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Residual stresses for a new class of transversely isotropic nonlinear elastic solid

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Abstract
We study the response of a class of transversely elastic bodies, wherein the Green–Saint Venant strain tensor is a function of the second Piola–Kirchhoff stress tensor, when the body is residually stressed. The notion of such non-Cauchy elastic bodies being transversely isotropic is defined in Rajagopal (Mech. Res. Commun. 64, 2015, 38–41), and by a body being residually stressed, we mean the interior of the body is not in a stress-free state although the boundary is free of traction as considered by Coleman and Noll (Arch. Ration. Mech. Anal. 15, 1964, 87–111) and by Hoger (Arch. Ration. Mech. Anal. 88, 1985, 271–289).

Keywords
Non-Green elastic solid, residually stressed hollow sphere, residual stresses for a cylindrical annulus, large elastic deformations, restrictions on residual stresses

1. Introduction

Since most bodies have been processed either by nature or humans they are invariably in a residually stressed state, and the usual assumption that the initial configuration of the body that is identified also as the reference configuration of the body as being stress free is mainly for mathematical simplicity in computations, and the wishful thinking that the stresses in the initial configuration are small enough to be ignored. It is apparent that in some bodies that such wishful thinking is misplaced and the initial/reference configuration from which the body is deformed is in a state of stress (such bodies are referred to as residually stressed bodies). It is well-known that a body in a residual state can only exhibit certain special material symmetries, i.e., the group of transformations to the reference configuration that allows its response to be the same have to be specific groups (see Coleman and Noll [1] and Hoger [2, 3]). For instance, an isotropic Cauchy elastic body (see Truesdell and Noll [4]) can only support a spherical state of residual stress and be traction-free on the boundary, and a transversely isotropic

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body can only support a very specific state of residual stress (see Hoger [3]) while being traction free on the boundary. However, there have been several papers wherein an inconsistent state of residual stress has been presumed for an anisotropic body. Recently, Rajagopal and Wineman [5] in addition to generalizing the result concerning the state of possible residual stress for Cauchy elastic bodies to the class of Simple Materials (see Noll [6]) pointed out the inconsistency in the assumption of the state of residual stress in Cauchy elastic bodies.

The notion of an elastic body, in the sense that the body cannot dissipate energy in a purely mechanical process, was extended by Rajagopal [7–9] wherein the Cauchy stress and the deformation gradient are related through an implicit relation. The class of Cauchy elastic bodies is a special sub-class of such implicit equations wherein the Cauchy stress is a function of the deformation gradient. Later, Rajagopal [10] extended the notion of material symmetry group (Peer group) introduced by Noll (see Truesdell and Noll [4] and Truesdell [11]) for bodies described by such implicit constitutive relations. In this note, we study the possible state of residual stress in a transversely isotropic elastic body described by a sub-class of implicit constitutive relation, wherein the Green–Saint Venant strain is given as a function of the second Piola–Kirchhoff tensor. Thus, we are not studying an implicit constitutive relation; we are studying the class of constitutive relations wherein the kinematics is given explicitly in terms of the stress. However, it is important to recognize that this relation might not be invertible, and hence, bodies that belong to the class of bodies that we are studying do not necessarily belong to the class of Cauchy elastic bodies.

The organization of this paper is as follows. In the next section, we provide a brief review of the basic kinematics and the balance laws that we need to establish the results. This is followed in section 3 wherein we delineate the condition that a member of the material symmetry group has to obey with regard to the reference configuration of the body when it is in a state of residual stress. In section 4, we assume that in the residually stressed reference configuration, the body is transversely isotropic and determines the nature of the residual stress. Section 5 is dedicated to determining the state of residual stress in a hollow sphere, while section 6 investigates the state of residual stress in a cylindrical annulus. In the final section, we consider the problem under the assumption that the displacement gradient of the body is sufficiently small that the nonlinear part of the Green–Saint Venant strain can be ignored.

2. Kinematics and balance laws

A particle X in a body $\mathscr{B}$ in the reference configuration occupies the point $\mathbf{X}$. The reference configuration is denoted $\kappa_{\mathrm{r}}(\mathscr{B})$. In the current configuration at time t, the point occupies the point $\mathbf{x}$ and the current configuration is denoted $\kappa_{\mathrm{t}}(\mathscr{B})$. It is assumed that there exist a one-to-one function $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The deformation gradient, the Green–Saint Venant strain tensor, the displacement field, and the linearized strain tensor are defined as:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}), \quad \mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}^{\mathrm{T}}\right), \tag{1}$$

where we assume that $J = \det \mathbf{F} > 0$.

Let ρ_{r} and ρ denote the density in the reference and current configuration, respectively. We will assume that the body is homogeneous, i.e., ρ_{r} is a constant over the body. The balance of mass is given by:

$$\rho_{\mathrm{r}} = \rho \det \mathbf{F}. \tag{2}$$

On introducing the Cauchy stress tensor $\mathbf{T}$, we can record the balance of linear momentum as:

$$\rho \ddot{\mathbf{x}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{3}$$

where $\mathbf{b}$ is the specific body force and $\ddot{\mathbf{x}}$ is the acceleration.

The second Piola–Kirchhoff stress tensor is denoted $\mathbf{S}$, and it is defined as:

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathrm{T}}. \tag{4}$$

The above definitions suffice for our study; more details concerning kinematic and the balance laws can be found in Truesdell and Toupin [11] and Truesdell [12].

3. Residual stresses for a new class of elastic solids

In this work, we are interested in studying residual stresses for the new class of elastic body (see Bustamante and Rajagopal [13]):

$$\mathbf{E} = \mathfrak{f}(\mathbf{S}), \tag{5}$$

where we assume that when there is no deformation $\mathbf{F} = \mathbf{I}$ (in which case $\mathbf{E} = \mathbf{0}$), there exists a stress $\mathbf{T}_{\mathrm{R}}$ such that $\mathbf{S} = \mathbf{T}_{\mathrm{R}}$ and[1]:

$$\mathbf{0} = \mathfrak{f}(\mathbf{T}_{\mathrm{R}}). \tag{6}$$

The stress tensor $\mathbf{T}_{\mathrm{R}}$ is called residual stress, and it must satisfy:

$$\mathrm{Div}\,\mathbf{T}_{\mathrm{R}} + \rho_{\mathrm{r}}\mathbf{b} = \mathbf{0}, \quad \mathbf{X} \in \kappa_{\mathrm{r}}(\mathscr{B}), \tag{7}$$

and

$$\mathbf{T}_{\mathrm{R}}\mathbf{N} = \mathbf{0}, \quad \mathbf{X} \in \partial\kappa_{\mathrm{r}}(\mathscr{B}), \tag{8}$$

where ρ_{r} is the material density in the reference configuration, $\mathbf{b}$ corresponds to the body force, and $\mathbf{N}$ is the unit normal vector to the surface of the body in the reference configuration $\partial\kappa_{\mathrm{r}}(\mathscr{B})$.

We study now a restriction for $\mathbf{T}_{\mathrm{R}}$ following the study in [5]. Let $\mathbf{H}$ belong to the symmetry group of the solid, then:

$$\mathbf{H}^{\mathrm{T}}\mathbf{E}\mathbf{H} = \mathfrak{f}(\mathbf{H}^{\mathrm{T}}\mathbf{S}\mathbf{H}). \tag{9}$$

In the case there is no deformation $\mathbf{F} = \mathbf{I}$, we have $\mathbf{E} = \mathbf{0}$ and $\mathbf{S} = \mathbf{T}_{\mathrm{R}}$, and from equation (9), we obtain:

$$\mathbf{0} = \mathfrak{f}(\mathbf{H}^{\mathrm{T}}\mathbf{T}_{\mathrm{R}}\mathbf{H}). \tag{10}$$

Let us assume that equation (6) has only one solution, then equation (10) is satisfied if and only if:

$$\mathbf{T}_{\mathrm{R}} = \mathbf{H}^{\mathrm{T}}\mathbf{T}_{\mathrm{R}}\mathbf{H} \quad \Leftrightarrow \quad \mathbf{H}\mathbf{T}_{\mathrm{R}} = \mathbf{T}_{\mathrm{R}}\mathbf{H}, \tag{11}$$

which is the same restriction studied in Rajagopal and Wineman [5] for Cauchy elastic solids.

If $\mathfrak{f}(\mathbf{S})$ is an isotropic function, i.e., the symmetry group is the full orthogonal group, then it follows [1–3] we have that the only residual stress that satisfies equation (11) is of the form $\mathbf{T}_{\mathrm{R}} = \alpha\mathbf{I}$, and from [1–3] that we have that such a stress will only satisfy equations (7) and (8) when $\alpha = 0$. If $\mathfrak{f}(\mathbf{S})$ is a transversely isotropic function (where $\mathbf{a}_0$ is the unit vector representing the direction the body is transversely isotropic), then for $\mathbf{H}$ belonging to that symmetry group, the residual stress tensor must be of the form (see Hoger [2, 3]):

$$\mathbf{T}_{\mathrm{R}} = \alpha\mathbf{I} + \beta\mathbf{a}_0 \otimes \mathbf{a}_0, \tag{12}$$

where α and β are the scalar functions that in general depend on $\mathbf{X}$. This case is more interesting and will be studied in subsequent next sections.

4. A transversely isotropic body with residual stresses

Following Bustamante and Rajagopal [13], we assume there exists a Gibbs potential $\mathcal{G} = \mathcal{G}(\mathbf{S}, \mathbf{a}_0)$ such that:

$$\mathbf{E} = \mathfrak{f}(\mathbf{S}, \mathbf{a}_0) = \frac{\partial\mathcal{G}}{\partial\mathbf{S}}. \tag{13}$$

Using the invariants of Rivlin and Spencer [14], we have $\mathcal{G}(\mathbf{S}, \mathbf{a}_0) = \mathcal{G}(I_1, I_2, I_3, I_4, I_5)$, where (see also Bustamante and Rajagopal [13]):

$$I_1 = \mathrm{tr}\,\mathbf{S}, \quad I_2 = \frac{1}{2}\mathrm{tr}(\mathbf{S}^2), \quad I_3 = \frac{1}{3}\mathrm{tr}(\mathbf{S}^3), \tag{14}$$

$$I_4 = \mathbf{a}_0 \cdot (\mathbf{S}\mathbf{a}_0), \quad I_5 = \mathbf{a}_0 \cdot (\mathbf{S}^2\mathbf{a}_0), \tag{15}$$

and from equation (13), we obtain:

$$\mathbf{E} = \mathcal{G}_1\mathbf{I} + \mathcal{G}_2\mathbf{S} + \mathcal{G}_3\mathbf{S}^2 + \mathcal{G}_4\mathbf{a}_0 \otimes \mathbf{a}_0 + \mathcal{G}_5[\mathbf{a}_0 \otimes (\mathbf{S}\mathbf{a}_0) + (\mathbf{S}\mathbf{a}_0) \otimes \mathbf{a}_0], \tag{16}$$

where $\mathcal{G}_i = \frac{\partial \mathcal{G}}{\partial I_i}$, $i = 1, 2, 3, 4, 5$.

Using equation (16) in equation (6), we get:

$$\mathbf{0} = \mathcal{G}_1 \mathbf{I} + \mathcal{G}_2 \mathbf{T}_\mathrm{R} + \mathcal{G}_3 \mathbf{T}_\mathrm{R}^2 + \mathcal{G}_4 \mathbf{a}_0 \otimes \mathbf{a}_0 + \mathcal{G}_5 [\mathbf{a}_0 \otimes (\mathbf{T}_\mathrm{R}\mathbf{a}_0) + (\mathbf{T}_\mathrm{R}\mathbf{a}_0) \otimes \mathbf{a}_0], \tag{17}$$

where $\mathcal{G}_i$ are evaluated from equations (14) and (15) using $\mathbf{T}_\mathrm{R}$ instead of $\mathbf{S}$. Taking the expression for the residual stress given in equation (12), we have:

$$\mathbf{0} = (\mathcal{G}_1 + \alpha\mathcal{G}_2 + \alpha^2\mathcal{G}_3)\mathbf{I} + [\beta\mathcal{G}_2 + \beta(2\alpha + \beta)\mathcal{G}_3 + \mathcal{G}_4 + 2(\alpha + \beta)\mathcal{G}_5]\mathbf{a}_0 \otimes \mathbf{a}_0, \tag{18}$$

which is satisfied for any $\mathbf{a}_0$ if and only if:

$$\mathcal{G}_1 + \alpha\mathcal{G}_2 + \alpha^2\mathcal{G}_3 = 0, \quad \beta\mathcal{G}_2 + \beta(2\alpha + \beta)\mathcal{G}_3 + \mathcal{G}_4 + 2(\alpha + \beta)\mathcal{G}_5 = 0, \tag{19}$$

where $I_1 = \mathrm{tr}\mathbf{T}_\mathrm{R}$, $I_2 = \frac{1}{2}\mathrm{tr}(\mathbf{T}_\mathrm{R}^2)$, $I_3 = \frac{1}{3}\mathrm{tr}(\mathbf{T}_\mathrm{R}^3)$, $I_4 = \mathbf{a}_0 \cdot (\mathbf{T}_\mathrm{R}\mathbf{a}_0)$, and $I_5 = \mathbf{a}_0 \cdot (\mathbf{T}_\mathrm{R}^2\mathbf{a}_0)$, and on using equation (12), these invariants become:

$$I_1 = 3\alpha + \beta, \quad I_2 = \frac{1}{2}[3\alpha^2 + \beta(2\alpha + \beta)], \quad I_3 = \frac{1}{3}[3\alpha^3 + \beta(3\alpha^2 + 3\alpha\beta + \beta^2)], \tag{20}$$

$$I_4 = \alpha + \beta, \quad I_5 = \alpha^2 + \beta(2\alpha + \beta). \tag{21}$$

For some of the calculations to be shown in the following section, we assume the following simplified expression for $\mathcal{G}(I_1, I_2, I_3, I_4, I_5)$:

$$\mathcal{G}(I_1, I_2, I_3, I_4, I_5) = c_1 I_1 + c_2 I_2 + c_3 I_3 + c_4 I_4 + c_5 I_5, \tag{22}$$

where c_1, c_2, c_3, c_4, and c_5 are the material parameters that do not depend on I_i, $i = 1, 2, 3, 4, 5$, but that can depend on $\mathbf{X}$ (inhomogeneous body).

5. Residual stresses for a hollow sphere

Let us study the problem of inflation of a hollow sphere with residual stresses. In the reference configuration, the sphere is defined through:

$$R_\mathrm{i} \leq R \leq R_\mathrm{o}, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq \Phi \leq \pi. \tag{23}$$

The unit basis vectors in spherical coordinates in the reference configuration are $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_\Phi\}$, and assuming that $\mathbf{a}_0 = \mathbf{E}_R$ from equation (12), we have:

$$\mathbf{T}_\mathrm{R} = \alpha(\mathbf{E}_R \otimes \mathbf{E}_R + \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \mathbf{E}_\Phi \otimes \mathbf{E}_\Phi) + \beta\mathbf{E}_R \otimes \mathbf{E}_R, \tag{24}$$

where we assume that $\alpha = \alpha(R)$ and $\beta = \beta(R)$.

Using this residual stress in equation (7) (assuming that $\mathbf{b} = \mathbf{0}$), we end up with the equation:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}R} + \frac{\mathrm{d}\beta}{\mathrm{d}R} + \frac{2\beta}{R} = 0, \tag{25}$$

while the boundary condition (8) for this problem implies that (taking into account (24)):

$$\alpha(R_\mathrm{i}) + \beta(R_\mathrm{i}) = 0, \quad \alpha(R_\mathrm{o}) + \beta(R_\mathrm{o}) = 0. \tag{26}$$

From equations (19) and (26), we have two equations, which must be satisfied by the functions $\alpha(R)$ and $\beta(R)$, they should also satisfy equation (8). In general, that is not possible; therefore, we solve the problem in a different manner. Let us assume the expression for $\mathcal{G}$ given in equation (22), assuming that $c_i = c_i(R)$, $i = 1, 2, 3, 4, 5$, and using this in equation (19), we obtain the algebraic equations:

$$c_1 + \alpha c_2 + \alpha^2 c_3 = 0, \quad \beta c_2 + \beta(2\alpha + \beta)c_3 + c_4 + 2(\alpha + \beta)c_5 = 0. \tag{27}$$

Now, there are two ways of proceeding. The first is to find forms of the material functions c_i, $i = 1, ..., 5$ (see equation (22)) that will allow specific structure for the functions $\alpha(R)$ and $\beta(R)$. The second is to assume c_i and determine the forms that are possible for $\alpha(R)$ and $\beta(R)$. In the first case, we are assuming the residual stress and determining the class of bodies wherein such a state of stress is possible. In the second case, we are assuming the constitutive relation for the material and determining what residual stresses it would support.

5.1. First case

Let us investigate the first case now. Let us suppose $c_i(R)$ are specified, and further supposed $c_3 = c_5 = 0$. Then, equation (27) reduces to:

$$c_1 + \alpha c_2 = 0, \quad \beta c_2 + c_4 = 0. \tag{28}$$

It immediately follows that:

$$\frac{\alpha}{\beta} = \frac{c_1}{c_4}. \tag{29}$$

Coupled with equation (25), this leads to two equations for determining $\alpha(R)$ and $\beta(R)$. Next, equation (25) can be expressed as:

$$\frac{\mathrm{d}\bar{\beta}}{\mathrm{d}R} = -\hat{P}\bar{\beta}, \tag{30}$$

where $\bar{\beta} = \beta\left(1 + \frac{c_1}{c_4}\right)$ and $\hat{P} = \frac{2}{R}\left(\frac{1}{1+\frac{c_1}{c_4}}\right)$. Let us introduce the integrating factor:

$$I_F(R) = \exp\left[\int \hat{P}(R)\,\mathrm{d}R\right], \tag{31}$$

then equation (30) can be expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[I_F(R)\bar{\beta}\right] = 0, \tag{32}$$

which implies that:

$$I_F(R)\left[\beta\left(1 + \frac{c_1}{c_4}\right)\right] = K = \text{constant}, \tag{33}$$

thus:

$$\beta(R) = \frac{K}{\left(1 + \frac{c_1}{c_4}\right) I_F(R)}. \tag{34}$$

Using (29) in (26) considering (34) we obtain $K/[I_F(R_\mathrm{i})] = 0$ and $K/[I_F(R_\mathrm{o})] = 0$, whose only solution is $K = 0$, which means for this case we obtain the trivial solution $\beta(R) = 0$ and $\alpha(R) = 0$.

We can try to find non-trivial solutions following a similar method as in this section, considering more general cases where c_3 and c_5 are not assumed to be zero, but for the sake of brevity we do not do that in this work.

5.2. Second case

In the second case, we can solve equation (25), e.g., for α in terms of β as:

$$\alpha(R) = C - \int_{R_\mathrm{i}}^{R}\left[2\frac{\beta(\xi)}{\xi} + \frac{\mathrm{d}\beta}{\mathrm{d}R}(\xi)\right]\mathrm{d}\xi, \tag{35}$$

where C is a constant. From equation (26), those boundary conditions become:

$$C + \beta(R_\mathrm{i}) = 0, \quad C - \int_{R_\mathrm{i}}^{R_\mathrm{o}}\left[2\frac{\beta(\xi)}{\xi} + \frac{\mathrm{d}\beta}{\mathrm{d}R}(\xi)\right]\mathrm{d}\xi + \beta(R_\mathrm{o}) = 0. \tag{36}$$

Let us assume, e.g., that the expression for $\beta(R)$ is given, and considering equations (35) and (27), we can find, e.g., $c_1 = c_1(R)$ and $c_4 = c_4(R)$ in terms of c_2, c_3, c_5, and β as:

$$c_1(R) = -\alpha(R)c_2 - \alpha(R)^2 c_3, \tag{37}$$

$$c_4(R) = -\beta(R)c_2 - \beta(R)(2\alpha(R) + \beta(R))c_3 - 2c_5(\alpha(R) + \beta(R)). \tag{38}$$

In the above expressions, we can assume that c_2, c_3, and c_5 are the constants, or more generally, we can assume that they are known functions in R (inhomogeneous body).

To make some more progress, let us assume a couple of specific expressions for $\beta(R)$.

5.2.1. First example for β. Let us assume that $\beta(R)$ is given as:

$$\beta(R) = \beta_0 + \beta_1 R, \tag{39}$$

where β_0 and β_1 are the constants. Using this in equation (35), we obtain:

$$\alpha(R) = C - 3\beta_1(R - R_{\mathrm{i}}) + 2\beta_0 \ln\left(\frac{R}{R_{\mathrm{i}}}\right), \tag{40}$$

and using this in equation (36), we have:

$$C + \beta_0 + \beta_1 R_{\mathrm{i}} = 0, \tag{41}$$

$$C - 3\beta_1(R_{\mathrm{o}} - R_{\mathrm{i}}) - 2\beta_0 \ln\left(\frac{R_{\mathrm{o}}}{R_{\mathrm{i}}}\right) + \beta_0 + \beta_1 R_{\mathrm{o}} = 0. \tag{42}$$

The above two equations can be solved, e.g., for C and β_0, or for C and β_1. When we solve for C and β_1, we have:

$$C = \beta_0\left[\frac{R_{\mathrm{i}} \ln\left(\frac{R_{\mathrm{o}}}{R_{\mathrm{i}}}\right)}{(R_{\mathrm{o}} - R_{\mathrm{i}})} - 1\right], \quad \beta_1 = -\frac{\beta_0 \ln\left(\frac{R_{\mathrm{o}}}{R_{\mathrm{i}}}\right)}{(R_{\mathrm{o}} - R_{\mathrm{i}})}. \tag{43}$$

Using equations (39) and (40) in equations (37) and (38) and using equation (43), we obtain:

$$c_1(R) = \left[3(R - R_{\mathrm{i}})\beta_1 - C + 2\beta_0 \ln\left(\frac{R}{R_{\mathrm{i}}}\right)\right]\left\{c_2 + c_3[C + 3\beta_1(R_{\mathrm{i}} - R)] - 2c_3\beta_0 \ln\left(\frac{R}{R_{\mathrm{i}}}\right)\right\}, \tag{44}$$

$$c_4(R) = -c_2(\beta_0 + \beta_1 R) - c_3(\beta_0 + \beta_1 R)\left[2C + \beta_0 - 5\beta_1 R + 6\beta_1 R_{\mathrm{i}} - 4\beta_0 \ln\left(\frac{R}{R_{\mathrm{i}}}\right)\right]$$
$$-2c_5\left[C + \beta_0 - 2\beta_1 R + 3\beta_1 R_{\mathrm{i}} - 2\beta_0 \ln\left(\frac{R}{R_{\mathrm{i}}}\right)\right]. \tag{45}$$

5.2.2. The inflation of a hollow sphere. In this section, we take the residually stressed sphere presented in the previous section, and we study a problem where we apply an external load to such a hollow sphere. The effects of the original residual stresses (when there is no external load applied on the sphere) can be studied by analyzing the strains and stresses that are produced when we apply such an external load.

Based on the above consideration for the residual stresses and the material parameters for the constitutive equation (22), we now study the problem of inflation of the above hollow sphere equation (23). We assume that on the inner surface of the sphere, we apply a traction P, and that on the outer surface of the sphere, there is no external load. We assume the symmetry of the sphere is maintained, then in the current configuration, we have:

$$r = r(R), \quad \theta = \Theta, \quad \phi = \Phi, \tag{46}$$

and thus, we obtain (the unit vector basis in the current configuration is $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$):

$$\mathbf{F} = r'(R)\mathbf{e}_r \otimes \mathbf{E}_R + \frac{r(R)}{R}(\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \mathbf{e}_\phi \otimes \mathbf{E}_\Phi), \tag{47}$$

thus:

$$\mathbf{E} = \frac{1}{2}[r'(R)^2 - 1]\mathbf{E}_R \otimes \mathbf{E}_R + \frac{1}{2}\left[\frac{r(R)^2}{R^2} - 1\right](\mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \mathbf{E}_\Phi \otimes \mathbf{E}_\Phi), \tag{48}$$

where $r'(R) = \frac{\mathrm{d}r}{\mathrm{d}R}$.

Let us now suppose that the stresses have the form (see, for example, [15]):

$$\mathbf{T} = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_\phi(r)\mathbf{e}_\phi \otimes \mathbf{e}_\phi. \tag{49}$$

On substituting this in equation (3), assuming there is no body force and $\ddot{\mathbf{x}} = \mathbf{0}$, we obtain $\sigma_\phi = \sigma_\theta$ and:

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}\sigma_r}{\mathrm{d}R}\frac{1}{r'} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0. \tag{50}$$

It follows from equations (47) and (49) that the second Piola–Kirchhoff stress tensor given in equation (4) is given by:

$$\mathbf{S} = \frac{\sigma_r r^2}{R^2 r'}\mathbf{E}_R \otimes \mathbf{E}_R + \sigma_\theta r'(\mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \mathbf{E}_\Phi \otimes \mathbf{E}_\Phi). \tag{51}$$

Using equations (48) and (51) in equation (16), appealing to equations (44) and (45) and recalling equation (43), we obtain:

$$\frac{1}{2}(r'^2 - 1) = c_1 + c_2\frac{\sigma_r r^2}{R^2 r'} + c_3\left(\frac{\sigma_r r^2}{R^2 r'}\right)^2 + c_4 + 2c_5\frac{\sigma_r r^2}{R^2 r'}, \tag{52}$$

$$\frac{1}{2}\left(\frac{r^2}{R^2} - 1\right) = c_1 + c_2\sigma_\theta r' + c_3(\sigma_\theta r')^2. \tag{53}$$

From equations (50), (52), and (53), we have three equations, which can be solved to find $\sigma_r(r) = \sigma_r(r(R)) = \sigma_r(R)$, $\sigma_\theta(r) = \sigma_\theta(r(R)) = \sigma_\theta(R)$ and $r(R)$. Regarding the boundary conditions, we have $\sigma_r(R_\mathrm{i}) = -P$ and $\sigma_r(R_\mathrm{o}) = 0$, where we recall again that here we have taken the residually stressed sphere analyzed in the previous section (such residual stresses corresponds to the case there is no external load on the sphere), to which we now are applying P.

5.2.3. Second expression for β. In this section, let us study briefly another possibility for $\beta(R)$, where we assume:

$$\beta(R) = \beta_0 + \frac{\beta_1}{R}, \tag{54}$$

where β_0 and β_1 are the constants. Using this in equation (35), we have:

$$\alpha(R) = C - 2\beta_0 \ln\left(\frac{R}{R_\mathrm{i}}\right) + \beta_1\left(\frac{1}{R} - \frac{1}{R_\mathrm{i}}\right). \tag{55}$$

Using equations (54) and (55) in equation (26) and solving again for C and β_1, we obtain:

$$C = \beta_0\left[\frac{R_\mathrm{o}\ln\left(\frac{R_\mathrm{o}}{R_\mathrm{i}}\right)}{R_\mathrm{o} - R_\mathrm{i}} - 1\right], \quad \beta_1 = -\frac{R_\mathrm{i}R_\mathrm{o}\beta_0\ln\left(\frac{R_\mathrm{o}}{R_\mathrm{i}}\right)}{(R_\mathrm{o} - R_\mathrm{i})}. \tag{56}$$

It follows from equations (54) and (55) and on taking into account the above results, from equations (37) and (38) we obtain:

$$c_1(R) = \left[\left(\frac{1}{R_\mathrm{i}} - \frac{1}{R}\right)\beta_1 - C + 2\beta_0\ln\left(\frac{R}{R_\mathrm{i}}\right)\right]\left\{c_2 + c_3\left[\left(\frac{1}{R_\mathrm{i}} - \frac{1}{R}\right)\beta_1 - C + 2\beta_0\ln\left(\frac{R}{R_\mathrm{i}}\right)\right]\right\}, \tag{57}$$

$$c_4(R) = -\frac{c_2}{R}(\beta_0 R + \beta_1) - 2c_5\left[C + \beta_0 + 2\frac{\beta_1}{R} - \frac{\beta_1}{R_\mathrm{i}} - 2\beta_0\ln\left(\frac{R}{R_\mathrm{i}}\right)\right] + \frac{c_3(\beta_0 R + \beta_1)}{R^2 R_\mathrm{i}}\left[(2R - 3R_\mathrm{i})\beta_1 - RR_\mathrm{i}(2C + \beta_0) + 4\beta_0 RR_\mathrm{i}\ln\left(\frac{R}{R_\mathrm{i}}\right)\right]. \tag{58}$$

These expressions can be used in equation (22), and we can solve the same problem of inflation of a hollow sphere presented in section 5.2.2.

6. Residual stresses for a cylindrical annulus

The problem of residual stresses for a cylindrical annulus considering transversely isotropic bodies is not as interesting as the previous example of the hollow sphere. In this section, we study this problem, briefly. Let us consider the cylindrical annulus defined in the reference configuration as:

$$R_{\rm i} \leq R \leq R_{\rm o}, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L. \tag{59}$$

The unit vector basis in cylindrical coordinates is $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$, and let us assume that $\mathbf{a}_0 = \mathbf{E}_R$, then from equation (12), we have:

$$\mathbf{T}_{\rm R} = \alpha(\mathbf{E}_R \otimes \mathbf{E}_R + \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \mathbf{E}_Z \otimes \mathbf{E}_Z) + \beta \mathbf{E}_R \otimes \mathbf{E}_R, \tag{60}$$

where we consider the case $\alpha = \alpha(R)$ and $\beta = \beta(R)$. Replacing this in equation (7), if $\mathbf{b} = \mathbf{0}$, we end up with the differential equation $\frac{{\rm d}\alpha}{{\rm d}R} + \frac{{\rm d}\beta}{{\rm d}R} + \frac{\beta}{R} = 0$, which can be solved, e.g., for α as $\alpha(R) = C - \int_{R_{\rm i}}^{R} \left[\frac{\beta(\xi)}{\xi} + \frac{{\rm d}\beta}{{\rm d}R}(\xi)\right] {\rm d}\xi$. Regarding the boundary condition (8), for this problem, we have the boundaries at $R = R_{\rm i}$, $R = R_{\rm o}$, but also the boundaries at $Z = 0$ and $Z = L$, then considering equation (60) and the above boundaries we have the same boundary conditions (26) (at $R = R_{\rm i}$ and $R = R_{\rm o}$), in addition to the boundary condition $\alpha = 0$ at $Z = 0$ and $Z = L$, which basically means $\alpha(R) = 0$ for any R, thus $\mathbf{T}_{\rm R} = \beta \mathbf{E}_R \otimes \mathbf{E}_R$, and if $\beta = \beta(R)$, equation (7) is satisfied if $\beta(R) = \beta_1/R$, where β_1 is a constant. But the above stress cannot satisfy the boundary conditions (26).

Considering the above results, as a second attempt, let us study the case $\mathbf{a}_0 = \mathbf{E}_R$ (where in equation (60) is valid), but where we assume $\alpha = \alpha(R, Z)$ and $\beta = \beta(R, Z)$. In such a case replacing equation (60) in equation (7) (assuming $\mathbf{b} = \mathbf{0}$), we obtain the two partial differential equations $\frac{\partial}{\partial R}(\alpha + \beta) + \frac{\beta}{R} = 0$ and $\frac{\partial \alpha}{\partial Z} = 0$. But this last equation has the solution $\alpha = \alpha(R)$, and considering the boundary conditions (8) (see equation (26)), we again end up with $\alpha = 0$ and $\mathbf{T}_{\rm R} = \beta \mathbf{E}_R \otimes \mathbf{E}_R$, which again cannot satisfy the boundary conditions (26).

In a third attempt, let us assume that $\mathbf{a}_0 = \mathbf{E}_Z$, thus from equation (12), we get $\mathbf{T}_{\rm R} = \alpha(\mathbf{E}_R \otimes \mathbf{E}_R + \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \mathbf{E}_Z \otimes \mathbf{E}_Z) + \beta \mathbf{E}_Z \otimes \mathbf{E}_Z$. Let us assume once again that $\alpha = \alpha(R, Z)$ and $\beta = \beta(R, Z)$, then from the equations of equilibrium (see equation (7) with $\mathbf{b} = \mathbf{0}$), we get $\frac{\partial \alpha}{\partial R} = 0$ and $\frac{\partial}{\partial Z}(\alpha + \beta) = 0$, which leads to $\alpha = \alpha(Z)$ and $\beta = -\alpha(Z) + \hat{\beta}(R)$, and using this in the boundary conditions (8), we have $\alpha = 0$ at $R = R_{\rm i}$ and $R = R_{\rm o}$, and hence, we conclude that $\alpha(Z) = 0$, and as a consequence of the boundary conditions at $Z = 0$ and $Z = L$, we obtain $\beta = 0$. Again, we do not find an interesting solution here, and we do not study this problem further.

7. The case when the norm of the gradient of the displacement field is small

In Bustamante and Rajagopal [16], a study is carried out for a body that is residually stressed when the linearized strain tensor is assumed to be a function of the Cauchy stress tensor. In that work, the restriction equation (12) (see Rajagopal and Wineman [5]) was not considered, and in this section, we want to study the same problem, starting from equation (5).

Let us assume that the norm of the gradient of the displacement field is very small, i.e., $|\nabla_{\rm r}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, where $\nabla_{\rm r}$ denotes $\frac{\partial}{\partial \mathbf{X}}$. In this case since $\mathbf{F} = \mathbf{I} + \nabla_{\rm r}\mathbf{u}$ and since by virtue of equation (1) $\mathbf{E} \approx \boldsymbol{\varepsilon}$, and as a consequence of equation (4), we have $\mathbf{S} \approx \mathbf{T}$, it follows from equation (5) that:

$$\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}), \tag{61}$$

which is the type of constitutive equation analyzed in Bustamante and Rajagopal [16]. Here, we have $\mathbf{0} = \mathfrak{f}(\mathbf{T}_{\rm R})$ (see equation (6)), thus $\mathbf{0} = \mathfrak{f}(\mathbf{H}^{\rm T}\mathbf{T}_{\rm R}\mathbf{H})$ (see equation (10)), as a result, the restriction equation (11) is also valid, i.e.:

$$\mathbf{0} = \mathfrak{f}(\mathbf{T}_{\rm R}), \quad \mathbf{H}\mathbf{T}_{\rm R} = \mathbf{T}_{\rm R}\mathbf{H}. \tag{62}$$

Since $|\nabla_{\rm r}\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, equation (7) becomes (approximately):

$$\operatorname{div} \mathbf{T}_{\rm R} + \rho \mathbf{b} = \mathbf{0}. \tag{63}$$

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Note

1. In Bustamante and Rajagopal [16], there is a analysis of residual stresses for a constitutive equation that is similar to equation (5), but that is valid for the case the gradient of the displacement field is very small, which implies **E** becomes the linearized strain tensor. In Bustamante and Rajagopal [16], the authors did not consider the restriction equation (11) for the residual stresses.

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