On the Convergence of Re-Centered Chen-Fliess Series

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***Abstract*—Chen-Fliess functional series provide a representation for a large class of nonlinear input-output systems. Like any infinite series, however, their applicability is limited by their radii of convergence. The goal of this letter is to present a computationally feasible method to re-center a Chen-Fliess series in order to expand its time horizon. It extends existing results in two ways. First, it takes a simpler combinatorial approach to the re-centering formula that draws directly on the analogous re-centering problem for Taylor series. Second, a convergence analysis is presented for the re-centered series. This information can be used to compute a lower bound on the radius of convergence for the output function and an estimate of the series truncation error. The method is demonstrated by simulation on a steering problem for a car-trailer system.**

***Index Terms*—Nonlinear systems, Chen-Fliess series, series convergence, formal power series.**

I. Introduction

CONTROL systems play a central role in most engineering applications like autonomous vehicles, power grids, and robotics. These systems exhibit nonlinear behavior and operate under dynamic and uncertain conditions [1], [2], [3], [4]. A key challenge in their analysis and design is to accurately describe their behavior over extended time horizons and for various input classes [5], [6]. This often involves a trade-off between analytical precision and computational feasibility.

Among the existing approaches for representing a nonlinear input-output system is the Chen-Fliess (CF) functional series [7], [8]. It provides a noncommutative formal power series representation which is coordinate free and suitable for many control applications [9]. It has been applied, for example, in adaptive control [10], for reachability analysis [6], and in optimization problems [11]. Like any infinite series, however, this model's applicability is limited by its radius of convergence. This places upper bounds on both the time horizon and the magnitude of applied inputs. Analytic continuation of functional series is in general a difficult problem [12]. If a state space model is available, one could re-center the series by recomputing its coefficients about a different initial condition, but this is computationally expensive. For systems without an underlying state space model, there is no existing way to re-center a CF series beyond the brute force approach proposed by two of the authors in [13].

The goal of this letter is to present a computationally feasible method to re-center a CF series in order to expand its time horizon. This extends the results in [13] in two ways. First, it takes a simpler combinatorial approach to the re-centering formula that draws directly on the analogous commutative re-centering problem for Taylor series. The key insight is that the binomial theorem used in the commutative case is replaced by Chen's identity in the noncommutative case [14], [15]. Second, a convergence analysis is presented for the re-centered series. Specifically, the coefficient growth parameters are computed as a function of the center point. This information can be used to compute a lower bound on the radius of convergence for the output function [5]. It can also be used to compute series truncation error. Truncation error analysis for CF series was originally presented in [16], [17]. In [13, Sec. II-B], the authors compute the time interval over which the truncation error stays below a preselected value. This is called the *execution time*. It is a function of the coefficient growth parameters. The results of this letter now permit one to compute the execution time as a function of the center point. This can be used to construct a grid of center points in a tracking problem so that the truncation error is uniformly bounded. Finally, the main results of this letter are demonstrated on two simple analytical examples followed by a simulation example involving a car-trailer steering system.

This letter is organized as follows. Section II summarizes some preliminary concepts regarding CF series that are used throughout this letter. Section III presents the new re-centering method. The convergence analysis is done in Section IV. Section V presents a collection of illustrative examples. The conclusions of this letter are given in the final section.

II. Notation and Preliminaries

An *alphabet* $X = \{x_0, x_1, \ldots, x_m\}$ is any nonempty finite set of symbols referred to as *letters*. A *word* $\eta = x_{i_1} \cdots x_{i_k}$ is a finite sequence of letters from X. The number of letters in a

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word η, written as $|\eta|$, is called its *length*. The empty word, $\emptyset$, is taken to have length zero. The collection of all words having length k is denoted by X^k. Define $X^* = \bigcup_{k\geq 0} X^k$, which is a monoid under the concatenation product.

A. Algebras of Formal Power Series

A formal power series is a mapping $c : X^* \to \mathbb{R}^\ell$. It is commonly expressed as the formal sum $c = \sum_{\eta\in X^*}(c,\eta)\eta$, where the coefficient (c,η) represents the image of $\eta \in X^*$ under c. The *support* of c, denoted by $\mathrm{supp}(c)$, is the collection of all words in X^* with nonzero coefficients. The $\mathbb{R}$-vector space of all noncommutative formal power series over the alphabet X is represented by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The subspace of series with finite support, i.e., polynomials, is denoted by $\mathbb{R}^\ell\langle X\rangle$. For $\xi \in X^*$, the *left-shift operator* is defined as $\xi^{-1} : X^* \to \mathbb{R}\langle X\rangle$ such that $\xi^{-1}(\eta) = \eta'$ when $\eta = \xi\eta'$ and zero otherwise. It is extended to $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ by linearity.

The linear spaces $\mathbb{R}^\ell\langle X\rangle$ and $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ are both associative $\mathbb{R}$-algebras under the concatenation or Cauchy product $(c,d)\mapsto cd$. It will also be useful to view this product as the $\mathbb{R}$-linear map:

$$\mathrm{cat} : \mathbb{R}\langle X\rangle \otimes \mathbb{R}\langle X\rangle \to \mathbb{R}\langle X\rangle,\ p\otimes q \mapsto pq.$$

Assume $\mathbb{R}\langle X\rangle$ is endowed with an $\mathbb{R}$-bilinear scalar-valued product defined by $(\eta,\xi) = 1$ when $\eta,\xi \in X^*$ are equal and zero otherwise. The adjoint $\mathrm{cat}^* : \mathbb{R}\langle X\rangle \to \mathbb{R}\langle X\rangle \otimes \mathbb{R}\langle X\rangle$ is defined so that

$$(\mathrm{cat}(p\otimes q), r) = (p\otimes q, \mathrm{cat}^*(r)),\ \ \forall p,q,r \in \mathbb{R}\langle X\rangle.$$

It is straightforward to verify that

$$\mathrm{cat}^*(r) = \sum_{\nu,\eta\in X^*}(r,\nu\eta)\,\nu\otimes\eta \tag{1}$$

[18]. The linear spaces $\mathbb{R}^\ell\langle X\rangle$ and $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ also become associative and commutative $\mathbb{R}$-algebras under the shuffle product, which is defined inductively by

$$(x_i\eta)\sqcup\!\sqcup(x_j\xi) = x_i(\eta\sqcup\!\sqcup(x_j\xi)) + x_j((x_i\eta)\sqcup\!\sqcup\xi),$$

where $x_i, x_j \in X$, $\eta,\xi \in X^*$, and with $\eta\sqcup\!\sqcup\emptyset = \emptyset\sqcup\!\sqcup\eta = \eta$. Given a language $L \subseteq X^*$, its *characteristic series* is $\mathrm{char}(L) := \sum_{\eta\in L}\eta$. If $L = X^k$, it can be shown that

$$\mathrm{char}(X^k) = \frac{\mathrm{char}(X)^{\sqcup\!\sqcup k}}{k!}, \tag{2}$$

where $p^{\sqcup\!\sqcup k}$ denotes the k-th shuffle power [9].

B. Chen and Chen-Fliess Series

Let $p \geq 1$ and $t_0 < t_1$. For any Lebesgue measurable function $u : [t_0,t_1] \to \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ represents the standard L_p-norm of a measurable real-valued function u_i on $[t_0,t_1]$. The set $L_p^m[t_0,t_1]$ consists of all measurable functions on $[t_0,t_1]$ with finite $\|\cdot\|_p$-norm. The bounded subset $B_p^m(R_u)[t_0,t_1]$ is defined as $\{u \in L_p^m[t_0,t_1] : \|u\|_p \leq R_u\}$. Let $C[t_0,t_1] \subset L_p[t_0,t_1]$ denote the set of continuous functions on $[t_0,t_1]$. For any $\eta \in X^*$, recursively define the map $E_\eta : L_p^m[t_0,t_1] \to C[t_0,t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\eta}[u](t,t_0) = \int_{t_0}^{t} u_i(\tau)E_\eta[u](\tau,t_0)\,d\tau,$$

where $x_i \in X$, and $u_0 = 1$. The *Chen series* for u is the exponential Lie series

$$P[u](t,t_0) := \sum_{\eta\in X^*} E_\eta[u](t,t_0)\eta.$$

This series satisfies the semigroup property

$$P[u](t,t_0) = P[u](t,\tau)P[u](\tau,t_0),\quad \tau\in[t_0,t], \tag{3}$$

which is also known as *Chen's identity* [14], [15].

For any generating series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can generalize the scalar product to being $\mathbb{R}^\ell$-valued and associate a causal m-input, ℓ-output system via the *Chen-Fliess series*

$$\begin{aligned} y(t) &= F_c[u](t,t_0) = (c, P[u](t,t_0)) \\ &:= \sum_{\eta\in X^*}(c,\eta)E_\eta[u](t,t_0). \end{aligned} \tag{4}$$

Convergence of this series is ensured if there exist constants $K, M \geq 0$ such that $|(c,\eta)| \leq KM^{|\eta|}|\eta|!$, $\forall\eta\in X^*$, where $|z| := \max_i |z_i|$ for $z\in\mathbb{R}^\ell$. Under these conditions, the series defining $y(t)$ converges absolutely and uniformly on $B_p^m(R_u)[t_0,t_0+T]$ for sufficiently small $R_u, T > 0$. The set of all such *locally convergent* generating series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. When c satisfies the more restrictive growth condition $|(c,\eta)| \leq KM^{|\eta|}$, $\forall\eta\in X^*$, the series (4) defines an operator on the extended space $L_p^m(t_0)$, where

$$\begin{aligned} L_p^m(t_0) := \ &\{u\colon [t_0,\infty) \to \mathbb{R}^m\colon u_{[t_0,t_1]} \in L_p^m[t_0,t_1], \\ &\forall t_1 \in (t_0,\infty)\}, \end{aligned}$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0,t_1]$. This set of *globally convergent* generating series is written as $\mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$. For $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ and $\mathrm{card}(X)$ the number of letters in X, the *radius of convergence* is defined to be

$$\mathfrak{r}(c) = \frac{1}{\limsup_{|\eta|\to\infty}\left|\frac{(c,\eta)}{|\eta|!}\right|^{1/|\eta|}\mathrm{card}(X)} = \frac{1}{M\mathrm{card}(X)}, \tag{5}$$

indicating that $y(t)$ can have a finite escape time [5]. If $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, then $\mathfrak{r}(c) = \infty$, and $y(t)$ is well-defined over any finite interval.

A CF series F_c defined on $B_p^m(R_u)[t_0,t_0+T]$ with $c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ is said to be *realizable* when there exists a state space realization

$$\dot z = g_0(z) + \sum_{i=1}^{m} g_i(z)u_i,\quad z(t_0) = z_0, \tag{6a}$$

$$y_j = h_j(z),\quad j = 1,\ldots,\ell \tag{6b}$$

with each g_i being a real analytic vector field expressed in local coordinates on some neighborhood W of $z_0 \in \mathbb{R}^n$, and each real-valued output function h_j is a real analytic function on W such that (6a) has a well-defined solution $z(t)$, $t\in[t_0,t_0+T]$ for any given input $u \in B_p^m(R_u)[t_0,t_0+T]$, and $y_j(t) = F_c[u](t) = h_j(z(t))$, $t\in[t_0,t_0+T]$. Denoting the *Lie derivative* of h_j with respect to g_i by $L_{g_i}h_j$, it was shown in [7] that for any word $\eta = x_{i_k}, \cdots x_{i_1} \in X^*$

$$(c_j, \eta) = L_{g_\eta} h_j(z_0) := L_{g_{i_1}}, \cdots L_{g_{i_k}} h_j(z_0). \tag{7}$$

III. Re-Centering the Chen-Fliess Series

This section describes the re-centering method for CF series. As motivation for the approach, the Taylor series version of the problem is given first.

Theorem 1: If f is a real analytic function on a neighborhood $U \subseteq \mathbb{R}$ of t_0, then its Taylor series about $\tau \in U$ is

$$f(t) = \sum_{k=0}^{\infty} f^{(k)}(\tau) \frac{(t-\tau)^k}{k!}$$

with

$$f^{(k)}(\tau) = \left(\sum_{n=k}^{\infty} \frac{f^{(n)}(t_0)}{(n-k)!} (\tau - t_0)^{n-k} \right).$$

Proof: The proof follows directly from the binomial theorem. That is,

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!} (t-t_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!} (t-\tau+\tau-t_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!} \sum_{k=0}^{n} \binom{n}{k} (t-\tau)^k (\tau-t_0)^{n-k} \\ &= \sum_{n=0}^{\infty} f^{(n)}(t_0) \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} \frac{(\tau-t_0)^{n-k}}{(n-k)!} \mathbb{1}(n-k), \end{aligned}$$

where $\mathbb{1}(n)$ denotes the Heaviside function. Hence,

$$f(t) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \frac{f^{(n)}(t_0)}{(n-k)!} (\tau-t_0)^{n-k} \right) \frac{(t-\tau)^k}{k!},$$

which completes the proof. ■

In light of the fact that $E_{x_0^k}[u](t, t_0) = (t - t_0)^k/k!$ for all $k \geq 0$, it is evident that the binomial theorem is a commutative version of Chen's identity (3). Hence, the following generalization is formulated for CF series.

Theorem 2: Let $F_c[u](t, t_0)$ be a CF series with $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ and $t_0 \in \mathbb{R}$. Select $\tau > t_0$ so that $F_c[u](\tau, t_0)$ converges. Then the CF series re-centered at τ is

$$F_c[u](t, \tau) = \sum_{\nu \in X^*} \left(\sum_{\eta \in X^*} (c, \eta) E_{\nu^{-1}(\eta)}[u](\tau, t_0) \right) E_\nu[u](t, \tau). \tag{8}$$

Proof: It follows from (1) and (3) that

$$\begin{aligned} &F_c[u](t, t_0) \\ &\quad = (c, P[u](t, t_0)) \\ &\quad = (c, P[u](t, \tau) P[u](\tau, t_0)) \\ &\quad = \left(\text{cat}^*(c), P[u](t, \tau) \otimes P[u](\tau, t_0)\right) \\ &\quad = \left(\sum_{\eta, \nu \in X^*} (c, \nu\eta)(\nu \otimes \eta), P[u](t, \tau) \otimes P[u](\tau, t_0) \right) \\ &\quad = \sum_{\eta, \nu \in X^*} (c, \nu\eta)((\nu \otimes \eta), P[u](t, \tau) \otimes P[u](\tau, t_0)) \\ &\quad = \sum_{\eta, \nu \in X^*} (c, \nu\eta)(P[u](t, \tau), \nu)(P[u](\tau, t_0), \eta) \\ &\quad = \sum_{\nu \in X^*} \sum_{\eta \in X^*} (c, \nu\eta) E_\nu[u](t, \tau) E_\eta[u](\tau, t_0). \end{aligned}$$

Making the change of variables $\bar{\eta} = \nu\eta$ so that $\eta = \nu^{-1}(\bar{\eta})$ yields (8). ■

Observe that the re-centered CF series has coefficients that are dependent on the new center point τ and the input over the interval $[t_0, \tau]$. That is, one could write

$$F_{c_\tau}[u](t, \tau) = \sum_{\nu \in X^*} (c_\tau, \nu) E_\nu[u](t, \tau),$$

where

$$\begin{aligned} (c_\tau, \nu) &:= \sum_{\eta \in X^*} (c, \eta) E_{\nu^{-1}(\eta)}[u](\tau, t_0) \\ &= F_{\nu^{-1}(c)}[u](\tau, t_0). \end{aligned} \tag{9}$$

The central question now is what are the convergence characteristics of the new CF series about the center point τ? This issue is addressed in the next section.

IV. Convergence Analysis

In this section, the convergence of a re-centered CF series is considered in detail. The global case is considered first and then the local case. For ease of notation and without loss of generality, it will be assumed throughout that $\ell = 1$. The first theorem states that the infinite radius of convergence is preserved under re-centering in the global case, but one of the growth parameters becomes dependent on the new center point.

Theorem 3: Let $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ with growth constants $K, M \geq 0$. Fix $u \in B_p^m(R_u)[t_0, t]$ and $\tau > t_0$. Then the re-centered CF series about τ has growth coefficients $K_\tau, M \geq 0$ satisfying

$$|(c_\tau, \nu)| \leq K_\tau M^{|\nu|}, \quad \forall \nu \in X^*$$

with $K_\tau = K \exp(M F_{\text{char}(X)}[u](\tau, t_0))$ and $F_{\text{char}(X)}[u]\ (\tau, t_0) = \sum_{x_i \in X} E_{x_i}(\tau, t_0)$.

Proof: The worst case scenario is where all the coefficients of c are growing at their maximum rate, i.e., $(c, \eta) = KM^{|\eta|}$ for all $\eta \in X^*$. It was shown in [5] that

$$F_c[u](\tau, t_0) = K \exp(M F_{\text{char}(X)}[u](\tau, t_0)).$$

Therefore, applying (9) and the identity $\nu^{-1}(c) = M^{|\nu|} c$ gives

$$\begin{aligned} (c_\tau, \nu) &= F_{\nu^{-1}(c)}[u](\tau, t_0) \\ &= M^{|\nu|} F_c[u](\tau, t_0) \\ &= K \exp(M F_{\text{char}(X)}[u](\tau, t_0)) M^{|\nu|} \\ &= K_\tau M^{|\nu|} \end{aligned}$$

as claimed. ■

The locally convergent case is addressed next. Here both growth constants of the re-centered CF series become dependent on the new center point τ. In which case, the finite radius of convergence also becomes τ dependent.

Theorem 4: Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ with growth constants $K, M \geq 0$. Fix $u \in B_p^m(R_u)[t_0, t]$ and assume $F_c[u]$ is convergent on $[t_0, t]$. Then the re-centered CF series about $\tau \in (t_0, t]$ has growth coefficients $K_\tau, M_\tau \geq 0$ satisfying

$$|(c_\tau, \nu)| \leq K_\tau M_\tau^{|\nu|}|\nu|!, \quad \forall \nu \in X^*$$

with

$$K_\tau = \frac{K}{1 - MF_{\mathrm{char}(X)}[u](\tau, t_0)} \tag{10a}$$

$$M_\tau = \frac{M}{1 - MF_{\mathrm{char}(X)}[u](\tau, t_0)}. \tag{10b}$$

Proof: The worst case scenario is where $(c, \eta) = KM^{|\eta|}|\eta|!$ for all $\eta \in X^*$. In order to apply (9), first observe that for any $\nu \in X^*$

$$\begin{aligned}\nu^{-1}(c) &= \sum_{\eta\in X^*} KM^{|\nu\eta|}|\nu\eta|!\,\eta \\ &= KM^{|\nu|}|\nu|! \sum_{\eta\in X^*} M^{|\eta|}\binom{|\nu| + |\eta|}{|\eta|}|\eta|!\,\eta \\ &= KM^{|\nu|}|\nu|! \sum_{k=0}^{\infty} M^k \binom{|\nu| + k}{k} k!\,\mathrm{char}(X^k).\end{aligned}$$

Therefore, from (2) it follows that

$$\begin{aligned}&(c_\tau, \nu) \\ &\quad = F_{\nu^{-1}(c)}[u](\tau, t_0) \\ &\quad = KM^{|\nu|}|\nu|! \sum_{k=0}^{\infty} M^k \binom{|\nu| + k}{k} k!\, F_{\mathrm{char}(X^k)}[u](\tau, t_0) \\ &\quad = KM^{|\nu|}|\nu|! \sum_{k=0}^{\infty} \binom{|\nu| + k}{k}\left(MF_{\mathrm{char}(X)}[u](\tau, t_0)\right)^k.\end{aligned}$$

From the identity $\sum_{k\geq 0}\binom{|\nu|+k}{k} r^k = 1/(1 - r)^{|\nu|+1}$, the closed form is

$$(c_\tau, \nu) = KM^{|\nu|}|\nu|! \frac{1}{\left(1 - MF_{\mathrm{char}(X)}[u](\tau, t_0)\right)^{|\nu|+1}},$$

which gives (10). ■

Observe that the radius of convergence of c_τ,

$$\mathfrak{r}(c_\tau) = \frac{1}{M_\tau \mathrm{card}(X)} = \frac{1 - MF_{\mathrm{char}(X)}[u](\tau, t_0)}{M\mathrm{card}(X)}, \tag{11}$$

is both τ and u dependent.

V. Examples

Three examples are presented in this section to demonstrate the results of the previous sections. First, a simple system having a globally convergent generating series is analyzed. Next, the theory is exercised on a simple system that is only locally convergent. Finally, a more practical example involving a car-trailer steering system is considered.

Example 1: Consider the single-input, single-output system

$$\dot{z} = zu, \quad z(t_0) = 1, \quad y = z. \tag{12}$$

It can be shown via the method of separation of variables that $y(t) = \exp(\int_{t_0}^{t} u(\theta)\, d\theta)$, $t \geq t_0$. Alternatively, letting $X = \{x_1\}$ and applying (7) gives the corresponding globally convergent generating series $c = \sum_{k\geq 0} x_1^k$ with growth constants $K = M = 1$. It follows directly that $F_c[u](t, t_0) = \sum_{k\geq 0} E_{x_1^k}[u](t, t_0)$ is exactly equal to $y(t)$ above. Applying Theorem 2 gives

$$\begin{aligned}(c_\tau, \nu) &= \sum_{\eta\in X^*} (c, \eta) E_{\nu^{-1}(\eta)}[u](\tau, t_0) \\ &= \sum_{k=|\nu|}^{\infty} E_{x_1^{k-|\nu|}}[u](\tau, t_0) = \exp\left(\int_{t_0}^{\tau} u(t)\, dt\right).\end{aligned}$$

From Theorem 3, the convergence constants of the re-centered CF series are $K_\tau = (c_\tau, \nu)$ and $M = 1$. As expected, global convergence is preserved under re-centering. The corresponding output about the center point τ is

$$\begin{aligned}y_\tau(t) &= \sum_{k=0}^{\infty}(c_\tau, x_1^k) E_{x_1^k}[u](t, \tau) \\ &= K_\tau \sum_{k=0}^{\infty} E_{x_1^k}[u](t, \tau) = K_\tau \exp\left(\int_{\tau}^{t} u(\theta)\, d\theta\right).\end{aligned}$$

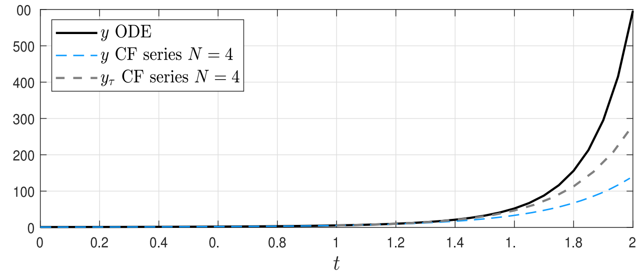

Fig. 1. Simulated outputs of globally convergent system in Example 1.

The key observation is that while $y(t) = y_\tau(t)$ for all $t \geq t_0$, the series y_τ is less sensitive to series truncation in a neighborhood of the new center point τ. This is illustrated by simulation in Fig. 1 when $u(t) = \mathrm{e}^t$, $t_0 = 0$, $\tau = 1$, and the CF series are truncated to words of length $N = 4$. Observe that near $\tau = 1$, the truncation error in y_1 as compared against y computed by numerically solving (12) is considerably less than the truncation error of the original series centered at $t_0 = 0$. In effect, this re-centering extends the duration over which the representation remains accurate without increasing the number of terms in the series.

Example 2: For the single-input, single-output system

$$\dot{z} = z^2 u, \quad z(t_0) = 1, \quad y = z, \tag{13}$$

the generating series is $c = \sum_{k\geq 0} k!\, x_1^k \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ with $K = M = 1$. It then follows from Theorem 2 that

$$\begin{aligned}(c_\tau, \nu) &= \sum_{k=|\nu|}^{\infty} k!\, E_{x_1^{k-|\nu|}}[u](\tau, t_0) \\ &= \sum_{k=|\nu|}^{\infty} \frac{k!}{(k - |\nu|)!}(E_{x_1}[u](\tau, t_0))^{k-|\nu|} \\ &= |\nu|! \sum_{k=0}^{\infty} \binom{k + |\nu|}{k}(E_{x_1}[u](\tau, t_0))^k \\ &= \frac{|\nu|!}{(1 - E_{x_1}[u](\tau, t_0))^{|\nu|+1}},\end{aligned}$$

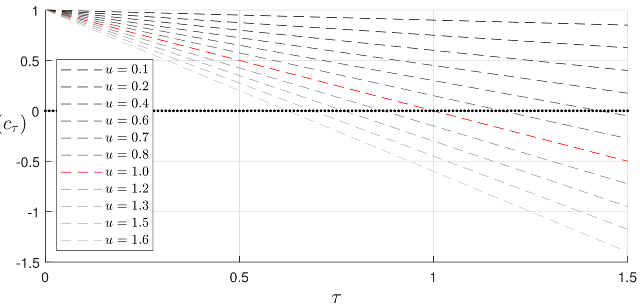

Fig. 2. $\mathfrak{r}(c_\tau)$ as a function of τ and u for the locally convergent series in Example 2.

which coincides with K_τ and M_τ in Theorem 4 when $X = \{x_1\}$. The re-centered output at τ is the locally convergent series

$$y_\tau(t) = \sum_{k=0}^{\infty} \frac{k!}{(1 - E_{x_1}[u](\tau, t_0))^{k+1}} E_{x_1^k}[u](t, \tau).$$

As a check, observe that

$$\begin{aligned} y_\tau(t) &= \frac{1}{1 - E_{x_1}[u](\tau, t_0)} \sum_{k=0}^{\infty} \left(\frac{E_{x_1}[u](t, \tau)}{1 - E_{x_1}[u](\tau, t_0)} \right)^k \\ &= \frac{1}{1 - E_{x_1}[u](\tau, t_0)} \frac{1}{1 - \frac{E_{x_1}[u](t,\tau)}{1 - E_{x_1}[u](\tau, t_0)}} \\ &= \frac{1}{1 - E_{x_1}[u](\tau, t_0) - E_{x_1}[u](t, \tau)} \\ &= \frac{1}{1 - E_{x_1}[u](t, t_0)}, \end{aligned}$$

which matches the solution of (13) derived using the separation of variables method. Now, the radius of convergence of $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ using (5) is $\mathfrak{r}(c) = 1$. The radius of convergence for the re-centered series is computed from (11) for any admissible input $u \in B_p(R_u)[t_0, \tau]$. For example, if $u_1(t) = u$ is a constant input, then it follows that

$$F_{\mathrm{char}(X)}[u](\tau, t_0) = E_{x_1}[u](\tau, t_0) = (\tau - t_0)u.$$

If $t_0 = 0$ and u is given, then the upper bound on τ is the value for which $\mathfrak{r}(c_\tau) = 0$, namely, $\tau = 1/u$. Fig. 2 shows $\mathfrak{r}(c_\tau)$ as a function of τ and u. As expected, all curves start at the value $\mathfrak{r}(c) = 1$ when $\tau = 0$ and decreases as u increases. Note that the red line for $u = 1$ crosses the zero line at $\tau = 1$, which is consistent with the analysis presented in Section IV.

Example 3: Consider the massless car-trailer system shown in Fig. 3. This device moves in the x-y plane with linear velocity u_1 and steering rate u_2 both applied only to the car portion of the system.

A control-affine description of the system with states $z_1 = x$, $z_2 = y$, $z_3 = u_2$, $z_4 = \theta$, and $z_5 = \theta_{\mathrm{trailer}}$ and outputs $y_i = z_i$,

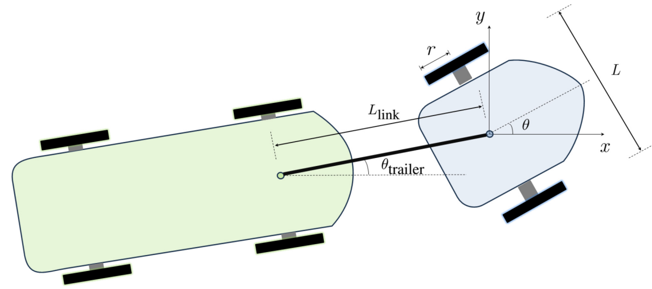

Fig. 3. Car-trailer steering system in Example 3.

$i = 1, 2$ gives the following two-input, two-output state space realization

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{pmatrix} = \begin{pmatrix} r\cos(z_3) \\ r\sin(z_3) \\ 0 \\ \frac{r}{L}\tan(z_3) \\ \frac{r}{L_{\mathrm{link}}}\sin(z_4 - z_5)\cos(z_3) \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ \frac{r}{L} \\ 0 \\ 0 \end{pmatrix} u_2 \tag{14a}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{14b}$$

where r is the radius of the car's wheels, L is the distance between the wheels, and L_{link} the distance between the car's axle and the trailer's front axle [19], [20]. Given that the trailer behaves passively, the choice of outputs makes the system non differentially flat since the inputs and states cannot be written in terms of the outputs and its derivatives [21].

The generating series, c, for the system can be computed directly from (7) using (14). If the initial condition is $z(t_0) = (1, 1, 0, 0, 0)^\top$, then its support is

$$\mathrm{supp}(c) = \{x_1 x_2^k \colon k \geq 0\} \cup \{\emptyset\}.$$

The series coefficients are

$$(c, \eta) = \begin{cases} (1, 1)^\top, & \text{if } \eta = \emptyset \\ (r, 0)^\top, & \text{if } \eta = x_1 \\ (r^{k+1}/L^k, 0)^\top, & \text{if } \eta = x_1 x_2^k,\ k \geq 1 \text{ odd} \\ (0, r^{k+1}/L^k)^\top, & \text{if } \eta = x_1 x_2^k,\ k \geq 2 \text{ even}. \end{cases}$$

This series is clearly in $\mathbb{R}^2_{GC}\langle\langle X \rangle\rangle$ with $X = \{x_1, x_2\}$.

Numerical simulations are shown next using the parameters $r = 1$, $L = 1$, and $L_{\mathrm{link}} = 0.5$. The applied inputs are $u_1(t) = 0.3\sin(1.5t)$ and $u_2(t) = 0.3$. First, the system was simulated numerically using (14) and MATLAB's ODE45 solver. This yielded the black curves shown in Fig. 4.[1] Next, the CF series was computed over the time interval $[0, 4.5]$ seconds by truncating to words of length $N = 4$ and 6 as shown by the blue and light blue dashed lines, respectively, in the same figure. Observe that the accuracy of these approximations begins to degrade beyond $t = 2$ seconds, particularly in the case of y_2 with $N = 4$. Increasing the number of terms in the approximation will improve the accuracy, but the computational cost grows exponentially. For

[1] In this example, non re-centered outputs are denoted by y_j, while re-centered outputs are written as $y_{j,\tau}$.

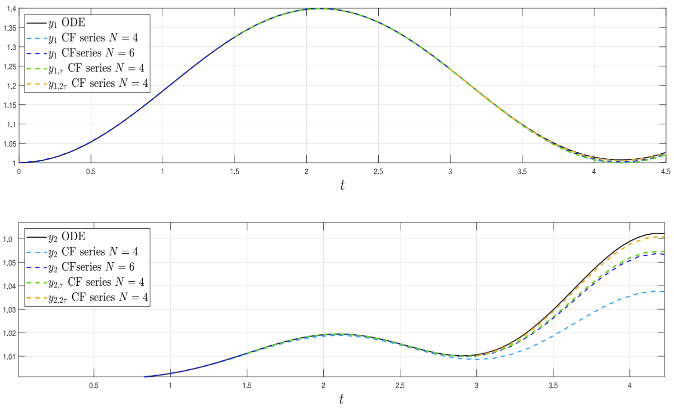

Fig. 4. Simulated outputs y_1 and y_2 in Example 3 using (14) compared against the truncated CF series, the truncated re-centered CF series $y_{1,\tau}$, $y_{2,\tau}$, $y_{1,2\tau}$, and $y_{2,2\tau}$ with $\tau = 1.5$ s.

TABLE I
COEFFICIENTS FOR y_1 BEFORE AND AFTER RE-CENTERING

	$(c\|_{t_0}, \eta)$	$(c\|_\tau, \eta)$	(c_τ, η)	$(c\|_{2\tau}, \eta)$	$(c_{2\tau}, \eta)$
$\emptyset$	1.00	1.33	1.33	1.24	1.24
x_1	1.00	1.00	1.00	1.00	1.00
x_1^2	0.00	-0.04	-0.04	0.04	0.03
x_1^3	0.00	-0.23	-0.23	-1.59	-1.45
$x_1^2x_2$	0.00	0.00	-0.11	0.00	0.05
$x_2x_1^2$	0.00	-0.11	0.00	0.08	0.00
x_1^4	0.00	0.01	0.01	-0.06	-0.00
$x_1^3x_2$	0.00	0.00	-1.18	0.00	-5.08
$x_1^2x_2x_1$	0.00	0.00	-0.59	0.00	-2.54
$x_1^2x_2^2$	0.00	0.00	-0.10	0.00	0.04
$x_1x_2x_1^2$	0.00	-0.59	0.00	-3.26	0.00
$x_2x_1^3$	0.00	-1.19	0.00	-6.52	0.00
$x_2^2x_1^2$	0.00	-0.11	0.00	0.21	0.00

example, truncating to word length $N = 6$ required 1092 terms to maintain an error below 0.06 units over the first 2 seconds. In contrast, computing c_τ via the re-centering formula in Theorem 2 for $\tau = 1.5$ seconds significantly improves the approximation accuracy when $N = 4$ as shown by the green curves in Fig. 4. This extends the time horizon of the CF series representation to at least $t = 3$ seconds. Applying a second re-centering at $2\tau = 3$ seconds further extends the time horizon to $t = 4.5$ seconds by keeping the accuracy to within 0.06 units as indicated by the orange curves in the same figures. These re-centered approximations required only 121 terms, which is significantly fewer than the 1092 terms required for the $N = 6$ approximations.

Finally, Table I presents the coefficients of $c|_{t_0}$, $c|_\tau$, c_τ, $c|_{2\tau}$, and $c_{2\tau}$ up to word length $N = 4$ for the output y_1, where $c|_t$ denotes the series with coefficients $(c|_t, \eta) = L_{g_\eta} h(z(t))$ as given in (7). Only words in the support of at least one series are shown. Note that the re-centered series differs from the series computed via Lie derivatives at the new center point. There is simply no reason to expect these two series to coincide. Specifically, they yield distinct input-output maps with different domains but happen to coincide on the intersection of their domains.

VI. CONCLUSION

A computationally feasible method to re-center a CF series was presented. It used a simple combinatorial calculation to derive the re-centering formula that draws directly on the analogous re-centering problem for Taylor series. Then, a convergence analysis was presented for the re-centered series. Both the global and local convergence cases were considered. Finally, two analytical examples were presented followed by a simulation example using a car-trailer steering system.

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