

More Information Is Not Always Better: Connections Between Zero-Sum Local Nash Equilibria in Feedback and Open-Loop Information Patterns

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Abstract—Noncooperative dynamic game theory provides a principled approach to modeling sequential decision-making among multiple noncommunicative agents. A key focus is on finding Nash equilibria in two-agent zero-sum dynamic games under various information structures. A well-known result states that in linear-quadratic games, unique Nash equilibria under feedback and open-loop information structures yield identical trajectories. Motivated by two key perspectives—(i) real-world problems extend beyond linear-quadratic settings and lack unique equilibria, making only *local* Nash equilibria computable, and (ii) local open-loop Nash equilibria (OLNE) are easier to compute than local feedback Nash equilibria (FBNE)—it is natural to ask whether a similar result holds for local equilibria in zero-sum games. To this end, we establish that for a broad class of zero-sum games with potentially nonconvex-nonconcave objectives and nonlinear dynamics: (i) the state/control trajectory of a local FBNE satisfies local OLNE first-order optimality conditions, and vice versa, (ii) a local FBNE trajectory

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satisfies local OLNE second-order necessary conditions, (iii) a local FBNE trajectory satisfying feedback sufficiency conditions also constitutes a local OLNE, and (iv) with additional hard constraints on agents' actuations, a local FBNE where strict complementarity holds satisfies local OLNE first-order optimality conditions, and vice versa.

Index Terms—Game theory, optimization, optimal control.

I. INTRODUCTION

IN CONTRAST to single-agent optimization problems, dynamic games require the additional specification of an *information structure*, which is the information available to each agent at every time step of the decision-making process. Information structures have two extremes: open-loop and feedback. The open-loop information structure assumes that, at each time t , every agent knows only the initial state of the game and nothing else about the state. On the other hand, the feedback information structure assumes that, at each time t , every agent knows the full state of all agents. The underlying information structure of a game can greatly affect the existence and expressivity of Nash equilibrium solutions. In particular, feedback strategies can encode complex behaviors such as delayed commitment which are not expressible in open-loop strategies [1], [2], [3]. However, outside of linear-quadratic settings, feedback Nash equilibria are generally far more complicated to compute than open-loop Nash solutions, cf. [4]. *Therefore, it is always valuable to know if for a game, the more computationally-intensive feedback equilibrium differs from an open-loop equilibrium.*

In the general-sum game setting, it is known that feedback Nash equilibria (FBNE) and open-loop Nash equilibria (OLNE) often diverge greatly [5]. In the two-agent zero-sum setting, while some results comparing FBNE and OLNE do exist, these results either (i) are restricted to linear-quadratic (LQ) games or (ii) assume the existence of a strongly unique Nash equilibrium.

However, we observe that these existing results cannot be applied to a large class of zero-sum games that have significant practical applications. In particular, many applications have nonlinear dynamics and more general nonquadratic costs. Such

scenarios are often solved by employing iterative algorithms that solve an approximated LQ game at every iteration [4], [6], [7], [8]. In such non-LQ settings, these methods can only find an *approximate local* FBNE/OLNE: it is generally intractable to find a global Nash equilibria, let alone one which is strongly unique.

Further, existing results that relax the LQ assumption are difficult to verify in potentially nonconvex-nonconcave settings such as generative adversarial network training [9], robust optimization [10], multi-agent reinforcement learning [11], etc. In such settings, a unique Nash equilibrium may not exist for any information structure, or may be intractable to find. In such zero-sum nonconvex-nonconcave settings, and without additional structural assumptions, all existing game-theoretic solvers can only find local Nash equilibria, if they exist [12], [13], [14].

Contributions: With this discussion in mind, it is pertinent to ask: *How are local FBNE and local OLNE related in zero-sum games that extend beyond linear-quadratic settings, and/or do not have a unique equilibrium?* To this end, we present the following contributions, which apply in a large class of zero-sum games with potentially nonconvex-nonconcave costs and nonlinear dynamics:

- 1) We show that any local FBNE also satisfies the first-order necessary conditions for a local OLNE of the game, and vice versa.
- 2) We show that any local FBNE also satisfy the second-order necessary conditions for a local OLNE of the game. Further, a local FBNE satisfying feedback second-order sufficiency conditions also constitutes a local OLNE of the game.
- 3) We show that, in the presence of additional constraints on agents' control variables, any local FBNE satisfying strict complementarity still satisfies the first-order necessary conditions for a local OLNE of the game, and vice versa.

II. RELATED EXISTING RESULTS

In the setting of general-sum dynamic games, it is well known that the state trajectories formed by control strategies corresponding to FBNE and OLNE often diverge greatly, and the extent of their divergence has been studied for general-sum linear-quadratic (LQ) dynamic games [5].

In zero-sum games however, the alignment of these equilibrium concepts becomes more nuanced. In continuous-time zero-sum pursuit-evasion games [15], for example, open-loop strategies are often used to synthesize feedback strategies [16, Ch. 8]. Moreover, the existing results in the discrete-time zero-sum setting hold only for a limited class of games. For a certain subclass of two-agent zero-sum LQ games, including convex-concave zero-sum LQ games, it is known that a unique FBNE exists and its open-loop realization is also an (not necessarily unique) OLNE [16, Th. 6.7]. Further, while the existence of a unique OLNE implies the existence of a unique FBNE in a two-agent zero-sum LQ game, the converse is not true [16, Proposition 6.2]. Thus, when they both exist, a unique OLNE and a unique FBNE generate the same state trajectory in a two-agent zero-sum LQ game [16, Remark 6.7].

Another existing result applies beyond the setting of LQ games, but in turn requires the game to have a *strongly* unique

FBNE. A strongly unique equilibrium is a unique equilibrium where each agent's equilibrium strategy is the unique best response to the other's. For a zero-sum dynamic game, if a strongly unique FBNE and an OLNE exist, then the OLNE is unique and generates the same state trajectory as the FBNE. On the other hand, if a strongly unique OLNE and a FBNE exist for the game, then the two also give the same state trajectory [16, Th. 6.9].

As mentioned in Section I, these existing results *do not cover* practical settings of interest, in which (i) the games are not linear-quadratic, or (ii) it is not possible to reason about or compute (strongly) unique equilibria, and only the notions of *local* FBNE and OLNE are computable.

III. PRELIMINARIES

For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, \dots, n\}$. Consider a two-agent zero-sum dynamic game with a fixed decision-making time horizon of K time steps, with dynamics given by $x_{t+1} = f_t(x_t, u_t^1, u_t^2)$, $t \in [K]$, where $x_t \in \mathbb{R}^n$ denotes the states of both agents at time t , and $u_t^i \in \mathbb{R}^{m_i}$ denotes the control action of agent i at time t . For agent i , we denote the other agent as $-i := \{1, 2\} \setminus \{i\}$. For $t \in [K]$, let $x_{t:K+1} := \{x_t, \dots, x_{K+1}\}$, and $u_{t:K} := \{u_t^1, \dots, u_K^1\}$. Let the set $x(u_{1:K}^1, u_{1:K}^2)$ denote the state trajectory $x_{1:K+1}$ obtained by unrolling the set of controls $u_{1:K}^1$ and $u_{1:K}^2$ according to the dynamics f_t , and an initial state x_1 . For $t \in [K-1]$, let $T_t = \{t, \dots, K\}$.

Without loss of generality, we assume that at time t , agent 1 minimizes a stage-wise cost $\ell_t(x_t, u_t^1, u_t^2) \in \mathbb{R}$ (and thus, agent 2 minimizes $-\ell_t$). For brevity, we denote the stagewise cost for agent i as ℓ_t^i and the dynamics as f_t , and $\ell^i(x_t, u_t^i, u_t^{-i}), f_t(x_t, u_t^i, u_t^{-i})$ denote the cost/dynamics functions with the appropriate order of control arguments. Further, let the terminal state cost for agent 1 be represented by $\ell_{K+1}(x_{K+1})$. Notably, we assume ℓ_t can be nonconvex (and nonquadratic), and that f_t can be nonlinear. The only assumption for ℓ_t^i is that it is K times differentiable, which is an implicit requirement for finding FBNE [4], [16]. We denote the cumulative cost for agent 1 as $J(x_1, u_{1:K}^1, u_{1:K}^2) = \sum_{t=1}^K \ell_t(x_t, u_t^1, u_t^2) + \ell_{K+1}(x_{K+1})$ where $x_t \in x(u_{1:K}^1, u_{1:K}^2)$. Let the initial state x_1 be known to both agents.

A. Open-Loop Zero-Sum Games

We begin by defining a local OLNE.

Definition 1 (Local OLNE): A local OLNE of the two-agent zero-sum game is a set of controls $u_{1:K}^{i,\text{OL}} = \{u_1^{i,\text{OL}}, \dots, u_K^{i,\text{OL}}\}$, $i = 1, 2$, and states $x(u_{1:K}^{1,\text{OL}}, u_{1:K}^{2,\text{OL}})$ such that

$$\begin{aligned} J(u_{1:K}^{1,\text{OL}}, u_{1:K}^{2,\text{OL}}) &\leq J(u_{1:K}^{1,\text{OL}}, u_{1:K}^{2,\text{OL}}) \leq J(u_{1:K}^{1,\text{OL}}, u_{1:K}^{2,\text{OL}}) \\ \forall u_{1:K}^1 \in \mathcal{N}(u_{1:K}^{1,\text{OL}}), u_{1:K}^2 \in \mathcal{N}(u_{1:K}^{2,\text{OL}}), \end{aligned} \quad (1)$$

where $\mathcal{N}(\cdot)$ denotes a feasible neighborhood. For unconstrained problems, a feasible neighborhood constitutes a ball of some finite size centered at the argument.

A local OLNE is said to be *strict* if the inequalities in (1) are strict. Note that strictness is different from and does not imply the uniqueness of a local Nash equilibrium. Finding a local OLNE of a two-agent zero-sum game with only

dynamics constraints amounts to locally solving the following equilibrium problem:

$$\underbrace{\min_{x_{2:K+1}, u_{1:K}^1} J(x_1, u_{1:K}^1, u_{1:K}^2)}_{\text{Agent 1}} \quad \underbrace{\min_{x_{2:K+1}, u_{1:K}^2} -J(x_1, u_{1:K}^1, u_{1:K}^2)}_{\text{Agent 2}},$$

s.t. $x_{t+1} = f_t(x_t, u_t^1, u_t^2)$, $t \in [K]$. (2)

1) Necessary Conditions for Local OLNE: The Karush-Kuhn-Tucker (KKT) conditions [17] for (2) give the first-order necessary conditions that the corresponding local OLNE must satisfy, under an appropriate constraint qualification. Introducing Lagrange multipliers λ_t^i , $t \in [K]$ for the constraints in (2), we define the Lagrangian for agent i as

$$\mathcal{L}^{i,OL} := \sum_{t=1}^{K+1} \ell_t^i - \sum_{t=1}^K \lambda_t^{i\top} (x_{t+1} - f_t), \quad (3)$$

which in turn yields the following KKT conditions:

$$\nabla_{x_t} \ell_t^i + \nabla_{x_t} f_t^\top \lambda_t^i - \lambda_{t-1}^i = 0, \quad \forall t \in T_2, \quad (4)$$

$$\nabla_{u_t} \ell_t^i + \nabla_{u_t} f_t^\top \lambda_t^i = 0, \quad \forall t \in [K], \quad (5)$$

$$\nabla_{x_{K+1}} \ell_{K+1}^i - \lambda_K^i = 0, \quad (6)$$

$$x_{t+1} - f_t(x_t, u_t^1, u_t^2) = 0, \quad \forall t \in [K]. \quad (7)$$

2) Sufficiency Conditions for Local OLNE: Nonlinear programming theory gives second-order sufficiency conditions for (2). If a set of states and controls satisfy these conditions in addition to the necessary KKT conditions (4)-(7), then they correspond to a local OLNE. Consider the stage-wise Lagrangian for agent i

$$\mathcal{L}_t^{i,OL} := \ell_t^i - \lambda_t^{i\top} (x_{t+1} - f_t). \quad (8)$$

Then the second-order sufficiency conditions are [17, Th. 12.6]

$$\begin{aligned} d_{u_1^i}^\top \nabla_{u_1^i}^2 \ell_1^i d_{u_1^i} + \sum_{t=2}^K \begin{bmatrix} d_{x_t} \\ d_{u_t^i} \end{bmatrix}^\top \nabla_{x_t, u_t^i}^2 \mathcal{L}_t^{i,OL} \begin{bmatrix} d_{x_t} \\ d_{u_t^i} \end{bmatrix} \\ + d_{x_{K+1}}^\top \nabla_{x_{K+1}}^2 \ell_{K+1}^i d_{x_{K+1}} > 0 \end{aligned} \quad (9)$$

$$\forall \{d_{u_1^i}, d_{x_t}, d_{u_t^i}, d_{x_{K+1}} \mid t \in T_2\} \text{ s.t.} \\ d_{x_{t+1}} - \nabla_{x_t} f_t d_{x_t} - \nabla_{u_t^i} f_t d_{u_t^i} = 0. \quad (10)$$

Here, $\begin{bmatrix} d_{u_1^i}^\top & d_{x_t}^\top & d_{u_t^i}^\top & d_{x_{K+1}}^\top \end{bmatrix}^\top \neq 0$, $t \in [K]$ represents a direction in the critical cone of the open-loop equilibrium problem (2).

B. Feedback Zero-Sum Games

A FBNE is defined in terms of mappings $\pi_t^i: \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}$, $t \in [K]$, $i = 1, 2$, value functions $V_t: \mathbb{R}^n \rightarrow \mathbb{R}$, $t \in [K+1]$, and control-value functions $Z_t: \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}$, $t \in [K+1]$. While V_t , Z_t correspond to agent 1, $-V_t$, $-Z_t$ correspond to agent 2. These relate to each other recursively

$$V(x_{K+1}) := \ell_{K+1}(x_{K+1}), \quad (11)$$

$$Z_t(x_t, u_t^1, u_t^2) := \ell_t(x_t, u_t^1, u_t^2) + V_{t+1}(f_t(x_t, u_t^1, u_t^2)), \quad (12)$$

$$V_t(x_t) := Z_t(x_t, \pi_t^1(x_t), \pi_t^2(x_t)), \quad \forall t \in [K]. \quad (13)$$

Definition 2 (Local FBNE): Let the controls and states at a local zero-sum FBNE be denoted by $u_{1:K}^{i,\text{FB}} =$

$\{u_1^{i,\text{FB}}, \dots, u_K^{i,\text{FB}}\}$, $i = 1, 2$, and $x(u_{1:K}^{1,\text{FB}}, u_{1:K}^{2,\text{FB}})$ respectively. Then local FBNE strategies correspond to maps $\pi_t^i(x_t)$, defined to return $u_t^{i,\text{FB}} = \pi_t^i(x_t)$ such that

$$\begin{aligned} Z_t(x_t, u_t^{1,\text{FB}}, u_t^{2,\text{FB}}) &\leq Z_t(x_t, u_t^{1,\text{FB}}, u_t^{2,\text{FB}}) \leq Z_t(x_t, u_t^1, u_t^{2,\text{FB}}) \\ \forall u_t^1 \in \mathcal{N}(u_t^{1,\text{FB}}), \quad u_t^2 \in \mathcal{N}(u_t^{2,\text{FB}}), \quad t \in [K]. \end{aligned} \quad (14)$$

If the inequalities in (14) are strict, the local FBNE is said to be strict. Definition 2 is the same as the local FBNE definition introduced in prior work [4, Definition 2.1], and is also equivalent to the traditional definition which encodes strong time-consistency [16, Definition 3.22]. Finding a local FBNE of a two-agent zero-sum game with only dynamics constraints amounts to locally solving the following optimization problem at each time $t \in [K]$ for agent i [4, Th. 2.2]:

$$\begin{aligned} \min_{u_{t:K}^i, u_{t+1:K+1}^{-i,\text{FB}}, x_{t+1:K+1}} \quad &\sum_{s=t}^K \ell_s(x_s, u_s^i, u_s^{-i,\text{FB}}) + \ell_{K+1}(x_{K+1}), \\ \text{s.t. } x_{s+1} - f_s(x_s, u_s^i, u_s^{-i,\text{FB}}) &= 0, \quad s \in T_t, \\ u_s^{-i,\text{FB}} - \pi_s^{-i}(x_s) &= 0, \quad s \in T_{t+1}. \end{aligned} \quad (15)$$

1) Necessary Conditions for Local FBNE: As before, we list the KKT conditions that states and controls corresponding to a local FBNE of (15) should satisfy. Introducing Lagrange multipliers λ_s^i , $s \in T_t$ and ψ_s^i , $s \in T_{t+1}$ for the constraints in (15), we define the Lagrangian for agent i as

$$\begin{aligned} \mathcal{L}^{i,\text{FB}} = \sum_{s=t}^{K+1} \ell_s^i - \sum_{s=t}^K \lambda_s^{i\top} (x_{s+1} - f_s) \\ - \sum_{s=t+1}^K \psi_s^{i\top} (u_s^{-i,\text{FB}} - \pi_t^{-i}), \end{aligned} \quad (16)$$

which in turn yields the following KKT conditions:

$$\nabla_{u_s^{-i,\text{FB}}} \ell_s^i + (\nabla_{u_s^{-i,\text{FB}}} f_s)^\top \lambda_s^i - \psi_s^i = 0, \quad s \in T_{t+1}, \quad (17)$$

$$\nabla_{u_t^i} \ell_t^i + (\nabla_{u_t^i} f_t)^\top \lambda_t^i = 0, \quad s \in T_t, \quad (18)$$

$$\nabla_{x_{K+1}} \ell_{K+1}^i - \lambda_K^i = 0, \quad (19)$$

$$x_{s+1} - f_s(x_s, u_s^i, u_s^{-i,\text{FB}}) = 0, \quad \forall s \in T_t, \quad (20)$$

$$u_s^{-i,\text{FB}} - \pi_t^{-i}(x_s) = 0, \quad \forall s \in T_{t+1}, \quad (21)$$

$$\begin{aligned} \nabla_{x_s} \ell_s^i - \lambda_{s-1}^i + (\nabla_{x_s} f_s)^\top \lambda_s^i + (\nabla_{x_s} \pi_s^{-i})^\top \psi_s^i \\ = 0, \quad \forall s \in T_{t+1}. \end{aligned} \quad (22)$$

2) Sufficiency Conditions for Local FBNE: The stage-wise Lagrangian for agent i in the feedback case becomes

$$\begin{aligned} \mathcal{L}_s^{i,\text{FB}} = \ell_s^i - \lambda_s^{i\top} (x_{s+1} - f_s) \\ - \psi_s^{i\top} (u_s^{-i,\text{FB}} - \pi_s^{-i}), \quad s \in T_{t+1}. \end{aligned} \quad (23)$$

in addition to the KKT conditions (17)-(22), using (23), the second order sufficiency condition for a local FBNE becomes

$$\begin{aligned} d_{u_t^i}^\top \nabla_{u_t^i}^2 \ell_t^i d_{u_t^i} + \sum_{s=t+1}^K \begin{bmatrix} d_{x_s} \\ d_{u_s^i} \\ d_{u_s^{-i}} \end{bmatrix}^\top \nabla_{x_s, u_s^i, u_s^{-i}}^2 \mathcal{L}_s^{i,\text{FB}} \begin{bmatrix} d_{x_s} \\ d_{u_s^i} \\ d_{u_s^{-i}} \end{bmatrix} \\ + d_{x_{K+1}}^\top \nabla_{x_{K+1}}^2 \ell_{K+1}^i d_{x_{K+1}} > 0 \end{aligned} \quad (24)$$

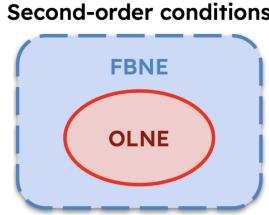


Fig. 1. For a zero-sum game with only dynamics-induced equality constraints, the second-order necessary/sufficiency conditions for a local OLNE are a subset of the corresponding conditions for a local FBNE.

$$\forall \{d_{u_t^i}, d_{x_s}, d_{u_s^i}, d_{u_s^{-i}}, d_{x_{K+1}}\}, s \in T_{t+1} \text{ s.t.}$$

$$d_{x_{s+1}} - \nabla_{x_s} f_s d_{x_{s+1}} - \nabla_{u_s^i} u_s^{-i} f_s \begin{bmatrix} d_{u_s^i} \\ d_{u_s^{-i}} \end{bmatrix} = 0 \quad \forall s \in T_{t+1}, \quad (25)$$

$$d_{x_{t+1}} - \nabla_{u_t^i} f_s d_{u_t^i} = 0, \quad (26)$$

$$d_{u_s^{-i}} - \nabla_{x_s} \pi_s^{-i} d_{x_s} = 0 \quad \forall s \in T_{t+1}. \quad (27)$$

Here, $\begin{bmatrix} d_{u_t^i}^\top & d_{x_s}^\top & d_{u_s^i}^\top & d_{u_s^{-i}}^\top & d_{x_{K+1}}^\top \end{bmatrix}^\top \neq 0$, $s \in T_{t+1}$ represents a direction in the critical cone of the feedback equilibrium problem (15).

Remark 1: For agent i , the feedback game (15) differs from the open-loop game (2) by having (i) an equilibrium problem at every time step, and (ii) an additional constraint forcing agent $-i$ to play a feedback policy. Consequently, the feedback KKT conditions (17)-(22) and second-order conditions (24)-(27) differ from the corresponding open-loop KKT and second-order conditions (4)-(7), (9)-(10). However, we observe that if the feedback policy constraints in the feedback game are *weakly active* for both agents at a local FBNE, i.e., $\psi_s^i = \psi_s^{-i} = 0$, and the feedback KKT conditions reduce to the open-loop KKT conditions. This insight motivates our results presented in Section IV. Moreover, we will show that, in this case, the set of all open-loop critical cone directions becomes a *subset* of the set of all feedback critical cone directions (cf. Theorem 1). A similar discussion holds for the second-order necessary conditions for local OLNE and local FBNE, which are similar to the sufficiency conditions with the exceptions of a nonstrict inequality and the requirement of an appropriate constraint qualification [17, Th. 12.6].

IV. MAIN RESULTS

In the following theorem, we show how state/control trajectories corresponding to any local FBNE policies are related to a local OLNE for the zero-sum game of the type described in Section III, i.e., containing no constraints apart from those due to the game dynamics.

Theorem 1: Consider a two-agent zero-sum dynamic game with K stages and dynamics constraints. Then:

- 1) The controls and states corresponding to any existing local FBNE also satisfy the first- and second-order necessary conditions for a local OLNE of the game.
- 2) The controls and states corresponding to any existing local OLNE also satisfy the first-order necessary conditions for a local FBNE of the game.
- 3) Assume that a local FBNE of the game exists, and that the corresponding local feedback equilibrium strategies

$u^{i*,FB} = \{\pi_t^{i*}(x_t), t \in [K]\}$, $i = 1, 2$, and state trajectory $x(u_{1:K}^{1*,FB}, u_{1:K}^{2*,FB}) = \{x_1^*, \dots, x_{K+1}^*\}$ satisfy the FBNE second-order sufficiency conditions. Then $u^{i*,FB}$ and $x(u_{1:K}^{1*,FB}, u_{1:K}^{2*,FB})$ also constitute a local OLNE of the game.

Proof: We prove the first part by showing that at any local FBNE of the game, the feedback policy constraints (21) are weakly active for both agents. The KKT condition for agent $-i$ corresponding to (19) is

$$\nabla_{x_{K+1}} \ell_{K+1}^{-i}(x_{K+1}^*) - \lambda_{K+1}^{-i} = 0. \quad (28)$$

Because the game is zero-sum, $\nabla_{x_{K+1}} \ell_{K+1}^{-i} = -\nabla_{x_{K+1}} \ell_{K+1}^i$, and from (28) combined with (19) evaluated at x_{K+1}^* we get

$$\lambda_K^{-i} = -\lambda_K^i. \quad (29)$$

Using the KKT condition (17) corresponding to agent $-i$ evaluated at time $s = K$, and (29), we get

$$-\nabla_{u_K^i} \ell_K^i - \left(\nabla_{u_K^i} \ell_K \right)^\top \lambda_K^i - \psi_K^{-i} = 0. \quad (30)$$

Using (18) evaluated at time $s = K$ and (30) we get

$$\psi_K^{-i} = 0. \quad (31)$$

Similarly, for the Lagrange multiplier for agent i in the KKT conditions of agent $-i$, we get

$$\psi_K^i = 0. \quad (32)$$

Using the KKT condition (22) for both agents $i, -i$ evaluated at time $s = K$, (29), (31) and (32), we get

$$\lambda_{K-1}^{-i} = -\lambda_{K-1}^i. \quad (33)$$

Equation (33) suggests the following recursive pattern, which can be verified by continuing the previous arguments: $\lambda_t^{-i} = -\lambda_t^i$, $\forall t \in [K]$. This yields

$$\psi_t^i = \psi_t^{-i} = 0 \quad \forall t \in T_2. \quad (34)$$

Thus, at a local FBNE, the constraints forcing both agents to play feedback policies are weakly active. From Remark 1 and (34), we can conclude that the local FBNE KKT conditions (17)-(22) imply the local OLNE KKT conditions (4)-(7) and thus $u^{1*,FB}$, $u^{2*,FB}$, $x(u_{1:K}^{1*,FB}, u_{1:K}^{2*,FB})$ also satisfy the first-order necessary conditions for a local OLNE.

Now consider the feedback second-order sufficiency conditions (24)-(27). Because the feedback policy constraints are weakly active, from Remark 1, the condition (27) can be dropped while considering the directions in the critical cone for which (24) should hold. Further, we observe that the directions such that

$$d_{u_t^{-i}} = 0 \quad (35)$$

$$d_{x_{t+1}} - \nabla_{u_t^i} f_s d_{u_t^i} = 0, \quad (36)$$

$$d_{x_{s+1}} - \nabla_{x_s} f_s d_{x_{s+1}} - \nabla_{u_s^i} f_s d_{u_s^i} = 0 \quad \forall s \in T_{t+1}, \quad (37)$$

all lie in the directions in the critical cone defined by (25), (26). We observe that (35)-(36) when applied to (24) exactly yield the open-loop second-order sufficiency conditions (9)-(10). *Thus the directions in the open-loop critical cone are a subset of the directions in the feedback critical cone.* From Remark 1 a similar relation holds for second order necessary conditions

between feedback and open-loop settings. Because the FBNE controls/states must be satisfying the feedback second-order necessary conditions, this implies that they also satisfy the OLNE second order necessary conditions. This proves the first part of the theorem.

To prove the second part of the theorem, let a local OLNE exist and $u_{1:K}^{1*,OL}, u_{1:K}^{2*,OL}, x_{1:K+1}^{*,OL} := x(u_{1:K}^{1*,OL}, u_{1:K}^{2*,OL})$ denote the corresponding controls and states. Thus, $u_{1:K}^{1*,OL}, u_{1:K}^{2*,OL}, x^{*,OL}$ satisfy the open-loop KKT conditions (4)-(7). Then it is trivial to observe that $u_{1:K}^{1*,OL}, u_{1:K}^{2*,OL}, x^{*,OL}$ also satisfy the feedback KKT conditions (17)-(22) for the choice of feedback policy constraints $\psi_t^i = \psi_t^{-i} = 0 \forall t \in [K]$. Note that the open-loop trajectory also satisfies (21), because the open-loop controls are derived by solving the KKT conditions backwards in time, which implies that the local OLNE controls satisfy (14).

For the third part of the theorem, if $u^{1*,FB}, u^{2*,FB}, x(u_{1:K}^{1*,FB}, u_{1:K}^{2*,FB})$ satisfy the feedback second-order sufficiency conditions (24)-(27), from the discussion above, we can conclude that $u^{1*,FB}, u^{2*,FB}, x(u_{1:K}^{1*,FB}, u_{1:K}^{2*,FB})$ also satisfy the second-order sufficiency conditions of (and thus constitute) a local OLNE of the game. ■

V. A BROADER CLASS OF ZERO-SUM DYNAMIC GAMES

The results in Section IV rely on a cascading pattern which ensures that the feedback policy constraints (21) are only weakly active at a local FBNE. While true for a zero-sum game with only dynamics constraints, this pattern may not hold true for zero-sum games with more arbitrary state/control constraints in general. In this section, however, we identify a broad class of games in which a similar property still arises.

Consider the class of two-agent zero-sum games with (i) dynamics constraints, and (ii) control bound (inequality) constraints for each agent. Such bound constraints often occur in practical applications, for example, in systems with hard actuator limits.

For such a game, the corresponding open-loop equilibrium problem for agent i is

$$\begin{aligned} \min_{x_{2:K+1}, u_{1:K}^i} \quad & \sum_{t=1}^K \ell_t^i(x_t, u_t^i, u_t^{-i}) + \ell_{K+1}^i(x_{K+1}) \\ \text{s.t.} \quad & x_{t+1} = f_t(x_t, u_t^1, u_t^2), \quad t \in [K], \\ & a_t^i \leq u_t^i \leq b_t^i, \quad t \in [K]. \end{aligned} \quad (38)$$

Using Lagrange multipliers $\tilde{\lambda}_t^i, \underline{v}_t^i, \bar{v}_t^i, t \in [K]$, the Lagrangian for agent i defined as

$$\begin{aligned} \tilde{\mathcal{L}}^{i,OL} := \sum_{t=1}^{K+1} \ell_t^i - \sum_{t=1}^K \left(\tilde{\lambda}_t^{i\top} (x_{t+1} - f_t) \right. \\ \left. + \underline{v}_t^{i\top} (u_t^i - a_t^i) + \bar{v}_t^{i\top} (b_t^i - u_t^i) \right). \end{aligned} \quad (39)$$

The feedback equilibrium problem for agent i becomes

$$\begin{aligned} \min_{u_{t:K}^i, u_{t+1:K+1}^{-i,FB}} \quad & \sum_{s=t}^K \ell_s(x_s, u_s^i, u_s^{-i,FB}) + \ell_{K+1}(x_{K+1}), \\ \text{s.t.} \quad & x_{s+1} - f_s(x_s, u_s^i, u_s^{-i,FB}) = 0, \quad s \in T_t, \end{aligned}$$

$$\begin{aligned} u_s^{-i,FB} - \pi_t^{-i}(x_s) &= 0, \quad s \in T_{t+1}, \\ a_s^i \leq u_s^{i,FB} \leq b_s^i, \quad & s \in T_t. \end{aligned} \quad (40)$$

Note that we do not encode agent $-i$'s control constraints in agent i 's problem at time t , as these constraints will be implicitly imposed through agent $-i$'s problem at future time steps. Using Lagrange multipliers $\tilde{\lambda}_s^i, \underline{v}_s^i, \bar{v}_s^i, s \in T_t$ and $\tilde{\psi}_s^i, s \in T_{t+1}$, the Lagrangian corresponding to (40) is defined as

$$\begin{aligned} \tilde{\mathcal{L}}^{i,FB} = \sum_{s=t}^{K+1} \ell_s^i - \sum_{s=t+1}^K \tilde{\psi}_s^{i\top} (u_s^{-i,FB} - \pi_t^{-i}) \\ - \sum_{s=t}^K \left(\tilde{\lambda}_s^{i\top} (x_{s+1} - f_s) + \underline{v}_s^{i\top} (u_s^i - a_s^i) + \bar{v}_s^{i\top} (b_s^i - u_s^i) \right). \end{aligned} \quad (41)$$

In such constrained games with nonconvex-nonconcave objectives and nonlinear dynamics, it is usually only possible to find solutions that (approximately) satisfy the FBNE first-order necessary conditions using existing game-theoretic feedback methods [4], [7]. These first-order feedback solutions also require much more computation than that required to find solutions satisfying OLNE first-order necessary conditions. Existing open-loop game-theoretic solvers can find exact solutions that satisfy OLNE first-order (and second-order) necessary conditions in such games [12], [13], [14]. Thus, it is valuable to know if some relation between local OLNE and local FBNE akin to Theorem 1 holds in such games as well.

To this end, we show that when strict complementarity holds (a standard assumption) in such zero-sum games (i.e., at a solution of (38) and (40), $u_t^i = a_t^i \implies \underline{v}_t^i > 0, u_t^i = b_t^i \implies \bar{v}_t^i > 0$), the local OLNE first-order necessary conditions are the same as the local FBNE first-order necessary conditions.

Theorem 2: Consider a two-agent zero-sum dynamic game with K stages, dynamic constraints, and control bound (inequality) constraints for each agent. Then:

- 1) The controls and states corresponding to any existing local FBNE where strict complementarity holds also satisfy the first-order necessary conditions for a local OLNE of the game.
- 2) The controls and states corresponding to any existing local OLNE where strict complementarity holds also satisfy the first-order necessary conditions for a local FBNE of the game.

Proof: Local OLNE first-order necessary conditions are

$$\nabla_{x_t} \ell_t^i + \nabla_{x_t} f_t^\top \tilde{\lambda}_t^i - \tilde{\lambda}_{t-1}^i = 0, \quad \forall t \in T_2, \quad (42)$$

$$\nabla_{u_t} \ell_t^i + \nabla_{u_t} f_t^\top \tilde{\lambda}_t^i - \underline{v}_t^i + \bar{v}_t^i = 0, \quad \forall t \in [K], \quad (43)$$

$$\nabla_{x_{K+1}} \ell_{K+1}^i - \tilde{\lambda}_K^i = 0, \quad (44)$$

$$x_{t+1} - f_t(x_t, u_t^1, u_t^2) = 0, \quad \forall t \in [K], \quad (45)$$

$$a_t^i \leq u_t^i \leq b_t^i \geq 0 \text{ and } 0 \leq \underline{v}_t^i \leq \bar{v}_t^i \leq b_t^i, \quad \forall t \in [K]. \quad (46)$$

Similarly, local FBNE first-order necessary conditions for (40) are

$$\nabla_{u_s^{-i,FB}} \ell_s^i + \left(\nabla_{u_s^{-i,FB}} f_s \right)^\top \tilde{\lambda}_s^i - \tilde{\psi}_s^i = 0, \quad s \in T_{t+1}, \quad (47)$$

$$\nabla_{u_s^i} \ell_s^i + \left(\nabla_{u_s^i} f_s \right)^\top \tilde{\lambda}_s^i - \underline{v}_s^i + \bar{v}_s^i = 0, \quad s \in T_t, \quad (48)$$

$$\nabla_{x_{K+1}} \ell_{K+1}^i - \tilde{\lambda}_K^i = 0, \quad (49)$$

$$x_{s+1} - f_s(x_s, u_s^i, u_s^{-i, \text{FB}}) = 0, \forall s \in T_t, \quad (50)$$

$$u_s^{-i, \text{FB}} - \pi_t^{-i}(x_s) = 0, \forall s \in T_{t+1}, \quad (51)$$

$$a_s^i \leq u_s^i \perp y_s^i \geq 0 \text{ and } 0 \leq \bar{v}_s^i \perp u_s^i \leq b_s^i, \forall s \in T_t, \quad (52)$$

$$\nabla_{x_s} \ell_s^i - \tilde{\lambda}_{s-1}^i + (\nabla_{x_s} f_s)^\top \tilde{\lambda}_s^i + (\nabla_{x_s} \pi_s^{-i})^\top \tilde{\psi}_s^i \\ = 0, \forall s \in T_{t+1}. \quad (53)$$

If none of the control bound constraints are active at a local FBNE/OLNE for both agents, then the result trivially follows from Theorem 1. Thus, we need to analyze the case when at least one control bound constraint is active for an agent.

To prove the first part, consider the feedback necessary conditions (47)-(53). From (49) and the corresponding condition for agent $-i$, we get $\tilde{\lambda}_K^i = -\tilde{\lambda}_K^{-i}$, which combined with (47) and agent $-i$'s condition corresponding to (48) at $s = K$ yields

$$\tilde{\psi}_K^i = \bar{v}_K^{-i} - y_K^{-i}, \tilde{\psi}_K^{-i} = \bar{v}_K^i - y_K^i. \quad (54)$$

Let $[q]_j$ denote the j^{th} component of some vector q , and let $[Q]_j$ denote the j^{th} row of some matrix Q . Consider $[u_K^{-i}]_j$ and $[\nabla_{x_K} \pi_K^{-i}]_j$, $j \in [m^{-i}]$. Due to strict complementarity, either one of the bound constraints on $[u_K^{-i}]_j$ is active and $[\nabla_{x_K} \pi_K^{-i}]_j = 0$, or both bounds on $[u_K^{-i}]_j$ are inactive and $[y_K^i]_j = [\bar{v}_K^i]_j = 0$. A similar argument holds for agent $-i$'s problem. Using (54), this implies at any local FBNE, $(\nabla_{x_K} \pi_K^{-i})^\top \tilde{\psi}_K^i = (\nabla_{x_K} \pi_K^i)^\top \tilde{\psi}_K^{-i} = 0$. Substituting this into (53) and using $\tilde{\lambda}_K^i = -\tilde{\lambda}_K^{-i}$, we get $\tilde{\lambda}_{K-1}^i = -\tilde{\lambda}_{K-1}^{-i}$, and a cascading pattern as before yields

$$(\nabla_{x_t} \pi_K^{-i})^\top \tilde{\psi}_K^i = (\nabla_{x_t} \pi_K^i)^\top \tilde{\psi}_K^{-i} = 0 \forall t \in T_2. \quad (55)$$

Equation (55) implies that at a local FBNE, the OLNE first-order necessary conditions (42)-(46) are a subset of the corresponding FBNE conditions (47)-(53).

To prove the second part, let a local OLNE exist and $u_{1:K}^{1*, \text{OL}}, u_{1:K}^{2*, \text{OL}}, x_{1:K+1}^{*, \text{OL}} := x(u_{1:K}^{1*, \text{OL}}, u_{1:K}^{2*, \text{OL}})$ denote the corresponding controls and states. Thus, $u_{1:K}^{1*, \text{OL}}, u_{1:K}^{2*, \text{OL}}$ and $x_{1:K+1}^{*, \text{OL}}$ satisfy the open-loop conditions (42)-(46). Then through the discussion above, it is trivial to observe that $u_{1:K}^{1*, \text{OL}}, u_{1:K}^{2*, \text{OL}}$ and $x_{1:K+1}^{*, \text{OL}}$ also satisfy the feedback first-order necessary conditions (47)-(53) for a choice of $\tilde{\psi}_t^i = \bar{v}_t^{-i} - y_t^{-i}, \tilde{\psi}_t^{-i} = \bar{v}_t^i - y_t^i, t \in T_2$. Thus, $u_{1:K}^{1*, \text{OL}}, u_{1:K}^{2*, \text{OL}}$ and $x_{1:K+1}^{*, \text{OL}}$ satisfy the first-order necessary conditions for a local FBNE of the game. ■

VI. CONCLUSION

In non-linear-quadratic dynamic games, a feedback Nash equilibrium (FBNE) is much more computationally expensive to compute than an open-loop equilibrium (OLNE). Thus, it is valuable to understand relationships between the two equilibrium types. To this end, we show that for a large class of zero-sum games with potentially nonconvex-nonconcave (smooth) agent costs and nonlinear dynamics:

- 1) Any local FBNE also satisfies the first-order necessary conditions for a local OLNE, and vice versa.
- 2) Any local FBNE also satisfies the second-order necessary conditions for a local OLNE of the game. Further, a local FBNE satisfying feedback sufficiency conditions also constitutes a local OLNE of the game.

- 3) In the presence of additional constraints on agents' control variables, any local FBNE (where strict complementarity holds) still satisfies the first-order necessary conditions for a local OLNE, and vice versa.

These results are immediately applicable—state of the art feedback game solvers [4], [7] are computationally expensive, yet are only guaranteed to find an *approximate* first-order local FBNE. Our results imply that one can instead find exact local OLNE solutions (e.g., via existing solvers [12], [13], [14]) and by implication, also *exact* first-order local FBNE solutions—at much less computational cost compared to the approximate solvers—leading to our claim that “More information is not always better”. We believe many important dynamic game applications in motion planning, energy management, finance, supply chain management, etc., will benefit from our results.

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