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Proof of the Goldberg–Seymour conjecture on edge–colorings of multigraphs

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Abstract

Given a multigraph $G = (V, E)$, the edge-coloring problem (ECP) is to color the edges of G with the minimum number of colors so that no two adjacent edges have the same color. This problem can be naturally formulated as an integer program, and its linear programming relaxation is referred to as the fractional edge-coloring problem (FECP). The optimal value of ECP (resp. FECP) is called the chromatic index (resp. fractional chromatic index) of G, denoted by $\chi'(G)$ (resp. $\chi^*(G)$). Let $\Delta(G)$ be the maximum degree of G and let $\Gamma(G)$ be the density of G, defined by

$$\Gamma(G) = \max\left\{\frac{2|E(U)|}{|U|-1} :\ U \subseteq V,\ |U| \geq 3 \ \text{and} \ \text{odd}\right\},$$

where $E(U)$ is the set of all edges of G with both ends in U. Clearly, $\max\{\Delta(G), \lceil\Gamma(G)\rceil\}$ is a lower bound for $\chi'(G)$. As shown by Seymour, $\chi^*(G) = \max\{\Delta(G), \Gamma(G)\}$. In the early 1970s Goldberg and Seymour independently conjectured that $\chi'(G) \leq \max\{\Delta(G)+1, \lceil\Gamma(G)\rceil\}$. Over the past five decades this conjecture, a cornerstone in modern edge-coloring, has been a subject of extensive research, and has stimulated an important body of work. In this paper we present a proof of this conjecture. Our result implies that, first, there are only two possible values for $\chi'(G)$, so an analogue to Vizing's theorem on edge-colorings of simple graphs holds for multigraphs; second, although it is *NP*-hard in general to determine $\chi'(G)$, we can approximate it within one of its true value, and find it exactly in polynomial time when $\Gamma(G) > \Delta(G)$; third, every multigraph G satisfies $\chi'(G) - \chi^*(G) \leq 1$, and thus FECP has a fascinating integer rounding property.

Keywords Multigraph · Edge-coloring · Chromatic index · Maximum degree · Density

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1 Introduction

Given a multigraph $G = (V, E)$, the *edge-coloring problem* (ECP) is to color the edges of G with the minimum number of colors so that no two adjacent edges have the same color. Its optimal value is called the *chromatic index* of G, denoted by $\chi'(G)$. In addition to its great theoretical interest, ECP arises in a variety of applications, so it has attracted tremendous research efforts in several fields, such as discrete mathematics, combinatorial optimization, and theoretical computer science. Holyer (1980) proved that it is *NP*-hard in general to determine $\chi'(G)$, even when restricted to a simple cubic graph, so there is no efficient algorithm for solving ECP exactly unless $NP = P$, and hence the focus of extensive research has been on near-optimal solutions to ECP or good estimates of $\chi'(G)$.

Let $\Delta(G)$ be the maximum degree of G. Clearly, $\chi'(G) \geq \Delta(G)$. There are two classical upper bounds on the chromatic index: the first of these, $\chi'(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor$, was established by Shannon (1949) in 1949, and the second, $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum multiplicity of edges in G, was proved independently by Vizing (1964) and Gupta (1967) in the 1960s. This second result is widely known as Vizing's theorem, which is particularly appealing when applied to a simple graph G, because it reveals that $\chi'(G)$ can take only two possible values: $\Delta(G)$ and $\Delta(G) + 1$. Nevertheless, in the presence of multiple edges, the gap between $\chi'(G)$ and these three bounds can be arbitrarily large. Therefore it is desirable to find some other graph theoretic parameters connected to the chromatic index.

Observe that each color class in an edge-coloring of G is a matching, so it contains at most $(|U| - 1)/2$ edges in $E(U)$ for any $U \subseteq V$ with $|U|$ odd, where $E(U)$ is the set of all edges of G with both ends in U. Hence the *density* of G, defined by

$$\Gamma(G) = \max \left\{ \frac{2|E(U)|}{|U| - 1} : \ U \subseteq V, \ |U| \geq 3 \ \text{ and } \ \text{odd} \right\},$$

provides another lower bound for $\chi'(G)$. It follows that $\chi'(G) \geq \max\{\Delta(G), \Gamma(G)\}$.

To facilitate better understanding of the parameter $\max\{\Delta(G), \Gamma(G)\}$, let A be the edge-matching incidence matrix of G (that is, each column of A is the incidence vector of a matching). Then ECP can be naturally formulated as an integer program, whose linear programming (LP) relaxation is exactly as given below:

$$\begin{array}{ll} \text{Minimize} & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \mathbf{1} \\ & \boldsymbol{x} \geq \mathbf{0}. \end{array}$$

This linear program is called the *fractional edge-coloring problem* (FECP), and its optimal value is called the *fractional chromatic index* of G, denoted by $\chi^*(G)$. As shown by Seymour (1979) using Edmonds' matching polytope theorem Edmonds (1965), it is always true that $\chi^*(G) = \max\{\Delta(G), \Gamma(G)\}$. Thus the preceding inequality actually states that $\chi'(G) \geq \chi^*(G)$.

As $\chi'(G)$ is integer-valued, we further obtain $\chi'(G) \geq \max\{\Delta(G), \lceil\Gamma(G)\rceil\}$. How good is this bound? In the early 1970s Goldberg (1973) and Seymour (1979) independently made the following conjecture.

Conjecture 1.1 *Every multigraph G satisfies* $\chi'(G) \leq \max\{\Delta(G)+1, \lceil\Gamma(G)\rceil\}$.

Over the past five decades Conjecture 1.1 has been a subject of extensive research, and has stimulated an important body of work; see McDonald (2015) for a survey on this conjecture and Stiebitz et al. (2012) for a comprehensive account of edge-colorings.

Several approximate results state that $\chi'(G) \leq \max\{\Delta(G)+\tau(G), \lceil\Gamma(G)\rceil\}$, where $\tau(G)$ is a positive number depending on G. Asymptotically, Kahn (1996) showed that $\tau(G) = o(\Delta(G))$. Scheide (2010) and Chen et al. (2009) independently proved that $\tau(G) \leq \sqrt{\Delta(G)/2}$. Chen et al. (2018) improved this to $\tau(G) \leq \sqrt[3]{\Delta(G)/2}$. Recently, Chen and Jing (2019) further strengthened this as $\tau(G) \leq \sqrt[3]{\Delta(G)/4}$.

There is another family of results, asserting that $\chi'(G) \leq \max\{\frac{m\Delta(G)+(m-3)}{m-1}, \lceil\Gamma(G)\rceil\}$, for increasing values of m. Such results have been obtained by Andersen (1977) and Goldberg (1973) for $m = 5$, Andersen (1977) for $m = 7$, Goldberg (1984) and Hochbaum et al. (1986) for $m = 9$, Nishizeki and Kashiwagi (1990) and Tashkinov (2000) for $m = 11$, Favrholdt et al. (2006) for $m = 13$, Scheide (2010) for $m = 15$, Chen et al. (2018) for $m = 23$, and Chen and Jing (2019) for $m = 39$. It is worthwhile pointing out that, when $\Delta(G) \leq 39$, the validity of Conjecture 1.1 follows instantly from Chen and Jing's result Chen and Jing (2019), because $\frac{39\Delta(G)+36}{38} < \Delta(G)+2$.

Haxell and McDonald (2012) obtained a different sort of approximation to Conjecture 1.1: $\chi'(G) \leq \max\{\Delta(G)+2\sqrt{\mu(G)\log\Delta(G)}, \lceil\Gamma(G)\rceil\}$. Another way to obtain approximations for Conjecture 1.1 is to incorporate the order n of G (that is, number of vertices) into a bound. In this direction, Plantholt (1999) proved that $\chi'(G) \leq \max\{\Delta(G), \lceil\Gamma(G)\rceil + 1 + \sqrt{n\log(n/6)}\}$ for any multigraph G with even order $n \geq 572$. In Plantholt (2013), he established an improved result that is applicable to all multigraphs.

Marcotte (1990) showed that $\chi'(G) = \max\{\Delta(G), \lceil\Gamma(G)\rceil\}$ for any multigraph G with no K_5^--minor, thereby confirming Conjecture 1.1 for this multigraph class. Recently, Haxell et al. (2019) established Conjecture 1.1 for random multigraphs.

The purpose of this paper is to present a proof of the Goldberg-Seymour conjecture.

Theorem 1.1 *Every multigraph G satisfies* $\chi'(G) \leq \max\{\Delta(G)+1, \lceil\Gamma(G)\rceil\}$.

Let r be a positive integer. A multigraph $G = (V, E)$ is called an *r-graph* if G is regular of degree r and for every $X \subseteq V$ with $|X|$ odd, the number of edges between X and $V - X$ is at least r. Note that if G is an r-graph, then $|V(G)|$ is even and $\Gamma(G) = r$. Seymour (1979) also proposed the following weaker version of Conjecture 1.1, which amounts to saying that $\chi'(G) \leq \max\{\Delta(G), \lceil\Gamma(G)\rceil\} + 1$ for any multigraph G.

Conjecture 1.2 *Every r-graph G satisfies* $\chi'(G) \leq r + 1$.

The following conjecture was posed by Gupta (1967) in 1967 and can be deduced from Conjecture 1.1, as shown by Scheide (2007).

Conjecture 1.3 *Let G be a multigraph such that* $\Delta(G)$ *cannot be expressed in the form* $2p\mu(G) - q$, *for any two integers p and q satisfying* $p > \lfloor (q+1)/2 \rfloor$ *and* $q \geq 0$. *Then* $\chi'(G) \leq \Delta(G) + \mu(G) - 1$.

A multigraph G is called *critical* if $\chi'(H) < \chi'(G)$ for any proper subgraph H of G. In edge-coloring theory, critical multigraphs are of special interest, because they have much more structural properties than arbitrary multigraphs. The following two conjectures are due to Jakobsen (1974, 1975) and were proved by Andersen (1977) to be weaker than Conjecture 1.1.

Conjecture 1.4 *Let G be a critical multigraph with* $\chi'(G) \geq \Delta(G) + 2$. *Then G contains an odd number of vertices.*

Conjecture 1.5 *Let G be a critical multigraph with* $\chi'(G) > \frac{m\Delta(G)+(m-3)}{m-1}$ *for an odd integer* $m \geq 3$. *Then G has at most* $m - 2$ *vertices.*

Motivated by Conjecture 1.1, Hochbaum et al. (1986) formulated the following conjecture concerning the approximability of ECP.

Conjecture 1.6 *There is a polynomial-time algorithm that colors the edges of any multigraph G using at most* $\max\{\chi'(G), \Delta(G) + 1\}$ *colors.*

Since Conjectures 1.2–1.5 are all weaker than the Goldberg-Seymour conjecture, the truth of them follows from Theorem 1.1 as corollaries.

Theorem 1.2 *Every r-graph G satisfies* $\chi'(G) \leq r + 1$.

Theorem 1.3 *Let G be a multigraph such that* $\Delta(G)$ *cannot be expressed in the form* $2p\mu(G) - q$, *for any two integers p and q satisfying* $p > \lfloor (q+1)/2 \rfloor$ *and* $q \geq 0$. *Then* $\chi'(G) \leq \Delta(G) + \mu(G) - 1$.

Theorem 1.4 *Let G be a critical multigraph with* $\chi'(G) \geq \Delta(G) + 2$. *Then G contains an odd number of vertices.*

Theorem 1.5 *Let G be a critical multigraph with* $\chi'(G) > \frac{m\Delta(G)+(m-3)}{m-1}$ *for an odd integer* $m \geq 3$. *Then G has at most* $m - 2$ *vertices.*

We have seen that FECP is intimately tied to ECP. For any multigraph G, the fractional chromatic index $\chi^*(G) = \max\{\Delta(G), \Gamma(G)\}$ can be determined in polynomial time by combining the Padberg-Rao separation algorithm for b-matching polyhedra Padberg and Rao (1982) (see also Letchford et al. 2008; Padberg and Wolsey 1984) with binary search. In Chen et al. (2019), Chen, Zang, and Zhao designed a combinatorial polynomial-time algorithm for finding the density $\Gamma(G)$ of any multigraph G, thereby resolving a problem posed in both Stiebitz et al. (2012) and Jensen and Toft (2015). Nemhauser and Park (1991) observed that FECP can be solved in polynomial time by an ellipsoid algorithm, because the separation problem of its LP dual is exactly the maximum-weight matching problem (see also Schrijver 2003, Theorem 28.6 on page 477). In Chen et al. (2019), Chen, Zang, and Zhao also came up with a combinatorial polynomial-time algorithm for FECP.

Our proof of Theorem 1.1 is not algorithmic in nature. It would be interesting to see if our proof can be adapted to yield a polynomial-time algorithm for finding an edge-coloring of any multigraph G with at most $\max\{\Delta(G) + 1, \lceil\Gamma(G)\rceil\}$ colors. A successful implementation would lead to an affirmative answer to Conjecture 1.6 as well.

Some remarks may help to put Theorem 1.1 in proper perspective.

First, by Theorem 1.1, there are only two possible values for the chromatic index of a multigraph G: $\max\{\Delta(G), \lceil\Gamma(G)\rceil\}$ and $\max\{\Delta(G) + 1, \lceil\Gamma(G)\rceil\}$. Thus an analogue to Vizing's theorem on edge-colorings of simple graphs, a fundamental result in discrete mathematics, holds for multigraphs.

Second, Theorem 1.1 exhibits a dichotomy on edge-coloring: While Holyer's theorem (Holyer 1980) tells us that it is *NP*-hard to determine $\chi'(G)$, we can approximate it within one of its true value, because $\max\{\Delta(G) + 1, \lceil\Gamma(G)\rceil\} - \chi'(G) \le 1$. Furthermore, if $\Gamma(G) > \Delta(G)$, then $\chi'(G) = \lceil\Gamma(G)\rceil$, so it can be found in polynomial time (Chen et al. 2019; Padberg and Rao 1982).

Third, by Theorem 1.1 and aforementioned Seymour's theorem on fractional chromatic index, every multigraph $G = (V, E)$ satisfies $\chi'(G) - \chi^*(G) \le 1$, which can be naturally extended to the weighted case. Let $w(e)$ be a nonnegative integral weight on each edge $e \in E$ and let $\boldsymbol{w} = (w(e) : e \in E)$. The *chromatic index* of $(G, \boldsymbol{w})$, denoted by $\chi'_w(G)$, is the minimum number of matchings in G such that each edge e is covered exactly $w(e)$ times by these matchings, and the *fractional chromatic index* of $(G, \boldsymbol{w})$, denoted by $\chi^*_w(G)$, is the optimal value of the following linear program:

$$\begin{array}{ll} \text{Minimize} & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{w} \\ & \ \ \boldsymbol{x} \ge \mathbf{0}, \end{array}$$

where A is again the edge-matching incidence matrix of G. Clearly, $\chi'_w(G)$ is the optimal value of the corresponding integer program. Let G_w be obtained from G by replacing each edge e with $w(e)$ parallel edges between the same ends. It is then routine to check that $\chi'_w(G) = \chi'(G_w)$ and $\chi^*_w(G) = \chi^*(G_w)$. So the inequality $\chi'_w(G) - \chi^*_w(G) \le 1$ holds for all nonnegative integral weight functions $\boldsymbol{w}$, and hence FECP has a fascinating integer rounding property (see Schrijver 1986, 2003). (The LP relaxation (LP) of a combinatorial optimization problem (IP) is said to have an integer rounding property if there exists an absolute positive constant c, such that the optimal value of LP differs from that of IP by at most c, for all weight functions. This property is of great interest in integer programming and combinatorial optimization.)

So far the most powerful and sophisticated technique for multigraph edge-coloring is the method of Tashkinov trees (Tashkinov 2000), which generalizes the earlier methods of Vizing fans (Vizing 1964) and Kierstead paths (Kierstead 1984). (These methods are named after the authors who invented them, respectively.) Most recent results described above Theorem 1.1 were obtained by using the method of Tashkinov trees. As remarked by McDonald (2015), the Goldberg-Seymour conjecture and ideas culminating in this method are two cornerstones in modern edge-coloring. Nevertheless, this method suffers some theoretical limitation when applied to prove the conjecture; see Asplund and Asplund and McDonald (2016) for detailed information. Despite various attempts to extend the Tashkinov trees (see, for instance, Chen et al. 2018; Chen and Jing 2019; Chen et al. 2009; Scheide 2010; Stiebitz et al. 2012), the difficulty encountered by the method remains unresolved and, undesirably, another problem emerges: it becomes very difficult to preserve the structure of an extended Tashkinov tree under Kempe changes (the most useful tool in edge-coloring theory). In this paper we introduce a new type of extended Tashkinov trees and develop an effective control mechanism over Kempe changes, which can overcome all the aforementioned difficulties. The reader is referred to Chen and Jing (2019) for a prototype of this control mechanism and its role in the derivation of the best approximate result on the Goldberg-Seymour conjecture presently available.

The remainder of this paper is organized as follows. In Sect. 2, we introduce some basic concepts and techniques of edge-coloring theory, and exhibit some important properties of stable colorings. In Sect. 3, we define the extended Tashkinov trees to be employed in subsequent proof, and give an outline of our proof strategy. In Sect. 4, we establish some auxiliary results on the extended Tashkinov trees and stable colorings, which ensure that this type of trees is preserved under some restricted Kempe changes. In Sect. 5, we develop an effective control mechanism over Kempe changes, the so-called good hierarchy of an extended Tashkinov tree, based on its prototype introduced by Chen and Jing in Chen and Jing (2019) (see Condition R2 therein). In Sect. 6, we derive some properties satisfied by the good hierarchies introduced in the preceding section. In Sect. 7, we present the last step of our proof based on these good hierarchies.

2 Preliminaries

This section presents some basic definitions, terminology, and notation used in our paper, along with some important properties and results.

2.1 Terminology and notation

Let $G = (V, E)$ be a multigraph. For each $X \subseteq V$, let $G[X]$ denote the subgraph of G induced by X, and let $G - X$ denote $G[V - X]$; we write $G - x$ for $G - X$ if

$X = \{x\}$. Moreover, we use $\partial(X)$ to denote the set of all edges with precisely one end in X, and write $\partial(x)$ for $\partial(X)$ if $X = \{x\}$. For each pair $x, y \in V$, let $E(x, y)$ denote the set of all edges between x and y. As it is no longer appropriate to represent an edge f between x and y by xy in a multigraph, we write $f \in E(x, y)$ instead. For each subgraph H of G, let $V(H)$ and $E(H)$ denote the vertex set and edge set of H, respectively, let $|H| = |V(H)|$, and let $G[H] = G[V(H)]$ and $\partial(H) = \partial(V(H))$.

Let e be an edge of G. A *tree-sequence* with respect to G and e is a sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 1$, consisting of distinct edges $e_1, e_2, \ldots, e_p$ and distinct vertices $y_0, y_1, \ldots, y_p$, such that $e_1 = e$ and each edge e_j with $1 \leq j \leq p$ is between y_j and some y_i with $0 \leq i < j$. In this paper a tree-sequence is treated as a tree. Given a tree-sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$, we can naturally associate a linear order $\prec$ with its vertices, such that $y_i \prec y_j$ if $i < j$. We write $y_i \preceq y_j$ if $i \leq j$. This linear order will be used repeatedly in subsequent sections. For each vertex y_j of T with $j \geq 1$, let $T(y_j)$ denote $(y_0, e_1, y_1, \ldots, e_j, y_j)$. Clearly, $T(y_j)$ is also a tree-sequence with respect to G and e. We call $T(y_j)$ the *segment* of T induced by y_j. Let T_1 and T_2 be two tree-sequences with respect to G and e. We write $T_2 - T_1$ for $E[T_2] - E[T_1]$; with a slight abuse of notation, we also use $T_2 - T_1$ to denote the subgraph of T_2 induced by $E[T_2] - E[T_1]$. Write $T_1 \subseteq T_2$ if T_1 is a segment of T_2, and write $T_1 \subset T_2$ if T_1 is a proper segment of T_2; that is, $T_1 \subseteq T_2$ and $T_1 \neq T_2$.

A *k-edge-coloring* of G is an assignment of k colors, $1, 2, \ldots, k$, to the edges of G so that no two adjacent edges have the same color. By definition, the chromatic index $\chi'(G)$ of G is the minimum k for which G has a k-edge-coloring. We use $[k]$ to denote the color set $\{1, 2, \ldots, k\}$, and use $\mathcal{C}^k(G)$ to denote the set of all k-edge-colorings of G. Note that every k-edge-coloring of G is a mapping from E to $[k]$.

Let φ be a k-edge-coloring of G. For each $\alpha \in [k]$, the edge set $E_{\varphi,\alpha} = \{e \in E : \varphi(e) = \alpha\}$ is called a *color class*, which is a matching in G. For any two distinct colors α and β in $[k]$, let H be the spanning subgraph of G with $E(H) = E_{\varphi,\alpha} \cup E_{\varphi,\beta}$. Then each component of H is either a path or an even cycle; we refer to such a component as an (α, β)-*chain* with respect to φ, and also call it an (α, β)-*path* (resp. (α, β)-*cycle*) if it is a path (resp. cycle). Possibly a component of H is an isolated vertex. We use $P_v(\alpha, \beta, \varphi)$ to denote the unique (α, β)-chain containing the vertex v. Clearly, for any two distinct vertices u and v, $P_u(\alpha, \beta, \varphi)$ and $P_v(\alpha, \beta, \varphi)$ are either identical or vertex-disjoint. Let C be an (α, β)-chain with respect to φ, and let φ' be the k-edge-coloring arising from φ by interchanging α and β on C. We say that φ' is obtained from φ by *recoloring* C, and write $\varphi' = \varphi/C$. This operation is called a *Kempe change*. We also say that this Kempe change is *rooted* at v if it has degree one in C.

Let F be an edge subset of G. As usual, $G - F$ stands for the multigraph obtained from G by deleting all edges in F; we write $G - f$ for $G - F$ if $F = \{f\}$. Let $\pi \in \mathcal{C}^k(G - F)$. For each $K \subseteq E$, define $\pi(K) = \cup_{e \in K-F} \{\pi(e)\}$. For each $v \in V$, define

$$\pi(v) = \pi(\partial(v)) \text{ and } \overline{\pi}(v) = [k] - \pi(v).$$

We call $\pi(v)$ the set of colors *present* at v and call $\overline{\pi}(v)$ the set of colors *missing* at v. For each $X \subseteq V$, define

$$\overline{\pi}(X) = \cup_{v \in X} \overline{\pi}(v).$$

We call X *elementary* with respect to π if $\overline{\pi}(u) \cap \overline{\pi}(v) = \emptyset$ for any two distinct vertices $u, v \in X$. We call X *closed* with respect to π if $\pi(\partial(X)) \cap \overline{\pi}(X) = \emptyset$; that is, no missing color of X appears on the edges in $\partial(X)$. Furthermore, we call X *strongly closed* with respect to π if X is closed with respect to π and $\pi(e) \neq \pi(f)$ for any two distinct colored edges $e, f \in \partial(X)$. For each subgraph H of G, write $\overline{\pi}(H)$ for $\overline{\pi}(V(H))$, and write $\pi\langle H \rangle$ for $\pi(E(H))$. Moreover, define

$$\partial_{\pi,\alpha}(H) = \{e \in \partial(H) : \pi(e) = \alpha\},$$

and define

$$I[\partial_{\pi,\alpha}(H)] = \{v \in V(H) : v \text{ is incident with an edge in } \partial_{\pi,\alpha}(H)\}.$$

For an edge $e \in \partial(H)$, we call its end in (resp. outside) H the *in-end* (resp. *out-end*) relative to H. Thus $I[\partial_{\pi,\alpha}(H)]$ consists of all in-ends (relative to H) of edges in $\partial_{\pi,\alpha}(H)$. If $|\partial_{\pi,\alpha}(H)| \geq 2$, we call α a *defective color* of H with respect to π, call each edge in $\partial_{\pi,\alpha}(H)$ a *defective edge* of H with respect to π, and call each vertex in $I[\partial_{\pi,\alpha}(H)]$ a *defective vertex* of H with respect to π. A color $\alpha \in \overline{\pi}(H)$ is called *closed* in H under π if $\partial_{\pi,\alpha}(H) = \emptyset$. For convenience, we say that H is *closed* (resp. *strongly closed*) with respect to π if $V(H)$ is closed (resp. strongly closed) with respect to π. Let α and β be two colors that are not assigned to $\partial(H)$ under π. We use $\pi/(G - H, \alpha, \beta)$ to denote the coloring π' obtained from π by interchanging α and β in $G - V(H)$. Since π belongs to $\mathcal{C}^k(G - F)$, so does π'.

2.2 Elementary multigraphs

Let $G = (V, E)$ be a multigraph. We call G an *elementary multigraph* if $\chi'(G) = \lceil \Gamma(G) \rceil$. With this notion, Conjecture 1.1 can be rephrased as follows.

Conjecture 2.1 *Every multigraph G with $\chi'(G) \geq \Delta(G) + 2$ is elementary.*

Recall that G is critical if $\chi'(H) < \chi'(G)$ for any proper subgraph H of G. As pointed out by Stiebitz et al. (2012) (see page 7), for a proof of Conjecture 2.1, it suffices to consider critical multigraphs. To see this, let G be an arbitrary multigraph with $\chi'(G) \geq \Delta(G) + 2$. Then G contains a critical multigraph H with $\chi'(H) = \chi'(G)$, which implies that $\chi'(H) \geq \Delta(H) + 2$. Note that if H is elementary, then so is G, because $\lceil \Gamma(G) \rceil \leq \chi'(G) = \chi'(H) = \lceil \Gamma(H) \rceil \leq \lceil \Gamma(G) \rceil$. Thus both inequalities hold with equalities, and hence $\chi'(G) = \lceil \Gamma(G) \rceil$.

To prove Conjecture 1.1, we shall actually establish the following statement.

Theorem 2.1 *Every critical multigraph G with $\chi'(G) \geq \Delta(G) + 2$ is elementary.*

In our proof we shall appeal to the following theorem, which reveals some intimate connection between elementary multigraphs and elementary sets. This result is implicitly contained in Andersen (1977) and Goldberg (1984), and explicitly stated in Stiebitz et al. (2012) (see Theorem 1.4 on page 8).

Theorem 2.2 *Let $G = (V, E)$ be a multigraph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If G is critical, then the following conditions are equivalent:*

(i) *G is an elementary multigraph.*
(ii) *For each edge $e \in E$ and each coloring $\varphi \in \mathcal{C}^k(G - e)$, the vertex set V is elementary with respect to φ.*
(iii) *There exists an edge $e \in E$ and a coloring $\varphi \in \mathcal{C}^k(G - e)$, such that the vertex set V is elementary with respect to φ.*
(iv) *There exists an edge $e \in E$, a coloring $\varphi \in \mathcal{C}^k(G - e)$, and a subset X of V, such that X contains both ends of e, and X is elementary as well as strongly closed with respect to φ.*

2.3 Stable colorings

In this subsection, we assume that *T* is a tree-sequence with respect to a multigraph $G = (V, E)$ and an edge *e*, *C* is a subset of [*k*], and φ is a coloring in $\mathcal{C}^k(G - e)$, where $k \geq \Delta(G) + 1$. We say that an edge *f* of *G* is *incident* to *T* if at least one end of *f* is contained in *T*; this definition applies to edges of *T* as well. Since our proof consists of a sophisticated sequence of Kempe changes, the concept of stable coloring introduced below will be employed to preserve some important coloring properties of *T*, such as, among others, the color on each edge and the set of colors missing at each vertex. Usually, *C* is the set of colors assigned to *E*(*T*) but not missing at any vertex of *T*.

To be specific, a coloring $\pi \in \mathcal{C}^k(G - e)$ is called a (T, C, φ)-*stable coloring* if the following two conditions are satisfied:
(*i*) $\pi(f) = \varphi(f)$ for any $f \in E$ incident to *T* with $\varphi(f) \in \overline{\varphi}(T) \cup C$; and
(*ii*) $\overline{\pi}(v) = \overline{\varphi}(v)$ for any $v \in V(T)$.
By convention, $\pi(e) = \varphi(e) = \emptyset$. The following lemma gives an equivalent definition of stable colorings.

Lemma 2.3 *A coloring $\pi \in \mathcal{C}^k(G - e)$ is (T, C, φ)-stable iff $\pi(f) = \varphi(f)$ for any $f \in E$ incident to T with $\varphi(f) \in \overline{\varphi}(T) \cup C$ or $\pi(f) \in \overline{\varphi}(T) \cup C$.*

Proof Let (i') stand for the condition specified in the "if" part. We propose to show that (i') holds iff both (*i*) and (*ii*) hold.

Trivially, (i') implies (*i*). If there exists $v \in V(T)$ such that $\overline{\pi}(v) \neq \overline{\varphi}(v)$, then some edge *f* incident to *v* satisfies $\pi(f) \in \overline{\varphi}(v)$ because $|\overline{\pi}(v)| = |\overline{\varphi}(v)|$. From (i') we deduce that $\pi(f) = \varphi(f)$ and hence $\varphi(f) \in \overline{\varphi}(v)$, a contradiction. So (i') implies (*ii*) as well.

Conversely, let $f \in E$ be an arbitrary edge incident to *T* with $\pi(f) \in \overline{\varphi}(T) \cup C$. We claim that $\varphi(f) = \pi(f)$. Assume the contrary: $\varphi(f) \neq \pi(f)$. Let $v \in V(T)$ be an end of *f*. By (*ii*), we have $\overline{\pi}(v) = \overline{\varphi}(v)$. So $\pi(v) = \varphi(v)$ and hence there exists an edge $g \in \partial(v) - \{f\}$ with $\varphi(g) = \pi(f)$. It follows that $\varphi(g) \in \overline{\varphi}(T) \cup C$. By (*i*), we obtain $\pi(g) = \varphi(g)$, which implies $\pi(f) = \pi(g)$, contradicting the hypothesis that $\pi \in \mathcal{C}^k(G - e)$. Our claim asserts that $\varphi(f) = \pi(f)$ for any $f \in E$ incident to *T* with $\pi(f) \in \overline{\varphi}(T) \cup C$. Combining this with (*i*), we conclude that (i') holds. □

From the definition and Lemma 2.3 we see that the following statements hold for a (T, C, φ)-stable coloring π:

- if $T' \subseteq T$ and $\overline{\varphi}(T') \cup C' \subseteq \overline{\varphi}(T) \cup C$, then π is also (T', C', φ)-stable;
- if a color $\alpha \in \overline{\varphi}(T)$ is closed in *T* under φ, then it is also closed in *T* under π; and
- if $\varphi\langle T\rangle \subseteq \overline{\varphi}(T) \cup C$, then $\pi(f) = \varphi(f)$ for all edges *f* on *T*.

Let us derive some further properties satisfied by stable colorings.

Lemma 2.4 *Being $(T, C, \cdot)$-stable is an equivalence relation on $\mathcal{C}^k(G - e)$.*

Proof From Lemma 2.3 and the above condition (*ii*), it is clear that being $(T, C, \cdot)$-stable is reflexive, symmetric, and transitive. So it defines an equivalence relation on $\mathcal{C}^k(G - e)$. □

Lemma 2.5 *Suppose T is closed but not strongly closed with respect to φ, with |V(T)| odd. If π is a (T, C, φ)-stable coloring, then T is also closed but not strongly closed with respect to π.*

Proof Let $X = V(T)$ and let *t* be the size of the set $[k] - \overline{\varphi}(X)$. By hypotheses, |*V*(*T*)| is odd and *T* is not strongly closed with respect to φ. Thus under the coloring φ each color in $[k] - \overline{\varphi}(X)$ is assigned to at least one edge in $\partial(T)$, and some color in $[k] - \overline{\varphi}(X)$ is assigned to at least two edges in $\partial(T)$. It follows that $|\partial(T)| \geq t + 1$. Since π is a (T, C, φ)-stable coloring, from Lemma 2.3 and the above condition (*ii*) we deduce that *T* is closed with respect to π and that $\overline{\pi}(X) = \overline{\varphi}(X)$ (so $[k] - \overline{\pi}(X)$ is also of size *t*). As only colors in $[k] - \overline{\pi}(X)$ can be assigned to edges in $\partial(T)$ under π, some of these colors is used at least twice by the Pigeonhole Principle. Hence *T* is not strongly closed with respect to π. □

Let *P* be a path in *G* whose edges are colored alternately by α and β in φ, with $|P| \geq 2$, and let *u* and *v* be the ends of *P* with $v \in V(T)$. We say that *P* is a *T-exit path* with respect to φ if $V(T) \cap V(P) = \{v\}$ and $\overline{\varphi}(u) \cap \{\alpha, \beta\} \neq \emptyset$; in this case, *v* is called a $(T, \varphi, \{\alpha, \beta\})$-*exit* and *P* is also called a $(T, \varphi, \{\alpha, \beta\})$-*exit path*. Note that possibly $\overline{\varphi}(v) \cap \{\alpha, \beta\} = \emptyset$; now *P* is a proper subpath of an (α, β)-path.

Lemma 2.6 *Suppose T is closed with respect to φ, and $f \in E(u, v)$ is an edge in $\partial(T)$ with $v \in V(T)$. If there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring π, such that $\overline{\pi}(u) \cap \overline{\pi}(T) \neq \emptyset$, then for any $\alpha \in \overline{\varphi}(v)$ there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring σ, such that v is a $(T, \sigma, \{\alpha, \varphi(f)\})$-exit and that the $(T, \sigma, \{\alpha, \varphi(f)\})$-exit path at v contains only one edge.*

Proof Let $\beta \in \overline{\pi}(u) \cap \overline{\pi}(T)$. By the definition of stable coloring, $\beta \in \overline{\varphi}(T)$. Since both α and β are closed in *T* under φ, they are also closed in *T* under π by Lemma 2.3. Define $\sigma = \pi/(G - T, \alpha, \beta)$. Clearly, σ is a $(T, C \cup \{\varphi(f)\}, \pi)$-stable coloring. By Lemma 2.4, σ is also a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring. Since $P_v(\alpha, \varphi(f), \sigma)$ consists of a single edge *f*, it is a *T*-exit path with respect to σ. Hence *v* is a $(T, \sigma, \{\alpha, \varphi(f)\})$-exit. □

2.4 Tashkinov trees

A multigraph *G* is called *k-critical* if it is critical and $\chi'(G) = k + 1$. Throughout this paper, by a *k-triple* we mean a *k*-critical multigraph $G = (V, E)$, where $k \geq \Delta(G) + 1$, together with an uncolored edge $e \in E$ and a coloring $\varphi \in \mathcal{C}^k(G - e)$; we denote it by (G, e, φ).

Let (G, e, φ) be a *k*-triple. A *Tashkinov tree* with respect to *e* and φ is a tree-sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with respect to *G* and *e*, such that for each edge e_j with $2 \leq j \leq p$, there is a vertex y_i with $0 \leq i < j$ satisfying $\varphi(e_j) \in \overline{\varphi}(y_i)$.

The following theorem is due to Tashkinov (2000); its proof can also be found in Stiebitz et al. (2012) (see Theorem 5.1 on page 116).

Theorem 2.7 *Let (G, e, φ) be a k-triple and let T be a Tashkinov tree with respect to e and φ. Then V(T) is elementary with respect to φ.*

Let $G = (V, E)$ be a *k*-critical multigraph *G* with $k \geq \Delta(G) + 1$. For each edge $e \in E$ and each coloring $\varphi \in \mathcal{C}^k(G - e)$, there is at least one Tashkinov tree *T* with respect to *e* and φ. The *Tashkinov order* of *G*, denoted by *t*(*G*), is the largest number of vertices contained in such a Tashkinov tree over all *e* and $\varphi \in \mathcal{C}^k(G - e)$. Scheide (2010) (see Proposition 4.5) has established the following result, which will be employed in our proof.

Theorem 2.8 *Let G be a critical multigraph G with $\chi'(G) \geq \Delta(G) + 2$. If $t(G) < 11$, then G is an elementary multigraph.*

Tashkinov trees have been used successfully to establish various approximate results on Conjecture 1.1. The crux of this approach is to capture the density $\Gamma(G)$ by exploring a sufficiently large Tashkinov tree (see Theorem 2.7). However, this target may become unreachable when the upper bound on $\chi'(G)$ (one wishes to derive) gets close to $\chi'(G)$, even if we allow for an unlimited number of Kempe changes; such an example has been found by Asplund and McDonald (2016). To carry out a proof of Conjecture 1.1, we introduce a type of extended Tashkinov trees in this paper by the procedure described below.

Definition 2.9 Given a *k*-triple (G, e, φ) and a tree-sequence *T* with respect to *G* and *e*, we say that a tree-sequence (*T*, *f*, *y*) is obtained from *T* by a *Tashkinov augmentation* (TA) under φ if $\varphi(f) \in \overline{\varphi}(T)$, one end *x* of *f* is contained in *T*, and the other end *y* of *f* is outside *T*. A *Tashkinov augmentation algorithm* (TAA) consists of a sequence of TAs under the same edge coloring. We call a tree-sequence T' a *closure* of *T* under

φ if T' arises from T by TAA and cannot grow further by TA under φ (equivalently, T' has become closed).

So a Tashkinov tree with respect to e and φ is a tree-sequence obtained from (y_0, e, y_1) by TAA, where y_0 and y_1 are two ends of e. We point out that, although there might be several ways to construct a closure of T under φ, the vertex set of these closures is unique.

In the next section we shall give a detailed description of an algorithm for constructing the aforementioned extended Tashkinov trees, and present the main result of this paper, which implies Theorem 2.1. In view of Theorem 2.2, to prove Conjecture 1.1, we may turn to finding a strongly closed elementary tree-sequence. Thus in our algorithm we introduce three types of extensions from a *closed* elementary tree-sequence T (say, a closed Tashkinov tree), which has defective edges under a coloring φ (so T is not strongly closed). Specifically, we consider the maximum defective vertex v in the order $\prec$ over all (T, C, φ)-stable colorings (here C contains all colors used to construct T but are not missing at vertices in T, so $C = \emptyset$ when T is a Tashkinov tree). Let f be a defective edge incident to v under φ, with $f \in E(u, v)$ and $\varphi(f) = \delta$.

If $T \cup \{u\}$ is elementary under all $(T, C \cup \{\delta\}, \varphi)$-stable colorings, we add this edge f to T and extend the resulting tree-sequence using TAA. This first type of extensions is called series extension (SE) in our algorithm.

Otherwise, we pick a color γ in $\overline{\varphi}(v)$; Lemma 2.6 guarantees the existence of a stable coloring such that v is the only common vertex of a (γ, δ)-path P and T. We then perform the second type of extensions, called parallel extension (PE). Each PE is followed by a sequence of the third type of extensions, called revisiting extension (RE) and performed whenever possible. During PE, we switch colors along P and apply TAA to T, which is no longer closed now. During REs, we repeatedly add an edge on some (γ, δ)-cycle intersecting T (the original T before the previous PE) and apply TAA, until the vertices of all (γ, δ)-cycles intersecting T are contained in the resulting tree-sequence. We may view each PE and its succeeding REs as a whole, in which PE is the primary extension while REs are auxiliary extensions.

Applying the above three extensions to closed elementary tree-sequences recursively, we shall end up with a strongly closed elementary tree-sequence, thereby proving the Goldberg-Seymour conjecture. The elementary property of such tree-sequences will be established in Sects. 5, 6 and 7 by further developing techniques introduced in Chen and Jing (2019), which essentially allow us to prove that if the elementary property of a closed tree-sequence T is preserved under adding a vertex using Algorithm 3.1, then this property can be extended even further to a closure of the resulting tree-sequence. To the best of our knowledge, no previous investigation has ever employed these three extensions; the weaker extension used in Chen and Jing (2019) (see Condition R1 therein) can only lead to some partial results despite several new techniques originating from this study.

It is worthwhile pointing out that, when encountering a non-elementary tree-sequence, almost all previous methods proceed by reducing the size of the non-elementary tree-sequence and eventually reach a contradiction by coloring the uncolored critical edge e. However, our algorithm goes the other way around: when $T \cup \{u\}$ is not elementary, it modifies the coloring and employs PE to construct a larger elementary tree-sequence while avoiding the edge *f*. Essentially our proof only requires the edge *e* to be critical as potential non-critical edges like *f* are bypassed.

3 Extended tashkinov trees

The purpose of this section is to present extended Tashkinov trees to be used in our proof and to give an outline of our proof strategy.

Given a *k*-triple (G, e, φ), a *Tashkinov series* constructed from it is a series of tuples $(T_n, \varphi_{n-1}, S_{n-1}, F_{n-1}, \Theta_{n-1})$ for $n = 1, 2, \ldots$ output by the following algorithm, where T_n is a closed tree-sequence with respect to φ_{n-1}, $S_{n-1} \subseteq [k]$, $F_{n-1} \subseteq E$, and Θ_{n-1} is a label holding information on how T_n is constructed. Furthermore, $T_n + f_n$ stands for the tree-sequence augmented from T_n by adding an edge f_n, and the definition of $(T, \varphi, \{\alpha, \beta\})$-*exit* can be found in the paragraph right above Lemma 2.6.

To help gain a clearer picture of our algorithm, we intentionally use a descriptive language and include Iteration 1, although it is contained in the general Iteration *n*.

Algorithm 3.1 Iteration 0. Let $(T_1, \varphi_0, S_0, F_0, \Theta_0)$ be the initial tuple, such that $\varphi_0 = \varphi$, T_1 is a closure of *e* under φ_0 (which is a closed Tashkinov tree with respect to *e* and φ_0), and $S_0 = F_0 = \Theta_0 = \emptyset$.

Iteration 1. If T_1 is strongly closed with respect to φ_0, stop. Else, we construct the tuple $(T_2, \varphi_1, S_1, F_1, \Theta_1)$ as follows. Set $D_0 = \emptyset$.

- Let v_1 be the maximum defective vertex,[1] in the order $\prec$ over all (T_1, D_0, φ_0)-stable colorings, let π_0 be a corresponding coloring, let f_1 be a defective edge (of T_1 with respect to π_0) incident to v_1, let u_1 be the other end of f_1, and let $\delta_1 = \pi_0(f_1)$.
  - If for every $(T_1, D_0 \cup \{\delta_1\}, \pi_0)$-stable coloring π, we have $\overline{\pi}(u_1) \cap \overline{\pi}(T_1) = \emptyset$, apply **SE** with $n = 1$.
  - Else, let γ_1 be an arbitrary color in $\overline{\pi}_0(v_1)$ and let π'_0 be a $(T_1, D_0 \cup \{\delta_1\}, \pi_0)$-stable coloring such that v_1 is a $(T_1, \pi'_0, \{\gamma_1, \delta_1\})$-exit (such π'_0 exists by Lemma 2.6), apply **PE** with $n = 1$.

Iteration n. If T_n is strongly closed with respect to φ_{n-1}, stop. Else, we construct the tuple $(T_{n+1}, \varphi_n, S_n, F_n, \Theta_n)$ as follows. Set $D_{n-1} = \cup_{i \le n-1} S_i - \overline{\varphi}_{n-1}(T_{n-1})$ (so $D_0 = \emptyset$).

- If there is a subscript $h \le n-1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that

[1] For each $(T_1, D_0\varphi_0)$-stable coloring π, let v_π be the largest defective vertex of T_1 with respect to π in the order $\prec$. Then v_1 is the largest vertex among all these vertices v_π in the order $\prec$. By definition, $v_1 = v_{\pi_0}$.

$\Theta_i = RE$ for all i with $h+1 \le i \le n-1$, if any, and that some (γ_h, δ_h)-cycle O with respect to φ_{n-1} intersects both $V(T_h)$ and $V(G) - V(T_n)$, apply **RE**. (Note that this case cannot occur when $n = 1$.)

- Else, let v_n be the maximum defective vertex in the order $\prec$ over all $(T_n, D_{n-1}, \varphi_{n-1})$-stable colorings, let π_{n-1} be a corresponding coloring, let f_n be a defective edge (of T_n with respect to π_{n-1}) incident to v_n, let u_n be the other end of f_n, and let $\delta_n = \pi_{n-1}(f_n)$.

  - If for every $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring π, we have $\overline{\pi}(u_n) \cap \overline{\pi}(T_n) = \emptyset$, apply **SE**.
  - Else, pick a color γ_n in $\overline{\pi}_{n-1}(v_n)$ as follows. If $v_n = v_i$ for some $1 \le i < n$ with $\Theta_i = PE$, let n' be the largest such i and let $\gamma_n = \delta_{n'}$. Otherwise, let γ_n be an arbitrary color in $\overline{\pi}_{n-1}(v_n)$. Let π'_{n-1} be an arbitrary $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring so that v_n is a $(T_n, \pi'_{n-1}, \{\gamma_n, \delta_n\})$ -exit (such π'_{n-1} exists by Lemma 2.6), apply **PE**.**RE.** Let f_n be an edge in $O \cap \partial(T_n)$ such that O contains a path L connecting f_n and $V(T_h)$ with $V(L) \subseteq V(T_n)$. Let $\varphi_n = \varphi_{n-1}$ and T_{n+1} be a closure of $T_n + f_n$ under φ_n. Set $\delta_n = \delta_h$, $\gamma_n = \gamma_h$, $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = RE$. We call this extension a **revisiting extension** (RE), call f_n an *RE connecting edge*, and call δ_n and γ_n *connecting colors*. Let v_n be the end of f_n in T_n. Note that v_n here is neither called an extension vertex nor called a supporting vertex.

SE. Let $\varphi_n = \pi_{n-1}$ and let T_{n+1} be a closure of $T_n + f_n$ under φ_n. Set $S_n = \{\delta_n\}$, $F_n = \{f_n\}$, and $\Theta_n = SE$. We call this extension a **series extension** (SE), call f_n an *SE connecting edge*, call δ_n a *connecting color*, and call v_n an *extension vertex*.

PE. Let $\varphi_n = \pi'_{n-1}/P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$. Note that $P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1}) \cap V(T_n) = \{v_n\}$ and $\delta_n \in \overline{\varphi}_n(v_n)$ is a defective color of T_n. So T_n is not closed under φ_n. Let T_{n+1} be a closure of T_n under φ_n. Set $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = PE$. We call this extension a **parallel extension** (PE), call f_n a *PE connecting edge*, call δ_n and γ_n *connecting colors*, and call v_n a *supporting vertex*. As f_n is the first edge along $P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$ and is colored by γ_n under φ_n, it is not necessarily contained in T_{n+1}.

Figure 1 shows the possible choices of extensions used in the construction of a Tashkinov series.

Definition 3.1 Let (G, e, φ) be a k-triple and let $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series constructed from (G, e, φ). A tree-sequence T is called an *extended Tashkinov tree* (ETT) constructed from $\mathcal{T}$ under φ_n if $T_n \subset T \subseteq T_{n+1}$. We say that φ_n is the *generating coloring* of T. If the Tashkinov series $\mathcal{T}$ is clear from the context, we may simply say that T is an ETT under φ_n.

We shall mainly work on an ETT T as defined above in the remainder of this paper. Such T is not necessarily closed under φ_n, while T_{n+1} is closed.

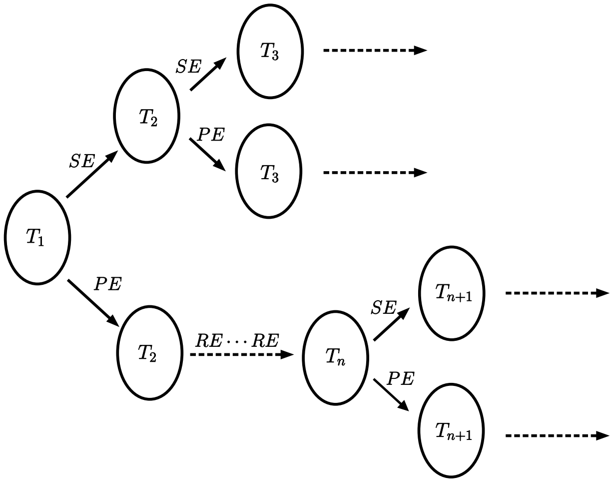


Fig. 1 Tashkinov series constructed by Algorithm 3.1

Throughout we reserve all symbols used for the same usage as in the algorithm. In particular, $D_i = \cup_{h \le i} S_h - \overline{\varphi}_i(T_i)$ for $i \ge 0$. To help understand the algorithm and ETTs better, let us make a few remarks and offer some simple observations.

(3.1) In our proof we shall always restrict our attention to the case when $|T_n|$ is odd. Suppose T_n is closed but not strongly closed with respect to φ_{n-1}. Then, by Lemma 2.5, the same property holds for T_n with respect to any $(T_n, D_{n-1}, \varphi_{n-1})$ -stable coloring π. Let v_π denote the largest defective vertex of T_n with respect to π in the order $\prec$. Note that v_n involved in SE and PE is the largest vertex among all these vertices v_π in the order $\prec$ and so it is uniquely determined by the triple $(T_n, D_{n-1}, \varphi_{n-1})$, while f_n involved in each extension might be selected in several ways.

(3.2) In the algorithm δ_n is a defective color of T_n with respect to φ_n when $\Theta_n = SE$ or PE (as $|\partial_{\pi_{n-1}, \delta_n}(T_n)| \ge 3$ when $|T_n|$ is odd), while γ_n is a defective color of T_n with respect to φ_n when $\Theta_n = RE$. Unlike PE or SE, v_n involved in RE may not be a maximum defective vertex. Moreover, the set $D_{n-1} = \cup_{i \le n-1} S_i - \overline{\varphi}_{n-1}(T_{n-1})$ is used to store colors employed in the construction of T_n but not missing at any vertex of T_{n-1} under φ_{n-1}.

(3.3) As described in the algorithm, after performing each PE, we grow the Tashkinov series by using RE, whenever possible. So revisiting extension (RE) has priority over both series and parallel extensions (SE and PE). If $\Theta_n = RE$, then all edges in $O \cap \partial(T_h)$ are colored with δ_h and all edges in $O \cap \partial(T_n)$ are colored with γ_h, because δ_h is a color missing at v_h under $\varphi_h = \varphi_{n-1}$ (thereby v_h is outside O), the only edge f_h in $\partial_{\varphi_h, \gamma_h}(T_h)$ (see Lemma 3.2(v) to be proved) is adjacent to v_h, and T_n is closed with respect to φ_{n-1}. Hence O contains at least one edge colored with γ_h in $G[T_h]$, at least two boundary edges of T_h colored with δ_h, and at least two boundary edges of T_n colored with γ_h.

RE is illustrated by the following figure, in which $O \cap \partial(T_n)$ contains four edges colored with γ_h: both the top and bottom edges are good candidates for f_n, but nei-

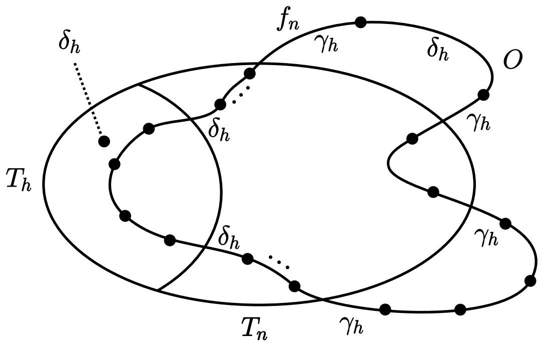


Fig. 2 Revisiting extension (RE)

ther of the middle edges can serve this purpose (Fig. 2), because O contains no path L connecting them and T_h, with $V(L) \subseteq V(T_n)$.

In all previous approaches to the Goldberg-Seymour conjecture using the method of Tashkinov trees, the trees involved in the proofs were constructed under a fix coloring. In sharp contrast, Algorithm 3.1 constructs tree-sequences and edge-colorings simultaneously as it progresses. Therefore, the structural property of an ETT embodied in the extension type and the corresponding coloring might be very fragile (see the next paragraph for details), even under stable colorings. This could cause a serious problem when we try to prove that an ETT T with $T_n \subset T \subseteq T_{n+1}$ is elementary.

For example, if T is not elementary, then we would apply Kempe changes to reduce the size of this counterexample to reach a contradiction while keeping the coloring (T_n, D_n, φ_n)-stable. However, if $\Theta_n = PE$ and the (γ_n, δ_n)-path starting at v_n has evolved to contain two or more vertices from T_n during the process, then the resulting tree-sequence may no longer be an ETT under the new coloring, because PE requires that the (γ_n, δ_n)-path starting at v_n share exactly one vertex with T_n in order to get T_{n+1}. Moreover, if $\Theta_n = RE$ and the edge f_n no longer belongs to any (γ_n, δ_n)-cycle during the process, then the resulting tree-sequence may not be an ETT under the new coloring anymore, because RE requires f_n to be an edge of a (γ_n, δ_n)-cycle.

To circumvent this problem, we introduce the concept of mod coloring (see Definition 3.7) and impose a maximum property (see Definition 3.8) on ETT, and then we can ensure that the ETT structure is preserved under stable colorings (see Theorem 3.10(vi) and Lemma 4.5). REs play an important role in the proof of Lemma 4.3 (Theorem 3.10(iv)), which in turn leads to Lemma 4.5 (Theorem 3.10(vi)). These technical results are essential to deriving the elementary property of the ETT we consider.

Let us look back at Algorithm 3.1. Clearly, PE is the only extension that involves a non-stable coloring (in which one missing color at the supporting vertex has been changed). Based on this observation, we can exhibit some basic coloring properties (Lemmas 3.2–3.6) satisfied by ETTs. Recall that $D_{n-1} = \cup_{i \leq n-1} S_i - \overline{\varphi}_{n-1}(T_{n-1})$ is used to store colors employed in the construction of T_n but not missing at any vertex of T_{n-1} under φ_{n-1}.

Lemma 3.2 *For $n \geq 1$, the following statements hold:*

(i) $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T_{n+1}) \cup D_n$.
(ii) *For any $i \le n$ and $v \in V(T_i)$, we have $\overline{\varphi}_{i-1}(v) = \overline{\varphi}_n(v)$ if v is not used as a supporting vertex at any iteration j with $i \le j \le n$ and $\Theta_j = PE$.*
(iii) *For any edge f incident to T_n, if $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$, then $\varphi_n(f) = \varphi_{n-1}(f)$, unless $\Theta_n = PE$ and $f = f_n$. So $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$ provided that $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$.*
(iv) *$\varphi_{n-1}\langle T_n\rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$ and $\varphi_n\langle T_n\rangle \subseteq \overline{\varphi}_n(T_n) \cup D_n$. So $\sigma_n(f) = \varphi_n(f)$ for any (T_n, D_n, φ_n)-stable coloring σ_n and any edge f on T_n.*
(v) *If $\Theta_n = PE$, then $\partial_{\varphi_n,\gamma_n}(T_n) = \{f_n\}$, and edges in $\partial_{\varphi_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. Furthermore, each color in $\overline{\varphi}_n(T_n) - \{\delta_n\}$ is closed in T_n under φ_n.*

Proof By definition, $D_{n-1} = \cup_{i\le n-1} S_i - \overline{\varphi}_{n-1}(T_{n-1})$. So $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} = \overline{\varphi}_{n-1}(T_n) \cup [\cup_{i\le n-1} S_i - \overline{\varphi}_{n-1}(T_{n-1})]$. Since $\overline{\varphi}_{n-1}(T_{n-1}) \subseteq \overline{\varphi}_{n-1}(T_n)$, we obtain

(1) $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} = \overline{\varphi}_{n-1}(T_n) \cup (\cup_{i\le n-1} S_i)$.

Similarly, we can prove that

(2) $\overline{\varphi}_n(T_n) \cup D_n = \overline{\varphi}_n(T_n) \cup (\cup_{i\le n} S_i)$.

(i) For any $\alpha \in \overline{\varphi}_{n-1}(T_n)$, from Algorithm 3.1 and definition of stable colorings we see that $\alpha \in \overline{\varphi}_n(T_n)$, unless $\Theta_n = PE$ and $\alpha = \gamma_n$; in this exceptional case, $\alpha \in S_n$. So $\overline{\varphi}_{n-1}(T_n) \subseteq \overline{\varphi}_n(T_n) \cup S_n$ and hence $\overline{\varphi}_{n-1}(T_n) \cup (\cup_{i\le n-1} S_i) \subseteq \overline{\varphi}_n(T_n) \cup (\cup_{i\le n} S_i)$. It follows from (1) and (2) that $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$. Clearly, $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T_{n+1}) \cup D_n$.

(ii) In Algorithm 3.1 we always work with stable colorings except during PEs, where only missing colors at supporting vertices are changed. So the desired statement follows.

(iii) Let *f* be an edge incident to T_n with $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. If $\Theta_n = RE$, then $\varphi_n = \varphi_{n-1}$ by Algorithm 3.1, which implies $\varphi_n(f) = \varphi_{n-1}(f)$. So we may assume that $\Theta_n \ne RE$. Let π_{n-1} be the $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring as specified in Algorithm 3.1. By the definition of stable colorings, we obtain $\pi_{n-1}(f) = \varphi_{n-1}(f)$. If $\Theta_n = SE$, then $\varphi_n(f) = \pi_{n-1}(f)$ by Algorithm 3.1. Hence $\varphi_n(f) = \varphi_{n-1}(f)$. It remains to consider the case when $\Theta_n = PE$. Let π'_{n-1} be the $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$ -stable coloring as specified in Algorithm 3.1. By Lemma 2.4, π'_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable. Hence $\pi'_{n-1}(f) = \varphi_{n-1}(f)$. Since $\varphi_n = \pi'_{n-1}/P_{v_n}(\delta_n, \gamma_n, \pi'_{n-1})$ and $P_{v_n}(\delta_n, \gamma_n, \pi'_{n-1})$ contains only one edge f_n incident to T_n (see Algorithm 3.1), we have $\varphi_n(f) = \pi'_{n-1}(f)$, unless $f = f_n$. It follows that $\varphi_n(f) = \varphi_{n-1}(f)$, unless $f = f_n$; in this exceptional case, $\varphi_{n-1}(f) = \delta_n$ and $\varphi_n(f) = \gamma_n \in S_n$. Hence $\varphi_n(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \cup S_n \subseteq \overline{\varphi}_n(T_n) \cup D_n \cup S_n = \overline{\varphi}_n(T_n) \cup D_n$ by (i) and (2), as desired.

(iv) Let us first prove the statement $\varphi_{n-1}\langle T_n\rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$ by induction on *n*. As the statement holds trivially when $n = 1$, we proceed to the induction step and assume that the statement has been established for $n - 1$; that is,

(3) $\varphi_{n-2}\langle T_{n-1}\rangle \subseteq \overline{\varphi}_{n-2}(T_{n-1}) \cup D_{n-2}$.

By (3) and (iii) (with $n-1$ in place of *n*), for each edge *f* on T_{n-1} we have $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_{n-1}) \cup D_{n-1} \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. For each edge $f \in T_n - T_{n-1}$, from Algorithm 3.1 and TAA we see that $\varphi_{n-1}(f) \in D_{n-1}$ if *f* is a connecting edge and $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n)$ otherwise. Combining these observations, we obtain $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. Hence $\varphi_{n-1}\langle T_n\rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$, which together with (iii) implies $\varphi_n\langle T_n\rangle \subseteq \overline{\varphi}_n(T_n) \cup D_n$.

It follows that for any edge *f* on T_n, we have $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$. Thus $\sigma_n(f) = \varphi_n(f)$ for any (T_n, D_n, φ_n)-stable coloring σ_n.

(v) From the definitions of π_{n-1}, vertex v_n (maximum defective vertex) and stable colorings, we see that edges in $\partial_{\pi_{n-1},\delta_n}(T_n)$ are all incident to $V(T_n(v_n))$, and each color in $\overline{\pi}_{n-1}(T_n)$ is closed in T_n under π_{n-1}. So, by the definitions of π'_{n-1} and stable colorings, edges in $\partial_{\pi'_{n-1},\delta_n}(T_n)$ are all incident to $V(T_n(v_n))$, and each color in $\overline{\pi}'_{n-1}(T_n)$ is closed in T_n under π'_{n-1}. Thus the desired statements follow instantly from the definition of φ_n in PE. □

The following lemma generalizes Lemma 3.2(iii) and ensures that colors on some edges incident to a tree remain intact if we grow it by using Algorithm 3.1.

Lemma 3.3 *For any $1 \le i \le n$ and any edge f incident to T_i, if $\varphi_{i-1}(f) \in \overline{\varphi}_{i-1}(T_i) \cup D_{i-1}$, then $\varphi_j(f) = \varphi_{i-1}(f)$ for any j with $i \le j \le n$, unless $f = f_h \in F_h$ for some h with $i \le h \le j$ and $\Theta_h = PE$. In particular, if f is an edge in $G[T_i]$ with $\varphi_{i-1}(f) \in \overline{\varphi}_{i-1}(T_i) \cup D_{i-1}$, then $\varphi_j(f) = \varphi_{i-1}(f)$ for any j with $i \le j \le n$.*

Proof By Lemma 3.2(i), we have $\overline{\varphi}_{h-1}(T_h) \cup D_{h-1} \subseteq \overline{\varphi}_h(T_{h+1}) \cup D_h$ for all $h \ge 1$. So to establish the first half, it suffices to prove the statement for $j = i$, which is exactly the same as Lemma 3.2(iii).

Note that if *f* is an edge in $G[T_i]$, then $f \notin \partial(T_h)$ for any *h* with $i \le h \le j$. Hence $f \ne f_h \in F_h$ for any *h* with $i \le h \le j$ and $\Theta_h = PE$. Thus the second half also holds. □

The lemma below describes some interesting properties satisfied by a sequence of PEs with the same supporting vertex.

Lemma 3.4 *Let u be a vertex of T_n and let B_n be the set of all iterations j with $1 \le j \le n$, such that $\Theta_j = PE$ and $v_j = u$. Suppose $B_n = \{i_1, i_2, \ldots, i_p\}$, where $1 \le i_1 < i_2 < \ldots < i_p \le n$. Then the following statements hold:*

(i) $\gamma_{i_2} = \delta_{i_1}, \gamma_{i_3} = \delta_{i_2}, \ldots, \gamma_{i_p} = \delta_{i_{p-1}}$;
(ii) $\overline{\varphi}_n(u) \cap (\cup_{j\in B_n} S_j) = \overline{\varphi}_{i_p}(u) \cap (\cup_{j\in B_n} S_j) = \{\delta_{i_p}\}$; *and*
(iii) $\overline{\varphi}_{i_1-1}(u) = (\overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}) \cup \{\gamma_{i_1}\}$ *and*
$\overline{\varphi}_{i_p}(u) = (\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\}) \cup \{\delta_{i_p}\}$.

Proof From the definition of B_n, we see that for any $1 \leq j \leq p-1$ and iteration h with $i_j + 1 \leq h \leq i_{j+1} - 1$, if $v_h = u$, then $\Theta_h = RE$ or *SE*. By Lemma 3.2(ii), we have

(1) $\overline{\varphi}_{i_j}(u) = \overline{\varphi}_{i_{j+1}-1}(u)$ for $1 \leq j \leq p-1$. Similarly, $\overline{\varphi}_{i_p}(u) = \overline{\varphi}_n(u)$.

According to the choice of γ_h in a general iteration h involving PE,

(2) $\gamma_{i_{j+1}} = \delta_{i_j}$ for $1 \leq j \leq p-1$, where $\gamma_{i_{j+1}} \in \overline{\varphi}_{i_{j+1}-1}(u)$ and $\delta_{i_j} \in \overline{\varphi}_{i_j}(u)$.

Thus (i) follows instantly from (2). Using (1) and (2), we obtain

(3) $\overline{\varphi}_{i_j}(u) - \{\delta_{i_j}\} = \overline{\varphi}_{i_{j+1}-1}(u) - \{\gamma_{i_{j+1}}\}$ for $1 \leq j \leq p-1$.

Since $\overline{\varphi}_{i_j}(u)$ is obtained from $\overline{\varphi}_{i_j-1}(u)$ by replacing γ_{i_j} with δ_{i_j},

(4) $\overline{\varphi}_{i_j-1}(u) - \{\gamma_{i_j}\} = \overline{\varphi}_{i_j}(u) - \{\delta_{i_j}\}$ for $1 \leq j \leq p$.

Combining (3) and (4), we deduce that

(5) the $2p$ sets $\overline{\varphi}_{i_j-1}(u) - \{\gamma_{i_j}\}$ and $\overline{\varphi}_{i_j}(u) - \{\delta_{i_j}\}$ for $1 \leq j \leq p$ are all equal.

By (5), the set $\cup_{j=1}^p \{\gamma_{i_j}, \delta_{i_j}\}$ (and hence $\cup_{j=1}^p S_{i_j}$) is disjoint from all the $2p$ sets displayed above. So $\overline{\varphi}_{i_p}(u) \cap (\cup_{j=1}^p S_{i_j}) = [(\overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}) \cap (\cup_{j=1}^p S_{i_j})] \cup \{\delta_{i_p}\} = \{\delta_{i_p}\}$, which together with (1) yields (ii).

Again by (5), $\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\} = \overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}$. As $\gamma_{i_1} \in \overline{\varphi}_{i_1-1}(u)$ and $\delta_{i_p} \in \overline{\varphi}_{i_p}(u)$, we get $\overline{\varphi}_{i_1-1}(u) = (\overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}) \cup \{\gamma_{i_1}\}$ and $\overline{\varphi}_{i_p}(u) = (\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\}) \cup \{\delta_{i_p}\}$. Therefore (iii) also holds. □

Lemma 3.5 $|D_n| \leq n$.

Proof Recall that $D_n = \cup_{i \leq n} S_i - \overline{\varphi}_n(T_n)$ (so $D_0 = \emptyset$). For $1 \leq i \leq n$, by Algorithm 3.1, we have $S_i = \{\delta_i\}$ if $\Theta_i = SE$ and $S_i = \{\delta_i, \gamma_i\}$ otherwise.

If $\Theta_n = RE$, then $\varphi_n = \varphi_{n-1}$ and $S_n = S_{n-1}$. So $D_n \subseteq D_{n-1}$. If $\Theta_n = SE$, then $S_n = \{\delta_n\}$ and $\overline{\varphi}_n(T_n) = \overline{\varphi}_{n-1}(T_n)$. It follows that $D_n \subseteq D_{n-1} \cup \{\delta_n\}$. It remains to consider the case when $\Theta_n = PE$. Now $\delta_n \notin \overline{\varphi}_{n-1}(T_n)$ and $(\overline{\varphi}_{n-1}(T_n) - \{\gamma_n\}) \cup \{\delta_n\} \subseteq \overline{\varphi}_n(T_n)$. So

$$\begin{aligned} D_n &= \cup_{i \leq n} S_i - \overline{\varphi}_n(T_n) \\ &\subseteq \cup_{i \leq n-1} S_i \cup \{\delta_n, \gamma_n\} - [(\overline{\varphi}_{n-1}(T_n) - \{\gamma_n\}) \cup \{\delta_n\}] \\ &\subseteq \cup_{i \leq n-1} S_i \cup \{\gamma_n\} - (\overline{\varphi}_{n-1}(T_n) - \{\gamma_n\}) \\ &\subseteq [\cup_{i \leq n-1} S_i - \overline{\varphi}_{n-1}(T_n)] \cup \{\gamma_n\} \\ &\subseteq D_{n-1} \cup \{\gamma_n\}. \end{aligned}$$

Combining the above three cases, we obtain $|D_n| \leq |D_{n-1}| + 1$ for $n \geq 1$. Hence $|D_n| \leq n$. □

Lemma 3.6 *Suppose $\Theta_n = PE$. Let σ_n be a (T_n, D_n, φ_n)-stable coloring and let $\sigma_{n-1} = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)$. If $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$, then σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable and hence is $(T_n, D_{n-1}, \varphi_{n-1})$-stable.*

Proof Let π'_{n-1} be as specified in Algorithm 3.1. Recall that

(1) π'_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable.

By definition, $\varphi_n = \pi'_{n-1}/P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$. So

(2) $\pi'_{n-1} = \varphi_n / P_{v_n}(\gamma_n, \delta_n, \varphi_n)$.
We propose to show that
(3) σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi'_{n-1})$-stable.
By the definition of σ_{n-1} and (2), we obtain
(4) σ_n and σ_{n-1} agree on every edge incident to T_n except f_n, for which $\sigma_n(f_n) = \gamma_n$ and $\sigma_{n-1}(f_n) = \delta_n$; and
(5) φ_n and π'_{n-1} agree on every edge incident to T_n except f_n, for which $\varphi_n(f_n) = \gamma_n$ and $\pi'_{n-1}(f_n) = \delta_n$.

Since $\{\gamma_n, \delta_n\} \subseteq \overline{\varphi}_n(T_n) \cup D_n$ and σ_n is (T_n, D_n, φ_n)-stable, (3) follows instantly from (4) and (5). Using (1), (3) and Lemma 2.4, we see that σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable. So σ_{n-1} is $(T_n, D_{n-1}, \pi_{n-1})$-stable. Since π_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable, from Lemma 2.4 we conclude that σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. □

Observe that an extended Tashkinov tree *T* (see Definition 3.1) has a built-in ladder-like structure. So we propose to call the sequence $T_1 \subset T_2 \subset \ldots \subset T_n \subset T$ the *ladder* of *T*, and call *n* the *rung number* of *T* and denote it by *r*(*T*) (so $r(T_{n+1}) = n$). Moreover, we call $(\varphi_0, \varphi_1, \ldots, \varphi_n)$ the *coloring sequence* of *T*, and call $\mathcal{T}$ the Tashkinov series *corresponding* to *T*.

In our proof we shall frequently work with stable colorings; the following concept will be used to keep track of the structures of ETTs.

Definition 3.7 Let $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series constructed from a *k*-triple (G, e, φ) by using Algorithm 3.1. A coloring $\sigma_n \in \mathcal{C}^k(G-e)$ is called φ_n mod T_n if every tree-sequence $T^* \supset T_n$ obtained from $T_n + f_n$ (resp. T_n) by TAA under σ_n when $\Theta_n =$ *RE* or *SE* (resp. when $\Theta_n = PE$) is an ETT under σ_n, with a corresponding Tashkinov series $\mathcal{T}^* = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, satisfying the following conditions for all *i* with $1 \le i \le n$:

$T_i^* = T_i$ and
σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G-e)$.

We call each T^* an ETT *corresponding* to (σ_n, T_n) (or simply *corresponding* to σ_n if no ambiguity arises).

Remark Comparing $\mathcal{T}^*$ with $\mathcal{T}$, we see that T^*_{i+1} in $\mathcal{T}^*$ is obtained from $T^*_i = T_i$ by using the same connecting edge, connecting color, and extension type as T_{i+1} in $\mathcal{T}$ for $1 \le i \le n$. However, T^*_{n+1} may be different from T_{n+1}. Furthermore, $T_1 \subset T_2 \subset \ldots \subset T_n \subset T^*$ is the ladder of T^* and $r(T^*) = n$. Since σ_i is a (T_i, D_i, φ_i)-stable coloring, by Lemma 3.2(iv), we have $\sigma_i(f) = \varphi_i(f)$ for any edge f on T_i and $1 \le i \le n$; this fact will be used repeatedly in our paper.

To ensure that the structures of ETTs are preserved under taking stable colorings, we impose some restrictions on such trees.

Definition 3.8 Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. We say that T has the *maximum property* (MP) under $(\varphi_0, \varphi_1, \ldots, \varphi_n)$ (or simply under φ_n if no ambiguity arises), if $|T_1|$ is maximum among all Tashkinov trees T'_1 with respect to an edge $e' \in E$ and a coloring $\varphi'_0 \in \mathcal{C}^k(G - e')$, and $|T_{i+1}|$ is maximum over all (T_i, D_i, φ_i)-stable colorings for any i with $1 \le i \le n-1$; that is, $|T_{i+1}|$ is maximum over all tree-sequences T'_{i+1}, which is a closure of $T_i + f_i$ (resp. T_i) under a (T_i, D_i, φ_i)-stable coloring φ'_i if $\Theta_i = RE$ or SE (resp. if $\Theta_i = PE$), where f_i is the connecting edge in F_i.

Notice that in the above definition $|T_{n+1}|$ is not required to be maximum over all (T_n, D_n, φ_n)-stable colorings. This relaxation allows us to proceed by induction in our proofs.

As described before, the tree-sequence structure generated by Algorithm 3.1 might be very fragile, because RE does not allow change of coloring and PE requires the supporting vertex to be the exit of an exit-path. At this point, it is natural to ask whether there exists an ETT with MP and with an arbitrarily given rung number or an arbitrarily given size. We shall demonstrate (see Corollary 3.11) that the answer is in the affirmative. The statement below follows instantly from the above two definitions and Lemma 2.4.

Lemma 3.9 *Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, let σ_n be a $\varphi_n \bmod T_n$ coloring, and let T^* be an ETT corresponding to (σ_n, T_n) (see Definition 3.7). If T satisfies MP under φ_n, then T^* satisfies MP under σ_n.* □

Let us introduce one more notation and two more concepts before presenting our main theorem. For each $v \in V(T)$, we use $m(v)$ to denote the minimum subscript i such that $v \in V(T_i)$. Let α and β be two colors in $[k]$. We say that α and β are *T-interchangeable* under φ_n if there is at most one (α, β)-path with respect to φ_n intersecting T (that is, the path and T have at least one vertex in common). Note that in this situation we can easily find an (α, β)-path disjoint from T and then switch colors along it while keeping the resulting coloring stable. So this concept is very helpful for deriving elementary property satisfied by an ETT. When T is closed (that is, $T = T_{n+1}$), we also say that T has the *interchangeability property* with respect to

φ_n if under any (T, D_n, φ_n)-stable coloring σ_n, any two colors α and β are T-interchangeable, provided that $\overline{\sigma}_n(T) \cap \{\alpha, \beta\} \neq \emptyset$ (equivalently $\overline{\varphi}_n(T) \cap \{\alpha, \beta\} \neq \emptyset$).

We aim to show, by induction on the rung number, that every ETT satisfying MP is elementary. To carry out the induction step, we need several auxiliary results concerning ETTs with MP. Thus what we are going to prove is a stronger theorem (containing six statements) given below, in which the undefined symbols and notations can all be found in Algorithm 3.1. Together with Theorem 2.2, statements (i) and (vi) will imply Theorem 2.1. Statements (ii)-(v) will be used in the proofs of (i) and (vi). Moreover, the proof of (iv) relies directly on MP and the design of RE, and the proofs of (iii) and (v) are based on the fact that the supporting and extension vertices involved in Algorithm 3.1 are maximum defective vertices over all stable colorings.

Theorem 3.10 *Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. If T has MP under φ_n, then the following statements hold:*

(i) *V(T) is elementary with respect to φ_n.*
(ii) *T_{n+1} has the interchangeability property with respect to φ_n.*
(iii) *For any $i \le n$, if v_i is a supporting vertex with $m(v_i) = j$, then every (T_i, D_i, φ_i)-stable coloring σ_i is $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable, so σ_i is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable. Furthermore, for any two distinct supporting vertices v_i and v_j with $i, j \le n$, if $m(v_i) = m(v_j)$, then $S_i \cap S_j = \emptyset$.*
(iv) *If $\Theta_n = PE$, then $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains precisely one vertex, v_n, from T_n for any (T_n, D_n, φ_n)-stable coloring σ_n.*
(v) *For any (T_n, D_n, φ_n)-stable coloring σ_n and any defective color δ of T_n with respect to σ_n, if v is a vertex but not the smallest one (in the order $\prec$) in $I[\partial_{\sigma_n,\delta}(T_n)]$, then $v \preceq v_i$ for any supporting or extension vertex v_i with $m(v) \le i$.*
(vi) *Every (T_n, D_n, φ_n)-stable coloring σ_n is a $\varphi_n \bmod T_n$ coloring. (So every ETT corresponding to (σ_n, T_n) (see Definition 3.7) satisfies MP under σ_n by Lemma 3.9.)*

Let us show that Theorem 2.1 can be deduced easily from statement (i) and the following corollary (which relies on statement (vi)).

Corollary 3.11 *Let $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series constructed from a k-triple (G, e, φ). Suppose T_{n+1} has MP under φ_n. Then there exists a Tashkinov series $\mathcal{T}^* = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, satisfying the following conditions for $1 \le i \le n$:*

(i) *$T_i^* = T_i$;*
(ii) *σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G-e)$; and*
(iii) *$|T_{i+1}^*|$ is maximum over all (T_i, D_i, σ_i)-stable colorings (note that Definition 3.8 only requires this for $i \le n-1$).*

Furthermore, if T^*_{n+1} is not strongly closed with respect to σ_n, then there exists a Tashkinov series $\{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+2\}$, such that $T^*_{n+1} \subset T^*_{n+2}$ and T^*_{n+2} satisfies MP under σ_{n+1}.

Proof Let σ_n be a (T_n, D_n, φ_n)-stable coloring such that a closure of T_n (resp. of $T_n + f_n$) under σ_n, denoted by T^*_{n+1}, has maximum size over all (T_n, D_n, φ_n)-stable colorings if $\Theta_n = PE$ (resp. if $\Theta_n = RE$ or *SE*). By Lemma 2.4, every (T_n, D_n, σ_n) -stable coloring is a (T_n, D_n, φ_n)-stable coloring. So $|T^*_{n+1}|$ is also maximum over all (T_n, D_n, σ_n)-stable colorings.

Since σ_n is (T_n, D_n, φ_n)-stable, it is $\varphi_n \bmod T_n$ by Theorem 3.10(vi). Thus Definition 3.7 guarantees the existence of a Tashkinov series $\mathcal{T}^* = \{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ that satisfies conditions (i) and (ii) as described above. By Lemma 3.9, $|T^*_{i+1}|$ is maximum over all (T_i, D_i, σ_i) -stable colorings as well for $1 \le i \le n-1$.

Suppose T^*_{n+1} is not strongly closed with respect to σ_n. Then we can construct a new tuple $(T^*_{n+2}, \sigma_{n+1}, S_{n+1}, F_{n+1}, \Theta_{n+1})$ by using Algorithm 3.1. Clearly, $T^*_{n+1} \subset T^*_{n+2}$ and T^*_{n+2} satisfies MP under σ_{n+1}. □

Proof of theorem 2.1 Let $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series constructed from a *k*-triple (G, e, φ), such that

(a) T_{n+1} satisfies MP under φ_n;
(b) subject to (a), $|T_{n+1}|$ is maximum over all (T_n, D_n, φ_n)-stable colorings; and
(c) subject to (a) and (b), the integer *n* is maximum.

Since *G* is finite, by Corollary 3.11, such a Tashkinov series $\mathcal{T}$ exists. Observe that T_{n+1} is strongly closed, for otherwise, Corollary 3.11 would enable us to further extend $\mathcal{T}$ to a longer Tashkinov series $\{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+2\}$ satisfying (a) and (b), contradicting (c). By Theorem 3.10(i), $V(T_{n+1})$ is elementary with respect to φ_n. From Theorem 2.2(i) and (iv), we thus deduce that *G* is an elementary multigraph. □The proof of Theorem 3.10 will take up the entire remainder of this paper.

4 Auxiliary results

We prove Theorem 3.10 by induction on the rung number $r(T) = n$. The present section is devoted to a proof of statement (ii) in Theorem 3.10 in the base case and proofs of statements (iii)–(vi) in the general case. A complete proof of (ii) is given in Sect. 7.2, which is the end of this paper, and the rest of the paper is devoted to proving (i).

For $n = 0$, statement (i) follows from Theorem 2.7, statements (iii)-(vi) hold trivially, and statement (ii) is a corollary of the following more general lemma (because T_1 is also a closed Tashkinov tree with respect to *e* and any (T_1, D_0, φ_0)-stable coloring σ_0).

Lemma 4.1 *Let (G, e, φ) be a k-triple, let T be a closed Tashkinov tree with respect to e and φ, and let α and β be two colors in [k] with $\overline{\varphi}(T) \cap \{\alpha, \beta\} \neq \emptyset$. Then there is at most one (α, β)-path with respect to φ intersecting T.*

Proof Assume the contrary: there are at least two (α, β)-paths Q_1 and Q_2 with respect to φ intersecting *T*. By Theorem 2.7, *V*(*T*) is elementary with respect to φ. So *T* contains at most two vertices *v* with $\overline{\varphi}(v) \cap \{\alpha, \beta\} \neq \emptyset$, which in turn implies that at least two ends of Q_1 and Q_2 are outside *T*. By hypothesis, *T* is closed with respect to φ. Hence precisely one of α and β, say α, is in $\overline{\varphi}(T)$. Thus we further deduce that at least three ends of Q_1 and Q_2 are outside *T*. Traversing Q_1 and Q_2 from these ends respectively, we can find at least three $(T, \varphi, \{\alpha, \beta\})$-exit paths P_1, P_2, P_3. We call the tuple $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$ a *counterexample* and use $\mathcal{K}$ to denote the set of all such counterexamples.

With a slight abuse of notation, let $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$ be a counterexample in $\mathcal{K}$ with the minimum $|P_1| + |P_2| + |P_3|$. For $i = 1, 2, 3$, let a_i and b_i be the ends of P_i with $b_i \in V(T)$, and f_i be the edge of P_i incident to b_i. Since *T* is closed, $\alpha \in \overline{\varphi}(T)$, and each $f_i \in \partial(T)$, we have $\varphi(f_i) \neq \alpha$. It follows that $\varphi(f_i) = \beta$ $(i = 1, 2, 3)$ and thus b_1, b_2, b_3 are distinct. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2 \prec b_3$. Let $\gamma \in \overline{\varphi}(b_3)$ and let $\sigma_1 = \varphi/(G - T, \alpha, \gamma)$. Then P_3 is a (γ, β)-path under σ_1. Clearly, $\sigma_1 \in \mathcal{C}^k(G - e)$ and *T* is also a Tashkinov tree with respect to *e* and σ_1. Furthermore, f_i is colored by β under both φ and σ_1 for $i = 1, 2, 3$.

Consider $\sigma_2 = \sigma_1/P_3$. Note that $\beta \in \overline{\sigma}_2(b_3)$. Let T' be obtained from $T(b_3)$ by adding f_1 and f_2 and let T'' be a closure of T' under σ_2. Obviously, both T' and T'' are Tashkinov trees with respect to *e* and σ_2. By Theorem 2.7, $V(T'')$ is elementary with respect to σ_2.

Observe that none of a_1, a_2, a_3 is contained in T'', for otherwise, let $a_i \in V(T'')$ for some i with $1 \leq i \leq 3$. Since $\{\beta, \gamma\} \cap \overline{\sigma}_2(a_i) \neq \emptyset$ and $\beta \in \overline{\sigma}_2(b_3)$, we obtain $\gamma \in \overline{\sigma}_2(a_i)$. Hence from TAA we see that P_1, P_2, P_3 are all entirely contained in $G[T'']$, which in turn implies $\gamma \in \overline{\sigma}_2(a_j)$ for $j = 1, 2, 3$. So $V(T'')$ is not elementary with respect to σ_2, a contradiction. Each P_i contains a subpath L_i, which is a T''-exit path with respect to σ_2. Since f_1 is not contained in L_1, we obtain $|L_1| + |L_2| + |L_3| < |P_1| + |P_2| + |P_3|$. Thus the existence of the counterexample $(\sigma_2, T'', \gamma, \beta, L_1, L_2, L_3)$ violates the minimality assumption on $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$. □

So Theorem 3.10 is true in the base case. Suppose we have established that

(4.1) Theorem 3.10 holds for all ETTs with at most $n - 1$ rungs and satisfying MP, for some $n \geq 1$.

Let us proceed to the induction step. We postpone the proof of Theorem 3.10(i) and (ii) to Sect. 7, and present a proof of Theorem 3.10(iii)–(vi) in this section. In our proof of the $(i + 2)$th statement in Theorem 3.10 for $2 \leq i \leq 4$, we further assume that

(4. *i*) the *j*th statement in Theorem 3.10 holds for all ETTs with at most *n* rungs and satisfying MP, for all *j* with $3 \leq j \leq i + 1$.

Note that $(4.i)$ corresponds to (4.2), (4.3) and (4.4), respectively, for $i = 2, 3$ and 4. For example, when we try to prove Theorem 3.10(v) (now $i = 3$), we assume (4.3), which says that both Theorem 3.10(iii) and Theorem 3.10(iv) hold for all ETTs with at most *n* rungs and satisfying MP.

We break the proof of the induction step into a series of lemmas. The following lemma derives some properties satisfied by supporting vertices and connecting colors.

Lemma 4.2 *(Assuming (4.1)) Theorem 3.10(iii) holds for all ETTs with n rungs and satisfying MP; that is, for any $i \le n$, if v_i is a supporting vertex with $m(v_i) = j$, then every (T_i, D_i, φ_i)-stable coloring σ_i is $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable, so σ_i is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable. Furthermore, for any two distinct supporting vertices v_i and v_j with $i, j \le n$, if $m(v_i) = m(v_j)$, then $S_i \cap S_j = \emptyset$.*

Proof By (4.1), Lemma 4.2 holds for all ETTs with at most $n-1$ rungs and satisfying MP. So we may assume that *T* is an ETT with the corresponding Tashkinov series $\mathcal{T} = \{(T_h, \varphi_{h-1}, S_{h-1}, F_{h-1}, \Theta_{h-1}) : 1 \le h \le n+1\}$ and satisfies MP under φ_n. Furthermore, $i = n$ throughout our proof.

In the first half of this lemma, $m(v_n) = j$ and σ_n is a (T_n, D_n, φ_n)-stable coloring. Write $T^* = T(v_n) - v_n$ (so $T^* \subseteq T_j$). As $j \le n$, repeated application of Lemma 3.2(i) yields $\overline{\varphi}_{j-1}(T_j) \cup D_{j-1} \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$. In particular, $D_{j-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$. Hence σ_n is a $(T^*, D_{j-1}, \varphi_n)$-stable coloring. By Lemma 2.4, to prove that σ_n is $(T^*, D_{j-1}, \varphi_{j-1})$-stable, it suffices to show that φ_n is $(T^*, D_{j-1}, \varphi_{j-1})$-stable.

If $j = n$, then v_n is the only supporting vertex contained inside T_n but outside T_{n-1}. Recall that in Algorithm 3.1 the coloring π'_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable and $\varphi_n = \pi'_{n-1}/P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$, where $P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1}) \cap V(T_n) = \{v_n\}$. So φ_n is a $(T^*, D_{n-1}, \pi_{n-1})$-stable coloring. By Lemma 2.4, it is also a $(T^*, D_{n-1}, \varphi_{n-1})$-stable coloring, because π_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. Thus we assume hereafter that $j < n$. As v_n is the largest vertex (in the order $\prec$) in $I[\partial_{\pi_{n-1}, \delta_n}(T_n)]$ (see Algorithm 3.1), with $\delta_n = \pi_{n-1}(f_n)$, and v_n is contained in $T_j \subseteq T_{n-1}$, we see that δ_n is a defective color of T_{n-1} with respect to π_{n-1}, and v_n is not the smallest vertex (in the order $\prec$) in $I[\partial_{\pi_{n-1}, \delta_n}(T_{n-1})]$. As π_{n-1} is also a $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable coloring, applying (4.1) and Theorem 3.10(v) to $v = v_n$ and π_{n-1}, we obtain $v_n \preceq v_h$ for any supporting vertex v_h with $j \le h \le n-1$. Thus $\overline{\varphi}_{j-1}(v) = \overline{\varphi}_n(v)$ for each vertex *v* of T^* by Lemma 3.2(ii). Furthermore, $\varphi_n(f) = \varphi_{j-1}(f)$ for each edge *f* incident to T^* with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T^*) \cup D_{j-1}$ by Lemma 3.3. Hence φ_n is $(T^*, D_{j-1}, \varphi_{j-1})$-stable, as desired.

Let us proceed to the second half. Now v_j is a supporting vertex with $j < n$ and $m(v_j) = m(v_n)$. To prove that $S_n \cap S_j = \emptyset$, we shall actually show that

(1) there are edges *f*, *g* in $G[T_n]$ incident to v_n with $\varphi_n(f) = \gamma_j$ and $\varphi_n(g) = \delta_j$.

Assuming (1), it follows instantly that $\gamma_j, \delta_j \notin S_n$, because $\varphi_n(f_n) = \gamma_n$, $f_n \in \partial(T_n)$ (so $f_n \notin G[T_n]$), and $\delta_n \in \overline{\varphi}_n(v_n)$ (see Algorithm 3.1).

To justify (1), recall that in Algorithm 3.1 coloring π'_{j-1} is $(T_j, D_{j-1} \cup \{\delta_j\}, \pi_{j-1})$-stable, $P = P_{v_j}(\gamma_j, \delta_j, \pi'_{j-1})$ contains only vertex v_j from T_j, and $\varphi_j = \pi'_{j-1}/P$. Write $Q = P_{v_n}(\gamma_j, \delta_j, \pi'_{j-1})$. Since $v_j \ne v_n$, *P* and *Q* are vertex-disjoint under π'_{j-1}. For convenience, we still use *P* and *Q* to denote the corresponding paths under φ_j. By

(4.1) and Theorem 3.10(ii), T_{j+1} has the interchangeability property with respect to φ_j. So P is the unique (γ_j, δ_j)-path intersecting T_{j+1} and Q is a (γ_j, δ_j)-cycle under φ_j. Let $r > j$ be the smallest subscript with $\Theta_r \neq RE$. Since $\Theta_n = PE$, we have $r \leq n$. From RE in Algorithm 3.1 we see that Q is fully contained in $G[T_r]$. Repeated application of Lemma 3.2(i) yields $\overline{\varphi}_j(T_{j+1}) \cup D_j \subseteq \overline{\varphi}_{r-1}(T_r) \cup D_{r-1}$. Since $\cup_{h=1}^{j} S_h \subseteq \overline{\varphi}_j(T_j) \cup D_j$, we have $S_j \subseteq \overline{\varphi}_{r-1}(T_r) \cup D_{r-1}$. So Q is also a (γ_j, δ_j)-cycle containing v_n under φ_n by Lemma 3.3 (with respect to T_r). Since $T_r \subseteq T_n$, we establish (1). □

The following lemma asserts that parallel extensions (PEs) used in Algorithm 3.1 are preserved under taking stable colorings. Its proof is perhaps the most difficult part of the whole paper. After reading the proof, we may fully understand why RE is introduced in the algorithm.

Lemma 4.3 *(Assuming (4.1) and (4.2)) Theorem 3.10(iv) holds for all ETTs with n rungs and satisfying MP; that is, if $\Theta_n = PE$, then $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains precisely one vertex, v_n, from T_n for any (T_n, D_n, φ_n)-stable coloring σ_n.*

Proof Assume the contrary: $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains at least two vertices from T_n for some (T_n, D_n, φ_n)-stable coloring σ_n. Let $j = m(v_n)$. By applying a series of Kempe changes to σ_n, we shall construct a certain $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring μ and a certain ETT T_j^{μ} corresponding to (μ, T_{j-1}) with ladder $T_1 \subset T_2 \subset \ldots \subset T_{j-1} \subset T_j^{\mu}$, such that either $|T_j^{\mu}| > |T_j|$ or $V(T_j^{\mu})$ is not elementary with respect to μ, which contradicts either the maximum property satisfied by T or the induction hypothesis (4.1) on Theorem 3.10(i). We divide the proof into five parts; the assumption on the intersection of $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ and T_n will only be used in the last part.

(I) In this part we exhibit some properties satisfied by supporting vertices a with $m(a) = j$ and corresponding connecting colors in $T_j - T_{j-1}$, which allow us to restore missing color sets of these vertices except v_n as under φ_{j-1} later.

Let L be the set of all subscripts s with $j \leq s \leq n$, such that $\Theta_s = PE$ and $m(v_s) \leq j$, where v_s is the supporting vertex involved in iteration s.

(1) For any $s, t \in L$ with $s < t$, we have $v_t \preceq v_s$. Consequently, $v_n \preceq v_s$ and $m(v_s) = j$ for all $s \in L$.

To justify this, let π_{t-1}, $S_t = \{\gamma_t, \delta_t\}$, and f_t be the $(T_t, D_{t-1}, \varphi_{t-1})$-stable coloring, the set of connecting colors, and the connecting edge, respectively, as specified in iteration t of Algorithm 3.1, with $\Theta_t = PE$. Recall that $\delta_t = \pi_{t-1}(f_t)$ is a defective color of T_t with respect to π_{t-1}, and v_t is the largest vertex (in the order $\prec$) in $I[\partial_{\pi_{t-1},\delta_t}(T_t)]$. Since $m(v_t) \leq j \leq s < t$, we have $v_t \in V(T_j) \subseteq V(T_{t-1})$. As π_{t-1} is a $(T_{t-1}, D_{t-1}, \varphi_{t-1})$-stable coloring and v_t is not the smallest vertex (in the order $\prec$) in $I[\partial_{\pi_{t-1},\delta_t}(T_{t-1})]$, applying (4.1) and Theorem 3.10(v) to π_{t-1}, T_{t-1}, and $v = v_t$, we obtain $v_t \preceq v_s$. Hence (1) holds.

(2) For any $s, t \in L$ with $s \leq t$, we have $\delta_t \notin \overline{\varphi}_{s-1}(T_s)$ (so $\delta_t \neq \gamma_s$). Consequently, $\delta_t \notin \overline{\varphi}_{j-1}(T_j)$ for all $t \in L$.

Assume the contrary: $\delta_t \in \overline{\varphi}_{s-1}(u)$ for some $u \in V(T_s)$. By Algorithm 3.1, $\delta_s \notin \overline{\varphi}_{s-1}(T_s)$. So $s < t$ and hence $V(T_s)$ is elementary with respect to φ_{s-1} by (4.1) and Theorem 3.10(i). Let *v* be an arbitrary vertex in $T_s - u$. Then $\delta_t \notin \overline{\varphi}_{s-1}(v)$, so *v* is incident to an edge *f* with $\varphi_{s-1}(f) = \delta_t$. As described in Algorithm 3.1, T_s is closed under φ_{s-1} and thus *f* is contained in $G[T_s]$. Hence $\varphi_{t-1}(f) = \varphi_{s-1}(f) = \delta_t$ by Lemma 3.3 (for $\delta_t \in \overline{\varphi}_{s-1}(u) \subseteq \overline{\varphi}_{s-1}(T_s)$). From Lemma 2.4 and the definitions of π_{t-1} and π'_{t-1} in Algorithm 3.1, we see that π'_{t-1} is $(T_t, D_{t-1}, \varphi_{t-1})$-stable. By Lemma 3.2(i), $\overline{\varphi}_{s-1}(T_s) \cup D_{s-1} \subseteq \overline{\varphi}_{t-1}(T_t) \cup D_{t-1}$. So $\pi'_{t-1}(f) = \varphi_{t-1}(f) = \delta_t$. Since *f* is contained in $G[T_s]$ and hence in $G[T_t]$, we have $v \notin I[\partial_{\pi'_{t-1},\delta_t}(T_t)]$ for any vertex *v* in $T_s - u$. In view of (1), $v_t \preceq v_s$, so $T_t(v_t) \subseteq T_s(v_s) \subseteq T_s$. Therefore v_t cannot be the supporting vertex of T_t with respect to φ_t and the connecting color δ_t (as v_t is the maximum defective vertex of T_t with corresponding defective color δ_t under π'_{t-1} in Algorithm 3.1); this contradiction implies that $\delta_t \notin \overline{\varphi}_{s-1}(T_s)$. Since $\gamma_s \in \overline{\varphi}_{s-1}(T_s)$, we conclude that $\delta_t \neq \gamma_s$. Finally, let *s* be the smallest subscript in *L*. Then $\overline{\varphi}_{j-1}(T_j) = \overline{\varphi}_{s-1}(T_j)$ by Algorithm 3.1 (see Lemma 3.2(ii)). So $\delta_t \notin \overline{\varphi}_{j-1}(T_j)$ and hence (2) is established.

We partition *L* into disjoint subsets $L_1, L_2, \ldots, L_\kappa$, such that two subscripts $s, t \in L$ are in the same subset iff $v_s = v_t$. For $1 \le i \le \kappa$, write $L_i = \{i_1, i_2, \ldots, i_{c(i)}\}$, where $i_1 < i_2 < \ldots < i_{c(i)}$, and let w_i denote the common supporting vertex corresponding to L_i. For each $t \in L$, we have $v_t \notin V(T_{j-1})$ because $m(v_t) = j$ by (1). It follows that $w_i \notin V(T_{j-1})$ for $1 \le i \le \kappa$. Renaming subscripts if necessary, we may assume that $w_1 \prec w_2 \prec \ldots \prec w_\kappa$. By (1), we obtain

(3) $v_n = w_1$ (so $n = 1_{c(1)}$) and $h_{c(h)} > h_{c(h)-1} > \ldots > h_1 > i_{c(i)} > i_{c(i)-1} > \ldots > i_1$ for any $1 \le h < i \le \kappa$.

From (2) and Lemma 3.4(i) it is clear that

(4) for any $1 \le i \le \kappa$, the colors in $\cup_{t \in L_i} S_t$ are

$$\gamma_{i_1}, \gamma_{i_2} = \delta_{i_1}, \gamma_{i_3} = \delta_{i_2}, \ldots, \gamma_{i_{c(i)}} = \delta_{i_{c(i)-1}}, \delta_{i_{c(i)}},$$

which are distinct.

From (4.2), Theorem 3.10(iii), (1), and (4) we deduce that

(5) for any $s, t \in L$ with $s < t$, the intersection $S_s \cap S_t \neq \emptyset$ iff *s* and *t* are two consecutive subscripts in the same L_i for some $1 \le i \le \kappa$; in this case, $S_s \cap S_t = \{\gamma_t\} = \{\delta_s\}$.

(II) In this part we derive some properties satisfied by σ_n and establish a result on the so-called strong interchangeability property, which enable us to keep the rest of T_j "stable" while restoring missing color sets of supporting vertices in $T_j - V(T_j(v_n))$ as under φ_{j-1}.

For each *t* with $1 \le t \le n-1$ and $\Theta_t = PE$, let $\epsilon(t)$ be the smallest subscript $r > t$ such that $\Theta_r \neq RE$. This $\epsilon(t)$ is well defined and $\epsilon(t) \le n$, as $\Theta_n = PE \neq RE$. Given a coloring φ and two colors α and β, we say that α and β are T_t*-strongly interchangeable* (T_t-SI) under φ if for each vertex *v* in $T_t - v_t$, the chain $P_v(\alpha, \beta, \varphi)$ is an (α, β)-cycle avoiding v_t and fully contained in $G[T_{\epsilon(t)}]$ (equivalently, $V(P_v(\alpha, \beta, \varphi)) \subseteq V(T_{\epsilon(t)})$).

Recall that α and β are called T_t-interchangeable under φ if there is at most one (α, β)-path with respect to φ intersecting T_t; that is, all (α, β)-chains intersecting T_t

are (α, β)-cycles, with possibly one exception. Therefore, if α and β are T_t-SI under φ, then they are T_t-interchangeable under φ.

The following observations reveal some connections between colorings σ_n and φ_{j-1}.

Claim 4.1 *The coloring σ_n satisfies the following properties:*

(a1)*σ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;*
(a2)*$\sigma_n(f) = \varphi_{j-1}(f)$ for all edges f in $G[T_j]$ with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T_j) \cup D_{j-1}$; in particular, this equality holds for all edges on T_j;*
(a3)*$\overline{\sigma}_n(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j) - \{w_1, w_2, \ldots, w_\kappa\}$;*
(a4)*$\overline{\sigma}_n(w_i) \cap (\cup_{t \in L_i} S_t) = \{\delta_{i_{c(i)}}\}$ and $\overline{\varphi}_{j-1}(w_i) = (\overline{\sigma}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ for each $i = 1, 2, \ldots, \kappa$;*
(a5)*for any $t \in L - \{n\}$, the colors γ_t and δ_t are T_t-SI under σ_n.*

To justify this claim, observe that $\overline{\varphi}_{j-1}(T_j) \cup D_{j-1} \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$ by Lemma 3.2(i) and that $\cup_{t \in L} S_t \subseteq \cup_{t \leq n} S_t \subseteq \overline{\varphi}_n(T_n) \cup D_n$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, it suffices to prove $(a1)$–$(a5)$ for φ_n (instead of σ_n).

Clearly, $(a1)$ follows from (4.2) and Theorem 3.10(iii), and $(a3)$ follows from Lemma 3.2(ii).

$(a2)$ By Lemma 3.3, we have $\varphi_n(f) = \varphi_{j-1}(f)$ for all edges $f \in G[T_j]$ with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T_j) \cup D_{j-1}$. By Lemma 3.2(iv), each edge f on T_j satisfies $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T_j) \cup D_{j-1}$, so the equality $\varphi_n(f) = \varphi_{j-1}(f)$ holds for all edges f on T_j.

$(a4)$ By Lemma 3.4(ii), we obtain $\overline{\varphi}_n(w_i) \cap (\cup_{t \in L_i} S_t) = \{\delta_{i_{c(i)}}\}$ (with w_i in place of u). Since L_i consists of all subscripts t with $j \leq t \leq n$, such that $v_t = w_i$ and $\Theta_t = PE$, there hold $\overline{\varphi}_{j-1}(w_i) = \overline{\varphi}_{i_1-1}(w_i)$ and $\overline{\varphi}_n(w_i) = \overline{\varphi}_{i_{c(i)}}(w_i)$ by Lemma 3.2(ii). Furthermore, $\overline{\varphi}_{i_1-1}(w_i) = (\overline{\varphi}_{i_{c(i)}}(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ by Lemma 3.4(iii) (with w_i in place of u). So $\overline{\varphi}_{j-1}(w_i) = (\overline{\varphi}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ for $1 \leq i \leq \kappa$.

$(a5)$ Let $t \in L - \{n\}$. Then $t < n$. By the induction hypothesis (4.1) on Theorem 3.10(ii), γ_t and δ_t are T_{t+1}- and hence T_t-interchangeable under φ_t. So all but at most one (γ_t, δ_t)-chains intersecting T_t under φ_t are (γ_t, δ_t)-cycles. According to Algorithm 3.1, $P_{v_t}(\gamma_t, \delta_t, \varphi_t)$ is a path containing only one vertex v_t from T_t. Hence, for each vertex v in $T_t - v_t$, $P_v(\gamma_t, \delta_t, \varphi_t)$ is a (γ_t, δ_t)-cycle avoiding v_t. Since *RE* has priority over *PE* and *SE* in Algorithm 3.1, $P_v(\gamma_t, \delta_t, \varphi_t)$ is fully contained in $G[T_{\epsilon(t)}]$, for otherwise, we would have $\Theta_{\epsilon(t)} = RE$, contradicting the definition of $\epsilon(t)$. It follows that γ_t and δ_t are T_t-SI under φ_t. By Lemma 3.3 (with respect to $T_{\epsilon(t)}$), we obtain $\varphi_t(f) = \varphi_n(f)$ for each edge f on $P_v(\gamma_t, \delta_t, \varphi_t)$, because $\{\gamma_t, \delta_t\} \subseteq \cup_{i \leq r-1} S_i \subseteq \overline{\varphi}_{r-1}(T_{r-1}) \cup D_{r-1} \subseteq \overline{\varphi}_{r-1}(T_r) \cup D_{r-1}$, where $r = \epsilon(t)$. Therefore γ_t and δ_t are T_t-SI under φ_n as well. This establishes Claim 4.1.

The following technical statement will be used repeatedly in our proof.

Claim 4.2 *Let $t \in L_i$ for some $1 \leq i \leq \kappa$ and let P be an arbitrary (γ_t, δ_t)-path. If the connecting colors γ_t, δ_t are T_t-SI under a coloring $\varphi \in \mathcal{C}(G - e)$ then, for any*

$s \in L_h$ with $h \neq i$ or $s < t$, the colors γ_s, δ_s are T_s-SI under $\varphi^* = \varphi/P$, provided that γ_s, δ_s are T_s-SI under φ.

To justify this, we assume that γ_s, δ_s are T_s-SI under coloring φ. For each $v \in V(T_s - v_s)$, we propose to show that $P_v(\gamma_s, \delta_s, \varphi^*) = P_v(\gamma_s, \delta_s, \varphi)$, which is a (γ_s, δ_s)-cycle avoiding v_s and fully contained in $G[T_{\epsilon(s)}]$. Consequently, γ_s, δ_s are also T_s-SI under coloring φ^*.

If $h \neq i$, then $\{\gamma_s, \delta_s\} \cap \{\gamma_t, \delta_t\} = \emptyset$ by (5). In this case, clearly $P_v(\gamma_s, \delta_s, \varphi^*) = P_v(\gamma_s, \delta_s, \varphi)$. So we assume that $h = i$ and $s < t$. Now $v_s = v_t = w_i$. Observe that P contains at most one vertex v_t from T_t, because γ_t, δ_t are T_t-SI under φ. Furthermore, $\epsilon(s) \leq t$, because $s < t$ and $\Theta_t = PE \neq RE$. As γ_s, δ_s are T_s-SI under coloring φ, the chain $P_v(\gamma_s, \delta_s, \varphi)$ is a cycle avoiding v_s and fully contained in $G[T_{\epsilon(s)}] \subseteq G[T_t]$, so it is disjoint from P. Thus $P_v(\gamma_s, \delta_s, \varphi)$ is still a (γ_s, δ_s)-cycle avoiding v_s and fully contained in $G[T_{\epsilon(s)}] \subseteq G[T_t]$ under φ^* and hence $P_v(\gamma_s, \delta_s, \varphi^*) = P_v(\gamma_s, \delta_s, \varphi)$, as desired.

(III) With the preparations made in the first two parts, now we can move on to the aforementioned restoration of missing color sets at certain supporting vertices.

Write $L^* = L - L_1$. By (3), the subscripts in L^* satisfies $h_{c(h)} > h_{c(h)-1} > \ldots > h_1 > i_{c(i)} > i_{c(i)-1} > \ldots > i_1$ for any $2 \leq h < i \leq \kappa$. So $2_{c(2)}$ (resp. κ_1) is the largest (resp. smallest) subscripts in L^*. Starting from σ_n and following the decreasing order of subscripts t in L^*, we perform a sequence of (γ_t, δ_t)-Kemple changes at v_t for all $t \in L^*$ and get a new coloring in $\mathcal{C}(G - e)$, under which each w_i, for $i \geq 2$, has the same set of missing colors as under φ_{j-1}. A detailed description of the algorithm is given below.

(A) Let $I = \emptyset$ and $\sigma = \sigma_n$. While $I \neq L^*$, do: let t be the largest member of $L^* - I$ and set

$$\text{A}(t): \qquad \sigma = \sigma/P_{v_t}(\gamma_t, \delta_t, \sigma) \qquad \text{and} \qquad I = I \cup \{t\}.$$

Let us make some observations about this algorithm.

(6) Let I, t, σ be as specified in Algorithm (A) before performing the iteration $A(t)$. Then $P_{v_t}(\gamma_t, \delta_t, \sigma)$ is a path containing precisely one vertex v_t from T_t, with $\delta_t \in \overline{\sigma}(v_t)$. Furthermore, let $\sigma' = \sigma/P_{v_t}(\gamma_t, \delta_t, \sigma)$ and $I' = I \cup \{t\}$ denote the objects generated in the iteration $A(t)$. Then for any $s \in L - \{n\} - I'$, the colors γ_s and δ_s are T_s-SI under the coloring σ'.

To justify this, recall that

(7) $\delta_{i_{c(i)}} \in \overline{\sigma}_n(w_i)$ for each $2 \leq i \leq \kappa$ by (a4) in Claim 4.1 and

(8) for any $s \in L - \{n\}$, the colors γ_s and δ_s are T_s-SI under σ_n by (a5) in Claim 4.1.

In particular, (8) holds for $t = 2_{c(2)}$, the largest subscript in L^*, which implies that now $P_{v_t}(\gamma_t, \delta_t, \sigma_n)$ is a path containing precisely one vertex $v_t = w_2$ from T_t, with $\delta_t \in \overline{\sigma}_n(v_t)$ by (7). Keep in mind that this $P_{v_t}(\gamma_t, \delta_t, \sigma_n)$ is the first path employed in Algorithm (A).

As the algorithm proceeds in the decreasing order of subscripts in L^*, using (4), (5), (7), (8) and applying Claim 4.2 repeatedly, we see that (6) is true.

Claim 4.3 *Let ϱ_1 denote the coloring σ output by Algorithm (A). Then the following statements hold:*

(b1)ϱ_1 *is* $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$*-stable;*
(b2)$\overline{\varrho}_1(v) = \overline{\varphi}_{j-1}(v)$ *for all* $v \in V(T_j - v_n)$*,* $\overline{\varrho}_1(v_n) = \overline{\sigma}_n(v_n)$*, and* $\varrho_1(f) = \sigma_n(f) = \varphi_{j-1}(f)$ *for all edges f on* T_j*;*
(b3)*for any edge* $f \in E(G)$*, if* $\varrho_1(f) \neq \sigma_n(f)$*, then f is not contained in* $G[T_j]$ *and* $\{\sigma_n(f), \varrho_1(f)\} \subseteq \cup_{t \in L^*} S_t$*; and*
(b4)*for any* $i \in L_1 - \{n\}$ *(so* $v_i = v_n$*), the colors* γ_i *and* δ_i *are* T_i*-SI under* ϱ_1*.*

To justify this claim, recall from (6) that

(9) at each iteration *A*(*t*) of Algorithm (*A*), the chain $P_{v_t}(\gamma_t, \delta_t, \sigma)$ is a path containing precisely one vertex v_t from T_t, with $\delta_t \in \overline{\sigma}(v_t)$.

By (3) and the definitions of L and w_i's, we have

(10) $v_n = w_1 \prec w_i$ for all $i \geq 2$. Besides, $v_n \prec v_t$ and $T_j \subset T_t$ for each iteration *A*(*t*) of Algorithm (*A*).

It follows from (9) and (10) that $\overline{\sigma}(v) = \overline{\sigma}_n(v)$ for each $v \in V(T_j(v_n) - v_n)$ and $\sigma(f) = \sigma_n(f)$ for all edges *f* incident to $T_j(v_n) - v_n$ during each iteration of Algorithm (A). So σ and hence ϱ_1 is a $(T_j(v_n) - v_n, D_{j-1}, \sigma_n)$-stable coloring. By (4.2) and Theorem 3.10(iii), σ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable. From Lemma 2.4 we deduce that ϱ_1 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable. So (*b*1) holds.

By (*a*4) in Claim 4.1, we have

(11) $\overline{\varphi}_{j-1}(w_i) = (\overline{\sigma}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ for each vertex w_i with $i \geq 2$.

Recall that $S_p \cap S_q = \emptyset$ whenever *p* and *q* are contained in different L_i's by (5). After executing Algorithm (*A*), using (4) and Lemma 3.4(iii) (more precisely, the same argument), we obtain $\overline{\varrho}_1(w_i) = (\overline{\sigma}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$, so $\overline{\varrho}_1(w_i) = \overline{\varphi}_{j-1}(w_i)$ for $i \geq 2$ by (11). Combining this with (*a*3) in Claim 4.1, we see that $\overline{\varrho}_1(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j - v_n)$. By (6), the path $P_{v_t}(\gamma_t, \delta_t, \sigma)$ involved in each iteration *A*(*t*) of Algorithm (A) is disjoint from $v_n = w_1$. So $\overline{\varrho}_1(v_n) = \overline{\sigma}_n(v_n) = \overline{\varphi}_n(v_n)$. In view of (9) and (10), we get $\sigma(f) = \sigma_n(f)$ for all edges *f* on T_j at each iteration *A*(*t*) of Algorithm (*A*). Hence $\varrho_1(f) = \sigma_n(f) = \varphi_{j-1}(f)$ for all edges *f* on T_j, where the second equality follows from (*a*2) in Claim 4.1. Thus (*b*2) is established.

Since the Kempe changes performed in Algorithm (*A*) only involve edges outside $G[T_j]$ and colors in $\cup_{t \in L^*} S_t$ by the first half of (6), we immediately get (*b*3). Clearly, (*b*4) follows from the second half of (6). This proves Claim 4.3.

By analyzing two cases in the last two parts, we now demonstrate that the desired coloring can indeed be obtained by making γ_{1_1} missing at a certain vertex *u* (to be introduced) outside T_j but inside a closure of $T_j(v_n)$.

Consider the coloring $\varrho_1 \in \mathcal{C}^k(G - e)$ described in Claim 4.3. By (*b*1), ϱ_1 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable, so it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a $\varphi_{j-1} \bmod T_{j-1}$ coloring by (4.1) and Theorem 3.10(vi), which implies that every ETT corresponding to (ϱ_1, T_{j-1}) satisfies MP. By (*b*2), we have $\overline{\varrho}_1(v) = \overline{\varphi}_{j-1}(v)$ for each $v \in V(T_j(v_n) - v_n)$ and $\varrho_1(f) = \varphi_{j-1}(f)$ for any edge *f* on $T_j(v_n)$. Thus $T_j(v_n)$ is an ETT satisfying MP under ϱ_1. Let T_j' be a closure of $T_j(v_n)$ under ϱ_1. (We point out that the first edge added to $T_j' - T_j(v_n)$ by TAA is incident to $V(T_n(v_n) - v_n)$ and colored with δ_n under ϱ_1 by Lemma 3.2(v), (*b*2) and (*b*3), though we do not need this in our proof.) Then

(12) T'_j is an ETT satisfying MP under ϱ_1. Hence $V(T'_j)$ is elementary with respect to ϱ_1 by (4.1) and Theorem 3.10(i) (as $j \le n$).

Depending on the intersection of $\overline{\varrho}_1(T'_j - v_n)$ and $\cup_{i\in L_1} S_i$, we consider two cases.

(IV) This part is devoted to the study of the situation when the intersection is nonempty.

Case 1. $\overline{\varrho}_1(T'_j - v_n) \cap (\cup_{i\in L_1} S_i) \ne \emptyset$.

Let *u* be the smallest vertex (in the order $\prec$) in $T'_j - v_n$ (so $u \ne v_n$), such that $\overline{\varrho}_1(u) \cap (\cup_{i\in L_1} S_i) \ne \emptyset$. By (12), $V(T'_j)$ is elementary with respect to ϱ_1. Since $\delta_n \in \overline{\varphi}_n(v_n) = \overline{\varrho}_1(v_n)$ by (*b*2), we obtain $\delta_n \notin \overline{\varrho}_1(T'_j - v_n)$; in particular, $\delta_n \notin \overline{\varrho}_1(u)$. Hence, by (4) and the definition of *u*, there exists a minimum member *r* (as an integer) of L_1, such that $\gamma_r \in \overline{\varrho}_1(u) \cap (\cup_{i\in L_1} S_i)$. Since $m(v_r) = j$ by (1), there holds $r \ge j$. We propose to show, by using γ_r, that

(13) $u \in V(T'_j) - V(T_j)$.

Indeed, if $r = 1_1$, then $\gamma_r \in \overline{\varphi}_{r-1}(v_n) = \overline{\varphi}_{j-1}(v_n)$ by Algorithm 3.1. Since $V(T_j)$ is elementary with respect to φ_{j-1} by (4.1) and Theorem 3.10(i) (for $j \le n$), we have $\gamma_r \notin \overline{\varphi}_{j-1}(T_j - v_n)$. If $r > 1_1$, then $\gamma_r = \delta_t$ for some $t \in L_1$ by (4). Note that $\delta_t \notin \overline{\varphi}_{j-1}(T_j)$ by (2). So we also have $\gamma_r \notin \overline{\varphi}_{j-1}(T_j - v_n)$. It follows from (*b*2) that $\gamma_r \notin \overline{\varrho}_1(T_j - v_n)$ in either subcase. As $u \ne v_n$ and $\gamma_r \in \overline{\varrho}_1(u)$, we obtain $u \notin V(T_j)$. This proves (13).

(14) $\overline{\varrho}_1(T'_j(u) - u) \cap (\cup_{i\in L_1} S_i - \{\delta_n\}) = \emptyset$.

By the minimality assumption on *u*, we have $\overline{\varrho}_1(T'_j(u) - \{v_n, u\}) \cap (\cup_{i\in L_1} S_i) = \emptyset$. Using Lemma 3.4(ii), we obtain $\overline{\varphi}_n(v_n) \cap (\cup_{i\in L_1} S_i) = \{\delta_n\}$. It follows from (*b*2) in Claim 4.3 that $\overline{\varrho}_1(v_n) \cap (\cup_{i\in L_1} S_i) = \{\delta_n\}$. Thus (14) holds.

Let *r* be the subscript as defined above (13). Then $r = 1_p$ for some $1 \le p \le c(1)$. By (4), we have $\gamma_r = \gamma_{1_p} = \delta_{1_{p-1}}$ if $p \ge 2$. Let $L_1^* = \{1_1, 1_2, \ldots, 1_{p-1}\}$ (so $L_1^* = \emptyset$ if $p = 1$). Since $1_{p-1} < 1_p = r \le n$, we have $n \notin L_1^*$. Observe that

(15) $\delta_n \notin \cup_{i\in L_1^*} S_i$ and $\overline{\varrho}_1(v_n) \cap (\cup_{i\in L_1^*} S_i) = \emptyset$.

Indeed, by (*b*2) in Claim 4.3 and Lemma 3.4(ii), we obtain $\overline{\varrho}_1(v_n) = \overline{\varphi}_n(v_n)$ and $\overline{\varphi}_n(v_n) \cap (\cup_{i\in L_1} S_i) = \{\delta_n\}$. As $n \notin L_1^*$, from (4) we see that $\delta_n \notin \cup_{i\in L_1^*} S_i$. So $\overline{\varphi}_n(v_n) \cap (\cup_{i\in L_1^*} S_i) = \emptyset$. Hence (15) follows.

We construct a new coloring from ϱ_1 by using the following algorithm.

(B) Let $I = \emptyset$ and $\varrho = \varrho_1$. While $I \ne L_1^*$, do: let *t* be the largest member of $L_1^* - I$ and set

$$\mathrm{B}(t): \qquad \varrho = \varrho / P_u(\gamma_t, \delta_t, \varrho) \quad \text{and} \quad I = I \cup \{t\}.$$

Let us exhibit some properties satisfied by this algorithm.

(16) Let I, t, ϱ be as specified in Algorithm (*B*) before performing the iteration *B*(*t*). Then $\delta_t \in \overline{\varrho}(u)$, and $P_u(\gamma_t, \delta_t, \varrho)$ is a path containing at most one vertex v_n from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \varrho)$. Furthermore, let $\varrho' = \varrho / P_u(\gamma_t, \delta_t, \varrho)$ and $I' = I \cup \{t\}$ denote the objects generated in the iteration *B*(*t*). Then for any $s \in L_1^* - I'$, the colors γ_s and δ_s are T_s-SI under the coloring ϱ'.

To justify this, recall that $\delta_{1_{p-1}} = \gamma_r \in \overline{\varrho}_1(u)$ and

(17) for any $i \in L_1 - \{n\}$ (so $v_i = v_n$), the colors γ_i and δ_i are T_i-SI under ϱ_1 by (*b*4).

In particular, (17) holds for $t = 1_{p-1}$, the largest subscript in L_1^*, which implies that now $P_u(\gamma_t, \delta_t, \varrho_1)$ is a path containing at most one vertex v_n from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \varrho_1)$ by (15). Keep in mind that this $P_u(\gamma_t, \delta_t, \varrho_1)$ is the first path employed in Algorithm (B).

Since the algorithm proceeds in the decreasing order of subscripts in L_1^*, using (4), (5), (15), (17), and applying Claim 4.2 repeatedly, we see that (16) is true.

Claim 4.4 *Let ϱ_2 denote the coloring ϱ output by Algorithm (B). Then the following statements hold:*

(c1)ϱ_2 *is* $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$*-stable;*
(c2)$\overline{\varrho}_2(v) = \overline{\varrho}_1(v)$ *for all* $v \in V(T_j \cup T'_j(u) - u)$ *and* $\varrho_2(f) = \varrho_1(f)$ *for all* $f \in E(T_j \cup T'_j(u))$*;*
(c3)$\gamma_{1_1} \in \overline{\varrho}_2(u)$.

To justify this claim, recall from (16) that

(18) at each iteration *B*(*t*), the path $P_u(\gamma_t, \delta_t, \varrho)$ contains at most one vertex v_n from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \varrho)$.

Since $T_j \subseteq T_t$, we have $\overline{\varrho}(v) = \overline{\varrho}_1(v)$ for each $v \in V(T_j(v_n) - v_n)$ and $\varrho(f) = \varrho_1(f)$ for each edge *f* incident to $T_j(v_n) - v_n$ during each iteration of Algorithm (B) by (18). It follows that ρ and hence ϱ_2 is a $(T_j(v_n) - v_n, D_{j-1}, \varrho_1)$-stable coloring. By ($b$1) in Claim 4.3, ϱ_1 is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. From Lemma 2.4 we see that (c1) holds.

Similarly, from (18) we deduce that $\overline{\varrho}_2(v) = \overline{\varrho}_1(v)$ for all $v \in V(T_j)$ and $\varrho_2(f) = \varrho_1(f)$ for all $f \in E(T_j)$. By (14) and (15), we also have $\overline{\varrho}_1(T'_j(u) - u) \cap (\cup_{i \in L_1^*} S_i) = \emptyset$. So $T'_j(u)$ does not contain the other end of $P_u(\gamma_t, \delta_t, \varrho)$ at each iteration *B*(*t*), and hence $\overline{\varrho}_2(v) = \overline{\varrho}_1(v)$ for each $v \in V(T'_j(u) - u)$. Since T'_j is a closure of $T_j(v_n)$ under ϱ_1, from TAA we deduce that $\varrho_1\langle T'_j(u) - T_j(v_n)\rangle \cap (\cup_{i \in L_1^*} S_i) = \emptyset$. It follows that $\varrho(f) = \varrho_1(f)$ for all edges *f* in $T'_j(u) - T_j(v_n)$ at each iteration *B*(*t*). So $\varrho_2(f) = \varrho_1(f)$ for all edges *f* in $T'_j(u) - T_j(v_n)$ and hence (c2) holds.

By (16), we have $\delta_t \in \overline{\varrho}(u)$ before each iteration *B*(*t*). So γ_t becomes a missing color at *u* after performing iteration *B*(*t*). It follows that $\gamma_{1_1} \in \overline{\varrho}_2(u)$ (see (4)). Hence (c3) and therefore Claim 4.4 is established.

By (c1) in Claim 4.4, ϱ_2 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable. So it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a $\varphi_{j-1} \bmod T_{j-1}$ coloring by (4.1) and Theorem 3.10(vi), which implies that every ETT corresponding to (ϱ_2, T_{j-1}) satisfies MP. By (b2) and (c2), we have $\varrho_2(f) = \varphi_{j-1}(f)$ for each edge *f* on $T_j(v_n)$. So $T_j(v_n)$ an ETT satisfying MP under ϱ_2. Since $T'_j(u)$ is obtained from $T_j(v_n)$ by TAA under ϱ_1, it can also be obtained from $T_j(v_n)$ by TAA under ϱ_2 by (c2). Thus $T'_j(u)$ is an ETT satisfying MP under ϱ_2 as well.

In view of (b2) and (c2), we have $\overline{\varrho}_2(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j - v_n)$, $\overline{\varrho}_2(v_n) = \overline{\varphi}_n(v_n)$,and$\varrho_2(f) = \varphi_{j-1}(f)$forall$f \in E(T_j)$.Moreover,byLemma3.4(iii) and (c3), we obtain $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1_1-1}(v_n) \subseteq \overline{\varphi}_{1_{c(1)}}(v_n) \cup \{\gamma_{1_1}\} = \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\}$ $= \overline{\varrho}_2(v_n) \cup \{\gamma_{1_1}\} \subseteq \overline{\varrho}_2(v_n) \cup \overline{\varrho}_2(u)$. Therefore we can further grow $T'_j(u)$ by adding all edges on T_j but outside $G[T'_j(u)]$ using TAA under ϱ_2; let T_j^1 denote the resulting tree-sequence. Clearly, T_j^1 is an ETT satisfying MP under ϱ_2 and $V(T_j \cup T_j'(u)) \subseteq V(T_j^1)$, which contradicts MP satisfied by T under φ_n, because $u \notin V(T_j)$ by (13).

(V) Let us give an analysis of the opposite situation, which is the last part of this long proof.

Case 2. $\overline{\varrho}_1(T_j' - v_n) \cap (\cup_{i \in L_1} S_i) = \emptyset$.

Recall that $L_1 = \{1_1, 1_2, ..., 1_{c(1)}\}$. Set $S' = \cup_{i \in L_1} S_i$. Let us make some simple observations about T_j', T_j and T_n.

(19) $\overline{\varrho}_1(T_j') \cap S' = \overline{\varrho}_1(v_n) \cap S' = \{\delta_n\}$ and $\varrho_1 \langle T_j' - T_j(v_n) \rangle \cap (S' - \{\delta_n\}) = \emptyset$.

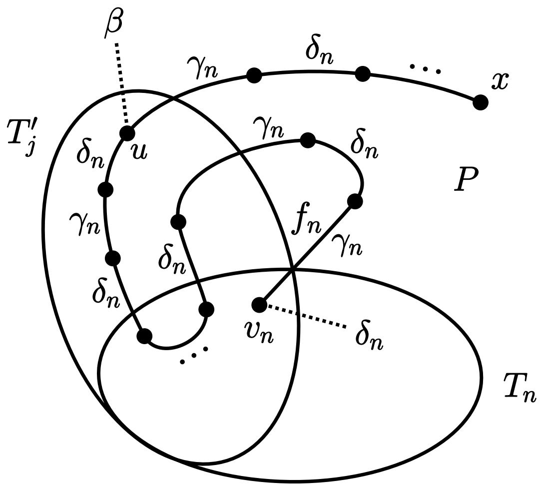
To justify this, note that $V(T_j')$ is elementary with respect to ϱ_1 by (12) and that $\overline{\varrho}_1(v_n) = \overline{\varphi}_n(v_n)$ by (b2). By Lemma 3.4(ii), we have $\overline{\varphi}_n(v_n) \cap S' = \{\delta_n\}$. So $\overline{\varrho}_1(v_n) \cap S' = \{\delta_n\}$ and hence $\delta_n \notin \overline{\varrho}_1(T_j' - v_n)$. By the hypothesis of the present case, we obtain $\overline{\varrho}_1(T_j') \cap S' = \overline{\varrho}_1(v_n) \cap S' = \{\delta_n\}$. Since T_j' is a closure of $T_j(v_n)$ under ϱ_1, from TAA we see that $\varrho_1 \langle T_j' - T_j(v_n) \rangle \cap (S' - \{\delta_n\}) = \emptyset$. Hence (19) holds.

(20) $\partial_{\varrho_1, \gamma_n}(T_n) = \{f_n\}$ and $\partial_{\varrho_1, \delta_n}(T_n \cup T_j') = \emptyset$.

To justify this, note from Lemma 3.2(v) that $\partial_{\varphi_n, \gamma_n}(T_n) = \{f_n\}$ and edges in $\partial_{\varphi_n, \delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. Since σ_n is (T_n, D_n, φ_n)-stable, by (b3) in Claim 4.3 and (5), we obtain $\partial_{\varrho_1, \alpha}(T_n) = \partial_{\sigma_n, \alpha}(T_n) = \partial_{\varphi_n, \alpha}(T_n)$ for $\alpha = \gamma_n, \delta_n$. In particular, $\partial_{\varrho_1, \gamma_n}(T_n) = \{f_n\}$. Since T_j' is a closure of $T_j(v_n)$ under ϱ_1 and $\delta_n \in \overline{\varphi}_n(v_n) = \overline{\varrho}_1(v_n)$ by (b2), from TAA we see that $\partial_{\varrho_1, \delta_n}(T_n \cup T_j') = \emptyset$. Hence (20) is true.

Consider the path $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ specified in the present lemma. By Algorithm 3.1, (4.1) and Theorem 3.10(i), we have $\delta_n, \gamma_n \notin \overline{\varphi}_{n-1}(T_n - v_n)$. So $\delta_n, \gamma_n \notin \overline{\varphi}_n(T_n - v_n)$ and hence $\delta_n, \gamma_n \notin \overline{\sigma}_n(T_n - v_n)$. It follows that the other end x of $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ is outside T_n. Let P denote $P_{v_n}(\gamma_n, \delta_n, \varrho_1)$. Then $P = P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ by (b3) and (5). From the hypothesis of the present case, we deduce that x is outside $T_j' - v_n$. Combining these two observations, we see that x is outside $T_n \cup T_j'$. Let u be the vertex of P such that the subpath $P[u, x]$ is a $(T_n \cup T_j')$-exit path with respect to ϱ_1. At the beginning of our proof, we assume that $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ (and hence P) contains

Fig. 3 The path P under ϱ_1



at least two vertices from T_n. So $u \neq v_n$. By (20), all edges in $E(P) \cap \partial(T_n \cup T_j')$ are colored by γ_n under ϱ_1 and f_n is the only edge in $\partial_{\varrho_1,\gamma_n}(T_n)$. Therefore *u* is not incident to f_n and furthermore

(21) $u \in V(T_j') - V(T_n)$.

Figure 3 gives an illustration of *P* under ϱ_1.

Let $\beta \in \overline{\varrho}_1(u)$. By the hypothesis of the present case, we have

(22) $\beta \notin S'$.

If $\beta \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$, let *z* be the smallest vertex in $T_j - V(T_j(v_n))$ in the order $\prec$ such that $\beta \in \overline{\varrho}_1(z)$; otherwise, let *z* be the largest vertex of T_j in the order $\prec$ (now $T_j(z) = T_j$).

(23) $\beta \notin \overline{\varrho}_1(T_j(z) - z)$ and $\beta \notin \varrho_1\langle T_j(z) - T_j(v_n)\rangle$.

By the definition of *z*, we have $\beta \notin \overline{\varrho}_1(T_j(z) - V(T_j(v_n)) - z)$. Since $\beta \in \overline{\varrho}_1(u)$, by (12) and (21) we obtain $\beta \notin \overline{\varrho}_1(T_j(v_n))$. So $\beta \notin \overline{\varrho}_1(T_j(z) - z)$. From (*b*2) it follows that $\beta \notin \overline{\varphi}_{j-1}(T_j(z) - z - v_n)$ and $\beta \notin \overline{\sigma}_n(v_n) = \overline{\varphi}_n(v_n)$. By Lemma 3.4(iii), $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1_1-1}(v_n) \subseteq \overline{\varphi}_{1_{c(1)}}(v_n) \cup \{\gamma_{1_1}\} = \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\}$. Since $\beta \neq \gamma_{1_1}$ by (22), we get $\beta \notin \overline{\varphi}_{j-1}(v_n)$. Hence $\beta \notin \overline{\varphi}_{j-1}(T_j(z) - z)$. As $T_j(z)$ is obtained from $T_j(v_n)$ by TAA under φ_{j-1}, $\beta \notin \varphi_{j-1}\langle T_j(z) - T_j(v_n)\rangle$. Therefore $\beta \notin \varrho_1\langle T_j(z) - T_j(v_n)\rangle$ by (*b*2). This justifies (23).

Claim 4.5 *There exists a coloring $\varrho_3 \in \mathcal{C}^k(G - e)$ with the following properties:*

(d1)ϱ_3 *is* $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$*-stable;*
(d2)$\overline{\varrho}_3(v) = \overline{\varrho}_1(v)$ *for all* $v \in V(T_j(z) \cup T_j'(u)) - \{u, z\}$ *and* $\varrho_3(f) = \varrho_1(f)$ *for all* $f \in E(T_j(z) \cup T_j'(u))$*. Furthermore,* $\delta_n \in \overline{\varrho}_3(z)$ *if* $\beta \in \overline{\varrho}_1(z)$*; and*
(d3)$\gamma_{1_1} \in \overline{\varrho}_3(u)$.

(Assuming Claim 4.5) By (*d*1) in Claim 4.5, ϱ_3 is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. So it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a $\varphi_{j-1} \bmod T_{j-1}$ coloring by (4.1) and Theorem 3.10(vi), which implies that every ETT corresponding to (ϱ_3, T_{j-1}) satisfies MP. By (*b*2) and (*d*2), we have $\varrho_3(f) = \varrho_1(f) = \varphi_{j-1}(f)$ for each edge *f* on $T_j(v_n)$. So $T_j(v_n)$ is an ETT satisfying MP under ϱ_3. Since $T_j'(u)$ is obtained from $T_j(v_n)$ by TAA under ϱ_1, it can also be obtained from $T_j(v_n)$ by TAA under ϱ_3 by (*d*2). Thus $T_j'(u)$ is an ETT satisfying MP under ϱ_3 as well.

In view of (*b*2) and (*d*2), we have $\overline{\varrho}_3(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j(z) - \{v_n, z\})$, $\overline{\varrho}_3(v_n) = \overline{\varphi}_n(v_n)$, and $\varrho_3(f) = \varphi_{j-1}(f)$ for all $f \in E(T_j(z))$. Moreover, by Lemma 3.4(iii) and (*d*3), we obtain $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1_1-1}(v_n) \subseteq \overline{\varphi}_{1_{c(1)}}(v_n) \cup \{\gamma_{1_1}\} = \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\} = \overline{\varrho}_3(v_n) \cup \{\gamma_{1_1}\} \subseteq \overline{\varrho}_3(v_n) \cup \overline{\varrho}_3(u)$. Therefore we can further grow $T_j'(u)$ by adding all edges on $T_j(z)$ but outside $G[T_j'(u)]$ using TAA under ϱ_3; let T_j^2 denote the resulting tree-sequence. Clearly, T_j^2 is an ETT satisfying MP. So $V(T_j^2)$ is elementary with respect to ϱ_3 by (4.1) and Theorem 3.10(i). If *z* is the largest vertex of T_j in the order $\prec$, then $V(T_j \cup T_j'(u)) = V(T_j(z) \cup T_j'(u)) \subseteq V(T_j^2)$, which contradicts MP satisfied by *T*, as $u \notin V(T_j)$ by (21); otherwise, $\delta_n \in \overline{\varrho}_3(z) \cap \overline{\varrho}_3(v_n)$ by ($d2$) and (19), which contradicts the elementary property satisfied by $V(T_j^2)$ under ϱ_3.

To prove Claim 4.5, we consider the coloring $\varrho_0 = \varrho_1/(G - T_j', \beta, \delta_n)$. Since T_j' is closed with respect to ϱ_1 and $\{v_n, u\} \subseteq V(T_j')$, no boundary edge of T_j' is colored by β or δ_n under ϱ_1 (see (19)). So ϱ_0 is $(T_j', D_{j-1}, \varrho_1)$-stable and hence is $(T_j(v_n) - v_n, D_{j-1}, \varrho_1)$-stable. Clearly, $P_u(\gamma_n, \beta, \varrho_0) = P_u(\gamma_n, \delta_n, \varrho_1)$. Thus u is the only vertex shared by $P_u(\gamma_n, \beta, \varrho_0)$ and $T_n \cup T_j'$. Define $\mu_0 = \varrho_0/P_u(\gamma_n, \beta, \varrho_0)$.

Claim 4.6 *The coloring μ_0 satisfies the following properties:*

(e1)μ_0 *is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring;*
(e2)$\overline{\mu}_0(v) = \overline{\varrho}_1(v)$ *for all $v \in V(T_j(z) \cup T_j'(u)) - \{u, z\}$ and $\mu_0(f) = \varrho_1(f)$ for all $f \in E(T_j(z) \cup T_j'(u))$. Furthermore, $\delta_n \in \overline{\mu}_0(z)$ if $\beta \in \overline{\varrho}_1(z)$;*
(e3)$\gamma_n = \delta_{1_{c(1)-1}} \in \overline{\mu}_0(u)$ *and $\beta \notin \overline{\mu}_0(u)$;*
(e4)*for any $t \in L_1 - \{n\}$, the colors γ_t and δ_t are T_t-SI under μ_0; and*
(e5)$\overline{\mu}_0(T_j' - u) \cap S' = \overline{\mu}_0(v_n) \cap S' = \{\delta_n\}$ *and*
$\mu_0\langle T_j' - T_j(v_n)\rangle \cap (S' - \{\delta_n\}) = \emptyset$.

To justify this, recall that ϱ_1 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable by ($b1$). By the definitions of ϱ_0 and μ_0, the transformation from ϱ_1 to μ_0 only changes colors on some edges disjoint from $V(T_j(v_n))$. So ($e1$) holds. Statement ($e3$) follows instantly from the definition of μ_0. Note that $\delta_n, \beta \notin \cup_{t\in L_1-\{n\}} S_t$ by (4), (5) and (22), and that $T_{\epsilon(t)} \subseteq T_n$ for each $t \in L_1 - \{n\}$. Furthermore, $P_u(\gamma_n, \beta, \varrho_0)$ is disjoint from $V(T_n)$. So ($e4$) can be deduced from ($b4$) immediately. Using (19) and the definitions of ϱ_0 and μ_0, we obtain ($e5$).

It remains to prove ($e2$). Recall from (23) that $\beta \notin \overline{\varrho}_1(T_j(z) - z)$ and $\beta \notin \varrho_1\langle T_j(z) - T_j(v_n)\rangle$. By (2), we obtain $\delta_n \notin \overline{\varphi}_{j-1}(T_j)$ and hence $\delta_n \notin \varphi_{j-1}\langle T_j(z) - T_j(v_n)\rangle$ by TAA. From ($b2$) we deduce that $\delta_n \notin \overline{\varrho}_1(T_j(z) - v_n)$ and $\delta_n \notin \varrho_1\langle T_j(z) - T_j(v_n)\rangle$. From the definition of ϱ_0 and μ_0, we see that ($e2$) holds. So Claim 4.6 is established.

Let $L_1^* = L_1 - \{n\}$. We construct a new coloring from μ_0 by using the following algorithm.

(C) Let $I = \emptyset$ and $\mu = \mu_0$. While $I \neq L_1^*$, do: let t be the largest member in $L_1^* - I$ and set

$$\text{C(t):} \qquad \mu = \mu/P_u(\gamma_t, \delta_t, \mu) \quad \text{and} \quad I = I \cup \{t\}.$$

Let ϱ_3 denote the coloring μ output by Algorithm (C). We aim to show that ϱ_3 is as described in Claim 4.5; our proof is based on the following statement.

(24) Let I, t, μ be as specified in Algorithm (C) before performing the iteration $C(t)$. Then $\delta_t \in \overline{\mu}(u)$, and $P_u(\gamma_t, \delta_t, \mu)$ is a path containing at most one vertex v_n from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \mu)$. Furthermore, let $\mu' = \mu/P_u(\gamma_t, \delta_t, \mu)$ and $I' = I \cup \{t\}$ denote the objects generated in the iteration $C(t)$. Then for any $s \in L_1^* - I'$, the colors γ_s and δ_s are T_s-SI under the coloring μ'.

To justify this, observe that
(25) $\overline{\mu}_0(v_n) \cap (\cup_{i\in L_1^*} S_i) = \emptyset$ by (4), (5) and ($e5$).
Furthermore,

(26) for any $s \in L_1^*$, the colors γ_s and δ_s are T_s-SI under μ_0 by ($e4$).

In particular, (26) holds for $t = 1_{c(1)-1}$, the largest subscript in L_1^*, which implies that now $P_u(\gamma_t, \delta_t, \mu_0)$ is a path containing at most one vertex $v_t = v_n$ from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \mu_0)$ by (25). In view of ($e3$), we have $\delta_t \in \overline{\mu}_0(u)$. Keep in mind that this $P_u(\gamma_t, \delta_t, \mu_0)$ is the first path employed in Algorithm (C).

As the algorithm proceeds in the decreasing order of subscripts in L_1^*, using (4), (5), (25), (26) and applying Claim 4.2 repeatedly, we see that (24) is true.

To justify Claim 4.5, recall from (24) that

(27) at each iteration *C*(*t*), the path $P_u(\gamma_t, \delta_t, \mu)$ contains at most one vertex $v_n = v_t$ from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \mu)$.

Since $T_j \subseteq T_t$, we have $\overline{\mu}(v) = \overline{\mu}_0(v)$ for each $v \in V(T_j(v_n) - v_n)$ and $\mu(f) = \mu_0(f)$ for each edge *f* incident to $T_j(v_n) - v_n$ during each iteration of Algorithm (C) by (27). It follows that μ and hence ϱ_3 is a $(T_j(v_n) - v_n, D_{j-1}, \mu_0)$-stable coloring. By ($e1$) in Claim 4.6, μ_0 is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. From Lemma 2.4 we see that ($d1$) holds.

Since $T_j(z) \subseteq T_t$, from ($e2$) and (27) we deduce that $\overline{\varrho}_3(v) = \overline{\mu}_0(v)$ for all $v \in V(T_j(z) - z)$ and $\varrho_3(f) = \mu_0(f)$ for all $f \in E(T_j(z))$. By ($e5$), we have $\overline{\mu}_0(T_j' - u) \cap S' = \overline{\mu}_0(v_n) \cap S' = \{\delta_n\}$ and $\mu_0\langle T_j' - T_j(v_n)\rangle \cap (S' - \{\delta_n\}) = \emptyset$. By (4) and (5), we obtain $\delta_n \notin \cup_{i \in L_1^*} S_i$. So at each iteration *C*(*t*) the path $P_u(\gamma_t, \delta_t, \mu)$ neither contains any edge from $T_j'(u)$ nor terminate at a vertex in $T_j'(u) - u$. It follows that $\overline{\varrho}_3(v) = \overline{\mu}_0(v)$ for all $v \in V(T_j'(u) - u)$ and $\varrho_3(f) = \mu_0(f)$ for all edges *f* in $T_j'(u) - T_j(v_n)$. Hence $\overline{\varrho}_3(v) = \overline{\mu}_0(v)$ for all $v \in V(T_j(z) \cup T_j'(u)) - \{u, z\}$ and $\varrho_3(f) = \mu_0(f)$ for all $f \in E(T_j(z) \cup T_j'(u))$. Combining this with ($e2$), we see that ($d2$) holds.

By (24), we have $\delta_t \in \overline{\mu}(u)$ before each iteration *C*(*t*). So γ_t becomes a missing color at *u* after performing iteration *C*(*t*). It follows that $\gamma_{1_1} \in \overline{\varrho}_3(u)$ (see (4)). Therefore ($d3$) is established. This completes the proof of Claim 4.5 and hence of Lemma 4.3. □

The following lemma asserts that supporting and extension vertices are subject to some order.

Lemma 4.4 *(Assuming (4.1) and (4.3)) Theorem 3.10(v) holds for all ETTs with n rungs and satisfying MP; that is, for any (T_n, D_n, φ_n)-stable coloring σ_n and any defective color δ of T_n with respect to σ_n, if v is a vertex but not the smallest one (in the order $\prec$) in $I[\partial_{\sigma_n,\delta}(T_n)]$, then $v \preceq v_i$ for any supporting or extension vertex v_i with $m(v) \leq i$.*

Proof By the hypothesis of Theorem 3.10, *T* is an ETT with the corresponding Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, and T satisfies MP under φ_n. Depending on the extension type Θ_n, we consider two cases.

Case 1. $\Theta_n = PE$. In this case, $\partial_{\varphi_n,\gamma_n}(T_n) = \{f_n\}$ by Lemma 3.2(v). Since σ_n is (T_n, D_n, φ_n)-stable and $\gamma_n \in S_n \subseteq \overline{\varphi}_n(T_n) \cup D_n$, we have $\partial_{\sigma_n,\gamma_n}(T_n) = \{f_n\}$. So $\delta \neq \gamma_n$.

By Theorem 3.10(iv), $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$. Define $\sigma_{n-1} = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)$. Then

(1) σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable by Lemma 3.6 and hence it is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. Furthermore, $\partial_{\sigma_n,\delta}(T_n) \subseteq \partial_{\sigma_{n-1},\delta}(T_n)$ (because $\delta \neq \gamma_n$ and possibly $\delta = \delta_n$).

If $i = n$, then $v \preceq v_n$ by (1), as v_n is the maximum defective vertex over all $(T_n, D_{n-1}, \varphi_{n-1})$-stable colorings. So we assume that $i < n$. Then $v \in T_{n-1}$ because $m(v) \leq i < n$. Since *v* is not the smallest vertex in $I[\partial_{\sigma_n,\delta}(T_n)]$ and $v \in T_{n-1}$, from (1) it can be seen that δ is a defective color of T_{n-1} with respect to σ_{n-1}, and *v* is not the smallest vertex in $I[\partial_{\sigma_{n-1},\delta}(T_{n-1})]$. Applying (4.3) and Theorem 3.10(v) to T_{n-1} and σ_{n-1} (see (1)), we obtain $v \preceq v_i$.

Case 2. $\Theta_n = RE$ or *SE*. In this case, φ_n is $(T_n, D_{n-1}, \varphi_{n-1})$-stable (see Algorithm 3.1). Since σ_n is (T_n, D_n, φ_n)-stable and $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$ by Lemma 3.2(i), σ_n is $(T_n, D_{n-1}, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. If $i = n$, then $v \preceq v_n$, because v_n the maximum defective vertex over all $(T_n, D_{n-1}, \varphi_{n-1})$-stable colorings. So we assume that $i < n$. Then $m(v) \leq i < n$. Since $v \in T_{n-1}$ and *v* is not the smallest vertex in $I[\partial_{\sigma_n,\delta}(T_n)]$, δ is a defective color of T_{n-1} with respect to σ_n, and *v* is not the smallest vertex in $I[\partial_{\sigma_n,\delta}(T_{n-1})]$. Since σ_n is $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable, from (4.3) and Theorem 3.10(v) we conclude that $v \preceq v_i$. □

By Definition 3.7, every $\varphi_n \bmod T_n$ coloring is a (T_n, D_n, φ_n)-stable coloring. The lemma below says that the converse also holds when MP is satisfied, so these two concepts are equivalent in this case.

Lemma 4.5 *(Assuming (4.1) and (4.4)) Theorem 3.10(vi) holds for all ETTs with n rungs and satisfying MP; that is, every (T_n, D_n, φ_n)-stable coloring σ_n is a $\varphi_n \bmod T_n$ coloring. (Thus every ETT corresponding to (σ_n, T_n) satisfies MP under σ_n by Lemma 3.9.)*

Proof By the hypothesis of Theorem 3.10, *T* is an ETT with the corresponding Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, and *T* satisfies MP under φ_n. Clearly, every tree-sequence T^* obtained from T_n (resp. $T_n + f_n$) by TAA under σ_n if $\Theta_n = PE$ (resp. if $\Theta_n = RE$ or *SE*) is a sub-sequence of some closure of T_n (resp. $T_n + f_n$) under σ_n. So to prove that σ_n is $\varphi_n \bmod T_n$, it suffices to show that, for an arbitrary closure T^*_{n+1} of T_n (resp. $T_n + f_n$) under σ_n, there exists a Tashkinov series $\mathcal{T}^* = \{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, satisfying the following conditions for all *i* with $1 \leq i \leq n$:

(1) $T^*_i = T_i$ and
(2) σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G - e)$.

For this purpose, we shall define a coloring σ_{n-1} based on σ_n, such that

(3) σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable.

By Lemma 3.2(iv), we have $\varphi_{n-1}\langle T_n \rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$, which together with (3) implies that $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ for every edge *f* on T_n and $\overline{\sigma}_{n-1}(v) = \overline{\varphi}_{n-1}(v)$ for every vertex *v* in T_n. Hence T_n can be obtained by TAA from T_{n-1} (resp.

$T_{n-1} + f_{n-1}$) under σ_{n-1} if $\Theta_{n-1} = PE$ (resp. if $\Theta_{n-1} = RE$ or *SE*) in the same way as it under φ_{n-1}. Moreover, since T_n is closed under φ_{n-1}, it is also a closure of T_{n-1} (resp. $T_{n-1} + f_{n-1}$) under σ_{n-1} if $\Theta_{n-1} = PE$ (resp. if $\Theta_{n-1} = RE$ or *SE*). By (3), (4.1) and Theorem 3.10(vi), σ_{n-1} is a $T_{n-1} \bmod \varphi_{n-1}$ coloring. Therefore

(4) there exists a Tashkinov series $\mathcal{T}' = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n\}$, which together with σ_n satisfies (1) and (2) for $1 \le i \le n$.

We shall then show, using Algorithm 3.1, that the desired Tashkinov series $\mathcal{T}^*$ can be built from $\mathcal{T}'$ by adding the tuple $(T_{n+1}^*, \sigma_n, S_n, F_n, \Theta_n)$.

Let us now give detailed descriptions. Depending on the extension type, we distinguish between two cases.

Case 1. $\Theta_n = RE$. In this case, define $\sigma_{n-1} = \sigma_n$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, so is σ_{n-1}. Recall that $\varphi_n = \varphi_{n-1}$ by RE in Algorithm 3.1 and that $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$ by Lemma 3.2(i). So σ_{n-1} satisfies (3) and hence (4) holds.

According to Algorithm 3.1, there is a subscript $h \le n-1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that $\Theta_i = RE$ for all *i* with $h+1 \le i \le n-1$, if any, and that some (γ_h, δ_h)-cycle *O* with respect to φ_{n-1} contains a sub-path *L* with $V(L) \subseteq V(T_n)$ connecting the edge f_n and $V(T_h)$. Note that $\varphi_h = \varphi_{h+1} = \ldots = \varphi_{n-1}$. Since v_h is an end of the exit-path $P_{v_h}(\gamma_h, \delta_h, \varphi_h) = P_{v_h}(\gamma_h, \delta_h, \varphi_{n-1})$, it is outside *O*. Take *w* in $V(L) \cap V(T_h)$. Then $w \ne v_h$. As σ_{n-1} is $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable by (3) and $\{\delta_h, \gamma_h\} \subseteq \cup_{i \le n-1} S_i \subseteq \overline{\varphi}_{n-1}(T_{n-1}) \cup D_{n-1}$, every edge of *L* is colored the same under σ_{n-1} as under φ_{n-1}.

Let O^* be the (γ_h, δ_h)-chain containing *L* under σ_{n-1}. Then O^* intersects T_h. By (4), $\mathcal{T}'$ is a Tashkinov series, $\Theta_h = PE$, and $\Theta_i = RE$ for all *i* with $h+1 \le i \le n-1$. From Algorithm 3.1 it follows that $\sigma_h = \sigma_{n-1}$, $\delta_h \in \overline{\sigma}_h(v_h)$, and $P_{v_h}(\gamma_h, \delta_h, \sigma_h) \cap V(T_h) = \{v_h\}$. Hence $\delta_h \in \overline{\sigma}_{n-1}(v_h)$, $P_{v_h}(\gamma_h, \delta_h, \sigma_{n-1}) = P_{v_h}(\gamma_h, \delta_h, \sigma_h)$, and O^* is disjoint from the (γ_h, δ_h)-path $P_{v_h}(\gamma_h, \delta_h, \sigma_{n-1})$ (because $w \in V(L) \subseteq V(O^*)$). Applying (4.1) and Theorem 3.10(ii) to T_n under σ_{n-1}, we see that there is at most one (γ_h, δ_h)-path intersecting T_n. So O^* must be a (γ_h, δ_h)-cycle containing *L* as a sub-path under σ_{n-1}. Therefore f_n can be chosen as an RE connecting edge for T_n under σ_{n-1}, and $\mathcal{T}^*$ can thus be built from $\mathcal{T}'$ by adding the tuple $(T_{n+1}^*, \sigma_n, S_n, F_n, \Theta_n)$ using RE of Algorithm 3.1.

Case 2. $\Theta_n = SE$ or *PE*. In this case, define $\sigma_{n-1} = \sigma_n$ if $\Theta_n = SE$ and $\sigma_{n-1} = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ if $\Theta_n = PE$. By (4.4) and Theorem 3.10(iv), we have

(5) $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap V(T_n) = \{v_n\}$ when $\Theta_n = PE$, because σ_n is (T_n, D_n, φ_n)-stable.

According to Algorithm 3.1, π_{n-1} is a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring whose largest defective vertex v_n is maximum over all $(T_n, D_{n-1}, \varphi_{n-1})$-stable colorings, and f_n is colored by δ_n under π_{n-1}. Observe that

(6) σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable and hence is $(T_n, D_{n-1}, \varphi_{n-1})$-stable.

Indeed, if $\Theta_n = SE$, then $\varphi_n = \pi_{n-1}$ by SE in Algorithm 3.1. Since $\sigma_{n-1} = \sigma_n$ is (T_n, D_n, φ_n)-stable and $\delta_n \in S_n \subseteq \overline{\varphi}_n(T_n) \cup D_n$, the desired statement (6) holds. If $\Theta_n = PE$, then (6) follows instantly from (5) and Lemma 3.6.

From (6) we see that both (3) and (4) hold true. Furthermore,

(7) $\partial_{\sigma_{n-1},\delta_n}(T_n) = \partial_{\pi_{n-1},\delta_n}(T_n)$.

By (7), we obtain $\sigma_{n-1}(f_n) = \pi_{n-1}(f_n) = \delta_n$. By (6) and Lemma 2.4, every $(T_n, D_{n-1}, \sigma_{n-1})$-stable coloring is also $(T_n, D_{n-1}, \varphi_{n-1})$-stable. So σ_{n-1} is a $(T_n, D_{n-1}, \sigma_{n-1})$-stable coloring whose largest defective vertex v_n is maximum over all $(T_n, D_{n-1}, \sigma_{n-1})$-stable colorings.

Again by (6) and Lemma 2.4, a coloring is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable iff it is $(T_n, D_{n-1} \cup \{\delta_n\}, \sigma_{n-1})$-stable. So the equality $\overline{\sigma}(u_n) \cap \overline{\sigma}(T_n) = \emptyset$ holds for every $(T_n, D_{n-1} \cup \{\delta_n\}, \sigma_{n-1})$-stable coloring σ iff the equality $\overline{\pi}(u_n) \cap \overline{\pi}(T_n) = \emptyset$ holds for every $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable colorings π, where u_n is the end of f_n outside T_n (see Algorithm 3.1). Moreover, if $\Theta_n = PE$, then v_n is also a $(T_n, \sigma_{n-1}, \{\gamma_n, \delta_n\})$-exit by the definition of σ_{n-1} and (5). From Algorithm 3.1 we thus deduce that if RE does not apply to the coloring σ_{n-1}, then we can construct $\mathcal{T}^*$ from $\mathcal{T}'$ (see (4)) by adding the tuple $(T^*_{n+1}, \sigma_n, S_n, F_n, \Theta_n)$ under σ_n, using the same extension type, SE or PE, as specified in Θ_n.

It remains to verify that indeed RE does not apply to the coloring σ_{n-1}. (Recall that RE has priority over both SE and PE in the construction of a Tashkinov series using Algorithm 3.1 (see (3.3)). That is why we need to check this.)

Assume the contrary: under σ_{n-1}, there exist an edge $f \in \partial_{\sigma_{n-1},\gamma_h}(T_n)$ and a (γ_h, δ_h)-cycle O containing a sub-path L with $V(L) \subseteq V(T_n)$ connecting the edge f and $V(T_h)$, where $\Theta_h = PE$, $S_h = \{\delta_h, \gamma_h\}$, and $\Theta_i = RE$ for all i with $h+1 \le i \le n-1$. Then $\varphi_h = \varphi_{h+1} = \ldots = \varphi_{n-1}$ by (4). Since $S_h \subseteq \overline{\varphi}_{n-1}(T_{n-1}) \cup D_{n-1}$, from (5) we see that $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ and that every edge of L is colored the same under σ_{n-1} as under $\varphi_{n-1} = \varphi_h$.

Let O^* be the (γ_h, δ_h)-chain containing L under φ_{n-1}. By an argument parallel to that used for Case 1 (see the paragraph right above Case 2), we can ensure that O^* is a (γ_h, δ_h)-cycle under φ_{n-1} containing the sub-path L connecting the edge f and $V(T_h)$. Therefore $\Theta_n = RE$ with respect to φ_{n-1} (see Algorithm 3.1), contradicting the hypothesis of the present case. □

5 Good hierarchies

It is well known that Kempe changes play a fundamental role in edge-coloring theory. To ensure that an ETT under a coloring remains to be an ETT under a new coloring arising from Kempe changes, in this section we develop an effective control mechanism over such operations, the so-called good hierarchy of an ETT, which will serve as a powerful tool in the proof of Theorem 3.10(i). As stated before, a prototype of this mechanism can be found in Chen and Jing (2019) (see Condition R2 therein). Throughout this section, we assume that

(5.1) Theorem 3.10(i) and (ii) hold for all ETTs with at most $n-1$ rungs and satisfying MP, and Theorem 3.10(iii)-(vi) hold for all ETTs with at most *n* rungs and satisfying MP.

In the case of $\Theta_n = PE$, let J_n be a closure of $T_n(v_n)$ under a (T_n, D_n, φ_n)-stable coloring σ_n. By Algorithm 3.1, $\delta_n \in \overline{\varphi}_n(v_n)$ and $|\partial_{\varphi_n,\delta_n}(T_n)| \geq 2$ (see (3.2)). By Lemma 3.2(v), edges in $\partial_{\varphi_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. Since $\bar{\sigma}_n(v_n) = \overline{\varphi}_n(v_n)$ and $\partial_{\sigma_n,\delta_n}(T_n) = \partial_{\varphi_n,\delta_n}(T_n)$, there holds $V(J_n) - V(T_n) \neq \emptyset$. We use $T_n \vee J_n$ to denote the tree-sequence obtained from T_n by adding all vertices in $V(J_n) - V(T_n)$ to T_n one by one, following the linear order $\prec$ in J_n, and using edges in J_n.

Lemma 5.1 *(Assuming (5.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. Suppose $\Theta_n = PE$ and T satisfies MP under φ_n. If J_n is a closure of $T_n(v_n)$ under a (T_n, D_n, φ_n)-stable coloring σ_n, then $V(T_n \vee J_n)$ is elementary with respect to σ_n.*

Proof Clearly, T_n is an ETT with corresponding Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}$ and satisfies MP under φ_{n-1}. Since $r(T_n) = n-1$, by (5.1) and Theorem 3.10(i), $V(T_n)$ is elementary with respect to φ_{n-1}. Let π_{n-1} and π'_{n-1} be as specified in Algorithm 3.1. Since π_{n-1} is a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring and π'_{n-1} is a $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$ -stable coloring, by definition $V(T_n)$ is also elementary with respect to π'_{n-1}. As $\varphi_n = \pi'_{n-1}/P_{v_n}(\delta_n, \gamma_n, \pi'_{n-1})$ and $\delta_n \notin \overline{\pi}'_{n-1}(T_n)$, we further obtain

(1) $V(T_n)$ is elementary with respect to φ_n and hence elementary with respect to σ_n.

Since σ_n is a (T_n, D_n, φ_n)-stable coloring, it follows from (5.1) and Theorem 3.10(iii) that σ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable and hence is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable, where $j = m(v_n)$. By Theorem 3.10(vi), σ_n is a $\varphi_{j-1} \bmod T_{j-1}$ coloring, so every ETT corresponding to (σ_n, T_{j-1}) satisfies MP. Using Lemma 3.2(iv) and Lemma 3.3, we obtain $\sigma_n(f) = \varphi_n(f) = \varphi_{j-1}(f)$ for each edge *f* on T_j. Hence J_n is a closure of $T_n(v_n) = T_j(v_n)$ under σ_n. Consequently, J_n is an ETT corresponding to (σ_n, T_{j-1}) and satisfies MP. Since $r(J_n) = j-1 \leq n-1$,

(2) $V(J_n)$ is elementary and closed with respect to σ_n by (5.1) and Theorem 3.10(i).

Suppose on the contrary that $V(T_n \vee J_n)$ is not elementary with respect to σ_n. Then $T_n \vee J_n$ contains two distinct vertices *u* and *v* such that $\overline{\sigma}_n(u) \cap \overline{\sigma}_n(v) \neq \emptyset$. By (1) and (2), we may assume that $u \in V(T_n) - V(J_n)$ and $v \in V(J_n) - V(T_n)$. So $v \neq v_n$. Let $\alpha \in \overline{\sigma}_n(u) \cap \overline{\sigma}_n(v)$. Then $\alpha \neq \delta_n$ by (2), because $\delta_n \in \overline{\varphi}_n(v_n) = \overline{\sigma}_n(v_n)$. Moreover, since $\gamma_n \in \overline{\varphi}_{n-1}(v_n)$ and $V(T_n)$ is elementary with respect to φ_{n-1}, from *PE* of Algorithm 3.1 and the definition of stable colorings, we deduce that $\gamma_n \notin \overline{\varphi}_n(T_n)$ and hence $\gamma_n \notin \overline{\sigma}_n(T_n)$. So $\alpha \neq \gamma_n$. Consequently,

(3) $\alpha \notin S_n$.

Since v_n is a maximum defective vertex according to Algorithm 3.1, $T_n(v_n)$ contains a vertex $w \neq v_n$. Note that *w* is contained in both T_n and J_n. Let $\beta \in \overline{\sigma}_n(w)$. Since $\delta_n \in \overline{\sigma}_n(v_n)$ and $\gamma_n \notin \overline{\sigma}_n(T_n)$, by (2) we obtain

(4) $\beta \notin S_n$ and the other end of $P_v(\alpha, \beta, \sigma_n)$ is *w*.

From (3), (4), and Algorithm 3.1, we see that $\partial(T_n)$ contains no edge colored by α or β under φ_n and hence under σ_n, because σ_n is (T_n, D_n, φ_n)-stable. Combining this with (1), we conclude that the other end of $P_u(\alpha, \beta, \sigma_n)$ is also *w*. Thus $P_w(\alpha, \beta, \sigma_n)$ terminates at both *u* and *v*, a contradiction. □

Let T be an ETT as specified in Theorem 3.10; that is, T is constructed from a *k*-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. To prove that *V*(*T*) is elementary with respect to φ_n, we shall turn to considering a restricted ETT T' with ladder $T_1 \subset T_2 \subset \ldots \subset T_n \subset T'$ and $V(T') = V(T_{n+1})$, and then show that $V(T')$ is elementary with respect to φ_n. For convenience, we may simply view T' as T_{n+1}.

In the remainder of this paper, we reserve the symbol R_n for a fixed closure of $T_n(v_n)$ under φ_n, if $\Theta_n = PE$. Let $T_n \vee R_n$ be the tree-sequence as defined above Lemma 5.1. We assume hereafter that

(5.2) T_{n+1} is a closure of $T_n \vee R_n$ under φ_n, which is a special closure of T_n under φ_n (see PE in Algorithm 3.1), when $\Theta_n = PE$.

By Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to φ_n, so we may further assume that

(5.3) $T \neq T_n \vee R_n$ if $\Theta_n = PE$, which together with (5.2) implies that $T_n \vee R_n$ is not closed with respect to φ_n.

(5.4) If $\Theta_n = PE$, then each color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$ is closed in $T_n \vee R_n$ with respect to φ_n. So $|T_n \vee R_n|$ is odd.

To justify this, note that each color in $\overline{\varphi}_n(R_n)$ is closed in R_n under φ_n because R_n is a closure. By Lemma 3.2(v), each color in $\overline{\varphi}_n(T_n) - \{\delta_n\}$ is closed in T_n under φ_n. Hence each color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \{\delta_n\}$ is closed in $T_n \vee R_n$ with respect to φ_n. Lemma 3.2(v) also asserts that edges in $\partial_{\varphi_n, \delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. So δ_n is closed in $T_n \vee R_n$ as well, because it is closed in R_n. Hence (5.4) follows.

To prove Theorem 3.10(i), we shall appeal to a *hierarchy* of *T* of the form

(5.5) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$, such that $T_n \vee R_n \subset T_{n,1}$ if $\Theta_n = PE$ and that $T_{n,i} = T(a_i)$ for $1 \leq i \leq q$, where $a_1 \prec a_2 \prec \ldots \prec a_q$ are some vertices in $T - V(T_n)$, called *dividers* of *T*. (So *T* has *q* dividers in total.)

As introduced before, $D_n = \cup_{h \leq n} S_h - \overline{\varphi}_n(T_n)$, where $S_h = \{\delta_h\}$ if $\Theta_h = SE$ and $S_h = \{\delta_h, \gamma_h\}$ otherwise. By Lemma 3.5, we have

(5.6) $|D_n| \leq n$.

Write $D_n = \{\eta_1, \eta_2, \ldots, \eta_{n'}\}$. In Definition 5.2 given below and the remainder of this paper,

- $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise, and $T^*_{n,j} = T_{n,j}$ if $j \geq 1$;
- $D_{n,j} = \cup_{h \leq n} S_h - \overline{\varphi}_n(T^*_{n,j})$ for $0 \leq j \leq q$ (so $D_{n,j} \subseteq D_n$);
- v_{η_h}, for $\eta_h \in D_n$, is defined to be the first vertex *u* of *T* in the order $\prec$ with

$\eta_h \in \overline{\varphi}_n(u)$, if

any, and defined to be the last vertex of T in the order $\prec$ otherwise;

- $T_{n,j}(v_{\eta_h}) = T_{n,j}$ if v_{η_h} is outside $T_{n,j}$ for $1 \le j \le q$ and $\eta_h \in D_n$; and
- $\Gamma^j = \cup_{\eta_h \in D_{n,j}} \Gamma^j_h$ for $0 \le j \le q$.

Let H be a subgraph of G and let C be a subset of $[k]$. We say that H is *C-closed* with respect to φ_n if $\partial_{\varphi_n,\alpha}(H) = \emptyset$ for any $\alpha \in C$, and say that H is *C^--closed* with respect to φ_n if it is $(\overline{\varphi}_n(H) - C)$-closed with respect to φ_n.

Definition 5.2 Hierarchy (5.5) of T is called *good* with respect to φ_n if for any j with $0 \le j \le q$ and any $\eta_h \in D_{n,j}$, there exists a 2-color subset $\Gamma^j_h = \{\gamma^j_{h_1}, \gamma^j_{h_2}\} \subseteq [k]$, such that

(i) $\Gamma^0_h \subseteq \overline{\varphi}_n(T_n) - \varphi_n\langle T_{n,1}(v_{\eta_h}) - T^*_{n,0}\rangle$ and $\Gamma^j_h \subseteq \overline{\varphi}_n(T_{n,j}) - \varphi_n\langle T_{n,j+1}(v_{\eta_h}) - T_{n,j}\rangle$ for $1 \le j \le q$ (so neither color in Γ^j_h can be used by edges on $T_{n,j+1} - T^*_{n,j}$ until after η_h becomes missing at the vertex v_{η_h} in $T_{n,j+1}$ for $0 \le j \le q$);
(ii) $\Gamma^j_g \cap \Gamma^j_h = \emptyset$ whenever η_g and η_h are two distinct colors in $D_{n,j}$;
(iii) for any j with $1 \le j \le q$, there exists precisely one color $\eta_g \in D_{n,j}$, such that $\Gamma^j_g \subseteq \overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$ (so $\Gamma^j_g \cap \Gamma^{j-1}_g = \emptyset$) and $\Gamma^j_h = \Gamma^{j-1}_h$ for all $\eta_h \in D_{n,j} - \{\eta_g\}$;
(iv) if $\Theta_n = PE$, then $T_n \vee R_n$ is not $(\Gamma^0)^-$-closed with respect to φ_n and, subject to this, $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0| \ge 4$; and
(v) $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h)^-$-closed with respect to φ_n for all j with $1 \le j \le q$.

The sets Γ^j_h are referred to as Γ-*sets* of the hierarchy (or of T) under φ_n.

At first glance, the concept of good hierarchies is very complicated. After reading the constructive proof of Theorem 5.4 shortly, one may realize that it is, nevertheless, fairly easy to understand. The following remarks may foster a better grasp of this concept.

(5.7) For $0 \le j \le q$ and $\eta_h \in D_{n,j}$, we have $\Gamma^j_h \subseteq \overline{\varphi}_n(T^*_{n,j})$ by Condition (i). So $\Gamma^j_h \cap D_{n,j} = \emptyset$ and hence $\Gamma^j \cap D_{n,j} = \emptyset$.

(5.8) Condition (iv) implies that $T_{n,1} \ne T_n \vee R_n$ if $\Theta_n = PE$.

(5.9) When $\Theta_n = RE$ or SE, the first edge added to $T_{n,1} - T_{n,0}$ is f_n (see (5.5) and Algorithm 3.1). For $1 \le j \le q$, by definitions, $D_{n,j} \subseteq D_{n,j-1}$, so Γ^{j-1}_h is well defined for any $\eta_h \in D_{n,j}$ and $\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h \subseteq \Gamma^{j-1}$. In view of Condition (v), the first edge added to $T_{n,j+1} - T_{n,j}$ is colored by a color α in Γ^{j-1}_g for some g with $\eta_g \in D_{n,j}$. From Condition (i) we see that $\alpha \notin \Gamma^j_g$. So $\Gamma^j_g \ne \Gamma^{j-1}_g$. According to Condition (iii), now Γ^j_g consists of two colors in $\overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$. Thus $\Gamma^{j-1}_g \cap \Gamma^j_g = \emptyset$ and hence $\alpha \notin \Gamma^j$.

(5.10) If a color $\alpha \in \overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$ for some j with $1 \le j \le q$, then $\alpha \notin \Gamma^{j-1}$ by Condition (i), and hence α is closed in $T_{n,j}$ with respect to φ_n by Condition (v). This simple observation will be used repeatedly in subsequent proofs.

(5.11) Note that not every ETT admits a good hierarchy. Suppose *T* does have such a hierarchy. To prove that *V*(*T*) is elementary with respect to φ_n, as usual, we shall perform a sequence of Kempe changes to reduce a minimum counterexample to an even smaller one, thereby reaching a contradiction. (The adjectives *minimum* and *smaller* used here are not meant with respect to the number of vertices. The rigorous definition of a minimum counterexample will be given in the next section; see (6.2)–(6.5).) Since interchanging with a color in $D_{n,j}$ by a Kempe change often results in a coloring which is not stable, in our proof we shall use colors in Γ^j_h as stepping stones to interchange with the color η_h in $D_{n,j}$ while maintaining stable colorings in subsequent proofs (such an interchange property indeed holds, as we shall see). So we may think of Γ^j_h as a color set exclusively reserved for η_h (see Condition (ii)) and think of a good hierarchy as a control mechanism over Kempe changes. We point out that Condition (i) can be used to preserve colors on edges of $T_{n,j}(v_{\eta_h}) - T^*_{n,j-1}$ under Kempe changes for η_h and a color in Γ^j_h. Condition (v) ensures that the aforementioned interchange property is satisfied by colors closed in $T_{n,j}$. Moreover, extending *T* by TAA while keeping condition (i) for $j = q$ leads to Condition (v). Unless *T* is already closed, Condition (iii) allows us to further extend *T* by TAA while keeping the good hierarchy property, provided that Condition (v) holds for $T = T_{n,q+1}$.

We break the proof of Theorem 3.10(i) into the following two theorems. Although the first theorem appears to be weaker than Theorem 3.10(i), the second one implies that they are actually equivalent. We only present a proof of the second theorem in this section, and will give a proof of the first one in the next two sections.

Theorem 5.3 *(Assuming (5.1)) Let T be an ETT constructed from a k-triple* (G, e, φ) *by using the Tashkinov series* $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. *Suppose T admits a good hierarchy and satisfies MP with respect to* φ_n. *Then V(T) is elementary with respect to* φ_n.

Theorem 5.4 *(Assuming (5.1)) Let T be an ETT constructed from a k-triple* (G, e, φ) *by using the Tashkinov series* $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. *If T satisfies MP under* φ_n, *then there exists a closed ETT* T' *corresponding to* (φ_n, T_n) *with* $V(T') = V(T_{n+1})$, *such that* T' *admits a good hierarchy and satisfies MP with respect to* φ_n.

Remark Our proof of Theorem 5.4 is based on Theorem 5.3, while the proof of theorem 5.3 is completely independent of Theorem 5.4.

Proof of theorem 5.4 By (5.1) and Theorem 3.10(i), $V(T_i)$ is elementary and closed with respect to φ_{i-1} for $1 \le i \le n$. So each $|T_i|$ is an odd number. Thus $|T_i| - |T_{i-1}| \ge 2$ for each $1 \le i \le n$. By Theorem 2.8, if $|T_1| \le 10$, then *G* is an elementary multigraph, thereby proving Theorem 2.1 in this case. So we may assume that $|T_1| \ge 11$. Hence

(1) $|T_i| \geq 2i + 9$ for $1 \leq i \leq n$.

We shall actually construct an ETT T' from T_n by using the same connecting edge, connecting color, and extension type as T, which has a good hierarchy:

(2) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T'$, such that $T_n \vee R_n \subset T_{n,1}$ if $\Theta_n = PE$ and such that $V(T') = V(T_{n+1})$.

Since $V(T_n)$ is elementary with respect to φ_{n-1}, by (1) we have $|\overline{\varphi}_{n-1}(T_n)| \geq 2n + 11$ (as e is uncolored). From Algorithm 3.1 we see that $|\overline{\varphi}_{n-1}(T_n)| = |\overline{\varphi}_n(T_n)|$. So

(3) $|\overline{\varphi}_n(T_n)| \geq 2n + 11$. Moreover, $|D_{n,0}| \leq |D_n| \leq n$ by (5.6).
(4) If $\Theta_n = PE$, then we can find a 2-color set $\Gamma^0_h = \{\gamma^0_{h_1}, \gamma^0_{h_2}\} \subseteq \overline{\varphi}_n(T_n)$ for each $\eta_h \in D_{n,0} = \cup_{h \leq n} S_h - \overline{\varphi}_n(T_n \vee R_n)$, such that $\Gamma^0_g \cap \Gamma^0_h = \emptyset$ whenever η_g and η_h are two distinct colors in $D_{n,0}$, and such that $T_n \vee R_n$ is not $(\Gamma^0)^-$-closed with respect to φ_n, where $\Gamma^0 = \cup_{\eta_h \in D_{n,0}} \Gamma^0_h$.

To justify this, let α be a color in $\overline{\varphi}_n(T_n \vee R_n)$ that is not closed in $T_n \vee R_n$ under φ_n; such a color exists by (5.3). In view of (3), $\overline{\varphi}_n(T_n) - \{\alpha\}$ contains at least $2n + 10$ colors. So (4) follows if we pick all colors in Γ^0 from $\overline{\varphi}_n(T_n) - \{\alpha\}$.

(5) If $\Theta_n = PE$, then there exists a 2-color set $\Gamma^0_h = \{\gamma^0_{h_1}, \gamma^0_{h_2}\} \subseteq \overline{\varphi}_n(T_n)$ for each $\eta_h \in D_{n,0}$ as described in (4), such that $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0| \geq 4$.

To justify this, let α be as specified in the proof of (4). Then $\alpha \notin \overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$ by (5.4). Since v_n is a maximum defective vertex and $v_n \in T_n \cap R_n$, the ends of the uncolored edge e are contained in both T_n and R_n. So $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)| \geq 4$. If we pick all colors in Γ^0 from $\overline{\varphi}_n(T_n) - \{\alpha\}$, with priority given to those in $\overline{\varphi}_n(T_n) - \overline{\varphi}_n(R_n)$, then $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0| \geq 4$ by (3), thereby establishing (5).

Thus Definition 5.2(iv) is satisfied by these sets Γ^0_h. Using (3), we can similarly get the following statement.

(6) If $\Theta_n \neq PE$, then we can find a 2-color set $\Gamma^0_h = \{\gamma^0_{h_1}, \gamma^0_{h_2}\} \subseteq \overline{\varphi}_n(T_n)$ for each $\eta_h \in D_{n,0} = D_n$, such that $\Gamma^0_g \cap \Gamma^0_h = \emptyset$ whenever η_g and η_h are two distinct colors in $D_{n,0}$.

So Definition 5.2(ii) is also satisfied by these sets Γ^0_h. Let us construct T' by the following Algorithms 5.5 and 5.6. Recall that v_{η_h} is defined to be the first vertex of T' in the order $\prec'$ for which $\eta_h \in \overline{\varphi}_n(v_{\eta_h})$, if any, and defined to be the last vertex of T' in the order $\prec'$ otherwise; and $T_{n,j+1}(v_{\eta_h}) = T_{n,j+1}$ if v_{η_h} is not contained in $T_{n,j+1}$ for $0 \leq j \leq q$.

Given $\{\Gamma^0_h : \eta_h \in D_{n,0}\}$, let us construct $T_{n,1}$ using the following procedure.

Algorithm 5.5 **Step 0.** Set $T_{n,1} = T_n \vee R_n$ if $\Theta_n = PE$ and $T_{n,1} = T_n + f_n$ otherwise, where f_n is the connecting edge used in Algorithm 3.1, depending on Θ_n.

Step 1. While there exists $f \in \partial(T_{n,1})$ with $\varphi_n(f) \in \overline{\varphi}_n(T_{n,1})$, do: set $T_{n,1} = T_{n,1} + f$ if the resulting $T_{n,1}$ satisfies $\Gamma_h^0 \cap \varphi_n\langle T_{n,1}(v_{\eta_h}) - T^*_{n,0}\rangle = \emptyset$ for all $\eta_h \in D_{n,0}$, where $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise.

Step 2. Return $T_{n,1}$.

Note that if $\Theta_n = PE$, then $T_n \vee R_n$ is not $(\Gamma^0)^-$-closed with respect to φ_n by (4) and (5). So Step 1 is applicable to $T_n \vee R_n$, and hence $T_{n,1} \neq T_n \vee R_n$. If $\Theta_n = RE$ or *SE*, then $T_{n,1} \neq T_n$ by the algorithm. For each $\eta_h \in D_{n,0}$, it follows from (5), (6), and Step 1 that $\Gamma_h^0 \subseteq \overline{\varphi}_n(T_n) - \varphi_n\langle T_{n,1}(v_{\eta_h}) - T^*_{n,0}\rangle$. So Definition 5.2(i) is satisfied. Moreover, $T_{n,1}$ is $(\cup_{\eta_h \in D_{n,1}} \Gamma_h^0)^-$-closed with respect to φ_n, as stated in Definition 5.2(v). To justify this, assume the contrary: there exists $f \in \partial(T_{n,1})$ with $\varphi_n(f) \in \overline{\varphi}_n(T_{n,1}) - (\cup_{\eta_h \in D_{n,1}} \Gamma_h^0)$. Then either $\varphi_n(f) \in \overline{\varphi}_n(T_{n,1}) - (\cup_{\eta_h \in D_{n,0}} \Gamma_h^0)$ or $\varphi_n(f) \in \Gamma_h^0$ for some $\eta_h \in D_{n,0}$ but $\eta_h \notin D_{n,1}$; in the latter case, η_h has become a missing color at the vertex v_{η_h} in $T_{n,1}$. Thus we can further grow $T_{n,1}$ by using *f* and Step 1 in either case, a contradiction. Since Definition 5.2(iii) starts with $j = 1$, $\{\Gamma_h^0 : \eta_h \in D_{n,0}\}$ and $T_{n,1}$ satisfy all the conditions specified in Definition 5.2.

Suppose we have constructed $\{\Gamma_h^{i-1} : \eta_h \in D_{n,i-1}\}$ and $T_{n,i}$ for all i with $1 \leq i \leq j$, which are as described in Definition 5.2. If $T_{n,j}$ is closed with respect to φ_n (equivalently $V(T_{n,j}) = V(T_{n+1})$), set $T' = T_{n,j}$. Otherwise, we proceed to the construction of $\{\Gamma_h^j : \eta_h \in D_{n,j}\}$ and $T_{n,j+1}$ using the following procedure.

Algorithm 5.6 **Step 0.** Set $\Gamma_h^j = \Gamma_h^{j-1}$ for each $\eta_h \in D_{n,j}$.

Step 1. Let *f* be an edge in $\partial(T_{n,j})$ with $\varphi_n(f) \in \Gamma_h^{j-1}$ for some $\eta_h \in D_{n,j}$, let $T_{n,j+1} = T_{n,j} + f$, and let $\{\gamma^j_{h_1}, \gamma^j_{h_2}\}$ be a 2-subset of $\overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$. Replace Γ_h^j by $\{\gamma^j_{h_1}, \gamma^j_{h_2}\}$.

Step 2. While there exists $f \in \partial(T_{n,j+1})$ with $\varphi_n(f) \in \overline{\varphi}_n(T_{n,j+1})$, do: set $T_{n,j+1} = T_{n,j+1} + f$ if the resulting $T_{n,j+1}$ satisfies $\Gamma_h^j \cap \varphi_n\langle T_{n,j+1}(v_{\eta_h}) - T_{n,j}\rangle = \emptyset$ for all $\eta_h \in D_{n,j}$.

Step 3. Return $\{\Gamma_h^j : \eta_h \in D_{n,j}\}$ and $T_{n,j+1}$.

Let us make some observations about this algorithm and its output.

As $T_{n,j}$ is not closed with respect to φ_n, $V(T_{n,j})$ is a proper subset of $V(T_{n+1})$. By Definition 5.2(v), $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1})^-$-closed with respect to φ_n. So there exists a color $\beta \in \cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1}$, such that $\partial_{\varphi_n,\beta}(T_{n,j}) \neq \emptyset$. Hence the edge *f* specified in Step 1 is available.

For $1 \leq i \leq j$, we have $|\overline{\varphi}_n(T_{n,i})| \geq |\overline{\varphi}_n(T_n)| \geq 2n + 11$ and $|D_{n,i}| \leq |D_{n,0}| \leq |D_n| \leq n$ by (3). So $\overline{\varphi}_n(T_{n,i}) - (\cup_{\eta_h \in D_{n,i}} \Gamma_h^{i-1}) \neq \emptyset$; let α be a color in this set. By Theorem 5.3 (see the remark right above the proof of this theorem), $V(T_{n,i})$ is elementary with respect to φ_n, which implies that $|T_{n,i}|$ is odd, because α is closed in $T_{n,i}$ under φ_n by Definition 5.2(v). By the definition of T_n and (5.4), $|T^*_{n,0}|$ is also odd. It follows that $|T_{n,j}| - |T^*_{n,j-1}| \geq 2$. So $\overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$ contains at least two distinct colors, and hence the 2-subset $\{\gamma^j_{h_1}, \gamma^j_{h_2}\}$ involved in

Step 1 exists. Thus Definition 5.2(iii) is satisfied. Since $T_{n,j}$ is elementary by Theorem 5.3 and Definition 5.2(ii) is satisfied by $T_{n,j-1}$, from Step 0 and Step 1 we see that Definition 5.2(ii) holds for $T_{n,j}$.

Note that each color in $\overline{\varphi}_n(T_{n,j+1}) - (\cup_{\eta_h \in D_{n,j+1}} \Gamma^j_h)$ is closed in $T_{n,j+1}$ with respect to φ_n, for otherwise, $T_{n,j+1}$ can be augmented further using Step 2 (see the paragraph succeeding Algorithm 5.5 for a proof). Thus $T_{n,j+1}$ is $(\cup_{\eta_h \in D_{n,j+1}} \Gamma^j_h)^-$ -closed with respect to φ_n, and hence Definition 5.2(v) holds. From the algorithm it follows that $\Gamma^j_h \subseteq \overline{\varphi}(T_{n,j}) - \varphi_n\langle T_{n,j+1}(v_{\eta_h}) - T^*_{n,j}\rangle$ for all $\eta_h \in D_{n,j}$, so Definition 5.2(i) holds. Thus $\{\Gamma^j_h : \eta_h \in D_{n,j}\}$ and $T_{n,j+1}$ satisfy all the conditions in Definition 5.2 and hence are as desired.

Repeating the process, we can eventually get a closed ETT T', with $V(T') = V(T_{n+1})$, that admits a good hierarchy with respect to φ_n. Clearly, T' also satisfies MP under φ_n. □

Consider the case when $\Theta_n = PE$. By the definition of hierarchy (see (5.5)), $T_n \vee R_n$ is fully contained in $T_{n,1}$. To maintain the structure of $T_n \vee R_n$ under Kempe changes, we need the following concept in subsequent proofs. A coloring $\sigma \in \mathcal{C}^k(G-e)$ is called a $(T_n \oplus R_n, D_n, \varphi_n)$-*stable coloring* if it is both (T_n, D_n, φ_n)-stable and $(R_n, \emptyset, \varphi_n)$-stable; that is, the following conditions are satisfied:

$\sigma(f) = \varphi_n(f)$ for any edge f incident to T_n with $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$;
$\sigma(f) = \varphi_n(f)$ for any edge f incident to R_n with $\varphi_n(f) \in \overline{\varphi}_n(R_n)$; and
$\overline{\sigma}(v) = \overline{\varphi}_n(v)$ for any $v \in V(T_n \cup R_n)$.

(5.12) If σ is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, then $\sigma(f) = \varphi_n(f)$ for any edge f on $T_n \cup R_n$, and R_n is also a closure of $T_n(v_n)$ under σ. To justify this, note that, for any edge f on T_n, this equality holds by Lemma 3.2(iv). For any edge f in $R_n - T_n$, we have $\varphi_n(f) \in \overline{\varphi}_n(R_n)$ by the definition of R_n and TAA. It follows from the above definition that $\sigma(f) = \varphi_n(f)$. Since σ is $(R_n, \emptyset, \varphi_n)$-stable, R_n is a closure of $T_n(v_n)$ under σ as well.

From Lemma 2.4 it is clear that being $(T_n \oplus R_n, D_n, \cdot)$-stable is also an equivalence relation on $\mathcal{C}^k(G-e)$. Moreover, every $(T_n \vee R_n, D_n, \varphi_n)$-stable coloring is $(T_n \oplus R_n, D_n, \varphi_n)$-stable, but the converse need not hold.

Observe that, in the case of $\Theta_n = PE$, even when $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a hierarchy of T (see (5.5)) under φ_n, and T remains an ETT under a (T_n, D_n, φ_n)-stable coloring σ_n, there is no guarantee that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a hierarchy of T under σ_n, because R_n may not be a closure of $T_n(v_n)$ under σ_n. Nonetheless, we can establish the following statement.

Lemma 5.7 *Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose $\Theta_n = PE$ and T satisfies MP under φ_n. Let $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ be a hierarchy of T under φ_n, and let σ_n be a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring. If*

T can be built from T_n by TAA under σ_n, then T is also an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of T under σ_n.

Proof By hypothesis, σ_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring. So $\sigma(f) = \varphi_n(f)$ for any edge f on $T_n \vee R_n$ and R_n is also a closure of $T_n(v_n)$ under σ_n by (5.12). Furthermore, σ_n is a (T_n, D_n, φ_n)-stable coloring and hence is a $\varphi_n \bmod T_n$ coloring by (5.1) and Theorem 3.10(vi). As T can be built from T_n by TAA under σ_n, it is an ETT corresponding to σ_n and satisfies MP under σ_n by Theorem 3.10(vi). In view of the hierarchy of T under φ_n, we obtain $T_n \vee R_n \subset T_{n,1}$. Hence $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of T under σ_n. □

From the above lemma we see that if $\Theta_n = PE$, σ_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, and T is also an ETT under σ_n, then each hierarchy of T under φ_n is also a hierarchy under σ_n. Thus, to check whether a good hierarchy of T remains good under a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring in subsequent proofs, we shall only check whether it satisfies Definition 5.2, without even stating that it is a hierarchy by Lemma 5.7.

We define one more term before proceeding. Let T be a tree-sequence with respect to G and e. A coloring $\pi \in \mathcal{C}^k(G-e)$ is called (T, φ_n)-*invariant* if $\pi(f) = \varphi_n(f)$ for any $f \in E(T-e)$ and $\overline{\pi}(v) = \overline{\varphi}_n(v)$ for any $v \in V(T)$. Clearly, being $(T, \cdot)$-invariant is also an equivalence relation on $\mathcal{C}^k(G-e)$. Note that for any subset C of $[k]$, a (T, C, φ_n)-stable coloring π is also (T, φ_n)-invariant, provided that $\pi\langle T\rangle \subseteq \overline{\varphi}_n(T) \cup C$. Thus, if a coloring σ_n is both (T, φ_n)-invariant and $(T_n \oplus R_n, D_n, \varphi_n)$-stable, then each hierarchy of T under φ_n is also a hierarchy under σ_n.

Lemma 5.8 *(Assuming (5.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose T satisfies MP under φ_n. Let σ_n be obtained from φ_n by recoloring some (α, β)-chains fully contained in $G - V(T)$. Then the following statements hold:*

- (i) *σ_n is (T, D_n, φ_n)-stable. In particular, σ_n is (T, φ_n)-invariant. Furthermore, if $\Theta_n = PE$ and $T_n \vee R_n \subseteq T$, then σ_n is $(T_n \oplus R_n, D_n, \varphi_n)$-stable.*
- (ii) *T is an ETT satisfying MP with respect to σ_n.*
- (iii) *If T admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ under φ_n, then this hierarchy of T remains good under σ_n, with the same Γ-sets (see Definition 5.2). Furthermore, if T is $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to φ_n, then T is also $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to σ_n.*

Proof Since the recolored (α, β)-chains are fully contained in $G - V(T)$, we have

(1) $\sigma_n(f) = \varphi_n(f)$ for each edge f incident to $V(T)$ and $\overline{\varphi}_n(v) = \overline{\sigma}_n(v)$ for each $v \in V(T)$.

Our proof relies heavily on this observation.

(i) From (1) and definitions, it is clear that σ_n is a (T, D_n, φ_n)-stable. In particular, σ_n is (T, φ_n)-invariant. Furthermore, if $\Theta_n = PE$ and $T_n \vee R_n \subseteq T$, then σ_n is $(T_n \vee R_n, D_n, \varphi_n)$-stable, which implies that σ_n is $(T_n \oplus R_n, D_n, \varphi_n)$-stable.

(ii) In view of (1), *T* can also be obtained by TAA from T_n (resp. $T_n + f_n$) under σ_n when $\Theta_n = PE$ (resp. Θ_n= *RE* or *SE*). Besides, σ_n is a (T_n, D_n, φ_n)-stable coloring. Hence, by Theorem 3.10(vi), *T* remains to be an ETT and satisfies MP under σ_n.

(iii) By (ii), *T* is also an ETT under σ_n. By hypothesis, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ is a good hierarchy of *T* under φ_n. Consider the Γ-sets specified in Definition 5.2 with respect to φ_n. Using (1) it is routine to check that these Γ-sets satisfy all the conditions in Definition 5.2 with respect to σ_n. So the given hierarchy of *T* remains good under σ_n, with the same Γ-sets. Furthermore, if *T* is $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to φ_n, then *T* is also $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to σ_n. □

In subsequent proofs, if we say that a hierarchy of an ETT under one coloring remains good under another coloring without giving the Γ-sets, we mean that it is a good hierarchy with the same Γ-sets.

6 Basic properties

As we have seen, Theorem 3.10(i) follows from Theorems 5.3 and 5.4. In the preceding section we have proved Theorem 5.4. The remainder of this paper is devoted to a proof of Theorem 5.3. In this section we make some technical preparations; the reader is referred to Chen and Jing (2019) for prototypes of some lemmas to be established herein.

Let *T* be an ETT that admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ and satisfies MP with respect to the generating coloring φ_n. To prove Theorem 5.3 (that is, *V*(*T*) is elementary with respect to φ_n), we apply induction on *q*; the induction base is Theorem 3.10(i) for T_n. For convenience, we view $T_{n,0}$ as an ETT with -1 divider and *n* rungs in the following assumption. Throughout this section we assume that

(6.1) In addition to (5.1), Theorem 5.3 holds for every ETT that admits a good hierarchy and satisfies MP, with *n* rungs and at most $q - 1$ dividers, where $q \geq 0$.

Let us first prove two technical lemmas that will be used in the proof of Theorem 5.3.

Lemma 6.1 *(Assuming (5.1)) Let T be an ETT constructed from a k-triple* (G, e, φ) *by using the Tashkinov series* $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. *Suppose* $\Theta_n = PE$ *and T satisfies MP under* φ_n. *Let* σ_n *be a* $(T_n \oplus R_n, D_n, \varphi_n)$ *-stable coloring and let* α *and* β *be two colors in [k]. Then the following statements hold:*

(i) α *and* β *are* R_n*-interchangeable under* σ_n *if* $\alpha \in \overline{\sigma}_n(R_n)$*;*
(ii) α *and* β *are* T_n*-interchangeable under* σ_n *if* $\alpha \in \overline{\sigma}_n(T_n)$*;*
(iii) α *and* β *are* $T_n \vee R_n$*-interchangeable under* σ_n *if* $\alpha \in \overline{\sigma}_n(T_n \vee R_n)$ *is closed*

in $T_n \vee R_n$ under σ_n; and

(iv) *α and β are $T_n \vee R_n$-interchangeable under σ_n if $\alpha \in \overline{\sigma}_n(T_n)$ and $\beta \in \overline{\sigma}_n(R_n)$.*

Proof Since σ_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, it is (T_n, D_n, φ_n)-stable by definition. Let $j = m(v_n)$. It follows from (5.1) and Theorem 3.10(iii) that σ_n is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. So σ_n is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable and hence, by (5.1) and Theorem 3.10(vi), it is a $\varphi_{j-1} \bmod T_{j-1}$ coloring, and every ETT corresponding to (σ_n, T_{j-1}) satisfies MP. Furthermore, $\sigma(f) = \varphi_n(f)$ for any edge f in $T_n \cup R_n$ by (5.12) and $\overline{\sigma}_n(v) = \overline{\varphi}_n(v)$ for all $v \in V(T_n \cup R_n)$.

(i) Since R_n is a closure of $T_n(v_n)$ under φ_n and σ_n is $(R_n, \emptyset, \varphi_n)$-stable, R_n is also a closure of $T_n(v_n)$ under σ_n. Since σ_n is $\varphi_{j-1} \bmod T_{j-1}$, R_n is an ETT corresponding to (σ_n, T_{j-1}) and satisfies MP under σ_n. Let α and β be as specified in the lemma. As $r(R_n) = j - 1 < n$, by (5.1) and Theorem 3.10(ii), there is at most one (α, β)-path with respect to σ_n intersecting R_n. Hence α and β are R_n-interchangeable under σ_n.

Let us make some observations before proving statements (ii) and (iii). By (5.4), each color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$ is closed in $T_n \vee R_n$ with respect to φ_n. Since σ_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, by definition we obtain

(1) each color in $\overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n)$ is closed in $T_n \vee R_n$ under σ_n.
(2) α and β are T_n-interchangeable under σ_n if $\alpha \in \overline{\sigma}_n(T_n)$, $\alpha \neq \delta_n$, and $\beta \neq \delta_n$.

To justify (2), note that $\alpha \neq \gamma_n$, because $\gamma_n \notin \overline{\varphi}_n(T_n) = \overline{\sigma}_n(T_n)$. So $\alpha \notin S_n$. Nevertheless, the case $\beta = \gamma_n$ may occur.

Let us first consider the case when $\beta \neq \gamma_n$. Since σ_n is (T_n, D_n, φ_n)-stable, $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$ by (5.1) and Theorem 3.10(iv). Define $\sigma_n' = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)$. By Lemma 3.6, σ_n' is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. From (5.1) and Theorem 3.10(ii) we deduce that α and β are T_n-interchangeable under σ_n'. So they are T_n-interchangeable under σ_n because $\{\alpha, \beta\} \cap S_n = \emptyset$.

It remains to consider the case when $\beta = \gamma_n$. In this case, f_n is the only edge in $\partial_{\sigma_n, \gamma_n}(T_n) = \partial_{\varphi_n, \gamma_n}(T_n)$ by Lemma 3.2(v). Since $V(T_n)$ is elementary with respect to φ_n, it is also elementary with respect to σ_n. Together with $\partial_{\sigma_n, \alpha}(T_n) = \emptyset$, we see that there is at most one (α, γ_n)-path with respect to σ_n intersecting T_n. So α and β are T_n-interchangeable under σ_n. Thus (2) is established.

By (1), δ_n is closed in $T_n \vee R_n$ with respect to σ_n. So statement (ii) follows instantly from (2) and statement (iii).

(iii) Assume the contrary: there are at least two (α, β)-paths P_1 and P_2 with respect to σ_n intersecting $T_n \vee R_n$. We may assume that

(3) $\alpha \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n)$.

To justify this, let A be the set of four ends of P_1 and P_2. Then at least two vertices from A are outside $T_n \vee R_n$ because, by Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to σ_n. Thus $P_1 \cup P_2$ contains two vertex-disjoint subpaths Q_1 and Q_2, which are two $T_n \vee R_n$-exit paths with respect to σ_n. Let $u \in V(T_n) \cap V(R_n)$, let $\eta \in \overline{\sigma}_n(u)$, and let $\sigma'_n = \sigma_n/(G - T_n \vee R_n, \alpha, \eta)$. By (1), η is closed in $T_n \vee R_n$ with respect to σ_n; so is α by hypothesis. Hence σ'_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, and Q_1 and Q_2 are two $T_n \vee R_n$-exit paths with respect to σ'_n. Since $P_u(\eta, \beta, \sigma'_n)$ contains at most one of Q_1 and Q_2, replacing σ_n and α by σ'_n and η, respectively, we obtain (3).

Let v be a vertex in $V(T_n) \cap V(R_n)$ with $\alpha \in \overline{\sigma}_n(v)$. Clearly, we may assume that $P_1 = P_v(\alpha, \beta, \sigma_n)$. By (i), we may further assume that P_2 is disjoint from R_n. So P_2 intersects T_n. Therefore α and β are not T_n-interchangeable under σ_n. Since $\gamma_n \notin \overline{\varphi}_n(T_n) = \overline{\sigma}_n(T_n)$, we have $\alpha \neq \gamma_n$. By (2), we may assume that $\alpha = \delta_n$ or $\beta = \delta_n$.

Suppose $\beta = \delta_n$. By Lemma 3.2(v) and the definition of stable colorings, edges in $\partial_{\sigma_n, \delta_n}(T_n)$ are all incident to $V(T_n) \cap V(R_n)$. Thus both P_1 and P_2 intersect $V(T_n) \cap V(R_n)$, contradicting statement (i).

Suppose $\alpha = \delta_n$. By (1), δ_n is closed in $T_n \vee R_n$ under σ_n. Since v_n is a maximum defective vertex, $V(T_n) \cap V(R_n)$ contains both ends of the uncolored edge e, so there exists a color $\theta \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n) - \{\delta_n\}$. Let $\sigma''_n = \sigma_n/(G - T_n \vee R_n, \delta_n, \theta)$. Then σ''_n is also $(T_n \oplus R_n, D_n, \varphi_n)$-stable. From the existence of P_1 and P_2, we see that θ and β are not $T_n \vee R_n$-interchangeable under σ''_n, contradicting our observation (2) above the case when $\alpha \neq \delta_n$ and $\beta \neq \delta_n$.

(iv) Assume the contrary: there are at least two (α, β)-paths P_1 and P_2 with respect to σ_n intersecting $T_n \vee R_n$. Let u be a vertex in T_n with $\alpha \in \overline{\sigma}_n(u)$ and let v be a vertex in R_n with $\beta \in \overline{\sigma}_n(v)$. By (ii) (resp. (i)), $P_u(\alpha, \beta, \sigma_n)$ (resp. $P_v(\alpha, \beta, \sigma_n)$) is the only (α, β)-path with respect to σ_n intersecting T_n (resp. R_n). Hence we may assume that $P_1 = P_u(\alpha, \beta, \sigma_n)$, $P_2 = P_v(\alpha, \beta, \sigma_n)$ (rename subscripts if necessary), and $P_u(\alpha, \beta, \sigma_n) \neq P_v(\alpha, \beta, \sigma_n)$. Moreover, neither $P_u(\alpha, \beta, \sigma_n)$ nor $P_v(\alpha, \beta, \sigma_n)$ has an end in $V(T_n) \cap V(R_n)$, which in turn implies that

(4) $u \in V(T_n) - V(R_n)$ and $v \in V(R_n) - V(T_n)$.

By (4) and statement (ii), $P_v(\alpha, \beta, \sigma_n)$ is disjoint from T_n. Let $\sigma'_n = \sigma_n/P_v(\alpha, \beta, \sigma_n)$. By Lemma 5.8, σ'_n is a (T_n, D_n, φ_n)-stable coloring. By Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to σ_n. Since $\alpha \in \overline{\sigma}_n(u)$ and $\beta \in \overline{\sigma}_n(v)$, from TAA we see that no edge in $R_n(v) - T_n(v_n)$ is colored by α or β under both φ_n and σ_n. Thus edges in $R_n(v) - T_n(v_n)$ are colored exactly the same under σ'_n as under σ_n and $\overline{\sigma}_n(x) = \overline{\sigma}'_n(x)$ for any $x \in V(R_n(v) - v)) \cup V(T_n)$. Let R'_n be a closure of $T_n(v_n)$ under σ'_n. Then $v \in V(R'_n)$. In view of Lemma 5.1, $V(T_n \vee R'_n)$ is elementary with respect to σ'_n. However, $\alpha \in \overline{\sigma}'_n(u) \cap \overline{\sigma}'_n(v)$, a contradiction. □

Lemma 6.2 *(Assuming (6.1)) Let T be an ETT satisfying MP constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. Suppose T has a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$. Let p be a subscript with $1 \leq p \leq q$, and let $\alpha \in \overline{\varphi}_n(T_{n,p})$ and $\beta \in [k] - \{\alpha\}$. If α is closed in $T_{n,p}$ under φ_n, then α and β are $T_{n,p}$-interchangeable under φ_n.*

Proof Assume the contrary: Let p be the smallest index such that there exist two (α,β)-paths Q_1 and Q_2 with respect to φ_n intersecting $T_{n,p}$. Let us make some simple observations about $T_{n,p}$ before proceeding. Since $T_{n,p}$ satisfies MP under φ_n and $p \le q$,

(1) $V(T_{n,p})$ is elementary with respect to φ_n by (6.1) and Theorem 5.3.

By hypthesis, $\alpha \in \overline{\varphi}_n(T_{n,p})$ is closed in $T_{n,p}$ with respect to φ_n, which together with (1) yields

(2) $|T_{n,p}|$ is odd.

As $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a good hierarchy,

(3) $T_{n,p}$ is $(\cup_{\eta_h \in D_{n,p}} \Gamma_h^{p-1})^-$-closed with respect to φ_n by Definition 5.2(v).

Depending on whether β is contained in $\overline{\varphi}_n(T_{n,p})$, we consider two cases.

Case 1. $\beta \in \overline{\varphi}_n(T_{n,p})$.

In this case, by (1) and (2), $|\partial_{\varphi_n,\beta}(T_{n,p})|$ is even. From the existence of Q_1 and Q_2, we see that G contains two vertex-disjoint $(T_{n,p}, \varphi_n, \{\alpha,\beta\})$-exit paths P_1 and P_2. For $i = 1, 2$, let a_i and b_i be the ends of P_i with $b_i \in V(T_{n,p})$. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2$. We distinguish between two subcases according to the location of b_2.

Subcase 1.1. $b_2 \in V(T_{n,p}) - V(T^*_{n,p-1})$.

Since the edge on P_2 incident to b_2 is a boundary edge of $T_{n,p}$ and is colored by β, we have $\beta \in \Gamma_h^{p-1}$ for some h with $\eta_h \in D_{n,p}$ by (3), which together with Definition 5.2(i) implies that $\beta \in \overline{\varphi}(T_{n,p-1})$. Let $\gamma \in \overline{\varphi}_n(b_2)$. By the assumption of the present subcase and Definition 5.2(i), we have $\gamma \notin \Gamma^{p-1}$. Hence γ is closed with respect to φ_n in $T_{n,p}$ by (3) (see (5.10) for details). So

(4) both α and γ are closed in $T_{n,p}$ under φ_n.

Let $\mu_1 = \varphi_n/(G - T_{n,p}, \alpha, \gamma)$. By Lemma 5.8,

(5) the given hierarchy of $T_{n,p}$ remains good under μ_1, with the same Γ-sets as those under φ_n (see Definition 5.2). Furthermore, $T_{n,p}$ is $(\cup_{\eta_h \in D_{n,p}} \Gamma_h^{p-1})^-$-closed under μ_1 and $\beta \in \overline{\mu}_1(T_{n,p-1})$.

Note that P_1 and P_2 are two $(T_{n,p}, \mu_1, \{\gamma,\beta\})$-exit paths. Let $\mu_2 = \mu_1/P_{b_2}(\gamma,\beta,\mu_1)$. Since $P_{b_2}(\gamma,\beta,\mu_1) \cap T_{n,p} = \{b_2\}$, all edges incident to $V(T_{n,p}(b_2) - b_2)$ are colored the same under μ_2 as under μ_1. So $\beta \in \overline{\mu}_2(T_{n,p-1})$. By (5) and Lemma 5.8, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subseteq T_{n,p}(b_2) - b_2$ is a good hierarchy of the ETT $T_{n,p}(b_2) - b_2$ (satisfying MP) under μ_2, with the same Γ-sets as $T_{n,p}$ under φ_n. So

(6) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T_{n,p}(b_2)$ is a good hierarchy of the ETT $T_{n,p}(b_2)$ (satisfying MP) under μ_2, with the same Γ-sets as $T_{n,p}$ under φ_n.

Thus from (6) and (6.1) on Theorem 5.3, we conclude that $V(T_{n,p}(b_2))$ is elementary with respect to μ_2. However, $\beta \in \overline{\mu}_2(T_{n,p-1}) \cap \overline{\mu}_2(b_2)$, a contradiction.

Subcase 1.2. $b_2 \in V(T^*_{n,p-1})$.

We propose to show that

(7) there exists a color $\theta \in \overline{\varphi}_n(T_n)$ that is closed in both $T^*_{n,0}$ and $T_{n,1}$ under φ_n if $p = 1$, and a color $\theta \in \overline{\varphi}_n(T_{n,p-1})$ that is closed in both $T_{n,p-1}$ and $T_{n,p}$ under φ_n if $p \geq 2$.

Our proof is based on the following simple observation (see (3) in the proof of Theorem 5.4).

(8) $|\overline{\varphi}_n(T_n)| \geq 2n + 11$ and $|D_{n,i}| \leq |D_n| \leq n$ for $0 \leq i \leq q$.

Let us first assume that $p = 1$. When $\Theta_n \neq PE$, let θ be a color in $\overline{\varphi}_n(T_n) - (\cup_{\eta_h \in D_{n,1}} \Gamma^0_h)$; such a color exists by (8). From Algorithm 3.1 we see that T_n is closed under φ_n. By (3), $T_{n,1}$ is $(\cup_{\eta_h \in D_{n,1}} \Gamma^0_h)^-$-closed under φ_n. So θ is as desired. When $\Theta_n = PE$, we have $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0| \geq 4$ by Definition 5.2(iv). Let $\theta \in \overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0 - \{\delta_n\}$. It follows from (5.4) that θ is closed in $T_n \vee R_n$ under φ_n. Since $T_{n,1}$ is $(\cup_{\eta_h \in D_{n,1}} \Gamma^0_h)^-$-closed with respect to φ_n, θ also closed in $T_{n,1}$ under φ_n as desired.

Next we assume that $p \geq 2$. By (8), we have $|\overline{\varphi}_n(T_{n,p-2})| \geq |\overline{\varphi}_n(T_n)| \geq 2n + 11$ and $|D_{n,p-1}| \leq |D_n| \leq n$. So there exists a color θ in $\overline{\varphi}_n(T_{n,p-2}) - (\cup_{\eta_h \in D_{n,p-1}} \Gamma^{p-2}_h)$. Since $\overline{\varphi}_n(T_{n,p-2}) \subseteq \overline{\varphi}_n(T_{n,p-1})$, we get $\theta \in \overline{\varphi}_n(T_{n,p-1}) - (\cup_{\eta_h \in D_{n,p-1}} \Gamma^{p-2}_h)$. By Definition 5.2(v), θ is closed in $T_{n,p-1}$ under φ_n. From the definition of θ and Definition 5.2(iii), it follows that $\theta \notin \Gamma^{p-1}$. So $\theta \in \overline{\varphi}_n(T_{n,p}) - \Gamma^{p-1} \subseteq \overline{\varphi}_n(T_{n,p}) - (\cup_{\eta_h \in D_{n,p}} \Gamma^{p-1}_h)$. By (3), θ is closed in $T_{n,p}$ under φ_n. Hence (7) is established.

Let $\mu_3 = \varphi_n/(G - T_{n,p}, \alpha, \theta)$. Since both α and θ are closed in $T_{n,p}$ with respect to φ_n, by Lemma 5.8, $T_{n,p}$ admits a good hierarchy and satisfies MP with respect to μ_3. Thus $T_{n,p-1}$ also admits a good hierarchy and satisfies MP with respect to μ_3 if $p \geq 2$. By (7), θ is closed in $T^*_{n,0}$ if $p = 1$ and closed in $T_{n,p-1}$ if $p \geq 2$ under μ_3. Note that both P_1 and P_2 are $(T^*_{n,p-1}, \mu_3, \{\theta, \beta\})$-exit paths. So θ and β are not $T^*_{n,0}$-interchangeable under μ_3 if $p = 1$ and not $T_{n,p-1}$-interchangeable under μ_3 if $p \geq 2$, which contradicts Lemma 6.1(iii) or the interchangeability property of T_n when $p = 1$, and the minimality assumption on p when $p \geq 2$.

Case 2. $\beta \notin \overline{\varphi}_n(T_{n,p})$.

In this case, $|\partial_{\varphi_n,\beta}(T_{n,p})|$ is odd and at least three by (1) and (2). From the existence of Q_1 and Q_2, we see that G contains at least three $(T_{n,p}, \varphi_n, \{\alpha, \beta\})$-exit paths P_1, P_2, P_3. For $i = 1, 2, 3$, let a_i and b_i be the ends of P_i with $b_i \in V(T_{n,p})$, and f_i be the edge of P_i incident to b_i. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2 \prec b_3$.

Subcase 2.1. $b_3 \in V(T_{n,p}) - V(T^*_{n,p-1})$.

For convenience, we call the tuple $(\varphi_n, T_{n,p}, \alpha, \beta, P_1, P_2, P_3)$ a *counterexample* and use $\mathcal{K}$ to denote the set of all such counterexamples. With a slight abuse of notation, we still use $(\varphi_n, T_{n,p}, \alpha, \beta, P_1, P_2, P_3)$ to denote a counterexample in $\mathcal{K}$ with the minimum $|P_1| + |P_2| + |P_3|$. Let $\gamma \in \overline{\varphi}(b_3)$. By the hypothesis of the present subcase and Definition 5.2(i), we have $\gamma \notin \Gamma^{p-1}$. So γ is closed in $T_{n,p}$ under φ_n by (3). Note that γ might be some $\eta_h \in D_n$.

Let $\mu_4 = \varphi_n/(G - T_{n,p}, \alpha, \gamma)$. By Lemma 5.8, $T_{n,p}$ admits a good hierarchy and satisfies MP under μ_4. Note that P_1, P_2, P_3 are three $(T_{n,p}, \mu_4, \{\gamma, \beta\})$-exit paths.

Consider $\mu_5 = \mu_4/P_{b_3}(\gamma, \beta, \mu_4)$. Clearly, $\beta \in \overline{\mu}_5(b_3)$ and $\beta \notin \Gamma^{p-1}$. Since $P_{b_3}(\gamma, \beta, \mu_4) \cap T_{n,p} = \{b_3\}$, it is easy to see that all edges incident to $V(T_{n,p}(b_3) - b_3)$ are colored the same under μ_5 as under μ_4. By Lemma 5.8, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T_{n,p}(b_3) - b_3$ is a good hierarchy of the ETT $T_{n,p}(b_3) - b_3$ satisfying MP under μ_5, with the same Γ-sets as $T_{n,p}$ under φ_n. So

(9) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T_{n,p}(b_3)$ is a good hierarchy of $T_{n,p}(b_3)$ under μ_5, with the same Γ-sets as $T_{n,p}$ under φ_n.

Let H be obtained from $T_{n,p}(b_3)$ by adding f_1 and f_2. Since $\beta \notin \Gamma^{p-1}$, it can be seen from (9) that

(10) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset H$ is a good hierarchy of H under μ_5, with the same Γ-sets as $T_{n,p}$ under φ_n.

By (5.1) and Theorem 3.10(vi), H satisfies MP under μ_5. Set $T' = H$. Let us grow T' by using the following algorithm:

(11) While there exists $f \in \partial(T')$ with $\mu_5(f) \in \overline{\mu}_5(T')$, do: set $T' = T' + f$ if the resulting T' satisfies $\Gamma_h^{p-1} \cap \mu_5\langle T'(v_{\eta_h}) - T_{n,p-1}\rangle = \emptyset$ for all $\eta_h \in D_{n,p-1}$.

Note that this algorithm is exactly the same as Step 2 in Algorithm 5.6. From (11) we see that

(12) T' is $(\cup_{\eta_h \in D'_{n,p}} \Gamma_h^{p-1})^-$-closed with respect to μ_5, where $D'_{n,p} = \cup_{h \le n} S_h - \overline{\mu}_5(T')$ (so $D'_{n,p} \subseteq D_{n,p-1}$).

In view of (10) and (11), we conclude that

(13) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'$ is a good hierarchy of T' under μ_5, with the same Γ-sets as $T_{n,p}$ under φ_n.

Clearly, T' satisfies MP under μ_5. By (13), (6.1), and Theorem 5.3, $V(T')$ is elementary with respect to μ_5. Observe that none of a_1, a_2, a_3 is contained in T', for otherwise, let $a_i \in V(T_2)$ for some i with $1 \le i \le 3$. Since $\{\beta, \gamma\} \cap \overline{\mu}_5(a_i) \ne \emptyset$ and $\beta \in \overline{\mu}_5(b_3)$, we obtain $\gamma \in \overline{\varphi}_2(a_i)$. Recall that $\beta, \gamma \notin \Gamma^{p-1}$. Hence from (11) we see that P_1, P_2, P_3 are all entirely contained in $G[T']$, which in turn implies $\gamma \in \overline{\varphi}_2(a_j)$ for $j = 1, 2, 3$. So $V(T')$ is not elementary with respect to μ_5, a contradiction. Therefore, each P_i contains a subpath L_i, which is a T'-exit path with respect to μ_5. Since f_1 is not contained in L_1, we obtain $|L_1| + |L_2| + |L_3| < |P_1| + |P_2| + |P_3|$. Thus, in view of (12), the existence of the counterexample $(\mu_5, T', \gamma, \beta, L_1, L_2, L_3)$ violates the minimality assumption on $(\varphi_n, T_{n,p}, \alpha, \beta, P_1, P_2, P_3)$.

Subcase 2.2. $b_3 \in V(T^*_{n,p-1})$.

The proof in this subcase is essentially the same as that in Subcase 1.2. Let θ be a color as described in (7). Consider $\mu_3 = \varphi_n/(G - T_{n,p}, \alpha, \theta)$. Then we can verify that θ and β are not $T^*_{n,0}$-interchangeable under μ_3 if $p = 1$ and not $T_{n,p-1}$-interchangeable under μ_3 if $p \geq 2$, which contradicts Lemma 6.1(iii) or the minimality assumption on *p*; for the omitted details, see the proof in Subcase 1.2. □

Let us make some further preparations before proving Theorem 5.3. Let $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ be a good hierarchy of *T* (see (5.5) and Definition 5.2). Recall that $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise, $T^*_{n,0} \subset T_{n,1}$ by (5.5), and $T^*_{n,i} = T_{n,i}$ if $i \geq 1$. Let *T* be constructed from $T^*_{n,q}$ using TAA by recursively adding edges $e_1, e_2, \ldots, e_p$ and vertices $y_1, y_2, \ldots, y_p$, where y_i is the end of e_i outside $T(y_{i-1})$ for $i \geq 1$, with $T(y_0) = T^*_{n,q}$. Write $T = T^*_{n,q} \cup \{e_1, y_1, e_2, ..., e_p, y_p\}$. The *path number* of *T*, denoted by *p*(*T*), is defined to be the smallest subscript $i \in \{1, 2, ..., p\}$ such that the sequence $(y_i, e_{i+1}, ..., e_p, y_p)$ corresponds to a path in *G*, where e_j is between y_{j-1} and y_j for $i + 1 \leq j \leq p$. Note that $p(T) = p$ if this path contains the vertex y_p only.

A coloring $\sigma_n \in \mathcal{C}^k(G - e)$ is called a $(T^*_{n,0}, D_n, \varphi_n)$-*weakly stable coloring* if it is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring when $\Theta_n = PE$ and is a (T_n, D_n, φ_n)-stable coloring when $\Theta_n \neq PE$. By Lemma 3.2(iv) and (5.12), every $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring is $(T^*_{n,0}, \varphi_n)$-invariant.

A coloring $\sigma_n \in \mathcal{C}^k(G - e)$ is called a $(T^*_{n,i}, D_n, \varphi_n)$-*weakly stable coloring*, with $1 \leq i \leq q$, if it is both a $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable and a $(T^*_{n,i}, \varphi_n)$-invariant coloring. By Lemma 3.2(iv), every $(T^*_{n,i}, D_n, \varphi_n)$-stable coloring is $(T^*_{n,i}, D_n, \varphi_n)$-weakly stable. From Theorem 3.10(vi) it is also clear that, under a $(T^*_{n,i}, D_n, \varphi_n)$-weakly stable coloring σ_n, $T^*_{n,i}$ is an ETT satisfying MP (this statement will frequently be used directly in subsequent proofs without even citing Theorem 3.10(vi)).

As stated before, our proof of Theorem 5.3 proceeds by induction on *q* (see (6.1)). The induction step will be carried out by contradiction. Throughout the remainder of this section and Subsection 7.1, (T, φ_n) stands for a minimum counterexample to Theorem 5.3; that is,

(6.2) *T* is an ETT that admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1}$
$= T$ and satisfies MP with respect to the generating coloring φ_n;

(6.3) subject to (6.2), *V*(*T*) is not elementary with respect to φ_n;

(6.4) subject to (6.2) and (6.3), *p*(*T*) is minimum; and

(6.5) subject to (6.2)–(6.4), $|T| - |T_{n,q}|$ is minimum.

Our objective is to find another counterexample (T', σ_n) to Theorem 5.3, which violates the minimality assumption (6.4) or (6.5) on (T, φ_n).

The following fact will be used frequently in subsequent proof.

(6.6) $V(T(y_{p-1}))$ is elementary with respect to φ_n.

Let us exhibit some basic properties satisfied by the minimum counterexample (T, φ_n) as specified above.

Lemma 6.3 *For* $0 \leq i \leq p - 1$, *the inequality*

$$|\overline{\varphi}_n(T(y_i)) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_i) - T^*_{n,q}\rangle| \geq 2n + 11$$

holds, where $T(y_0) = T^*_{n,q}$*. Furthermore, if*

$$|\overline{\varphi}_n(T(y_i)) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_i) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})| \leq 4,$$

then there exist 7 distinct colors $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_i))$ *such that* $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_i) - T^*_{n,q}\rangle = \emptyset$*, where* Γ^q *and* Γ^q_h *are introduced in Definition 5.2.*

Proof By (6.6), $T(y_{p-1})$ is elementary with respect to φ_n. Since the number of vertices in $T(y_i) - V(T^*_{n,q})$ is *i*, and the number of edges in $T(y_i) - T^*_{n,q}$ is also *i*, we obtain $|\overline{\varphi}_n(T(y_i) - V(T^*_{n,q}))| \geq |\varphi_n\langle T(y_i) - T^*_{n,q}\rangle|$. Hence

$$\begin{aligned}
& |\overline{\varphi}_n(T(y_i)) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_i) - T^*_{n,q}\rangle| \\
\geq\ & |\overline{\varphi}_n(T(y_i))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_i) - T^*_{n,q}\rangle| \\
\geq\ & |\overline{\varphi}_n(T(y_i))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\overline{\varphi}_n(T(y_i) - V(T^*_{n,q}))| \\
=\ & |\overline{\varphi}_n(T^*_{n,q})| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| \\
\geq\ & |\overline{\varphi}_n(T^*_{n,0})| - |\overline{\varphi}_n(T^*_{n,0}) - \overline{\varphi}_n(T_n)| \\
=\ & |\overline{\varphi}_n(T_n)| \\
\geq\ & 2n + 11,
\end{aligned}$$

where the last inequality can be found in the proof of Theorem 5.4 (see (3) therein). So the first inequality is established.

Suppose the second inequality also holds. Then these two inequalities guarantee the existence of at least $2n + 7$ colors in the intersection C of $\overline{\varphi}_n(T(y_i)) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_i) - T^*_{n,q}\rangle$ and $\Gamma^q \cup D_{n,q}$. By (5.6), we have $|D_{n,q}| \leq |D_n| \leq n$ and $|\Gamma^q| \leq 2|D_{n,q}| \leq 2n$. So $|\Gamma^q \cup D_{n,q}| \leq 3n$. Since $|C| \leq |\Gamma^q \cup D_{n,q}|$, it follows that $2n + 7 \leq 3n$, which implies $n \geq 7$. Note that $C = \cup_{\eta_h \in D_{n,q}}(\Gamma^q_h \cup \{\eta_h\}) \cap C$ and $|(\Gamma^q_h \cup \{\eta_h\}) \cap C| \leq 3$ for any η_h in $D_{n,q}$. Since $|C| \geq 2n + 7$ and $n \geq 7$, by the Pigeonhole Principle, there exist at least 7 distinct colors η_h in $D_{n,q}$, such that $|(\Gamma^q_h \cup \{\eta_h\}) \cap C| = 3$, or equivalently, $\Gamma^q_h \cup \{\eta_h\} \subseteq C$. For each of these η_h, clearly $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_i))$ and $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_i) - T^*_{n,q}\rangle = \emptyset$ by the definition of *C*. □

Let *v* be a vertex of *T* and $T' \subset T$. By $T' \prec v$ we mean that $u \prec v$ for any $u \in V(T')$. Given a color $\alpha \in [k]$, we use v_α to denote the first vertex *u* of *T* in the order $\prec$ for which $\alpha \in \overline{\varphi}_n(u)$, if any, and defined to be the last vertex of *T* in the order $\prec$ otherwise.

Lemma 6.4 *Suppose* $q \geq 1$ *and* $\alpha \in \overline{\varphi}_n(T_{n,q})$*. If there exists a subscript i with* $0 \leq i \leq q$*, such that* α *is closed in* $T^*_{n,i}$ *with respect to* φ_n*, then* $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$*, where r is the largest such i. If there is no such subscript i,* *then* $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1} \subseteq \Gamma^{j-1}$ *for* $1 \le j \le q$, $\Theta_n = PE$, $v_\alpha \in V(T_n) - V(R_n)$, *and* $\alpha \notin \varphi_n\langle T_{n,q} - T_n\rangle$.

Proof Recall that T has a good hierarchy by (6.2). Let us first assume the existence of a subscript i with $0 \le i \le q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. By definition, r is the largest such i. As the statement holds trivially when $r = q$, we may assume that $r < q$. Let s be an arbitrary index with $r + 1 \le s \le q$. From the definition of r, we see that α is not closed in $T_{n,s}$ with respect to φ_n. It follows from Definition 5.2(v) that $\alpha \in \Gamma_h^{s-1}$ for some $\eta_h \in D_{n,s} \subseteq D_{n,s-1}$. By definition, $D_{n,s} = \cup_{h \le n} S_h - \overline{\varphi}_n(T_{n,s})$, so $\eta_h \notin \overline{\varphi}_n(T_{n,s})$ and hence $T_{n,s}(v_{\eta_h}) = T_{n,s}$ (see paragraphs above Definition 5.2 for the notation $T_{n,s}(v_{\eta_h})$). Since $\alpha \in \Gamma_h^{s-1}$, Definition 5.2(i) (with $j = s - 1$) implies $\alpha \notin \varphi_n\langle T_{n,s}(v_{\eta_h}) - T^*_{n,s-1}\rangle = \varphi_n\langle T_{n,s} - T^*_{n,s-1}\rangle$. As this property holds for all s with $r + 1 \le s \le q$, we get $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$.

Next we assume that there exists no subscript i with $0 \le i \le q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. Since $\alpha \in \overline{\varphi}_n(T_{n,q})$, it follows from (5.10) that $\alpha \in \overline{\varphi}_n(T^*_{n,0})$. By Definition 5.2(v), we obtain

(1) $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1} \subseteq \Gamma^{j-1}$ for $1 \le j \le q$.

Hence $\alpha \in \Gamma^j$ for all $0 \le j \le q - 1$. From the definition of Γ^0, we see that $v_\alpha \in V(T_n)$. If $\Theta_n \ne PE$, then α would be closed in $T_n = T^*_{n,0}$ under φ_n, a contradiction. So $\Theta_n = PE$. Moreover, since α is not closed in $T^*_{n,0}$, by (5.4), we have $v_\alpha \in V(T_n) - V(R_n)$. Since R_n is a closure of $T_n(v_n)$ under φ_n, using (6.6) and TAA we obtain

(2) $\alpha \notin \overline{\varphi}_n(R_n - V(T_n))$ and $\alpha \notin \varphi_n\langle R_n - T_n\rangle$.
(3) $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,0}\rangle$.

Let s be an arbitrary index with $1 \le s \le q$. By (1), we have $\alpha \in \Gamma_h^{s-1}$ for some $\eta_h \in D_{n,s} \subseteq D_{n,s-1}$. As $D_{n,s} = \cup_{h \le n} S_h - \overline{\varphi}_n(T_{n,s})$, there holds $\eta_h \notin \overline{\varphi}_n(T_{n,s})$. So $T_{n,s}(v_{\eta_h}) = T_{n,s}$. From Definition 5.2(i) (with $j = s - 1$), we deduce that $\alpha \notin \varphi_n\langle T_{n,s}(v_{\eta_h}) - T^*_{n,s-1}\rangle = \varphi_n\langle T_{n,s} - T^*_{n,s-1}\rangle$. Since this property is valid for all s with $1 \le s \le q$, we establish (3).

Combining (2) and (3), we conclude that $\alpha \notin \varphi_n\langle T_{n,q} - T_n\rangle$. □

Our proof of Theorem 5.3 relies heavily on the following two technical lemmas.

Lemma 6.5 *Let* α *and* β *be two colors in* $\overline{\varphi}_n(T(y_{p-1}))$. *Suppose both* $v_\alpha \prec v_\beta$ *and* $\alpha \notin \varphi_n\langle T(v_\beta) - T^*_{n,q}\rangle$ *hold if* $\{\alpha, \beta\} - \overline{\varphi}_n(T^*_{n,q}) \ne \emptyset$. *Then* $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$ *if one of the following cases occurs:*

(i) $q \ge 1$, *and* $\alpha \in \overline{\varphi}_n(T_{n,q})$ *or* $\{\alpha, \beta\} \cap D_{n,q} = \emptyset$;
(ii) $q = 0$, *and* $\alpha \in \overline{\varphi}_n(T_n)$ *or* $\{\alpha, \beta\} \cap D_n = \emptyset$; *and*
(iii) $\alpha \in \overline{\varphi}_n(T^*_{n,q})$ *and is closed in* $T^*_{n,q}$ *with respect to* φ_n.

Furthermore, in Case (iii), $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting $T^*_{n,q}$.

Proof Let $a = v_\alpha$ and $b = v_\beta$. We distinguish among three cases according to the locations of a and b.

Case 1. $\{a, b\} \subseteq V(T^*_{n,q})$.

By (6.6), $V(T^*_{n,q})$ is elementary with respect to φ_n. So a (resp. b) is the only vertex in $T^*_{n,q}$ missing α (resp. β). If both α and β are closed in $T^*_{n,q}$ with respect to φ_n, then no boundary edge of $T^*_{n,q}$ is colored by α or β. Hence $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$ is the only path intersecting $T^*_{n,q}$. So we may assume that α or β is not closed in $T^*_{n,q}$ with respect to φ_n. It follows that if $q = 0$, then $\Theta_n = PE$, for otherwise, Algorithm 3.1 would imply that both α and β are closed in $T_n = T^*_{n,0}$, a contradiction. Therefore

(1) $T^*_{n,0} = T_n \vee R_n$ if $q = 0$.

Let us first assume that precisely one of α and β is closed in $T^*_{n,q}$ with respect to φ_n. In this subcase, by Lemma 6.2 if $q \geq 1$ and by (1) and Lemma 6.1(iii) if $q = 0$, colors α and β are $T^*_{n,q}$-interchangeable under φ_n, so $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$ is the only path intersecting $T^*_{n,q}$.

Next we assume that neither α nor β is closed in $T^*_{n,q}$ with respect to φ_n. In this subcase, we only need to show that $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$. Symmetry allows us to assume that $a \prec b$. Let r be the subscript with $\beta \in \overline{\varphi}_n(T^*_{n,r} - V(T^*_{n,r-1}))$, where $0 \leq r \leq q$ and $T^*_{n,-1} = \emptyset$. Then $a, b \in V(T^*_{n,r})$. By (6.2), $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a good hierarchy of T. If $r \geq 1$, then β is closed in $T_{n,r}$ with respect to φ_n by Definition 5.2 (see (5.10)). From the above discussion about $T^*_{n,q}$ (with r in place of q), we similarly deduce that $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$. So we may assume that $r = 0$. If $\Theta_n \neq PE$, then both α and β are closed in T_n with respect to φ_n (see Algorithm 3.1), so $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$ by (6.6). If $\Theta_n = PE$, then it follows from Lemma 6.1(i), (ii) and (iv) that $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$.

Case 2. $\{a, b\} \cap V(T^*_{n,q}) = \emptyset$.

By the hypotheses of the present case and the present lemma, we have $\{\alpha, \beta\} \cap D_{n,q} = \emptyset$ if $q \geq 1$ and $\{\alpha, \beta\} \cap D_n = \emptyset$ if $q = 0$. So

(2) $\alpha, \beta \notin D_{n,q} \cup \overline{\varphi}_n(T^*_{n,q})$ if $q \geq 1$ and $\alpha, \beta \notin D_n \cup \overline{\varphi}_n(T^*_{n,0})$ if $q = 0$.

By the definitions of D_n and $D_{n,q}$, we have $D_n \cup \overline{\varphi}_n(T_n) \subseteq D_{n,q} \cup \overline{\varphi}_n(T^*_{n,q})$. By Lemma 3.2(iv) and Algorithm 3.1, we obtain $\varphi_n\langle T(a)\rangle \subseteq D_n \cup \overline{\varphi}_n(T(a) - a)$ and $\varphi_n\langle T(b)\rangle \subseteq D_n \cup \overline{\varphi}_n(T(b) - b)$. Since $T^*_{n,q} \prec a \prec b$ and $\alpha \notin \varphi_n\langle T(b) - T^*_{n,q}\rangle$ (by the hypotheses of the present case and the present lemma), from (2) we see that

(3) $\alpha, \beta \notin \varphi_n\langle T(b)\rangle$.

Suppose on the contrary that $P_a(\alpha, \beta, \varphi_n) \neq P_b(\alpha, \beta, \varphi_n)$. Consider $\sigma_n = \varphi_n / P_b(\alpha, \beta, \varphi_n)$. By (2), σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-stable coloring, so it is also a $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable coloring. Thus, by (3) and (6.6), every edge of $T(b)$ is colored under σ_n the same as under φ_n. So $T(b)$ is still an ETT satisfying MP with respect to σ_n. Moreover, from (2), (3), and (6.6), we deduce that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is still a good hierarchy of $T(b)$ under σ_n, with the same Γ-sets as T under φ_n (see Definition 5.2). As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the pair $(T(b), \sigma_n)$ is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on (T, φ_n).

Case 3. $a \in V(T^*_{n,q})$ and $b \notin V(T^*_{n,q})$.

By the hypotheses of the present case and the present lemma, (6.6) and TAA, we obtain

(4) $\alpha \notin \varphi_n\langle T(b) - T^*_{n,q}\rangle$ and $\beta \notin \overline{\varphi}_n(T(b) - b)$. So β is not used by any edge in $T(b) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\beta \in D_n$).

Let us first assume that α is closed in $T^*_{n,q}$ with respect to φ_n. By Lemma 6.2 if $q \geq 1$ and by Lemma 6.1(iii) or Theorem 3.10(ii) (see (5.1)) if $q = 0$, colors α and β are $T^*_{n,q}$-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path intersecting $T^*_{n,q}$. Suppose on the contrary that $P_a(\alpha, \beta, \varphi_n) \neq P_b(\alpha, \beta, \varphi_n)$. Then $P_b(\alpha, \beta, \varphi_n)$ is vertex-disjoint from $T^*_{n,q}$ and hence contains no edge incident to $T^*_{n,q}$.

Consider $\sigma_n = \varphi_n/P_b(\alpha, \beta, \varphi_n)$. It is routine to check that σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable coloring, and *T*(*b*) is an ETT satisfying MP with respect to σ_n. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of *T*(*b*) under σ_n, with the same Γ-sets as *T* under φ_n, by (4). As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the pair $(T(b), \sigma_n)$ is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on (T, φ_n).

So we assume hereafter that

(5) α is not closed in $T^*_{n,q}$ with respect to φ_n.

Hence our objective is to show that $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$. Assume the contrary: $P_a(\alpha, \beta, \varphi_n) \neq P_b(\alpha, \beta, \varphi_n)$. We distinguish between two subcases according to the value of *q*.

Subcase 3.1. $q = 0$.

By the hypothesis of the present lemma, $\alpha \in \overline{\varphi}_n(T_n)$ or $\{\alpha, \beta\} \cap D_n = \emptyset$. So $\alpha \notin D_n$. From (5) and Algorithm 3.1 we deduce that $T^*_{n,0} \neq T_n$. Hence

(6) $\Theta_n = PE$, which together with (5) and (5.4) yields $a \notin V(T_n) \cap V(R_n)$.

Consider $\sigma_n = \varphi_n/P_b(\alpha, \beta, \varphi_n)$. We claim that

(7) σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring.

To justify this, note that if $a \in V(T_n) - V(R_n)$, then $\alpha, \beta \notin \overline{\varphi}_n(R_n)$ by (6.6) and the hypothesis of the present case. By definition, σ_n is $(R_n, \emptyset, \varphi_n)$-stable. In view of Lemma 6.1(ii), $P_b(\alpha, \beta, \varphi_n)$ is disjoint from T_n and hence contains no edge incident to T_n. So σ_n is (T_n, D_n, φ_n)-stable. Hence (7) holds. Suppose $a \in V(R_n) - V(T_n)$. By the hypothesis of the present lemma, $\{\alpha, \beta\} \cap D_n = \emptyset$. By (6.6), we also have $\alpha, \beta \notin \overline{\varphi}_n(T_n)$. Thus $\alpha, \beta \notin \overline{\varphi}_n(T_n) \cup D_n$. By definition, σ_n is (T_n, D_n, φ_n)-stable. Using Lemma 6.1(i), $P_b(\alpha, \beta, \varphi_n)$ is disjoint from R_n and hence contains no edge incident to R_n. By definition, σ_n is $(R_n, \emptyset, \varphi_n)$-stable. Therefore (7) is true.

From (4), (7) and (6.6) we see that $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(b) - b)$ (recall that every $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring is $(T^*_{n,0}, \varphi_n)$-invariant). Furthermore, *T*(*b*) is an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T(b)$ is a good hierarchy of *T*(*b*) under σ_n, with the same Γ-sets as *T* under φ_n. As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the pair $(T(b), \sigma_n)$ is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on (T, φ_n).

Subcase 3.2. $q \geq 1$.

Let us first assume that α is closed in $T^*_{n,i}$ with respect to φ_n for some *i* with $0 \leq i \leq q$. Let *r* be the largest subscript *i* with this property. Then $r \leq q - 1$ by (5). Note that $\alpha \in \overline{\varphi}_n(T^*_{n,r})$ since α is closed in $T^*_{n,r}$ and $|T^*_{n,r}|$ is odd (because it is elementary and closed for colors in $\overline{\varphi}_n(T^*_{n,r} - V(T^*_{n,r-1})))$. By Lemma 6.4, we have $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$, which together with (4) yields

(8) $\alpha \notin \varphi_n\langle T(b) - T^*_{n,r}\rangle$.

By Lemma 6.2 if $r \geq 1$ and by Theorem 3.10(ii) or Lemma 6.1(iii) if $r = 0$, colors α and β are $T^*_{n,r}$-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β) -path with respect to φ_n intersecting $T^*_{n,r}$. Hence $P_b(\alpha, \beta, \varphi_n)$ is vertex-disjoint from $T^*_{n,r}$ and therefore contains no edge incident to $T^*_{n,r}$. Let $\sigma_n = \varphi_n/P_b(\alpha, \beta, \varphi_n)$. By Lemma 5.8, σ_n is a $(T^*_{n,r}, D_n, \varphi_n)$-weakly stable coloring, and $T^*_{n,r}$ is an ETT having a good hierarchy and satisfying MP with respect to σ_n. By (4) and TAA, β is not used by any edge in $T(b) - T^*_{n,r}$, except possibly e_1 when $r = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\beta \in D_n$). Since σ_n is (T_n, D_n, φ_n)-stable, it follows from (8) and (6.6) that $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(b) - b)$. So *T*(*b*) is an ETT satisfying MP with respect to σ_n. Moreover,

(9) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of *T*(*b*) under σ_n, with the same Γ-sets as *T* under φ_n.

Since $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(b) - b)$, to justify (9), it suffices to verify that Definition 5.2(v) is satisfied with respect to σ_n; that is, $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h)^-$-closed with respect to σ_n for $1 \leq j \leq q$. As the statement holds trivially if $P_b(\alpha, \beta, \varphi_n)$ is vertex-disjoint from $T_{n,j}$, we may assume that $P_b(\alpha, \beta, \varphi_n)$ intersects $T_{n,j}$. Thus $r + 1 \leq j \leq q$. Observe that $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h$, for otherwise, α is closed in $T_{n,j}$ with respect to φ_n by Definition 5.2(v), contradicting the definition of *r*. By (6.6), we also obtain $\beta \notin \overline{\varphi}_n(T_{n,j})$. Consequently, $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h)^-$-closed with respect to σ_n. (Note that α may become closed in $T_{n,j}$ with respect to σ_n. Yet, even in this situation the desired statement is true.) This proves (9).

As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the existence of $(T(b), \sigma_n)$ contradicts the minimality assumption (6.5) on (T, φ_n).

Next we assume that α is not closed in $T^*_{n,i}$ with respect to φ_n for any *i* with $0 \leq i \leq q$. By the hypothesis of the present subcase, $q \geq 1$. In view of Lemma 6.4, we obtain

(10) $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h \subseteq \Gamma^{j-1}$ for $1 \leq j \leq q$, $\Theta_n = PE$, $a \in V(T_n) - V(R_n)$, and $\alpha \notin \varphi_n\langle T_{n,q} - T_n\rangle$.

It follows from (4), (10) and TAA that

(11) $\alpha \notin \varphi_n\langle T(b) - T_n\rangle$ and $\beta \notin \varphi_n\langle T(b) - T^*_{n,0}\rangle$.

Since R_n is a closure of $T_n(v_n)$ under φ_n, using (10), (6.6) and TAA we obtain

(12) $\alpha, \beta \notin \overline{\varphi}_n(R_n)$ and $\beta \notin \varphi_n\langle R_n - T_n\rangle$.

By Lemma 6.1(ii), colors α and β are T_n-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting T_n. Hence $P_b(\alpha, \beta, \varphi_n)$ is vertex-disjoint from T_n and therefore contains no edge incident to T_n. Consider $\sigma_n = \varphi_n/P_b(\alpha, \beta, \varphi_n)$. By Lemma 5.8, σ_n is a (T_n, D_n, φ_n)-stable coloring, and T_n is an ETT satisfying MP with respect to σ_n. From (11) and (12) we further deduce that σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring, $\sigma_n(f) = \varphi_n(f)$ for each

$f \in E(T(b))$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(b) - b)$. So *T*(*b*) is an ETT satisfying MP with respect to σ_n. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of *T*(*b*) under σ_n, with the same Γ-sets as *T* under φ_n (see (10) and the proof of (9) for omitted details). As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the existence of $(T(b), \sigma_n)$ contradicts the minimality assumption (6.5) on (T, φ_n). □

Lemma 6.6 *Let α and β be two colors in $\overline{\varphi}_n(T(y_{p-1}))$, let Q be an (α, β)-chain with respect to φ_n, and let $\sigma_n = \varphi_n/Q$. Suppose one of the following cases occurs:*

1) *$q \geq 1$, $\alpha \in \overline{\varphi}_n(T_{n,q})$, and Q is an (α, β)-path disjoint from $P_{v_\alpha}(\alpha, \beta, \varphi_n)$;*
2) *$q = 0$, $\alpha \in \overline{\varphi}_n(T_n)$, or $\alpha \in \overline{\varphi}_n(T^*_{n,0})$ with $\alpha, \beta \notin D_n$, and Q is an (α, β)-path disjoint from $P_{v_\alpha}(\alpha, \beta, \varphi_n)$; and*
3) *$T^*_{n,q} \prec v_\alpha \prec v_\beta$, $\alpha, \beta \notin D_{n,q}$, $\alpha \notin \varphi_n\langle T(v_\beta) - T(v_\alpha)\rangle$, and Q is an arbitrary (α, β)-chain.*

Then the following statements hold:

(i) *σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable coloring;*
(ii) *$T^*_{n,q}$ is an ETT satisfying MP with respect to σ_n; and*
(iii) *if $q \geq 1$, then $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ is a good hierarchy of $T_{n,q}$ under σ_n, with the same Γ-sets (see Definition 5.2) as T under φ_n, and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$-closed with respect to σ_n.*

Furthermore, in Case 3, *T* is also an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of *T* under σ_n, with the same Γ-sets (see Definition 5.2) as *T* under φ_n.

Remark In the proof of Theorem 5.3, frequently we need to check whether a "smaller" counterexample T' with $T_{n,q} \subset T'$ has a good hierarchy with the same Γ-sets under σ_n as *T* under φ_n. Lemma 6.6 is established to fulfill such needs: We shall use the above Statement (iii) to ensure that Definition 5.2(i)–(v) are satisfied by $T_{n,q}$ and that Definition 5.2(v) is satisfied by T'. Since the Γ-sets used under σ_n are the same as those under φ_n, Definition 5.2(ii)–(iv) are automatically satisfied by T'. One technical question remains unanswered: How can we verify that Definition 5.2(i) is satisfied by T'? It is only a straightforward matter, as we shall see.

Proof of Lemma 6.6 Write $a = v_\alpha$ and $b = v_\beta$. Let us consider the three cases described in the lemma separately.

Case 1. $q \geq 1$, $\alpha \in \overline{\varphi}_n(T_{n,q})$, and *Q* is an (α, β)-path disjoint from $P_a(\alpha, \beta, \varphi_n)$. We distinguish between two subcases according to the location of *b*.

Subcase 1.1. $b \in V(T_{n,q})$.

Let us first assume that there exists a subscript *i* with $0 \leq i \leq q$, such that α or β is closed in $T^*_{n,i}$ with respect to φ_n. Let *r* be the largest such *i*. By (5.10) and Lemma 6.4, we have

(1) $\{a, b\} \subseteq V(T^*_{n,r})$ and $\alpha, \beta \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$.
(2) α and β are $T^*_{n,r}$-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n)$.

To justify this, note that if $r \geq 1$, then (2) holds by Lemma 6.2. So we assume that $r = 0$. Then α or β is closed in $T^*_{n,0}$ with respect to φ_n. Hence, by Lemma 6.1(iii) if $\Theta_n = PE$ and by (5.1) and Theorem 3.10(ii) otherwise, α and β are $T^*_{n,0}$-interchangeable under φ_n. This proves (2).

It follows from (2) that Q is vertex-disjoint from $T^*_{n,r}$ and hence contains no edge incident to $T^*_{n,r}$. By Lemma 5.8, $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,r}, D_n, \varphi_n)$-weakly stable coloring, and $T^*_{n,r}$ is an ETT satisfying MP with respect to σ_n. By (1) and (6.6), we obtain $\sigma_n(f) = \varphi_n(f)$ for each edge f of $T_{n,q}$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each vertex u of $T_{n,q}$. Therefore σ_n is a $(T_{n,q}, D_n, \varphi_n)$-weakly stable coloring. By the definition of r, for any $r+1 \leq j \leq q$ and $\theta \in \{\alpha, \beta\}$, we have $\partial_{\varphi_n,\theta}(T_{n,j}) \neq \emptyset$, so $\theta \in \cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1}$ by Definition 5.2(v). It is then routine to check that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ is a good hierarchy of $T_{n,q}$ under σ_n, with the same Γ-sets as T under φ_n[2], and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$-closed with respect to σ_n.

Next we assume that there exists no subscript i with $0 \leq i \leq q$, such that α or β is closed in $T^*_{n,i}$ with respect to φ_n. By Lemma 6.4, we have

(3) $\alpha, \beta \in \cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1} \subseteq \Gamma^{j-1}$ for $1 \leq j \leq q$, $\Theta_n = PE$, $a, b \in V(T_n) - V(R_n)$, and $\alpha, \beta \notin \varphi_n\langle T_{n,q} - T_n\rangle$.

Since R_n is a closure of $T_n(v_n)$ under φ_n, using (6.6) and TAA we obtain

(4) $\alpha, \beta \notin \overline{\varphi}_n(R_n)$.

By Lemma 6.1(ii), colors α and β are T_n-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting T_n. Hence Q is vertex-disjoint from T_n and therefore contains no edge incident to T_n. By Lemma 5.8, $\sigma_n = \varphi_n/Q$ is a (T_n, D_n, φ_n)-stable coloring, and T_n is an ETT satisfying MP with respect to σ_n. By (3), (4) and (6.6), we further deduce that σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-stable coloring, $\sigma_n(f) = \varphi_n(f)$ for each edge f of $T_{n,q}$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each vertex u of $T_{n,q}$. It is then routine to check that the desired statements hold.

Subcase 1.2. $b \notin V(T_{n,q})$.

Let us first assume that there exists a subscript i with $0 \leq i \leq q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. Let r be the largest such i. By (5.10), Lemma 6.4 and TAA, we have

(5) $a \subseteq V(T^*_{n,r})$ and $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$. Furthermore, no edge in $T_{n,q} - T^*_{n,r}$ is colored by β, except possibly e_1 when $r = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\beta \in D_n$).

Using the same argument as that of (2), we obtain

(6) α and β are $T^*_{n,r}$-interchangeable under φ_n.

[2] See the justification of (9) in the proof of Lemma 6.5 for omitted details. Note that α or β may become closed in $T_{n,j}$ with respect to σ_n for some j with $r+1 \leq j \leq q$. Yet, even in this situation Definition 5.2(v) remains valid with respect to σ_n.

It follows from (6) that Q is vertex-disjoint from $T^*_{n,r}$ and hence contains no edge incident to $T^*_{n,r}$. By Lemma 5.8, $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,r}, D_n, \varphi_n)$-weakly stable coloring, and $T^*_{n,r}$ is an ETT satisfying MP with respect to σ_n. Using (5), we obtain $\sigma_n(f) = \varphi_n(f)$ for each edge f of $T_{n,q}$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each vertex u of $T_{n,q}$. Therefore σ_n is a $(T_{n,q}, D_n, \varphi_n)$-weakly stable coloring, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ is a good hierarchy of $T_{n,q}$ under σ_n, with the same Γ-sets as T under φ_n, and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma^{q-1}_h)^-$-closed with respect to σ_n (see the justification of (9) in the proof of Lemma 6.5 for omitted details).

Next we assume that there exists no subscript i with $0 \le i \le q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. By Lemma 6.4, we have

(7) $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h \subseteq \Gamma^{j-1}$ for $1 \le j \le q$, $\Theta_n = PE$, $a \in V(T_n) - V(R_n)$, and $\alpha \notin \varphi_n\langle T_{n,q} - T_n\rangle$.

It follows that (4) also holds. By Lemma 6.1(ii), colors α and β are T_n-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting T_n. Hence Q is vertex-disjoint from T_n and therefore contains no edge incident to T_n. By Lemma 5.8, $\sigma_n = \varphi_n/Q$ is a (T_n, D_n, φ_n)-stable coloring, and T_n is an ETT satisfying MP with respect to σ_n. Since $b \notin V(T_{n,q})$, no edge in $T_{n,q} - T^*_{n,0}$ is colored by β by TAA, because $T^*_{n,0} = T_n \vee R_n$ by (7). Using (4) and (7), it is routine to check that the desired statements hold.

Case 2. $q = 0$, $\alpha \in \overline{\varphi}_n(T_n)$, or $\alpha \in \overline{\varphi}_n(T^*_{n,0})$ with $\alpha, \beta \notin D_n$, and Q is an (α, β) -path disjoint from $P_{v_\alpha}(\alpha, \beta, \varphi_n)$.

Let us first assume that α or β is closed in $T^*_{n,0}$ with respect to φ_n. By Lemma 6.1(iii) or Theorem 3.10(ii) (see (5.1)), colors α and β are $T^*_{n,0}$-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path intersecting $T^*_{n,0}$, and hence Q is vertex-disjoint from $T^*_{n,0}$. It is then routine to check that $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring, and $T^*_{n,0}$ is an ETT satisfying MP with respect to σ_n by Theorem 3.10(vi). So we assume hereafter that

(8) neither α nor β is closed in $T^*_{n,0}$ with respect to φ_n.

By the hypothesis of the present case, $\alpha \in \overline{\varphi}_n(T_n)$ or $\{\alpha, \beta\} \cap D_n = \emptyset$. So $\alpha \notin D_n$. From (8) and Algorithm 3.1 we deduce that $T^*_{n,0} \neq T_n$. Hence

(9) $\Theta_n = PE$, which together with (5.4) yields $a, b \notin V(T_n) \cap V(R_n)$.

Let us show that

(10) $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring.

To justify this, note that if one of a and b is contained in $V(T_n) - V(R_n)$ and the other is contained in $V(R_n) - V(T_n)$, then α and β are $T^*_{n,0}$-interchangeable under φ_n by Lemma 6.1(iv). So Q is vertex-disjoint from $T^*_{n,0}$ and hence (10) holds. In view of (9), we may assume that

(11) if $a, b \in V(T^*_{n,0})$, then either $a, b \in V(T_n) - V(R_n)$ or $a, b \in V(R_n) - V(T_n)$.

Let us first assume that $a \in V(T_n) - V(R_n)$. Then $\alpha \notin \overline{\varphi}_n(R_n)$ by (6.6) and $b \in V(T_n) - V(R_n)$ if $b \in V(T^*_{n,0})$ by (11). So α and β are T_n-interchangeable under φ_n by Lemma 6.1(ii) and $\beta \notin \overline{\varphi}_n(R_n)$ by (6.6). It follows that Q is vertex-disjoint from T_n and that $\sigma_n(f) = \varphi_n(f)$ for any edge f incident to R_n with $\varphi_n(f) \in \overline{\varphi}_n(R_n)$. Hence (10) holds.

Next we assume that $a \in V(R_n) - V(T_n)$. Then $\alpha \notin \overline{\varphi}_n(T_n)$ by (6.6) and $b \in V(R_n) - V(T_n)$ if $b \in V(T^*_{n,0})$ by (11). So α and β are R_n-interchangeable under φ_n by Lemma 6.1(i) and $\beta \notin \overline{\varphi}_n(T_n)$ by (6.6). It follows that Q is vertex-disjoint from R_n. By the hypothesis of the present case, $\{\alpha, \beta\} \cap D_n = \emptyset$. So $\alpha, \beta \notin \overline{\varphi}_n(T_n) \cup D_n$ and hence (10) holds.

From (10) we deduce that $T^*_{n,0}$ is an ETT satisfying MP with respect to σ_n.

Case 3. $T^*_{n,q} \prec v_\alpha \prec v_\beta, \alpha, \beta \notin D_{n,q}, \alpha \notin \varphi_n\langle T(v_\beta) - T(v_\alpha)\rangle$, and Q is an arbitrary (α, β)-chain.

By (6.6), $V(T(y_{p-1}))$ is elementary with respect to φ_n. So $\alpha, \beta \notin \overline{\varphi}_n(T^*_{n,q})$. By hypothesis, $\alpha, \beta \notin D_{n,q}$. Hence

(12) $\quad \alpha, \beta \notin \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}.$

By the definitions of D_n and $D_{n,q}$, we have $D_n \cup \overline{\varphi}_n(T_n) \subseteq D_{n,q} \cup \overline{\varphi}_n(T^*_{n,q})$. So $\alpha, \beta \notin \overline{\varphi}_n(T_n) \cup D_n$. From Lemma 3.2(iv), TAA and the hypothesis of the present case, we further deduce that

(13) $\quad \alpha, \beta \notin \varphi_n\langle T(b)\rangle.$

In view of Lemma 6.5, we obtain

(14) $\quad P_a(\alpha, \beta, \varphi) = P_b(\alpha, \beta, \varphi)$. (Possibly Q is this path.)

Since $T^*_{n,q} \prec a \prec b$, using (12)-(14), it is straightforward to verify that $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,q}, D_n, \varphi_n)$-stable coloring, so σ_n is also $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable.

From (12) and (13) we also see that $T(b)$ can be obtained from $T^*_{n,q}$ by using TAA, no matter whether $Q = P_a(\alpha, \beta, \varphi)$. Thus T is an ETT corresponding to (σ_n, T_n). As neither α nor β is contained in any Γ-set, it is clear that T also satisfies MP under σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as T under φ_n. □

7 Elementariness and interchangeability

In Sect. 5 we have developed a control mechanism over Kempe changes; that is, a good hierarchy of an ETT. In Sect. 6 we have derived some properties satisfied by such hierarchies. Now we are ready to present a proof of Theorem 5.3 by using Kempe changes based on these hierarchies, whose origin can be traced back to Tashkinov's proof of Theorem 2.7 (Tashkinov 2000) (see Stiebitz et al. 2012 for an English version).

7.1 Proof of theorem 5.3

By hypothesis, T is an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Furthermore, T admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ and satisfies MP with respect to φ_n. Our objective is to show that $V(T)$ is elementary with respect to φ_n.

As introduced in the preceding section, $T = T^*_{n,q} \cup \{e_1, y_1, e_2, ..., e_p, y_p\}$, where y_i is the end of e_i outside $T(y_{i-1})$ for $i \ge 1$, with $T(y_0) = T^*_{n,q}$. Suppose on the contrary that $V(T)$ is not elementary with respect to φ_n. Then

(7.1) $\overline{\varphi}_n(T(y_{p-1})) \cap \overline{\varphi}_n(y_p) \ne \emptyset$ by (6.6).

For ease of reference, recall that (see (3) in the proof of Theorem 5.4)

(7.2) $|\overline{\varphi}_n(T_n)| \ge 2n + 11$ and $|D_{n,j}| \le |D_n| \le n$ for $0 \le j \le q$.

In our proof, by $A \cap B = \emptyset$ we mean A and B are vertex-disjoint, provided that A is a path and B is a tree. We shall frequently make use of a coloring $\sigma_n \in \mathcal{C}^k(G - e)$ with properties (i)-(iii) as described in Lemma 6.6; that is,

(7.3) σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable coloring, and $T^*_{n,q}$ is an ETT satisfying MP with respect to σ_n. Furthermore, if $q \ge 1$, then $T_{n,q}$ admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ under σ_n, with the same Γ-sets (see Definition 5.2) as T under φ_n, and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma^{q-1}_h)^-$-closed with respect to σ_n (see the remark succeeding Lemma 6.6).

Claim 7.1 $p \ge 2$.

Assume the contrary: $p = 1$; that is, $T = T^*_{n,q} \cup \{e_1, y_1\}$. Then

(1) there exists a color α in $\overline{\varphi}_n(T^*_{n,q}) \cap \overline{\varphi}_n(y_1)$ by (7.1).

We consider two cases according to the value of q.

Case 1. $q = 0$. In this case, from (1) and Algorithm 3.1 we see that $\Theta_n \ne SE$. Let us first assume that $\Theta_n = RE$. Let δ_n, γ_n be as specified in RE of Algorithm 3.1. Since $\alpha, \delta_n \in \overline{\varphi}_n(T_n)$, both of them are closed in T_n with respect to φ_n. Hence $P_{y_1}(\alpha, \delta_n, \varphi_n)$ is vertex-disjoint from T_n. Let $\sigma_n = \varphi_n / P_{y_1}(\alpha, \delta_n, \varphi_n)$. Then $\delta_n \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(y_1)$. By Lemma 5.8, σ_n is a (T_n, D_n, φ_n)-stable coloring and hence, by Theorem 3.10(vi), it is a $\varphi_n \bmod T_n$ coloring. In view of Definition 3.7, $f_n = e_1$ is still an RE connecting edge under σ_n. From Algorithm 3.1 we see that $q \ge 1$ and e_1 is contained in a (δ_n, γ_n)-cycle under σ_n, which is impossible because $\delta_n \in \overline{\sigma}_n(y_1)$.

So we may assume that $\Theta_n = PE$. Let $\beta = \varphi_n(e_1)$. From TAA we see that $\beta \in \overline{\varphi}_n(T^*_{n,0})$. Let $\theta \in \overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$. Then θ is closed in $T^*_{n,0}$ under φ_n by (5.4). By Lemma 6.1(iii), $P_{v_\theta}(\alpha, \theta, \varphi_n)$ is the only (α, θ)-path intersecting $T^*_{n,0}$. Thus $P_{y_1}(\alpha, \theta, \varphi_n) \cap T^*_{n,0} = \emptyset$. Let $\sigma_n = \varphi_n / P_{y_1}(\alpha, \theta, \varphi_n)$. Then θ is also closed in $T^*_{n,0}$ with respect to σ_n, and σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring by Lemma 5.8. In view of Lemma 6.1(iii), β and θ are $T^*_{n,0}$-interchangeable under σ_n. As $P_{y_1}(\theta, \beta, \sigma_n) \cap T^*_{n,0} \ne \emptyset$ and $\theta, \beta \in \overline{\sigma}_n(T^*_{n,0})$, there are at least two (θ, β)-paths with respect to σ_n intersecting $T^*_{n,0}$, a contradiction.

Case 2. $q \ge 1$. In this case, by Definition 5.2(v), we have

(2) $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$-closed with respect to φ_n

So e_1 is colored by some color γ in $\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1}$. By Definition 5.2(i) and (5.9), we have $\gamma \notin \Gamma^q$. Let $\theta \in \overline{\varphi}_n(T_{n,q}) - \overline{\varphi}_n(T^*_{n,q-1})$. Then $\theta \notin \Gamma^{q-1}$ (so $\theta \neq \gamma$) by Definition 5.2(i). Furthermore, θ is closed in $T_{n,q}$ under φ_n by (2). In view of Lemma 6.2, α and θ are $T_{n,q}$-interchangeable under φ_n. So $P_{v_\theta}(\alpha, \theta, \varphi_n) = P_{v_\alpha}(\alpha, \theta, \varphi_n)$ is the unique (α, θ)-path intersecting $T_{n,q}$. Hence $P_{y_1}(\alpha, \theta, \varphi_n) \cap T_{n,q} = \emptyset$. Let $\sigma_n = \varphi_n / P_{y_1}(\alpha, \theta, \varphi_n)$. Then σ_n satisfies all the properties described in (7.3) by Lemma 6.6. Since e_1 is still colored by $\gamma \in \Gamma^{q-1}$ under σ_n and $\gamma \notin \Gamma^q$, we can obtain *T* from $T_{n,q}$ by TAA under σ_n, so *T* is an ETT satisfying MP under σ_n. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). As $P_{y_1}(\theta, \gamma, \sigma_n) \cap T_{n,q} \neq \emptyset$ and $\theta, \gamma \in \overline{\sigma}_n(T_{n,q})$, there are at least two (θ, γ)-paths with respect to σ_n intersecting $T_{n,q}$, contradicting Lemma 6.5(iii) (with σ_n in place of φ_n), because $\theta, \gamma \in \overline{\sigma}_n(T_{n,q})$ and θ is also closed in $T_{n,q}$ under σ_n by (2). Hence Claim 7.1 is justified.

Recall that the path number *p*(*T*) of *T* is the smallest subscript $i \in \{1, 2, ..., p\}$, such that the sequence $(y_i, e_{i+1}, ..., e_p, y_p)$ corresponds to a path in *G*, where $p \geq 2$ by Claim 7.1. Depending on the value of *p*(*T*), we distinguish among three situations, labeled as Situations 7.1, 7.2, and 7.3.

Situation 7.1 $p(T) = 1$. Now $T - V(T^*_{n,q})$ is a path obtained by using TAA under φ_n.

Claim 7.2 *We may assume that* $\overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p) \neq \emptyset$ *for some i with* $1 \leq i \leq p - 1$.

To justify this, let $\alpha \in \overline{\varphi}_n(T(y_{p-1})) \cap \overline{\varphi}_n(y_p)$ (see (7.1)). If $\alpha \in \overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p)$ for some *i* with $1 \leq i \leq p - 1$, we are done. So we assume that

(1) $\alpha \in \overline{\varphi}_n(T^*_{n,q}) \cap \overline{\varphi}_n(y_p)$ and $\alpha \notin \overline{\varphi}_n(y_i)$ for all $1 \leq i \leq p - 1$.
(2) If $\Theta_n = PE$ and $q = 0$, then we may further assume that $\alpha \in \overline{\varphi}_n(T_n)$.

Let us justify (2). By (1), we have $\alpha \in \overline{\varphi}_n(T^*_{n,0})$. Suppose $\alpha \in \overline{\varphi}_n(R_n - V(T_n))$. Then $\alpha \notin \Gamma^0$ by Definition 5.2(i). In view of (7.2), we have $|\overline{\varphi}_n(T_n)| \geq 11 + 2n$ and $|\Gamma^0| \leq 2|D_{n,0}| \leq 2n$. So there exists $\beta \in \overline{\varphi}_n(T_n) - \Gamma^0$. By Lemma 6.1(iv), α and β are $T^*_{n,0}$-interchangeable under φ_n. Thus $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$ and $P_{y_p}(\alpha, \beta, \varphi_n)$ is disjoint from $T^*_{n,0}$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.6 (the second case), σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-weakly stable coloring, and $T^*_{n,0}$ is an ETT satisfying MP with respect to σ_n. Note that *T* can also be obtained from $T^*_{n,0}$ by TAA under σ_n, because $\alpha, \beta \in \overline{\sigma}_n(T^*_{n,0})$. Hence *T* is an ETT satisfying MP under σ_n as well. Since $\alpha, \beta \notin \Gamma^0$ and $\alpha, \beta \notin \overline{\varphi}_n(T(y_{p-1}) - V(T^*_{n,0}))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). As $\beta \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(y_p)$, replacing φ_n by σ_n and α by β if necessary, we see that (2) holds.

Depending on whether α is used by edges in $T - T^*_{n,q}$, we consider two cases.

Case 1. $\alpha \notin \varphi_n\langle T - T^*_{n,q}\rangle$. In this case, let $\beta \in \overline{\varphi}_n(y_{p-1})$. Then β is not used by any edge in $T - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \beta \in D_n$). By (1) and (2), we have $\alpha \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\alpha \in \overline{\varphi}_n(T_n)$ if $q = 0$. It follows from Lemma 6.5 that $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{y_{p-1}}(\alpha, \beta, \varphi_n)$. So $P_{y_p}(\alpha, \beta, \varphi_n)$ is disjoint from $P_{v_\alpha}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \beta \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha, \beta, \varphi_n)$. So $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. Thus T can be obtained from $T^*_{n,q} + e_1$ by TAA and is an ETT satisfying MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). As $\beta \in \overline{\sigma}_n(y_{p-1}) \cap \overline{\sigma}_n(y_p)$, replacing φ_n by σ_n if necessary, we see that Claim 7.2 is true.

Case 2. $\alpha \in \varphi_n\langle T - T^*_{n,q}\rangle$. In this case, let e_j be the edge with the smallest subscript in $T - T^*_{n,q}$ such that $\varphi(e_j) = \alpha$. We distinguish between two subcases according to the value of j.

Subcase 2.1. $j \geq 2$. In this subcase, let $\beta \in \overline{\varphi}_n(y_{j-1})$. Then β is not used by any edge in $T(y_j) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \beta \in D_n$). By (1) and (2), we have $\alpha \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\alpha \in \overline{\varphi}_n(T_n)$ if $q = 0$. It follows from Lemma 6.5 that $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{y_{j-1}}(\alpha, \beta, \varphi_n)$. So $P_{y_p}(\alpha, \beta, \varphi)$ is disjoint from $P_{v_\alpha}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \beta \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha, \beta, \varphi_n)$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ_n and hence is an ETT satisfying MP under σ_n.

Note that $\beta \notin \Gamma^q$ by Definition 5.2(i) and that $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$ by (6.6). If $\alpha \notin \Gamma^q$, then clearly $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. If $\alpha \in \Gamma^q$, say $\alpha \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$, then Definition 5.2(i) implies that $\eta_h \in \overline{\varphi}_n(w)$ for some $w \preceq y_{j-1}$. Since only edges outside $T(w)$ may change colors between α and β as we transform φ_n into σ_n, it follows that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\beta \in \overline{\sigma}_n(y_{j-1}) \cap \overline{\sigma}_n(y_p)$, replacing φ_n by σ_n if necessary, we see that Claim 7.2 holds.

Subcase 2.2. $j = 1$. In this subcase, $\alpha = \varphi(e_1)$. Note that $\alpha \notin \Gamma^q$ by Definition 5.2(i) and (5.9). We propose to show that

(3) there exists a color γ in $\overline{\varphi}_n(T_{n,q}) - \Gamma^q$ if $q \geq 1$ and in $\overline{\varphi}_n(T_n) - \Gamma^0$ if $q = 0$, such that γ is closed in $T^*_{n,q}$ with respect to φ_n.

Let us first assume that $q \geq 1$. By (7.2), we obtain $|\overline{\varphi}_n(T_{n,q})| \geq |\overline{\varphi}_n(T_n)| \geq 2n + 11$ and $|\Gamma^{q-1}| \leq 2|D_{n,q-1}| \leq 2n$. So $|\overline{\varphi}_n(T_{n,q}) - \Gamma^{q-1}| \geq 11$. By Definition 5.2(iii), we have $|\Gamma^q - \Gamma^{q-1}| = 2$. So $|\overline{\varphi}_n(T_{n,q}) - (\Gamma^{q-1} \cup \Gamma^q)| \geq 9$. Let γ be a color in $\overline{\varphi}_n(T_{n,q}) - (\Gamma^{q-1} \cup \Gamma^q)$. By Definition 5.2(v), γ is closed in $T_{n,q}$ with respect to φ_n.

Next we assume that $q = 0$. Again, by (7.2), we have $|\overline{\varphi}_n(T_n)| \geq 2n + 11$ and $|\Gamma^0| \leq 2|D_{n,0}| \leq 2|D_n| \leq 2n$. Let γ be a color in $\overline{\varphi}_n(T_n) - \Gamma^0$ if $\Theta_n \neq PE$ and a color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0$ if $\Theta_n = PE$ (see Definition 5.2(iv)). By Algorithm 3.1 and (5.4), γ is closed in $T^*_{n,0}$ with respect to φ_n. So (3) holds.

By (3) and Lemma 6.5, $P_{v_\alpha}(\alpha, \gamma, \varphi_n) = P_{v_\gamma}(\alpha, \gamma, \varphi_n)$ is the only (α, γ)-path intersecting $T^*_{n,q}$. So $P_{y_p}(\alpha, \gamma, \varphi_n)$ is disjoint from $T^*_{n,q}$ and hence it does not contain e_1. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \gamma, \varphi_n)$. Then σ_n satisfies all the properties described in (7.3) by Lemma 6.6. Moreover, $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for all $u \in V(T(y_{p-1}))$ by (6.6). Since $\alpha, \gamma \in \overline{\varphi}_n(T^*_{n,q})$, we have $\alpha, \gamma \in \overline{\sigma}_n(T^*_{n,q})$. Hence we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n, so T is an ETT satisfying MP under σ_n. Since $\alpha, \gamma \notin \Gamma^q$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since e_1 is outside $P_{y_p}(\alpha, \gamma, \varphi_n)$, we have $\sigma_n(e_1) = \alpha$. As $\gamma \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(v)$ for some $v \in V(T_{n,q})$ and $\alpha \neq \gamma$, the present subcase reduces to Case 1 if $\gamma \notin \sigma_n\langle T - T^*_{n,q}\rangle$ or to Subcase 2.1 if $\gamma \in \sigma_n\langle T - T^*_{n,q}\rangle$. This proves Claim 7.2.

Claim 7.3 *We may assume that* $\overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p) \neq \emptyset$.

To justify this, let $\mathcal{K}$ be the set of all minimum counterexamples (T, φ_n) to Theorem 5.3 (see (6.2)–(6.5)), and let i be the largest subscript with $1 \leq i \leq p-1$, such that there exists a member (T, μ_n) of $\mathcal{K}$ with $\overline{\mu}_n(y_i) \cap \overline{\mu}_n(y_p) \neq \emptyset$; this i exists by Claim 7.2. We aim to show that $i = p-1$. Thus Claim 7.3 follows by replacing φ_n with μ_n, if necessary.

With a slight abuse of notation, we assume that $\overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p) \neq \emptyset$ and assume, on the contrary, that $i \leq p-2$. Let $\alpha \in \overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p)$. Using (6.6) and TAA, we obtain

(1) $\alpha \notin \overline{\varphi}_n(T(y_{i-1}))$, where $T(y_0) = T^*_{n,q}$. So α is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \alpha \in D_n$).

Recall that Definition 5.2 involves $\Gamma^q_h = \{\gamma^q_{h_1}, \gamma^q_{h_2}\}$ for each $\eta_h \in D_{n,q}$. Nevertheless, the proof of this claim only involves one $\eta_h \in D_{n,q}$. For simplicity, we abbreviate its corresponding $\gamma^q_{h_j}$ to γ_j for $j = 1, 2$. By Definition 5.2(i) and (5.9), we have

(2) $\gamma_j \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\gamma_j \in \overline{\varphi}_n(T_n)$ if $q = 0$. Moreover, if $\eta_h \in \overline{\varphi}_n(y_t)$ for some $t \geq 1$, then $\gamma_j \notin \varphi_n\langle T(y_t) - T^*_{n,q}\rangle$ for $j = 1, 2$.

Depending on whether $\alpha \in D_{n,q}$, we consider two cases.

Case 1. $\alpha \notin D_{n,q}$. In this case, let $\theta \in \overline{\varphi}_n(y_{i+1})$. From TAA and (6.6) it follows that

(3) $\theta \notin \overline{\varphi}_n(T(y_i))$, so θ is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \theta \in D_n$).

If $\theta \notin D_{n,q}$, then $\{\alpha, \theta\} \cap D_{n,q} = \emptyset$. By the definitions of D_n and $D_{n,q}$, we have $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$, which together with (1) and (3) implies

$\{\alpha, \theta\} \cap D_n = \emptyset$. Hence $P_{y_i}(\alpha, \theta, \varphi_n) = P_{y_{i+1}}(\alpha, \theta, \varphi_n)$ by Lemma 6.5. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \theta, \varphi_n)$. Since both y_i and y_{i+1} are contained in $T - V(T^*_{n,q})$ and (1) holds, by Lemma 6.6 (the third case), σ_n satisfies all the properties described in (7.3). Furthermore, T is also an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\theta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(y_{i+1})$, we reach a contradiction to the maximality assumption on i.

So we may assume that $\theta \in D_{n,q}$. Let $\theta = \eta_h \in D_{n,q}$. In view of (2) and Lemma 6.5, we obtain $P_{v_{\gamma_1}}(\alpha, \gamma_1, \varphi_n) = P_{y_i}(\alpha, \gamma_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \gamma_1, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \gamma_1, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \alpha \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha, \gamma_1, \varphi_n)$. By (6.6), (1) and (2), we have $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma_n(f) = \varphi_n(f)$ for each edge f in $T(y_{i+1})$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ_n, and hence is an ETT satisfying MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), with $\gamma_1 \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T_{n,q})$.

Using (2) and Lemma 6.5, we obtain $P_{v_{\gamma_1}}(\eta_h, \gamma_1, \sigma_n) = P_{y_{i+1}}(\eta_h, \gamma_1, \sigma_n)$, which is disjoint from $P_{y_p}(\eta_h, \gamma_1, \sigma_n)$. Let $\sigma'_n = \sigma_n / P_{y_p}(\eta_h, \gamma_1, \sigma_n)$. By Lemma 6.6, σ'_n satisfies all the properties described in (7.3) (with σ'_n in place of σ_n). In particular, if $e_1 = f_n$ and $\sigma_n(e_1) = \eta_h \in D_n$, then $\sigma'_n(e_1) = \sigma_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_h, \gamma_1, \sigma_n)$. By (6.6), (2) and (3), we have $\overline{\sigma}'_n(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma'_n(f) = \sigma_n(f)$ for each edge f in $T(y_{i+1})$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ'_n, and hence is an ETT satisfying MP under σ'_n. Furthermore, since $\eta_h \in \overline{\sigma}'_n(y_{i+1})$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ'_n, with the same Γ-sets as those under φ_n. Therefore (T, σ'_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\eta_h \in \overline{\sigma}'_n(y_p) \cap \overline{\sigma}'_n(y_{i+1})$, we reach a contradiction to the maximality assumption on i.

Case 2. $\alpha \in D_{n,q}$. In this case, let $\alpha = \eta_h \in D_{n,q}$. Then $\Gamma^q_h = \{\gamma_1, \gamma_2\}$ (see the paragraph above (2)). Renaming subscript if necessary, we may assume that $\varphi_n(e_{i+1}) \neq \gamma_1$. By (1) and (2), we have

(4) $\gamma_1 \notin \varphi_n\langle T(y_{i+1}) - T^*_{n,q}\rangle$ and η_h is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

By (4) and Lemma 6.5, we obtain $P_{v_{\gamma_1}}(\eta_h, \gamma_1, \varphi_n) = P_{y_i}(\eta_h, \gamma_1, \varphi_n)$, which is disjoint from the path $P_{y_p}(\eta_h, \gamma_1, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\eta_h, \gamma_1, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_h \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_h, \gamma_1, \varphi_n)$. By (6.6) and (4), we have $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma_n(f) = \varphi_n(f)$ for each edge f in $T(y_{i+1})$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ_n, and hence is an ETT satisfying MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), with $\gamma_1 \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T_{n,q})$. Let $\theta \in \overline{\sigma}_n(y_{i+1})$. From TAA we see that

(5) θ is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$ under σ_n, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\sigma_n(e_1) = \theta \in D_n$).

By (6.6), we have $\theta \neq \gamma_1$. Using (4) and Lemma 6.5, we get $P_{v_{\gamma_1}}(\theta, \gamma_1, \sigma_n) = P_{y_{i+1}}(\theta, \gamma_1, \sigma_n)$. Let $\sigma'_n = \sigma_n / P_{y_p}(\theta, \gamma_1, \sigma_n)$. By Lemma 6.6, σ'_n satisfies all the properties described in (7.3) (with σ'_n in place of σ_n). In particular, if $e_1 = f_n$ and $\sigma_n(e_1) = \theta \in D_n$, then $\sigma'_n(e_1) = \sigma_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\theta, \gamma_1, \sigma_n)$. From (6.6) and (4) we deduce that $\overline{\sigma}'_n(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$, and $\sigma'_n(f) = \sigma_n(f)$ for each edge f in $T(y_{i+1})$. So T can also be obtained from $T^*_{n,q} + e_1$ by TAA under σ'_n, and hence is an ETT satisfying MP under σ'_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of those under σ'_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ'_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\theta \in \overline{\sigma}'_n(y_p) \cap \overline{\sigma}'_n(y_{i+1})$, we reach a contradiction to the maximality assumption on i. Hence Claim 7.3 is established.

By Claim 7.1, $p \geq 2$. By Claim 7.3, $\overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p) \neq \emptyset$. Let $\alpha \in \overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p)$ and $\beta = \varphi_n(e_p)$. Let σ_n be obtained from φ_n by recoloring e_p with α and let $T' = T(y_{p-1})$. Then $\beta \in \overline{\sigma}_n(y_{p-1}) \cap \overline{\sigma}_n(T'(y_{p-2}))$ and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under σ_n. So (T', σ_n) is a counterexample to Theorem 5.3 (see (6.2)–(6.4)), which violates the minimality assumption (6.5) on (T, φ_n). This completes our discussion about Situation 7.1.

Situation 7.2 $p(T) = p$. Now e_p is not incident to y_{p-1}.

By (7.1), there exists a color $\alpha \in \overline{\varphi}_n(T(y_{p-1})) \cap \overline{\varphi}_n(y_p)$. We divide this situation into 3 cases and further into 6 subcases (see Figure 4), depending on whether $v_\alpha = y_{p-1}$ or $\alpha \in D_{n,q}$. Our proof of Subcase 1.1 is self-contained. Yet, in our discussion Subcase 1.2 may be redirected to Subcase 1.1 and Subcase 2.1, and Subcase 2.1 may be redirected to Subcase 1.1, etc. Figure 4 illustrates such redirections (note that no cycling occurs).

Throughout this situation we reserve the symbol θ for $\varphi_n(e_p)$. Clearly, $\theta \neq \alpha$.

Case 1. $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_{p-1})$ and $\alpha \in D_{n,q}$.

Let $\alpha = \eta_m \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma^q_{m_1}$ and $\gamma^q_{m_2}$ in Γ^q_m (see Definition 5.2) to γ_1 and γ_2, respectively. Since $\eta_m \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_{p-1})$, from TAA and Definition 5.2(i) we see that

(1) $\gamma_1, \gamma_2 \notin \varphi_n\langle T(y_{p-1}) - T^*_{n,q}\rangle$ and η_m is not used by any edge in $T - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_m \in D_{n,q} \subseteq D_n$).

By (1) and Lemma 6.5 (with respect to (T, φ_n)), we have

(2) $P_{v_{\gamma_j}}(\eta_m, \gamma_j, \varphi_n) = P_{y_{p-1}}(\eta_m, \gamma_j, \varphi_n)$ for $j = 1, 2$.

Let us consider two subcases according to whether $\theta \in \overline{\varphi}_n(y_{p-1})$.

Subcase 1.1. $\theta \notin \overline{\varphi}_n(y_{p-1})$.

In our discussion about this subcase, we shall appeal to the following two tree-sequences:

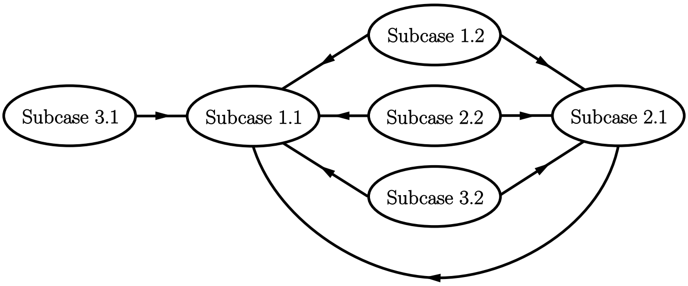


Fig. 4 Redirections

- $T^- = (T^*_{n,q}, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)$ and
- $T^* = (T^*_{n,q}, e_1, y_1, e_2, \ldots, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1})$.

Note that T^- is obtained from T by deleting y_{p-1} and T^* arises from T by interchanging the order of (e_{p-1}, y_{p-1}) and (e_p, y_p). We propose to show that both T^- and T^* are ETTs corresponding to φ_n. By the hypothesis of the present subcase, $\varphi_n(e_p) = \theta \notin \overline{\varphi}_n(y_{p-1})$. Thus if $T(y_{p-2}) \neq T_n$, then $\varphi_n(e_{p-1})$ and $\varphi_n(e_p)$ are in $\overline{\varphi}_n(T(y_{p-2}))$, and therefore both T^- and T^* can be obtained from $T(y_{p-2})$ by using TAA under φ_n. So we assume that $T(y_{p-2}) = T_n$. If $\Theta_n = RE$ or *SE*, we must have that e_p is incident to T_n and therefore $\theta \in \overline{\varphi}_n(y_{p-1})$, because T_n is closed under φ_n. Thus we deduce that $\Theta_n = PE$ because $\theta \notin \overline{\varphi}_n(y_{p-1})$. Hence $\varphi_n(e_{p-1})$ and $\varphi_n(e_p)$ are in $\overline{\varphi}_n(T_n)$ following Algorithm 3.1, and therefore both T^- and T^* can be obtained from $T(y_{p-2})$ by using TAA under φ_n as well. Therefore both T^- and T^* are ETTs corresponding to φ_n. In view of the maximum property enjoyed by T, we further conclude that both T^- and T^* are ETTs satisfying MP with respect to φ_n.

Let us first assume that $\theta \notin \Gamma^q$. Now it is easy to see that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^-$ is a good hierarchy of T^- under φ_n, with the same Γ-sets (see Definition 5.2) as T. (If $\theta \in \Gamma^q$, say $\theta \in \Gamma^q_h$, and $\eta_h \in \overline{\varphi}_n(y_{p-1})$, then T^- no longer satisfies Definition 5.2(i).) Observe that $\gamma_1 \notin \overline{\varphi}_n(y_p)$, for otherwise, γ_1 is missing at two vertices in T^-. Thus (T^-, φ_n) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, φ_n). Let us turn to considering T^*. Since $\theta \notin \overline{\varphi}_n(y_{p-1})$ and $\theta \notin \Gamma^q$, it is clear that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^*$ is a good hierarchy of T^* under φ_n, with the same Γ-sets as T. Moreover, by (1), we have $\gamma_1 \notin \varphi_n\langle T^*(y_p) - T^*_{n,q}\rangle$. It follows from Lemma 6.5 (with respect to (T^*, φ_n)) that $P_{v_{\gamma_1}}(\eta_m, \gamma_1, \varphi_n) = P_{y_p}(\eta_m, \gamma_1, \varphi_n)$, contradicting (2).

Next we assume that $\theta \in \Gamma^q$. Then $\theta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$. If $\eta_h \notin \overline{\varphi}_n(y_{p-1})$, then $\eta_h \in \overline{\varphi}_n(T(y_{p-2}))$ by Definition 5.2(i). So we can still ensure that both T^- and T^* have good hierarchies under φ_n. Thus, using the same argument as employed in the preceding paragraph, we can reach a contradiction. Hence we may assume that $\eta_h \in \overline{\varphi}_n(y_{p-1})$.

Clearly, $\theta \neq \gamma_1$ or γ_2. Renaming subscripts if necessary, we may assume that

(3) $\theta \neq \gamma_2$.

Since $P_{v_{\gamma_2}}(\eta_m, \gamma_2, \varphi_n) = P_{y_{p-1}}(\eta_m, \gamma_2, \varphi_n)$ by (2), this path is disjoint from $P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_m \in D_n$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. By (1) and (3), we have $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1; thereby T is an ETT satisfying MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which γ_2 is missing at two vertices.

By Lemma 6.3, we have $|\overline{\mu}_1(T(y_{p-2})) - \overline{\mu}_1(T^*_{n,0} - V(T_n)) - \mu_1\langle T(y_{p-2}) - T^*_{n,q}\rangle| \geq 2n + 11$, where $T(y_0) = T^*_{n,q}$. It follows that $|\overline{\mu}_1(T(y_{p-2})) - \overline{\mu}_1(T^*_{n,0} - V(T_n)) - \mu_1\langle T - T^*_{n,q}\rangle| \geq 2n + 9$. As $|\Gamma^q| \leq 2|D_{n,q}| \leq 2|D_n| \leq 2n$ by Lemma 3.5, using (6.6) we obtain

(4) there exists a color β in $\overline{\mu}_1(T(y_{p-2})) - \overline{\mu}_1(T^*_{n,0} - V(T_n)) - \mu_1\langle T - T^*_{n,q}\rangle - \Gamma^q$.

By Lemma 6.5 (with γ_2 in place of α), $P_{v_{\gamma_2}}(\beta, \gamma_2, \mu_1) = P_{v_\beta}(\beta, \gamma_2, \mu_1)$, so it is disjoint from $P_{y_p}(\beta, \gamma_2, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\beta, \gamma_2, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). By (1), (3) and (4), we have $\beta, \gamma_2 \notin \mu_1\langle T(y_p) - T^*_{n,q}\rangle$. So $\mu_2(f) = \mu_1(f)$ for each $f \in E(T)$ and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. Hence we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_2; thereby T satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which β is missing at two vertices. Since $\theta \in \Gamma^q_h$ and $\eta_h \in \overline{\varphi}_n(y_{p-1}) = \overline{\mu}_1(y_{p-1}) = \overline{\mu}_2(y_{p-1})$, we obtain

(5) $\theta \notin \mu_2\langle T(y_{p-1}) - T^*_{n,q}\rangle$.

By (4), we also have

(6) $\beta \notin \mu_2\langle T - T^*_{n,q}\rangle$.

It follows from (5) and Lemma 6.5 (with θ in place of α) that $P_{v_\theta}(\beta, \theta, \mu_2) = P_{v_\beta}(\beta, \theta, \mu_2)$, so it is disjoint from $P_{y_p}(\beta, \theta, \mu_2)$. Finally, set $\mu_3 = \mu_2 / P_{y_p}(\beta, \theta, \mu_2)$. By Lemma 6.6, μ_3 satisfies all the properties described in (7.3) (with μ_3 in place of σ_n). From (5) and (6) we see that T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_3. Hence T is an ETT satisfying MP under μ_3. Note that $\mu_3(f) = \mu_2(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_3(e_p) = \beta$, and $\overline{\mu}_3(u) = \overline{\mu}_2(u)$ for each $u \in V(T(y_{p-1}))$. Moreover, $\beta \notin \Gamma^q$ by (4). It is a routine matter to check that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_3, with the same Γ-sets as those under μ_2. Since $\mu_3(e_p) = \beta \notin \Gamma^q$ and $v_\beta \prec y_{p-1}$, we see that T^- has a good hierarchy and is an ETT satisfying MP with respect to μ_3. As θ is missing at two vertices in T^-, we conclude that (T^-, μ_3) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which contradicts the minimality assumption (6.4) or (6.5) on (T, φ_n).

Subcase 1.2. $\theta \in \overline{\varphi}_n(y_{p-1})$.

In this subcase, from (6.6) and TAA we see that

(7) $\theta \notin \overline{\varphi}_n(T(y_{p-2}))$, so $\theta \notin \Gamma^q$ and hence $\theta \neq \gamma_1, \gamma_2$. Furthermore, θ is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \theta \in D_n$).

Since $P_{v_{\gamma_1}}(\eta_m, \gamma_1, \varphi_n) = P_{y_{p-1}}(\eta_m, \gamma_1, \varphi_n)$ by (2), this path is disjoint from $P_{y_p}(\eta_m, \gamma_1, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\eta_m, \gamma_1, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_m \in D_n$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_m, \gamma_1, \varphi_n)$. By (1) and (6.6), we have $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1, and hence T satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which γ_1 is missing at two vertices.

From (1) and the definition of μ_1, we see that

(8) $\gamma_1 \notin \mu_1\langle T - T^*_{n,q}\rangle$.

From (8) and Lemma 6.5 (with γ_1 in place of α), we deduce that $P_{v_{\gamma_1}}(\theta, \gamma_1, \mu_1) = P_{y_{p-1}}(\theta, \gamma_1, \mu_1)$, which is disjoint from $P_{y_p}(\theta, \gamma_1, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\theta, \gamma_1, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \theta \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\theta, \gamma_1, \mu_1)$. In view of (7), (8) and (6.6), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_2(e_p) = \gamma_1$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. Moreover, $\theta \notin \Gamma^q$. So T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_2, and hence is an ETT satisfying MP under μ_2. It is a routine matter to check that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\theta \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$ and $\mu_2(e_p) = \gamma_1 \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 1.1 if $\theta \in D_{n,q}$ and reduces to Subcase 2.1 (to be discussed below) if $\theta \notin D_{n,q}$.

Case 2. $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_{p-1})$ and $\alpha \notin D_{n,q}$.

By the definitions of D_n and $D_{n,q}$, we have $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$. Using (6.6) and this set inclusion, we obtain

(9) $\alpha \notin \overline{\varphi}_n(T(y_{p-2}))$ and $\alpha \notin D_n$. So $\alpha \notin \varphi_n\langle T - T^*_{n,q}\rangle$ by TAA (see, for instance, (1)).

Recall that $T(y_0) = T^*_{n,q}$ and $\theta = \varphi_n(e_p)$. We consider two subcases according to whether $\theta \in \overline{\varphi}_n(y_{p-1})$.

Subcase 2.1. $\theta \notin \overline{\varphi}_n(y_{p-1})$.

In our discussion about this subcase, we shall also appeal to the following two tree-sequences:

- $T^- = (T^*_{n,q}, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)$ and
- $T^* = (T^*_{n,q}, e_1, y_1, e_2, \ldots, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1})$.

As stated in Subcase 1.1, T^- is obtained from T by deleting y_{p-1} and T^* arises from T by interchanging the order of (e_{p-1}, y_{p-1}) and (e_p, y_p). Furthermore, both T^- and T^* are ETTs satisfying MP with respect to φ_n. Observe that

(10) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T^*$ is a good hierarchy of T^* under φ_n, unless $\theta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$ such that $\eta_h \in \overline{\varphi}_n(y_{p-1})$.

Let us first assume that the exceptional case in (10) does not occur; that is, there exists no $\eta_h \in D_{n,q}$ such that $\eta_h \in \overline{\varphi}_n(y_{p-1})$ and $\theta \in \Gamma^q_h$. It is easy to see that now $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^-$ is a good hierarchy of T^- under φ_n.

By Lemma 6.3, we have $|\overline{\varphi}_1(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_1\langle T(y_{p-2}) - T^*_{n,q}\rangle| \geq 2n + 11$ holds, where $T(y_0) = T^*_{n,q}$. Since $|D_n| \leq n$ by Lemma 3.5, using (6.6) we obtain

(11) there exists a color β in $\overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - D_n$.

Note that $\beta \notin \overline{\varphi}_n(y_p)$, for otherwise, (T^-, φ_n) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, σ_n). Since $\alpha, \beta \notin \varphi_n\langle T - T^*_{n,q}\rangle$ and $\alpha, \beta \notin D_n$ by (9) and (11), applying Lemma 6.5 (with α, β switched in the lemma) to (T, φ_n) and (T^*, φ_n), respectively, we obtain $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{p-1}}(\alpha, \beta, \varphi_n)$ and $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_p}(\alpha, \beta, \varphi_n)$, a contradiction.

So we assume that the exceptional case in (10) occurs; that is, there exists $\eta_h \in D_{n,q}$ such that $\eta_h \in \overline{\varphi}_n(y_{p-1})$ and $\theta \in \Gamma^q_h$. For simplicity, we abbreviate the two colors $\gamma^q_{h_1}$ and $\gamma^q_{h_2}$ in Γ^q_h (see Definition 5.2) to γ_1 and γ_2, respectively. Renaming subscripts if necessary, we may assume that $\theta = \gamma_1$. By Definition 5.2(i) and TAA, we have

(12) $\gamma_2 \notin \varphi_n\langle T - T^*_{n,q}\rangle$ and η_h is not used by any edge in $T - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

By (12) and Lemma 6.5 (with α in place of β), we obtain $P_{v_{\gamma_2}}(\alpha, \gamma_2, \varphi_n) = P_{y_{p-1}}(\alpha, \gamma_2, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \gamma_2, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\alpha, \gamma_2, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Since $\alpha, \gamma_2 \notin \varphi_n\langle T(y_p) - T^*_{n,q}\rangle$ by (9) and (12), we have $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1, and hence T is an ETT satisfying MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which γ_2 is missing at two vertices.

If $\eta_h \in \overline{\mu}_1(y_p)$, then $\eta_h \in \overline{\mu}_1(y_p) \cap \overline{\mu}_1(y_{p-1})$, $\eta_h \in D_{n,q}$, and $\mu_1(e_p) = \gamma_1 \notin \overline{\varphi}_n(y_{p-1})$. Thus the present subcase reduces to Subcase 1.1. So we may assume that $\eta_h \notin \overline{\mu}_1(y_p)$. By (12) and the definition of μ_1, we have

(13) $\gamma_2 \notin \mu_1\langle T - T^*_{n,q}\rangle$ and η_h is not used by any edge in $T - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_h \in D_n$).

By (13) and Lemma 6.5 (with γ_2 in place of α), we obtain $P_{v_{\gamma_2}}(\eta_h, \gamma_2, \mu_1) = P_{y_{p-1}}(\eta_h, \gamma_2, \mu_1)$, which is disjoint from $P_{y_p}(\eta_h, \gamma_2, \mu_1)$. Let

$\mu_2 = \mu_1/P_{y_p}(\eta_h, \gamma_2, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \eta_h \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_h, \gamma_2, \mu_1)$. By (13), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T)$ and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_2, and hence T is an ETT satisfying MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\eta_h \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_h \in D_{n,q}$, and $\mu_2(e_p) = \gamma_1 \notin \overline{\mu}_2(y_{p-1})$. Thus the present subcase reduces to Subcase 1.1.

Subcase 2.2. $\theta \in \overline{\varphi}_n(y_{p-1})$.

Let us first assume that $\theta \in D_{n,q}$; that is, $\theta = \eta_m$ for some $\eta_m \in D_{n,q}$. For simplicity, we use ε_1 and ε_2 to denote the two colors $\gamma^q_{m_1}$ and $\gamma^q_{m_2}$ in Γ^q_m (see Definition 5.2), respectively. By Definition 5.2(i) and TAA, we have

(14) $\varepsilon_1, \varepsilon_2 \notin \varphi_n\langle T - T^*_{n,q}\rangle$ and η_m is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_m \in D_n$).

By (14) and Lemma 6.5, we obtain $P_{v_{\varepsilon_1}}(\alpha, \varepsilon_1, \varphi_n) = P_{y_{p-1}}(\alpha, \varepsilon_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. Let $\mu_1 = \varphi_n/P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). By (9) and (14), we have

(15) $\alpha, \varepsilon_1 \notin \mu_1\langle T - T^*_{n,q}\rangle$ and η_m is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_m \in D_n$).

So $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. Thus T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_1, and hence is an ETT satisfying MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which ε_1 is missing at two vertices.

By (15) and Lemma 6.5 (with ε_1 in place of α), we obtain $P_{v_{\varepsilon_1}}(\eta_m, \varepsilon_1, \mu_1) = P_{y_{p-1}}(\eta_m, \varepsilon_1, \mu_1)$, which is disjoint from $P_{y_p}(\eta_m, \varepsilon_1, \mu_1)$. Let $\mu_2 = \mu_1/P_{y_p}(\eta_m, \varepsilon_1, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \eta_m \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_m, \varepsilon_1, \mu_1)$. In view of (15), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_2(e_p) = \varepsilon_1$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. So T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_2, and hence satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\eta_m \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_m \in D_{n,q}$, and $\mu_2(e_p) = \varepsilon_1 \notin \overline{\mu}_2(y_{p-1})$. Thus the present subcase reduces to Subcase 1.1.

Next we assume that $\theta \notin D_{n,q}$. Set $T(y_0) = T^*_{n,q}$. We propose to show that

(16) there exists a color $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - D_{n,q}$, such that either $\beta \notin \Gamma^q$ or $\beta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{p-2}))$.

To justify this, note that if $|\overline{\varphi}_n(T(y_{p-2}))-\overline{\varphi}_n(T^*_{n,0}-V(T_n))-\varphi_n\langle T(y_{p-2})-T^*_{n,q}\rangle -(\Gamma^q\cup D_{n,q})|\geq 5$, then $|\overline{\varphi}_n(T(y_{p-2}))-\overline{\varphi}_n(T^*_{n,0}-V(T_n))-\varphi_n\langle T-T^*_{n,q}\rangle -(\Gamma^q\cup D_{n,q})|\geq 3$, because $T-T(y_{p-2})$ contains precisely two edges. Thus there exists a color $\beta\in\overline{\varphi}_n(T(y_{p-2}))-\overline{\varphi}_n(T^*_{n,0}-V(T_n))-\varphi_n\langle T-T^*_{n,q}\rangle-D_{n,q}$, such that $\beta\notin\Gamma^q$.

So we assume that $|\overline{\varphi}_n(T(y_{p-2}))-\overline{\varphi}_n(T^*_{n,0}-V(T_n))-\varphi_n\langle T(y_{p-2})-T^*_{n,q}\rangle -(\Gamma^q\cup D_{n,q})|\leq 4$. By Lemma 6.3, there exist 7 distinct colors $\eta_h\in D_{n,q}\cap\overline{\varphi}_n(T(y_{p-2}))$ such that $(\Gamma^q_h\cup\{\eta_h\})\cap\varphi_n\langle T(y_{p-2})-T^*_{n,q}\rangle=\emptyset$. Let β be an arbitrary color in such a Γ^q_h. From Definition 5.2, we see that $\Gamma^q_h\subseteq\overline{\varphi}_n(T^*_{n,q})\subseteq\overline{\varphi}_n(T(y_{p-2}))$, $\Gamma^q_h\cap\overline{\varphi}_n(T^*_{n,0}-V(T_n))=\emptyset$, and $\Gamma^q_h\cap D_{n,q}=\emptyset$ (see (5.7)). So $\beta\in\overline{\varphi}_n(T(y_{p-2})) -\overline{\varphi}_n(T^*_{n,0}-V(T_n))-\varphi_n\langle T(y_{p-2})-T^*_{n,q}\rangle-D_{n,q}$. Since $T-T(y_{p-2})$ contains precisely two edges, there exists $\beta\in\overline{\varphi}_n(T(y_{p-2}))-\overline{\varphi}_n(T^*_{n,0}-V(T_n))-\varphi_n \langle T-T^*_{n,q}\rangle-D_{n,q}$, such that $\beta\in\Gamma^q_h$ for some $\eta_h\in D_{n,q}\cap\overline{\varphi}_n(T(y_{p-2}))$. Hence (16) is established.

By the definitions of D_n and $D_{n,q}$, we have $\overline{\varphi}_n(T_n)\cup D_n\subseteq\overline{\varphi}_n(T^*_{n,q})\cup D_{n,q}$. By (16), $\beta\notin\overline{\varphi}_n(T^*_{n,0}-V(T_n))\cup D_{n,q}$. It follows from these two observations that

(17) if $q\geq 1$, then $\beta\in\overline{\varphi}_n(T^*_{n,q})$ or $\beta\notin D_n$; if $q=0$, then $\beta\in\overline{\varphi}_n(T_n)$ or $\beta\notin D_n$.

By (9), (16), (17) and Lemma 6.5, we obtain $P_{v_\beta}(\alpha,\beta,\varphi_n)=P_{y_{p-1}}(\alpha,\beta,\varphi_n)$, which is disjoint from $P_{y_p}(\alpha,\beta,\varphi_n)$. Let $\mu_3=\varphi_n/P_{y_p}(\alpha,\beta,\varphi_n)$. By Lemma 6.6, μ_3 satisfies all the properties described in (7.3) (with μ_3 in place of σ_n). By (9) and (16), we have $\alpha,\beta\notin\varphi_n\langle T-T^*_{n,q}\rangle$. So

(18) $\alpha,\beta\notin\mu_3\langle T-T^*_{n,q}\rangle$,

$\mu_3(f)=\varphi_n(f)$ for each $f\in E(T)$, and $\overline{\mu}_3(u)=\overline{\varphi}_n(u)$ for each $u\in V(T(y_{p-1}))$. Thus we can obtain T from $T^*_{n,q}+e_1$ by using TAA under μ_3, and hence T is an ETT satisfying MP under μ_3. Furthermore, $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be a good hierarchy of T under μ_3, with the same Γ-sets as those under φ_n. Therefore, (T,μ_3) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which β is missing at two vertices.

Since $\theta\in\overline{\varphi}_n(y_{p-1})$, it follows from (6.6) that $\theta\notin\overline{\varphi}_n(T^*_{n,q})$. By assumption, $\theta\notin D_{n,q}$. As $\overline{\varphi}_n(T_n)\cup D_n\subseteq\overline{\varphi}_n(T^*_{n,q})\cup D_{n,q}$, we obtain

(19) $\theta\notin D_n$ and hence $\theta\notin\mu_3\langle T(y_{p-1})-T^*_{n,q}\rangle$ by TAA.

By (17)-(19) and Lemma 6.5, we obtain $P_{v_\beta}(\theta,\beta,\mu_3)=P_{y_{p-1}}(\theta,\beta,\mu_3)$, which is disjoint from $P_{y_p}(\theta,\beta,\mu_3)$. Let $\mu_4=\mu_3/P_{y_p}(\theta,\beta,\mu_3)$. By Lemma 6.6, μ_4 satisfies all the properties described in (7.3) (with μ_4 in place of σ_n). By (18) and (19), we have $\mu_4(f)=\mu_3(f)$ for each $f\in E(T(y_{p-1}))$ and $\overline{\mu}_4(u)=\overline{\mu}_3(u)$ for each $u\in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q}+e_1$ by using TAA under μ_4, and hence T is an ETT satisfying MP under μ_4. Since either $\beta\notin\Gamma^q$ or $\beta\in\Gamma^q_h$ for some $\eta_h\in D_{n,q}\cap\overline{\mu}_3(T(y_{p-2}))$ by (16), it follows that $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be a good hierarchy of T under μ_4, with the same Γ-sets as those under μ_3. Therefore, (T,μ_4) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\theta\in\overline{\mu}_4(y_p)\cap\overline{\mu}_4(y_{p-1})$, $\theta\notin D_{n,q}$, and $\mu_4(e_p)=\beta\notin\overline{\mu}_4(y_{p-1})$. Thus the present subcase reduces to Subcase 2.1.

Case 3. $\alpha\in\overline{\varphi}_n(y_p)\cap\overline{\varphi}_n(v)$ for some vertex $v\prec y_{p-1}$.

Set $T(y_0)=T^*_{n,q}$. Let us first impose some restrictions on α.

(20) We may assume that $\alpha \in \overline{\varphi}_n(T(y_{p-2})) - \varphi_n\langle T - T^*_{n,q}\rangle$, such that either $\alpha \notin D_{n,q} \cup \Gamma^q$ if $q \geq 1$ and $\alpha \notin D_n \cup \Gamma^0$ if $q = 0$, or α is some $\eta_h \in D_{n,q}$ satisfying $\Gamma^q_h \cap \varphi_n\langle T - T^*_{n,q}\rangle = \emptyset$.

To justify this, note that if $|\overline{\varphi}_n(T(y_{p-2}))\overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n \langle T(y_{p-2}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})| \geq 5$, then $|\overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})| \geq 3$, because $T - T(y_{p-2})$ contains precisely two edges. Thus there exists a color $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$. Clearly, $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \varphi_n\langle T - T^*_{n,q}\rangle$ and $\beta \notin D_{n,q} \cup \Gamma^q$ if $q \geq 1$ and $\beta \notin D_n \cup \Gamma^0$ if $q = 0$ (note that $\beta \notin D_n$ because $\beta \notin \overline{\varphi}_n(T^*_{n,0} - V(T_n)) \cup D_{n,0}$).

If $|\overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{p-2}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})| \leq 4$, then, by Lemma 6.3, there exist 7 distinct colors $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{p-2}))$ such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_{p-2}) - T^*_{n,q}\rangle = \emptyset$. Since $T - T(y_{p-2})$ contains precisely two edges, there exists one of these η_h, denoted by β, such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T - T^*_{n,q}\rangle = \emptyset$.

Combining the above observations, we conclude that

(21) there exists $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \varphi_n\langle T - T^*_{n,q}\rangle$, such that either $\beta \notin D_{n,q} \cup \Gamma^q$ if $q \geq 1$ and $\beta \notin D_n \cup \Gamma^0$ if $q = 0$, or β is some $\eta_h \in D_{n,q}$ satisfying $\Gamma^q_h \cap \varphi_n\langle T - T^*_{n,q}\rangle = \emptyset$.

If $\beta \in \overline{\varphi}_n(y_p)$, then (20) holds by replacing α with β (recall the hypothesis of the present case). So we assume hereafter that $\beta \notin \overline{\varphi}_n(y_p)$. Let $Q = P_{y_p}(\alpha, \beta, \varphi_n)$ and let $\sigma_n = \varphi_n/Q$. We propose to show that one of the following statements (a) and (b) holds:

(a) σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable coloring, T is also an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of T under σ_n, with the same Γ-sets (see Definition 5.2) as those under φ_n. Moreover, (20) holds with respect to (T, σ_n).
(b) There exists an ETT T' satisfying MP with respect to φ_n, such that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. Moreover, $V(T')$ is not elementary with respect to φ_n and $p(T') < p(T)$.

Note that if (b) holds, then (T', φ_n) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) on (T, φ_n).

Let us first assume that Q is vertex-disjoint from $T(y_{p-1})$. By Lemma 5.8, σ_n is both $(T(y_{p-1}), D_n, \varphi_n)$-stable and $(T(y_{p-1}), \varphi_n)$-invariant. If $\Theta_n = PE$, then σ_n is also $(T_n \oplus R_n, D_n, \varphi_n)$-stable. Furthermore, $T(y_{p-1})$ is an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(y_{p-1})$ is a good hierarchy of $T(y_{p-1})$, with the same Γ-sets as T under σ_n. By definition, σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable coloring. By the hypothesis of Case 3 and assumption on β, we have $\varphi_n(e_p) \neq \alpha, \beta$. Thus it is clear that (a) is true, and (20) follows if we replace φ_n by σ_n and α by β.

Next we assume that Q and $T(y_{p-1})$ have vertices in common. Let u be the first vertex of Q contained in $T(y_{p-1})$ as we traverse Q from y_p. Define $T' = T(y_{p-1}) \cup Q[u, y_p]$ if $u = y_{p-1}$ and $T' = T(y_{p-2}) \cup Q[u, y_p]$ otherwise. By the hypothesis of Case 3 and (21), we have $\alpha, \beta \in \overline{\varphi}_n(T(y_{p-2}))$. So T' can be obtained from $T(y_{p-2})$ by using TAA under φ_n, with $p(T') < p(T)$. It follows that T' is an ETT satisfying MP with respect to φ_n.

By Definition 5.2, we have $D_{n,q} \cap \Gamma^q = \emptyset$ (see (5.7)). Thus

(22) $\beta \notin \Gamma^q$ by (21).

Let us proceed by considering three possibilities for α.

- $\alpha \notin \Gamma^q$. Since both α and β are outside Γ^q (see (22)), it is easy to see that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. Hence (b) holds.
- $\alpha \in \Gamma^q \cap \varphi_n\langle T - T^*_{n,q}\rangle$. Let $\alpha \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$. Since $\varphi(e_p) \neq \alpha$, we have $\alpha \in \varphi_n\langle T(y_{p-1}) - T^*_{n,q}\rangle$. Hence $\eta_h \in \overline{\varphi}_n(T(y_{p-2}))$ by Definition 5.2(i). Furthermore, $\beta \in \overline{\varphi}_n(T(y_{p-2}))$ and $\beta \notin \Gamma^q$ by (21) and (22). Therefore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. Hence (b) holds.
- $\alpha \in \Gamma^q - \varphi_n\langle T - T^*_{n,q}\rangle$. By the definition of Γ^q, we have $\alpha \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\alpha \in \overline{\varphi}_n(T_n)$ if $q = 0$. It follows from Lemma 6.5 that $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$, which is disjoint from Q. By Lemma 6.6, $\sigma_n = \varphi_n/Q$ satisfies all the properties described in (7.3). Since $\alpha, \beta \notin \varphi_n\langle T - T^*_{n,q}\rangle$ by the assumption on α, (21) and (6.6), we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n, and hence T is an ETT satisfying MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which β is missing at two vertices. So (a) holds and therefore (20) is established by replacing φ_n with σ_n and β with α.

Let α be a color as specified in (20). Recall that $\theta = \varphi_n(e_p)$. We consider two subcases according to whether $\theta \in \overline{\varphi}_n(y_{p-1})$.

Subcase 3.1. $\theta \notin \overline{\varphi}_n(y_{p-1})$.

Consider the tree-sequence $T^- = (T^*_{n,q}, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)$. As stated in Subcase 1.1, T^- arises from T by deleting y_{p-1}, and T^- is an ETT satisfying MP with respect to φ_n. Observe that

(23) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^-$ is a good hierarchy of T^- under φ_n, unless $\theta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ such that $\eta_m \in \overline{\varphi}_n(y_{p-1})$.

It follows that the exceptional case stated in (23) must occur, for otherwise, (T^-, φ_n) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, φ_n). So $\theta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ such that $\eta_m \in \overline{\varphi}_n(y_{p-1})$.

Since $\alpha \in \overline{\varphi}_n(T(y_{p-2}))$, we have $\alpha \neq \eta_m$ by (6.6). From Definition 5.2(i), we see that

(24) $\theta \notin \varphi_n\langle T(y_{p-1}) - T^*_{n,q}\rangle$.

By the definition of Γ^q, we have $\theta \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\theta \in \overline{\varphi}_n(T_n)$ if $q = 0$. Thus, by (20), (24) and Lemma 6.5, we obtain $P_{v_\alpha}(\alpha, \theta, \varphi_n) = P_{v_\theta}(\alpha, \theta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \theta, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\alpha, \theta, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Using (20) and (24), we get

(25) $\alpha, \theta \notin \mu_1 \langle T(y_{p-1}) - T^*_{n,q} \rangle$,

$\mu_1(f) = \varphi_n(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_1(e_p) = \alpha \notin \Gamma^q$ (see (20)), and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1 and hence T is an ETT satisfying MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which θ is missing at two vertices.

By (25) and Lemma 6.5, we obtain $P_{v_\theta}(\eta_m, \theta, \mu_1) = P_{y_{p-1}}(\eta_m, \theta, \mu_1)$, which is disjoint from $P_{y_p}(\eta_m, \theta, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\eta_m, \theta, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). Note that η_m is not used by any edge in $T - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_m \in D_n$). So e_1 is outside $P_{y_p}(\eta_m, \theta, \mu_1)$. Hence $\mu_2(f) = \mu_1(f)$ for each $f \in E(T)$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. It follows that T can be obtained from $T^*_{n,q} + e_1$ by using TAA and hence is an ETT satisfying MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\eta_m \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_m \in D_{n,q}$, and $\mu_2(e_p) = \alpha \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 1.1.

Subcase 3.2. $\theta \in \overline{\varphi}_n(y_{p-1})$.

We first assume that $\theta \in D_{n,q}$. Let $\theta = \eta_m \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma^q_{m_1}$ and $\gamma^q_{m_2}$ in Γ^q_m (see Definition 5.2) to γ_1 and γ_2, respectively. By (20) and Definition 5.2(i), we have

(26) $\{\alpha, \gamma_1, \gamma_2\} \cap \varphi_n \langle T - T^*_{n,q} \rangle = \emptyset$.

By (26) and Lemma 6.5, we obtain $P_{v_\alpha}(\alpha, \gamma_1, \varphi_n) = P_{v_{\gamma_1}}(\alpha, \gamma_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \gamma_1, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\alpha, \gamma_1, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Since $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$, and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1 and hence T is an ETT satisfying MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under μ_1. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which γ_1 is missing at two vertices. In view of (26) and Definition 5.2(i), we get

(27) $\{\alpha, \gamma_1, \gamma_2\} \cap \mu_1 \langle T - T^*_{n,q} \rangle = \emptyset$, and η_m is not used by any edge in $T - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_m \in D_{n,q} \subseteq D_n$).

By (27) and Lemma 6.5, we obtain $P_{v_{\gamma_1}}(\gamma_1, \eta_m, \mu_1) = P_{y_{p-1}}(\gamma_1, \eta_m, \mu_1)$, which is disjoint from $P_{y_p}(\gamma_1, \eta_m, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\gamma_1, \eta_m, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \eta_m \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\gamma_1, \eta_m, \mu_1)$. Since $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$ by (27), and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_2 and hence T is an ETT satisfying MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\eta_m \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_m \in D_{n,q}$, and $\mu_2(e_p) = \gamma_1 \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 1.1.

Next we assume that $\theta \notin D_{n,q}$. By (6.6) and the hypothesis of the present subcase, we have $\theta \notin \overline{\varphi}_n(T^*_{n,q})$. So $\theta \notin \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$, which implies $\theta \notin \overline{\varphi}_n(T_n) \cup D_n$. In particular,

(28) $\theta \notin D_{n,q} \cup \Gamma^q$ if $q \geq 1$ and $\theta \notin D_n \cup \Gamma^0$ if $q = 0$. Furthermore, θ is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$ by TAA (see, for instance, (1)).

We proceed by considering two possibilities for α.

- $\alpha \notin D_{n,q}$. Now it follows from (20) that

(29) $\alpha \notin D_{n,q} \cup \Gamma^q$ if $q \geq 1$ and $\alpha \notin D_n \cup \Gamma^0$ if $q = 0$.

By (20) and Lemma 6.5, we obtain $P_{v_\alpha}(\alpha, \theta, \varphi_n) = P_{y_{p-1}}(\alpha, \theta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \theta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \theta, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). Since $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_{p-1}))$ by (20) and (28), and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T is an ETT satisfying MP under σ_n. In view of (28) and (29), $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\theta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(y_{p-1})$, $\theta \notin D_{n,q}$, and $\sigma_n(e_p) = \alpha \notin \overline{\sigma}_n(y_{p-1})$, the present subcase reduces to Subcase 2.1.

- $\alpha \in D_{n,q}$. Let $\alpha = \eta_h \in D_{n,q}$. For simplicity, we use ε_1 and ε_2 to denote the two colors $\gamma^q_{h_1}$ and $\gamma^q_{h_2}$ in Γ^q_h (see Definition 5.2), respectively. By (20), we have

(30) $\{\alpha, \varepsilon_1, \varepsilon_2\} \cap \varphi_n \langle T - T^*_{n,q} \rangle = \emptyset$.

By (30) and Lemma 6.5, we obtain $P_{v_\alpha}(\alpha, \varepsilon_1, \varphi_n) = P_{v_{\varepsilon_1}}(\alpha, \varepsilon_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Since $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ by (30), and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1 and hence T is an ETT satisfying MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which ε_1 is missing at two vertices. From (30) and Definition 5.2(i) we see that

(31) $\varepsilon_1 \notin \mu_1 \langle T - T^*_{n,q} \rangle$.

By (31) and Lemma 6.5, we obtain $P_{v_{\varepsilon_1}}(\theta, \varepsilon_1, \mu_1) = P_{y_{p-1}}(\theta, \varepsilon_1, \mu_1)$, which is disjoint from $P_{y_p}(\theta, \varepsilon_1, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\theta, \varepsilon_1, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In view of (28) and (31), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$ and

$\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So T can be obtained from $T^*_{n,q} + e_1$ by using TAA and hence is an ETT satisfying MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)). Since $\theta \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\theta \notin D_{n,q}$, and $\mu_2(e_p) = \varepsilon_1 \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 2.1. This completes our discussion about Situation 7.2.

Situation 7.3 $2 \leq p(T) \leq p - 1$.

Recall that $T = T^*_{n,q} \cup \{e_1, y_1, e_2, ..., e_p, y_p\}$, and the path number $p(T)$ of T is the smallest subscript $t \in \{1, 2, ..., p\}$ such that the sequence $(y_t, e_{t+1}, ..., e_p, y_p)$ corresponds to a path in G. Set $I_{\varphi_n} = \{1 \leq t \leq p-1 : \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_t) \neq \emptyset\}$. We use $\max(I_{\varphi_n})$ to denote the maximum element of I_{φ_n} if $I_{\varphi_n} \neq \emptyset$. For convenience, set $\max(I_{\varphi_n}) = -1$ if $I_{\varphi_n} = \emptyset$.

If $\max(I_{\varphi_n}) \geq p(T)$, then we may assume that $\max(I_{\varphi_n}) = p - 1$ (the proof is exactly the same as that of Claim 7.3). Let $\alpha \in \overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p)$ and $\beta = \varphi_n(e_p)$. Let σ_n be obtained from φ_n by recoloring e_p with α and let $T' = T(y_{p-1})$. Then $\beta \in \overline{\sigma}_n(y_{p-1}) \cap \overline{\sigma}_n(T')$ and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under σ_n. So (T', σ_n) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, φ_n).

So we may assume hereafter that $\max(I_{\varphi_n}) < p(T)$. Let $i = \max(I_{\varphi_n})$ if $I_{\varphi_n} \neq \emptyset$, and let $j = p(T)$. Then e_j is not incident to y_{j-1}. In our proof we reserve y_0 for the maximum vertex (in the order $\prec$) in $T^*_{n,q}$.

Claim 7.4 *We may assume that there exists* $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(T(y_{j-2}))$*, such that either* $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$ *or* $\alpha \in \Gamma^q_m$ *for some* $\eta_m \in D_{n,q}$ *with* $v_{\eta_m} \preceq y_{j-2}$.

To establish this statement, we consider two cases, depending on whether I_φ is empty.

Case 1. $I_\varphi \neq \emptyset$.

By assumption, $\max(I_{\varphi_n}) < p(T)$. So $i \leq j - 1$. Let $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_i)$. By (6.6), we obtain

(1) $\alpha \notin \overline{\varphi}_n(T^*_{n,q})$. So $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$.

If $i \leq j - 2$, then $\alpha \in \overline{\varphi}_n(T(y_{j-2}))$, as desired. Thus we may assume that $i = j - 1$.

(2) There exists a color $\beta \in \overline{\varphi}_n(T(y_{j-2}))\overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$ or a color $\beta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ and $(\Gamma^q_m \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle = \emptyset$.

Tojustifythis,notethatif$|\overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-2}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})| \geq 5$, then there exists a color β in $\overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$, because $T(y_{j-1}) - T(y_{j-2})$ contains only one edge.

If $|\overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-2}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})| \leq 4$, then, by Lemma 6.3, there exist 7 distinct colors $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{j-2}))$ such that $(\Gamma_h^q \cup \{\eta_h\}) \cap \varphi_n\langle T(y_{j-2}) - T_{n,q}^*\rangle = \emptyset$. Since $T(y_{j-1}) - T(y_{j-2})$ contains only one edge, there exists at least one of these η_h, say η_m, such that $(\Gamma_m^q \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T_{n,q}^*\rangle = \emptyset$. So (2) is true.

Depending on whether α is contained in $D_{n,q}$, we distinguish between two subcases.

Subcase 1.1. $\alpha \in D_{n,q}$. In this subcase, let $\alpha = \eta_h \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma_{h_1}^q$ and $\gamma_{h_2}^q$ in Γ_h^q (see Definition 5.2) to γ_1 and γ_2, respectively. Since $\eta_h \in \overline{\varphi}_n(y_{j-1})$, by Definition 5.2(i) and TAA, we have

(3) $\gamma_1, \gamma_2 \notin \varphi_n\langle T(y_{j-1}) - T_{n,q}^*\rangle$, and η_h is not used by any edge in $T(y_{j-1}) - T_{n,q}^*$, except possibly e_1 when $q = 0$ and $T_{n,0}^* = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

By (3) and Lemma 6.5, we obtain $P_{v_{\gamma_1}}(\gamma_1, \eta_h, \varphi_n) = P_{y_{j-1}}(\gamma_1, \eta_h, \varphi_n)$, which is disjoint from $P_{y_p}(\gamma_1, \eta_h, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\gamma_1, \eta_h, \varphi_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_h \in D_n$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\gamma_1, \eta_h, \varphi_n)$. Using (3) and (6.6), we get $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T(y_{j-1}))$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T_{n,q}^* + e_1$ by using TAA under μ_1, and hence T is an ETT satisfying MP under μ_1. Furthermore, since $\eta_h \in \overline{\mu}_1(y_{j-1})$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_1, with the same Γ-sets as those under μ_1. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which γ_1 is missing at two vertices.

From (3) we see that

(4) $\gamma_1, \gamma_2 \notin \mu_1\langle T(y_{j-1}) - T_{n,q}^*\rangle$, and η_h is not used by any edge in $T(y_{j-1}) - T_{n,q}^*$ under μ_1, except possibly e_1 when $q = 0$ and $T_{n,0}^* = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

Let β be a color as specified in (2). Note that

(5) $\beta \notin \mu_1\langle T(y_{j-1}) - T_{n,q}^*\rangle$, $\beta \notin D_{n,q}$, and $\beta \neq \eta_h = \alpha$.

Since $\gamma_1 \in \overline{\mu}_1(T_{n,q})$ if $q \geq 1$ and $\gamma_1 \in \overline{\mu}_1(T_n)$ if $q = 0$, from (4) and Lemma 6.5 we deduce that $P_{v_{\gamma_1}}(\gamma_1, \beta, \mu_1) = P_{v_\beta}(\gamma_1, \beta, \mu_1)$, which is disjoint from $P_{y_p}(\gamma_1, \beta, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\gamma_1, \beta, \mu_1)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). By (4), (5) and (6.6), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{j-1}))$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T_{n,q}^* + e_1$ by using TAA under μ_2 and hence T is an ETT satisfying MP under μ_2. If $\beta \notin \Gamma^q$, then clearly $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. So we assume that $\beta \in \Gamma^q$. By (2), we have $\beta \in \Gamma_m^q$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ and $(\Gamma_m^q \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T_{n,q}^*\rangle = \emptyset$. It follows that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is still a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\beta \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(T(y_{j-2}))$. From (2) and the definitions of μ_1 and μ_2, we see that either $\beta \notin \Gamma^q \cup \overline{\varphi}_n(T_{n,0}^* - V(T_n))$ or $\beta \in \Gamma_m^q$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$. Thus Claim 7.4 holds by replacing φ_n with μ_2 and α with β.

Subcase 1.2. $\alpha \notin D_{n,q}$. In this subcase, using (1) and the set inclusion $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$, we get

(6) $\alpha \notin D_n$. So α is not used by any edge in $T(y_{j-1}) - T^*_{n,q}$ by TAA.

Let β be a color as specified in (2). Then there are two possibilities for β.

•$\beta \in \overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$. Now it follows from Lemma 6.5 that $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{j-1}}(\alpha, \beta, \varphi_n)$, so this path is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). By (6), the assumption on β and (6.6), we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_{j-1}))$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T satisfies MP under σ_n. Since $\alpha, \beta \notin \Gamma^q$ (see (1)), the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T(y_{j-2}))$. Thus Claim 7.4 holds by replacing φ_n with σ_n and α with β.

• $\beta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ and $(\Gamma^q_m \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle = \emptyset$. Note that $\eta_m \in \overline{\varphi}_n(T(y_{j-2}))$ and hence $\alpha \neq \eta_m$ by (6.6). In view of Lemma 6.5, we obtain $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{j-1}}(\alpha, \beta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). By (6), the assumption on β and (6.6), we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_{j-1}))$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T is an ETT satisfying MP under σ_n. Since $\alpha \notin \Gamma^q$ (see (1)) and $\eta_m \in \overline{\varphi}_n(T(y_{j-2}))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T(y_{j-2}))$. Thus Claim 7.4 holds by replacing φ_n with σ_n and α with β.

Case 2. $I_\varphi = \emptyset$.

Let $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(T(y_{p-1}))$. By the hypothesis of the present case, we have $\alpha \in \overline{\varphi}_n(T^*_{n,q})$. If $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$, we are done. So we assume that $\alpha \in \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$.

Subcase 2.1. $\alpha \in \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \Gamma^q$. Let us first show that

(7) there exists a color $\beta \in \overline{\varphi}_n(T^*_{n,q}) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \Gamma^q$.

Indeed, since $V(T^*_{n,q})$ is elementary with respect to φ_n, we have $|\overline{\varphi}_n(T^*_{n,q}) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \Gamma^q| \geq |\overline{\varphi}_n(T^*_{n,0}) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \Gamma^q| = |\overline{\varphi}_n(T_n) - \Gamma^q|$. In view of (7.2), we obtain $|\overline{\varphi}_n(T_n)| \geq 2n + 11$ and $|\Gamma^q| \leq 2|D_{n,q}| \leq 2n$. So $|\overline{\varphi}_n(T^*_{n,q}) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \Gamma^q| \geq 11$, which implies (7).

By (7) and Lemma 6.5, we obtain $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). Since $\alpha, \beta \in \overline{\varphi}_n(T^*_{n,q})$, we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T^*_{n,q})$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T is an ETT satisfying MP under σ_n. As $\alpha, \beta \notin \Gamma^q$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore,

(T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T(y_{j-2}))$. Thus Claim 7.4 holds by replacing φ_n with σ_n and α with β.

Subcase 2.2. $\alpha \in \Gamma^q$. Let $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$. Depending on whether η_m is contained in $\overline{\varphi}_n(T(y_{p-1}))$, we consider two possibilities.

• $\eta_m \notin \overline{\varphi}_n(T(y_{p-1}))$. By Definition 5.2(i), we have $\alpha \notin \varphi_n\langle T - T^*_{n,q}\rangle$. Let β be a color in $\overline{\varphi}_n(y_{p-1})$. By Lemma 6.5, we obtain $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). Since $\beta \in \overline{\varphi}_n(y_{p-1})$, we see that $\beta \neq \varphi_n(e_p)$ as e_p is incident with y_{p-1}. Because $T(y_{p-1})$ is elementary (see (6.6)), by the construction of T, β was not used by any edge on $T - T^*_{n,q}$ under φ_n, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \beta \in D_{n,q} \subseteq D_n$). In particular, in the case that $e_1 = f_n$ and $\varphi_n(e_1) = \beta \in D_n$, Lemma 6.6 guarantees $\sigma_n(e_1) = \varphi_n(e_1)$. Moreover, since $\alpha \notin \varphi_n\langle T - T^*_{n,q}\rangle$ and $\alpha, \beta \in \overline{\varphi}_n(T(y_{p-2}))$, we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T)$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T is an ETT satisfying MP under σ_n. Furthermore, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(y_(p-1))$. However, by recoloring e_p with β, we have reached a smaller counterexample than T, which violates the minimality assumption (6.4) or (6.5) on (T, φ_n).

• $\eta_m \in \overline{\varphi}_n(T(y_{p-1}))$. Note that $\eta_m \notin \overline{\varphi}_n(T^*_{n,q})$ because $\eta_m \in D_{n,q}$. So $\eta_m \in \overline{\varphi}_n(y_t)$ for some $1 \leq t \leq p-1$. If $t \leq j-2$, then Claim 7.4 holds. Thus we may assume that $t \geq j-1$. Since $\eta_m \in \overline{\varphi}_n(y_t)$, it is not used by any edge in $T(y_t) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_m \in D_{n,q} \subseteq D_n$). Since $\alpha \in \Gamma^q_m$, by Definition 5.2(i), α is not used by any edge in $T(y_t) - T^*_{n,q}$. It follows from Lemma 6.5 that $P_{v_\alpha}(\alpha, \eta_m, \varphi_n) = P_{y_t}(\alpha, \eta_m, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \eta_m, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \eta_m, \varphi_n)$. By Lemma 6.6, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_m \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha, \eta_m, \varphi_n)$. Since $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_t))$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n, so T is an ETT satisfying MP under σ_n. Furthermore, As $\alpha, \eta_m \in \overline{\sigma}_n(T(y_t))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\eta_m \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(y_t)$. Thus the present subcase reduces to the case when $\max(I_{\sigma_n}) \geq p(T)$ if $j \preceq t$ (see the paragraphs above Claim 7.4), and reduces to Case 1 (where $I_{\sigma_n} \neq \emptyset$) if $t = j-1$. This proves Claim 7.4.

Let α be a color as specified in Claim 7.4; that is, $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(T(y_{j-2}))$, such that either $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$ or $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$. Since $T(y_j) - T(y_{j-2})$ contains precisely two edges, Lemma 6.3 guarantees the existence of a color β in $\overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_j) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$ or a color

$\beta = \eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{j-2}))$ such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_j) - T^*_{n,q}\rangle = \emptyset$. Note that

(8) $\beta \notin \varphi_n\langle T(y_j) - T^*_{n,q}\rangle \cup \Gamma^q$.

Let $Q = P_{y_p}(\alpha, \beta, \varphi_n)$. We consider two cases, depending on whether Q intersects $T(y_{j-1})$.

Case 1. Q and $T(y_{j-1})$ have vertices in common. Let u be the first vertex of Q contained in $T(y_{j-1})$ as we traverse Q from y_p. Define $T' = T(y_{j-1}) \cup Q[u, y_p]$ if $u = y_{j-1}$ and $T' = T(y_{j-2}) \cup Q[u, y_p]$ otherwise. By the choices of α and β, we have $\alpha, \beta \in \overline{\varphi}_n(T(y_{j-2}))$. So T' can be obtained from $T(y_{j-2})$ by using TAA under φ_n. It follows that T' is an ETT satisfying MP with respect to φ_n, with $p(T') < p(T)$. If $\alpha \notin \Gamma^q$, then both α and β are outside Γ^q (see (8)), so $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. If $\alpha \in \Gamma^q$, then $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ by Claim 7.4. Since $\alpha, \eta_m \in \overline{\varphi}_n(T(y_{j-2}))$ and $\beta \notin \Gamma^q$, it is clear that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is also a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. So (T', φ_n) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) on (T, φ_n).

Case 2. Q is vertex-disjoint from $T(y_{j-1})$. Let $\sigma_n = \varphi_n/Q$. By Lemma 5.8, σ_n is $(T(y_{j-1}), D_n, \varphi_n)$-stable. In particular, σ_n is $(T(y_{j-1}), \varphi_n)$-invariant. If $\Theta_n = PE$, then σ_n is also $(T_n \oplus R_n, D_n, \varphi_n)$-stable. Furthermore, $T(y_{j-1})$ is an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(y_{j-1})$ is a good hierarchy of $T(y_{j-1})$ under σ_n, with the same Γ-sets as T under φ_n. By definition, σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-weakly stable coloring. If $\alpha \notin \Gamma^q$, then both α and β are outside Γ^q (see (8)), so $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T$ is a good hierarchy of T under σ_n, with the same Γ-sets as T under φ_n. If $\alpha \in \Gamma^q$, then $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ by Claim 7.4. Since $\alpha, \eta_m \in \overline{\varphi}_n(T(y_{j-2}))$ and $\beta \notin \Gamma^q$, it is clear that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T$ is also a good hierarchy of T under σ_n, with the same Γ-sets as T under φ_n. So (T, σ_n) is a counterexample to Theorem 5.3, in which β is missing at two vertices.

From the choice of β above (8) and the definition of σ_n, we see that

(9) either $\beta \notin \overline{\sigma}_n(T^*_{n,0} - V(T_n)) \cup \sigma_n\langle T(y_j) - T^*_{n,q}\rangle \cup (\Gamma^q \cup D_{n,q})$ or $\beta = \eta_h \in D_{n,q} \cap \overline{\sigma}_n(T(y_{j-2}))$, such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \sigma_n\langle T(y_j) - T^*_{n,q}\rangle = \emptyset$.

Let $\theta \in \overline{\sigma}_n(y_j)$. Then $\theta \notin \Gamma^q$. We proceed by considering two subcases.

Subcase 2.1. $\theta \notin D_{n,q}$. In this subcase, using (6.6) and the set inclusion $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$, we obtain

(10) $\theta \notin \overline{\sigma}_n(T(y_{j-1}))$ and $\theta \notin D_n$. So θ is not assigned to any edge in $T(y_j) - T^*_{n,q}$ by TAA.

As described in (9), there are two possibilities for β.

- $\beta \notin \overline{\sigma}_n(T^*_{n,0} - V(T_n)) \cup \sigma_n\langle T(y_j) - T^*_{n,q}\rangle \cup (\Gamma^q \cup D_{n,q})$. Observe that $\beta \notin D_n$ if $q = 0$. By Lemma 6.5, we obtain $P_{v_\beta}(\beta, \theta, \sigma_n) = P_{y_j}(\beta, \theta, \sigma_n)$, which is disjoint from $P_{y_p}(\beta, \theta, \sigma_n)$. Let $\mu_1 = \sigma_n/P_{y_p}(\beta, \theta, \sigma_n)$. By Lemma 6.6, μ_1 satisfies all the properties described in (7.3). By (10), the assumption on β and (6.6), we have $\mu_1(f) = \sigma_n(f)$ for each $f \in E(T(y_j))$ and $\overline{\mu}_1(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1 and hence T is an ETT satisfying MP under μ_1. As $\beta, \theta \notin \Gamma^q$, the hierarchy

$T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_1, with the same Γ-sets as those under σ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\theta \in \overline{\mu}_1(y_p) \cap \overline{\mu}_1(y_j)$. Thus the present subcase reduces to the case when $\max(I_{\mu_1}) \geq p(T)$ (see the paragraphs above Claim 7.4).

• $\beta = \eta_h \in D_{n,q} \cap \overline{\sigma}_n(T(y_{j-2}))$, such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \sigma_n\langle T(y_j) - T^*_{n,q}\rangle = \emptyset$. For simplicity, we abbreviate the two colors $\gamma^q_{h_1}$ and $\gamma^q_{h_2}$ in Γ^q_h (see Definition 5.2) to γ_1 and γ_2, respectively. By Lemma 6.5, we obtain $P_{v_\beta}(\beta, \gamma_1, \sigma_n) = P_{v_{\gamma_1}}(\beta, \gamma_1, \sigma_n)$, which is disjoint from $P_{y_p}(\beta, \gamma_1, \sigma_n)$. Let $\mu_2 = \sigma_n / P_{y_p}(\beta, \gamma_1, \sigma_n)$. By Lemma 6.6, μ_2 satisfies all the properties described in (7.3). By the assumption on β, neither β nor γ_1 is used by any edge in $T(y_j) - T^*_{n,q}$. So $\mu_2(f) = \sigma_n(f)$ for each $f \in E(T(y_j))$. By (6.6), we get $\overline{\mu}_2(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$. It follows that T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_2 and hence T is an ETT satisfying MP under μ_2. Furthermore, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_2, with the same Γ-sets as those under σ_n. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which γ_1 is missing at both y_p and v_{γ_1}.

From the assumption on β and the definition of μ_2, we deduce that (11) $\beta = \eta_h \in D_{n,q} \cap \overline{\mu}_2(T(y_{j-2}))$, such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \mu_2\langle T(y_j) - T^*_{n,q}\rangle = \emptyset$.

By (11) and Lemma 6.5, we obtain $P_{v_{\gamma_1}}(\theta, \gamma_1, \mu_2) = P_{y_j}(\theta, \gamma_1, \mu_2)$, which is disjoint from $P_{y_p}(\theta, \gamma_1, \mu_2)$. Let $\mu_3 = \mu_2 / P_{y_p}(\theta, \gamma_1, \mu_2)$. By Lemma 6.6, μ_3 satisfies all the properties described in (7.3). By (10), (11) and (6.6), we have $\mu_3(f) = \mu_2(f)$ for each $f \in E(T(y_j))$ and $\overline{\mu}_3(u) = \overline{\mu}_2(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_3 and hence T is ETT satisfying MP under μ_3. Furthermore, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_3, with the same Γ-sets as those under μ_2. Therefore, (T, μ_3) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which θ is missing at both y_p and y_j. Thus the present subcase reduces to the case when $\max(I_{\mu_3}) \geq p(T)$ (see the paragraphs above Claim 7.4).

Subcase 2.2. $\theta \in D_{n,q}$. Let $\theta = \eta_t \in D_{n,q}$. For simplicity, we use ε_1 and ε_2 to denote the two colors $\gamma^q_{t_1}$ and $\gamma^q_{t_2}$ in Γ^q_t (see Definition 5.2), respectively. Then

(12) $\varepsilon_1, \varepsilon_2 \notin \sigma_n\langle T(y_j) - T^*_{n,q}\rangle$ and η_t is not used by any edge in $T(y_j) - T^*_{n,q}$ under σ_n, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\sigma_n(e_1) = \eta_t \in D_{n,q} \subseteq D_n$).

By (12) and Lemma 6.5 (with ε_1 in place of α), we obtain $P_{v_{\varepsilon_1}}(\varepsilon_1, \beta, \sigma_n) = P_{v_\beta}(\varepsilon_1, \beta, \sigma_n)$, which is disjoint from $P_{y_p}(\varepsilon_1, \beta, \sigma_n)$. Let $\mu_4 = \sigma_n / P_{y_p}(\varepsilon_1, \beta, \sigma_n)$. By Lemma 6.6, μ_4 satisfies all the properties described in (7.3). By (9), we have $\beta \notin \sigma_n\langle T(y_j) - T^*_{n,q}\rangle$, which together with (12) and (6.6) implies $\mu_4(f) = \sigma_n(f)$ for each $f \in E(T(y_j))$ and $\overline{\mu}_4(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_4 and hence T is an ETT satisfying MP under μ_4. Since $\beta \notin \Gamma^q$ by (9) and $\eta_t \in \overline{\mu}_4(y_j)$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_4, with the same Γ-sets as those under σ_n. Therefore, (T, μ_4) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which ε_1 is missing at both y_p and v_{ε_1}.

From (12) and (6.6) it can be seen that

(13) $\varepsilon_1, \varepsilon_2 \notin \mu_4\langle T(y_j) - T^*_{n,q}\rangle$ and $\eta_t \notin \overline{\mu}_4(T(y_{j-1}))$. So η_t is not used by any edge in $T(y_j) - T^*_{n,q}$ under μ_4, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_4(e_1) = \eta_t \in D_{n,q} \subseteq D_n$).

By (13) and Lemma 6.5, we obtain $P_{v_{\varepsilon_1}}(\varepsilon_1, \eta_t, \mu_4) = P_{y_j}(\varepsilon_1, \eta_t, \mu_4)$, which is disjoint from $P_{y_p}(\varepsilon_1, \eta_t, \mu_4)$. Let $\mu_5 = \mu_4/P_{y_p}(\varepsilon_1, \eta_t, \mu_4)$. By Lemma 6.6, μ_5 satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\mu_4(e_1) = \eta_t \in D_n$, then $\mu_5(e_1) = \mu_4(e_1)$, which implies that e_1 is outside $P_{y_p}(\varepsilon_1, \eta_t, \mu_4)$. By (13) and (6.6), we have $\mu_5(f) = \mu_4(f)$ for each $f \in E(T(y_j))$ and $\overline{\mu}_5(u) = \overline{\mu}_4(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain *T* from $T^*_{n,q} + e_1$ by using TAA under μ_5 and hence *T* is an ETT satisfying MP under μ_5. Since $\eta_t, \varepsilon_1 \in \overline{\mu}_5(T(y_j))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_5, with the same Γ-sets as those under μ_4. Therefore, (T, μ_5) is also a minimum counterexample to Theorem 5.3 (see (6.2)–(6.5)), in which $\theta = \eta_t$ is missing at both y_p and y_j. Thus the present subcase reduces to the case when $\max(I_{\mu_5}) \geq p(T)$ (see the paragraphs above Claim 7.4).

This completes our discussion about Situation 7.3 and hence our proof of Theorem 5.3. □

7.2 Proof of theorem 3.10(ii)

In the preceding subsection we have proved Theorem 5.3 and hence Theorem 3.10(i). To complete the proof of Theorem 3.10, we still need to establish the interchangeability property as described in Theorem 3.10(ii).

Lemma 7.1 *Suppose Theorem 3.10(i), (iv), and (vi) hold for all ETTs with n rungs and satisfying MP, and suppose Theorem 3.10(ii) holds for all ETTs with $n - 1$ rungs and satisfying MP. Then Theorem 3.10(ii) holds for all ETTs with n rungs and satisfying MP; that is, T_{n+1} has the interchangeability property with respect to φ_n.*

Proof Let $T = T_{n+1}$, let σ_n be a (T, D_n, φ_n)-stable coloring, and let α and β be two colors in [*k*] with $\alpha \in \overline{\sigma}_n(T)$ (equivalently $\alpha \in \overline{\varphi}_n(T)$). We aim to prove that α and β are *T*-interchangeable under σ_n. Assume the contrary: there are at least two (α, β)-paths Q_1 and Q_2 with respect to σ_n intersecting *T*. By Theorem 3.10(i), *V*(*T*) is elementary with respect to φ_n, so it is also elementary with respect to σ_n. Since $T = T_{n+1}$ is closed with respect to φ_n, it is also closed with respect to σ_n. As $\alpha \in \overline{\sigma}_n(T)$, it follows that |*V*(*T*)| is odd and $\beta \notin \overline{\sigma}_n(T)$. From the existence of Q_1 and Q_2, we see that *G* contains at least three $(T, \sigma_n, \{\alpha, \beta\})$-exit paths P_1, P_2, P_3.

We call the tuple $(\sigma_n, T, \alpha, \beta, P_1, P_2, P_3)$ a *counterexample* and use $\mathcal{K}$ to denote the set of all such counterexamples. With a slight abuse of notation, let $(\sigma_n, T, \alpha, \beta, P_1, P_2, P_3)$ be a counterexample in $\mathcal{K}$ with the minimum $|P_1| + |P_2| + |P_3|$. For $i = 1, 2, 3$, let a_i and b_i be the ends of P_i with $b_i \in V(T)$, and f_i be the edge of P_i incident to b_i. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2 \prec b_3$. We propose to show that

(1) $b_2 \notin V(T_n)$

Otherwise, $b_2 \in V(T_n)$. Let γ be a color in $\overline{\sigma}_n(T_n) - \{\delta_n\}$ if $\Theta_n = PE$ and a color in $\overline{\sigma}_n(T_n)$ otherwise. Since $T = T_{n+1}$ is closed with respect to σ_n, both α and γ are closed in T with respect to σ_n. Let $\mu_1 = \sigma_n/(G - T, \alpha, \gamma)$. Then P_1 and P_2 are two $(T_n, \mu_1, \{\gamma, \beta\})$-exit paths. By Lemma 5.8, μ_1 is a (T, D_n, σ_n)-stable coloring, so it is also (T, D_n, φ_n)-stable. As $T_n \subset T$, μ_1 is a (T_n, D_n, φ_n)-stable coloring.

If $\Theta_n = SE$ or RE then, by Algorithm 3.1 and Lemma 3.2(i), μ_1 is $(T_n, D_{n-1}, \varphi_{n-1})$-stable and hence is $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. By Theorem 3.10(vi) and TAA, T_n is an ETT corresponding to μ_1 (see Definition 3.7) and satisfies MP under μ_1, with $n-1$ rungs. Since P_1 and P_2 are two $(T_n, \mu_1, \{\gamma, \beta\})$ -exit paths and $\gamma \in \overline{\mu}_1(T_n) = \overline{\sigma}_n(T_n)$, there are at least two (γ, β)-paths with respect to μ_1 intersecting T_n. Hence γ and β are not T_n-interchangeable under μ_1, contradicting Theorem 3.10(ii) because T_n has $n-1$ rungs.

So we assume that $\Theta_n = PE$. Since T_n is an ETT under φ_{n-1}, $V(T_n)$ is elementary under φ_{n-1} by Theorem 3.10(i), and hence $\delta_n \in \overline{\varphi}_n(T_n)$ and $\gamma_n \notin \overline{\varphi}_n(T_n)$ by Algorithm 3.1. Since σ_n is (T, D_n, φ_n)-stable, $\overline{\varphi}_n(T_n) = \overline{\sigma}_n(T_n)$ and $\partial_{\varphi_n,\gamma_n}(T_n) = \partial_{\sigma_n,\gamma_n}(T_n)$. As $\gamma \in \overline{\sigma}_n(T_n) - \delta_n$, $\delta_n \in \overline{\sigma}_n(T_n)$, and $\gamma_n, \beta \notin \overline{\sigma}_n(T_n)$, we have $\gamma \notin \{\gamma_n, \delta_n\}$ and $\beta \neq \delta_n$. In view of Lemma 3.2(v), we obtain $|\partial_{\varphi_n,\gamma_n}(T_n)| = 1$. So $|\partial_{\sigma_n,\gamma_n}(T_n)| = 1$, which implies $\beta \neq \gamma_n$, because $\{f_1, f_2\} \subseteq \partial_{\sigma_n,\beta}(T_n)$ (as $b_2 \in V(T_n)$). Therefore $\{\beta, \gamma\} \cap \{\gamma_n, \delta_n\} = \emptyset$. Since μ_1 is (T_n, D_n, φ_n)-stable, $P_{v_n}(\gamma_n, \delta_n, \mu_1) \cap T_n = \{v_n\}$ by Theorem 3.10(iv). Let $\mu_2 = \mu_1/P_{v_n}(\gamma_n, \delta_n, \mu_1)$. Then μ_2 is $(T_n, D_{n-1}, \varphi_{n-1})$-stable by Lemma 3.6. As $\{\beta, \gamma\} \cap \{\gamma_n, \delta_n\} = \emptyset$, we see that P_1 and P_2 are two $(T_n, \mu_2, \{\gamma, \beta\})$-exit paths and $\gamma \in \overline{\mu}_2(T_n)$. So there are at least two (γ, β)-paths with respect to μ_2 intersecting T_n. Thus γ and β are not T_n-interchangeable under μ_2, contradicting Theorem 3.10(ii) because T_n has $n-1$ rungs. Therefore (1) is established.

Let $\gamma \in \overline{\sigma}_n(b_3)$ and let $\mu_3 = \sigma_n/(G - T, \alpha, \gamma)$. By Lemma 5.8, μ_3 is (T, D_n, φ_n) -stable and hence is (T_n, D_n, φ_n)-stable. By Theorem 3.10(vi), T is an ETT corresponding to μ_3 and satisfies MP under μ_3. Furthermore, f_i is colored by β under both μ_3 and σ_n, for $i = 1, 2, 3$, and $P_3 = P_{b_3}(\beta, \gamma, \mu_3)$.

Consider $\mu_4 = \mu_3/P_{b_3}(\beta, \gamma, \mu_3)$. Clearly, $\beta \in \overline{\mu}_4(b_3)$. Since $P_{b_3}(\beta, \gamma, \mu_3) \cap T = \{b_3\}$, by Lemma 5.8, μ_4 is $(T(b_3) - b_3, D_n, \varphi_n)$-stable and $T(b_3) - b_3$ is an ETT corresponding to μ_4 and satisfies MP under μ_4. Since $b_2 \prec b_3$, it is contained in $T(b_3) - b_3$. So (1) implies that b_3 is not the first vertex added to T_n in the construction of T. According to Algorithm 3.1, b_3 is added to $T(b_3) - b_3$ by TAA under φ_n. Since colors on the edges of $T(b_3)$ are not affected under this Kempe change and μ_3 is (T, D_n, φ_n)-stable, b_3 can still be added to $T(b_3) - b_3$ by TAA under μ_4. Hence $T(b_3)$ is still an ETT satisfying MP under μ_4 by Theorem 3.10(vi). Let T' be a closure of $T(b_3)$ under μ_4. Then T' is an ETT satisfying MP under μ_4. Since both f_1 and f_2 are colored by β under μ_4 and $\beta \in \overline{\mu}_4(b_3)$, the ends of f_1 and f_2 are all contained in T'. By Theorem 3.10(i), $V(T')$ is elementary with respect to μ_4, because T' has *n* rungs.

Observe that none of a_1, a_2, a_3 is contained in T', for otherwise, let $a_i \in V(T')$ for some i with $1 \le i \le 3$. Since $\{\beta, \gamma\} \cap \overline{\mu}_4(a_i) \neq \emptyset$ and $\beta \in \overline{\mu}_4(b_3)$, we obtain $\gamma \in \overline{\mu}_4(a_i)$. Hence from TAA we see that P_1, P_2, P_3 are all entirely contained in $G[T']$, which in turn implies $\gamma \in \overline{\mu}_4(a_j)$ for $j = 1, 2, 3$. So $V(T')$ is not elementary with respect to μ_4, a contradiction. Thus each P_i contains a subpath L_i, which is a T'

-exit path with respect to μ_4. Since both ends of f_1 are contained in T', f_1 is outside L_1. It follows that $|L_1| + |L_2| + |L_3| < |P_1| + |P_2| + |P_3|$. Therefore the existence of the counterexample $(\mu_4, T', \gamma, \beta, L_1, L_2, L_3)$ violates the minimality assumption on $(\sigma_n, T, \alpha, \beta, P_1, P_2, P_3)$. This completes our proof of Lemma 7.1 and hence the whole proof of Theorem 3.10. □

Subject index

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