A NONNEGATIVE WEAK SOLUTION TO THE PHASE FIELD CRYSTAL MODEL WITH DEGENERATE MOBILITY

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Abstract. Phase field crystal is a model used to describe the behavior of crystalline materials at the mesoscale. In this study, we investigate the well-posedness of a phase field crystal equation subject to a degenerate mobility $M(u)$ that equals zero for $u \leq 0$. First, we prove the existence of a weak solution to a phase field crystal equation with non-degenerate cutoff mobility. Then, assuming that the initial data $u_0(x)$ is positive, we establish the existence of a nonnegative weak solution to the degenerate case. Such solution is the limit of solutions corresponding to non-degenerate mobilities. We also verify that such a weak solution satisfies an energy dissipation inequality.

1. **Introduction.** In this paper, we consider a phase field crystal (PFC) model. Let $u : \Omega \to [0, \infty)$ be an Ω–periodic unary atom density, where $\Omega = [0, 2\pi]^d \subset \mathbb{R}^d$, $1 \leq d \leq 3$. The free energy for the state defined by u is

$$\mathcal{F}(u) = \int_\Omega \left[W(u) + \kappa \left(\frac{1}{2} u^2 - |\nabla u|^2 + \frac{1}{2} |\Delta u|^2 \right) \right] dx, \qquad (1.1)$$

where $W : [0, \infty) \to [W_0, \infty)$ is a homogeneous free energy density, $W_0 > -\infty$, and $\kappa > 0$ is a parameter. The PFC equation is a conserved gradient flow with respect to the PFC energy functional $\mathcal{F}(u)$ and is written as

$$u_t = \nabla \cdot (M(u)\nabla\omega), \quad \text{in } \Omega_T := \Omega \times (0, T), \qquad (1.2)$$

$$\omega = W'(u) + \kappa(u + 2\Delta u + \Delta^2 u), \quad \text{in } \Omega_T, \qquad (1.3)$$

where $T > 0$ is the final time.

From a mathematical point of view, this equation and its variants have been analyzed in [17, 22, 14, 15, 6] and references therein. In particular, Miranville studied the existence and uniqueness of variational solutions for a PFC model with constant mobility $M(u) \equiv 1$ and logarithmic (singular) nonlinear terms [17]. Wu and Zhu theoretically and numerically analyzed the well-posedness of a square PFC model in the three-dimensional case [22]. The significant difference between our result and theirs is that we are studying the PFC equation subject to a non-constant mobility $M(u)$ that is dependent on u.

1.1. **Derivation.** In this section, we quickly re-derive the PFC model to motivate the form that we investigate. See, for example, the nice paper by Archer et al. [1] for all of the possible modeling choices. The phase field crystal (PFC) model was introduced in [10, 11] as continuum description of solidification in a unary material. It was formulated as a mass conservative version of the classical Swift-Hohenberg equation, but, later, the model was re-derived, via certain reasonable simplifications, from the dynamical density functional theory (DDFT) [4, 12]. (See [1] for a more in-depth derivation from the DDFT framework.) In particular, assuming a constant, uniform temperature field, one expresses the dimensionless Helmholtz free energy density via

$$F(\rho) = \int_\Omega \left[f(\rho) + \frac{\kappa}{2}(\rho - 1)\mathcal{C}(\rho - 1) \right] dx, \qquad (1.4)$$

where Ω is some spatial domain of interest; $\rho : \Omega \to [0, \infty)$ is the number density field of the unary material in Ω; $\kappa > 0$ is a positive dimensionless constant; f is the homogeneous Helmholtz free energy density; and $\mathcal{C}$ is a symmetric, potentially nonlocal, two-point correlation operator. Here we have taken the state $\rho = \rho_o = 1$ as the dimensionless reference density.

The homogeneous free energy density, f, is often taken to satisfy an "ideal gas" model:

$$f(\rho) = \rho \ln (\rho) - \rho. \qquad (1.5)$$

Note that this energy density has a global minimum at the reference state $\rho = 1$. Often, one makes a (Taylor) polynomial approximation of the logarithmic term about the reference density to make the model more tractable. However, the singular nature of the logarithmic term contributes to the positivity of the solutions, and this is an important feature in the numerical and PDE analyses.

The designation of the lower-order terms of the density functional free energy expansion (1.4) as the "ideal gas" part is standard in the literature. See, for example, [4, Section 4.7]. This does not imply that the material being modeled is a gas, but that the lower-order free energy has the form of that used in the modeling of an ideal gas. The typical derivation of the free energy of a fluid starts with the statistical mechanical ensemble of a large number of identical particles, where the Hamiltonian of the collection has a kinetic energy (ideal gas) part and a long-range interaction potential (non-ideal gas) part. The kinetic energy part results in the logarithmic form for $f(\rho)$ employed in (1.5). The higher-order correction terms in the free energy (1.4) result from the long-range interactions of particles and distinguish the model from that of an ideal gas.

At constant temperature, one can argue that the dynamics of the model should satisfy a diffusion-dominated mass conservation equation of the form [1]

$$\partial_t \rho = -\nabla \cdot \mathbf{J}, \quad \mathbf{J} = -\rho \nabla \mu, \qquad (1.6)$$

where $\mathbf{J}$ is the diffusion flux; and μ is the chemical potential:

$$\mu := \delta_\rho F = \log(\rho) + \kappa \mathcal{C}(\rho - 1), \qquad (1.7)$$

where we have assumed, for simplicity, that the boundary conditions are periodic. Observe that

$$\nabla \cdot (\rho \nabla (\log(\rho))) = \Delta \rho,$$

which suggests that the model incorporates standard particle diffusion at leading order. For this derivation, we will assume that $\mathcal{C}$ is a differential operator of the

form

$$\mathcal{C}(\varrho) = \gamma\varrho + 2\Delta\varrho + \Delta^2\varrho,$$

where $\gamma \in \mathbb{R}$ is a dimensionless parameter. This chosen form promotes the formation of spatially oscillatory density fields and was originally suggested by Elder and co-workers [10, 11, 12]. To sum up, the chemical potential is modeled as

$$\mu = \log(\rho) + \kappa\gamma(\rho - 1) + \kappa\left(2\Delta\rho + \Delta^2\rho\right). \qquad (1.8)$$

As a consequence of our model assumptions, the total free energy is dissipated as the system evolves toward equilibrium, and the dissipation rate is

$$d_t F = -\int_\Omega \rho|\nabla\mu|^2 \, dx \le 0.$$

Of course, is be necessary to justify the property that $\rho > 0$ (or at least $\rho \ge 0$) point-wise for the model to make sense.

As we indicated, it is most typical in the physics literature to use a Taylor polynomial approximation of the logarithmic free energy density, around the dimensionless reference state $\rho = \rho_o = 1$, of the form [12]

$$\rho\ln(\rho) - \rho \approx -1 + \frac{1}{2}(\rho - 1)^2 - \frac{1}{6}(\rho - 1)^3 + \frac{1}{12}(\rho - 1)^4. \qquad (1.9)$$

This approximation regularizes the singular nature of the ideal gas law, but also it removes the singular free energy barrier against states that have negative density regions.

Note that we are keeping the degenerate mobility and approximating the ideal gas law with a polynomial. Archer et al. [1] have argued that if, on the other hand, one approximates the degenerate mobility with a positive constant, then one is absolutely forced to replace the logarithmic potential (the ideal gas law) by the approximating Taylor polynomial. Otherwise, the model predictions are unphysical.

With the simplifications described above, we have the following PFC model. For each Ω–periodic state function $u : \Omega \to [0, \infty)$, the free energy is

$$\mathcal{F}(u) = \int_\Omega \left[f_0(u) + \frac{\kappa}{2}(u - 1)\left(\gamma(u - 1) + 2\Delta(u - 1) + \Delta^2(u - 1)\right) \right] dx, \qquad (1.10)$$

where

$$f_0(u) = -1 + \frac{1}{2}(u - 1)^2 - \frac{1}{6}(u - 1)^3 + \frac{1}{12}(u - 1)^4.$$

The conserved gradient flow describing the evolution of the state function u is

$$\partial_t u = -\nabla \cdot (u\nabla\omega),$$

$$\omega = f_0'(u) + \kappa\left(\gamma(u - 1) + 2\Delta u + \Delta^2 u\right),$$

which matches the form that we introduced earlier, after defining $W(u)$ to be

$$W(u) = f_0(u) + \frac{\kappa\gamma}{2}(u - 1)^2 - \frac{\kappa}{2}u^2.$$

1.2. **Main result.** In this paper, we model the mobility $M(u)$ as

$$M(u) = \begin{cases} u, & u > 0, \\ 0, & u \le 0, \end{cases} \qquad (1.11)$$

which degenerates when $u \le 0$. We assume that the potential $W \in C^2(\mathbb{R})$ satisfies the following growth conditions:

$$b_1 z^{2m} - b_2 \le W(z) + \frac{\epsilon}{2}z^2 \le b_3 z^{2m} + b_4, \qquad (1.12)$$

$$|W'(z)| \le b_3 |z|^{2m-1} + b_4, \tag{1.13}$$

$$b_1 z^{2m-2} - b_2 \le W''(z) \le b_3 z^{2m-2} + b_4, \tag{1.14}$$

for all $z \in \mathbb{R}$, where $m > 1$ is an integer, $0 < \epsilon < \kappa$, and b_1, b_2, b_3, b_4 are positive constants. A simple example of such a potential W is

$$W(u) = \frac{1}{4}(u-1)^4 - \frac{\epsilon}{2}(u-1)^2,$$

where $\epsilon \in \mathbb{R}$. This satisfies the above growth conditions with $m = 2$.

Our analysis used a framework similar to [13, 7, 9]. First, we approximate the degenerate mobility $M(u)$ using a nondegenerate mobility $M_\theta(u)$ defined by

$$M_\theta(u) = \begin{cases} u, & u > \theta, \\ \theta, & u \le \theta, \end{cases} \tag{1.15}$$

for any $\theta \in (0,1)$, and show that the equation (1.2)–(1.3) with the mobility $M_\theta(u)$ has a sufficiently regular weak solution u_θ, that is, we prove the following theorem.

Theorem 1.1. *Let $u_0 \in H^2(\Omega)$. For any given constant $T > 0$, there exists a function u_θ that satisfies the following conditions:*

(i) *For any $0 < \alpha < 1/2$,*

$$u_\theta \in L^2(0,T; H^5(\Omega)) \cap L^2(0,T; C^{3,\alpha}(\bar{\Omega}))$$
$$\cap C([0,T]; H^1(\Omega)) \cap C([0,T]; C^{0,\alpha}(\bar{\Omega})),$$

(ii) *$\partial_t u_\theta \in L^2(0,T; H^{-2}(\Omega))$.*

(iii) *$u_\theta(x,0) = u_0(x)$, for all $x \in \Omega$.*

(iv) *u_θ solves the PFC equation (1.2)–(1.3) in the following weak sense:*

$$\int_0^T \langle \partial_t u_\theta, \xi \rangle_{H^{-2}(\Omega), H^2(\Omega)} \, dt$$
$$= -\int_0^T \int_\Omega M_\theta(u_\theta)(W''(u_\theta)\nabla u_\theta + \kappa\nabla u_\theta + 2\kappa\nabla\Delta u_\theta + \kappa\nabla\Delta^2 u_\theta) \cdot \nabla\xi \, dxdt, \tag{1.16}$$

for all $\xi \in L^2(0,T; H^2(\Omega))$. In addition, for any $t \ge 0$, the following energy inequality holds:

$$\int_\Omega W(u_\theta(x,t)) + \kappa\left(\frac{1}{2}|u_\theta(x,t)|^2 - |\nabla u_\theta(x,t)|^2 + \frac{1}{2}|\Delta u_\theta(x,t)|^2\right) dx$$
$$+ \int_0^t \int_\Omega M_\theta(u_\theta(x,\tau)) \Big| W''(u_\theta(x,\tau))\nabla u_\theta(x,\tau)$$
$$+ \kappa\left(\nabla u_\theta(x,\tau) + 2\nabla\Delta u_\theta(x,\tau) + \nabla\Delta^2 u_\theta(x,\tau)\right) \Big|^2 dxd\tau$$
$$\le \int_\Omega W(u_0) + \kappa\left(\frac{1}{2}u_0^2 - |\nabla u_0|^2 + \frac{1}{2}|\Delta u_0|^2\right) dx. \tag{1.17}$$

(v) *If $u_0(x) > 0$, for all $x \in \Omega$, then*

$$\operatorname*{ess\,sup}_{0 \le t \le T} \int_\Omega |(u_\theta(x,t))_- + \theta|^2 \, dx \le C(\theta^2 + \theta + \theta^{1/2}), \tag{1.18}$$

where $(u_\theta)_- = \min\{u_\theta, 0\}$, and C is a generic positive constant that may depend on $d, T, \Omega, b_1, b_2, b_3, b_4, m, \kappa, \epsilon$ and u_0, but not on θ.

Next, we consider the limit of $\{u_\theta\}$ as $\theta \to 0$. We show that the limiting function u of $\{u_\theta\}$ exists and is a weak solution to the PFC equation (1.2)–(1.3) with the mobility $M(u)$ defined by (1.11). Moreover, if the initial data $u_0(x)$ is positive in Ω, using the estimate (1.18) in part (v) of Theorem 1.1, we can show that such weak solution is nonnegative in Ω_T. We will prove the following result.

Theorem 1.2. *Assume that $u_0 \in H^2(\Omega)$. For any given constant $T > 0$, there exists a function u that satisfies the following conditions:*

(i) *For any $0 < \alpha < 1/2$,*

$$u \in L^\infty(0, T; H^2(\Omega)) \cap C([0, T]; H^1(\Omega)) \cap C([0, T]; C^{0,\alpha}(\bar{\Omega})).$$

(ii) $\partial_t u \in L^2(0, T; H^{-2}(\Omega))$.

(iii) $u(x, 0) = u_0$.

(iv) *u can be considered as a weak solution for the PFC equation (1.2)–(1.3) in the following weak sense:*

(a) *Let P be the set where $M(u)$ is not degenerate, that is,*

$$P := \{(x, t) \in \Omega_T : u(x, t) > 0\}. \tag{1.19}$$

There exist a set $B \subset \Omega_T$ with $|\Omega_T \backslash B| = 0$ and a function $\Psi : \Omega_T \to \mathbb{R}^d$ satisfying $\chi_{B \cap P} M(u)\Psi \in L^2(0, T; L^{2d/(d+2)}(\Omega; \mathbb{R}^d))$, where $\chi_{B \cap P}$ is the characteristic function of $B \cap P$, such that

$$\int_0^T \langle \partial_t u, \xi \rangle_{H^{-2}(\Omega), H^2(\Omega)} \, dt = -\int_{B \cap P} M(u)\Psi \cdot \nabla\xi \, dx \, dt, \tag{1.20}$$

for all $\xi \in L^2(0, T; H^2(\Omega))$.

(b) *Let $\nabla\Delta^2 u$ be the generalized derivative of u in terms of distributions. If $\nabla\Delta^2 u \in L^q(U_T)$, where $U_T = U \times (0, T)$, for some open set $U \subset \Omega$ and some $q > 1$ that may depend on U, then we have*

$$\Psi = W''(u)\nabla u + \kappa\nabla u + 2\kappa\nabla\Delta u + \kappa\nabla\Delta^2 u \quad \text{in } U_T.$$

(c) *For any $t \geq 0$, the following energy inequality holds:*

$$W(u(x, t)) + \kappa \left(\frac{1}{2}|u(x, t)|^2 - |\nabla u(x, t)|^2 + \frac{1}{2}|\Delta u(x, t)|^2 \right) dx$$

$$+ \int_{\Omega_t \cap B \cap P} M(u(x, \tau))|\Psi(x, \tau)|^2 dx \, d\tau$$

$$\leq \int_\Omega W(u_0) + \kappa \left(\frac{1}{2}u_0^2 - |\nabla u_0|^2 + \frac{1}{2}|\Delta u_0|^2 \right) dx. \tag{1.21}$$

(v) *In addition, if $u_0(x) > 0$, for all $x \in \Omega$, then $u(x, t) \geq 0$, for a.e. $x \in \Omega$ and all $t \in [0, T]$, and $u(x, t)$ is not always zero in Ω_T.*

1.3. **Notation.** In this paper, we use C to denote a generic positive constant that may depend on $d, T, \Omega, b_1, b_2, b_3, b_4, m, \kappa, \epsilon$ and u_0, but nothing else, in particular not on θ and N. We also use C_θ to denote a generic positive constant that may depend on $d, T, \Omega, b_1, b_2, b_3, b_4, m, \kappa, \epsilon, u_0$ and θ, but not on N.

The reamainder of this paper is organized as follows. In Section 2, we prove Theorem 1.1 using the Galerkin approximation. In Section 3, we prove Theorem 1.2, which shows the existence of a weak solution to the PFC equation (1.2)–(1.3) with the degenerate mobility (1.11).

2. **Proof of theorem 1.1: Weak solution for the positive mobility case.** In this section we prove the existence of a weak solution to the PFC equation (1.2)–(1.3) with the nondegenerate mobility $M_\theta(u)$ defined by (1.15).

2.1. **Galerkin approximation.** Let $\{\phi_j : j = 1, 2, ...\}$ be the normalized eigenfunctions in $L^2(\Omega)$, that is, $\|\phi_j\|_{L^2(\Omega)} = 1$, of the eigenvalue problem:

$$-\Delta u = \lambda u \text{ in } \Omega, \text{ and } u \text{ is } \Omega\text{–periodic.}$$

The eigenfunctions ϕ_j are orthogonal with respect to the $H^2(\Omega)$ and $L^2(\Omega)$ inner products. We assume that $\lambda_1 = 0$, which implies that $\phi_1(x) \equiv (2\pi)^{-d/2}$.

We consider the Galerkin approximation for the PFC equation (1.2)–(1.3):

$$u^N(x, t) = \sum_{j=1}^N c_j^N(t)\phi_j(x), \quad \omega^N(x, t) = \sum_{j=1}^N d_j^N(t)\phi_j(x), \tag{2.1}$$

$$\int_\Omega \partial_t u^N \phi_j dx = -\int_\Omega M_\theta(u^N)\nabla\omega^N \cdot \nabla\phi_j dx \tag{2.2}$$

$$\int_\Omega \omega^N \phi_j dx = \int_\Omega (W'(u^N)\phi_j + \kappa u^N \phi_j - 2\kappa\nabla u^N \cdot \nabla\phi_j + \kappa\Delta u^N \Delta\phi_j) dx \tag{2.3}$$

$$u^N(x, 0) = \sum_{j=1}^N \left(\int_\Omega u_0 \phi_j dx\right) \phi_j(x). \tag{2.4}$$

This system is equivalent to the following system of ordinary differential equations for $c_1^N, ..., c_N^N$:

$$\partial_t c_j^N = -\sum_{k=1}^N d_k^N \int_\Omega M_\theta\left(\sum_{i=1}^N c_i^N \phi_i\right) \nabla\phi_k \cdot \nabla\phi_j dx, \tag{2.5}$$

$$d_j^N = \int_\Omega W'\left(\sum_{k=1}^N c_k^N \phi_k\right) \phi_j dx + \kappa c_j^N - 2\lambda_j c_j^N + \kappa\lambda_j^2 c_j^N, \tag{2.6}$$

$$c_j^N(0) = \int_\Omega u_0 \phi_j dx. \tag{2.7}$$

Because the right hand side of (2.5) depends continuously on $c_1^N, ..., c_N^N$, according to the Peano existence theorem [18, 19], the initial value problem (2.5)–(2.7) has a local solution. Lemma 2.2 gives the uniform bound for $c_1^N, ..., c_N^N$, therefore, by the Picard–Lindelöf theorem (see [5], page 12, Theorem 3.1), a global solution for the initial value problem (2.5)–(2.7) exists.

2.2. **A priori estimates.** We will establish some necessary prior estimates in this section.

Lemma 2.1. *Let u^N be a solution of the system (2.2)–(2.4). Then, for any $0 \leq t \leq T$,*

$$\int_\Omega u^N(x, t)dx = \int_\Omega u^N(x, 0)dx. \tag{2.8}$$

Proof. Taking $j = 1$ in (2.2) gives $\int_\Omega \partial_t u^N dx = 0$. Thus, $\int_\Omega u^N(x, t)dx = \int_\Omega u^N(x, 0)dx$, for any $0 \leq t \leq T$. $\qquad\square$

Lemma 2.2. *Let u^N be a solution of the system (2.2)–(2.4), then, for any $0 \leq t \leq T$,*

$$\|u^N\|_{L^\infty(0,T;H^2(\Omega))} \leq C, \tag{2.9}$$

$$\int_0^t \int_\Omega M_\theta(u^N(x,\tau))|\nabla\omega^N(x,\tau)|^2 dx d\tau \leq C. \tag{2.10}$$

Proof. For any $0 \leq t \leq T$, since

$$\frac{d}{dt}\mathcal{F}(u^N(x,t)) = -\int_\Omega M_\theta(u^N(x,t))|\nabla\omega^N(x,t)|^2 dx,$$

integrating in time over $(0,t)$, we obtain

$$\int_\Omega W(u^N(x,t)) + \kappa\left(\frac{1}{2}|u^N(x,t)|^2 - |\nabla u^N(x,t)|^2 + \frac{1}{2}|\Delta u^N(x,t)|^2\right) dx$$

$$+ \int_0^t \int_\Omega M_\theta(u^N(x,\tau))|\nabla\omega^N(x,\tau)|^2 dx d\tau$$

$$= \int_\Omega W(u^N(x,0)) + \kappa\left(\frac{1}{2}|u^N(x,0)|^2 - |\nabla u^N(x,0)|^2 + \frac{1}{2}|\Delta u^N(x,0)|^2\right) dx. \tag{2.11}$$

Using the growth condition (1.12) and Sobolev embedding for $H^2(\Omega)$ in $\mathbb{R}^d (d = 1,2,3)$, we obtain the bound for the right hand side of (2.11):

$$\int_\Omega W(u^N(x,0)) + \kappa\left(\frac{1}{2}|u^N(x,0)|^2 - |\nabla u^N(x,0)|^2 + \frac{1}{2}|\Delta u^N(x,0)|^2\right) dx$$

$$\leq \int_\Omega \left(b_3|u^N(x,0)|^{2m} + b_4 + \frac{\kappa-\epsilon}{2}|u^N(x,0)|^2 - \kappa|\nabla u^N(x,0)|^2 + \frac{\kappa}{2}|\Delta u^N(x,0)|^2\right) dx$$

$$\leq C\left(\|u^N(x,0)\|_{H^2(\Omega)}^{2m} + \|u^N(x,0)\|_{H^2(\Omega)}^2 + 1\right) dx$$

$$\leq C\left(\|u_0\|_{H^2(\Omega)}^{2m} + \|u_0\|_{H^2(\Omega)}^2 + 1\right)$$

$$\leq C. \tag{2.12}$$

Now, let us consider the first integral on the left hand side of (2.11). For any number $a > 0$, applying the AM–GM inequality (see [3], page 457, Theorem 17), we obtain

$$b_1|u^N|^{2m} + (m-1)a \geq m(b_1|u^N|^{2m}a^{m-1})^{1/m} = mb_1^{1/m}a^{(m-1)/m}|u^N|^2. \tag{2.13}$$

We choose the number a so that $mb_1^{1/m}a^{(m-1)/m} = \kappa/2$, that is,

$$a = a_0 := \left(\frac{\kappa}{2m}\right)^{m/(m-1)} b_1^{1/(m-1)}. \tag{2.14}$$

So, with the choice $a = a_0$, (2.13) implies

$$\int_\Omega b_1|u^N|^{2m} dx \geq \int_\Omega \frac{\kappa}{2}|u^N|^2 dx - (m-1)a_0|\Omega|. \tag{2.15}$$

In addition, using integration by parts and Young's inequality, for any number $\beta > 0$, we obtain

$$-\kappa\int_\Omega |\nabla u^N|^2 dx = \kappa\int_\Omega u^N \Delta u^N dx \geq -\frac{\kappa}{2\beta}\int_\Omega |u^N|^2 dx - \frac{\kappa\beta}{2}\int_\Omega |\Delta u^N|^2 dx. \tag{2.16}$$

Combining (1.12), (2.15) and (2.16) we obtain

$$\int_\Omega W(u^N) + \kappa \left(\frac{1}{2}|u^N|^2 - |\nabla u^N|^2 + \frac{1}{2}|\Delta u^N|^2 \right) dx$$

$$\geq \int_\Omega \left(b_1|u^N|^{2m} - b_2 + \frac{\kappa - \epsilon}{2}|u^N|^2 - \kappa|\nabla u^N|^2 + \frac{\kappa}{2}|\Delta u^N|^2 \right) dx$$

$$\geq \left(\kappa - \frac{\epsilon}{2} - \frac{\kappa}{2\beta} \right) \int_\Omega |u^N|^2 dx + \frac{\kappa}{2}(1 - \beta) \int_\Omega |\Delta u^N|^2 dx - (b_2 + (m - 1)a_0)|\Omega|.$$

(2.17)

We choose the number β so that $\kappa - \frac{\epsilon}{2} - \frac{\kappa}{2\beta} > 0$ and $1 - \beta > 0$, that is, $\frac{\kappa}{2\kappa - \epsilon} < \beta$ and $\beta < 1$. Since $0 < \epsilon < \kappa$, then $\frac{\kappa}{2\kappa - \epsilon} < 1$, hence, we can always find β_0 such that $\frac{\kappa}{2\kappa - \epsilon} < \beta_0 < 1$. Define

$$\gamma_0 := \min \left\{ \kappa - \frac{\epsilon}{2} - \frac{\kappa}{2\beta_0}, \frac{\kappa}{2}(1 - \beta_0) \right\} > 0,$$

then we have

$$\int_\Omega W(u^N) + \kappa \left(\frac{1}{2}|u^N|^2 - |\nabla u^N|^2 + \frac{1}{2}|\Delta u^N|^2 \right) dx$$

$$\geq \gamma_0 \left(\int_\Omega |u^N|^2 dx + \int_\Omega |\Delta u^N|^2 dx \right) - (b_2 + (m - 1)a_0)|\Omega|.$$

(2.18)

Combining (2.11), (2.12) and (2.18) we get

$$\gamma_0 \left(\int_\Omega |u^N(x,t)|^2 dx + \int_\Omega |\Delta u^N(x,t)|^2 dx \right)$$

$$+ \int_0^t \int_\Omega M_\theta(u(x,\tau))|\nabla \omega^N(x,\tau)|^2 dx d\tau \leq C,$$

(2.19)

for a.e. $t \in [0,T]$. The estimate (2.10) is established.

From (2.19), using integration by parts and Young's inequality again, we get

$$\int_\Omega |\nabla u^N|^2 dx = - \int_\Omega u^N \Delta u^N dx \leq \frac{1}{2} \int_\Omega |u^N|^2 dx + \frac{1}{2} \int_\Omega |\Delta u^N|^2 dx \leq C. \quad (2.20)$$

Then we obtain the estimate (2.9) by combining (2.19) and (2.20). $\square$

Using the growth conditions (1.12)–(1.14) and Sobolev embedding for $H^2(\Omega)$ in $\mathbb{R}^d$ ($1 \leq d \leq 3$), we obtain the following corollary.

Corollary 2.1. *Let u^N be a solution of the system (2.2)–(2.4). Then, we have*

$$\|u^N\|_{L^\infty(0,T;C^{0,\alpha}(\bar\Omega))} \leq C, \text{ for any } 0 < \alpha < 1/2, \qquad (2.21)$$

$$\|M_\theta(u^N)\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C, \qquad (2.22)$$

$$\|W'(u^N)\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C, \qquad (2.23)$$

$$\|W''(u^N)\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C. \qquad (2.24)$$

Lemma 2.3. *Let u^N be a solution of the system (2.2)–(2.4). Then, we have*

$$\|\omega^N\|_{L^2(0,T;H^1(\Omega))} \leq C_\theta. \qquad (2.25)$$

Proof. Since $M_\theta(u) \geq \theta$, by (2.10) we have

$$\int_0^T \int_\Omega |\nabla \omega^N|^2 dx dt \leq \frac{C}{\theta}, \qquad (2.26)$$

which implies (2.25) by using Poincaré's inequality. $\qquad\square$

Lemma 2.4. *Let u^N be a solution of the system (2.2)–(2.4). Then, we have*

$$\|u^N\|_{L^2(0,T;H^5(\Omega))} \le C_\theta. \tag{2.27}$$

Proof. Since $\Delta u^N = \sum_{j=1}^N c_j^N(t)\Delta\phi_j = -\sum_{j=1}^N c_j^N(t)\lambda_j\phi_j$, then from (2.3) we get

$$\int_\Omega \omega^N \Delta u^N = \int_\Omega (W'(u^N)\Delta u^N + \kappa u^N \Delta u^N - 2\kappa\nabla u^N \cdot \nabla\Delta u^N + \kappa\Delta u^N \Delta^2 u^N)dx$$
$$= \int_\Omega (W'(u^N)\Delta u^N - \kappa|\nabla u^N|^2 + 2\kappa|\Delta u^N|^2 - \kappa|\nabla\Delta u^N|^2)dx. \tag{2.28}$$

Then, using (2.9), (2.23) and (2.25), we obtain

$$\int_0^T \int_\Omega |\nabla\Delta u^N|^2 dxdt$$
$$= \frac{1}{\kappa}\int_0^T \int_\Omega (W'(u^N)\Delta u^N - \omega^N\Delta u^N - \kappa|\nabla u^N|^2 + 2\kappa|\Delta u^N|^2)dxdt$$
$$\le \frac{1}{\kappa}(\|W'(u^N)\|_{L^2(\Omega_T)}^2 + \|\omega^N\|_{L^2(0,T;L^2(\Omega))}^2 + 2\kappa)\|\Delta u^N\|_{L^2(0,T;L^2(\Omega))}^2$$
$$\le C_\theta. \tag{2.29}$$

By (2.3), we have

$$\omega^N = W'(u^N) + \kappa u^N + 2\kappa\Delta u^N + \kappa\Delta^2 u^N. \tag{2.30}$$

So by (2.9), (2.23) and (2.25), we obtain

$$\int_0^T \int_\Omega |\Delta^2 u^N|^2 dxdt$$
$$\le \frac{4}{\kappa^2}\int_0^T \int_\Omega (|W'(u^N)|^2 + \kappa^2|u^N|^2 + 4\kappa^2|\Delta u^N|^2 + |\omega^N|^2)dxdt$$
$$\le C_\theta. \tag{2.31}$$

Taking the derivatives with respect to x on both sides of (2.30), we get

$$\nabla\omega^N = W''(u^N)\nabla u^N + \kappa\nabla u^N + 2\kappa\nabla\Delta u^N + \kappa\nabla\Delta^2 u^N. \tag{2.32}$$

Then, by (2.9), (2.24), (2.25) and (2.29), we obtain

$$\int_0^T \int_\Omega |\nabla\Delta^2 u^N|^2 dxdt$$
$$\le \frac{4}{\kappa^2}\int_0^T \int_\Omega (|\nabla\omega^N|^2 + |W''(u^N)|^2|\nabla u^N|^2 + \kappa^2|\nabla u^N|^2 + 4\kappa^2|\nabla\Delta u^N|^2)dxdt$$
$$\le C_\theta. \tag{2.33}$$

Combining (2.9), (2.29), (2.31) and (2.33), we obtain (2.27). $\qquad\square$

Lemma 2.5. *Let u^N be a solution of the system (2.2)–(2.4), then we have*

$$\|\partial_t u^N\|_{L^2(0,T;H^{-2}(\Omega))} \le C. \tag{2.34}$$

Proof. First, the estimate (2.19) implies that $\|\sqrt{M_\theta(u^N)}\nabla\omega^N\|_{L^2(0,T;L^2(\Omega))} \leq C$. Then, for any $\phi \in L^2(0,T;H^2(\Omega))$, by (2.2) and the generalized Hölder's inequality, we have

$$
\begin{aligned}
\left|\int_\Omega \partial_t u^N \phi dx\right| &= \left|\int_\Omega M_\theta(u^N)\nabla\omega^N\nabla\phi dx\right| \\
&\leq \left\|\sqrt{M_\theta(u^N)}\right\|_{L^d(\Omega)} \left\|\sqrt{M_\theta(u^N)}\nabla\omega^N\right\|_{L^2(\Omega)} \|\nabla\phi\|_{L^{2d/(d-2)}(\Omega)} dt \\
&\leq C\left\|\sqrt{M_\theta(u^N)}\nabla\omega^N\right\|_{L^2(\Omega)} \|\nabla\phi\|_{L^{2d/(d-2)}(\Omega)}.
\end{aligned}
\tag{2.35}
$$

Since $H^1(\Omega) \subset\subset L^{2d/(d-2)}(\Omega)$, by (2.22) and Hölder's inequality, we have

$$
\begin{aligned}
\left|\int_0^T \int_\Omega \partial_t u^N \phi dxdt\right| &\leq C\int_0^T \left\|\sqrt{M_\theta(u^N)}\nabla\omega^N\right\|_{L^2(\Omega)} \|\nabla\phi\|_{L^{2d/(d-2)}(\Omega)} dt \\
&\leq C\left\|\sqrt{M_\theta(u^N)}\nabla\omega^N\right\|_{L^2(0,T;L^2(\Omega))} \|\nabla\phi\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))} \\
&\leq C\|\nabla\phi\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\|\phi\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}
\tag{2.36}
$$

This implies $\|\partial_t u^N\|_{L^2(0,T;H^{-2}(\Omega))} \leq C$. $\qquad\square$

2.3. Convergence of u^N and the existence of a weak solution.

In $\mathbb{R}^d$ with $1 \leq d \leq 3$, we have $H^2(\Omega) \subset\subset H^1(\Omega) \subset\subset C^{0,\alpha}(\bar{\Omega}) \hookrightarrow H^{-2}(\Omega)$ and $H^5(\Omega) \subset\subset H^l(\Omega) \subset\subset C^{3,\alpha}(\bar{\Omega}) \hookrightarrow H^{-2}(\Omega)$, for any integer $1 \leq l \leq 4$ and any $0 < \alpha < 1/2$. Hence, by the Aubin–Lions lemma (see [2, 21]), we have

$$
\begin{aligned}
\{f \in L^\infty(0,T;H^2(\Omega)) : \partial_t f \in L^2(0,T;H^{-2}(\Omega))\} &\subset\subset C([0,T];H^1(\Omega)), \\
\{f \in L^\infty(0,T;H^2(\Omega)) : \partial_t f \in L^2(0,T;H^{-2}(\Omega))\} &\subset\subset C([0,T];C^{0,\alpha}(\bar{\Omega})), \\
\{f \in L^2(0,T;H^5(\Omega)) : \partial_t f \in L^2(0,T;H^{-2}(\Omega))\} &\subset\subset L^2(0,T;H^l(\Omega)), \\
\{f \in L^2(0,T;H^5(\Omega)) : \partial_t f \in L^2(0,T;H^{-2}(\Omega))\} &\subset\subset L^2(0,T;C^{3,\alpha}(\bar{\Omega})),
\end{aligned}
$$

for any integer $1 \leq l \leq 4$ and any $0 < \alpha < 1/2$. Combining with the weak compactness in $L^2(0,T;H^5(\Omega))$ and $L^2(0,T;H^{-2}(\Omega))$, by (2.9), (2.27) and (2.34), there exist a subsequence of $\{u^N\}$ (not relabeled) and a function

$$
u_\theta \in L^2(0,T;H^5(\Omega)) \cap L^2(0,T;C^{3,\alpha}(\bar{\Omega})) \cap C([0,T];H^1(\Omega)) \cap C([0,T];C^{0,\alpha}(\bar{\Omega}))
$$

such that as $N \to \infty$,

$$
u^N \rightharpoonup u_\theta \quad \text{weakly in } L^2(0,T;H^5(\Omega)),
\tag{2.37}
$$

$$
\partial_t u^N \rightharpoonup \partial_t u_\theta \quad \text{weakly in } L^2(0,T;H^{-2}(\Omega)),
\tag{2.38}
$$

$$
u^N \to u_\theta \quad \text{strongly in } C([0,T];H^1(\Omega)),
\tag{2.39}
$$

$$
u^N \to u_\theta \quad \text{strongly in } C([0,T];C^{0,\alpha}(\bar{\Omega})),
\tag{2.40}
$$

$$
u^N \to u_\theta \quad \text{strongly in } L^2(0,T;H^l(\Omega)) \text{ and a.e. in } \Omega_T,
\tag{2.41}
$$

$$
u^N \to u_\theta \quad \text{strongly in } L^2(0,T;C^{3,\alpha}(\bar{\Omega})),
\tag{2.42}
$$

for any $1 \le l \le 4$ and any $0 < \alpha < 1/2$. In addition, using (2.9) and (2.34), we have the following bounds for u_θ:

$$\|u_\theta\|_{L^\infty(0,T;H^2(\Omega))} \le C, \tag{2.43}$$

$$\|\partial_t u_\theta\|_{L^2(0,T;H^{-2}(\Omega))} \le C. \tag{2.44}$$

From (2.25), there exist a subsequence of $\{\omega^N\}$ (not relabeled) and a function $\omega_\theta \in L^2(0,T;H^1(\Omega))$ such that as $N \to \infty$,

$$\omega^N \rightharpoonup \omega_\theta \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{2.45}$$

By (2.39), $M_\theta(u^N) \to M_\theta(u_\theta)$ strongly in $C([0,T];L^d(\Omega))$, hence

$$M_\theta(u^N)\nabla\omega^N \rightharpoonup M_\theta(u_\theta)\nabla\omega_\theta \quad \text{weakly in } L^2(0,T;L^{2d/(d+2)}(\Omega,\mathbb{R}^d)) \text{ as } N \to \infty. \tag{2.46}$$

For any $\gamma(t) \in L^2(0,T)$, because $\gamma(t)\nabla\phi_j \in L^2(0,T;L^{2d/(d-2)}(\Omega))$, multiplying both sides of (2.2) by $\gamma(t)$, integrating over the time interval $(0,T)$, and taking the limits as $N \to \infty$, by (2.38) and (2.46), we have

$$\int_0^T \langle \partial_t u_\theta, \gamma(t)\phi_j(x) \rangle_{H^{-2}(\Omega),H^2(\Omega)} \, dt = -\int_0^T \int_\Omega M_\theta(u_\theta)\nabla\omega_\theta \cdot \gamma(t)\nabla\phi_j(x) dx dt. \tag{2.47}$$

For any function $\xi \in L^2(0,T;H^2(\Omega))$, since its Fourier series $\sum_{j=1}^\infty a_j(t)\phi_j(x)$ converges strongly to ξ in $L^2(0,T;H^2(\Omega))$, then $\sum_{j=1}^\infty a_j(t)\nabla\phi_j(x)$ converges strongly to $\nabla\xi$ in $L^2(0,T;L^{2d/(d-2)}(\Omega,\mathbb{R}^d))$. Hence, by (2.47), we have

$$\int_0^T \langle \partial_t u_\theta, \xi \rangle_{H^{-2}(\Omega),H^2(\Omega)} \, dt = -\int_0^T \int_\Omega M_\theta(u_\theta)\nabla\omega_\theta \cdot \nabla\xi dx dt, \tag{2.48}$$

for all $\xi \in L^2(0,T;H^2(\Omega))$,.

By (2.40) and (2.23), we have

$$W'(u^N) \to W'(u_\theta) \quad \text{strongly in } C([0,T];L^q(\Omega)) \text{ as } N \to \infty, \tag{2.49}$$

for any $1 \le q < \infty$. Then, for any $\gamma(t) \in L^2(0,T)$, because $\gamma(t)\phi_j \in L^2(0,T;C(\Omega))$, multiplying both sides of (2.3) by $\gamma(t)$, integrating over the time interval $(0,T)$, and taking the limits as $N \to \infty$, by (2.37) and (2.49), we obtain

$$\int_0^T \int_\Omega \omega_\theta \gamma(t)\phi_j dx dt$$
$$= \int_0^T \int_\Omega W'(u_\theta)\gamma(t)\phi_j + \kappa u_\theta \gamma(t)\phi_j - 2\kappa\nabla u_\theta \cdot \gamma(t)\nabla\phi_j + \kappa\Delta u_\theta \gamma(t)\Delta\phi_j \, dx dt. \tag{2.50}$$

For any function $\xi \in L^2(0,T;H^2(\Omega))$, since its Fourier series $\sum_{j=1}^\infty a_j(t)\phi_j(x)$ converges strongly to ξ in $L^2(0,T;H^2(\Omega))$, then $\sum_{j=1}^\infty a_j(t)\nabla\phi_j(x)$ converges strongly to $\nabla\xi$ in $L^2(0,T;L^{2d/(d-2)}(\Omega,\mathbb{R}^d))$ and $\sum_{j=1}^\infty a_j(t)\Delta\phi_j(x)$ converges strongly to $\Delta\xi$ in $L^2(0,T;L^{2d/(d-2)}(\Omega))$. Hence, by (2.50), we obtain

$$\int_0^T \int_\Omega \omega_\theta \xi dx dt = \int_0^T \int_\Omega W'(u_\theta)\xi + \kappa u_\theta \xi - 2\kappa\nabla u_\theta \cdot \nabla\xi + \kappa\Delta u_\theta \Delta\xi \, dx dt, \tag{2.51}$$

for all $\xi \in L^2(0, T; H^2(\Omega))$. Since $u_\theta \in L^2(0, T; H^5(\Omega))$, then $\omega_\theta \in L^2(0, T; H^3(\Omega))$, hence, from (2.51) we obtain

$$\omega_\theta = W'(u_\theta) + \kappa u_\theta + 2\kappa \Delta u_\theta + \kappa \Delta^2 u_\theta \quad \text{a.e. in } \Omega_T, \tag{2.52}$$

and

$$\nabla \omega_\theta = W''(u_\theta) \nabla u_\theta + \kappa \nabla u_\theta + 2\kappa \nabla \Delta u_\theta + \kappa \nabla \Delta^2 u_\theta \quad \text{a.e. in } \Omega_T. \tag{2.53}$$

Combining (2.53) and (2.48) we get (1.16).

For the initial value, by (2.4), we have

$$u^N(x, 0) \to u_0(x) \quad \text{strongly in } L^2(\Omega) \text{ as } N \to \infty.$$

Combining with (2.41), we have $u_\theta(x, 0) = u_0(x)$, for a.e. $x \in \Omega$.

2.4. **Energy inequality.** From (2.39), we have $\sqrt{M_\theta(u^N)} \to \sqrt{M_\theta(u_\theta)}$ strongly in $C([0, T]; L^2(\Omega))$ as $N \to \infty$. Combining with (2.45), we get

$$\sqrt{M_\theta(u^N)} \nabla \omega^N \rightharpoonup \sqrt{M_\theta(u_\theta)} \nabla \omega_\theta \quad \text{weakly in } L^2(0, T; L^1(\Omega; \mathbb{R}^d)) \text{ as } N \to \infty. \tag{2.54}$$

On the other hand, by (2.10), there exist a subsequence of $\{\sqrt{M_\theta(u^N)} \nabla \omega^N\}$ (not relabeled) and a function $\Lambda_\theta \in L^2(0, T; L^2(\Omega; \mathbb{R}^d))$ such that

$$\sqrt{M_\theta(u^N)} \nabla \omega^N \rightharpoonup \Lambda_\theta \quad \text{weakly in } L^2(0, T; L^2(\Omega; \mathbb{R}^d)) \text{ as } N \to \infty. \tag{2.55}$$

By the uniqueness of weak limits, we have $\Lambda_\theta = \sqrt{M_\theta(u_\theta)} \nabla \omega_\theta$, which implies

$$\sqrt{M_\theta(u^N)} \nabla \omega^N \rightharpoonup \sqrt{M_\theta(u_\theta)} \nabla \omega_\theta \quad \text{weakly in } L^2(0, T; L^2(\Omega; \mathbb{R}^d)) \text{ as } N \to \infty. \tag{2.56}$$

Also, by (2.40), we have

$$W(u^N) \to W(u_\theta) \quad \text{strongly in } C([0, T]; L^q(\Omega)) \text{ as } N \to \infty, \tag{2.57}$$

for any $1 \le q < \infty$.

Because u^N and ω^N satisfy the energy identity (2.11), using (2.37)–(2.42), (2.56) and (2.57), and the weak lower semicontinuity of norms, by taking the limits as $N \to \infty$ in (2.11), we obtain the energy inequality (1.17).

2.5. **Positive initial data.** In this section, we assume that the initial data $u_0(x) > 0$, for all $x \in \Omega$. For any $0 < \theta < 1$, we define the entropy densities $\Phi : (0, \infty) \to [0, \infty)$ and $\Phi_\theta : \mathbb{R} \to [0, \infty)$ as

$$\Phi''(z) = \frac{1}{M(z)}, \quad \Phi(1) = \Phi'(1) = 0,$$

and

$$\Phi_\theta''(z) = \frac{1}{M_\theta(z)}, \quad \Phi_\theta(1) = \Phi_\theta'(1) = 0.$$

Using the definitions of $M(u)$ and $M_\theta(u)$ in (1.11) and (1.15), we obtain

$$\Phi(z) = z \ln(z) - z + 1 \quad \text{for } z > 0, \tag{2.58}$$

and

$$\Phi_\theta(z) = \begin{cases} z \ln(z) - z + 1 & , z > \theta, \\ \frac{1}{2\theta} z^2 + (\ln(\theta) - 1)z + 1 - \frac{\theta}{2} & , z \le \theta. \end{cases} \tag{2.59}$$

We see that $\Phi_\theta \in C^2(\mathbb{R})$, $0 \le \Phi_\theta(z) \le \Phi(z)$, for all $z > 0$, and $\Phi_\theta(z) = \Phi(z)$, for all $z \ge \theta$.

Claim 1. *For any* $t \in [0, T]$,

$$\int_\Omega \Phi_\theta(u_\theta(x,t))dx - \int_\Omega \Phi_\theta(u_0(x))dx = -\int_0^t \int_\Omega \nabla \omega_\theta \cdot \nabla u_\theta dx d\tau. \tag{2.60}$$

Proof. For any $\epsilon > 0$, let $\Phi_{\theta,\epsilon}$ be the mollification of Φ_θ. Since $\Phi_\theta \in C^2(\mathbb{R})$, then $\Phi_{\theta,\epsilon} \to \Phi_\theta, \Phi'_{\theta,\epsilon} \to \Phi'_\theta$ and $\Phi''_{\theta,\epsilon} \to \Phi''_\theta$ uniformly on compact subsets of $\mathbb{R}$ as $\epsilon \to 0$. We will show that

$$\int_\Omega \Phi_{\theta,\epsilon}(u_\theta(x,t))dx - \int_\Omega \Phi_{\theta,\epsilon}(u_0(x))dx = -\int_0^t \int_\Omega M_\theta(u_\theta)\Phi''_{\theta,\epsilon}(u_\theta)\nabla\omega_\theta \cdot \nabla u_\theta dx d\tau. \tag{2.61}$$

For any $h > 0$, define

$$u_{\theta,h}(x,t) := \frac{1}{h}\int_{t-h}^t u_\theta(x,\tau)d\tau, \tag{2.62}$$

where we set $u_\theta(x,t) = u_0(x)$ for $t \le 0$. Since $H^3(\Omega) \subset\subset H^2(\Omega) \hookrightarrow H^{-3}(\Omega)$, by the Aubin–Lions lemma,

$$\{f \in L^2(0,T;H^3(\Omega)) : \partial_t f \in L^2(0,T;H^{-3}(\Omega))\} \subset\subset L^2(0,T;H^2(\Omega)).$$

Since $u_\theta \in L^\infty(0,T;H^5(\Omega))$ and $\Phi'_{\theta,\epsilon}, \Phi''_{\theta,\epsilon}, \Phi'''_{\theta,\epsilon}, \Phi^{(4)}_{\theta,\epsilon}$ are bounded, we have

$$\sup_{h>0} \|\Phi'_{\theta,\epsilon}(u_{\theta,h})\|_{L^2(0,T;H^3(\Omega))} \le C_\theta \quad \text{and} \quad \partial_t \Phi'_{\theta,\epsilon}(u_{\theta,h}) \in L^2(0,T;H^{-3}(\Omega)),$$

for any $h > 0$. Hence, there exists a subsequence of $\{\Phi'_{\theta,\epsilon}(u_{\theta,h})\}_{h>0}$ (not relabeled) such that

$$\Phi'_{\theta,\epsilon}(u_{\theta,h}) \to \Phi'_{\theta,\epsilon}(u_\theta) \quad \text{strongly in } L^2(0,T;H^2(\Omega)) \text{ as } h \to 0. \tag{2.63}$$

On the other hand, define $\mathbf{J}_\theta := -M_\theta(u_\theta)\nabla\omega_\theta$. Since $\omega_\theta \in L^2(0,T;H^3(\Omega))$, $\mathbf{J}_\theta \in L^2(0,T;H^2(\Omega;\mathbb{R}^d))$. Hence, for any $\xi \in L^2(0,T;H^2(\Omega))$,

$$\left| \langle \partial_t u_{\theta,h} - \partial_t u_\theta, \xi \rangle_{(L^2(0,T;H^{-2}(\Omega)),L^2(0,T;H^2(\Omega)))} \right|$$

$$= \frac{1}{h} \left| \int_0^T \left\langle \int_{t-h}^t (\partial_t u_\theta(\tau) - \partial_t u_\theta(t))d\tau, \xi \right\rangle_{(H^{-2}(\Omega),H^2(\Omega))} dt \right|$$

$$= \frac{1}{h} \left| \int_0^T \left\langle \int_{-h}^0 (\partial_t u_\theta(t+s) - \partial_t u_\theta(t))ds, \xi \right\rangle_{(H^{-2}(\Omega),H^2(\Omega))} dt \right|$$

$$\le \frac{1}{h} \int_{-h}^0 \left| \int_0^T \int_\Omega \nabla\xi \cdot (\mathbf{J}_\theta(t+s) - \mathbf{J}_\theta(t))dx dt \right| ds$$

$$\le \|\xi\|_{L^2(0,T;H^2(\Omega))} \sup_{-h \le s \le 0} \|\mathbf{J}_\theta(\cdot + s) - \mathbf{J}_\theta(\cdot)\|_{L^2(0,T;H^2(\Omega))}, \tag{2.64}$$

which implies that

$$\|\partial_t u_{\theta,h} - \partial_t u_\theta\|_{L^2(0,T;H^{-2}(\Omega))} \le \sup_{-h \le s \le 0} \|\mathbf{J}_\theta(\cdot + s) - \mathbf{J}_\theta(\cdot)\|_{L^2(0,T;H^2(\Omega))}. \tag{2.65}$$

Since $\sup_{-h \le s \le 0} \|\mathbf{J}_\theta(\cdot + s) - \mathbf{J}_\theta(\cdot)\|_{L^2(0,T;H^2(\Omega))} \to 0$ as $h \to 0$, then

$$\partial_t u_{\theta,h} \to \partial_t u_\theta \quad \text{strongly in } L^2(0,T;H^{-2}(\Omega)) \text{ as } h \to 0. \tag{2.66}$$

Since $\Phi'_{\theta,\epsilon}(u_{\theta,h})$ and $\partial_t u_{\theta,h}$ are both in $L^2(\Omega_T)$, we have

$$
\begin{aligned}
\int_0^t \left\langle \partial_t u_{\theta,h}, \Phi'_{\theta,\epsilon}(u_{\theta,h}) \right\rangle_{(H^{-2}(\Omega), H^2(\Omega))} d\tau &= \int_0^t \int_\Omega \Phi'_{\theta,\epsilon}(u_{\theta,h}) \partial_t u_{\theta,h} dx d\tau \\
&= \int_\Omega \int_0^t \partial_t \Phi_{\theta,\epsilon}(u_{\theta,h}(\tau,x)) d\tau dx \\
&= \int_\Omega \Phi_{\theta,\epsilon}(u_{\theta,h}(x,t)) dx - \int_\Omega \Phi_{\theta,\epsilon}(u_0(x)) dx,
\end{aligned}
\tag{2.67}
$$

for a.e. $t \in [0,T]$. Passing to the limit as $h \to 0$ and combining with (2.63) and (2.66), we get

$$
\int_0^t \left\langle \partial_t u_\theta, \Phi'_{\theta,\epsilon}(u_\theta) \right\rangle_{(H^{-2}(\Omega), H^2(\Omega))} d\tau = \int_\Omega \Phi_{\theta,\epsilon}(u_\theta(x,t)) dx - \int_\Omega \Phi_{\theta,\epsilon}(u_0(x)) dx.
\tag{2.68}
$$

Since $\Phi''_{\theta,\epsilon}$ and $\Phi'''_{\theta,\epsilon}$ are bounded, then $\Phi'_{\theta,\epsilon}(u_\theta) \in L^2(0,T; H^2(\Omega))$. So $\Phi'_{\theta,\epsilon}(u_\theta)$ is an admissible test function for the equation (2.48). Hence, for any $t \in [0,T]$,

$$
\begin{aligned}
\int_0^t \left\langle \partial_t u_\theta, \Phi'_{\theta,\epsilon}(u_\theta) \right\rangle_{(H^{-2}(\Omega), H^2(\Omega))} d\tau &= -\int_0^t \int_\Omega M_\theta(u_\theta) \nabla \omega_\theta \cdot \nabla(\Phi'_{\theta,\epsilon}(u_\theta)) dx d\tau \\
&= -\int_0^t \int_\Omega M_\theta(u_\theta) \Phi''_{\theta,\epsilon}(u_\theta) \nabla \omega_\theta \cdot \nabla u_\theta dx d\tau.
\end{aligned}
\tag{2.69}
$$

Combining (2.68) and (2.69) we get (2.61).

Now, for each $t \in [0,T]$, since $u_\theta \in C([0,T]; C^{3,\alpha}(\bar{\Omega}))$, then $u_\theta(\bar{\Omega}, t) := \{u_\theta(x,t) : x \in \bar{\Omega}\}$ is a compact subset of $\mathbb{R}$, so $\Phi_{\theta,\epsilon} \to \Phi_\theta$ and $\Phi''_{\theta,\epsilon} \to \Phi''_\theta$ uniformly in $u_\theta(\bar{\Omega}, t)$ as $\epsilon \to 0$, hence, $\Phi_{\theta,\epsilon}(u_\theta) \to \Phi_\theta(u_\theta)$ and $\Phi''_{\theta,\epsilon}(u_\theta) \to \Phi''_\theta(u_\theta)$ uniformly in $\bar{\Omega}$ as $\epsilon \to 0$. Also, $u_\theta(\overline{\Omega_T}) := \{u_\theta(x,\tau) : x \in \bar{\Omega}, \tau \in [0,T]\}$ is a compact subset of $\mathbb{R}$, so $\Phi''_{\theta,\epsilon} \to \Phi''_\theta$ uniformly in $u_\theta(\overline{\Omega_T})$ as $\epsilon \to 0$, hence, $\Phi''_{\theta,\epsilon}(u_\theta) \to \Phi''_\theta(u_\theta)$ uniformly in $\overline{\Omega_T}$ as $\epsilon \to 0$. Using Lemma 2.2, we obtain

$$
\|M_\theta(u_\theta)\|_{L^\infty(\Omega_T)} < \infty \quad \text{and} \quad \left\| \sqrt{M_\theta(u_\theta)} \nabla \omega_\theta \right\|_{L^2(\Omega_T)} < \infty.
\tag{2.70}
$$

Since $u_\theta \in C([0,T]; C^{3,\alpha}(\bar{\Omega}))$, then $\|\nabla u_\theta\|_{L^\infty(\Omega_T)} < \infty$. Thus,

$$
\left| \int_0^t \int_\Omega M_\theta(u_\theta) \nabla \omega_\theta \cdot \nabla u_\theta \left(\Phi''_{\theta,\epsilon}(u_\theta) - \Phi''_\theta(u_\theta) \right) dx d\tau \right|
$$
$$
\leq \|M_\theta(u_\theta)\|_{L^\infty(\Omega_T)}^{1/2} \|\nabla u_\theta\|_{L^\infty(\Omega_T)} \|\sqrt{M_\theta(u_\theta)} \nabla \omega_\theta\|_{L^2(\Omega_T)} \|\Phi''_{\theta,\epsilon}(u_\theta) - \Phi''_\theta(u_\theta)\|_{L^2(\Omega_T)}
$$
$$
\to 0 \quad \text{as } \epsilon \to 0.
\tag{2.71}
$$

So passing to limits as $\epsilon \to 0$ on both sides of (2.61) and using $\Phi''_\theta(u_\theta) = 1/M_\theta(u_\theta)$, we get (2.60). $\qquad \square$

Claim 2. *Let* $(u_\theta)_- := \min\{u_\theta, 0\}$, *then, for any* $0 < \theta < 1$,

$$
\operatorname*{ess\,sup}_{0 \leq t \leq T} \int_\Omega |(u_\theta)_- + \theta|^2 dx \leq C(\theta^2 + \theta + \theta^{1/2}).
\tag{2.72}
$$

Proof. For any $z \leq 0$, we rewrite $\Phi_\theta(z)$ as

$$\Phi_\theta(z) = \frac{1}{2\theta}(z+\theta)^2 + (\ln\theta - 2)z + (1-\theta). \tag{2.73}$$

Since $0 < \theta < 1$, (2.73) implies

$$(z+\theta)^2 \leq 2\theta\Phi_\theta(z) \quad \text{for all } z \leq 0. \tag{2.74}$$

Hence, for any $t \in [0,T]$, since $\Phi_\theta(z) \geq 0$, for all $z \in \mathbb{R}$, we have

$$\int_\Omega |(u_\theta(x,t))_- + \theta|^2 dx \leq 2\theta \int_\Omega \Phi_\theta(u_\theta(x,t)_-)dx$$

$$\leq 2\theta \left(\int_{\{u_\theta \leq 0\}} \Phi_\theta(u_\theta(x,t))dx + \int_{\{u_\theta > 0\}} \Phi_\theta(0)dx \right)$$

$$\leq 2\theta \left(\int_\Omega \Phi_\theta(u_\theta(x,t))dx + \int_\Omega \Phi_\theta(0)dx \right)$$

$$\leq 2\theta \left[\int_\Omega \Phi_\theta(u_\theta(x,t))dx + \left(1 - \frac{\theta}{2}\right)|\Omega| \right] \tag{2.75}$$

From (2.26) and (2.9), we have $\|\nabla\omega_\theta\|_{L^2(\Omega_T)} \leq C/\theta^{1/2}$ and $\|\nabla u_\theta\|_{L^2(\Omega_T)} \leq C$. Thus, by (2.60) and Hölder's inequality, we have

$$\left| \int_\Omega \Phi_\theta(u_\theta(x,t))dx \right| \leq \left| \int_\Omega \Phi_\theta(u_0)dx \right| + \left| \int_0^t \int_\Omega \nabla\omega_\theta(x,\tau) \cdot \nabla u_\theta(x,\tau)dxd\tau \right|$$

$$\leq \int_\Omega \Phi_\theta(u_0)dx + \|\nabla\omega_\theta\|_{L^2(\Omega_T)}\|\nabla u_\theta\|_{L^2(\Omega_T)}$$

$$\leq \int_\Omega \Phi(u_0)dx + \frac{C}{\theta^{1/2}}, \tag{2.76}$$

for any $t \in [0,T]$. This inequality and (2.75) implies (2.72). $\qquad\square$

3. Proof of theorem 1.2: Weak solutions for the degenerate mobility case.

In this section we prove the existence of a nonnegative weak solution to the PFC equation (1.2)–(1.3) with the degenerate mobility $M(u)$ defined by (1.11).

3.1. Convergence of $\{u_\theta\}$ as $\theta \to 0$.
Fix $u_0 \in H^2(\Omega)$ and a sequence of positive numbers $\{\theta_i\}_{i=1}^\infty$ that monotonically decreases to 0 as $i \to \infty$. For each θ_i, according to Theorem 1.1, there exists a function

$$u_{\theta_i} \in L^2(0,T;H^5(\Omega)) \cap L^2(0,T;C^{3,\alpha}(\bar\Omega)) \cap C([0,T];H^1(\Omega)) \cap C([0,T];C^{0,\alpha}(\bar\Omega)),$$

for any $0 < \alpha < 1/2$, whose weak derivative is

$$\partial_t u_{\theta_i} \in L^2(0,T;H^{-2}(\Omega)),$$

such that, for any $\xi \in L^2(0,T;H^2(\Omega))$,

$$\int_0^T \langle \partial_t u_{\theta_i}, \xi \rangle_{H^{-2}(\Omega),H^2(\Omega)} dt = - \int_0^T \int_\Omega M_{\theta_i}(u_{\theta_i})\nabla\omega_{\theta_i} \cdot \nabla\xi dxdt, \tag{3.1}$$

$$\omega_{\theta_i} = W'(u_{\theta_i}) + \kappa u_{\theta_i} + 2\kappa\nabla u_{\theta_i} + \kappa\Delta u_{\theta_i}, \tag{3.2}$$

$$u_{\theta_i}(x,0) = u_0(x), \text{ for all } x \in \Omega. \tag{3.3}$$

For simplicity, we use the notations $u_i := u_{\theta_i}, \omega_i := \omega_{\theta_i}$ and $M_i := M_{\theta_i}$. By the proofs of the lemmas in Section 2.2, the bounds on the right hand sides of (2.9),

(2.10) and (2.34) depend only on $d, T, \Omega, b_1, b_2, b_3, b_4, m, \kappa, \epsilon$ and u_0, but not on θ. Therefore, there exists a constant $C > 0$ independent on $\{\theta_i\}_{i=1}^{\infty}$ such that

$$\|u_i\|_{L^\infty(0,T;H^2(\Omega))} \le C, \tag{3.4}$$

$$\|\partial_t u_i\|_{L^2(0,T;H^{-2}(\Omega))} \le C, \tag{3.5}$$

$$\left\|\sqrt{M_i(u_i)}\nabla\omega_i\right\|_{L^2(\Omega_T)} \le C. \tag{3.6}$$

By the Aubin–Lions lemma, we have

$$\{f \in L^\infty(0,T;H^2(\Omega)) : \partial_t f \in L^2(0,T;H^{-2}(\Omega))\} \subset\subset C([0,T];H^1(\Omega)),$$

and

$$\{f \in L^\infty(0,T;H^2(\Omega)) : \partial_t f \in L^2(0,T;H^{-2}(\Omega))\} \subset\subset C([0,T];C^{0,\alpha}(\bar\Omega)),$$

for any $0 < \alpha < 1/2$. Combining with the weak compactness in $L^\infty(0,T;H^2(\Omega))$ and $L^2(0,T;H^{-2}(\Omega))$, there exist a subsequence of $\{u_i\}$ (not relabeled) and a function

$$u \in L^\infty(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega)) \cap C([0,T];C^{0,\alpha}(\bar\Omega))$$

such that as $i \to \infty$,

$$u_i \rightharpoonup u \quad \text{weakly--* in } L^\infty(0,T;H^2(\Omega)), \tag{3.7}$$

$$\partial_t u_i \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;H^{-2}(\Omega)), \tag{3.8}$$

$$u_i \to u \quad \text{strongly in } C([0,T];C^{0,\alpha}(\bar\Omega)), \text{ for any } 0 < \alpha < 1/2, \tag{3.9}$$

$$u_i \to u \quad \text{strongly in } C([0,T];H^1(\Omega)) \text{ and a.e. in } \Omega_T. \tag{3.10}$$

From (3.9) and (3.10), the uniform convergence of $M_i \to M$ and $\sqrt{M_i} \to \sqrt{M}$ as $i \to \infty$, and the General Lebesgue Dominated Convergence Theorem (see [20], page 89, Theorem 19) we have

$$M_i(u_i) \to M(u) \quad \text{strongly in } C([0,T];L^{d/2}(\Omega)), \tag{3.11}$$

$$\sqrt{M_i(u_i)} \to \sqrt{M(u)} \quad \text{strongly in } C([0,T];L^d(\Omega)), \tag{3.12}$$

as $i \to \infty$. By (3.6), there exist a subsequence of $\{\sqrt{M_i(u_i)}\nabla\omega_i\}$ (not relabeled) and a function $\Lambda \in L^2(\Omega_T;\mathbb{R}^d)$ such that

$$\sqrt{M_i(u_i)}\nabla\omega_i \rightharpoonup \Lambda \quad \text{weakly in } L^2(\Omega_T;\mathbb{R}^d) \text{ as } i \to \infty. \tag{3.13}$$

Combining with (3.12) we get

$$M_i(u_i)\nabla\omega_i \rightharpoonup \sqrt{M(u)}\Lambda \quad \text{weakly in } L^2(0,T;L^{2d/(d+2)}(\Omega;\mathbb{R}^d)) \text{ as } i \to \infty. \tag{3.14}$$

So, taking the limits as $i \to \infty$ in (3.1), we obtain

$$\int_0^T \langle\partial_t u, \xi\rangle_{H^{-2}(\Omega),H^2(\Omega)} dt = -\int_0^T \int_\Omega \sqrt{M(u)}\Lambda \cdot \nabla\xi dx dt \tag{3.15}$$

for all $\xi \in L^2(0,T;H^2(\Omega))$. For the initial data, from (3.3) and (3.9), we have $u(x,0) = u_0(x)$, for all $x \in \Omega$.

Next, we will show that the function u solves the PFC equation (1.2)–(1.3) in any open subset of Ω_T in which u has sufficient regularity. Moreover, the set in which u does not have sufficient regularity is contained in the set in which $M(u)$ is degenerate and another set of Lebesgue measure zero.

3.2. **Weak solution to the degenerate PFC equation.** Choose a sequence of positive numbers $\{\delta_j\}_{j=1}^{\infty}$ that monotonically decreases to 0. For each δ_j, by (3.10) and Egorov's theorem, there exists a subset $B_j \subset \Omega_T$ with $|\Omega_T \backslash B_j| < \delta_j$ such that

$$u_i \to u \quad \text{uniformly in } B_j \text{ as } i \to \infty. \tag{3.16}$$

We may choose B_j's so that $B_1 \subset B_2 \subset ... \subset B_j \subset B_{j+1} \subset ... \subset \Omega_T$. Let

$$B := \bigcup_{j=1}^{\infty} B_j,$$

then $|\Omega_T \backslash B| = 0$. We also define

$$P_j := \{(x,t) \in \Omega_T : u(x,t) > \delta_j\},$$

then $P_1 \subset P_2 \subset ... \subset P_j \subset P_{j+1} \subset ... \subset \Omega_T$. Let

$$P := \bigcup_{j=1}^{\infty} P_j = \{(x,t) \in \Omega_T : u(x,t) > 0\}.$$

For each j, B_j can be split into two parts:

$$B_j \cap P_j, \text{ where } u > \delta_j \text{ and } u_i \to u \text{ uniformly as } i \to \infty,$$

$$B_j \backslash P_j, \text{ where } u \leq \delta_j \text{ and } u_i \to u \text{ uniformly as } i \to \infty.$$

By the choice of B_j and P_j, we have

$$(B_1 \cap P_1) \subset (B_2 \cap P_2) \subset ... \subset (B_j \cap P_j) \subset (B_{j+1} \cap P_{j+1}) \subset ... \subset (B \cap P), \tag{3.17}$$

and

$$B \cap P = \bigcup_{j=1}^{\infty} (B_j \cap P_j). \tag{3.18}$$

For any $\Phi \in L^2(0,T; L^{2d/(d-2)}(\Omega; \mathbb{R}^d))$ and for each j, we have

$$\int_{\Omega_T} M_i(u_i) \nabla \omega_i \cdot \Phi dx dt = \int_{\Omega_T \backslash B_j} M_i(u_i) \nabla \omega_i \cdot \Phi dx dt + \int_{B_j \cap P_j} M_i(u_i) \nabla \omega_i \cdot \Phi dx dt$$

$$+ \int_{B_j \backslash P_j} M_i(u_i) \nabla \omega_i \cdot \Phi dx dt. \tag{3.19}$$

As $i \to \infty$, by (3.14), the left hand side of (3.19) has the limit

$$\lim_{i \to \infty} \int_{\Omega_T} M_i(u_i) \nabla \omega_i \cdot \Phi dx dt = \int_{\Omega_T} \sqrt{M(u)} \Lambda \cdot \Phi dx dt. \tag{3.20}$$

For the first term on the right hand side of (3.19), since $\lim_{j \to \infty} |\Omega_T \backslash B_j| = |\Omega_T \backslash B| = 0$, we have

$$\lim_{j \to \infty} \lim_{i \to \infty} \int_{\Omega_T \backslash B_j} M_i(u_i) \nabla \omega_i \cdot \Phi dx dt = \lim_{j \to \infty} \int_{\Omega_T \backslash B_j} \sqrt{M(u)} \Lambda \cdot \Phi dx dt = 0. \tag{3.21}$$

We now analyze the second term on the right hand side of (3.19). Because $u_i \to u$ uniformly in B_1 as $i \to \infty$, there exists an integer $N_1 > 0$ such that, for all $i \geq N_1$,

$$u_i > \frac{\delta_1}{2} \text{ in } B_1 \cap P_1, \quad \text{and} \quad u_i \leq 2\delta_1 \text{ in } B_1 \backslash P_1.$$

Then by (3.6), for any $i \geq N_1$, we have

$$\frac{\delta_1}{2} \int_{B_1 \cap P_1} |\nabla \omega_i|^2 dx dt \leq \int_{B_1 \cap P_1} M_i(u_i) |\nabla \omega_i|^2 dx dt$$

$$\leq \int_{\Omega_T} M_i(u_i)|\nabla\omega_i|^2 dxdt \leq C, \tag{3.22}$$

which implies $\{\nabla\omega_i\}_{i=N_1}^{\infty}$ is bounded in $L^2(B_1 \cap P_1; \mathbb{R}^d)$. So there exist a subsequence $\{\nabla\omega_{1,k}\}_{k=1}^{\infty}$ of $\{\nabla\omega_i\}_{i=N_1}^{\infty}$ and a function $\Psi_1 \in L^2(B_1 \cap P_1; \mathbb{R}^d)$ such that

$$\nabla\omega_{1,k} \rightharpoonup \Psi_1 \quad \text{weakly in } L^2(B_1 \cap P_1; \mathbb{R}^d) \text{ as } k \to \infty.$$

We also write $\{u_{1,k}\}_{k=1}^{\infty}$ as the subsequence of $\{u_i\}_{i=N_1}^{\infty}$ corresponding to $\{\nabla\omega_{1,k}\}_{k=1}^{\infty}$. Using the same process, for each $j = 1, 2, ...$, we obtain a subsequence $\{\nabla\omega_{j,k}\}_{k=1}^{\infty}$ of $\{\nabla\omega_{j-1,k}\}_{k=1}^{\infty}$ and a function $\Psi_j \in L^2(B_j \cap P_j; \mathbb{R}^d)$ such that

$$\nabla\omega_{j,k} \rightharpoonup \Psi_j \quad \text{weakly in } L^2(B_j \cap P_j; \mathbb{R}^d) \text{ as } k \to \infty,$$

and we also write $\{u_{j,k}\}_{k=1}^{\infty}$ as the subsequence of $\{u_{j-1,k}\}_{k=1}^{\infty}$ corresponding to $\{\nabla\omega_{j,k}\}_{k=1}^{\infty}$. Moreover, for any $j, k = 1, 2, ...$, we have

$$u_{j,k} > \frac{\delta_j}{2} \text{ in } B_j \cap P_j, \quad \text{and} \quad u_{j,k} \leq 2\delta_j \text{ in } B_j \backslash P_j. \tag{3.23}$$

Also, by (3.17) and (3.18), $\Psi_j = \Psi_{j-1}$ a.e. in $B_{j-1} \cap P_{j-1}$. Moreover, we can extend each function $\Psi_j \in L^2(B_j \cap P_j; \mathbb{R}^d)$ to a function $\widehat{\Psi}_j \in L^2(B \cap P; \mathbb{R}^d)$ by defining

$$\widehat{\Psi}_j(x,t) = \begin{cases} \Psi_j(x,t) & , \text{ if } (x,t) \in B_j \cap P_j, \\ 0 & , \text{ if } (x,t) \in (B \cap P)\backslash(B_j \cap P_j). \end{cases}$$

With this definition, $\lim_{j\to\infty} \widehat{\Psi}_j(x,t)$ exists for a.e. $(x,t) \in B \cap P$. Define

$$\Psi(x,t) := \lim_{j\to\infty} \widehat{\Psi}_j(x,t) \quad \text{for a.e. } (x,t) \in B \cap P,$$

then $\Psi(x,t) = \Psi_j(x,t)$, for a.e. $(x,t) \in B_j \cap P_j$ and for any $j = 1, 2, ...$

Using a standard diagonal argument, we can extract a subsequence $\{\nabla\omega_{k,N_k}\}_{k=1}^{\infty}$ such that, for each $j = 1, 2, ...$,

$$\nabla\omega_{k,N_k} \rightharpoonup \Psi \quad \text{weakly in } L^2(B_j \cap P_j; \mathbb{R}^d) \text{ as } k \to \infty. \tag{3.24}$$

Combining with (3.12), for each $j = 1, 2, ...$, we have

$$\chi_{B_j \cap P_j}\sqrt{M_{k,N_k}(u_{k,N_k})}\nabla\omega_{k,N_k} \rightharpoonup \chi_{B_j \cap P_j}\sqrt{M(u)}\Psi \text{ weakly in } L^2(0,T;L^{\frac{2d}{d+2}}(\Omega;\mathbb{R}^d)) \tag{3.25}$$

as $k \to \infty$, where $\chi_{B_j \cap P_j}$ is the characteristic function of $B_j \cap P_j \subset \Omega_T$. Then, combining with (3.13), we have $\Lambda = \sqrt{M(u)}\Psi$ in every set $B_j \cap P_j$, which implies that

$$\Lambda = \sqrt{M(u)}\Psi \quad \text{in } B \cap P. \tag{3.26}$$

Consequently, by (3.14), we have, as $k \to \infty$,

$$\chi_{B \cap P}M_{k,N_k}(u_{k,N_k})\nabla\omega_{k,N_k} \rightharpoonup \chi_{B \cap P}M(u)\Psi \text{ weakly in } L^2(0,T;L^{2d/(d+2)}(\Omega;\mathbb{R}^d)). \tag{3.27}$$

For the third term on the right hand side of (3.19), from (3.6) and (3.23), and the generalized Hölder's inequality, we have

$$\left|\int_{B_j \backslash P_j} M_{k,N_k}(u_{k,N_k})\nabla\omega_{k,N_k} \cdot \Phi dxdt\right|$$

$$\leq \left(\sup_{B_j \backslash P_j} \sqrt{M_{k,N_k}(u_{k,N_k})} \right) \left\| \sqrt{M_{k,N_k}(u_{k,N_k})} \nabla \omega_{k,N_k} \right\|_{L^2(B_j \backslash P_j)} \|\Phi\|_{L^2(B_j \backslash P_j)}$$

$$\leq \max\{\sqrt{2\delta_j}, \sqrt{\theta_{k,N_k}}\} \left\| \sqrt{M_{k,N_k}(u_{k,N_k})} \nabla \omega_{k,N_k} \right\|_{L^2(\Omega_T)} \|\Phi\|_{L^2(0,T;L^2(\Omega))}$$

$$\leq \max\{\sqrt{2\delta_j}, \sqrt{\theta_{k,N_k}}\} \left\| \sqrt{M_{k,N_k}(u_{k,N_k})} \nabla \omega_{k,N_k} \right\|_{L^2(\Omega_T)} \|\Phi\|_{L^2(0,T;L^{2d/(d-2)}(\Omega))} |\Omega|^{1/d}$$

$$\leq C \max\{\sqrt{2\delta_j}, \sqrt{\theta_{k,N_k}}\}. \tag{3.28}$$

Since $\lim_{j \to \infty} \delta_j = 0$ and $\lim_{k \to \infty} \theta_{k,N_k} = 0$, (3.28) implies that

$$\lim_{j \to \infty} \lim_{k \to \infty} \int_{B_i \backslash P_j} M_{k,N_k}(u_{k,N_k}) \nabla \omega_{k,N_k} \cdot \Phi dx dt = 0 \tag{3.29}$$

Now, in (3.19), replacing u_i with the above subsequence u_{k,N_k} and taking the limits first as $k \to \infty$ and then as $j \to \infty$, by (3.20) (3.21), (3.27) and (3.29), we have

$$\int_{\Omega_T} \sqrt{M(u)} \Lambda \cdot \Phi dx dt = \lim_{j \to \infty} \int_{B_j \cap P_j} M(u)\Psi \cdot \Phi dx dt$$

$$= \int_{B \cap P} M(u)\Psi \cdot \Phi dx dt, \tag{3.30}$$

for any $\Phi \in L^2(0,T;L^{2d/(d-2)}(\Omega;\mathbb{R}^d))$. Combining this equation with (3.15), we see that u and Ψ satisfy the weak formulation

$$\int_0^T \langle \partial_t u, \xi \rangle_{H^{-2}(\Omega),H^2(\Omega)} dt = -\int_{B \cap P} M(u)\Psi \cdot \nabla \xi dx dt, \tag{3.31}$$

for all $\xi \in L^2(0,T;H^2(\Omega))$.

3.3. **The relationship between Ψ and u.** From (1.2)–(1.3) and (3.31), we expect Ψ to be

$$\Psi = \nabla \omega = W''(u)\nabla u + \kappa \nabla u + 2\kappa \nabla \Delta u + \kappa \nabla \Delta^2 u,$$

if the weak solution u has a sufficient regularity. However, given the known regularities of u, the terms $\nabla \Delta u$ and $\nabla \Delta^2 u$ are only defined in the sense of distributions and may not even be functions. Therefore, we need higher regularity conditions on u.

Claim. *For any open set $U \subset \Omega$ such that $\nabla \Delta^2 u \in L^q(U_T)$, for some $q > 1$ (q may depend on U), where $U_T = U \times (0,T)$, we have*

$$\Psi = W''(u)\nabla u + \kappa \nabla u + 2\kappa \nabla \Delta u + \kappa \nabla \Delta^2 u \quad in \ U_T. \tag{3.32}$$

Proof. Let $U \subset \Omega$ be an open set such that $\nabla \Delta^2 u \in L^q(U_T)$, for some $q > 1$. Let us consider the limit of

$$\nabla \omega_{k,N_k} = W''(u_{k,N_k})\nabla u_{k,N_k} + \kappa \nabla u_{k,N_k} + 2\kappa \nabla \Delta u_{k,N_k} + \kappa \nabla \Delta^2 u_{k,N_k} \tag{3.33}$$

as $k \to \infty$. Since $\nabla \Delta^2 u \in L^q(U_T)$ and $u \in L^\infty(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega)) \cap C([0,T];C^{0,\alpha}(\bar{\Omega}))$, for any $0 < \alpha < 1/2$, then we have

$\Delta^2 u \in L^q(0,T;W^{1,q}(U)), \quad \nabla \Delta u \in L^q(0,T;W^{2,q}(U)),$

$\Delta u \in L^q(0,T;W^{3,q}(U)) \cap L^\infty(0,T;L^2(U)),$

$\nabla u \in L^q(0,T;W^{4,q}(U)) \cap L^\infty(0,T;H^1(U)) \cap C([0,T];L^2(U)),$

$$u \in L^q(0,T;W^{5,q}(U)) \cap L^\infty(0,T;H^2(U)) \cap C([0,T];H^1(U)) \cap C([0,T];C^{0,\alpha}(\bar{U})),$$

for any $0 < \alpha < 1/2$.

By (3.7), we have, as $k \to \infty$,

$$\nabla u_{k,N_k} \rightharpoonup \nabla u \quad \text{weakly--* in } L^\infty(0,T;H^1(U)), \tag{3.34}$$

$$\nabla \Delta u_{k,N_k} \rightharpoonup \nabla \Delta u \quad \text{weakly--* in } L^\infty(0,T;(H^1(U))'), \tag{3.35}$$

$$\nabla \Delta^2 u_{k,N_k} \rightharpoonup \nabla \Delta^2 u \quad \text{weakly--* in } L^\infty(0,T;(H^3(U))'). \tag{3.36}$$

We see that the bound on the right hand side of (2.24) depends only on d, T, Ω, b_1, b_2, b_3, b_4, m, κ, ϵ and u_0, but not on θ. Combining with (3.10), we obtain

$$W''(u_{k,N_k}) \to W''(u) \quad \text{strongly in } C([0,T];L^2(U)) \text{ as } k \to \infty. \tag{3.37}$$

Combining with (3.34), we have

$$W''(u_{k,N_k})\nabla u_{k,N_k} \rightharpoonup W''(u)\nabla u \quad \text{weakly--* in } L^\infty(0,T;L^1(U)) \text{ as } k \to \infty. \tag{3.38}$$

Then, using (3.33), (3.34), (3.35), (3.36) and (3.38), we obtain

$$\nabla \omega_{k,N_k} \rightharpoonup W''(u)\nabla u + \kappa\nabla u + 2\kappa\nabla\Delta u + \kappa\nabla\Delta^2 u \quad \text{weakly--* in } L^\infty(0,T;(H^3(U))')$$

as $k \to \infty$. Combining this with (3.24), by the uniqueness of the weak limit, we get

$$\Psi = W''(u)\nabla u + \kappa\nabla u + 2\kappa\nabla\Delta u + \kappa\nabla\Delta^2 u \quad \text{in } B \cap P \cap U_T.$$

Because Ψ is originally defined only in $B \cap P$, we may extend it to U_T by defining

$$\Psi := W''(u)\nabla u + \kappa\nabla u + 2\kappa\nabla\Delta u + \kappa\nabla\Delta^2 u \quad \text{in } U_T \backslash (B \cap P).$$

The claim is established. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Now define the set

$$\mathcal{A} := \bigcup \{U_T = U \times (0,T) : U \text{ is open in } \Omega \text{ and } \nabla\Delta^2 u \in L^q(U_T),$$
$$\text{for some } q > 1 \text{ that may depend on } U\},$$

then $\mathcal{A}$ is open in Ω_T and

$$\Psi = W''(u)\nabla u + \kappa\nabla u + 2\kappa\nabla\Delta u + \kappa\nabla\Delta^2 u \quad \text{in } \mathcal{A}.$$

So Ψ is defined in $(B \cap P) \cup \mathcal{A}$. To extend Ψ to Ω_T, notice that

$$\Omega_T \backslash ((B \cap P) \cup \mathcal{A}) \subset \Omega_T \backslash (B \cap P) = (\Omega_T \backslash B) \cup (\Omega_T \backslash P).$$

Since $|\Omega_T \backslash B| = 0$ and $M(u) = 0$ in $\Omega_T \backslash P$, the value of Ψ outside of $(B \cap P) \cup \mathcal{A}$ does not contribute to the integral on the right hand side of (3.31), so we may define $\Psi \equiv 0$ in $\Omega_T \backslash ((B \cap P) \cup \mathcal{A})$.

3.4. **Energy inequality.** By the energy inequality (1.17), for any $j, k = 1, 2, \ldots$ and any $t \in [0,T]$, we have

$$\int_\Omega W(u_{k,N_k}(x,t)) + \kappa \left(\frac{1}{2}|u_{k,N_k}(x,t)|^2 - |\nabla u_{k,N_k}(x,t)|^2 + \frac{1}{2}|\Delta u_{k,N_k}(x,t)|^2 \right) dx$$

$$+ \int_{\Omega_t \cap B_j \cap P_j} M_\theta(u_{k,N_k}(x,\tau))|\nabla\omega_{k,N_k}(x,\tau))|^2 dx d\tau$$

$$\leq \int_\Omega W(u_0) + \kappa \left(\frac{1}{2}u_0^2 - |\nabla u_0|^2 + \frac{1}{2}|\Delta u_0|^2 \right) dx. \tag{3.39}$$

Using (3.9), (3.10) and (3.25), by taking the limits as $k \to \infty$ first and then $j \to \infty$ in (3.39), we obtain the energy inequality (1.21).

3.5. **Nonnegative weak solution with positive initial data.** Assume that the initial data $u_0(x) > 0$, for all $x \in \Omega$. By (1.18), there exists a constant C independent on $\{\theta_i\}_{i=1}^{\infty}$ such that, for each $i = 1, 2, ...,$

$$\underset{0 \leq t \leq T}{\text{ess sup}} \int_{\Omega} |(u_i(x, t))_- + \theta_i|^2 dx \leq C(\theta_i^2 + \theta_i + \theta_i^{1/2}). \tag{3.40}$$

Passing to the limits as $i \to \infty$ in (3.40), by the convergence in (3.9) and (3.10), we get $u \geq 0$ a.e. in Ω_T. Moreover, since $u_0 > 0$ in Ω, u is not always zero in Ω_T owing to the continuity of u in Ω_T. This completes the proof of Theorem 1.2.

4. **Conclusions.** We have derived the phase field crystal (PFC) equation by making standard approximations in the equations of the dynamic density function theory (DDFT) of freezing. We have shown that the inclusion of a degenerate mobility in the phase field crystal model is a natural carryover from the DDFT context. We then used a Galerkin approximation method to build solutions to a PFC model with positive mobility obtained via the degenerate mobility but using a cutoff. We then showed that we can shrink the cutoff to zero an thereby establish the existence of a weak, non-negative solution, provided that the initial data are positive.

In the future, we wish to analyze the a version of the PFC model with degenerate mobility and singular lower order free energy coming from the logarithmic "ideal gas" component of the free energy. We also want to perform a numerical analysis of the model, along the lines of the work in [16], where a degenerate mobility is analyzed in the Poisson-Nernst-Planck model. We also plan to perform extensive simulations of the degenerate PFC model along the lines of the work done by Shibin Dai and Qiang Du, for example, [8], which was done for a degenerate Cahn-Hilliard model. We point out that Archer and co-workers [1] have already done some preliminary simulations for a model similar to one considered here.

Conflict of interest. The authors declare that they have no known conflicts of interest relevant to this article.

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