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Stable and non-symmetric pitchfork bifurcations

In Memory of Professor Shantao Liao

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Abstract In this paper, we present a criterion for pitchfork bifurcations of smooth vector fields based on a topological argument. Our result expands Rajapakse and Smale's result [15] significantly. Based on our criterion, we present a class of families of non-symmetric vector fields undergoing a pitchfork bifurcation.

Keywords pitchfork bifurcation, index, non-symmetric vector field, center manifold

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1 Introduction

In this paper, we consider the bifurcation of isolated equilibria of locally defined vector fields in $\mathbb{R}^n$. This well-studied subject has recently had some fresh observations by Rajapakse and Smale [15] concerning the pitchfork bifurcation and its relevance for biology. It is our intention to expand on their treatment by generalizing the hypotheses and uncovering significant new subtleties.

First, let us recall the context: when locally defined vector fields and their bifurcations are used to model a phenomenon in the observable world, the fact that the phenomenon is observable at all speaks to its stability under small perturbation. The dogma of perturbation and bifurcation theory reasonably asserts that the aspects of the dynamics of the vector fields and their bifurcations used to explain the phenomenon should be stable as well. The only generic and stable simple non-hyperbolic bifurcation with one-dimensional parameters is the saddle-node bifurcation, in which zeros of adjacent indices are created or canceled.

While the pitchfork bifurcation is not generally stable, it is stable under a certain additional hypothesis such as symmetry (namely equivariant branching) or the vanishing of a certain second derivative at the bifurcation point (see [12] and [9, Theorem 7.7]). The stability and the symmetry of the pitchfork bifurcation is usually expressed in terms of its normal form $\dot{u} = u\varepsilon - u^3$. This family of vector fields is invariant under the involution $u \to -u$. Rajapakse and Smale [13–15] were most interested in the case where one stable equilibrium gives rise to two new stable equilibria after the bifurcation and without

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symmetry. They argue that if the state of a cell is modeled as a stable equilibrium, then the cellular division should give rise to two new stable equilibria after division. They model this phenomenon with a non-symmetric pitchfork bifurcation in which one stable equilibrium gives rise to three, two new stable and one unstable. We generalize their results significantly and supply complete proofs.

Consider a one-parameter family of C^2 vector fields in $\mathbb{R}^n$ given by $\dot{x} = V(x, \varepsilon)$ where $\varepsilon \in \mathbb{R}^1$. A point (x_0, ε_0) is *simple non-hyperbolic* if $D_x V(x_0, \varepsilon_0)$ has a simple eigenvalue $\lambda = 0$ and all other eigenvalues are not on the imaginary axis. A fixed point $(x, \varepsilon) = (x_0, \varepsilon_0)$ is said to undergo a $1 \to$ *many bifurcation*, if the flow has one and only one fixed point in a neighborhood of x_0 for any sufficient close $\varepsilon \leqslant \varepsilon_0$ while the flow has many fixed points around x_0 for any sufficient close $\varepsilon > \varepsilon_0$. A fixed point $(x, \varepsilon) = (x_0, \varepsilon_0)$ is said to undergo a *many* $\to 1$ *bifurcation*, if the flow has many fixed points in a neighborhood of x_0 for any sufficient close $\varepsilon \leqslant \varepsilon_0$ while the flow has one and only one fixed point around x_0 for any sufficient close $\varepsilon > \varepsilon_0$. We say that the bifurcation is of *pitchfork type* if there is a neighborhood of x_0 such that x_0 is the unique non-hyperbolic zero in the neighborhood for any sufficient close $\varepsilon \leqslant \varepsilon_0$, x_0 continues smoothly to x_ε as one of the equilibrium points for any sufficiently close $\varepsilon > \varepsilon_0$ and the number of the equilibrium points for any sufficient close $\varepsilon > \varepsilon_0$ is greater than or equal to three. Moreover it is called a *pitchfork bifurcation* if the number of new equilibria for a pitchfork-type bifurcation is exactly three.

In the literature, the hypotheses to guarantee the existence of a pitchfork bifurcation generally contain one of the following two types of assumptions:

Type a. The set of zeros (x_t, ε_t) consists of one stable equilibrium of index -1 for $t < 0$ which continues smoothly to (x_t, ε_t) of index 1 for $t > 0$ when an eigenvalue at the zeros changes from negative to positive (see [3–5, 10, 15]).

Type b. The equation has some symmetry which is frequently exhibited by a normal form with respect to a center manifold which is assumed to be explicitly known (see [9, Chapter 7], [18, Chapter 19] and the references therein).

In this paper, we assume neither of these scenarios. We work with n-dimensional vector fields and prove that Type a follows from our hypotheses. Our hypotheses are much easier to check compared with the hypotheses of Type a or Type b. We also give examples without symmetry and counterexamples to show that if any of our hypotheses fails there may not be a pitchfork-type or a pitchfork bifurcation.

An essential part of our treatment relies on a topological argument. We refer to [8, 11, 16, 17] for some work in the literature using topological approaches in dealing with bifurcation problems. Here we consider under which conditions the bifurcation of an isolated simple non-hyperbolic equilibrium with non-zero index gives rise to many equilibria with non-zero index (see Subsection 4.1 for the definition of index). We are interested in the bifurcation of stable equilibria which are interior to the basin of attraction. Our criteria for the bifurcation are multidimensional (see (P0)–(P2) below) and are expressed in terms of the derivative at the bifurcation point. We do not invoke the explicit form of a reduction to the center manifold, even for (P3). Based on our criterion, we give an example of a family of vector fields without symmetry which undergoes a pitchfork bifurcation.

Fix (x_0, ε_0). Denote by $\mathcal{F}$ the set of one-parameter C^∞ vector fields V which satisfy the following conditions:

(P0) $V(x, \varepsilon_0)$ has an isolated simple non-hyperbolic equilibrium x_0 with non-zero index;

(P1) $\frac{\partial V}{\partial \varepsilon}|_{(x_0, \varepsilon_0)} \in \operatorname{image}(D_x V)$;

(P2) there exists $\omega = (\omega_1, \ldots, \omega_{n+1})^{\mathrm{T}}$ such that

$$DV(x_0, \varepsilon_0)\omega = 0 \quad \text{and} \quad \omega_{n+1} \neq 0, \quad \text{and} \quad D(\det(D_x V))(x_0, \varepsilon_0)\omega \neq 0.$$

Our conditions are easy to check. Here we make some comments.

- (P0) is immediate if there exists a small ball $B(x_0)$ around x_0 such that for any $\varepsilon < \varepsilon_0$ close enough to ε_0, there is one and only one zero inside the small ball $B(x_0)$ and it is transversal, i.e., 0 is not an eigenvalue of the derivative. Then the index of the zero at $\varepsilon = \varepsilon_0$ is either 1 or -1 (see Subsection 2.1).
- (P1) is verified if the rank of the derivative $DV(x_0, \varepsilon_0)$ is $n - 1$.
- In the coordinates $(x_1, \ldots, x_n)$ given by the eigenspaces of $D_x V(x_0, \varepsilon_0)$ with $(1, 0, \ldots, 0)$ the zero eigenvector, (P2) is true if and only if $\frac{\partial^2 V_1}{\partial x \partial \varepsilon} \neq 0$.

Theorem 1.1 (Bifurcation). *Every $V \in \mathcal{F}$ undergoes a pitchfork-type bifurcation, i.e., it is a $1 \to k$ or $k \to 1, 3 \leqslant k \leqslant +\infty$ bifurcation at (x_0, ε_0).*

Theorem 1.1 implies Rajapakse and Smale's result [15].

Corollary 1.2 (See [15]). *Suppose the following conditions*:

(1) $\frac{dx}{dt} = V(x,\varepsilon), x \in X, V(x_0,\varepsilon) = 0$ *for* $\varepsilon \leqslant \varepsilon_0$ *and the determinant of the Jacobian of* V *at* (x_0,ε_0) *is zero.*

(2) *The eigenvalues of the Jacobian matrix satisfy*

$$\mathrm{real}(\lambda_i) < 0, \quad i > 1; \quad \lambda_1 = 0 \quad \textit{and} \quad \frac{d\lambda_1}{d\varepsilon}\bigg|_{(x_0,\varepsilon_0)} > 0.$$

(3) *The multiplicity of* $V(x,\varepsilon_0)$ *at* x_0 *is three and the Poincarè index is* $(-1)^n$ *relative to a disk* B_r^n *about* x_0.

These are sufficient conditions for the pitchfork bifurcation.

The condition (P1) is trivial in Corollary 1.2, i.e., $\frac{\partial V}{\partial \varepsilon}(\varepsilon_0, x_0) = 0$ since V is constant in ε at x_0. Conditions (P0) and (P2) are also trivial in Corollary 1.2. The multiplicity assumption in Corollary 1.2 implies the bifurcation given in Theorem 1.1 is exactly one to three. Hence Theorem 1.1 is much more general. We refer to Section 2 for an example, where (P0)–(P2) are not trivial while Theorem 1.1 applies. Moreover, Corollary 1.2 may be false if the conditions are not satisfied (see Section 3).

As one may have noticed, one of the key points in Theorem 1.1 is that we consider the derivative of the determinant of DV, instead of V, DV and D^2V as the classical argument goes. The proof of theorem 1.1 goes in two steps. Here is an outline:

Step 1. From the fact that the equilibrium is a simple non-hyperbolic point, it follows that there is a center manifold normally hyperbolic associated with it. Moreover, it is shown that the index property can be reduced to the index restricted to the center manifold.

Step 2. (P0) and (P1) guarantee a continuation of the zero to any parameter value near the bifurcation parameter. This follows from considering the dynamics of vector fields along the center manifolds and then proving the fact that (P0) and (P1) are carried over to the dynamics along the center manifold. Moreover, (P2), i.e., the condition on the derivative of the determinant implies that the bifurcation is of pitchfork type, meaning that at least two new zeros with different indices arise after the bifurcation.

A natural question is to consider how many equilibria appear in Theorem 1.1. The following theorem gives a criterion for the existence of a pitchfork bifurcation which does not depend on the multiplicity hypothesis in Corollary 1.2. Let $\mathcal{G} \subset \mathcal{F}$ be such that any $V \in \mathcal{G}$ is of the form

$$\begin{cases} \dot{u} = F(u,y,\varepsilon), \\ \dot{y} = My + G(u,y,\varepsilon), \end{cases}$$

where $u \in \mathbb{R}^1$ and $y \in \mathbb{R}^{n-1}$, the square matrix M has eigenvalues with only non-zero real parts and

$$F(0,0,0) = 0, \quad DF(0,0,0) = 0, \quad G(0,0,0) = 0, \quad DG(0,0,0) = 0,$$

satisfy one extra condition

(P3) $D_{uu} \det DV\,|_{(0,0,0)} - D_y(\det DV)(M^{-1}G_{uu})\,|_{(0,0,0)} \neq 0$.

We give a comment here about (P3).

- In the case where the center manifold is explicitly known, with the presence of the condition (P0), the conditions (P1)–(P3) are equivalent to the usual hypotheses that the third derivative of V restricted to the center manifold is not zero. But we re-emphasize that (P3) does not require knowledge of the center manifold which might be difficult to compute.

Theorem 1.3 (Pitchfork bifurcation). *Every $V \in \mathcal{G}$ undergoes a $1 \to 3$ or $3 \to 1$ bifurcation.*

Endow $\mathcal{F}$ with the usual topology of C^∞ maps. Based on Theorem 1.3, we obtain the genericity of the pitchfork bifurcation.

Theorem 1.4. *There exists an open and dense subset of vector fields with the pitchfork bifurcation in $\mathcal{F}$.*

It is worth noting that there is a Banach space version of these theorems where the index refer to the index in the finite dimensional center manifold.

The rest of this paper is organized as follows. In Section 2, we give some examples: one of them shows the lack of stability of the pitchfork bifurcation under general perturbation and one shows the existence of a pitchfork bifurcation without symmetry. In Section 3, we provide examples that show if any of the assumptions fails, there may not be a pitchfork bifurcation. In Section 4, we introduce some preliminaries. As a preparation for the proof of Theorem 1.1, we give some discussions on the index of fixed points in Section 5. Then in Section 6 we deliver some observations for the one-dimensional case. In Section 7, we present Theorem 1.1 based on center reduction techniques and the product property of the index of fixed points. We also give the proof of Corollary 1.2. In Section 8, we give the proofs of Theorems 1.3 and 1.4 based on an analysis of graph transform.

To finish Section 1, we give some comments and comparison with some results in the literature. We give necessary and sufficient conditions for those pitchfork bifurcations which can be put in the normal form. Our conditions are in terms of the Taylor expansion of V at the point (x_0, ε_0) alone. This makes the conditions significantly easier to check. In previous studies of the pitchfork bifurcation, for example the one provided by Crandall and Rabinowitz [4, 5] and explained in Subsection 6.6 of the book *Methods of Bifurcation* by Chow and Hale [3], it is explicitly assumed that for any parameter nearby the bifurcation point there is at least one zero, i.e., there is a branch of solutions through $(0, 0)$. In particular, that hypothesis is not assumed in our paper. Moreover, it is shown in Example 3.3 that even assuming there is a branch of solutions, if the other condition (P2) is not satisfied then it could happen that there is no bifurcation. Also, Example 3.2 shows that the conditions provided in [13] is not enough to guarantee a bifurcation if the zero of the initial vector field is allowed to move.

Our work shares some similarities with [10, Theorem 2.1], where the authors also give a criterion for the existence of transcritical and pitchfork bifurcations without known center manifolds and a line of known solutions. The condition (F1) there also assumes simple eigenvalues, as part of our condition (P0) here; (F2′) there is the same as (P1) here. Besides (F1) and (F2′), the work in [10] requires $\det(H_0) \neq 0$ (see Formula (2.2) in [10] for the definition of H_0) which is different from the other conditions in our paper. Our result does apply to some cases when $\det(H_0) = 0$. Such an example can always be constructed by varying the coefficients of the x^2, y^2, ϵ^2 terms in Example 2.3 in Section 2. In addition, the techniques used here are different from the techniques used there.

An excellent detailed account of the pitchfork bifurcation with a Taylor series description may be found in [6, p. 25]. The discussion proceeds via Liapunov-Schmidt reduction and extends to group symmetries and universal unfolding. The main difference with the treatment here is a difference of perspective. The index assumption is crucial for us and does not appear in [6]; moreover, the center manifold reduction also gives a clear idea of the dynamics after the bifurcation which is not immediately clear from the singularity perspective.

2 Examples

In this section, we will use our method to detect bifurcations. Compared with the classical method, the normal form, our method tends to be more efficient. We also give a construction of a one-parameter family of vector fields without symmetry which undergoes a pitchfork bifurcation.

Example 2.1 (Revisiting the Rajapakse-Smale example). Consider the vector field X with

$$\begin{cases} \dot{x} = y^2 - (\varepsilon + 1)y - x, \\ \dot{y} = x^2 - (\varepsilon + 1)x - y \end{cases}$$

near the equilibrium point $(x, y) = (0, 0)$ at the bifurcation parameter $\varepsilon = 0$. We use Theorem 1.1 to verify the existence of bifurcations: for $\varepsilon < 0$, around $(0, 0)$ there is one and only one real equilibrium:

$(x_0, y_0) = (0, 0)$. Moreover,

$$\det(D_x V) = 1 - (2x - (\varepsilon + 1))(2y - (\varepsilon + 1)).$$

Here, we use D_x to stand for $D_{(x,y)}$. Then

$$\frac{\partial}{\partial \varepsilon} \det(D_x V)\,|_{((x,y),\varepsilon)=((0,0),0)} = -2.$$

Now let us verify the condition (P2): $\frac{\partial V}{\partial \varepsilon}(0,0) = (0,0)$. Hence we have all of the conditions in Theorem 1.1 for Example 2.1. Since the multiplicity of $(0,0)$ is three (one zero far away from $(0,0)$), we have a pitchfork bifurcation.

Here, we also give the argument by using the classical method, the normal form, as a comparison. Under the change of the coordinates

$$\begin{cases} u = x - y, \\ v = x + y, \end{cases}$$

we get

$$\begin{cases} \dot{u} = \varepsilon u - uv, \\ \dot{v} = -(2+\varepsilon)v + \dfrac{u^2 + v^2}{2}. \end{cases}$$

For ε near 0, we reduce this vector field to a parametrized equation along the local center manifold, i.e., $\dot{u} = \varepsilon u - uh(u, \varepsilon)$, where $v = h(u, \varepsilon)$ satisfies that $h(0, 0) = 0$, $D_{(u,\varepsilon)} h(0, 0) = 0$, and

$$\partial_u h(u, \varepsilon)[\varepsilon u - uh(u, \varepsilon)] = -(2 + \varepsilon)h(u, \varepsilon) + \frac{u^2 + (h(u, \varepsilon))^2}{2}.$$

Taking $\varepsilon = 0$, and expanding $h(u, 0) = h_2 u^2 + O(u^3)$, we get $h_2 = \frac{1}{4}$. Therefore, we obtain $\dot{u} = \varepsilon u - \frac{1}{4}u^3$. By Lemma 6.4, this vector field experiences a pitchfork bifurcation.

Even though Example 2.1 does not have $(x, y) \to (-x, -y)$ symmetry, it does have center symmetry, i.e., $(x, y) \to (y, x)$. Here, we add a small perturbation of the Rajapakse-Smale example to destroy the symmetry. We recall the definition of symmetry for a vector field.

Definition 2.2 (See [9, p. 278])**.** We say that the vector field $\dot{x} = V(x, \varepsilon)$, $x \in \mathbb{R}^n$, $\varepsilon \in \mathbb{R}$, has *symmetry* if there exists a matrix transformation $R : x \mapsto Rx$ such that

$$RV(x, \varepsilon) = V(Rx, \varepsilon), \quad R^2 = I.$$

Example 2.3 (Pitchfork bifurcation without symmetry)**.** Consider the 2-D ordinary differential equation (ODE)

$$\begin{cases} \dot{x} = y^2 - (\varepsilon + 1)y - x, \\ \dot{y} = x^2 - (\varepsilon + 1)x - y + \varepsilon^2 \end{cases}$$

near the equilibrium point $(x, y) = (0, 0)$ at the bifurcation parameter $\varepsilon = 0$. We note here that Corollary 1.2 does not apply to this example. We use Theorem 1.1 to verify the pitchfork bifurcation: for $\varepsilon < 0$, there are only one equilibrium $(x_0, y_0) = (0, 0)$ in a neighborhood of $(0, 0)$. Moreover,

$$\det(D_x V) = 1 - (2x - (\varepsilon + 1))(2y - (\varepsilon + 1)).$$

Hence $\det(D_x V) = 0$ at $((0, 0), 0)$ and $\frac{\partial}{\partial \varepsilon} \det(D_x V)\,|_{((x,y),\varepsilon)} = -2$. Hence we have all of the conditions in Theorem 1.1 for Example 2.3. Since the multiplicity of $(0, 0)$ is three (one zero far away from $(0, 0)$), the pitchfork bifurcation follows.

Example 2.4 (Perturbation of the pitchfork bifurcation)**.** Consider the following family of vector fields:

$$\begin{cases} \dot{x} = y^2 - (\varepsilon + 1)y - x, \\ \dot{y} = (1 + \varepsilon_0)x^2 - (\varepsilon + 1)x - y. \end{cases}$$

Let $y^2-(\varepsilon+1)y-x=0$ and $(1+\varepsilon_0)x^2-(\varepsilon+1)x-y=0$. Then we have $x=y^2-(\varepsilon+1)y$. Plugging it into the second one at $\varepsilon=0$ gives $(1+\varepsilon_0)(y^2-y)^2-(y^2-y)-y=0$, i.e., $y^2((1+\varepsilon_0)y^2-2(1+\varepsilon_0)y+\varepsilon_0)=0$. Hence as long as $\varepsilon_0\neq 0$, we have four zeros $y=0$, $y=0$ and $y=\frac{1+\varepsilon_0\pm\sqrt{1+\varepsilon_0}}{1+\varepsilon_0}$. Hence the vector field can only undergo a saddle-node bifurcation at $(0,0)$ while for $\varepsilon_0=0$, we already know it undergoes a pitchfork bifurcation. We can view this as the perturbation of the Rajapakse and Smale example. As long as $\varepsilon_0\neq 0$, the vector field undergoes a saddle-node bifurcation which may be hard to see numerically. When $\varepsilon_0=0$, it undergoes a pitchfork bifurcation. This shows clearly that the pitchfork bifurcation is not stable. Actually, because the derivative at the bifurcation point in (x,ε) has a two-dimensional kernel, the bifurcation cannot be transversal to the zero section. Hence the pitchfork bifurcation is not stable. This is also clearly visible from the fact that the zero set is not locally a manifold.

3 Necessity of the conditions provided

In the present section, we show through examples that if any of our conditions is not satisfied then the pitchfork bifurcation may not happen.

Example 3.1 (Missing (P0): Transcritical bifurcation)**.** The vector field

$$\dot{x}=\varepsilon x+x^2+x^3$$

has the transcritical bifurcation locally. The zeros are given by $x=0$ and $x=\frac{-1\pm\sqrt{1-4\varepsilon}}{2}$. We miss (P0) because the index of $x=0$ is zero. Even though the other conditions (P1)–(P3) are all satisfied, we do not have a pitchfork bifurcation in this example.

Example 3.2 (Missing (P1): No bifurcation)**.** The vector field

$$\dot{x}=\varepsilon-\varepsilon x+x^3$$

has no bifurcation. It is easy to see that (P0), (P2) and (P3) all hold, but (P1) does not. There is only one solution for small ε. This is because if (P1) does not hold, then the zeros lie on a smooth curve through (x_0,ε_0). So there is no bifurcation. In this example, the eigenvalues of the zeros go from positive to zero to positive.

Example 3.3 (Missing (P2): A moving center manifold)**.** Consider

$$\begin{cases}\dot{x}=x\varepsilon+2xy+x^3,\\ \dot{y}=2y+\varepsilon.\end{cases}$$

We have $\frac{\partial V}{\partial\varepsilon}=(0,1)$ which is transversal to the center direction $(1,0)$. However, this is not enough. Also

$$\frac{\partial\det(DV)}{\partial\varepsilon}(0,0)=\frac{\partial(\varepsilon+2y+3x^2)}{\partial\varepsilon}=2\neq 0.$$

However, there is no bifurcation. The only equilibrium is $(0,-\frac{\varepsilon}{2})$. This is because the (P2) condition is not satisfied: the kernel of DV is generated by $(1,0,0)$ and $(0,-\frac{1}{2},1)$. So

$$D(\det DV)(0,1,0)=0,\quad D(\det DV)\left(0,-\frac{1}{2},1\right)=0.$$

Example 3.4 (Missing (P3): $1\to k, k>3$ bifurcation)**.** Consider the vector fields

$$\dot{x}=x\left(\varepsilon-x^2\sin^2\frac{1}{x}-x^4\right).$$

We claim that this example satisfies the conditions in our main theorem. Now let us prove this claim. When $\varepsilon=0$, $V(x,0)=-x^3\sin^2\frac{1}{x}-x^5$. Since $x^2\sin^2\frac{1}{x}+x^4>0, \forall\, x\neq 0$, we have

$$V(x,0)=-x\left(x^2\sin^2\frac{1}{x}+x^4\right)<0,\quad \forall\, x>0.$$

Similarly, we have $V(x,0) > 0, \forall x < 0$. Hence the index of $(0,0)$ is -1. We have $\frac{\partial V(x,\varepsilon)}{\partial \varepsilon}(0,0) = 0$ and $\frac{\partial V(x,\varepsilon)}{\partial x} = (\varepsilon - x^2 \sin^2 \frac{1}{x} - x^4) + x(4x^3 - 2\sin\frac{1}{x}\cos\frac{1}{x} + 2x\sin^2\frac{1}{x})$. Hence $\frac{\partial V(x,\varepsilon)}{\partial x}(0,0) = 0$, and $\frac{\partial^2 V(x,\varepsilon)}{\partial \varepsilon \partial x} = 1 \neq 0$. So it satisfies (P0)–(P2) but not (P3). It undergoes a $1 \to k, k > 3$ bifurcation. One direct way to prove it is to compute the zeros for the vector field numerically.

Example 3.5 (Missing "half of (P3)": $1 \to k, k > 3$ bifurcation)**.** Consider

$$\begin{cases} \dot{x} = 2x^3 - xy + xy^2 - 4x^5 + x\left(\varepsilon - x^4 \sin^2 \dfrac{1}{x} - x^6\right), \\ \dot{y} = 2y - 4x^2. \end{cases}$$

This vector field has the same zeros as $\dot{x} = x(\varepsilon - x^4 \sin^2 \frac{1}{x} - x^6)$ which undergoes a $1 \to k$, $k > 3$ bifurcation. Even though it satisfies $D_{uu}(\det DV)(0) = 8$ which is positive definite, it does not satisfy (P3). This is because $(0, -M^{-1}G_{uu}(0)) = (0,4)$ and $D(\det(D_{(x,y)}V(0))) = (0,-2)$. Hence

$$D_{uu}(\det DV)(0) + D_y(\det(D_{(x,y)}V))(0, -M^{-1}G_{uu}(0))^{\mathrm{T}} = 0.$$

4 Preliminaries

4.1 An index property for vector fields

Given a map $\phi : S^n \to S^n$, the degree of ϕ denoted by degϕ is the unique integer such that for any $x \in H_n S^n$, $\phi_*(x) = \deg\phi \cdot x$. Here, ϕ_* is the induced homomorphism in integral homology. Suppose that x_0 is an isolated zero of the vector field V. Pick a closed disk D centered at x_0, so that x_0 is the only zero of V in D. Then we define the index of x_0 for V, $\mathrm{ind}_{x_0}(V)$, to be the degree of the map $\phi : \partial D^n \to S^{n-1}$, $\phi(x) = \frac{V(z)}{|V(z)|}$. The following theorem is a well-known result on the index of vector fields (see, for example, [1]).

Theorem 4.1. *Consider a smooth vector field* $\frac{dx}{dt} = V(x)$. *If* D *is a disk containing finitely many zeros* $x_1, \ldots, x_k$ *of* V, *then the degree of* $\frac{V(x)}{\|V(x)\|}$ *on* ∂D *is equal to the sum of the indices of* V *at the* x_i. *Moreover, when* x_i *are all non-degenerate,*

$$\sum_{V(x)=0, x\in D} \mathrm{sign}(\det(J))(x) = Q,$$

where J *is the Jacobian of* V *at* x *and* Q *is the degree of the map* $\frac{V(x)}{\|V(x)\|}$ *from the boundary of* D *to the* $n-1$ *sphere.*

4.2 Center manifold

Theorem 4.2 (See [7] and [2, p. 16])**.** *Let* E *be an open subset of* $\mathbb{R}^n$ *containing the origin and consider the non-linear system* $\dot{x} = V(x)$, *i.e.,*

$$\begin{cases} \dot{x} = Cx + F(x,y), \\ \dot{y} = My + G(x,y), \end{cases} \tag{4.1}$$

where the square matrix C *has c-eigenvalues with zero real parts and the square matrix* M *has only eigenvalues with non-zero real parts and*

$$F(0,0) = 0, \quad DF(0,0) = 0; \quad G(0,0) = 0, \quad DG(0,0) = 0.$$

Then there exist a $\delta > 0$ *and a function* $h \in C^r(B_\delta(0))$, $h(0) = 0$, $Dh(0) = 0$ *that defines the local center manifold*

$$W^c_{\mathrm{loc}}(0) = \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u \mid y = h(x) \text{ for } \|x\| \leqslant \delta\}$$

and satisfies $Dh(x)[Cx + F(x,h(x))] - Mh(x) - G(x,h(x)) = 0, |x| \leqslant \delta$ *and the flow on the center manifold* $W^c(0)$ *is defined by* $\dot{u} = Cu + F(x,h(u))$.

Theorem 4.3 (See [9, p. 155]). *The flow given by the vector field* (4.1) *is locally topologically equivalent near the origin to the product system*

$$\begin{cases} \dot{x} = Cx + F(x, h(x)), \\ \dot{y} = My, \end{cases} \tag{4.2}$$

i.e., there exists a homeomorphism h mapping orbits of the first system onto orbits of the second system, preserving the direction of time.

5 Index of the fixed points

As a preparation for the proof of Theorem 1.1, in this section we present a product property for the index of the fixed points. Let us consider the vector field $\dot{x} = V(x)$, with $V(x_0) = 0$ and the eigenvalues of $DV(x_0)$ have non-zero real parts except for one eigenvalue. Here, we assume x_0 is an isolated zero point for V. Let $U \subset \mathbb{R}^n$ be a small neighborhood of x_0 such that $V(x) \neq 0$. Let D^n be a homeomorphic image of the n-ball with the natural orientation and $x_0 \in D^n \subset \overline{D^n} \subset U$. According to the definition of the index at x_0 of V, the index of the zero x_0 for V is given by the degree of the map $\xi_V(x) = \frac{V(x)}{\|V(x)\|}, x \in \partial D^n$ where ∂D^n is a ball around x_0.

The following lemma builds a relation between the index of the fixed points x_0 for the vector field V and the index of x_0 as a zero for the map $V(x)$.

Lemma 5.1. *The index of the zero point x_0 of V equals the index of x_0 as a fixed point of the locally defined flow ϕ_t for $t > 0$ sufficiently small.*

Proof. Let $U \subset \mathbb{R}^n$ be a small neighborhood of x_0 such that $V(x) \neq 0$ and $\phi^t(x) \neq x$ for all $x \in U \backslash \{x_0\}$. Let D^n be a homeomorphic image of the n-ball with the natural orientation and $x_0 \in D^n \subset \overline{D^n} \subset U$. According to the definition of the index at x_0 of V, it suffices to prove that $\xi_V(x) = \frac{V(x)}{\|V(x)\|}, x \in \partial D^n$ and $\phi_{\phi^t}(x) = \frac{x - \phi^t(x)}{\|x - \phi^t(x)\|}$, $x \in \partial D^n$ have the same degree. Denote

$$\delta := \min\{\inf\{\|V(x)\| \mid x \in \partial D^n\}, \inf\{\|x - \phi^t(x)\| \mid x \in \partial D^n\}\}.$$

Since the eigenvalues of $DV(x_0)$ have non-zero real parts except for one eigenvalue, there is no small periodic orbits in U. Hence $\delta > 0$. As long as t is sufficiently small, we have

$$\|V(x) - x - \phi^t(x)\| = \|V(x) - x - \phi^t(x)\| \leqslant \|V(x) - tV(x)\| \leqslant (1-t)\delta$$

on ∂D^n, since ϕ^t is differentiable at x_0 and $V(x)$ is its differential. Hence ξ_V and ϕ_{ϕ^t} are never antipodal, and thus straight-line homotopic via $\frac{t\xi_V + (1-t)\phi_{\phi^t}}{\|t\xi_V + (1-t)\phi_{\phi^t}\|}$. Thus $\deg \xi_V = \deg \phi_{\phi^t}$. □

We note here that the vector fields V and $A^{-1}V(A)$ have the same index at the fixed point x_0 and $A^{-1}x_0$, respectively, where A is a linear isomorphism. This follows immediately from the independence of the definition of index on the coordinates. Please refer to [1, Chapter 7] for a proof. Under suitable coordinates, we assume the vector field V can be written as

$$\begin{cases} \dot{x} = Cx + F(x, y), \\ \dot{y} = My + G(x, y), \end{cases}$$

where the square matrix C has c-eigenvalues with zero real parts and the square matrix M has only eigenvalues with non-zero real parts and $F(0,0) = 0, DF(0,0) = 0; G(0,0) = 0, DG(0,0) = 0$. By Theorem 4.2, there exist a $\delta > 0$ and a function $h \in C^r(B_\delta(0))$, $h(0) = 0, Dh(0) = 0$ such that the vector field on the center manifold is defined by

$$\dot{u} = V^c := Cu + F(x, h(u)).$$

Lemma 5.2. *The product property* $\mathrm{ind}_V(0) = \mathrm{ind}_{V^c}(0) \times (-1)^{\sharp\{i \,|\, \lambda_i > 0\}}$ *holds, where* λ_i *are the non-zero eigenvalues for* DV.

Proof. On the one hand, by Theorem 4.3, the index of $(0,0)$ for the flow ϕ_V^t given by V is the same as the index of $(0,0)$ for the flow $\phi_{V_1}^t$. On the other hand, by Lemma 5.1, we obtain ϕ_V^t and $\phi_{V_1}^t$ have the same index at $(0,0)$. Therefore, the two vector fields

$$V = \begin{cases} \dot{x} = Cx + F(x,y), \\ \dot{y} = My + G(x,y) \end{cases}$$

and $V_1 = (Cx + F(x, h(x)), My)$ have the same index for the zero $(0,0)$. Finally, by the fact that the index of a product map is the product of the index along each direction, we finish the proof. □

6 Observations on the one-dimensional case

Before we delve into the proof of Theorem 1.1, let us turn our attention to the one-dimensional case first. Theorem 4.3 and Lemma 5.2 show that the one-dimensional center direction can reflect the bifurcation properties and the index around the fixed point of an arbitrary-dimensional vector field. Following this idea, a classical argument will be the method of center reduction. By doing center reduction, one can change the high-dimensional problems to one-dimensional problems. In this section, we study some observations for the one-dimensional case.

Lemma 6.1. *Consider the family of smooth functions* $V(u,\varepsilon), u \in \mathbb{R}^1, \varepsilon \in \mathbb{R}^1$. *Let* $u = 0$ *be an isolated non-hyperbolic zero with non-zero index for* $V(u,0)$. *Assume*

$$\frac{\partial V}{\partial \varepsilon}(0,0) = 0 \quad \textit{and} \quad \frac{\partial^2 V}{\partial u \partial \varepsilon}(0,0) \neq 0.$$

Then for any ε *sufficiently close to zero, we have* u_ε *as zeros for* $V(\cdot,\varepsilon)$ *inside* $B_{\varepsilon^{1-\delta}}(0)$, *for any sufficiently small number* $\delta > 0$. *Moreover, the index of* u_ε *for* $V(\cdot,\varepsilon)$ *has different signs for* $\varepsilon > 0$ *and* $\varepsilon < 0$.

Proof. We shall use Newton's method to find the zero point u_ε. By the assumption that $u = 0$ is a non-hyperbolic zero for $V(u,0)$, we get $V(0,0) = 0$, $\frac{\partial V}{\partial u}(0,0) = 0$. Since the index of $u = 0$ is non-zero, we know the first k such that $\frac{\partial^k V}{\partial u^k}(0,0) \neq 0$ should be odd. Hence $\frac{\partial^2 V}{\partial u^2}(0,0) = 0$. Fix an arbitrary small number ε. Denote $V_\varepsilon(u) := V(u,\varepsilon)$. Consider the following sequence of iterations given in Newton's argument: $u_n = u_{n-1} - \frac{V_\varepsilon(u_{n-1})}{V_\varepsilon'(u_{n-1})}$. Then the fixed point of the following map will be the zero points for V_ε: $F_\varepsilon(u) = u - \frac{V_\varepsilon(u)}{V_\varepsilon'(u)}$. We claim that F_ε is a contracting map on the disc $B_{\varepsilon^{1-\delta}}(0)$. Actually, we have

$$F_\varepsilon'(u) = 1 - \frac{V_\varepsilon'(u)^2 - V_\varepsilon(u)V_\varepsilon''(u)}{V_\varepsilon'(u)^2} = \frac{V_\varepsilon(u)V_\varepsilon''(u)}{(V_\varepsilon'(u))^2}.$$

Denote $\frac{\partial^2 V}{\partial u \partial \varepsilon}(0,0) = c \neq 0$. The denominator $V_\varepsilon'(u)$ satisfies

$$\begin{aligned} |V_\varepsilon'(u)| &= \left| \frac{\partial V}{\partial u}(u,\varepsilon) - \frac{\partial V}{\partial u}(0,\varepsilon) + \frac{\partial V}{\partial u}(0,\varepsilon) - \frac{\partial V}{\partial u}(0,0) \right| \\ &\geqslant -\left| \frac{\partial V}{\partial u}(u,\varepsilon) - \frac{\partial V}{\partial u}(0,\varepsilon) \right| + \left| \frac{\partial V}{\partial u}(0,\varepsilon) - \frac{\partial V}{\partial u}(0,0) \right| \\ &\geqslant \frac{\partial^2 V}{\partial u \partial \varepsilon}(0,\tilde{\varepsilon})\varepsilon - \frac{\partial^2 V}{\partial u^2}(\tilde{u},\varepsilon)u \geqslant C_0(|c + \tilde{\varepsilon}|)\varepsilon - |\tilde{u}u| \geqslant Cc\varepsilon \end{aligned}$$

on the ball $B_{\varepsilon^{1-\delta}}(0)$, where C and C_0 are constant numbers (in the following argument we shall use C for all constant numbers). Similarly, the numerator satisfies $V_\varepsilon(u)V_\varepsilon''(u) \leqslant Cc\varepsilon^{3-2\delta}$. Therefore we have

$$F_\varepsilon'(u) = 1 - \frac{V_\varepsilon'(u)^2 - V_\varepsilon(u)V_\varepsilon''(u)}{V_\varepsilon'(u)^2} = \frac{V_\varepsilon(u)V_\varepsilon''(u)}{(V_\varepsilon'(u))^2} \leqslant \frac{Cc\varepsilon^{3-2\delta}}{Cc^2\varepsilon^2} \leqslant C\varepsilon^{1-2\delta},$$

where C is a constant number. Hence we finish the proof of the claim. On the other hand, since $|F_\varepsilon(u)| \leqslant C\varepsilon^{2-3\delta} \leqslant \varepsilon^{1-\delta}$, we have $F_\varepsilon(B_{\varepsilon^{1-\delta}}(0)) \subset B_{\varepsilon^{1-\delta}}(0)$ for small $\delta > 0$. It follows that there is one and only one fixed point inside $B_{\varepsilon^{1-\delta}}(0)$. At u_ε, we have $\frac{\partial V(u_\varepsilon,\varepsilon)}{\partial u}$ which has the same sign as $c\varepsilon$. Since there is a change of signs for $c\varepsilon$ with the variation of ε from negative to positive, there is a change of signs for $\frac{\partial V(u_\varepsilon,\varepsilon)}{\partial u}$ with the variation of ε from negative to positive. □

The following lemma shows that the vector field has one and only one equilibrium at one side of the bifurcation time.

Lemma 6.2 (Uniqueness). *Consider the family of one-dimensional vector fields*: $\dot{u} = V(u,\varepsilon)$, $u \in \mathbb{R}^1$, $\varepsilon \in \mathbb{R}^1$. *Assume* $u = 0$ *to be an isolated non-hyperbolic zero with non-zero index for* $V(u,0)$. *Assume*

$$\frac{\partial V}{\partial \varepsilon}(0,0) = 0 \quad \textit{and} \quad \frac{\partial^2 V}{\partial \varepsilon \partial u}(0,0) \neq 0.$$

Then there exist a neighborhood $U \subset \mathbb{R}^1$ *of* $x = 0$ *and a small number* $\varepsilon_0 > 0$ *such that there is one and only one zero* $u_\varepsilon \in U$ *for any* $\varepsilon \in [0,\varepsilon_0]$ *or any* $\varepsilon \in [-\varepsilon_0, 0]$.

Proof. By the implicit function theorem and the assumption $\frac{\partial^2 V}{\partial \varepsilon \partial u}(0,0) \neq 0$, there exists $(u,\varepsilon(u))$ as the graph of $\frac{\partial V}{\partial u}(u,\varepsilon(u)) = 0$. We claim that there exists a small neighborhood $(-r,r)$ such that either $\varepsilon(u) > 0$, for any $u \in (-r,r)$ or $\varepsilon(u) < 0$, for any $u \in (-r,r)$. Now let us prove this claim. Since the index of $V(u,0)$ at $u = 0$ is 1 (the argument for the index -1 case is similar), there exists a small neighborhood $(-r,r)$ such that $V(u,0) > 0$, $\forall\, u \in (0,r)$ and $V(u,0) < 0$, $\forall\, u \in (-r,0)$. Hence by the mean value theorem, for any $0 < r_1 < r$ there exist $u_1 \in (0,r_1)$ such that $\frac{\partial V(u,\varepsilon)}{\partial u}|_{(u_1,0)} > 0$ and $u_2 \in (0,r_1)$ such that $\frac{\partial V(u,\varepsilon)}{\partial u}|_{(u_2,0)} > 0$. On the other hand, the graph of $\frac{\partial V}{\partial u}(u,\varepsilon) = 0$ will cut the (u,ε) space into two connected regions $A_1 = \{(u,\varepsilon) \mid \frac{\partial V}{\partial u}(u,\varepsilon) > 0\}$ and $A_2 = \{(u,\varepsilon) \mid \frac{\partial V}{\partial u}(u,\varepsilon) < 0\}$. Hence the vertical line $([-r,r],0)\backslash\{(0,0)\}$ can only lie in A_1. So $(u,\varepsilon(u))$ cannot go across the vertical line $([-r,r],0)$ and that finishes the claim.

By the definition of index, we have any zero of $V(u,\varepsilon) = 0$ lying in A_1 has index 1, any zero of $V(u,\varepsilon) = 0$ lying in A_1 has index -1 and any zero of $V(u,\varepsilon) = 0$ lying in $(u,\varepsilon(u))$ can only have index $1,-1$ or 0. By Theorem 4.1, we have for sufficiently small $|\varepsilon|$, $\sum_{V(u,\varepsilon)=0} \mathrm{index}(u) = 1$. If $\varepsilon(u) > 0$, we have for any $\varepsilon < 0$ sufficiently close to zero, there are no zero points on $(u,\varepsilon(u))$. Hence there is one and only one zero $u(\varepsilon)$ for $\varepsilon < 0$. If $\varepsilon(u) < 0$, we have for any $\varepsilon > 0$ sufficiently close to zero, there are no zero points on $(u,\varepsilon(u))$. Hence there is one and only one zero $u(\varepsilon)$ for $\varepsilon > 0$. □

Corollary 6.3 (Bifurcation). *Consider the family of one-dimensional vector fields*: $\dot{u} = V(u,\varepsilon)$, $u \in \mathbb{R}^1$, $\varepsilon \in \mathbb{R}^1$. *Assume* $u = 0$ *to be an isolated non-hyperbolic zero with non-zero index for* $V(u,0)$. *Assume*

$$\frac{\partial V}{\partial \varepsilon}(0,0) = 0 \quad \textit{and} \quad \frac{\partial^2 V}{\partial \varepsilon \partial u}(0,0) \neq 0.$$

Then V *undergoes a* $1 \to k$ *or* $k \to 1$, $k \geqslant 3$ *bifurcation in a neighborhood of* (u_0,ε_0).

Proof. By Lemma 6.1, there always exists zero x_ε for $V(\cdot,\varepsilon)$. By Lemma 6.2, there exists a neighborhood U of $x = 0$ such that x_ε is the only zero for $V(\cdot,\varepsilon)$, either for negative ε sufficiently close to zero or for positive ε sufficiently close to zero. Assume it holds for negative ε. By Lemma 6.1 again, the index of u_ε changes signs when ε varies from negative to positive. Hence there must be at least two other zeros inside U for $\varepsilon > 0$. □

Finally, let us give a criterion for the $1 \to 3$ or $3 \to 1$ bifurcation. The condition $\frac{\partial^3 V}{\partial u^3}(0,0) \neq 0$ in the following corollary plays the role of the multiplicity assumption in Corollary 1.2.

Corollary 6.4 (Pitchfork bifurcation). *Consider the family of one-dimensional vector fields*: $\dot{u} = V(u,\varepsilon), u \in \mathbb{R}^1, \varepsilon \in \mathbb{R}^1$. *Assume* $u = 0$ *to be an isolated non-hyperbolic zero with non-zero index for* $V(u,0)$. *Assume*

$$\frac{\partial V}{\partial \varepsilon}(0,0) = 0, \quad \frac{\partial^2 V}{\partial \varepsilon \partial u}(0,0) \neq 0, \quad \textit{and} \quad \frac{\partial^3 V}{\partial u^3}(0,0) \neq 0.$$

Then $V(x,\varepsilon)$ *undergoes a* $1 \to 3$ *or* $3 \to 1$ *bifurcation around* $(0,0)$.

Proof. Since $\frac{\partial^3 V}{\partial u^3}(0,0) \neq 0$, locally the maximal number of zeros is 3. By Corollary 6.3, it undergoes a $1 \to 3$ or $3 \to 1$ bifurcation. We finish the proof. □

7 Undergoing of bifurcations

In this section, we present the proof of the undergoing of bifurcations under the assumptions (P0)–(P2), i.e., the proof of Theorem 1.1. First of all, let us study the invariance of (P0)–(P2) under the change of coordinates. In the following argument, we shall use an equivalent condition for (P1):

(P1′) $v_l \frac{\partial V}{\partial \varepsilon} = 0$, where $v_l D_x V(x_0, \varepsilon_0) = 0$.

The following lemma shows that the assumption (P2) makes sense.

Lemma 7.1. *For the vector field $V(x,\varepsilon)$ with the conditions* (P0) *and* (P1)*, there exists*

$$\omega = (\omega_1, \ldots, \omega_n, \omega_{n+1})^{\mathrm{T}}$$

such that $DV(x_0,\varepsilon_0)\omega = 0$ and $\omega_{n+1} \neq 0$.

Proof. Denote by v_l and v_r the vectors such that

$$v_l D_x V(x_0,\varepsilon_0) = 0 \quad \text{and} \quad D_x V(x_0,\varepsilon_0) v_r = 0.$$

It is straightforward that $(v_l, 0)DV = 0$. Assume the extended vector fields to be $\dot{x} = V(x,\varepsilon)$, $\dot{\varepsilon} = 0$. Differentiating the extended vector field, we have

$$\begin{bmatrix} D_x V & D_\varepsilon V \\ 0 & 0 \end{bmatrix}$$

with $(v_l, 0)$ and $(0, 1)$ as two left eigenvectors for the eigenvalue zero. Since the dimensions of the left null space and the right null space are the same, there exists a vector $\omega = (\omega_1, \ldots, \omega_n, \omega_{n+1})^{\mathrm{T}}$ such that $DV(x_0,\varepsilon_0)\omega = 0$ and $\omega_{n+1} \neq 0$. □

Lemma 7.2. *For a family of vector fields $\dot{x} = V(x,\varepsilon)$, the following conditions*:

(P0) *$V(x,\varepsilon)$ has an isolated simple non-hyperbolic equilibrium (x_0,ε_0) with non-zero index*; *denote by v_l and v_r the unique left eigenvectors for the eigenvalue* 0*, i.e., $v_l D_x V(x_0,\varepsilon_0) = 0$ and $D_x V(x_0,\varepsilon_0) v_r = 0$*;

(P1) $v_l \frac{\partial V}{\partial \varepsilon}\,|_{(x_0,\varepsilon_0)} = 0$;

(P2) $D(\det(D_x V))(x_0,\varepsilon_0)\omega \neq 0$, *for any $\omega = (\omega_1, \ldots, \omega_{n+1})^{\mathrm{T}}$ such that*

$$DV(x_0,\varepsilon_0)\omega = 0 \quad \textit{and} \quad \omega_{n+1} \neq 0,$$

are invariant under the linear change of the coordinates

$$\tilde{A} = \begin{bmatrix} A & * \\ 0 & 1 \end{bmatrix}, \quad \textit{where } * \textit{ is an arbitrary } n \times 1 \textit{ vector}.$$

Proof. Consider the following linear change of coordinates: $(\tilde{x}, \tilde{\varepsilon})^{\mathrm{T}} = \tilde{A}(x,\varepsilon)^{\mathrm{T}}$, where

$$\tilde{A} = \begin{bmatrix} A & * \\ 0 & 1 \end{bmatrix}.$$

Denote by $\tilde{A}^{-1}$ the inverse matrix of $\tilde{A}$. Since the inverse of the upper triangular matrix is still upper triangular, we have

$$\tilde{A}^{-1} = \begin{bmatrix} A^{-1} & * \\ 0 & 1 \end{bmatrix}.$$

Under the new coordinates, the vector field becomes $(\dot{\tilde{x}})^{\mathrm{T}} = AV(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon})^{\mathrm{T}}) =: \tilde{V}(\tilde{x},\tilde{\varepsilon})$. Denote by

$$\omega^1 := \begin{bmatrix} v_r \\ 0 \end{bmatrix}, \quad \omega = (\omega_1,\ldots,\omega_n,\omega_n)^{\mathrm{T}}$$

the base for the kernel of $D_{(x,\varepsilon)}V(x_0,\varepsilon_0)$. Since $D_{(\tilde{x},\tilde{\varepsilon})}\tilde{V}(\tilde{x}_0,\tilde{\varepsilon}_0) = ADV(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon})^{\mathrm{T}})\tilde{A}^{-1}$, we have

$$ADV(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon})^{\mathrm{T}})\tilde{A}^{-1}\tilde{A}\begin{bmatrix} v_r \\ 0 \end{bmatrix} = ADV(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon})^{\mathrm{T}})\begin{bmatrix} v_r \\ 0 \end{bmatrix} = 0$$

and $ADV(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon})^{\mathrm{T}})\tilde{A}^{-1}\tilde{A}\omega = ADV(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon})^{\mathrm{T}})\omega = 0$. Hence the base for the center direction of the kernel $D_{(\tilde{x},\tilde{\varepsilon})}\tilde{V}(\tilde{x}_0,\tilde{\varepsilon}_0)$ is

$$\left\{\tilde{A}\omega^1 = \begin{bmatrix} Av_r \\ 0 \end{bmatrix}, \tilde{A}\omega\right\}.$$

For the vector field $\tilde{V}$, let us check the conditions (P0)–(P2). Assume $(\tilde{x}_0,\varepsilon_0)$ to be the fixed points. Actually, the first condition (P0) $\mathrm{index}(\tilde{x}_0) = \mathrm{index}(x_0)$ holds, since the index is topological invariant.

Let us verify (P1). First of all, the left eigenvector of $\tilde{V}$ for the eigenvalue zero is given by v_lA^{-1}. Hence we have

$$v_lA^{-1}D_{\tilde{\varepsilon}}\tilde{V}\,|_{\tilde{x}_0} = v_lA^{-1}AD_{\tilde{\varepsilon}}(V(\tilde{A}^{-1}(\tilde{x},\varepsilon))\tilde{A}^{-1})(0,1)^{\mathrm{T}} = 0.$$

Now let us check the condition (P2) for the vector field $\tilde{V}$. For this vector field, we have $D_{\tilde{x}}\tilde{V} = AD_xV(\tilde{A}^{-1}(\tilde{x},\varepsilon))A^{-1}$. Moreover, it follows that

$$\det(D_{\tilde{x}}\tilde{V}) = \det(AD_xV(\tilde{A}^{-1}(\tilde{x},\varepsilon))A^{-1}) = \det D_xV(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon})).$$

Hence we have $D(\det(D_{\tilde{x}}(\tilde{V}))(\tilde{x},\tilde{\varepsilon})) = D\det(D_xV)(\tilde{A}^{-1}(\tilde{x},\tilde{\varepsilon}))\tilde{A}^{-1}$. For any $\tilde{\omega} = (\tilde{\omega}_0,\ldots,\tilde{\omega}_{n+1})^{\mathrm{T}}$, we have

$$\omega = \tilde{A}^{-1}\tilde{\omega} = \begin{bmatrix} A^{-1}(\omega_1,\ldots,\omega_n)^{\mathrm{T}} + \omega_{n+1}* \\ \omega_{n+1} \end{bmatrix}.$$

So $\omega_{n+1} \neq 0$ if and only if $\tilde{\omega}_{n+1} \neq 0$. On the other hand, we have

$$\begin{aligned} D(\det(D_{\tilde{x}}\tilde{V})(\tilde{x}_0,\tilde{\varepsilon}_0))\tilde{\omega} &= D\det(D_xV)(\tilde{A}^{-1}(\tilde{x}_0,\tilde{\varepsilon}_0))\tilde{A}^{-1}(\tilde{A}\omega - t\tilde{A}v_r) \\ &= D\det(D_xV)(\tilde{A}^{-1}(\tilde{x}_0,\tilde{\varepsilon}_0))\omega \\ &= D\det(D_xV)(x_0,\varepsilon_0)\omega \neq 0. \end{aligned}$$

Hence (P2) still holds. □

Now we are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem 4.1, a continuous deformation would not change the total index in U, i.e., $\mathrm{index}(V(\cdot,\varepsilon),U) = \mathrm{index}(x_0)$. By Lemma 7.2, we can assume the vector field V is of the form

$$\begin{cases} \dot{u} = F(u,y,\varepsilon), \\ \dot{y} = My + G(u,y,\varepsilon), \end{cases}$$

where $u \in \mathbb{R}^1$ and $y \in \mathbb{R}^{n-1}$, the square matrix M has eigenvalues with only non-zero real parts and

$$F(0,0,0) = 0, \quad D_{(u,y)}F(0,0,0) = 0, \quad G(0,0,0) = 0, \quad DG(0,0,0) = 0$$

with the conditions (P0)–(P2). The left center direction for V now is $v_l = (1,0)$. By (P1), we have $v_lD_\varepsilon V(0,0,0) = D_\varepsilon F(0,0,0) = 0$. Hence we have $DF(0,0,0) = 0$. So we can apply Theorem 4.2 to the extended vector field by adding $\dot{\varepsilon} = 0$ as one direction. By Theorem 4.2, there exists a smooth function $h(u,\varepsilon)$ which represents the center manifold for V. The vector field along the center becomes

$\dot{u} = F(u, h(u, \varepsilon), \varepsilon)$. By Lemma 5.2, the index of $(u, y) = (0, 0)$ for $V(u, y, 0)$ is non-zero if and only if the index of $u = 0$ is non-zero for $F(u, h(u, 0), 0)$. Hence by the assumption (P0), it follows that the first k such that $\frac{\partial^k F}{\partial u^k}(0, 0) \neq 0$ is an odd number and $k \geqslant 3$.

Claim 1. $\frac{\partial^2 F}{\partial u \partial \varepsilon}(0, 0) \neq 0$. First of all, let us give some discussions on DV. We have

$$D_{(u,y)}V(u, y, \varepsilon) = \begin{bmatrix} D_u F(u, y, \varepsilon) & D_y F(u, y, \varepsilon) \\ D_u G(u, y, \varepsilon) & M + D_y G(u, y, \varepsilon) \end{bmatrix}.$$

By Jacobi's formula,

$$\begin{aligned} &\frac{\partial \det(D_{(u,y)}V(u, 0, 0))}{\partial u} \\ &= \mathrm{tr}\left(\mathrm{adj} \begin{bmatrix} D_u F(u, 0, 0) & D_y F(u, 0, 0) \\ D_u G(u, 0, 0) & M + D_y G(u, 0, 0) \end{bmatrix} \begin{bmatrix} D_{uu} F(u, 0, 0) & D_{yu} F(u, 0, 0) \\ D_{uu} G(u, 0, 0) & D_{yu} G(u, 0, 0) \end{bmatrix} \right) \\ &= \mathrm{tr}\left(\mathrm{adj} \begin{bmatrix} D_u F(u, 0, 0) & D_y F(u, 0, 0) \\ D_u G(u, 0, 0) & M + D_y G(u, 0, 0) \end{bmatrix} \begin{bmatrix} D_{uu} F(u, 0, 0) & D_{yu} F(u, 0, 0) \\ D_{uu} G(u, 0, 0) & D_{yu} G(u, 0, 0) \end{bmatrix} \right). \end{aligned}$$

Hence at $u = 0$, we have

$$\begin{aligned} \frac{\partial \det(D_{(u,y)}V(0, 0, 0))}{\partial u} &= \mathrm{tr}\left(\mathrm{adj} \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & D_{yu} F(0, 0, 0) \\ D_{uu} G(0, 0, 0) & D_{yu} G(0, 0, 0) \end{bmatrix} \right) \\ &= \mathrm{tr}\left(\begin{bmatrix} \det M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & D_{yu} F(0, 0, 0) \\ D_{uu} G(0, 0, 0) & D_{yu} G(0, 0, 0) \end{bmatrix} \right) \\ &= 0. \end{aligned}$$

It is easy to see that the kernel of $DV(u, y, \varepsilon)$ has $(1, 0, 0)$ and $(0, 0, 1)$ as a base. Since along the direction $(1, 0, 0)$, we have

$$\begin{aligned} D_{(u,y,\varepsilon)} \det(D_{(u,y)}V)\,|_{(0,0,0)}(1, 0, 0)^{\mathrm{T}} &= \frac{\partial \det(D_{(u,y)}V)}{\partial u}\bigg|_{(0,0,0)} \\ &= \frac{\partial \det(D_{(u,y)}V(u, 0, 0))}{\partial u}\bigg|_{u=0} = 0, \end{aligned}$$

by the assumption (P2), we have

$$D_{(u,y,\varepsilon)} \det(D_{(u,y)}V)\,|_{(0,0,0)}(0, 0, 1)^{\mathrm{T}} = \frac{\partial \mathrm{Det} D_x V}{\partial \varepsilon}\bigg|_{(x_0,\varepsilon_0)} \neq 0.$$

On the other hand, $\mathrm{Det} D_{(u,y)}V(0, 0, \varepsilon) = \frac{\partial F}{\partial u}(0, 0, \varepsilon)(\mathrm{Det}(M + \frac{\partial G}{\partial y}(0, 0, \varepsilon)))$. Hence

$$\frac{\partial \mathrm{Det} D_x V\,|_{(x_0,\varepsilon_0)}}{\partial \varepsilon} = \frac{\partial^2 F}{\partial u \partial \varepsilon}(0, 0, 0)\left(\mathrm{Det}\left(M + \frac{\partial G}{\partial y}(0, 0, 0)\right)\right) = \frac{\partial^2 F}{\partial u \partial \varepsilon} \cdot \mathrm{Det} M \neq 0.$$

So we get $\frac{\partial^2 F}{\partial u \partial \varepsilon} \neq 0$. We complete the proof of Claim 1. Hence the function $V^c(u, \varepsilon) := F(u, h(u, \varepsilon), \varepsilon)$ satisfies all of the conditions in Corollary 6.3. By Corollary 6.3, V undergoes a $1 \to k$ or $k \to 1$ bifurcation. □

Remark 7.3. Fix $V \in \mathcal{F}$. For $\varepsilon > 0$, the number of zeros is less than or equal to the first non-vanishing jet of $V(x, 0)$ restricted to the center manifold.

At the end of this section, we would like to prove Corollary 1.2 from Theorem 1.1.

Proof of Corollary 1.2. Since $\frac{d\lambda_1}{d\varepsilon}(x_0,\varepsilon_0) > 0$, $\lambda_1(x_0,\varepsilon_0) = 0$, we have $\lambda_1(x_0,\varepsilon) < 0$, for ε less than ε_0 and close enough to ε_0. Hence the index of $\lambda_1(x_0,\varepsilon)$ does not equal to 0. By the isolated requirement on the fixed points (x_0,ε) for $\varepsilon < \varepsilon_0$ and the stability of the indices of fixed points, we know that the index of (x_0,ε_0) is non-zero. By $\lambda_1(x_0,\varepsilon_0) = 0$ again, we know $\frac{\partial V}{\partial \varepsilon}(x_0,\varepsilon_0) = 0$. Then the condition (P1) follows. By $\frac{d\lambda_1}{d\varepsilon}(x_0,\varepsilon_0) > 0$ and $\lambda_1(x_0,\varepsilon_0) = 0$, we know $D_x(\det D_xV) = 0$ and $D_\varepsilon(\det D_xV) \neq 0$. So (P2) holds. Hence we have all of the conditions required in Theorem 1.1. It follows that there exists a $1 \to k$, $k \geqslant \infty$ bifurcation. By the assumption on the multiplicity, there are at most three fixed points showing up. So it is the pitchfork bifurcation. We finish the proof of this corollary. □

8 Pitchfork bifurcation and its genericity

In this section, we prove a criterion for a pitchfork bifurcation and its genericity. To do this, we would like to state an equivalent condition first.

Lemma 8.1. *For any vector field V, assume $(c_1(u),\dots,c_n(u))$ to be the center manifold. The following condition*

(P3′) $(\det(D_xV(c_1(u),\dots,c_n(u))))''\,|_{u_0} \neq 0$ *where u_0 is the bifurcation point*

is equivalent to

(P3″) $(N^{\mathrm{T}}D_x^2\det(D_xV)N + D_x\det(D_xV)N')\,|_{(x_0)} \neq 0$ *where $N' = (c_1''(u),\dots,c_n''(u))$.*

Proof. This is basically due to the chain rule. Denote $N = (1,0,\dots,0)^{\mathrm{T}}$. It follows from

$$\begin{aligned}
&(\det(D_xV(c_1(u),\dots,c_n(u))))''\\
&\quad= (D\det(D_xV(c_1(u),\dots,c_n(u)))(c_1',\dots,c_n')^{\mathrm{T}})'\\
&\quad= (c_1',\dots,c_n')D^2\det(D_xV(c_1(u),\dots,c_n(u)))(c_1',\dots,c_n')^{\mathrm{T}}\\
&\qquad+ D\det(D_xV(c_1(u),\dots,c_n(u)))(c_1'',\dots,c_n'')^{\mathrm{T}}\\
&\quad= (N^{\mathrm{T}}D_x^2\det(D_xV)N + D_x\det(D_xV)N')\,|_{(x_0,\varepsilon_0)} \neq 0.
\end{aligned}$$

This completes the proof. □

Lemma 8.2. *For the vector field V of the form*

$$V := \begin{cases} \dot{u} = F(u,y,\varepsilon),\\ \dot{y} = My + G(u,y,\varepsilon), \end{cases}$$

where $u \in \mathbb{R}^1$ and $y \in \mathbb{R}^{n-1}$, the square matrix M has only eigenvalues with non-zero real parts and $F(0,0,0) = 0$, $D_{(u,y)}F(0,0,0) = 0$, $G(0,0,0) = 0$, $DG(0,0,0) = 0$. Then (P3) *is equivalent to* (P3″) $(N^{\mathrm{T}}D_x^2\det(D_xV)N + D_x\det(D_xV)N')\,|_{(x_0)} \neq 0$ *with the center manifold $(u,c_2(u),\dots,c_n(u))$, $N = (1,0,0)$ and $N' = (0,c_2''(u),\dots,c_n''(u))$.*

Proof. First of all, it is easy to see that $N = (1,0,0)$ is the center direction at $(0,0)$. Hence we can assume the center manifold for V to be $c(u) = (u,c_2(u),\dots,c_n(u))$. Denote $N'(u) = (c_1''(u),c_2''(u),\dots,c_n''(u))$. Then $N' = N'(u_0)$ where u_0 is the point such that $c(u_0) = 0$. By the local center manifold theorem, we have

$$V(c(u)) = a(u)c'(u), \tag{8.1}$$

where $a(u): \mathbb{R}^1 \to \mathbb{R}^1$ is the scaling. At $u = u_0$, we obtain $a(u_0) = 0$. Differentiating (8.1), we have

$$DV(c(u))c'(u) = a'(u)c'(u) + a(u)c''(u). \tag{8.2}$$

Hence at $u = u_0$, we have $a'(u_0) = 0$. Moreover, differentiating (8.2), we obtain

$$c'(u)^{\mathrm{T}}D^2V(c(u))c'(u) + DV(c(u))c''(u) = a''(u)c'(u) + a'(u)c'(u) + a'(u)c''(u) + a(u)c'''(u).$$

Hence at $u = u_0$, we obtain $N^{\mathrm{T}} D^2 V(0) N + DV(0) N' = a''(u_0) N$. Hence

$$DV(0)N' = a''(u_0)N - N^{\mathrm{T}} D^2 V(0) N. \tag{8.3}$$

Plugging V into (8.3), we obtain $N' = (0, M^{-1} G_{uu}(0))$ and

$$N^{\mathrm{T}} D^2 \det(D_x V) N = D_{uu}(\det(DV)).$$

Hence we know that (P3) is equivalent to (P3$''$). □

Now let us give the proof of Theorem 1.3.

Proof of Theorem 1.3. Due to the conditions (P0)–(P2), we have

$$D_\varepsilon V(x_0, \varepsilon_0) = 0 \quad \text{and} \quad D_\varepsilon(\det(D_x V)(x_0, \varepsilon_0)) \neq 0. \tag{8.4}$$

Now let us consider the solution of the implicit function: $\det(D_x V)(x, \varepsilon(x)) = 0$. By (8.4) and the implicit function theorem, we have

$$D_x(\varepsilon(x_0)) = -\frac{D_x(\det(D_x V)(x_0, \varepsilon_0))}{D_\varepsilon(\det(D_x V)(x_0, \varepsilon_0))}.$$

Denote the graph of the center manifold by $c(u) = (u, c_2(u), \dots, c_n(u))$. Then the restriction of $(x, \varepsilon(x))$ to the center manifold becomes: $(c(u), \varepsilon(c(u)))$. We claim two facts:

Claim 1. $D_u(\varepsilon(c(u)))\,|_{c(u)=x_0} = 0$.

Claim 2. $D_u^2(\varepsilon(c(u)))\,|_{c(u)=x_0} \neq 0$.

Claim 1 follows from the following equality:

$$D_u(\varepsilon(c(u)))\,|_{c(u)=x_0} = D_x(\varepsilon(c(u))) D_u(c(u)) = \frac{D_x(\det(D_x V)(x_0, \varepsilon_0))}{D_\varepsilon(\det(D_x V)(x_0, \varepsilon_0))} D_u(c(u)) = 0,$$

where the second equality holds because of the index assumption (as Claim 1 in the proof of Theorem 1.1). Moreover, Claim 2 holds because of Lemma 8.1. By Claims 1 and 2, we know locally the graph of $\varepsilon(c(u))$ satisfies either $\varepsilon(c(u)) > 0$, or $\varepsilon(c(u)) < 0$. Without lose of generality, we assume $\varepsilon(c(u)) > 0$. Hence for sufficiently small $\varepsilon > 0$, there exist at most two points on the center manifold such that $\det(D_x V(c(u), \varepsilon)) = 0$. Moreover, for sufficiently small $\varepsilon > 0$, there are at most three zeros for $V(c(u), \varepsilon)$. Otherwise by the mean value theorem, there will be more than three points with $\det(D_x V(c(u), \varepsilon)) = 0$ which leads to a contradiction. By Theorem 1.1, there exists at least three points. Hence we have exactly a $1 \to 3$ bifurcation, i.e., a pitchfork bifurcation. Hence the proof is complete. □

Lemma 8.3. *(P3$'$) in Lemma 8.1 is invariant under the linear change $x = A\tilde{x}$.*

Proof. Assume the change of coordinates to be $x = A\tilde{x}$. Then the vector field $\dot{x} = V(x)$ becomes $\dot{\tilde{x}} = A^{-1} V(A\tilde{x}) =: \tilde{V}(\tilde{x}, \varepsilon)$. For this vector field, we have $D_{\tilde{x}} \tilde{V} = A^{-1} D_x V(A\tilde{x}) A$. Moreover, it follows that $\det(D_{\tilde{x}} \tilde{V}) = \det(A^{-1} D_x V(A\tilde{x}, \varepsilon) A) = \det D_x V(A\tilde{x})$. Assume $(c_1(u), \dots, c_n(u))$ to be the center manifold for V. Then it follows directly from the invariance of the center manifold, the center manifold after changing of coordinates becomes $A^{-1}(c_1(u), \dots, c_n(u))$. Hence

$$\det(D_x \tilde{V}(A^{-1}(c_1(u), \dots, c_n(u)))) = \det(D_x V(c_1(u), \dots, c_n(u))).$$

So we have (P3$'$) is invariant under changing of coordinates. □

The proof of Theorem 1.4. Based on Theorem 1.3 and Lemma 8.3, we only need to prove that the vector fields with the condition (P3) are open and dense inside $\mathcal{F}$. Since $(0, M^{-1} G_{uu}(0))$ is decided by $(D_x V, D_x^2 V)$ (order two terms in the expansion) of the vector field $V(x)$ at (x_0, ε_0), we denote $x = (u, y)$. Besides, $D_x(\det(D_x V))\,|_{(x_0, \varepsilon_0)}$ is also determined by $(D_x V, D_x^2 V)$ at (x_0, ε_0). On the other hand, since $D_x^2 \det D_x V(x)$ is decided by $(DV, D^2 V, D^3 V)$, we can perturb V by changing $D^3 V$ and keeping $DV, D^2 V$ so that the condition (P3) holds. Hence we know that the maps with (P3) are open and dense inside $\mathcal{F}$. □

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