces to comparing the associativity map for T^M to the associativity map on $T_G^{\mathcal{C}_1}$. For this it is enough to compare the associativity map for $\bar{T}_{*,*}^{\bar{M}}$ and $\bar{T}_{*,*}$, where $\bar{T}_{*,*}$ is defined in Construction 11.1. Compatibility of these maps is easily seen from the commuting diagram

$$\begin{array}{ccc}
 \bar{T}_{i_1,j_1}^{\bar{M}} X \wedge \bar{T}_{i_2,j_2}^{\bar{M}} Y & \longrightarrow & \bar{T}_{i,j}^{\bar{M}}(X \wedge Y) \\
 \downarrow & & \downarrow \\
 \bar{\mathcal{C}}_1(2)_+ \wedge \bar{T}_{i_1,j_1} X \wedge \bar{T}_{i_2,j_2} Y & \longrightarrow & \bar{T}_{i,j}(X \wedge Y)
 \end{array}$$

where the vertical map on the left is induced by

$$\begin{aligned}
 \bar{T}_{i_1,j_1}^{\bar{M}} X \wedge \bar{T}_{i_2,j_2}^{\bar{M}} Y &= (\bar{T}_{i_1,j_1} X \wedge \mathbb{R}_+^{>0}) \wedge (\bar{T}_{i_2,j_2} Y \wedge \mathbb{R}_+^{>0}) \\
 &\cong (\mathbb{R}^{>0} \times \mathbb{R}^{>0})_+ \wedge \bar{T}_{i_1,j_1} X \wedge \bar{T}_{i_2,j_2} Y \\
 &\longrightarrow \bar{\mathcal{C}}_1(2)_+ \wedge \bar{T}_{i_1,j_1} X \wedge \bar{T}_{i_2,j_2} Y
 \end{aligned}$$

(with the last map induced by the map μ_2 of Construction 11.3), and the vertical map on the right is induced by the collapse map $\mathbb{R}^{>0} \rightarrow *$. \square

Next we move on to compare the maps

$$T^M X \wedge_{T^M A}^{\mathbf{L}} T^M Y \longrightarrow T^M(X \wedge_A^{\mathbf{L}} Y)$$

and

$$JT_G^{\mathcal{L}} X \wedge_{JT_G^{\mathcal{L}} A}^{\mathbf{L}} JT_G^{\mathcal{L}} Y \longrightarrow JT_G^{\mathcal{L}} (X \wedge_A^{\mathbf{L}} Y).$$

First we need to compare the objects, and we follow roughly same strategy as above, transporting the constructions across the maps of operads. Following the ideas of [25], we construct the balanced smash product of operadic modules.

The diagram preceding the previous theorem compares the $n = 2$ case of the structure maps

$$(18.2) \quad \overline{\mathcal{O}}(n)_+ \wedge T_G^{\mathcal{O}} X_1 \wedge \cdots \wedge T_G^{\mathcal{O}} X_n \longrightarrow T_G^{\mathcal{O}} (X_1 \wedge \cdots \wedge X_n)$$

for $\mathcal{O} = \overline{\mathcal{C}}_1, \mathcal{L} \times \overline{\mathcal{C}}_1, \mathcal{L} \times \mathcal{O}_1^{\Xi}$, or \mathcal{L} , where $\overline{\mathcal{O}}(n)$ denotes the corresponding n th space (for the E_∞ operads) or identity permutation subspace (for the A_∞ operads). We also write $\overline{\mathcal{O}}$ for the non- Σ A_∞ operad corresponding to \mathcal{O} : For one of the E_∞ operads $\overline{\mathcal{O}} = \mathcal{O}$ with the permutations forgotten, and for one of the A_∞ operads \mathcal{O} is the identity permutation component in each arity. It is clear from the general case of the structure map and comparison diagram that for an associative ring orthogonal G -spectrum A , $T_G^{\mathcal{O}} A$ inherits the structure of a (non- Σ) $\overline{\mathcal{O}}$ -algebra and the maps of operads $\mathcal{O} \rightarrow \mathcal{O}'$ over \mathcal{O}_1^{Ξ} induce maps of $\overline{\mathcal{O}}$ -algebras $T_G^{\mathcal{O}} A \rightarrow T_G^{\mathcal{O}'} A$. In addition, for Y a left A -module, $T_G^{\mathcal{O}} Y$ inherits the structure of a left $T_G^{\mathcal{O}} A$ -module over $\overline{\mathcal{O}}$: It has structure maps of the form

$$\overline{\mathcal{O}}(n+1)_+ \wedge (T_G^{\mathcal{O}} A)^{(n)} \wedge T_G^{\mathcal{O}} Y \longrightarrow T_G^{\mathcal{O}} Y$$

(for $n \geq 0$) satisfying the usual unit and associativity conditions with respect to the operadic multiplication on A . The maps of operads $\mathcal{O} \rightarrow \mathcal{O}'$ above induce maps of left $T_G^{\mathcal{O}} A$ -modules over $\overline{\mathcal{O}}$. Analogous observations apply to right modules.

For an $\overline{\mathcal{O}}$ -algebra B , the category of left B -modules over $\overline{\mathcal{O}}$ is equivalent to the category of left modules for an associative ring symmetric spectrum $U_L^{\mathcal{O}} B$, called the *left enveloping algebra*. Concretely $U_L^{\mathcal{O}} B$ can be constructed as the coequalizer

$$\bigvee_{n, m_1, \dots, m_n} (\overline{\mathcal{O}}(n+1) \times (\overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n)))_+ \wedge B^{(m)} \rightrightarrows \bigvee_n \overline{\mathcal{O}}(n+1)_+ \wedge B^{(n)}$$

for $n \geq 0$, $m_1, \dots, m_n \geq 0$, and $m = m_1 + \cdots + m_n$, where one map is induced by the operadic multiplication

$$\begin{aligned} \overline{\mathcal{O}}(n+1) \times (\overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n)) \\ \cong \overline{\mathcal{O}}(n+1) \times (\overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n) \times \{1\}) \longrightarrow \overline{\mathcal{O}}(m+1) \end{aligned}$$

and the other by the $\overline{\mathcal{O}}$ -action

$$\overline{\mathcal{O}}(m_i)_+ \wedge B^{(m_i)} \longrightarrow B.$$

The unit is induced by the map

$$\mathbb{S} \cong \{1\}_+ \wedge \mathbb{S} = \{1\}_+ \wedge B^{(0)} \longrightarrow \overline{\mathcal{O}}(1) \wedge B^{(0)}$$

and the multiplication is induced by the map

$$\begin{aligned} (\overline{\mathcal{O}}(n+1)_+ \wedge B^{(n)}) \wedge (\overline{\mathcal{O}}(n'+1)_+ \wedge B^{(n')}) \\ \cong (\overline{\mathcal{O}}(n+1) \times \overline{\mathcal{O}}(n'+1))_+ \wedge B^{(n+n')} \longrightarrow \overline{\mathcal{O}}(n+n'+1)_+ \wedge B^{(n+n')} \end{aligned}$$

induced by the operadic multiplication \circ_{n+1} . Analogously, the category of right B -modules over $\overline{\mathcal{O}}$ is the category of right modules for the *right enveloping algebra*

$U_R^{\overline{\mathcal{O}}} B$, which may be constructed as an analogous coequalizer with identical formulas except that it uses the map

$$\begin{aligned} \overline{\mathcal{O}}(n+1) \times (\overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n)) \\ \cong \overline{\mathcal{O}}(n+1) \times (\{1\} \times \overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n)) \longrightarrow \overline{\mathcal{O}}(m+1) \end{aligned}$$

in the construction and the map

$$\begin{aligned} (\overline{\mathcal{O}}(n+1)_+ \wedge B^{(n)}) \wedge (\overline{\mathcal{O}}(n'+1)_+ \wedge B^{(n')}) \\ \cong (\overline{\mathcal{O}}(n'+1) \times \overline{\mathcal{O}}(n+1))_+ \wedge B^{(n+n')} \longrightarrow \overline{\mathcal{O}}(n+n'+1)_+ \wedge B^{(n+n')} \end{aligned}$$

induced by \circ_1 (leaving the factors of B in the same order) in the multiplication.

CONSTRUCTION 18.3. We construct a right $(U_R^{\overline{\mathcal{O}}} B)^{\text{op}} \wedge (U_L^{\overline{\mathcal{O}}} B)$ -module $\text{Bal}^{\overline{\mathcal{O}}}(B)$ as follows. The underlying orthogonal spectrum is the coequalizer

$$\bigvee_{n, m_1, \dots, m_n} (\overline{\mathcal{O}}(n+2) \times (\overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n)))_+ \wedge B^{(m)} \rightrightarrows \bigvee_n \overline{\mathcal{O}}(n+2)_+ \wedge B^{(n)}$$

where one map is induced by the operadic multiplication

$$\begin{aligned} \overline{\mathcal{O}}(n+2) \times (\overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n)) \\ \cong \overline{\mathcal{O}}(n+2) \times (\{1\} \times \overline{\mathcal{O}}(m_1) \times \cdots \times \overline{\mathcal{O}}(m_n) \times \{1\}) \longrightarrow \overline{\mathcal{O}}(m+2) \end{aligned}$$

and the other by the $\overline{\mathcal{O}}$ -action

$$\overline{\mathcal{O}}(m_i)_+ \wedge B^{(m_i)} \longrightarrow B.$$

The left $U_R^{\overline{\mathcal{O}}} B$ -action is induced by the map

$$\begin{aligned} (\overline{\mathcal{O}}(n+1)_+ \wedge B^{(n)}) \wedge (\overline{\mathcal{O}}(n'+2)_+ \wedge B^{(n')}) \\ \cong (\overline{\mathcal{O}}(n'+2) \times \overline{\mathcal{O}}(n+1))_+ \wedge B^{(n)} \wedge B^{(n')} \longrightarrow \overline{\mathcal{O}}(n'+n+2)_+ \wedge B^{(n+n')} \end{aligned}$$

induced by \circ_1 (where the factors of B remain in the same order). The right $U_L^{\overline{\mathcal{O}}} B$ -action is induced by the map

$$\begin{aligned} (\overline{\mathcal{O}}(n'+2)_+ \wedge B^{(n')}) \wedge (\overline{\mathcal{O}}(n+1)_+ \wedge B^{(n)}) \\ \cong (\overline{\mathcal{O}}(n'+2) \times \overline{\mathcal{O}}(n+1))_+ \wedge B^{(n')} \wedge B^{(n)} \longrightarrow \overline{\mathcal{O}}(n'+n+2)_+ \wedge B^{(n'+n)} \end{aligned}$$

induced by $\circ_{n'+2}$.

DEFINITION 18.4. Let $\overline{\mathcal{O}}$ be a non- Σ A_∞ operad, let B be an $\overline{\mathcal{O}}$ -algebra, let M be a right B -module over $\overline{\mathcal{O}}$, and let N be a left B -module over $\overline{\mathcal{O}}$. Define the *point-set balanced smash product* of M and N over B and $\overline{\mathcal{O}}$ to be the orthogonal spectrum

$$M \wedge_B^{\overline{\mathcal{O}}} N := \text{Bal}^{\overline{\mathcal{O}}}(B) \wedge_{(U_R^{\overline{\mathcal{O}}} B)^{\text{op}} \wedge (U_L^{\overline{\mathcal{O}}} B)} (M \wedge N),$$

an enriched bifunctor

$$\mathcal{M}\text{od}_{U_R^{\overline{\mathcal{O}}} B}^r \wedge \mathcal{M}\text{od}_{U_L^{\overline{\mathcal{O}}} B}^\ell \longrightarrow \mathcal{S}.$$

This definition relates to the example of $T_G^{\mathcal{O}}$ as follows. When A is an associative ring orthogonal G -spectrum and X and Y are right and left A -modules, the structure maps (18.2)

$$\overline{\mathcal{O}}(n+2)_+ \wedge T_G^{\mathcal{O}} X \wedge (T_G^{\mathcal{O}} A)^{(n)} \wedge T_G^{\mathcal{O}} Y \longrightarrow T_G^{\mathcal{O}}(X \wedge A^{(n)} \wedge Y) \longrightarrow T_G^{\mathcal{O}}(X \wedge_A Y)$$

fit together to define a map

$$(18.5) \quad T_G^{\mathcal{O}} X \wedge_{T_G^{\mathcal{O}} A}^{\overline{\mathcal{O}}} T_G^{\mathcal{O}} Y \longrightarrow T_G^{\mathcal{O}}(X \wedge_A Y),$$

naturally in A , X , Y , and \mathcal{O} .

We have a derived version of the balanced smash product that is easiest to define if we restrict to the non- Σ A_∞ operads $\overline{\mathcal{C}}_1$, $\mathcal{L} \times \overline{\mathcal{C}}_1$, $\mathcal{L} \times \mathcal{O}_1^\Xi$, and \mathcal{L} involved in our comparison. These operads have the following special property that we prove in the next section.

THEOREM 18.6. *Let $\overline{\mathcal{O}} = \overline{\mathcal{C}}_1$, $\mathcal{L} \times \overline{\mathcal{C}}_1$, $\mathcal{L} \times \mathcal{O}_1^\Xi$, or \mathcal{L} , and let B be any $\overline{\mathcal{O}}$ -algebra. The maps*

$$\begin{aligned} \overline{\mathcal{O}}(2)_+ \wedge B &\longrightarrow U_L^{\overline{\mathcal{O}}} B \\ \overline{\mathcal{O}}(2)_+ \wedge B &\longrightarrow U_R^{\overline{\mathcal{O}}} B \\ \overline{\mathcal{O}}(3)_+ \wedge B &\longrightarrow \text{Bal}^{\overline{\mathcal{O}}} B \end{aligned}$$

in the defining colimits are weak equivalences.

REMARK 18.7. For more general non- Σ A_∞ operads, we do not expect the enveloping algebras to always have the correct homotopy type for arbitrary $\overline{\mathcal{O}}$ -algebras B ; they will, however, have the correct homotopy types for cofibrant $\overline{\mathcal{O}}$ -algebras, and so cofibrant replacement of the algebra B is necessary to obtain the correct derived categories of B -modules in this setting. This presents no real practical difficulties except to complicate definitions and statements of theorems. To avoid these complications, we define derived functors only for $\overline{\mathcal{O}} = \overline{\mathcal{C}}_1$, $\mathcal{L} \times \overline{\mathcal{C}}_1$, $\mathcal{L} \times \mathcal{O}_1^\Xi$, or \mathcal{L} .

DEFINITION 18.8. For $\overline{\mathcal{O}} = \overline{\mathcal{C}}_1$, $\mathcal{L} \times \overline{\mathcal{C}}_1$, $\mathcal{L} \times \mathcal{O}_1^\Xi$, or \mathcal{L} , define the *derived balanced smash product* $\text{Tor}^{B, \overline{\mathcal{O}}}(M, N)$ as the left derived enriched bifunctor [21, 5.3] of the point-set balanced product functor.

As a special case of [21, 8.2], the derived balanced smash product may be constructed by using a cofibrant left $(U_R^{\overline{\mathcal{O}}} B)^{\text{op}} \wedge (U_L^{\overline{\mathcal{O}}} B)$ -module approximation of $M \wedge N$; one good way to choose such an approximation is to smash a cofibrant right $U_R^{\overline{\mathcal{O}}} B$ -module approximation of M with a cofibrant left $U_L^{\overline{\mathcal{O}}} B$ -module approximation of N .

We now need to compare the balanced smash product of Definition 18.4 with the Blumberg-Hill EKMM smash product in $\mathcal{L}(1)$ -spectra in orthogonal spectra and the balanced smash product of modules over an associative ring orthogonal spectrum. The first of these is the following theorem, the proof of which is given in the next section. Recall that we use J to denote the functor $(-) \wedge_{\mathcal{L}} \mathbb{S}$ from $\mathcal{L}(1)$ -spectra in orthogonal spectra to EKMM \mathbb{S} -modules in orthogonal spectra.

THEOREM 18.9. *Let B be an associative ring EKMM \mathbb{S} -module in orthogonal spectra, M a right B -module, and N a left B -module (for the EKMM smash product). Then the Blumberg-Hill EKMM smash product $M \wedge_B N$ is canonically isomorphic to $J(M \wedge_B^{\mathcal{L}} N)$ for the balanced smash product constructed in Definition 18.4.*

The following is now an immediate consequence of Theorems 18.6 and 18.9.

COROLLARY 18.10. *Let A be an associative ring orthogonal G -spectrum, let X be a right A -module and let Y be a left A -module. Then the natural transformations of balanced smash products*

$$\begin{aligned} T_G^{\mathcal{C}_1} X \wedge_{T_G^{\mathcal{C}_1} A}^{\bar{\mathcal{C}}_1} T_G^{\mathcal{C}_1} Y &\longleftarrow T_G^{\mathcal{L} \times \mathcal{C}_1} X \wedge_{(T_G^{\mathcal{L} \times \mathcal{C}_1} A)}^{\mathcal{L} \times \bar{\mathcal{C}}_1} T_G^{\mathcal{L} \times \mathcal{C}_1} Y \longrightarrow \\ T_G^{\mathcal{L} \times \mathcal{O}_1^{\bar{\Xi}}} X \wedge_{(T_G^{\mathcal{L} \times \mathcal{O}_1^{\bar{\Xi}}} A)}^{\mathcal{L} \times \mathcal{O}_1^{\bar{\Xi}}} T_G^{\mathcal{L} \times \mathcal{O}_1^{\bar{\Xi}}} Y &\longleftarrow T_G^{\mathcal{L}} X \wedge_{T_G^{\mathcal{L}} A}^{\mathcal{L}} T_G^{\mathcal{L}} Y \\ &\longleftarrow J(T_G^{\mathcal{L}} X \wedge_{J T_G^{\mathcal{L}} A}^{\mathcal{L}} J T_G^{\mathcal{L}} Y) \cong J T_G^{\mathcal{L}} X \wedge_{J T_G^{\mathcal{L}} A}^{\mathcal{L}} J T_G^{\mathcal{L}} Y \end{aligned}$$

induce isomorphism of derived balanced smash products.

We still need to compare the balanced smash product $T_G^{\mathcal{C}_1} X \wedge_{T_G^{\mathcal{C}_1} A}^{\bar{\mathcal{C}}_1} T_G^{\mathcal{C}_1} Y$ in the context of $\bar{\mathcal{C}}_1$ -algebras and modules with the balanced smash product $T^{\bar{M}} X \wedge_{T^{\bar{M}} A} T^{\bar{M}} Y$ in the context of associative orthogonal spectra and modules. As in Section 11, for any $\bar{\mathcal{C}}_1$ -algebra B , we can form the Moore construction B^M as the pushout

$$B^M := (\mathbb{S} \wedge \mathbb{R}_+^{>0}) \cup_{\mathbb{S} \wedge \mathbb{R}_+^{>0}} (B \wedge \mathbb{R}_+^{>0}),$$

which has the natural structure of an associative ring orthogonal spectrum and also has the property that the canonical map $B^M \rightarrow B$ (induced by the map collapsing \mathbb{R} to a point) is a homotopy equivalence of orthogonal spectra. Given X and Y right and left B -modules over $\bar{\mathcal{C}}_1$, the Moore construction $(-)^{\bar{M}} := (-) \wedge \mathbb{R}_+^{>0}$ converts X and Y to B^M -modules. We then have a natural map

$$X^{\bar{M}} \wedge_{B^M} Y^{\bar{M}} \cong X^{\bar{M}} \wedge_{B^M} B^M \wedge_{B^M} Y^{\bar{M}} \longrightarrow X \wedge_B^{\bar{\mathcal{C}}_1} Y$$

induced by the map μ_3 (from Construction 11.3) interpreted as a map

$$(X \wedge \mathbb{R}_+^{>0}) \wedge (B \wedge \mathbb{R}_+^{>0}) \wedge (Y \wedge \mathbb{R}_+^{>0}) \longrightarrow \bar{\mathcal{C}}_1(3)_+ \wedge X \wedge B \wedge Y$$

and the unital extension of μ_3 viewed as a map

$$(X \wedge \mathbb{R}_+^{>0}) \wedge (\mathbb{S} \wedge \mathbb{R}_+^{>0}) \wedge (Y \wedge \mathbb{R}_+^{>0}) \longrightarrow \bar{\mathcal{C}}_1(2)_+ \wedge X \wedge Y$$

composed with the maps in the defining colimit

$$\bar{\mathcal{C}}_1(3)_+ \wedge X \wedge B \wedge Y \longrightarrow X \wedge_B^{\bar{\mathcal{C}}_1} Y \quad \text{and} \quad \bar{\mathcal{C}}_1(2)_+ \wedge X \wedge Y \longrightarrow X \wedge_B^{\bar{\mathcal{C}}_1} Y.$$

THEOREM 18.11. *For B a $\bar{\mathcal{C}}_1$ -algebra and X and Y right and left B -modules over $\bar{\mathcal{C}}_1$, the canonical map*

$$X^{\bar{M}} \wedge_{B^M} Y^{\bar{M}} \longrightarrow X \wedge_B^{\bar{\mathcal{C}}_1} Y$$

induces an isomorphism on the derived balanced smash products.

PROOF. The proof follows the usual induction up the cellular filtration argument. We may as well take X and Y to be cofibrant. Let $Y' \rightarrow Y^M$ be a cofibrant approximation; we want to show that the composite point-set map

$$X^{\bar{M}} \wedge_{B^M} Y' \longrightarrow X^{\bar{M}} \wedge_{B^M} Y^{\bar{M}} \longrightarrow X \wedge_B^{\bar{\mathcal{C}}_1} Y$$

is a weak equivalence. Without loss of generality, we can assume that Y is a cellular left B -module, $Y = \text{colim } Y_n$ where each Y_n is formed as the pushout of cell attachments using the generating cofibrations [24, 12.1] in the model category of $U_L^{\mathcal{C}_1} B$ -modules as cells. We can likewise arrange that $Y' = \text{colim } Y'_n$ where each $Y'_n \rightarrow Y'_{n+1}$ is the inclusion of a subcomplex, and we have a system of compatible

weak equivalences $Y'_n \rightarrow Y_n^M$. Since both $X^{\bar{M}} \wedge_{B^M} (-)$ and $X \wedge_B^{\bar{C}_1} (-)$ preserve h -cofibrations and all colimits, it suffices to show that the maps

$$X^{\bar{M}} \wedge_{B^M} (Y'_n / Y'_{n-1}) \longrightarrow X^{\bar{M}} \wedge_{B^M} (Y_n^{\bar{M}} / Y_{n-1}^{\bar{M}}) \longrightarrow X \wedge_B^{\bar{C}_1} (Y_n / Y_{n-1})$$

are weak equivalences, where we understand $Y'_{-1} = Y_{-1} = *$. Since both functors $X^{\bar{M}} \wedge_{B^M} (-)$ and $X \wedge_B^{\bar{C}_1} (-)$ preserve weak equivalences between cofibrant objects and coproducts, it suffices to consider the case when $Y_n / Y_{n-1} = B \wedge F_n S^m$ (for some $m, n \in \mathbb{N}$) and this case is clear. \square

COROLLARY 18.12. *Let A be an associative ring orthogonal G -spectrum, let X be a right A -module and let Y be a left A -module. Then the natural transformation of balanced smash products*

$$T^{\bar{M}} X \wedge_{T^M A} T^{\bar{M}} Y \longrightarrow T_G^{\mathcal{C}_1} X \wedge_{T_G^{\mathcal{C}_1} A} T_G^{\mathcal{C}_1} Y$$

induces an isomorphism on derived balanced smash products.

If the underlying orthogonal G -spectra of X and Y come with structure maps $\mathbb{S} \rightarrow X$ and $\mathbb{S} \rightarrow Y$, then the natural transformation of balanced smash products

$$T^{\bar{M}} X \wedge_{T^M A} T^{\bar{M}} Y \longrightarrow T^M X \wedge_{T^M A} T^M Y$$

induces a weak equivalence of derived balanced smash products. This completes a comparison of the derived balanced smash product for our filtered lax monoidal model and our (point-set) lax symmetric monoidal model. For the comparison of the maps

$$T^M X \wedge_{T^M A}^{\mathbf{L}} T^M Y \longrightarrow T^M (X \wedge_A^{\mathbf{L}} Y)$$

and

$$JT_G^{\mathcal{L}} X \wedge_{JT_G^{\mathcal{L}} A}^{\mathbf{L}} JT_G^{\mathcal{L}} Y \longrightarrow JT_G^{\mathcal{L}} (X \wedge_A^{\mathbf{L}} Y)$$

we combine with the natural maps on derived functors induced by the maps (18.5) to obtain the following commuting diagram, where all vertical arrows are isomorphisms in the stable category.

$$\begin{array}{ccc}
T^{\bar{M}} X \wedge_{T^M A}^{\mathbf{L}} T^{\bar{M}} Y & \longrightarrow & T^{\bar{M}} (X \wedge_A^{\mathbf{L}} Y) \\
\downarrow \simeq & & \downarrow \simeq \\
\text{Tor}^{T_G^{\mathcal{C}_1} A, \bar{C}_1} (T_G^{\mathcal{C}_1} X, T_G^{\mathcal{C}_1} Y) & \longrightarrow & T_G^{\mathcal{C}_1} (X \wedge_A^{\mathbf{L}} Y) \\
\uparrow \simeq & & \uparrow \simeq \\
\text{Tor}^{T_G^{\mathcal{L} \times \mathcal{C}_1} A, \mathcal{L} \times \bar{C}_1} (T_G^{\mathcal{L} \times \mathcal{C}_1} X, T_G^{\mathcal{L} \times \mathcal{C}_1} Y) & \longrightarrow & T_G^{\mathcal{L} \times \mathcal{C}_1} (X \wedge_A^{\mathbf{L}} Y) \\
\downarrow \simeq & & \downarrow \simeq \\
\text{Tor}^{T_G^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} A, \mathcal{L} \times \mathcal{O}_1^{\Xi}} (T_G^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} X, T_G^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} Y) & \longrightarrow & T_G^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} (X \wedge_A^{\mathbf{L}} Y) \\
\uparrow \simeq & & \uparrow \simeq \\
JT_G^{\mathcal{L}} X \wedge_{JT_G^{\mathcal{L}} A}^{\mathbf{L}} JT_G^{\mathcal{L}} Y & \longrightarrow & JT_G^{\mathcal{L}} (X \wedge_A^{\mathbf{L}} Y)
\end{array}$$

19. Identification of the enveloping algebras and Bal

In this section, we identify in more concrete terms the enveloping algebras $U_L^{\bar{\mathcal{O}}} B$, $U_R^{\bar{\mathcal{O}}} B$, and balanced product object $\text{Bal}^{\bar{\mathcal{O}}} B$ for an algebra B over one of the particular non- Σ A_∞ operads $\bar{\mathcal{O}} = \bar{\mathcal{C}}_1$, $\mathcal{L} \times \bar{\mathcal{C}}_1$, $\mathcal{L} \times \bar{\mathcal{O}}_1^{\bar{\Xi}}$, and \mathcal{L} . As a consequence, we deduce Theorems 18.6 and 18.9. The theorems only refer to the underlying orthogonal spectra, but we have included a full description of the ring and module structures for completeness. We work one operad at a time. In each of the following subsections, we use B to denote an arbitrary $\bar{\mathcal{O}}$ -algebra for the given non- Σ operad $\bar{\mathcal{O}}$.

The operad $\bar{\mathcal{C}}_1$. For this operad, the left enveloping algebra was identified in [25, 2.5] and we only need to review its description.

Let D denote the subspace of $\bar{\mathcal{C}}_1(1)$ where the interval does not start at 0; then $U_L^{\bar{\mathcal{C}}_1} B$ is the pushout

$$U_L^{\bar{\mathcal{C}}_1} B = (\bar{\mathcal{C}}_1(1)_+ \wedge \mathbb{S}) \cup_{D_+ \wedge \mathbb{S}} (D_+ \wedge B)$$

with unit induced by the map

$$\mathbb{S} \cong \{1\}_+ \wedge \mathbb{S} \longrightarrow \bar{\mathcal{C}}_1(1)_+ \wedge \mathbb{S} \longrightarrow U_L^{\bar{\mathcal{C}}_1} B$$

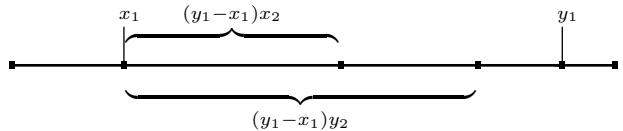
and product induced as follows. Given elements $[x_1, y_1]$ and $[x_2, y_2]$ of D , we get an element

$$[0, x_1/(x_1 + (y_1 - x_1)x_2)], [x_1/(x_1 + (y_1 - x_1)x_2), 1]$$

of $\bar{\mathcal{C}}_1(2)$ and an element

$$[x_1 + (y_1 - x_1)x_2, x_1 + (y_1 - x_1)y_2]$$

of D . This can also be expressed in terms of \circ_2 where we view D as in [25, §2] as the subset of $\bar{\mathcal{C}}_1(2)$ where the first interval starts at 0 and ends at the start of the second interval, which is the given interval that does not start at 0. The picture of $\{[0, x_1], [x_1, y_1]\} \circ_2 \{[0, x_2], [x_2, y_2]\}$ is



The first two segments translated and rescaled to begin at 0 and end at 1 give the element of $\bar{\mathcal{C}}_1(2)$ and the third segment gives the element of D . We use the element of $\bar{\mathcal{C}}_1(2)$ to multiply the copies of B and the element of D above for the new element of D . More formally, writing $f: D \times D \rightarrow \bar{\mathcal{C}}_1(2)$ and $g: D \times D \rightarrow \bar{\mathcal{C}}_1(1)$, the product is induced by

$$(D_+ \wedge B) \wedge (D_+ \wedge B) \cong (D \times D)_+ \wedge B \wedge B \xrightarrow{(g \times f) \wedge \text{id}} D_+ \wedge \bar{\mathcal{C}}_1(2)_+ \wedge B \wedge B \xrightarrow{\text{id} \wedge \xi_2} D_+ \wedge B$$

where ξ is the action of $\bar{\mathcal{C}}_1$ on B .

The right enveloping algebra has a similar description except that we use the subspace D' of $\bar{\mathcal{C}}_1(1)$ of intervals that do not end at 1. The underlying orthogonal spectrum is then

$$U_R^{\bar{\mathcal{C}}_1} B = (\bar{\mathcal{C}}_1(1)_+ \wedge \mathbb{S}) \cup_{D'_+ \wedge \mathbb{S}} (D'_+ \wedge B)$$

with unit induced by the map

$$\mathbb{S} \cong \{1\}_+ \wedge \mathbb{S} \longrightarrow \bar{\mathcal{C}}_1(1)_+ \wedge \mathbb{S} \longrightarrow U_R^{\bar{\mathcal{C}}_1} B$$

and product induced by the maps $D' \times D' \rightarrow \bar{\mathcal{C}}_1(2)$ and $D' \times D' \rightarrow D'$ that send the pair of elements $[x_1, y_1]$ and $[x_2, y_2]$ of D to the element

$$\left[0, \frac{(y_2 - x_2)(1 - y_1)}{(y_2 - x_2)(1 - y_1) + (1 - y_2)} \right], \left[\frac{(y_2 - x_2)(1 - y_1)}{(y_2 - x_2)(1 - y_1) + (1 - y_2)}, 1 \right]$$

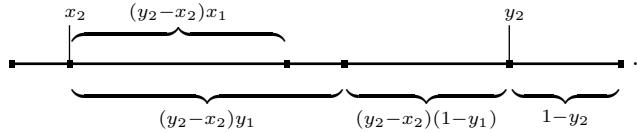
of $\bar{\mathcal{C}}_1(2)$ and the element

$$[x_2 + (y_2 - x_2)x_1, x_2 + (y_2 - x_2)y_1]$$

of D' , respectively. Here we are using the \circ_1 product

$$\{[x_2, y_2], [y_2, 1]\} \circ_1 \{[x_1, y_1], [y_1, 1]\},$$

whose picture is



The last two segments translated and rescaled to start at 0 and end at 1 give the element of $\bar{\mathcal{C}}_1(2)$ and the third segment (whose length is $(y_2 - x_2)(y_1 - x_1)$) gives the element of D' .

To identify $\text{Bal}^{\bar{\mathcal{C}}_1} B$, let C be the subset of $\bar{\mathcal{C}}_1(2)$ consisting of those pairs of intervals $[a, b], [c, d]$ with $b < c$. Let Z be the pushout

$$Z = (\bar{\mathcal{C}}_1(2)_+ \wedge \mathbb{S}) \cup_{C_+ \wedge \mathbb{S}} (C_+ \wedge B).$$

We define the right $U_L^{\bar{\mathcal{C}}_1} B$ -module structure essentially using \circ_2 . Specifically, we have maps

$$C \times D \longrightarrow \bar{\mathcal{C}}_1(2) \quad \text{and} \quad C \times D \longrightarrow C$$

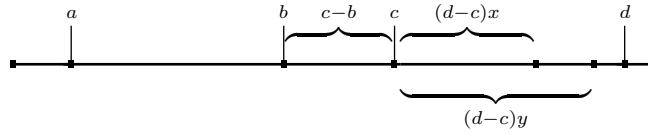
sending the elements $[a, b], [c, d]$ in C and $[x, y]$ in D to

$$[0, (c - b)/(c - b + x(d - c))], [(c - b)/(c - b + x(d - c)), 1]$$

in $\bar{\mathcal{C}}_1(2)$ and

$$[a, b], [c + (d - c)x, c + (d - c)y]$$

in C , respectively; we use the element of $\bar{\mathcal{C}}_1(2)$ to multiply the two factors of B and the element of C as the new element of C . Pictorially,



the element in $\bar{\mathcal{C}}_1(2)$ is the third and fourth segments translated and rescaled to start at 0 and end at 1, while the element of C is the second and fifth segments. We also need to describe what the action of the $\bar{\mathcal{C}}_1(1)_+ \wedge \mathbb{S}$ part does; the same formulas apply (the first formula meaning to use the isomorphism $B \wedge \mathbb{S} \cong B$), and so it is easy to see that this describes a well-defined pairing $Z \wedge U_L^{\bar{\mathcal{C}}_1} B \rightarrow Z$. Examining the formula for $\{[0, 1]\}_+ \wedge \mathbb{S}$ shows that the pairing is unital, and an tedious arithmetic check shows that it is associative.

The left action of $U_R^{\bar{\mathcal{C}}_1} B$ is similar, using \circ_1 (on C) instead of \circ_2 : We have maps

$$D' \times C \longrightarrow \bar{\mathcal{C}}_1(2) \quad \text{and} \quad D' \times C \longrightarrow C$$

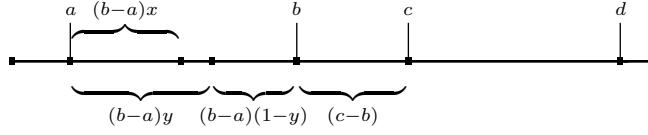
that take the pair of elements $[x, y]$ in D' and $[a, b], [c, d]$ in C (with $b < c$) to the element

$$\left[0, \frac{(b-a)(1-y)}{(b-a)(1-y) + c-b}\right], \left[\frac{(b-a)(1-y)}{(b-a)(1-y) + c-b}, 1\right]$$

in $\bar{\mathcal{C}}_1(2)$ and the element

$$[a + (b-a)x, a + (b-a)y], [c, d]$$

of C . Pictorially,



the element of $\bar{\mathcal{C}}_1(2)$ is the fourth and fifth segments translated and rescaled to start at 0 and end at 1, and the element of C is the third and sixth segments.

To see that the two actions commute, we note that the two maps

$$U_R^{\bar{\mathcal{C}}_1} B \wedge Z \wedge U_L^{\bar{\mathcal{C}}_1} B$$

can each be written in terms of maps

$$D' \times C \times D \longrightarrow \bar{\mathcal{C}}_1(3) \quad \text{and} \quad D' \times C \times D \longrightarrow C$$

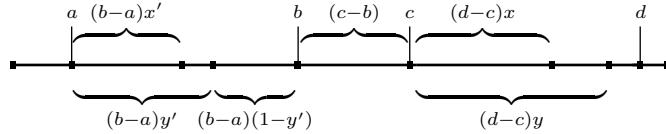
using the element of $\bar{\mathcal{C}}_1(3)$ to multiply the three factors of B . Writing $[x', y']$ for the element of D' , $[a, b], [c, d]$ for the element of C , and $[x_2, y_2]$ for the element of D , both maps give the same element

$$\left[0, \frac{(b-a)(1-y')}{\ell}\right], \left[\frac{(b-a)(1-y')}{\ell}, \frac{(b-a)(1-y') + (b-c)}{\ell}\right], \left[\frac{(b-a)(1-y') + (b-c)}{\ell}, 1\right]$$

of $\bar{\mathcal{C}}_1(3)$, where $\ell = (b-a)(1-y') + c-b + (d-c)x$, and both maps give the same element

$$[a + (b-a)x', a + (b-a)y'], [c + (d-c)x, c + (d-c)y]$$

of C . Pictorially,



the element of $\bar{\mathcal{C}}_1(3)$ for both maps consists of the fourth, fifth, and sixth segments translated and rescaled to start at 0 and end at 1, and the element of C consists of the third and seventh segments.

The map $C \rightarrow \bar{\mathcal{C}}_1(3)$ together with the maps $\bar{\mathcal{C}}_1(3)_+ \wedge B \rightarrow \text{Bal}^{\bar{\mathcal{C}}_1} B$ and $\bar{\mathcal{C}}_1(2)_+ \wedge \mathbb{S} \rightarrow \text{Bal}^{\bar{\mathcal{C}}_1} B$ induces a map $Z \rightarrow \text{Bal}^{\bar{\mathcal{C}}_1} B$.

THEOREM 19.1. *The map $Z \rightarrow \text{Bal}^{\bar{\mathcal{C}}_1} B$ is an isomorphism of right $(U_R^{\bar{\mathcal{C}}_1} B)^{\text{op}} \wedge U_L^{\bar{\mathcal{C}}_1} B$ -modules.*

PROOF. We construct a map $\text{Bal}^{\bar{\mathcal{C}}_1} B \rightarrow Z$ by constructing compatible maps $\bar{\mathcal{C}}_1(n+2)_+ \wedge B^{(n)} \rightarrow Z$ as follows. For $n = 0$, we use the map $\bar{\mathcal{C}}_1(2)_+ \wedge \mathbb{S} \rightarrow Z$ from the construction of Z . For $n > 0$, we have an isomorphism

$$\bar{\mathcal{C}}_1(n+2) \longrightarrow C \times \bar{\mathcal{C}}_1(n)$$

defined by taking the element

$$[x_0, y_0], \dots, [x_{n+1}, y_{n+1}]$$

in $\bar{\mathcal{C}}_1(n+2)$ (with $y_i \leq x_{i+1}$) to the element $[x_0, y_0], [x_{n+1}, y_{n+1}]$ of C and the element

$$\left[\frac{x_1 - y_0}{x_{n+1} - y_0}, \frac{y_1 - y_0}{x_{n+1} - y_0} \right], \dots, \left[\frac{x_n - y_0}{x_{n+1} - y_0}, \frac{y_n - y_0}{x_{n+1} - y_0} \right]$$

in $\bar{\mathcal{C}}_1(n)$. We then get a map $\bar{\mathcal{C}}_1(n+2)_+ \wedge B^{(n)} \rightarrow Z$ using the given element of C and the element of $\bar{\mathcal{C}}_1(n)$ to multiply the factors of B . This obviously factors through the coequalizer to define a map $\text{Bal}^{\bar{\mathcal{C}}_1} B \rightarrow Z$. Looking at the composite maps

$$\bar{\mathcal{C}}_1(2)_+ \wedge \mathbb{S} \longrightarrow \text{Bal}^{\bar{\mathcal{C}}_1} B \longrightarrow Z \quad \text{and} \quad C_+ \wedge B \longrightarrow \bar{\mathcal{C}}_1(3)_+ \wedge B \longrightarrow \text{Bal}^{\bar{\mathcal{C}}_1} B \longrightarrow Z,$$

we see that the composite map on Z is the identity. Likewise, looking at the surjection $\circ_2: \bar{\mathcal{C}}_1(3) \times \bar{\mathcal{C}}_1(n) \rightarrow \bar{\mathcal{C}}_1(n+2)$ for $n > 0$ (which induces the inverse of the isomorphism above when restricted to $C \times \bar{\mathcal{C}}_1(n)$), we can see that the composite on $\text{Bal}^{\bar{\mathcal{C}}_1} B$ is the identity. To see that $\text{Bal}^{\bar{\mathcal{C}}_1} B \rightarrow Z$ is a map of bimodules, it suffices to check each of the module structures separately. The left action of $U_R^{\bar{\mathcal{C}}_1} B$ on $\text{Bal}^{\bar{\mathcal{C}}_1} B$ is induced by

$$\begin{aligned} & (\bar{\mathcal{C}}_1(m+1)_+ \wedge B^{(m)}) \wedge (\bar{\mathcal{C}}_1(n+2)_+ \wedge B^{(n)}) \\ & \cong (\bar{\mathcal{C}}_1(n+2) \times \bar{\mathcal{C}}_1(m+1))_+ \wedge B^{(m+n)} \xrightarrow{\circ_1 \wedge \text{id}} \bar{\mathcal{C}}_1(m+n+2)_+ \wedge B^{(m+n)}. \end{aligned}$$

It is now easy to check in the case $m = 0, 1, n = 0, 1$ that the composite map to Z is the same as the composite

$$(\bar{\mathcal{C}}_1(m+1)_+ \wedge B^{(m)}) \wedge (\bar{\mathcal{C}}_1(n+2)_+ \wedge B^{(n)}) \longrightarrow U_R^{\bar{\mathcal{C}}_1} B \wedge Z$$

with the action map $U_R^{\bar{\mathcal{C}}_1} B \wedge Z$ defined above, and this suffices to show the result. The case of the right action by $U_L^{\bar{\mathcal{C}}_1} B$ is similar. \square

We can now prove the case of Theorem 18.6 for $\bar{\mathcal{O}} = \bar{\mathcal{C}}_1$, which we state as the following proposition.

PROPOSITION 19.2. *The maps $\bar{\mathcal{C}}_1(2)_+ \wedge B \rightarrow U_L^{\bar{\mathcal{C}}_1} B$, $\bar{\mathcal{C}}_1(2)_+ \wedge B \rightarrow U_R^{\bar{\mathcal{C}}_1} B$, and $\bar{\mathcal{C}}_1(3)_+ \wedge B \rightarrow \text{Bal}^{\bar{\mathcal{C}}_1} B$ are homotopy equivalences of orthogonal spectra.*

PROOF. The case of the enveloping algebras are essentially proved in [25, 1.1], but it is no extra work to include those cases here. We have maps

$$U_L^{\bar{\mathcal{C}}_1} B \longrightarrow B, \quad U_R^{\bar{\mathcal{C}}_1} B \longrightarrow B, \quad \text{Bal}^{\bar{\mathcal{C}}_1} B \longrightarrow B$$

induced by the maps $\bar{\mathcal{C}}_1(1) \rightarrow *$, $\bar{\mathcal{C}}_1(2) \rightarrow *$, and $\bar{\mathcal{C}}_1(3) \rightarrow *$. Choosing elements in $\bar{\mathcal{C}}_1(2)$ and $\bar{\mathcal{C}}_1(3)$, we then get composite maps

$$U_L^{\bar{\mathcal{C}}_1} B \longrightarrow \bar{\mathcal{C}}_1(2)_+ \wedge B, \quad U_R^{\bar{\mathcal{C}}_1} B \longrightarrow \bar{\mathcal{C}}_1(2)_+ \wedge B, \quad \text{Bal}^{\bar{\mathcal{C}}_1} B \longrightarrow \bar{\mathcal{C}}_1(3)_+ \wedge B$$

and we see that the composites on $\bar{\mathcal{C}}_1(2)_+ \wedge B$ and $\bar{\mathcal{C}}_1(3)_+ \wedge B$ are homotopic to the identity since $\bar{\mathcal{C}}_1(2)$ and $\bar{\mathcal{C}}_1(3)$ are contractible. Likewise, it is straightforward to write explicit formulas for contractions on the pairs $(\bar{\mathcal{C}}_1(1), D)$, $(\bar{\mathcal{C}}_1(1), D')$, and $(\bar{\mathcal{C}}_1(2), C)$ that produce the homotopies for the composites on $U_L^{\bar{\mathcal{C}}_1} B$, $U_R^{\bar{\mathcal{C}}_1} B$, and $\text{Bal}^{\bar{\mathcal{C}}_1} B$. \square

The operad \mathcal{L} . For the \mathcal{L} we also need to prove Theorem 18.9 in addition to Theorem 18.6. For this operad, we adapt the techniques of EKMM [10, §L.5]. We begin by identifying the enveloping algebras.

PROPOSITION 19.3. *The underlying orthogonal spectra of $U_L^{\mathcal{L}} B$ and $U_R^{\mathcal{L}} B$ are naturally isomorphic to the pushout*

$$(\mathcal{L}(1)_+ \wedge \mathbb{S}) \cup_{\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} \mathbb{S}} \mathcal{L}(2) \wedge_{\mathcal{L}(1)} B$$

where for $U_L^{\mathcal{L}} B$ the action of $\mathcal{L}(1)$ on $\mathcal{L}(2)$ is via $\mathcal{L}(1) \times \{1\} \subset \mathcal{L}(1) \times \mathcal{L}(1)$ and for $U_R^{\mathcal{L}} B$ the action of $\mathcal{L}(1)$ on $\mathcal{L}(2)$ is via $\{1\} \times \mathcal{L}(1) \subset \mathcal{L}(1) \times \mathcal{L}(1)$.

PROOF. We treat the case of $U_L^{\mathcal{L}} B$ as the other case is entirely similar. For the purpose of the proof, denote the pushout as P . The defining coequalizer for $U_L^{\mathcal{L}} B$ induces the map $P \rightarrow U_L^{\mathcal{L}} B$. Using Hopkins' Lemma [10, I.5.4], we have isomorphisms

$$\mathcal{L}(n+1)_+ \wedge B^{(n)} \cong \mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} (\mathcal{L}(n)_+ \wedge B^{(n)})$$

from which the \mathcal{L} -action on B induces a map to P . These maps glue over the coequalizer to construct a map $U_L^{\mathcal{L}} B \rightarrow P$. We see that the composite map is the identity on P by looking at the composite maps $\mathcal{L}(2)_+ \wedge B \rightarrow P$ and $\mathcal{L}(1)_+ \wedge \mathbb{S} \rightarrow P$. Likewise the map

$$(\mathcal{L}(1)_+ \wedge B) \vee \bigvee \mathcal{L}(2)_+ \wedge (\mathcal{L}(n)_+ \wedge B^{(n)}) \longrightarrow U_L^{\mathcal{L}} B$$

is unchanged by composing with the composite $U_L^{\mathcal{L}} B \rightarrow P \rightarrow U_L^{\mathcal{L}} B$, and we conclude that the composite is the identity on $U_L^{\mathcal{L}} B$ (since the coequalizer description of $U_L^{\mathcal{L}} B$ plus Hopkins' Lemma implies that the displayed map is an epimorphism). \square

For the pushout description of $U_L^{\mathcal{L}} B$ in the previous proposition, the unit $\mathbb{S} \rightarrow U_L^{\mathcal{L}} B$ is induced by the map

$$\mathbb{S} \cong \{1\}_+ \wedge \mathbb{S} \longrightarrow \mathcal{L}(1)_+ \wedge \mathbb{S} \longrightarrow U_L^{\mathcal{L}} B.$$

The product is induced by the map

$$\begin{aligned} (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \wedge (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) &\longrightarrow (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \wedge_{\mathcal{L}(1)} (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \\ &\cong \mathcal{L}(3)_+ \wedge_{\mathcal{L}(1) \times \mathcal{L}(1)} B \wedge B \cong \mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1) \times \mathcal{L}(1)} B \wedge B) \\ &\longrightarrow \mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B \end{aligned}$$

and the straightforward modifications using $\mathcal{L}(1)_+ \wedge \mathbb{S}$ in place of one or both copies of $\mathcal{L}(2)_+ \wedge B$ with $\mathcal{L}(1)_+ \wedge \mathbb{S}$. (The latter is the map

$$(\mathcal{L}(1)_+ \wedge \mathbb{S}) \wedge (\mathcal{L}(1)_+ \wedge \mathbb{S}) \cong (\mathcal{L}(1) \times \mathcal{L}(1))_+ \wedge \mathbb{S} \longrightarrow \mathcal{L}(1)_+ \wedge \mathbb{S}$$

induced by the operadic multiplication.) The product for $U_R^{\mathcal{L}} B$ is similar except that the operad factors (but not the B factors) transpose, plugging the left $\mathcal{L}(2)$ into the right $\mathcal{L}(2)$ with respect to the operadic multiplication.

We have the following concrete description of $\text{Bal}^{\mathcal{L}} B$. The proof uses the same techniques as the proof of the previous proposition.

PROPOSITION 19.4. *The underlying orthogonal spectrum of $\text{Bal}^{\mathcal{L}} B$ is computed by the pushout*

$$(\mathcal{L}(2)_+ \wedge \mathbb{S}) \cup_{\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} \mathbb{S}} (\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B)$$

where the action of $\mathcal{L}(1)$ on $\mathcal{L}(3)$ is via $\{1\} \times \mathcal{L}(1) \times \{1\} \subset \mathcal{L}(1) \times \mathcal{L}(1) \times \mathcal{L}(1)$.

The right action of $U_L^{\mathcal{L}} B$ is induced by the map

$$(\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B) \wedge (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \longrightarrow \mathcal{L}(4)_+ \wedge_{\mathcal{L}(1) \times \mathcal{L}(1)} (B \wedge B) \longrightarrow \mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B$$

again multiplying $\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1) \times \mathcal{L}(1)} B \wedge B \rightarrow B$ on the inner factors of B , using Hopkins' Lemma (with similar formulas for $\mathcal{L}(2)_+ \wedge \mathbb{S}$ and/or $\mathcal{L}(1)_+ \wedge \mathbb{S}$). The left action of $U_R^{\mathcal{L}} B$ is similar but transposing the operad spaces so that the operad space for $U_R^{\mathcal{L}} B$ plugs in to the operad space for $\text{Bal}^{\mathcal{L}} B$ for the operadic multiplication.

Turning to Theorem 18.9, recall that $J(-) = (-) \wedge_{\mathcal{L}} \mathbb{S}$ denotes the functor from $\mathcal{L}(1)$ -spectra in orthogonal spectra to EKMM \mathbb{S} -modules in orthogonal spectra. One feature it has is that J turns the maps $\mathcal{L}(n)_+ \wedge_{\mathcal{L}} \mathbb{S} \rightarrow \mathcal{L}(n-1)_+ \wedge \mathbb{S}$ into isomorphisms (another application of Hopkins' Lemma). Because of this we then have natural isomorphisms

$$\begin{aligned} JU_L^{\mathcal{L}} B &\cong J(\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \\ (19.5) \quad JU_R^{\mathcal{L}} B &\cong J(\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \\ J \text{Bal}^{\mathcal{L}} B &\cong J(\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B) \end{aligned}$$

When B is already an EKMM \mathbb{S} -module in orthogonal spectra, we can omit the J on the right side. We now prove Theorem 18.9.

PROOF OF THEOREM 18.9. It suffices to construct an isomorphism between $J(M \wedge_B^{\mathcal{L}} N)$ and $M \wedge_B B \wedge_B N$, where the smash products without the superscript denote smash in the category of EKMM \mathbb{S} -modules in orthogonal spectra. With the simplification of (19.5), we have $J(M \wedge_B^{\mathcal{L}} N)$ as the coequalizer of

$$\begin{array}{c} (\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B) \wedge (U_R^{\mathcal{L}} B)^{\text{op}} \wedge U_L^{\mathcal{L}} B \wedge M \wedge N \\ \Downarrow \\ (\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B) \wedge M \wedge N. \end{array}$$

Because of the maps $\mathcal{L}(1)_+ \wedge \mathbb{S} \rightarrow (U_R^{\mathcal{L}} B)^{\text{op}}$ and $\mathcal{L}(1)_+ \wedge \mathbb{S} \rightarrow U_L^{\mathcal{L}} B$, we can replace the $(\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B) \wedge M \wedge N$ with $(\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)^3} (M \wedge B \wedge N))$. Also using (19.5) to replace the enveloping algebras with $\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B$, we then identify $J(M \wedge_B^{\mathcal{L}} N)$ as the coequalizer of

$$\begin{array}{c} (\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B) \wedge (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \wedge (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \wedge M \wedge N \\ \Downarrow \\ \mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)^3} (M \wedge B \wedge N). \end{array}$$

We can simplify the above coequalizer diagram to

$$\begin{array}{c} \mathcal{L}(5)_+ \wedge_{\mathcal{L}(1)^5} (M \wedge B \wedge B \wedge B \wedge N) \\ \Downarrow \\ \mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)^3} (M \wedge B \wedge N) \end{array}$$

using the categorical epimorphism

$$\begin{aligned} & (\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B) \wedge (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \wedge (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \wedge M \wedge N \\ & \longrightarrow \mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)^3} ((\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B) \wedge B \wedge (\mathcal{L}(2)_+ \wedge_{\mathcal{L}(1)} B)) \wedge M \wedge N \\ & \qquad \qquad \qquad \cong \mathcal{L}(5)_+ \wedge_{\mathcal{L}(1)^5} (M \wedge B \wedge B \wedge B \wedge N). \end{aligned}$$

The latter coequalizer is easily seen (using Hopkins' Lemma) to be $M \wedge_B B \wedge_B N$. \square

The following proposition gives the instance of Theorem 18.6 for the non- Σ A_∞ operad $\overline{\mathcal{O}} = \mathcal{L}$.

PROPOSITION 19.6. *For any non- Σ \mathcal{L} -algebra B , the maps $\mathcal{L}(2)_+ \wedge B \rightarrow U_L^\mathcal{L} B$, $\mathcal{L}(2)_+ \wedge B \rightarrow U_R^\mathcal{L} B$, and $\mathcal{L}(3)_+ \wedge B \rightarrow \text{Bal}^\mathcal{L} B$ are weak equivalences of orthogonal spectra.*

PROOF. It suffices to check that these maps are weak equivalences after applying J . Using (19.5), each map is an instance of the map

$$\mathcal{L}(n)_+ \wedge_{\mathcal{L}(1)} \mathbb{S} \cong (\mathcal{L}(n) \times_{\mathcal{L}(1)} \mathcal{L}(0))_+ \wedge \mathbb{S} \longrightarrow \mathcal{L}(n-1)_+ \wedge \mathbb{S}.$$

This map is a weak equivalence; see [10, XI.2.2]. \square

The operad $\mathcal{L} \times \overline{\mathcal{C}}_1$. This operad combines the features of the previous two cases. See the case of $\overline{\mathcal{C}}_1$ for the definition of D , D' , and C .

PROPOSITION 19.7. *The underlying orthogonal spectrum of $U_L^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$ is*

$$((\mathcal{L}(1) \times \overline{\mathcal{C}}_1)_+ \wedge \mathbb{S}) \cup_{((\mathcal{L}(2) \times D)_+ \wedge_{\mathcal{L}(1)} \mathbb{S})} ((\mathcal{L}(2) \times D)_+ \wedge B).$$

The underlying orthogonal spectrum of $U_R^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$ is

$$((\mathcal{L}(1) \times \overline{\mathcal{C}}_1)_+ \wedge \mathbb{S}) \cup_{((\mathcal{L}(2) \times D')_+ \wedge_{\mathcal{L}(1)} \mathbb{S})} ((\mathcal{L}(2) \times D')_+ \wedge B).$$

The underlying orthogonal spectrum of $\text{Bal}^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$ is

$$((\mathcal{L}(2) \times \overline{\mathcal{C}}_1(2))_+ \wedge \mathbb{S}) \cup_{((\mathcal{L}(3) \times C)_+ \wedge_{\mathcal{L}(1)} \mathbb{S})} ((\mathcal{L}(3) \times C)_+ \wedge_{\mathcal{L}(1)} B).$$

The proof is to use Hopkins' Lemma as in the previous subsection and the homeomorphisms

$$D \times \overline{\mathcal{C}}_1(n) \xrightarrow{\cong} \overline{\mathcal{C}}_1(n+1), \quad D' \times \overline{\mathcal{C}}_1(n) \xrightarrow{\cong} \overline{\mathcal{C}}_1(n+1), \quad \text{and} \quad C \times \overline{\mathcal{C}}_1(n) \xrightarrow{\cong} \overline{\mathcal{C}}_1(n+2)$$

(for $n > 0$) induced by the maps $D \rightarrow \overline{\mathcal{C}}_1(2)$, $D' \rightarrow \overline{\mathcal{C}}_1(2)$, and $C \rightarrow \overline{\mathcal{C}}_1(3)$ and the operadic products \circ_1, \circ_2 , and \circ_2 , respectively, as in the first subsection. The unit and product on this model of $U_L^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$ and $U_R^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$ are the evident generalization of the structure described in the previous two subsections, as is the action on $\text{Bal}^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$.

The following proposition is the instance of Theorem 18.6 for the operad $\overline{\mathcal{O}} = \mathcal{L} \times \overline{\mathcal{C}}_1$.

PROPOSITION 19.8. *The maps $(\mathcal{L}(2) \times \overline{\mathcal{C}}_1(2))_+ \wedge B \rightarrow U_L^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$, $(\mathcal{L}(2) \times \overline{\mathcal{C}}_1(2))_+ \wedge B \rightarrow U_R^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$, and $(\mathcal{L}(3) \times \overline{\mathcal{C}}_1(3))_+ \wedge B \rightarrow \text{Bal}^{\mathcal{L} \times \overline{\mathcal{C}}_1} B$ are weak equivalences of orthogonal spectra.*

PROOF. We treat the case of $\text{Bal}^{\mathcal{L} \times \bar{\mathcal{C}}_1} B$; the remaining cases are similar. The left action of $\mathcal{L}(1)$ on $\mathcal{L}(n) \times \bar{\mathcal{C}}_1(n)$ (as $\mathcal{L}(1) \times \{1\}$) and the models given in the previous proposition allow us to view $\text{Bal}^{\mathcal{L} \times \bar{\mathcal{C}}_1}$ as $\mathcal{L}(1)$ -spectra; we can then apply the functor J to convert to EKMM \mathbb{S} -modules (in orthogonal spectra). Applying J to the pushout

$$(19.9) \quad ((\mathcal{L}(3) \times \bar{\mathcal{C}}_1(2)) \wedge_{\mathcal{L}(1)} \mathbb{S}) \cup_{((\mathcal{L}(3) \times \bar{\mathcal{C}}_1(3))_+ \wedge_{\mathcal{L}(1)} \mathbb{S})} ((\mathcal{L}(3) \times \bar{\mathcal{C}}_1(3))_+ \wedge_{\mathcal{L}(1)} B)$$

is isomorphic to $J(\text{Bal}^{\mathcal{L} \times \bar{\mathcal{C}}_1} B)$, and so suffices to show that the map from $\mathcal{L}(3)_+ \wedge B$ to (19.9) is a weak equivalence. Using the weak equivalence

$$\mathcal{L}(3)_+ \wedge B \longrightarrow \mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B$$

(see [10, XI.2.2]), it suffices to check that the inclusion of $(\mathcal{L}(3) \times \bar{\mathcal{C}}_1(3))_+ \wedge_{\mathcal{L}(1)} B$ in the pushout (19.9) is a weak equivalence. From here the proof is identical to the proof of Proposition 19.2 using $\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} B$ in place of B and $\mathcal{L}(3)_+ \wedge_{\mathcal{L}(1)} \mathbb{S}$ in place of \mathbb{S} . \square

The operad $\mathcal{L} \times \mathcal{O}_1^{\Xi}$. The operad \mathcal{O}_1^{Ξ} satisfies $\mathcal{O}_1^{\Xi}(n) = \mathcal{O}_1^{\Xi}(1)^n$, the operad built from the monoid $\mathcal{O}_1^{\Xi}(1)$ using the diagonal multiplication. In any reasonable symmetric monoidal category tensored over spaces, a non- Σ \mathcal{O}_1^{Ξ} -algebra A is just an associative monoid object together with a left action of the monoid $\mathcal{O}_1^{\Xi}(1)$ on its underlying object such that the multiplication is $\mathcal{O}_1^{\Xi}(1)$ -equivariant and the unit is $\mathcal{O}_1^{\Xi}(1)$ -fixed. Likewise a left or right A -module M over \mathcal{O}_1^{Ξ} is just a left or right A -module (in the usual sense) together with a left action of $\mathcal{O}_1^{\Xi}(1)$ for which the module structure map is $\mathcal{O}_1^{\Xi}(1)$ -equivariant. In particular, we see that the left enveloping algebra of A is $\mathcal{O}_1^{\Xi}(1)_+ \wedge A$ (where we have written $(-)_+ \wedge (-)$ for the tensor with a space) with $\mathcal{O}_1^{\Xi}(1)$ acting diagonally. Likewise, the right enveloping algebra is $\mathcal{O}_1^{\Xi}(1)^{\text{op}} \wedge A$. (Recall that our convention is for the right enveloping algebra to act on the right; the alternative convention of having it act on the left would yield $\mathcal{O}_1^{\Xi}(1)_+ \wedge A^{\text{op}}$ for the right enveloping algebra.) Working in the weak symmetric monoidal category of $\mathcal{L}(1)$ -spectra in orthogonal spectra, we obtain the following analogous statement.

PROPOSITION 19.10. *For any non- Σ $(\mathcal{L} \times \mathcal{O}_1^{\Xi})$ -algebra B , we have natural isomorphisms of associative ring orthogonal spectra*

$$\begin{aligned} U_L^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} B &\cong \mathcal{O}_1^{\Xi}(1)_+ \wedge U_L^{\mathcal{L}} B \\ U_R^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} B &\cong \mathcal{O}_1^{\Xi}(1)^{\text{op}} \wedge U_R^{\mathcal{L}} B \end{aligned}$$

and an isomorphism of bimodules

$$\text{Bal}^{\mathcal{L} \times \mathcal{O}_1^{\Xi}} B \cong (\mathcal{O}_1^{\Xi}(1) \times \mathcal{O}_1^{\Xi}(1))_+ \wedge \text{Bal}^{\mathcal{L}} B$$

(where the action has both factors of $\mathcal{O}_1^{\Xi}(1)$ in Bal always left for the multiplication on $\mathcal{O}_1^{\Xi}(1)$).

The instance of Theorem 18.6 for $\mathcal{L} \times \mathcal{O}_1^{\Xi}$ now follows from Proposition 19.6.

20. A topologically enriched lax symmetric monoidal fibrant replacement functor for equivariant orthogonal spectra
(Proof of Lemma 2.1)

In this section we construct a topologically enriched lax symmetric monoidal fibrant replacement functor for the positive stable model category of orthogonal G -spectra. We use an equivariant version of the construction of Kro [20, 3.3].

Before beginning the construction and the argument, it is useful to be slightly more precise about the homotopy groups of an orthogonal G -spectrum. Recall that a *complete G -universe* is an infinite dimensional G -inner product space containing a representative of each finite dimensional G -representation. We use the notation $V < U$ to denote that V is a finite dimensional G -linear subspace of U . Given a complete G -universe U , $V < U$, and W an arbitrary finite dimensional G -inner product space, for $H < G$ and X a orthogonal G -spectrum, define

$$\pi_{W,V < U}^H X = \operatorname{colim}_{V < Z < U} [S^{W \oplus (Z-V)}, X(Z)]^H,$$

where $[-, -]^H$ denotes the set of homotopy classes of maps of based H -spaces and $Z - V$ denotes the orthogonal complement of V in Z . The following facts are well-established.

- (i) $\pi_{W,V < U}^H$ has the natural structure of an abelian group.
- (ii) If U' is a complete G -universe and $f: U \rightarrow U'$ is a G -equivariant linear isometry (not necessarily isomorphism), then the induced map

$$\pi_{W,V < U}^H X \longrightarrow \pi_{W,f(V) < U'}^H X$$

is an isomorphism.

- (iii) For any finite-dimensional G -inner product space W' , the map

$$\pi_{W,V < U}^H X \longrightarrow \pi_{W \oplus W', V \oplus W' < U \oplus W'}^H X$$

(induced by $(-) \wedge S^{W'}$ and the structure map on X) is an isomorphism.

- (iv) A map $X \rightarrow Y$ of orthogonal G -spectra is a stable equivalence if and only if the induced maps on $\pi_{W,V < U}^H$ are isomorphisms for all $H < G$, $V < U$, and W .

Indeed, the group $\pi_{W,V < U}^H X$ defined above is a specific model for the $RO(G)$ -graded homotopy group $\pi_{[W]-[V]}^H X$; (ii) and (iii) are some minimal invariance properties easily proved by comparison of colimit arguments, while (iv) follows from the fact that a map in the stable category induces an isomorphism on integer-graded homotopy groups if and only if it induces an isomorphism on $RO(G)$ -graded homotopy groups. Another useful observation is that when U is a complete G -universe and W is any non-trivial finite dimensional G -inner product space, $U \otimes W$ is also a complete G -universe [23, IV.3.9].

For W any finite dimensional G -inner product space, define

$$(R_G X)(W) = \operatorname{hocolim}_{V < U} \Omega^{V \otimes W} X((\mathbb{R} \oplus V) \otimes W)$$

where for $V < V'$ the map in the hocolim system is induced by the structure map for X

$$\begin{aligned} X((\mathbb{R} \oplus V) \otimes W) \wedge S^{(V' - V) \otimes W} \\ \longrightarrow X(((\mathbb{R} \oplus V) \otimes W) \oplus ((V' - V) \otimes W)) \cong X((\mathbb{R} \oplus V') \otimes W) \end{aligned}$$

and the canonical isomorphism $V \oplus (V' - V) \cong V'$. The structure map

$$(R_G X)(W) \wedge S^{W'} \longrightarrow (R_G X)(W \oplus W')$$

is induced at the V spot in the hocolim

$$\Omega^{V \otimes W} X((\mathbb{R} \oplus V) \otimes W) \wedge S^{W'} \longrightarrow \Omega^{V \otimes (W \oplus W')} X((\mathbb{R} \oplus V) \otimes (W \oplus W'))$$

as the adjoint under the $(\Sigma^{W \oplus W'}, \Omega^{W \oplus W'})$ -adjunction to the map

$$\begin{aligned} X((\mathbb{R} \oplus V) \otimes W) \wedge S^{W'} \wedge S^{V \otimes W'} &\cong X((\mathbb{R} \oplus V) \otimes W) \wedge S^{(\mathbb{R} \oplus V) \otimes W'} \\ &\longrightarrow X((\mathbb{R} \oplus V) \otimes W \oplus (\mathbb{R} \oplus V) \otimes W') \cong X((\mathbb{R} \oplus V) \otimes (W \oplus W')) \end{aligned}$$

coming from the structure map for X . The check that the structure map is well-defined works exactly as in the non-equivariant case, and this together with the evident G and $\mathcal{I}(W, W')$ -action maps make R_G into an endofunctor on orthogonal G -spectra. The inclusion of $X(W)$ as $\Omega^{V \otimes W} X((\mathbb{R} \oplus 0) \otimes W)$ induces a natural transformation $\text{Id} \rightarrow R_G$.

PROPOSITION 20.1. *For any orthogonal G -spectrum X , $R_G X$ is a positive G - Ω -spectrum and $X \rightarrow R_G X$ is a stable equivalence.*

PROOF. The formula for $(R_G X)(W)$ gives a canonical isomorphism

$$\begin{aligned} \pi_n^H((R_G X)(W)) &\cong \text{colim}_{(V < U)} [S^{\mathbb{R}^n \oplus (V \otimes W)}, X((\mathbb{R} \oplus V) \otimes W)]^H \\ &\cong \text{colim}_{(\mathbb{R} \otimes W < Z < (\mathbb{R} \oplus U) \otimes W)} [S^{\mathbb{R}^n \oplus (Z - \mathbb{R} \otimes W)}, X(Z)]^H \end{aligned}$$

with the second isomorphism by cofinality. This gives a canonical isomorphism

$$\pi_n^H(R_G X(W)) \cong \pi_{\mathbb{R}^n, \mathbb{R} \otimes W < (\mathbb{R} \oplus U) \otimes W}^H X$$

when W is non-trivial (we have not defined the righthand side when W is trivial). Similarly, we have a canonical isomorphism

$$\pi_n^H(\Omega^{W'}(R_G X(W \oplus W'))) \cong \pi_{\mathbb{R}^n \oplus W', \mathbb{R} \otimes (W \oplus W') < (\mathbb{R} \oplus U) \otimes (W \oplus W')}^H X$$

and the adjoint structure map $R_G X(W) \rightarrow \Omega^{W'}(R_G X(W \oplus W'))$ is a weak equivalence whenever W is nontrivial as an instance of properties (ii) and (iii) of homotopy groups listed above. It follows from this calculation (and property (iv) of homotopy groups above) that R_G takes stable equivalences of orthogonal G -spectra to positive level equivalences. In particular, to prove that the natural transformation $\text{Id} \rightarrow R_G$ is a stable equivalence, it suffices to check that it is a positive level equivalence on a positive G - Ω -spectrum X . In this case, for W nontrivial, $\pi_n^H(X(W)) \rightarrow \pi_{\mathbb{R}^n, W < U}^H X$ is an isomorphism, and we can identify the induced map on homotopy groups $\pi_n^H(X(W)) \rightarrow \pi_n^H(R_G X(W))$ as an isomorphism once again using property (ii) of homotopy groups above. \square

The functor R_G is clearly continuous on mapping spaces. Thus, to complete the proof of Lemma 2.1, we need to construct the lax symmetric monoidal structure on R_G and prove that $\text{Id} \rightarrow R_G$ is a symmetric monoidal transformation. This works essentially just as in the non-equivariant case, using internal sum $+$ of finite dimensional subspaces of U in place of max of natural numbers. (Of course, $\mathbb{R}^k + \mathbb{R}^\ell < \mathbb{R}^\infty$ is $\mathbb{R}^{\max(k, \ell)} < \mathbb{R}^\infty$.) Denoting by \mathcal{I}_U the partially ordered set of $V < U$, $+$ defines a functor $\mathcal{I}_U \times \mathcal{I}_U \rightarrow \mathcal{I}_U$ that is strictly symmetric ($V + V' = V' + V$) and strictly associative ($(V + V') + V'' = V + (V' + V'')$). For specified orthogonal

G -spectra X and X' and finite dimensional G -inner product spaces W and W' , we have a natural transformation

$$\begin{aligned} & \Omega^{V \otimes W} X((\mathbb{R} \oplus V) \otimes W) \wedge \Omega^{V' \otimes W'} X((\mathbb{R} \oplus V') \otimes W') \\ & \quad \downarrow \\ & \Omega^{(V+V') \otimes W} X((\mathbb{R} \oplus (V+V')) \otimes W) \wedge \Omega^{(V+V') \otimes W'} X'((\mathbb{R} \oplus (V+V')) \otimes W') \\ & \quad \downarrow \\ & \Omega^{(V+V') \otimes (W \oplus W')} ((X \wedge X')((\mathbb{R} \oplus (V+V')) \otimes (W \oplus W'))) \end{aligned}$$

subordinate to $+$ and inducing a map of based G -spaces

$$R_G X(W) \wedge R_G X'(W') \longrightarrow (R_G(X \wedge X'))(W \oplus W').$$

The check that this assembles to a natural transformation

$$R_G X \wedge R_G X' \longrightarrow R_G(X \wedge X')$$

is now straightforward and essentially the same as in the non-equivariant case, as are the remaining checks of the associativity and symmetry properties and the check of lax symmetric monoidality of the natural transformation $\text{Id} \rightarrow R_G$.

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