A BORDERED HF^{-} ALGEBRA FOR THE TORUS

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ABSTRACT. We describe a weighted A_∞-algebra associated to the torus. We give a combinatorial construction of this algebra, and an abstract characterization. The abstract characterization also gives a relationship between our algebra and the wrapped Fukaya category of the torus. These algebras underpin the (unspecialized) bordered Heegaard Floer homology for three-manifolds with torus boundary, which will be constructed in forthcoming work.

CONTENTS

2020 *Mathematics Subject Classification.* Primary 57R58; Secondary 18G70, 53D40, 57M27.
RL was supported by NSF grant DMS-1810893.
PSO was supported by NSF grants DMS-1708284 and DMS-2104536.
DPT was supported by NSF grant DMS-2110143.

1. Introduction

This is a tale of an algebra. In fact, it is a tale of several algebras:

- $\widehat{\mathcal{A}}$, the algebra introduced in [LOT18] governing $\widehat{HF}$ for 3-manifolds with torus boundary. (In [LOT18], $\widehat{\mathcal{A}}$ is denoted $\mathcal{A}(T^2)$.) The algebra $\widehat{\mathcal{A}}$ is a finite-dimensional, associative algebra over $\mathbb{F}_2$ with a grading by a particular group G with a distinguished central element λ.
- $\mathcal{A}_-^{0,\mathrm{as}}$, an associative algebra over $\mathbb{F}_2$ containing $\widehat{\mathcal{A}}$ as a subalgebra. The algebra $\mathcal{A}_-^{0,\mathrm{as}}$ is also graded by G. The algebra $\mathcal{A}_-^{0,\mathrm{as}}$ is infinite-dimensional, but is finitely generated and is finite-dimensional in each grading. The definition of $\mathcal{A}_-^{0,\mathrm{as}}$ is an unsurprising extension of the definition of $\widehat{\mathcal{A}}$.
- $\mathcal{A}_-^0$, an A_∞-algebra over $\mathbb{F}_2[U]$, with trivial differential ($m_1 = 0$). The algebra $\mathcal{A}_-^0$ is free over $\mathbb{F}_2[U]$, and the A_∞-operations are U-equivariant. Since $m_1 = 0$, $\mathcal{A}_-^0$ has an underlying associative algebra; this algebra is $\mathcal{A}_-^{0,\mathrm{as}}[U] \coloneqq \mathcal{A}_-^{0,\mathrm{as}} \otimes \mathbb{F}_2[U]$.
- $\mathcal{A}_- = \mathcal{A}_-^A$, the main algebra of interest for studying HF^- for 3-manifolds with torus boundary. This is what we call a *weighted A_∞-algebra* (see Section 2 and [LOT20]). The weight-zero part of $\mathcal{A}_-^A$ is $\mathcal{A}_-^0$.

Although weighted A_∞-algebras are discussed in detail in Section 2, perhaps a few words are in order now. In brief, a weighted A_∞-algebra is a curved A_∞-algebra $A[[t]] = A \otimes_{\mathbb{F}_2} \mathbb{F}_2[[t]]$ over $\mathbb{F}_2[[t]]$, such that the curvature lies in the ideal $tA[[t]]$. The operations $\mu\colon T^*A[[t]] \to A[[t]]$ are determined by the maps $\mu_n^k\colon A^{\otimes n} \to t^k A$, which we call the *weight k part* of μ_n. Weighted A_∞-algebras are also called one-parameter deformations of A_∞-algebras (see, e.g., [Sei15, Section 3b]).

While this paper is self contained, we explain briefly how the algebra fits into the broader bordered context from [LOT18, LOT23]. The algebra $\widehat{\mathcal{A}}$ is an algebra of Reeb chords that are not permitted to cross a certain basepoint adjacent to the boundary; multiplication is encoded by collisions of curves at "east infinity". The algebra $\mathcal{A}_-^{0,\mathrm{as}}$ is a larger algebra, consisting of all Reeb chords. The A_∞ deformation $\mathcal{A}_-^0$ is needed to account for boundary deformations, i.e. holomorphic curves whose boundary lie entirely in the α-tori. Indeed, the deformation $\mathcal{A}_-^0$ is the one which counts the disk that covers the torus once. (See [LOT23, Figure 7].) The weighted deformation $\mathcal{A}_-$ is the algebraic object which also encodes the Reeb orbits. (See [LOT23, Figure 8].)

The main goal of this paper is to define $\mathcal{A}_-$. In fact, we give two paths to defining $\mathcal{A}_-$. The first construction is based on the combinatorics of certain kinds of planar graphs. We give this construction in Section 3. Interpreting the planar graphs as coverings of the torus, one can think of this definition as an almost trivial case of the theory of pseudoholomorphic curves; see Section 3.5.

The second path to defining $\mathcal{A}_-$ is more indirect. As mentioned above, it is easy to define the algebra $\mathcal{A}_-^{0,\mathrm{as}}$. The algebra $\mathcal{A}_-^{0,\mathrm{as}}$ turns out to have a unique nontrivial A_∞-deformation over $\mathbb{F}_2[U]$ respecting the gradings, up to A_∞-isomorphism; see Theorem 5.45. This deformation is $\mathcal{A}_-^0$. (Theorem 5.45 follows from a computation of the Hochschild cohomology of $\mathcal{A}_-^{0,\mathrm{as}}$.) Similarly, $\mathcal{A}_-^0$ turns out to have a unique extension to a weighted A_∞-algebra subject to the conditions that the curvature is a particular element (the sum of the "length 4 chords") and respecting particular gradings; see Theorem 5.71. (Again, Theorem 5.71 follows from computing the Hochschild cohomology of

$\mathcal{A}_-^0$.) This extension is $\mathcal{A}_-$. In particular, the uniqueness theorem implies that the two definitions of $\mathcal{A}_-$ agree.

Auroux showed that $\widehat{\mathcal{A}}$ is derived equivalent to a certain partially-wrapped version of the Fukaya category of the torus [Aur10a, Aur10b]. We show that $\mathcal{A}_-^0$ is equivalent to the fully wrapped Fukaya category of the torus.

We note that various versions of the Fukaya categories of Riemann surfaces have been studied extensively. For instance, Abouzaid explicitly computed K_0 of (a particular variant of the) Fukaya category of a surface [Abo08] (see also [AB22]). In a previous paper [LOT13], we gave an explicit description of the mapping class group action on a partially-wrapped Fukaya category of a surface, and showed this action was faithful and captures the dilatation, a point rediscovered by Dimitrov-Haiden-Katzarkov-Kontsevich [DHKK14]. Lekili-Perutz studied a different variant of the Fukaya category of the torus, giving an explicit description of it and showing it is not formal [LP11]. A reformulation of bordered $\widehat{HF}$ with torus boundary in terms of (a version of) the Fukaya category of the torus was given by Hanselman-Rasmussen-Watson [HRW24]. There have been many papers about mirror symmetry for Riemann surfaces (e.g., [Sei11, Efi12, LP17, PS19, AS20], and many others). There has also been substantial interest in the Fukaya category from the representation theory community (e.g., [KS02, CS20]).

This paper is organized as follows. In Section 2 we introduce weighted A_∞-algebras, the algebraic structure underlying $\mathcal{A}_-$. In Section 3 we define $\mathcal{A}_-^{0,\mathrm{as}}$, $\mathcal{A}_-^0$ and $\mathcal{A}_-$; the gradings on these algebras are deferred to Section 4. Section 5 uses computations of Hochschild homology groups to prove uniqueness theorems for $\mathcal{A}_-^0$ and $\mathcal{A}_-$. Section 6 gives a Fukaya-categorical interpretation of $\mathcal{A}_-^0$. In Section 7, we explain how to lift the constructions from this paper to arbitrary characteristic.

Acknowledgements. We thank Rumen Zarev for helpful conversations. We also thank Mohammed Abouzaid, Tim Perutz, and Nick Sheridan, for helpful conversations, and Tom Hockenhull for corrections to an earlier version. We also thank the referee for many helpful comments and corrections.

2. Weighted A_∞ algebras and modules

2.1. Definition and first properties.

The HF^- extension of bordered Floer homology associates to the torus T^2 an object $\mathcal{A}_- = \mathcal{A}_-(T^2)$, having a particular algebraic structure, which we will call a *weighted A_∞-algebra*. (Such objects have appeared in the literature already in several forms; see Remarks 2.10, 2.11 and 2.12.) A more detailed discussion can be found in our previous paper [LOT20, Section 4].

For now, we will discuss ungraded weighted A_∞-algebras, and defer the discussion of gradings to Section 2.2.

Fix a unital commutative ring R of characteristic 2.

Definition 2.1. *A curved* A_∞-algebra over R *consists of R-bimodule maps*

$$\mu_n\colon A^{\otimes_R n} \to A$$

for $n \geq 0$ satisfying the structure equation

$$\sum_{k=-1}^{n-1}\sum_{i=1}^{n-k} \mu_{n-k}(a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu_{k+1}(a_i \otimes \cdots \otimes a_{i+k}) \otimes a_{i+k+1} \otimes \cdots \otimes a_n) = 0.$$

for all n and all $a_1, \dots, a_n \in A$. (In the graded setting, the map μ_n decreases the homological grading by $2-n$.)

In the special case $n = 0$, $A^{\otimes_R 0} = R$, so $\mu_0\colon R \to A$, so μ_0 is determined by $\mu_0(1) =: \mu_0$, which is the curvature *of the curved A_∞-algebra.*

A curved A_∞-algebra $\mathcal{A} = (A, \{\mu_n\})$ over R is strictly unital *if there is an element $1 \in A$ so that for all $a \in A$, $\mu_2(a,1) = \mu_2(1,a) = a$ and for all $n \neq 2$, and $a_1, \dots, a_n \in A$, if some $a_i = 1$ then $\mu_n(a_1, \dots, a_n) = 0$.*

Definition 2.2. *A* weighted A_∞-algebra *over R is a curved A_∞-algebra $(A[[t]], \{\mu_n : A[[t]]^{\otimes n} \to A[[t]]\}_{n=0}^{\infty})$ over $R[[t]]$ (where $A[[t]] = A \otimes_R R[[t]]$ for some R-module A) such that the curvature μ_0 lies in $tA[[t]] \subset A[[t]]$.*

Convention 2.3. *Henceforth, all curved or weighted A_∞-algebras will be assumed strictly unital. We will identify elements $r \in R$ with their images $r \cdot 1 \in A$.*

Lemma 2.4. *A weighted A_∞-algebra over R is specified by an R-bimodule A and maps $\mu_n^w : A^{\otimes_R n} \to A$ for $n, w \in \mathbb{Z}_{\geq 0}$ such that:*

(1) $\mu_0^0 = 0$.
(2) For any $r \in R$, $\mu_2^0(a, r) = ar = ra = \mu_2^0(r, a)$.
(3) For any $(n, w) \neq (2, 0)$, $\mu_n^w(a_1, \dots, a_n) = 0$ if some $a_i \in R$.
(4) For each $n, w \in \mathbb{Z}_{\geq 0}$ and $a_1, \dots, a_n \in A$,

$$\sum_{\substack{p+q=n+1\\u+v=w}} \sum_{i=0}^{n-q} \mu_p^u(a_1 \otimes \cdots \otimes a_i \otimes \mu_q^v(a_{i+1} \otimes \cdots \otimes a_{i+q}) \otimes a_{i+q+1} \otimes \cdots \otimes a_n) = 0. \tag{2.5}$$

Proof. Immediate from the definitions. □

Definition 2.6. *Given a weighted A_∞-algebra $\mathcal{A} = (A, \{\mu_n^k\})$, the operations $\{\mu_n^0\}$ make A into an ordinary A_∞-algebra. We call $(A, \{\mu_n^0\})$ the* undeformed A_∞-algebra *of $\mathcal{A}$.*

Definition 2.7. *Given weighted A_∞-algebras $\mathcal{A}$ and $\mathcal{B}$, a* homomorphism *$f : \mathcal{A} \to \mathcal{B}$ is a homomorphism of curved A_∞-algebras (i.e., a sequence of maps $f_n : A[[t]]^{\otimes n} \to B[[t]]$, $n = 0, \dots, \infty$, over $R[[t]]$, satisfying the A_∞-algebra homomorphism relations) so that $f_0 \in tB[[t]] \subset B[[t]]$. A homomorphism is called* uncurved *if $f_0 = 0$ (which implies that $f_1(\mu_0^{\mathcal{A}}) = \mu_0^{\mathcal{B}}$).*

The identity homomorphism of $\mathcal{A}$ is given by $f_1 = \mathbb{I}$ and $f_n = 0$ for $n \neq 1$. The composition of homomorphisms of weighted A_∞-algebras is induced by the usual composition of homomorphisms of curved A_∞-algebras, i.e.,

$$(g \circ f)_n = \sum_{\substack{k_1+\cdots+k_m=n\\k_i \geq 0}} g_m \circ (f_{k_1} \otimes \cdots \otimes f_{k_m}).$$

Convergence of this sum follows from the fact that $f_0 \in tB[[t]]$. Also, since

$$(g \circ f)_0 = g_0 + g_1(f_0) + g_2(f_0, f_0) + \cdots,$$

a composition of uncurved homomorphisms is uncurved.

An isomorphism *of weighted A_∞-algebras is an invertible homomorphism.*

Given a weighted A_∞-homomorphism $f : \mathcal{A} \to \mathcal{B}$ we let $f_n^k : A^{\otimes n} \to B$ be the coefficient of t^k in f_n (restricted to $A^{\otimes n} \subset A[[t]]^{\otimes n}$). So, for example, the identity homomorphism is given by $\mathbb{I}_1^0(a) = a$ and $\mathbb{I}_n^k = 0$ for $(n, k) \neq (1, 0)$.

Lemma 2.8. *[LOT20, Lemma 4.19] If $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism of weighted A_∞-algebras so that f_1^0 is an isomorphism of R-modules then f is an isomorphism.*

Definition 2.9. *An* augmentation *of a weighted A_∞-algebra $\mathcal{A}$ over R is a A_∞-homomorphism $f : A[[t]] \to R[[t]]$, where $R[[t]]$ has $\mu_n = 0$ for $n \neq 2$ and μ_2 the usual multiplication.*

Notice, in particular, that any augmentation $f : A[[t]] \to R[[t]]$ sends the curvature μ_0 to 0, i.e., $f_1(\mu_0) = 0$.

With these properties established, we turn to some alternate formulations of weighted A_∞-algebras.

Remark 2.10. A weighted A_∞-algebra $\mathcal{A} = (A, \{\mu_n^k\})$ is the same as a one-parameter deformation of $(A, \{\mu_n^0\})$, in the sense of (for instance) [Sei15, Section 3b], except that we allow curved deformations. See also Section 5.4.

Remark 2.11. A weighted A_∞-algebra is a special case of a gapped, filtered A_∞-algebra [FOOO09].

Remark 2.12. Let $\mathcal{A} = (A, \{\mu_n\})$ be an uncurved A_∞-algebra so that (A, μ_1, μ_2) is an associative differential algebra, and let $\omega \in \mathcal{A}$ be an element such that:

- ω is central with respect to μ_2.
- $A = B \oplus \omega B$ is a free $\mathbb{F}_2[\omega]/(\omega^2)$-module.
- If $a_1, \dots, a_n \in B \cup \{\omega\}$ and $n \geq 2$ then $\mu_n(a_1, \dots, a_n) \in B$.

Then there is an associated weighted A_∞-algebra $(B, \{\mu_n^k\})$ where

$$\mu_n^k(a_1, \dots, a_n) = \mu_{n+k}(\overbrace{\omega, \dots, \omega}^{k}, a_1, \dots, a_n) + \mu_{n+k}(\overbrace{\omega, \dots, \omega}^{k-1}, a_1, \omega, a_2, \dots, a_n) + \cdots$$

is the sum of all ways of inserting k copies of ω into the sequence $(a_1, \dots, a_n)$ and then applying the operation μ_{n+k}. In particular, $\mu_0^k = \mu_k(\omega, \dots, \omega)$.

Let T be a rooted, planar tree with n distinguished *input* leaves, and one distinguished output, the root. We call the vertices of T other than the inputs and the output *internal.* (Note that this definition allows for some leaves that are neither inputs nor the output.) A *weight function* on T is a function w from the internal vertices of T to the non-negative integers, with the property that any internal vertex v with valence 1 or 2 has weight $w(v) \geq 1$. A *weighted tree* is a rooted, planar tree T, a choice of input leaves for T, and a weight function on the internal vertices of T. The *dimension* of a weighted tree T with n inputs, i internal vertices, and total weight w is

$$\dim(T) = n + 2w - i - 1. \tag{2.13}$$

A weighted A_∞ algebra structure on A associates a map

$$\mu(T)\colon A^{\otimes n} \to A$$

to a weighted tree T with n inputs. Specifically, the operation μ_n^w specifies the action of the n input corolla (planar graph with one internal vertex) with weight w, Ψ_w^n; actions of more complicated trees are obtained from these actions by suitable compositions. In particular, the valence-1 internal vertices correspond to operations of the form μ_0^w.

If S and T are two rooted, planar, weighted trees, and i is some input to S, let $S \circ_i T$ denote the rooted, planar, weighted tree obtained by joining the output to T to the i^{th} input to S. The weighted A_∞ relation can now be phrased as the identities (indexed by pairs of non-negative integers w and n)

$$\sum_{\substack{a+b=n+1\\ u+v=w}} \sum_{i=1}^{a} \mu(\Psi_a^u \circ_i \Psi_b^v) = 0.$$

2.2. **Gradings.** Fix a set S. By an *S-graded R-module* we mean an R-module M together with a decomposition $M = \bigoplus_{s \in S} M_s$. Elements of M_s are called *homogeneous of grading s.* (The element $0 \in M$ is viewed as being homogeneous of all gradings.)

Now, fix a group G and central elements λ_d and λ_w in G. By a *$(G, \lambda_d, \lambda_w)$-graded weighted A_∞-algebra* we mean a weighted A_∞-algebra $\mathcal{A} = (A, \{\mu_m^w\})$ and a G-grading of A so that μ_m^w is graded of degree $\lambda_d^{m-2}\lambda_w^w$. More explicitly, if $a_1, \dots, a_m$ are homogeneous of gradings $g_1, \dots, g_m$ then $\mu_m^w(a_1, \dots, a_m)$ is homogeneous of grading $\lambda_d^{m-2}\lambda_w^w g_1 \cdots g_m$. Equivalently, viewing $\mathcal{A}$ as a curved A_∞-algebra over $\mathbb{F}_2[[t]]$, t has grading λ_w^{-1} and μ_m has degree λ_d^{m-2}.

Remark 2.14. In the notation of our previous paper [LOT20], where we considered only integer-valued gradings, λ_d was -1 and λ_w was denoted κ.

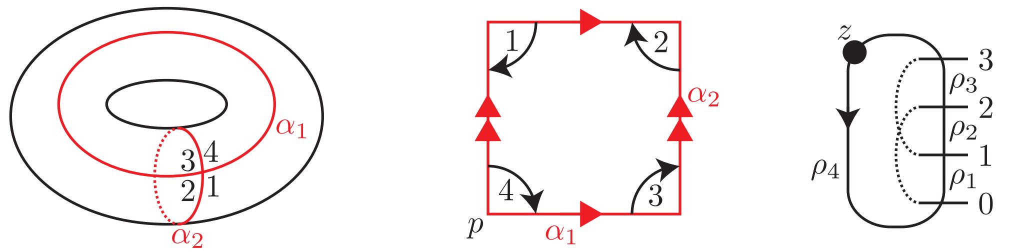
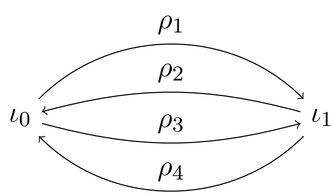


FIGURE 1. **The torus.** Left: the torus T^2 with α_1 and α_2 drawn on it and the corners $1, \dots, 4$ labeled. Center: the same torus, as an identification space of the rectangle S. Right: the pointed matched circle for the torus and labeling of the points, matching, and indecomposable chords.

3. Definitions of the algebras via planar graphs

3.1. **The associative algebra $\mathcal{A}_-^{0,\mathrm{as}}$.** The associative algebra $\mathcal{A}_-^{0,\mathrm{as}}$ is the path algebra with relations

(3.1)



$$/\big(\rho_2\rho_1 = \rho_3\rho_2 = \rho_4\rho_3 = \rho_1\rho_4 = 0\big)$$

over the ground ring $\mathbb{F}_2$.

The idempotents ι_0, ι_1 and elements

$$\rho_{i,i+1,\dots,i+n} = \rho_i\rho_{i+1}\cdots\rho_{i+n},$$

$i \in \mathbb{Z}/4\mathbb{Z}$, $n \in \mathbb{Z}_{>0}$ form a $\mathbb{F}_2$-basis for $\mathcal{A}_-^{0,\mathrm{as}}$. We call these elements *basic elements* of the algebra; we call the elements of the form $\rho = \rho_i\cdots\rho_{i+n}$ *Reeb elements*. to distinguish them from ι_0 and ι_1. The *length* of $\rho_{i,\dots,i+n}$ is $n+1$ and is denoted $|\rho_{i,\dots,i+n}|$; the idempotents are defined to have length 0:

$$|\iota_0| = |\iota_1| = 0.$$

The idempotents ι_0 and ι_1 generate a 2-dimensional subalgebra $\Bbbk \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \subset \mathcal{A}_-^{0,\mathrm{as}}$.

To put this algebra into a wider context, recall from [LOT18] the *pointed matched circle* $\mathcal{Z} = (Z, \mathbf{a}, M, z)$ for T^2; see Figure 1. Then each ρ_i is a chord in $(Z, \mathbf{a})$ (as indicated in the figure), and the $\rho_{i,\dots,i+n}$ are chords of length $n+1$. In particular, each chord $\rho_{i,\dots,i+n}$ has a *support* in $H_1(Z, \mathbf{a})$. Explicitly, identify $Z = [0,4]/(0 \sim 4)$ and $\mathbf{a} = \{0,1,2,3\}$, so that ρ_1 corresponds to the interval $[1,2]$, ρ_2 to the interval $[2,3]$, and so on. Declare that the support of ρ_i is $[i, i+1]$ and that the support of a product $\rho\rho'$ is the sum of the support of ρ and the support of ρ'. We will write the support of ρ as $[\rho] = (n_1(\rho), n_2(\rho), n_3(\rho), n_4(\rho))$. So, for instance, $[\rho_1] = (1,0,0,0)$, $[\rho_2] = (0,1,0,0)$, and so on. We have

$$|\rho| = n_1(\rho) + n_2(\rho) + n_3(\rho) + n_4(\rho)$$

Remark 3.2. Let I be the 2-sided ideal of $\mathcal{A}_-^{0,\mathrm{as}}$ generated by ρ_4. Then $\mathcal{A}_-^{0,\mathrm{as}}/I$ is the algebra $\mathcal{A}(T^2)$ associated to the torus in [LOT18].

We will extend the length grading on $\mathbb{F}_2[U] \otimes \mathcal{A}_-^{0,\mathrm{as}}$, by declaring $|U| = 4$.

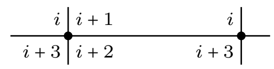


FIGURE 2. **Conventions on a valid labeling.** The labels are in $\{1,\dots,4\}$ (mod 4).

3.2. **The definition of $\mathcal{A}_-$.** We will define a weighted A_∞-algebra structure $\mathcal{A}_-$ on $\mathcal{A}_-^{0,\mathrm{as}}[U] \coloneqq \mathbb{F}_2[U]\otimes\mathcal{A}_-^{0,\mathrm{as}}$. The higher operations will be defined in terms of combinatorial objects associated to certain planar graphs, as follows. These are interpreted as certain maps of the disk into the torus in Section 3.5. (Note that these graphs play a different role from the rooted, planar trees in Section 2.1.)

Definition 3.3. *By a* rooted, planar graph*, we mean a graph Γ, together with an embedding of Γ into the disk $\mathbb{D}$, so that Γ meets $\partial\mathbb{D}$ in its leaves, together with a choice of a distinguished leaf of Γ. This distinguished leaf is called the* root *(and we will no longer refer to it as a "leaf"). Let Γ be a rooted, planar, graph with the following properties:*

- *Γ is connected.*
- *Γ has at least one internal vertex.*
- *Each internal vertex in Γ has valence 4.*
- *Each component of $\mathbb{D}\setminus\Gamma$ either meets $\partial\mathbb{D}$, or it meets exactly 4 edges and vertices of Γ. In the second case, these embedded, 4-edged cycles in Γ are called* short cycles.

Let Q denote the set of sectors (quadrants) around all of the internal vertices. A valid labelling *on Γ is a function $\Lambda\colon Q\to\{1,\dots,4\}$ with the following properties:*

- *If q_1,q_2,q_3,q_4 are the four quadrants around some vertex, labelled in clockwise order (around that vertex), then, up to cyclic reordering,*
$$(\Lambda(q_1),\dots,\Lambda(q_4)) = (1,2,3,4).$$
- *Let v_1 and v_2 be two vertices that are connected by an edge e. Orient e, and let q_1 and q_2 be the two sectors around v_1 and v_2 to the right of e, labelled in the order given by the orientation of e. Then, $\Lambda(q_1)+1\equiv\Lambda(q_2) \pmod 4$. (This is required to hold for both orientations of e.)*

A centered tiling pattern *is a rooted planar graph with the above properties, equipped with a valid labelling Λ.*

Observe that a valid labelling is uniquely determined by its value on any single sector. The conventions of a valid labelling are summarized in Figure 2.

We will extend slightly our graphs:

Definition 3.4. *Let Γ be a centered tiling pattern as in Definition 3.3. Let e be the edge of Γ adjacent to the root; orient e so it points towards the root. Enlarge the graph to obtain a new graph Γ' by inserting a sequence of 2-valent vertices along e. A* valid labeling *on Γ' is now a labelling on certain distinguished sectors, as follows. Near each 2 valent vertex, there are two sectors. For a* left-extended *(respectively* right-extended*) tiling the sectors to the left (respectively right) of e are distinguished; in both cases, we will also think of all the quadrants as being distinguished sectors. A* valid labelling *of Γ' is a function Λ from the distinguished sectors to $\{1,\dots,4\}$ satisfying the conditions of Definition 3.3. An* extended tiling pattern *is a planar, rooted graph Γ' as above, together with a valid labelling. A* tiling pattern *is either a centered tiling pattern or an extended tiling pattern. For a tiling pattern, the underlying graph has first homology $H_1(\Gamma)$ isomorphic to $\mathbb{Z}^w$ for some $w\geq 0$. This integer w is called the* weight *of the tiling pattern.*

Definition 3.5. *Let Γ be a tiling pattern. The graph Γ divides $\partial\mathbb{D}$ into k intervals $I_1,\dots,I_k$, starting and ending at the root, labelled in the order they appear with respect to the boundary orientation.*

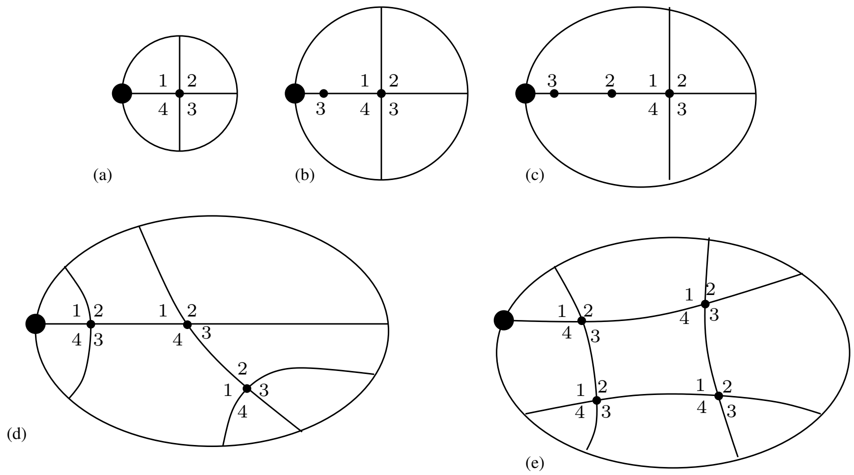


FIGURE 3. **Tiling patterns.** When these are extended (as in (b) and (c) above), we indicate the distinguished sectors by writing the labels in those sectors. The root is indicated by the large dot on the boundary.

Let $(q_1,\dots,q_m)$ be those distinguished sectors which are visible from the boundary, ordered as one traverses the boundary with its orientation, starting at the root vertex. There is a sequence $1 = n_1 < n_2 < \cdots < n_k < m$, with the property that $q_{n_i},\dots,q_{n_{i+1}-1}$ are the sectors visible along I_i. The chord sequence *of a labelled tiling pattern is defined to be the element of $\left(\mathcal{A}_-^{0,\mathrm{as}}\right)^{\otimes_{\Bbbk} m}$ given by*

$$\left(\prod_{i=1}^{n_2-1}\rho_{\Lambda(q_i)}\right)\otimes\cdots\otimes\left(\prod_{i=n_\ell}^{n_{\ell+1}-1}\rho_{\Lambda(q_i)}\right)\otimes\cdots\otimes\left(\prod_{i=n_k}^{m}\rho_{\Lambda(q_i)}\right).$$

Lemma 3.6. *If Γ is a tiling pattern, its chord sequence $\rho^1\otimes\cdots\otimes\rho^n$ is non-zero; and moreover for each $i=1,\dots,n-1$, we have that $\rho^i\cdot\rho^{i+1}=0$.*

Proof. This is immediate from the conventions on a valid labelling, and the definition of the chord sequence. ☐

Lemma 3.7. *If Γ is a tiling pattern, then length k of its chord sequence is even.*

Proof. The boundary of Γ cuts $\partial\mathbb{D}$ into arcs. The number of arcs is the length of the chord sequence. Since the graph underlying Γ is 4-valent, we can think of this graph as a union of ℓ arcs embedded in $\mathbb{D}$, which intersect transversely. The boundary of each arc meets $\partial\mathbb{D}$ in a pair of points; so Γ meets $\partial\mathbb{D}$ in 2ℓ points. Clearly, $k=2\ell$. ☐

Definition 3.8. *Given an extended tiling pattern, let d denote the number of valence 4 vertices. The* output element *in $\mathcal{A}_-^{0,\mathrm{as}}[U]$ of the tiling pattern is U^d times the following element of $\mathcal{A}_-^{0,\mathrm{as}}$:*

- *If the tiling pattern is centered, the element is the left idempotent of $\rho_{\Lambda(q_1)}$, which coincides with the right idempotent of $\rho_{\Lambda(q_m)}$.*
- *If the tiling pattern is extended, the element is the product of $\rho_{\Lambda(q)}$, taken over all the distinguished valence 2 sectors (taken in their natural order).*

Example 3.9. Consider Figure 3. We have illustrated five tiling patterns: (a), (d), and (e) are centered; (b) is right extended; (c) is left-extended. The weights of (a)–(d) are 0; (e) has weight 1. We have chord sequences: (a) $\rho_4\otimes\rho_3\otimes\rho_2\otimes\rho_1$; (b) $\rho_{34}\otimes\rho_3\otimes\rho_2\otimes\rho_1$; (c) $\rho_4\otimes\rho_3\otimes\rho_2\otimes\rho_{123}$; (d) $\rho_4\otimes\rho_{341}\otimes\rho_4\otimes\rho_3\otimes\rho_{23}\otimes\rho_2\otimes\rho_{12}\otimes\rho_1$; and (e) $\rho_{41}\otimes\rho_4\otimes\rho_{34}\otimes\rho_3\otimes\rho_{23}\otimes\rho_2\otimes\rho_{12}\otimes\rho_1$. The outputs are (a) $U\iota_0$; (b) $U\rho_3$; (c) $U\rho_{23}$; (d) $U^3\iota_0$; and (e) $U^4\iota_0$.

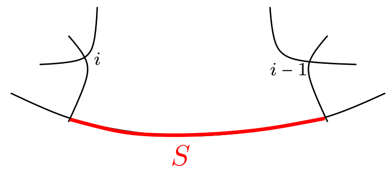


FIGURE 4. **Composite labelling convention.** We require that Λ changes as shown, as we cross S between the two components Γ_1 and Γ_2.

Definition 3.10. *As an $\mathbb{F}_2[U]$-module, the weighted A_∞-algebra $\mathcal{A}_-$ is $\mathcal{A}_-^{0,\mathrm{as}}[U]$. The operations $\mu_n^w\colon (\mathcal{A}_-)^{\otimes n} \to \mathcal{A}_-$ are defined as follows. First, the operations are U-multilinear, so it suffices to define them on basic elements of $\mathcal{A}_-^{0,\mathrm{as}}$. The operation μ_2^0 is given by the multiplication on $\mathcal{A}_-^{0,\mathrm{as}}$, and $\mu_n^w(a_1,\dots,a_n) = 0$ if $(n,w) \neq (2,0)$ and some $a_i \in \{\iota_0,\iota_1\}$. So, it remains to define $\mu_n^w(\rho^1,\dots,\rho^n)$, where the ρ^i are Reeb elements.*

The element μ_0^1 is

$$\mu_0^1 = \rho_{1234} + \rho_{2341} + \rho_{3412} + \rho_{4123}, \tag{3.11}$$

the sum of the length-4 chords. For all $w \neq 1$, we have $\mu_0^w = 0$.

For the remaining pairs (n,w), the function $\mu_n^w(\rho^1,\dots,\rho^n)$ is the sum of all the output elements (in the sense of Definition 3.8) of labelled tiling patterns which have weight w and chord sequence $\rho^1 \otimes \cdots \otimes \rho^n$.

Interpreted in terms of the algebra, Lemma 3.7 states that $\mu_n^w = 0$ if n is odd.

In Sections 5 and 6 we will also be interested in the ordinary A_∞-algebra underlying $\mathcal{A}_-$:

Definition 3.12. *Let $\mathcal{A}_-^0$ denote the undeformed A_∞-algebra underlying $\mathcal{A}_-$, i.e., $\mathcal{A}_-^0 = (\mathcal{A}_-^{0,\mathrm{as}}[U], \{\mu_n^0\})$, where the operations μ_n^0 are as in Definition 3.10.*

In other words, the operations on $\mathcal{A}_-^0$ count tile patterns whose underlying graph Γ is a tree.

3.3. Verifying the A_∞ relations.

The aim of this section is to prove that the operations from Definition 3.10 make $\mathcal{A}_-$ into a weighted A_∞-algebra. To establish this property, it is helpful to have a graphical representation of the composition of tiling patterns.

Definition 3.13. *A* centered composite pattern *consists of the following data:*

- *a rooted, planar graph Γ satisfying the conditions of Definition 3.4, except that now rather than being connected, the underlying graph is required to have exactly two components, labelled $\Gamma_1 \coprod \Gamma_2$; and each component is required to have at least one 4-valent vertex.*
- *An arc S on the boundary of $\mathbb{D}$, with the following property. If we cut $\mathbb{D}$ along $\Gamma_1 \cup \Gamma_2$, there is a distinguished region $\mathcal{D}$ that meets both Γ_1 and Γ_2. This region meets the boundary in two arcs. The arc S is required to be one of those two arcs.*
- *A valid labelling Λ as in Definition 3.3, which is also required to satisfy a compatibility condition at the distinguished edge e, as shown in Figure 4.*

A composite pattern is called extremal *if the arc S meets the root vertex; otherwise it is called* generic. *An* extended composite pattern *is defined similarly, but has a sequence of valence two vertices adjacent to the root vertex, with distinguished sectors, all of which lie on the same side of the arc from the root to the first 4-valent vertex.*

See Figure 5 for an example.

Let $\Gamma_1 \subset \mathbb{D}_1$ be a tiling pattern with chord sequence $\rho^1 \otimes \cdots \otimes \rho^n$, fix some $i \in \{1,\dots,n\}$, and let $\Gamma_2 \subset \mathbb{D}_2$ be an extended tiling pattern with chord sequence $\sigma^1 \otimes \cdots \otimes \sigma^m$ whose output element is ρ^i. We will compose these two tiling patterns, to obtain a composite in the sense of Definition 3.13, as follows. Let $J \subset \partial\mathbb{D}_2$ be the boundary segment that meets the distinguished 2-valent sectors: i.e.

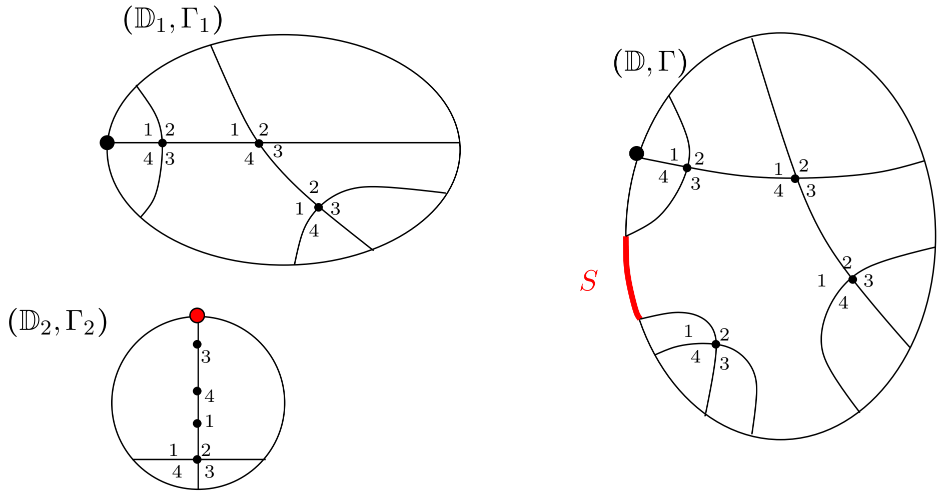


FIGURE 5. **Example of a composite pattern.** Here, Γ_1 and Γ_2 are two tiling patterns; we can form their composition $\Gamma_1 \#_2 \Gamma_2$ to obtain the composite pattern Γ on the right. The red dot is the root vertex of Γ_2, which is to be joined to the 2^{nd} boundary arc of Γ_1.

this is immediately after the root if Γ_2 is left-extended, and it is immediately before the root if Γ_2 is right-extended. Let Γ_2' be obtained from Γ_2 by removing all the 2-valent vertices. The composition $\Gamma_1 \#_i \Gamma_2$ is the boundary connected sum of $\mathbb{D}_1$ with $\mathbb{D}_2$, gluing I_i to J, which we think of as a disk $\mathbb{D}$, equipped with the graph $\Gamma = \Gamma_1 \coprod \Gamma_2'$. Let $S \subset \partial\mathbb{D}$ be the interval that connects the root vertex of Γ_2' with some vertex in Γ_1. See Figure 5 for an illustration.

Each composite pattern Γ can be uniquely decomposed as $\Gamma_1 \#_i \Gamma_2$ for suitable choices of Γ_1 and Γ_2. Explicitly, cut $\mathbb{D}$ in two along an arc A that intersects S exactly once and is disjoint from Γ. Let $(\mathbb{D}_1, \Gamma_1)$ be the component that contains the root of Γ; the other component is $(\mathbb{D}_2, \Gamma_2')$, equipped with the root vertex induced from S. The sequence of 2-valent vertices needed to reconstruct Γ_2 from Γ_2' can be read off the Γ_1 side.

Definition 3.14. *The* chord sequence *of a composite pattern Γ is defined as in Definition 3.5, with the understanding that the factor in the tensor product of chords coming from the sectors adjacent to the interval S is dropped. Let d denote the number of valence 4 vertices in Γ. The* output element *of the composite pattern Γ is U^d times the product of the chords associated to the distinguished valence 2 sectors. Equivalently, if $\Gamma = \Gamma_1 \#_i \Gamma_2$, the output element of Γ is the output element of Γ_1 times U^{d_2}, where d_2 denotes the number of valence 4 vertices in Γ_2.*

For example, consider the picture on the right of Figure 5. Reading the chords in order as they are seen from the boundary (starting at the root) gives the sequence

$$\rho_4 \otimes \rho_{3412} \otimes \rho_1 \otimes \rho_4 \otimes \rho_3 \otimes \rho_{2341} \otimes \rho_4 \otimes \rho_3 \otimes \rho_{23} \otimes \rho_2 \otimes \rho_{12} \otimes \rho_1.$$

For the chord sequence, though, we drop the second tensor factor ρ_{3412}, since that is the term visible from S; thus, thus the chord sequence associated to the composite is:

$$\rho_4 \otimes \rho_1 \otimes \rho_4 \otimes \rho_3 \otimes \rho_{2341} \otimes \rho_4 \otimes \rho_3 \otimes \rho_{23} \otimes \rho_2 \otimes \rho_{12} \otimes \rho_1.$$

By construction, the chord sequence of $\Gamma_1 \#_i \Gamma_2$ is obtained from the chord sequence for Γ_1 by replacing the i^{th} tensor factor with the chord sequence for Γ_2.

Lemma 3.15. *Fix a sequence of Reeb elements $\rho^1, \dots, \rho^n$ of $\mathcal{A}_-^{0,\mathrm{as}}$, with the property that $\rho^1 \otimes \cdots \otimes \rho^n \in (\mathcal{A}_-^{0,\mathrm{as}})^{\otimes_{\Bbbk} n}$ is non-zero and $\rho^k \rho^{k+1} = 0$ for all k (cf. Lemma 3.6). Fix some $1 \le i \le n$ and fix a factorization $\rho^i = \sigma_1 \cdot \sigma_2$ into two Reeb elements. Consider the following sets.*

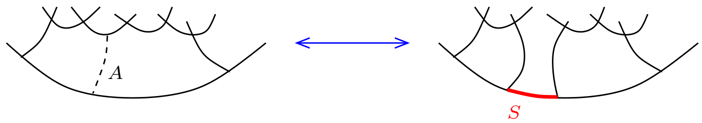


FIGURE 6. **Pushing out the edge.** Illustration of the $1:1$ correspondence from Lemma 3.15

- *The set $\mathfrak{S}_1$ of tiling patterns with weight w and chord sequence $\rho^1 \otimes \cdots \otimes \rho^n$.*
- *The set $\mathfrak{P}$ of composite patterns with weight w and chord sequence*

$$\rho^1 \otimes \cdots \otimes \rho^{i-1} \otimes \sigma_1 \otimes \sigma_2 \otimes \rho^{i+1} \otimes \cdots \otimes \rho^n.$$

These composite patterns are always generic in the sense of Definition 3.13.
- *The set $\mathfrak{T}$ of tiling patterns with weight $w-1$ and chord sequence*

$$\rho^1 \otimes \cdots \otimes \rho^{i-1} \otimes \sigma_1 \otimes \tau \otimes \sigma_2 \otimes \rho^{i+1} \otimes \cdots \otimes \rho^n,$$

where τ has length 4.
- *(When $i=1$) the set $\mathfrak{L}$ of centered or left-extended tiling patterns Γ with weight w and chord sequence*

$$\sigma_2 \otimes \rho^2 \otimes \cdots \otimes \rho^n.$$

If Γ has output element $U^d\rho$, then when viewed as an element of $\mathfrak{L}$ we take its associated output element to be $U^d\sigma_1 \cdot \rho$.
- *(When $i=n$) the set $\mathfrak{R}$ of centered or right-extended tiling patterns Γ with weight w and chord sequence*

$$\rho^1 \otimes \rho^2 \otimes \cdots \otimes \rho^{n-1} \otimes \sigma_1.$$

If Γ has output element $U^d\rho$, then when viewed as an element of $\mathfrak{R}$ we take its associated output element to be $U^d\rho \cdot \sigma_2$.

Then there is a one-to-one correspondence between $\mathfrak{S}_1$ and $\mathfrak{P} \cup \mathfrak{T} \cup \mathfrak{L} \cup \mathfrak{R}$ which preserves the associated output elements

Proof. Fix an element of $\mathfrak{S}_1$. The factorization $\rho^i = \sigma_1 \cdot \sigma_2$ corresponds to an arc A whose interior is disjoint from Γ, with one endpoint on some edge e in Γ and the other endpoint on $\partial\mathbb{D}$. Assume first that $i \notin \{1,n\}$. The one-to-one correspondence from $\mathfrak{S}_1$ to $\mathfrak{P} \cup \mathfrak{T}$ is obtained by pushing e out to the boundary along A, as shown in Figure 6.

If pushing out e disconnects Γ, we label the newly introduced arc on $\partial\mathbb{D}$ by S, and the result is a composite pattern, which is in $\mathfrak{P}$. If pushing out e does not disconnect Γ (i.e., if e is part of a cycle in Γ), the result is another tiling pattern, which is in $\mathfrak{T}$. Examples are illustrated in Figure 7.

In the case $i=1$, pushing out e from Γ may result in a disconnected graph, one of whose components has no valence 4 vertices. In this case, the operation of pushing e out to the boundary along A results in a configuration which is not a composite pattern in the sense of Definition 3.13, as one of the two components Γ_1 and Γ_2 consists of a chain of 2-valent vertices. Deleting that component, and placing the new root as for composite patterns, gives a tiling pattern which is either left-extended or centered, with chord sequence $\sigma_2 \otimes \rho^2 \otimes \dots \rho^n$. View the result as an element of $\mathfrak{L}$. With this addition, we obtain the desired one-to-one correspondence between $\mathfrak{S}_1$ and $\mathfrak{L} \cup \mathfrak{P} \cup \mathfrak{T}$.

The case where $i=n$ is analogous to the case when $i=1$, except in that case the tiling pattern is either centered or right-extended, and we view it now as an element of $\mathfrak{R}$. This gives the desired one-to-one correspondence between $\mathfrak{S}_1$ and $\mathfrak{R} \cup \mathfrak{P} \cup \mathfrak{T}$. ☐

Remark 3.16. If $\mathfrak{T}$ is non-empty, the newly-introduced length 4 chord ρ appearing in its chord sequence is determined by σ_1: it is the length four chord for which $\sigma_1 \otimes \rho \neq 0$ (i.e., with matching idempotents), and for which $\sigma_1 \cdot \rho = \rho \cdot \sigma_2 = 0$.

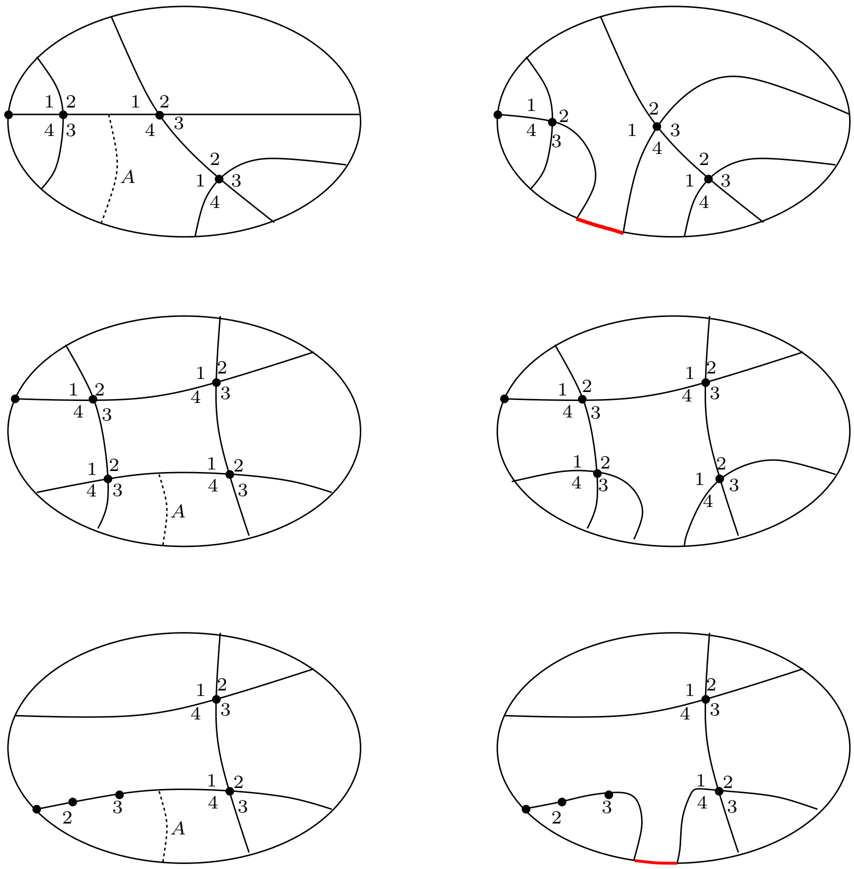


FIGURE 7. **Pushing out the edge: examples.** On the upper left is an arc A connecting an edge to the boundary, so that if we push the edge out along A, we obtain a composite pattern, as illustrated on the upper right. On the middle left is an arc connected to a non-disconnecting edge; pushing out along that arc gives another tiling pattern (with one smaller weight), as illustrated on the middle right. On the bottom left is a left-extended tiling pattern and an arc A which disconnects the graph; but pushing out the edge, as illustrated on the lower right, does not give a composite pattern. Rather, if we delete the component with no 4-valent vertices, we obtain a (centered) tiling pattern of type $\mathfrak{L}$.

Lemma 3.17. *Let Γ be a composite pattern, and write $\Gamma = \Gamma_1 \#_i \Gamma_2$. Let n be the length of the chord sequence for Γ_1. Then Γ is generic (in the sense of Definition 3.13) if and only if one of the following conditions holds:*

- $1 < i < n$;
- $i = 1$ *and Γ_2 is left-extended; or*
- $i = n$ *and Γ_2 is right-extended.*

The composite pattern Γ is extremal, and its distinguished 2-valent sectors, if any, lie in the region $\mathcal{D}$ (see Definition 3.13) if and only if one of the following conditions holds:

- $i = 1$, *Γ_1 is centered or left-extended, and Γ_2 is right-extended; or*
- $i = n$, *Γ_1 is centered or right-extended, and Γ_2 is left-extended.*

The composite pattern Γ is extremal, extended, and its distinguished 2-valent sectors lie in a region other than $\mathcal{D}$ if and only if one of the following conditions holds:

- $i = 1$, *Γ_1 is right-extended, and Γ_2 is right-extended; or*

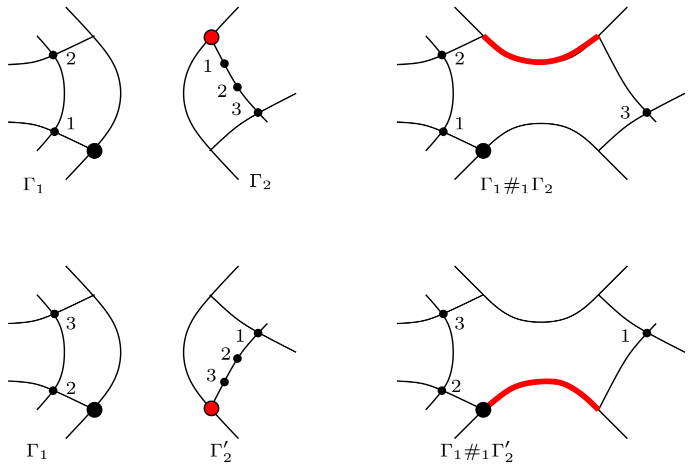


FIGURE 8. **Generic and extremal compositions.** We have drawn here compositions $\Gamma_1 \#_1 \Gamma_2$ and $\Gamma_1 \#_1 \Gamma_2'$. In the top line, we use a left-extended Γ_2 and the result is generic; while in the second line, we use a right-extended Γ_2', and the result is extremal.

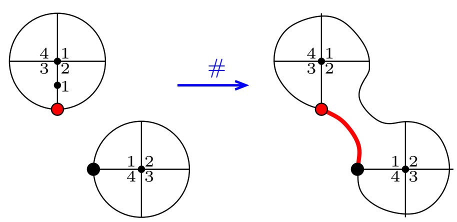


FIGURE 9. **An extremal composite pattern.** In this case, Γ_1 is centered and Γ_2 is left-extended.

- *$i = n$, Γ_1 is left-extended, and Γ_2 is left-extended.*

Proof. It is clear from the definitions that $\Gamma_1 \#_i \Gamma_2$ is extremal precisely when $i = 1$ and Γ_2 is right-extended or $i = n$ and Γ_2 is left-extended. See Figure 9. The case of an extremal composite pattern with no distinguished 2-valent sectors is illustrated in Figure 8. □

Lemma 3.18. *Let $\mathfrak{S}_2$ denote the set of extremal composite patterns whose distinguished 2-valent sectors are in the region $\mathcal{D}$. This set $\mathfrak{S}_2$ admits a fixed-point-free involution which preserves the chord sequence, output element, and total weight.*

Proof. The involution is obtained by placing the root vertex on the other endpoint of S and then moving the 2-valent sectors (if there are any) so that they remain adjacent to the root vertex. See Figure 10. (In the bottom row, we have an example where 2-valent vectors need to be moved). □

Theorem 3.19. *The operations from Definition 3.10 give $\mathcal{A}_-$ the structure of a weighted A_∞-algebra.*

Proof. We must verify the A_∞ relation for each fixed input sequence of algebra elements and fixed weight. This relation is a sum of contributions of weighted, rooted trees with two internal vertices. (Note that we consider a leaf with positive weight internal.)

The A_∞ relations with ≤ 2 inputs or with 3 inputs and weight 0 are easy consequences of the following facts:

- $\mu_0^w = 0$ except when $w = 1$.
- $\mu_1^w = 0$ for all $w \geq 0$.

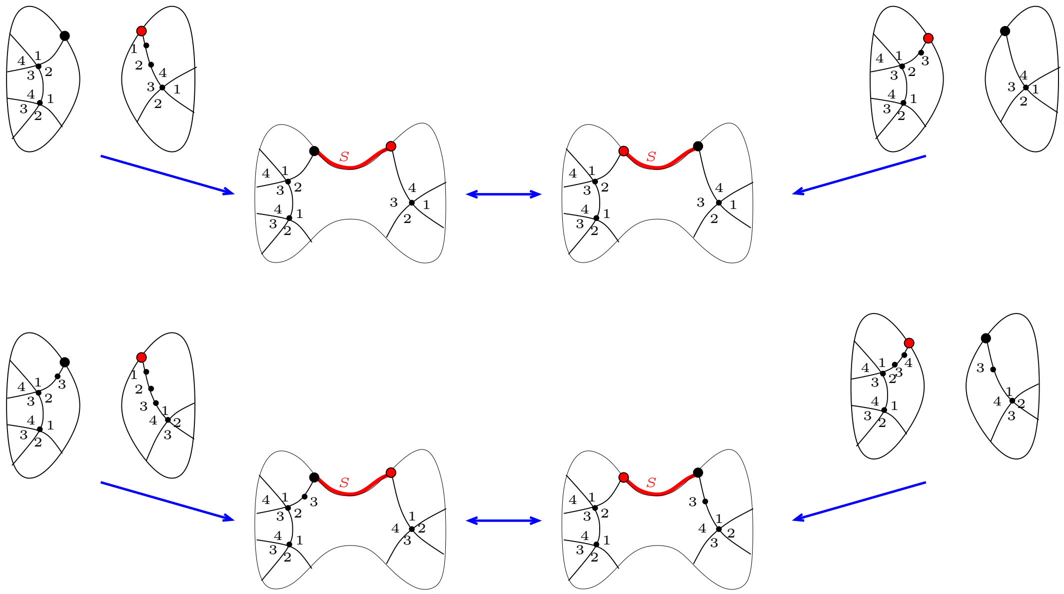


FIGURE 10. **Involution from Lemma 3.18.** The involution is given by moving the root vertex from one endpoint of S to the other (as indicated by the double arrows). In the second row, we also had to move a 2-valent vertex during this operation. We have also included compositions that give rise to the composites. As before, we use the conventions on compositions: the red dot lies on the tree Γ_2, which is joined to some input of Γ_1.

- $\mu_2^w = 0$ for all $w > 0$.
- $\mathcal{A}_-^{0,\mathrm{as}}$ is an associative algebra.
- μ_0^1 is a central element of $\mathcal{A}_-^{0,\mathrm{as}}$.

Consider an A_∞ relation, then, with at least 4 inputs or with 3 inputs and weight ≥ 1. We describe these terms according to the type of the two A_∞ operations involved, with the outer μ_n^w written first, using a symbol

- 0 for the curvature μ_0^1,
- 2 for a multiplication μ_2^0,
- L for a left-extended μ_n^w with $n \geq 4$,
- C for a centered μ_n^w with $n \geq 4$, and
- R for a right-extended μ_n^w with $n \geq 4$.

In addition, we add a final symbol $+$, $-$, or g describing whether the inner μ_n^w is fed into the leftmost input, rightmost input, or any other input of the outer μ_n^w. We will also use a $*$ for a wild-card any of these symbols.

We thus have terms of the following types, classified according to the number of inputs of the smaller μ_n^w.

- **Terms of type** (0) involve μ_0^1, which is necessarily the inner A_∞ operation. The other operation is associated to a tiling pattern Γ. The overall term is written, for example, $(C0+)$ for a term where Γ is centered and the curvature is fed into the first input of the corresponding μ_n^w.
- **Terms of type** (2) involve μ_2^0, which can be the inner or outer operation, giving terms like $(2L-)$, from a left-extended tiling pattern Γ fed into the second input of the μ_2^0, or $(R2g)$, from a μ_2^0 fed into a generic input of a right-extended tiling pattern. Note that if the 2 comes first, then the last symbol cannot be a g, since μ_2^0 has no generic inputs.

Term	Cancellation	Term	Cancellation	Term	Cancellation
$(L0+)$	Cancels $(R0-)$	$(L2+)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2L+)$	See Table 2
$(L0g)$	$\mathfrak{T}$ in Lem. 3.15	$(L2g)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2L-)$	$\mathfrak{L}$ in Lem. 3.15
$(L0-)$	See Table 2	$(L2-)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2C-)^{\times}$	$\mathfrak{L}$ in Lem. 3.15
$(C0+)$	Cancels $(C0-)$	$(C2+)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2C-)^{0}$	See Table 2
$(C0g)$	$\mathfrak{T}$ in Lem. 3.15	$(C2g)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2C+)^{\times}$	$\mathfrak{R}$ in Lem. 3.15
$(C0-)$	Cancels $(C0+)$	$(C2-)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2C+)^{0}$	See Table 2
$(R0+)$	See Table 2	$(R2+)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2R+)$	$\mathfrak{R}$ in Lem. 3.15
$(R0g)$	$\mathfrak{T}$ in Lem. 3.15	$(R2g)$	$\mathfrak{S}_1$ in Lem. 3.15	$(2R-)$	See Table 2
$(R0-)$	Cancels $(L0+)$	$(R2-)$	$\mathfrak{S}_1$ in Lem. 3.15		
$[LL+]$	$\mathfrak{P}$ in Lem. 3.15	$[CL+]$	$\mathfrak{P}$ in Lem. 3.15	$[RL+]$	$\mathfrak{P}$ in Lem. 3.15
$[LLg]$	$\mathfrak{P}$ in Lem. 3.15	$[CLg]$	$\mathfrak{P}$ in Lem. 3.15	$[RLg]$	$\mathfrak{P}$ in Lem. 3.15
$[LL-]$	See Table 2	$[CL-]$	$\mathfrak{S}_2$ in Lem. 3.18	$[RL-]$	$\mathfrak{S}_2$ in Lem. 3.18
$[LR+]$	$\mathfrak{S}_2$ in Lem. 3.18	$[CR+]$	$\mathfrak{S}_2$ in Lem. 3.18	$[RR+]$	See Table 2
$[LRg]$	$\mathfrak{P}$ in Lem. 3.15	$[CRg]$	$\mathfrak{P}$ in Lem. 3.15	$[RRg]$	$\mathfrak{P}$ in Lem. 3.15
$[LR-]$	$\mathfrak{P}$ in Lem. 3.15	$[CR-]$	$\mathfrak{P}$ in Lem. 3.15	$[RR-]$	$\mathfrak{P}$ in Lem. 3.15

TABLE 1. Types of terms in A_∞ relations not involving idempotents, and how they cancel.

- **Terms of type** [] involve two higher multiplications μ_n^w with $n > 2$. These correspond to composite patterns in the sense of Definition 3.13, and are written with square brackets to distinguish them from types (0) and (2). So, for example, $[CR+]$ means a right-extended operation feeding into the first input of a centered operation. The second letter cannot be C: the output of a centered operation is a power of U times an idempotent, and if such an output is channelled into another μ_n^w with $(w, n) \neq (0, 2)$, the result vanishes.

Finally, we make one further distinction: we divide terms of type $(2C-)$ into two types:

- $(2C-)^0$, where $\rho^1 \cdot \rho^2 = 0$, and
- $(2C-)^\times$, where $\rho^1 \cdot \rho^2 \neq 0$.

We similarly distinguish $(2C+)$ into $(2C+)^0$, where $\rho^{n-1} \cdot \rho^n = 0$, and $(2C+)^\times$, where $\rho^{n-1} \cdot \rho^n \neq 0$.

Our goal is to explain how these terms cancel in the A_∞ relations. Before proceeding to the main verification, we make some remarks about the sequence of incoming algebra elements. By linearity, it suffices to verify the A_∞ relation when the sequence of incoming algebra elements consists of basic algebra elements. Moreover, the case where at least one of those elements is an idempotent can be handled easily. Since the $\mu_m^w(a_1, \dots, a_m) = 0$ if some a_i is an idempotent and $(n, w) \neq (2, 0)$, if $(a_1, \dots, a_n)$ is a sequence of inputs to a non-zero term in the A_∞ relation and a_i is an idempotent, then either $i = 1$, in which case we have two cancelling terms in the A_∞ relation of types $(2{*}-)$ and a $({*}2+)$; or $i = n$, in which case we have two cancelling terms of types $(2{*}+)$ and $({*}2-)$. For example, in the weight 0 A_∞ relation with inputs $\iota_1 \otimes \rho_4 \otimes \rho_3 \otimes \rho_2 \otimes \rho_1$, there are two non-zero terms of type $(2C+)$ and $(C2-)$ respectively:

$$\mu_0^2(\iota_1, \mu_0^4(\rho_4, \rho_3, \rho_2, \rho_1)) = \mu_0^4(\mu_0^2(\iota_1, \rho_4), \rho_3, \rho_2, \rho_1)),$$

We now proceed to the verification of the A_∞ relation, assuming that the input sequence consists of Reeb elements $\rho^1 \otimes \cdots \otimes \rho^n$, with $n \geq 4$ or $n = 3$ and $w \geq 1$. See Table 1 for a listing of the term types, and how they cancel in the following proof.

Terms of types $(L0+)$ and $(R0-)$ cancel, as follows. A term of type $(L0+)$ is determined by a left-extended tiling pattern Γ, which has a string of at least 1 but at most 3 distinguished 2-valent sectors after the root vertex. The corresponding term of type $(R0-)$ is obtained from Γ by moving the root vertex of Γ to the next position (with respect to the boundary orientation) of the

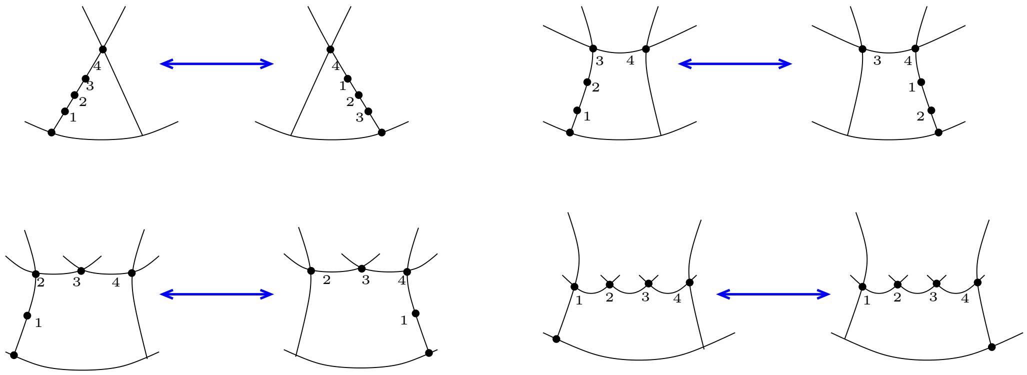


FIGURE 11. **Cancellation of type** $(L0+)$ **and** $(R0-)$**; and** $(C0+)$ **and** $(C0-)$. The first three pictures illustrate cancellations of $(L0+)$ and $(R0-)$, while the last one is a cancellation of $(C0+)$ with $(C0-)$. For each picture there are three other cases obtained by cyclically permuting the subscripts.

intersection of Γ with $\partial\mathbb{D}$, and moving around the valence 2 vertices as needed, as illustrated in Figure 11. For example, in the weight 1 A_∞ relation with inputs $\rho_3 \otimes \rho_2 \otimes \rho_1$, there are two non-zero terms, of type $(L0+)$ and $(R0-)$, respectively:

$$\mu_4^0(\mu_0^1, \rho_3, \rho_2, \rho_1) = \mu_4^0(\rho_{1234}, \rho_3, \rho_2, \rho_1) = U\rho_{123}$$
$$\mu_4^0(\rho_3, \rho_2, \rho_1, \mu_0^1) = \mu_4^0(\rho_3, \rho_2, \rho_1, \rho_{4123}) = U\rho_{123}.$$

Another pair of cancelling terms of type $(L0+)$ and $(R0-)$ is given by

$$\mu_6^0(\rho_{3412}, \rho_1, \rho_4, \rho_{34}, \rho_3, \rho_2) = U^2\rho_{34} = \mu_6^0(\rho_1, \rho_4, \rho_{34}, \rho_3, \rho_2, \rho_{1234}).$$

There is a similar cancellation of terms of types (C0+) and (C0-), as shown in the last picture of Figure 11. An example is provided by the cancellation of

$$\mu_{10}^0(\rho_{1234}, \rho_3, \rho_2, \rho_{12}, \rho_1, \rho_{41}, \rho_4, \rho_{34}, \rho_3, \rho_2) = U^4 \cdot \iota_0 = \mu_{10}^0(\rho_3, \rho_2, \rho_{12}, \rho_1, \rho_{41}, \rho_4, \rho_{34}, \rho_3, \rho_2, \rho_{1234}).$$

The patterns of type $\mathfrak{S}_1$ from Lemma 3.15 contribute terms of types $(*2*)$. According to that lemma, these cancel against the following types of terms:

- generic composition patterns (i.e., the terms in $\mathfrak{P}$) which, by Lemma 3.17, are the terms of type $[**g]$, $[*L+]$, and $[*R-]$;
- terms corresponding to configurations in $\mathfrak{T}$, which are of type $(*0g)$;
- configurations in $\mathfrak{L}$ from the notation of Lemma 3.15, which in turn correspond to all terms of type $(2L-)$ and $(2C-)^\times$; and
- configurations in $\mathfrak{R}$, which in turn correspond to all terms of type of type $(2R+)$ and $(2C+)^\times$.

For example, we have the following cancellations from Lemma 3.15. The terms

$$\mu_8^0(\rho_4, \mu_2^0(\rho_3, \rho_{41}), \rho_4, \rho_3, \rho_{23}, \rho_2, \rho_{12}, \rho_1) \qquad \text{and} \qquad \mu_4^0(\rho_4, \rho_3, \mu_6^0(\rho_{41}, \rho_4, \rho_3, \rho_{23}, \rho_2, \rho_{12}), \rho_1),$$

are of types $(C2g)$ and $[CRg]$ respectively; they correspond to the diagrams on the top row of Figure 7. Likewise,

$$\mu_8^1(\rho_{41}, \rho_4, \mu_2^0(\rho_3, \rho_4), \rho_3, \rho_{23}, \rho_2, \rho_{12}, \rho_1) \qquad \text{and} \qquad \mu_{10}^0(\rho_{41}, \rho_4, \rho_3, \mu_0^1, \rho_4, \rho_3, \rho_{23}, \rho_2, \rho_{12}, \rho_1)$$

are of types $(C2g)$ and $(C0g)$ respectively; they correspond to the diagrams on the middle row of Figure 7. Finally,

$$\mu_6^1(\mu_2^0(\rho_{23}, \rho_4), \rho_3, \rho_{23}, \rho_2, \rho_1, \rho_{41}) \qquad \text{and} \qquad \mu_2^0(\rho_{23}, \mu_6^0(\rho_4, \rho_3, \rho_{23}, \rho_2, \rho_4, \rho_{41}))$$

are of types $(L2+)$ and $(2C-)^\times$ respectively; they correspond to the diagrams from the bottom row of Figure 7.

	$\lvert\rho\rvert < \lvert\rho^1\rvert$	$\lvert\rho\rvert = \lvert\rho^1\rvert$	$\lvert\rho\rvert > \lvert\rho^1\rvert$
$\lvert\rho\rvert < \lvert\rho^n\rvert$	$[LL-]$, $(L0-)$; $[RR+]$, $(R0+)$	$(2C-)^0$; $[RR+]$, $(R0+)$	$(2R-)$; $[RR+]$ $(R0+)$
$\lvert\rho\rvert = \lvert\rho^n\rvert$	$[LL-]$, $(L0-)$; $(2C+)^0$	$(2C-)^0$; $(2C+)^0$	$(2R-)$; $(2C+)^0$
$\lvert\rho\rvert > \lvert\rho^n\rvert$	$[LL-]$, $(L0-)$; $(2L+)$	$(2C-)^0$; $(2L+)$	$(2R-)$; $(2L+)$

TABLE 2. Remaining cancellation of terms

By Lemma 3.17, the following terms correspond to extremal composite patterns, all of whose distinguished 2-valent sectors lie in the region $\mathcal{D}$ from Definition 3.13: $[LR+]$, $[CR+]$, $[CL-]$, and $[RL-]$. By Lemma 3.18, these terms cancel each other. For example, the terms

$$\mu_6^0(\rho_1, \rho_4, \rho_{34}, \rho_3, \rho_2, \mu_4^0(\rho_{123}, \rho_2, \rho_1, \rho_4)) \qquad \text{and} \qquad \mu_4^0(\mu_6^0(\rho_1, \rho_4, \rho_{34}, \rho_3, \rho_2, \rho_{123}), \rho_2, \rho_1, \rho_4)$$

are of types $[CL-]$ and $[CR+]$, and cancel by Lemma 3.18; they correspond to the diagrams on the top of Figure 10. Likewise,

$$\mu_6^0(\rho_1, \rho_4, \rho_{34}, \rho_3, \rho_2, \mu_4^0(\rho_{1234}, \rho_3, \rho_2, \rho_1)) \qquad \text{and} \qquad \mu_4^0(\mu_6^0(\rho_1, \rho_4, \rho_{34}, \rho_3, \rho_2, \rho_{1234}), \rho_3, \rho_2, \rho_1)$$

are of types $[RL-]$ and $[LR+]$, and correspond to the diagrams on the bottom of Figure 10.

The remaining possible terms are of types $(2L+)$, $(2R-)$, $(2C+)^0$, $(2C-)^0$, $[LL-]$, and $[RR+]$. For cancellations among these remaining terms, we take a closer look at the sequence of algebra elements $(\rho^1 \otimes \cdots \otimes \rho^n)$ entering the A_∞ relation. For all of these terms, we have that $\rho^i \cdot \rho^{i+1} = 0$ for all $i = 1, \dots, n-1$; this follows from Lemma 3.6.

We will now construct a bijection M between terms of types $[LL-]$, $(L0-)$, $(2C-)^0$, and $(2R-)$ with terms of types $[RR+]$, $(R0+)$, $(2C+)^0$, and $(2L+)$, preserving the input sequence and output element. Indeed, for each given chord sequence $(\rho^1 \otimes \cdots \otimes \rho^n)$ and output element $U^n \otimes \rho$, exactly one of $[LL-] \cup (L0-)$, $(2C-)^0$, and $(2R-)$ can be non-empty; similarly, exactly one of $[RR+] \cup (R0+)$, $(2C+)^0$, and $(2L+)$ is non-empty. These possibilities are subdivided according to these according to the relative lengths of ρ, ρ^1, and ρ^n, as follows.

For a left-extended operation $[L{*}{*}]$ or $(L{*}{*})$, the output chord is shorter than the first input; whereas for a μ_2^0 operation $(2{*}-)$, the output chord is at least as long as the first input. Thus, if $|\rho| < |\rho^1|$, there can be no terms of type $(2{*}-)$; if $|\rho| \geq |\rho^1|$, there can be no terms of types $[LL-]$ or $(L0-)$. Symmetrically, $|\rho| < |\rho^n|$ excludes terms of type $(2{*}+)$, and $|\rho| \geq |\rho^n|$ excludes terms of types $[RR+]$ or $(R0+)$. Thus, the relative sizes of $(|\rho|, |\rho^1|)$ and $(|\rho|, |\rho^n|)$ exclude all but the types of remaining terms as shown in Table 2.

The map M is defined as follows. Suppose the input to M is an element of type $(2C-)^0$ or $(2R-)$ (so that $|\rho| \geq |\rho^1|$). Let Γ be the tiling pattern appearing in the configuration. When $|\rho| \geq |\rho^n|$, the map M gives a result of type $(2C+)^0$ or $(2L+)$, whose tiling pattern is obtained from Γ by moving the root vertex one spot back (with respect to the boundary orientation) and reshuffling the 2-valent vertices, as needed. When $|\rho| < |\rho^n|$, there is an arc A that connects some edge in Γ to the a point on $\partial\mathbb{D}$, corresponding to the factorization $\rho^n = \tau \cdot \rho$ for some suitable choice of τ (in the sense that one endpoint of A is on the edge corresponding to the factorization). In that case, M is defined by pushing A to the boundary, moving the root vertex back, and reshuffling the 2-valent vertices. In that case, the result of M is a term of type $[RR+]$ or $(R0+)$ (depending on whether or not the edge pushed out disconnects). In all these cases, the ability to do the necessary reshuffling depends on the fact that $\rho^1 \cdot \rho^2 = 0$. See Figures 12 and 13 for illustrations.

Starting from a composite pattern Γ of type $[LL-]$, we proceed similarly. Provided that $|\rho| \geq |\rho^n|$, we pull in the arc S and then move the root vertex one spot backwards, to find cancelling terms of type $(2C+)^0$ or $(2R+)$. When $|\rho| < |\rho^n|$, we find an arc A to push out to the boundary (again, as before). Now, when we pull in S and push out along A, we obtain the cancelling term of type $[RR+]$ or $(R0+)$. When the input is a configuration of $(L0-)$, we perform the same operations

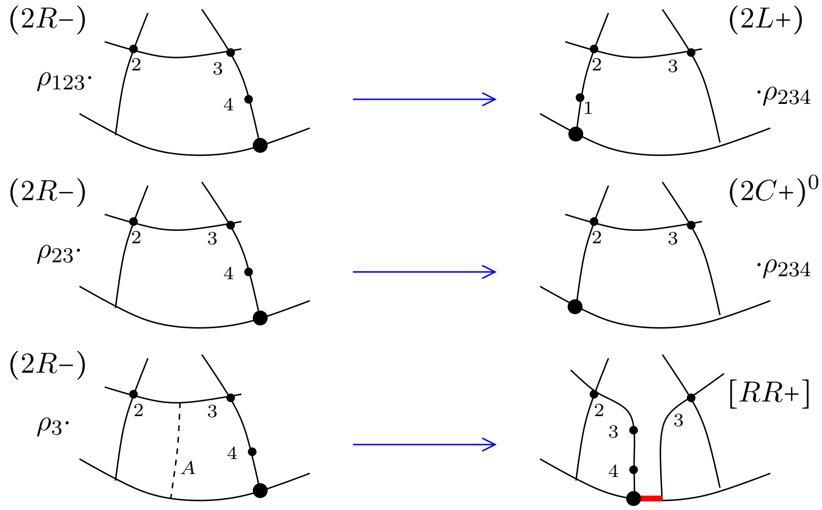


FIGURE 12. **Defining** $M(2R-)$**.** At the left are elements of $(2R-)$; the image of the term under M is indicated on the right. In the first line, $\rho = \rho_{1234}$; in the second, $\rho = \rho_{234}$; and in the third, $\rho = \rho_{34}$. We have also listed the types of the result. In all cases, $\rho^n = \rho_{234}$.

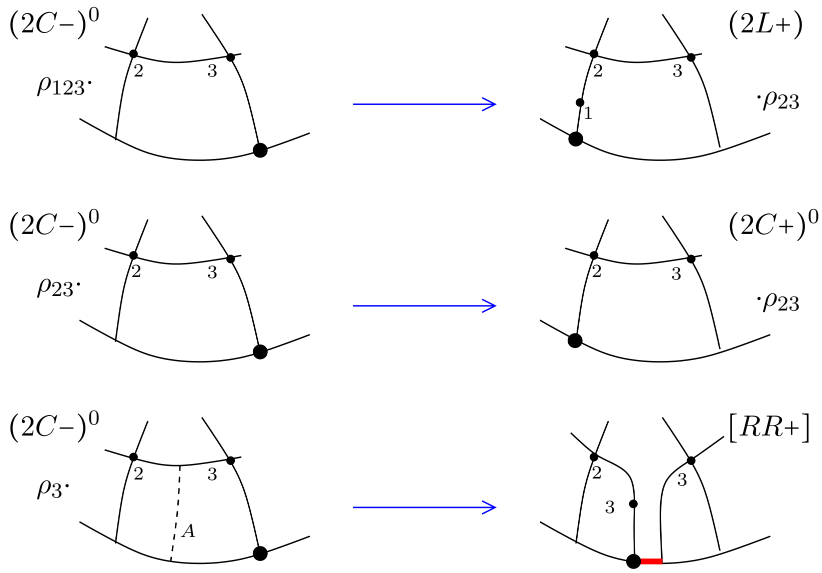


FIGURE 13. **Defining** $M(2C-)^0$**.** The element of type $(2C-)^0$ is on the left column. In the first line, $\rho = \rho_{123}$; in the second, ρ_{23}; and in the third, $\rho = \rho_3$. In all cases, $\rho^n = \rho_{23}$.

as in the case of $[LL-]$, thinking of S as the last interval before the root vertex. See Figure 14. Examples are provided by the cancelling terms

$$\mu_4^0(\rho_{34}, \rho_3, \rho_2, \mu_4^0(\rho_{12}, \rho_1, \rho_4, \rho_3)) \qquad \text{and} \qquad \mu_2^0(\mu_6^0(\rho_{34}, \rho_3, \rho_2, \rho_{12}, \rho_1, \rho_4), \rho_3) \tag{3.20}$$

which are of type $[LL-]$ and $(2C+)^0$ respectively. Cancelling terms

$$\mu_4^0(\rho_{234}, \rho_3, \rho_2, \mu_4^0(\rho_{12}, \rho_1, \rho_4, \rho_3)) \qquad \text{and} \qquad \mu_2^0(\mu_6^0(\rho_{234}, \rho_3, \rho_2, \rho_{12}, \rho_1, \rho_4), \rho_3) \tag{3.21}$$

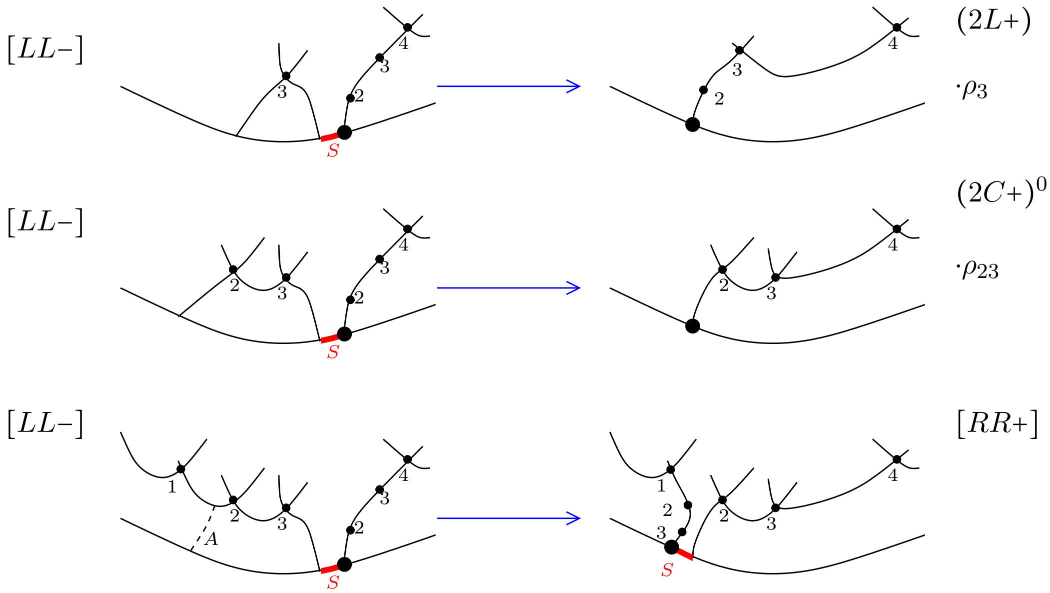


FIGURE 14. **Defining $M[LL-]$.** In these pictures, $\rho = \rho_{23}$.

are of types $[LL-]$ and $(2L+)$ respectively. Finally, terms

$$\mu_4^0(\rho_{34}, \rho_3, \rho_2, \mu_6^0(\rho_{12}, \rho_1, \rho_{41}, \rho_4, \rho_3, \rho_{23})) \qquad \text{and} \qquad \mu_4^0(\mu_6^0(\rho_{34}, \rho_3, \rho_2, \rho_{12}, \rho_1, \rho_{41}), \rho_4, \rho_3, \rho_{23}) \tag{3.22}$$

are of types $[LL-]$ and $[RR+]$ respectively. The map M in these three cases is illustrated in Figure 15.

An inverse to M is constructed by pushing out along an arc A and then moving the vertex one spot forwards. In cases where $|\rho| < |\rho^n|$, we must first pull an interval on the boundary in before pushing out along A: when the configuration is of type $[RR+]$, the interval to be pulled in is the interval S; when the configuration is of type $(R0+)$, the interval to be pulled in is the first interval after the root vertex. ☐

3.4. First properties of $\mathcal{A}_-$.

Lemma 3.23. *If $a^1, \dots, a^k$ are basic algebra elements, then for every non-zero operation,*

$$|\mu_k^w(a^1 \otimes \cdots \otimes a^k)| = 4w + \sum_{i=1}^k |a^i|.$$

Proof. The cases μ_2^0 and μ_0^1 are immediate. Otherwise, let Γ be a graph that contributes to the operation. From Definition 3.10, we see that $|\mu_k^w(a^1 \otimes \cdots \otimes a^k)|$ is the total number of distinguished sectors in Γ. On the other hand, the number of distinguished sectors in Γ is given by $4w + \sum_{i=1}^k |a^i|$, since the sectors visible from the boundary are the ones in $\sum_{i=1}^k |a^i|$, and the ones that are not occur 4 times in each short cycle, and there are w short cycles. ☐

The following properties of $\mathcal{A}_-$ will be useful when studying the gradings:

Lemma 3.24. *For any $n+2w > 4$, if $\rho^1, \dots, \rho^n$ are Reeb elements with $\mu_n^w(\rho^1, \dots, \rho^n) \neq 0$ then there exists an i so that ρ^i factors nontrivially (i.e., $\rho^i = \rho\rho'$ for some Reeb elements ρ, ρ'). In fact, there are at least two such integers i.*

Proof. If $n + 2w > 4$, the tile pattern contains at least 2 internal vertices. There is some internal edge connecting 2 distinct vertices that is visible from the boundary. That edge corresponds to a

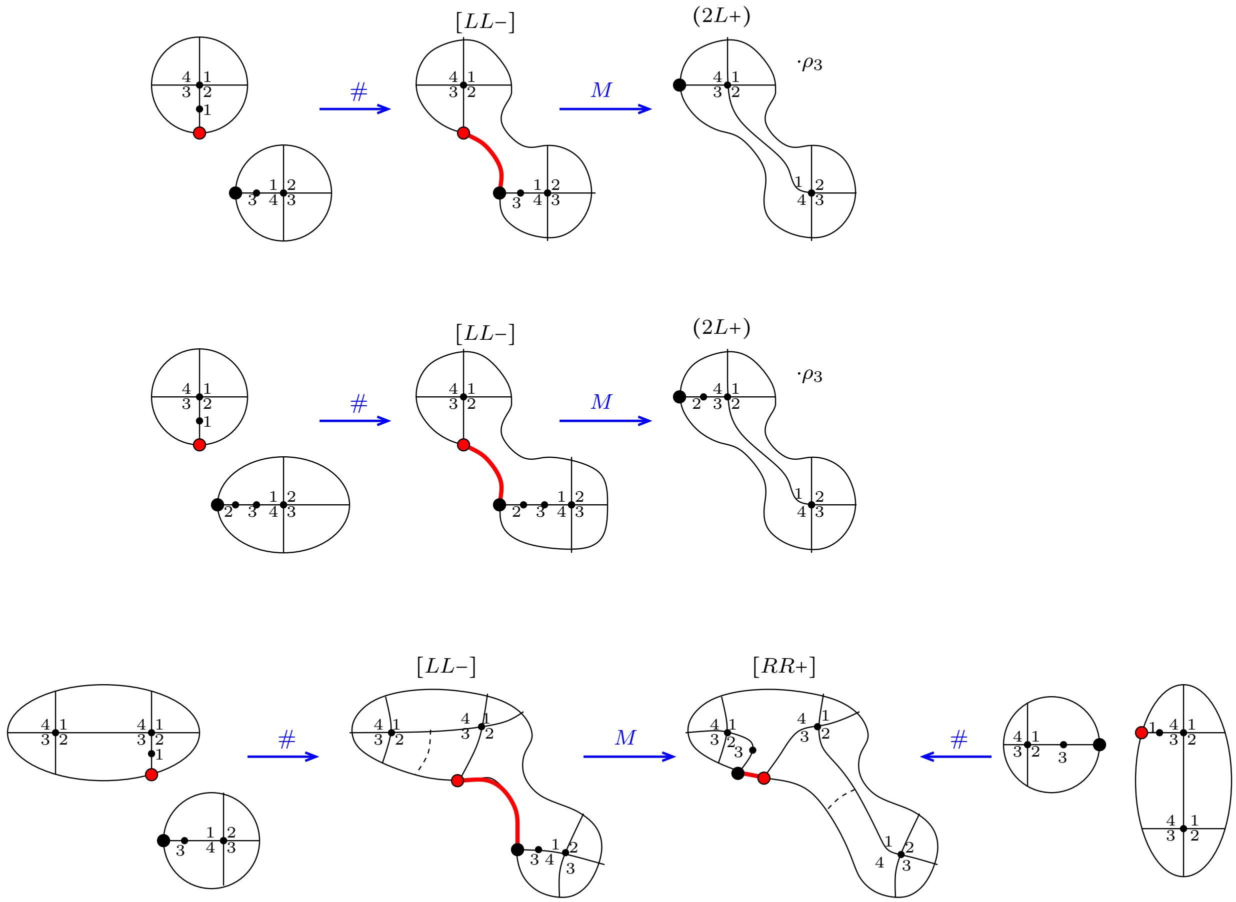


FIGURE 15. **Cancellation of terms of type $[LL-]$.** The three lines represent the cancellations from Equation (3.20), (3.21), and (3.22) respectively.

factorization of $\rho^i = \rho\rho'$. Further, there is either more than one such internal edge or there is one internal edge visible from the boundary in two different places; so, there are at least two integers i so that ρ^i factors. □

Lemma 3.25. *Suppose that $\rho^1, \dots, \rho^n$ are Reeb elements so that $\mu_n^w(\rho^1, \dots, \rho^n) = b \neq 0$ for some $w > 0$. Then there is an i and a factorization $\rho^i = \rho\rho'$ so that $\mu_{n+2}^{w-1}(\rho^1, \dots, \rho^{i-1}, \rho, \mu_0^1, \rho', \rho^{i+1}, \dots, \rho^n)$ has b as a term.*

Proof. Some edge on a short cycle is visible from the boundary. That edge corresponds to the desired factorization. □

Finally, we note a boundedness property of the algebra, which is useful for working with twisted complexes (type D structures) over it, and for defining certain kinds of bimodules over it.

Definition 3.26. *Recall that a weighted tree T with n inputs, i internal vertices, and total weight w has a dimension $\dim(T) = n - 1 + 2w - i$ (Equation (2.13)). A weighted algebra $\mathcal{A}$ is called* bonsai *if there is an integer N with the property that for all weighted trees T with $\dim(T) > N$, $\mu(T) = 0$. A weighted algebra $\mathcal{A}$ over $\mathbb{F}_2[U]$ is called* pre-filtered bonsai *if, for every m, the quotient $\mathcal{A}/U^m\mathcal{A}$ is bonsai.*

If $\mathcal{A}$ is pre-filtered bonsai, then its completion with respect to the sequence of (weighted A_∞) ideals

$$U\mathcal{A} \supset U^2\mathcal{A} \supset U^3\mathcal{A} \supset \dots$$

is filtered bonsai, in the sense of [LOT20, Definition 9.3].

The algebra $\mathcal{A}_-$ constructed here satisfies these hypotheses, according to the following:

Proposition 3.27. *If T is a weight w operation tree with n inputs then $\mu(T)$ maps $A^{\otimes n}$ into $U^{\dim(T)/2}A$. In particular, the weighted algebra $\mathcal{A}_-$ is pre-filtered bonsai.*

Proof. We start by verifying that μ^w_n maps $A^{\otimes n}$ to $U^{w+n/2-1}A$; this verifies the lemma when $T = \Psi^w_n$. This is vacuous for μ^0_2 and μ^1_0. Let Γ be a tiling pattern that contributes to μ^w_n. e the number of edges of Γ, v the number of vertices, x the number of valence 2 internal vertices, and d the number of valence 4 internal vertices. Then $v = n + x + d$, $2e = 2x + 4d + n$, and the Euler characteristic of Γ is $1 - w$; so $1 - w = (n + x + d) - (x + 2d + n/2) = n/2 - d$; and the operation contributes $U^d = U^{w+n/2-1}$, as desired.

Suppose now that T is a weighted operation tree with n inputs, e edges, i internal vertices, and weight w, where an internal vertex v has n_v inputs, weight w_v, and contribution u_v to the U power. It is elementary to see that

$$n - 1 = \sum_{\text{internal vertices } v} (n_v - 1).$$

Since we have already verified that $u_v = w_v + (n_v - 1)/2 - 1/2$, it follows that the total U contribution is

$$\sum u_v = w + \frac{n-1}{2} - \frac{i}{2} = \frac{\dim(T)}{2}. \qquad \square$$

3.5. **From tiling patterns to immersions.** The weighted algebra operations have an interpretation in terms of immersions of the disk into the torus, as follows.

Mark the torus, drawing a pair of curves α_1, α_2 on the torus so that $\alpha_1 \pitchfork \alpha_2$ in a single point p. Label the four corners near p by $1, 2, 3, 4$ in *clockwise* order around p. (The reason for this ordering is that we will sometimes think of p as a puncture; then this is the orientation induced on the circle, thought of as the circle at infinity on $T^2 \setminus \{p\}$.) Cutting along $\alpha_1 \cup \alpha_2$, we obtain a square with opposite sides identified, as shown in Figure 1, whose four corners are labelled $1, \dots, 4$.

Given a centered tiling pattern (Γ, Λ), there is an immersion of the disk to the torus (with possible branching at the corners), defined as follows. To each vertex v in Γ, we associate a copy of the standard square $S(v)$ with labelled corners (as in Figure 1). If Γ has an edge from v_1 to v_2, we perform an identification of $S(v_1)$ with $S(v_2)$ along a shared edge. The result of these identifications is a topological disk $\Delta(\Gamma)$, equipped with a tiling by squares. The map which projects each tile to T^2 induces a map from $\Delta(\Gamma)$ to T^2, which is an immersion away from the corners of $\Delta(\Gamma)$.

An inverse operation is given as follows. Given a square-tiled disk with an immersion as above, let $\alpha_1^\vee$ (respectively $\alpha_2^\vee$) be a disjoint, isotopic translate of α_2 (respectively α_1) so that $\alpha_i \pitchfork \alpha_i^\vee$ and $\alpha_1^\vee \pitchfork \alpha_2^\vee$ in a single point each. Then Γ is the preimage of $\alpha_1^\vee \cup \alpha_2^\vee$ under the map $u\colon \mathbb{D} \to T^2$.

If the tiling pattern has d internal vertices, then the degree of the corresponding immersion is $d/4$. The weight of the operation is the number of preimages of p in the interior of $\Delta(\Gamma)$.

This compelling geometric interpretation of the A_∞ operations, as counts of immersions, is closely connected to pseudo-holomorphic curve theory; but we will not make further use of it in this paper.

4. Gradings

This section is devoted to the gradings on $\mathcal{A}_-$. As in the $\widehat{HF}$ case, the algebra $\mathcal{A}_-$ is graded by a non-commutative group. (See Section 2.2 for a discussion of gradings of weighted A_∞-algebras by non-commutative groups, and [LOT18, Section 2.5] for a more leisurely discussion of gradings of A_∞-algebras by non-commutative groups.) Also as in the $\widehat{HF}$ case, there are three different groups that can be used to grade $\mathcal{A}_-$. Consider the pointed matched circle $(Z, \mathbf{a}, M, z)$ for the torus (Figure 1). The largest of the grading groups, denoted G', is a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow G' \longrightarrow H_1(Z, \mathbf{a}) \cong \mathbb{Z}^4 \longrightarrow 0. \tag{4.1}$$

(This is analogous to $G'(4)$ from [LOT18, Section 3.3.1].) The smallest of the grading groups, denoted $G(\mathbb{T})$, is a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow G(\mathbb{T}) \longrightarrow H_1(T^2) \cong \mathbb{Z}^2 \longrightarrow 0. \tag{4.2}$$

(This is analogous to $G(\mathcal{Z})$ from [LOT18, Section 3.3.2].) An intermediate grading group G is described in Section 4.2 (analogous to [LOT18, Section 11.1]). In Section 5, we will primarily work with this intermediate group.

The gradings by G and $G(\mathbb{T})$ are inherited from the grading by G', but through different processes. The grading by G is induced by a homomorphism $G' \twoheadrightarrow G$ (Section 4.2). The grading by $G(\mathbb{T})$ is defined using grading refinement data (Section 4.3).

We also define two other gradings, the winding number grading and total weight grading, in Section 4.4, and a (rather boring) mod-2 grading coming from the grading by $G(\mathbb{T})$ with respect to suitable choices, in Section 4.5.

4.1. **The big grading group.** Elements of $H_1(Z,\mathbf{a})$ are linear combinations of connected components of $Z \smallsetminus \mathbf{a}$. There is a map $m\colon H_1(Z,\mathbf{a}) \otimes H_0(\mathbf{a}) \to \mathbb{Z}$ given by defining, for I a component of $Z \smallsetminus \mathbf{a}$ and $p \in \mathbf{a}$, a multiplicity

$$m(I,p) = \begin{cases} 1/2 & \text{if } p \text{ is the terminal endpoint of } I \\ -1/2 & \text{if } p \text{ is the initial endpoint of } I \\ 0 & \text{if } p \text{ is disjoint from the closure of } I \end{cases}$$

and extending bilinearly. We can use m to define a linking pairing $L\colon H_1(Z,\mathbf{a}) \otimes H_1(Z,\mathbf{a}) \to \frac{1}{2}\mathbb{Z}$ by setting $L(\alpha_1,\alpha_2) = m(\alpha_2,\partial\alpha_1) = -m(\alpha_1,\partial\alpha_2)$.

The big grading group G' is the central extension as in Formula (4.1) with commutation relation $gh = \lambda^{2L([g],[h])}hg$ where $[g]$ denotes the image of g in $\mathbb{Z}^4$ and λ is a generator of the central $\mathbb{Z}$. Explicitly, consider the set of quintuples $(m;a,b,c,d) \in \big((\frac{1}{2}\mathbb{Z}) \times \mathbb{Z}^4\big)$. Define a multiplication by setting

$$\begin{aligned}(m;a,b,c,d)\cdot(m';a',b',c',d')& \\ &= \Big(m+m'+\tfrac{1}{2}\left|\begin{smallmatrix} a & b \\ a' & b'\end{smallmatrix}\right| + \tfrac{1}{2}\left|\begin{smallmatrix} b & c \\ b' & c'\end{smallmatrix}\right| + \tfrac{1}{2}\left|\begin{smallmatrix} c & d \\ c' & d'\end{smallmatrix}\right| + \tfrac{1}{2}\left|\begin{smallmatrix} d & a \\ d' & a'\end{smallmatrix}\right|, a+a', b+b', c+c', d+d'\Big)\end{aligned}$$

Lemma 4.3. *This operation makes $(\frac{1}{2}\mathbb{Z}) \times \mathbb{Z}^4$ into a group. Further, the elements $(-1/2;1,0,0,0)$, $(-1/2;0,1,0,0)$, $(-1/2;0,0,1,0)$ and $(-1/2;0,0,0,1)$ and $\lambda = (1;0,0,0,0)$ generate an index 2 subgroup isomorphic to G'.*

Proof. Straightforward; see also [LOT18, Proposition 3.37] (which also makes the index 2 subgroup explicit). ☐

Given an element $g = (m;a,b,c,d) \in G'$ we refer to m as *the Maslov component* of g and (a,b,c,d) as *the* spin^c *component* of g.

To define the grading of $\mathcal{A}_-$ by G', recall from Section 3.1 that the algebra $\mathcal{A}_-^{0,\mathrm{as}}$ has an $\mathbb{F}_2$-basis given by idempotents ι_0, ι_1 and chords $\rho_{i,i+1,\dots,i+n}$ for some $i \in \{1,2,3,4\}$ and $n \geq 0$. Each $\rho_{i,\dots,i+n}$ has a support $[\rho_{i,\dots,i+n}] \in H_1(Z,\mathbf{a})$.

Define $\mathrm{gr}'(\iota_0) = \mathrm{gr}'(\iota_1) = 0$. Define the grading of $\rho_{i,\dots,i+n}$ to be

$$\mathrm{gr}'(\rho_{i,\dots,i+n}) = \begin{cases} (-\frac{n+1}{4};[\rho_{i,\dots,i+n}]) & 4 \mid n+1 \\ (-1/2 - \lfloor \frac{n+1}{4} \rfloor;[\rho_{i,\dots,i+n}]) & 4 \nmid n+1. \end{cases} \tag{4.4}$$

For example:

$$\begin{aligned}
\mathrm{gr}'(\rho_1) &= (-1/2;1,0,0,0) & \mathrm{gr}'(\rho_2) &= (-1/2;0,1,0,0)\\
\mathrm{gr}'(\rho_3) &= (-1/2;0,0,1,0) & \mathrm{gr}'(\rho_4) &= (-1/2;0,0,0,1)\\
\mathrm{gr}'(\rho_{12}) &= (-1/2;1,1,0,0) & \mathrm{gr}'(\rho_{23}) &= (-1/2;0,1,1,0)\\
\mathrm{gr}'(\rho_{34}) &= (-1/2;0,0,1,1) & \mathrm{gr}'(\rho_{41}) &= (-1/2;1,0,0,1)\\
\mathrm{gr}'(\rho_{123}) &= (-1/2;1,1,1,0) & \mathrm{gr}'(\rho_{234}) &= (-1/2;0,1,1,1) & \dots\\
\mathrm{gr}'(\rho_{1234}) &= (-1;1,1,1,1) & \mathrm{gr}'(\rho_{2341}) &= (-1;1,1,1,1) & \dots\\
\mathrm{gr}'(\rho_{12341}) &= (-3/2;2,1,1,1) & &\dots.
\end{aligned}$$

Lemma 4.5. *Formula (4.4) defines a grading on the associative algebra $\mathcal{A}_-^{0,\mathrm{as}}$ by G'.*

Proof. First, note that $\mathrm{gr}'(\rho_{i,i+1,i+2,i+3}) = (-1;1,1,1,1)$ is central, and the grading satisfies

$$\mathrm{gr}'(\rho_{i,\dots,i+3})\,\mathrm{gr}'(\rho_{i,\dots,i+n}) = \mathrm{gr}'(\rho_{i,\dots,i+n+4}).$$

So, it suffices to verify that if $n, m \le 2$ and $\rho_{i,\dots,i+n}\cdot\rho_{j,\dots,j+m} \neq 0$ then

$$\mathrm{gr}'(\rho_{i,\dots,i+n})\,\mathrm{gr}'(\rho_{j,\dots,j+m}) = \mathrm{gr}'(\rho_{i,\dots,i+n}\cdot\rho_{j,\dots,j+m}).$$

By symmetry, we may assume $i = 1$. Further, it suffices to check the cases $n = 1$ or $m = 1$, as we can factor any chord into length 1 chords. So, we check:

$$\begin{aligned}
\mathrm{gr}'(\rho_1)\,\mathrm{gr}'(\rho_2) &= (-1/2;1,0,0,0)(-1/2;0,1,0,0) = (-1/2;1,1,0,0) = \mathrm{gr}'(\rho_{12})\\
\mathrm{gr}'(\rho_{12})\,\mathrm{gr}'(\rho_3) &= (-1/2;1,1,0,0)(-1/2;0,0,1,0) = (-1/2;1,1,1,0) = \mathrm{gr}'(\rho_{123})\\
\mathrm{gr}'(\rho_{123})\,\mathrm{gr}'(\rho_4) &= (-1/2;1,1,1,0)(-1/2;0,0,0,1) = (-1;1,1,1,1) = \mathrm{gr}'(\rho_{1234})\\
\mathrm{gr}'(\rho_1)\,\mathrm{gr}'(\rho_{23}) &= (-1/2;1,0,0,0)(-1/2;0,1,1,0) = (-1/2;1,1,1,0) = \mathrm{gr}'(\rho_{123})\\
\mathrm{gr}'(\rho_1)\,\mathrm{gr}'(\rho_{234}) &= (-1/2;1,0,0,0)(-1/2;0,1,1,1) = (-1;1,1,1,1) = \mathrm{gr}'(\rho_{1234}).
\end{aligned}$$

This proves the result. □

Next, to grade $\mathcal{A}_-^0$, define

$$\mathrm{gr}'(U) = (-1;1,1,1,1). \tag{4.6}$$

Proposition 4.7. *Formulas (4.4) and (4.6) define a grading on the A_∞-algebra $\mathcal{A}_-^0$ by G' with*

$$\lambda_d = \lambda = (1;0,0,0,0). \tag{4.8}$$

Proof. Since μ_n is U-equivariant and $\mathrm{gr}'(U)$ is central, it suffices to prove that for any n, basic elements $a_1,\dots,a_n$ and term $b \in \mu_n(a_1,\dots,a_n)$,

$$\mathrm{gr}'(b) = \lambda^{n-2}\,\mathrm{gr}'(a_1)\cdots\mathrm{gr}'(a_n). \tag{4.9}$$

Lemma 4.5 implies Equation (4.9) when $n = 2$.

The operation μ_3 vanishes identically. For μ_4, note that

$$\begin{aligned}
\mathrm{gr}'(\mu_4(\rho_4,\rho_3,\rho_2,\rho_1)) &= \mathrm{gr}'(U) = (-1;1,1,1,1)\\
\lambda^2\,\mathrm{gr}'(\rho_4)\,\mathrm{gr}'(\rho_3)\,\mathrm{gr}'(\rho_2)\,\mathrm{gr}'(\rho_1) &= \lambda^2(-3/2;0,0,1,1)(-3/2;1,1,0,0)\\
&= (2;0,0,0,0)(-3;1,1,1,1) = (-1;1,1,1,1).
\end{aligned}$$

Similar computations hold for cyclic permutations of the indices.

We prove Equation (4.9) in general by induction on the total length of the inputs $L = \sum_{i=1}^n |a_i|$.

If $L > 4$, there is some i so that a_i factors as a product $a_i = a_i'\cdot a_i''$ of Reeb elements. This is obvious when $n = 4$ and it follows from Lemma 3.24 when $n > 4$. Consider the A_∞ relation with inputs $(a_1,\dots,a_{i-1},a_i',a_i'',a_{i+1},\dots,a_n)$. One term in the relation is $\mu_n(a_1,\dots,\mu_2(a_i',a_i''),\dots,a_n) = b$. By

Lemma 3.6, the only other non-zero terms have the form $\mu_{n-k+2}(a_1,\dots,a_i',\mu_k(a_i'',\dots,a_{i+k}),\dots,a_n)$ or $\mu_{n-k+2}(a_1,\dots,\mu_k(a_{i-k+1},\dots,a_i'),a_i'',\dots,a_n)$, for some $k>2$; and there must be a non-zero term of at least one of these two forms. (In the language of Lemma 3.15, in the unweighted case, we cancel $\mathfrak{S}_1$ against patterns of type $\mathfrak{P}$, $\mathfrak{L}$, and $\mathfrak{R}$, all of which are of the above form.) So, in view of Lemma 3.23, the inductive hypothesis ensures that

$$\mathrm{gr}'(U^\ell)=\lambda^{n-k}\lambda^{k-2}\,\mathrm{gr}'(a_1)\cdots\mathrm{gr}'(a_i')\,\mathrm{gr}'(a_i'')\cdots\mathrm{gr}'(a_n)=\lambda^{n-2}\,\mathrm{gr}'(a_1)\cdots\mathrm{gr}'(a_n),$$

as desired. □

Finally, to define a grading on $\mathcal{A}_-$, we need to specify an element λ_w, so that μ_m^k has degree $\lambda_d^{m-2}\lambda_w^k$ (or equivalently, the formal variable t has grading λ_w^{-1}); see Section 2.2. Define:

$$\lambda_w=(1;1,1,1,1). \tag{4.10}$$

Theorem 4.11. *Formulas (4.4), (4.6), (4.8), and (4.10) define a grading on the weighted A_∞-algebra $\mathcal{A}_-$ by G'.*

Proof. The proof is by induction on the weight. The case of weight 0 is Proposition 4.7. Next, suppose that we know the statement for weight $k-1$, and that $\mu_n^k(a_1,\dots,a_n)=b\neq 0$. By Lemma 3.25, we can find an i and a factorization $a_i=a_i'a_i''$ so that $\mu_{n+2}^{k-1}(a_1,\dots,a_i',\mu_0^1,a_i'',\dots,a_n)$ has b as a term. Then

$$\begin{aligned}\mathrm{gr}'(b)&=\lambda_w^{k-1}\lambda_d^{n}\,\mathrm{gr}'(a_1)\cdots\mathrm{gr}'(a_i')\,\mathrm{gr}'(\mu_0^1)\,\mathrm{gr}'(a_i'')\cdots\mathrm{gr}'(a_n)\\&=\lambda_w^{k-1}\lambda_d^{n-1}\,\mathrm{gr}'(\mu_0^1)\,\mathrm{gr}'(a_1)\cdots\mathrm{gr}'(a_n).\end{aligned}$$

So, the result follows from the fact that

$$\begin{aligned}\lambda_w^{k-1}\lambda_d^{n-1}\,\mathrm{gr}'(\mu_0^1)&=(k-1;k-1,k-1,k-1,k-1)(n;0,0,0,0)(-1;1,1,1,1)\\&=(k+n-2;k,k,k,k)=\lambda_w^k\lambda_d^{n-2}.\end{aligned}$$

□

4.2. **The intermediate grading group.** As in the case of bordered $\widehat{HF}$ with torus (but not higher genus) boundary [LOT18, Section 11.1], there is a grading group G between G' and $G(\mathbb{T})$ which admits a homomorphism from G'. That is, let the *intermediate grading group* be

$$G=\left\{(m;a,b)\in(\tfrac{1}{2}\mathbb{Z})^3\;\middle|\;a+b\in\mathbb{Z},\,m+\frac{(2a+1)(a+b+1)+1}{2}\in\mathbb{Z}\right\}. \tag{4.12}$$

The second arithmetic condition for m, a, and b is equivalent to: $m\in\mathbb{Z}$ if and only if $a,b\in\mathbb{Z}$ and $a\equiv b \pmod 2$. The multiplication on G is

$$(m;a,b)\cdot(n;c,d)=(m+n+ad-bc;a+c,b+d). \tag{4.13}$$

The homological grading element is

$$\lambda_d=(1;0,0).$$

Lemma 4.14. *This operation makes G into a group.*

Proof. The only nontrivial part is verifying that G is closed under multiplication. The condition that $a+b\in\mathbb{Z}$ is certainly closed under multiplication. If we let

$$f(m;a,b)=m+\frac{(2a+1)(a+b+1)+1}{2}$$

then

$$f((m;a,b)\cdot(n;c,d))-f(m;a,b)-f(m;c,d)=ad-bc+\frac{4ac+2ad+2bc-2}{2}\equiv 2a(c+d)\pmod 1$$

which is an integer, since $c+d$ is. □

There is a homomorphism $G' \to G$ defined by

$$(j;a,b,c,d) \mapsto \left(j-d; \frac{a+b-c-d}{2}, \frac{-a+b+c-d}{2}\right). \tag{4.15}$$

(See also [LOT18, Section 11.1].) Composing with this homomorphism allows us to turn the grading by G' into a grading by G, with

$$\begin{aligned}
\mathrm{gr}(\rho_1) &= (-1/2;1/2,-1/2) & \mathrm{gr}(\rho_2) &= (-1/2;1/2,1/2) \\
\mathrm{gr}(\rho_3) &= (-1/2;-1/2,1/2) & \mathrm{gr}(\rho_4) &= (-3/2;-1/2,-1/2) \\
\mathrm{gr}(\rho_{12}) &= (-1/2;1,0) & \mathrm{gr}(\rho_{23}) &= (-1/2;0,1) \\
\mathrm{gr}(\rho_{34}) &= (-3/2;-1,0) & \mathrm{gr}(\rho_{41}) &= (-3/2;0,-1) \\
\mathrm{gr}(\rho_{123}) &= (-1/2;1/2,1/2) & \mathrm{gr}(\rho_{234}) &= (-3/2;-1/2,1/2) & \dots \\
\mathrm{gr}(\rho_{1234}) &= (-2;0,0) & \mathrm{gr}(\rho_{2341}) &= (-2;0,0) & \dots \\
\mathrm{gr}(\rho_{12341}) &= (-5/2;1/2,-1/2) & & \dots \\
\mathrm{gr}(U) &= (-2;0,0) & \lambda_w &= (0;0,0).
\end{aligned}$$

In particular, having the term $j-d$ instead of just j in Formula (4.15) ensured that $\lambda_w = (0;0,0)$.

4.3. **Grading refinements and the small grading group.** The small grading group $G(\mathbb{T})$ is a central extension as in Formula (4.2) with commutation relation

$$gh = \lambda^{2[g]\cdot[h]}hg$$

where $[g]$ denotes the image of g in $H_1(F)$ and $\cdot$ is the intersection pairing; that is, the central extension corresponding to the 2-cocycle on $H_1(T^2)$ given by the intersection form (not twice it). We have several explicit models for this group:

Lemma 4.16. *The group $G(\mathbb{T})$ is isomorphic to the following:*

(SG-1) The subgroup $\{(m;a,b,c,d) \in G' \mid b = a+c,\ d = 0\} \subset G'$.
(SG-2) The subquotient $\{(m;a,b,c,d) \in G'/(1;1,1,1,1) \mid a+c = b+d\} \subset G'/\langle(1;1,1,1,1)\rangle$.
(SG-3) The subgroup $\{(m;a,b) \in G \mid a,b \in \mathbb{Z}\} \subset G$.

Proof. Model (SG-1) for the small grading group is the one given in our first paper and is identified with the abstract definition there [LOT18, Section 3.3.2]. The isomorphism between model (SG-1) and (SG-2) is clear. The isomorphism between model (SG-1) and (SG-3) is given by $(m;a,b) \mapsto (m;a,a+b,b,0)$. ☐

We will most often use the third of these ways of realizing $G(\mathbb{T})$, viewing elements of $G(\mathbb{T})$ as triples $(m;a,b) \in \frac{1}{2}\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with $m + \frac{a+b}{2} \in \mathbb{Z}$, equipped with group law

$$(m;a,b)\cdot(m';a',b') = \left(m+m'+\left|\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right|; a+a', b+b'\right).$$

The grading elements are, again,

$$\lambda_d = \lambda = (1;0,0) \qquad\qquad \lambda_w = (0;0,0).$$

Note that the surjection $G' \to G$ does not send $G(\mathbb{T}) \subset G'$ to $G(\mathbb{T}) \subset G$.

To define a grading on $\mathcal{A}_-$ by G, we use the notion of grading refinement data, which we review first.

4.3.1. *Grading refinement data in general.* The following is an abstract reformulation of material from our earlier papers [LOT18, Section 3.3.2], [LOT15, Section 3.2.1].

Recall that a groupoid is a category in which every morphism is invertible. All groupoids relevant later will also be *connected*, i.e., $\mathrm{Hom}(i,j)\neq\varnothing$ for all objects i,j. Since we will think of a groupoid as a group with many objects, given a groupoid $\mathcal{G}$ and morphisms $g\in\mathrm{Hom}_{\mathcal{G}}(x,y)$, $h\in\mathrm{Hom}_{\mathcal{G}}(y,z)$, let

$$g\cdot h=h\circ g.$$

From here on, we will drop the subscript $\mathcal{G}$ from $\mathrm{Hom}_{\mathcal{G}}$ when it will not cause confusion.

Definition 4.17. *A* central element *of $\mathcal{G}$ consists of an element $\lambda_x\in\mathrm{Hom}(x,x)$ for each object $x\in\mathrm{Obj}(\mathcal{G})$ so that for any $g\in\mathrm{Hom}(x,y)$, $\lambda_x\cdot g=g\cdot\lambda_y$. We will typically denote the collection of elements λ_x by λ.*

Let $\Bbbk=\bigoplus_{x\in X}\mathbb{F}_2$ for some finite set X and let $\mathcal{A}$ be a strictly unital weighted algebra over $\Bbbk$. So, each $\mathbf{x}\in X$ corresponds to some idempotent $I_{\mathbf{x}}=\mu_2^0(\mathbf{x},1)\in\mathcal{A}$.

Definition 4.18. *Let $\mathcal{G}$ be a groupoid and λ_d and λ_w central elements of G. A* grading on $\mathcal{A}$ by $\mathcal{G}$ *consists of a map of sets $\pi\colon X\to\mathrm{Obj}(\mathcal{G})$ and, for each $\mathbf{x},\mathbf{y}\in X$, a decomposition*

$$I_{\mathbf{x}}\cdot\mathcal{A}\cdot I_{\mathbf{y}}=\bigoplus_{g\in\mathrm{Hom}(\pi(\mathbf{x}),\pi(\mathbf{y}))}(I_{\mathbf{x}}\cdot\mathcal{A}\cdot I_{\mathbf{y}})_g$$

satisfying the following property. For each pair of integers $w,m\geq 0$, $(w,m)\neq(0,0)$, sequence $\mathbf{x}_0,\dots,\mathbf{x}_m\in X$, elements $g_i\in\mathrm{Hom}(\pi(\mathbf{x}_{i-1}),\pi(\mathbf{x}_i))$ for $i=1,\dots,m$, and elements $a_i\in(I_{\mathbf{x}_{i-1}}\cdot\mathcal{A}\cdot I_{\mathbf{x}_i})_{g_i}$, we have

$$\mu_m^w(a_1,\dots,a_m)\in(I_{\mathbf{x}_0}\cdot\mathcal{A}\cdot I_{\mathbf{x}_m})_{\lambda_d^{m-2}\lambda_w^w g_1\cdots g_m}.$$

This generalizes the notion of group-valued gradings. If G' is a group, there is an associated groupoid with one object. Any G'-graded algebra $\mathcal{A}$ can be viewed as graded by this associated groupoid.

Definition 4.19. *With notation as in Definition 4.18,* grading refinement data *consists of an element $\mathbf{x}_0\in X$, called the* base idempotent*, and for every object $\mathbf{x}\in X$ an element $\psi(\mathbf{x})\in\mathrm{Hom}(\mathbf{x}_0,\mathbf{x})$.*

Definition 4.20. *Let $\mathcal{A}$ be a $\mathcal{G}$-graded algebra, and let $G=\mathrm{Hom}_{\mathcal{G}}(\mathbf{x}_0,\mathbf{x}_0)$. Given grading refinement data $\{\psi\}$, there is an induced G-grading on $\mathcal{A}$, denoted $\mathcal{A}_\psi$, specified by*

$$\mathrm{gr}_\psi(I_{\mathbf{x}}\cdot a\cdot I_{\mathbf{y}})=\psi(\mathbf{x})\cdot\mathrm{gr}(a)\cdot\psi(\mathbf{y})^{-1}$$

for each $\mathcal{G}$-homogeneous element a with $a=I_{\mathbf{x}}\cdot a\cdot I_{\mathbf{y}}$. We call this the G-valued grading the refined grading with respect to ψ.

We can also un-refine gradings:

Lemma 4.21. *Let $\mathcal{A}$ be a weighted A_∞-algebra over $\Bbbk$, $(\mathcal{G},\lambda_d,\lambda_w)$ be a groupoid with distinguished central elements, and $(\mathbf{x}_0,\{\psi_{\mathbf{x}}\})$ be grading refinement data. Suppose $\mathcal{A}$ is graded by $(G=\mathrm{Hom}_{\mathcal{G}}(\mathbf{x}_0,\mathbf{x}_0),\lambda_d,\lambda_w)$. Let gr_ψ denote this G-valued grading. Then setting*

$$\mathrm{gr}(a)=\psi(\iota_i)^{-1}\,\mathrm{gr}_\psi(a)\psi(\iota_j)\qquad\text{if } \iota_i a\iota_j=a$$

defines a grading on $\mathcal{A}$ by $\mathcal{G}$.

Proof. This is immediate from the definitions. ☐

4.3.2. *Grading refinement data for the torus algebra.* Consider the groupoid $\mathcal{G}$ with two objects, 0 and 1, and

$$\mathrm{Hom}(i,j) = \left\{(m;a,b) \in G \;\middle|\; a + \frac{i+j}{2} \in \mathbb{Z}\right\}.$$

Lemma 4.22. *Multiplication in G makes $\mathcal{G}$ into a groupoid.*

Proof. It is straightforward to verify that $\mathcal{G}$ is closed under multiplication (composition) and inverses. □

Observe that for $i \in \{0,1\}$, we have that $\mathrm{Hom}(i,i) = G(\mathbb{T})$, the smallest of the three grading groups.

The element $\lambda \in G$ induces an element $\lambda \in \mathrm{Hom}_{\mathcal{G}}(i,i)$, $i \in \{0,1\}$, forming a central element of $\mathcal{G}$.

Lemma 4.23. *The G-grading on $\mathcal{A}_-$ induces a $\mathcal{G}$-grading on $\mathcal{A}_-$.*

Proof. It suffices to verify that the grading of each algebra element a with $\iota_i a \iota_j = a$ $(i,j \in \{0,1\})$ lies in $\mathrm{Hom}(i,j)$. To see this, it suffices to check the result for ρ_1, ρ_2, ρ_3, and ρ_4, which is straightforward. □

We define grading refinement data for the torus algebra. Choose the base idempotent $x_0 = \iota_0$, and let $\psi(\iota_0) = e$ and $\psi(\iota_1) = \mathrm{gr}(\rho_1)$. This induces a grading gr_ψ on $\mathcal{A}_-$ by $G(\mathbb{T})$. Explicitly, we have:

$$\begin{aligned}
\mathrm{gr}_\psi(\rho_1) &= (0;0,0) & \mathrm{gr}_\psi(\rho_2) &= (-1/2;1,0) & \\
\mathrm{gr}_\psi(\rho_3) &= (0;-1,1) & \mathrm{gr}_\psi(\rho_4) &= (-5/2;0,-1) & \\
\mathrm{gr}_\psi(\rho_{12}) &= (-1/2;1,0) & \mathrm{gr}_\psi(\rho_{23}) &= (1/2;0,1) & \\
\mathrm{gr}_\psi(\rho_{34}) &= (-3/2;-1,0) & \mathrm{gr}_\psi(\rho_{41}) &= (-5/2;0,-1) & \\
\mathrm{gr}_\psi(\rho_{123}) &= (1/2;0,1) & \mathrm{gr}_\psi(\rho_{234}) &= (-2;0,0) & \dots\\
\mathrm{gr}_\psi(\rho_{1234}) &= (-2;0,0) & \mathrm{gr}_\psi(\rho_{2341}) &= (-2;0,0) & \dots\\
\mathrm{gr}_\psi(\rho_{12341}) &= (-2;0,0) & \mathrm{gr}_\psi(\rho_{23412}) &= (-5/2;1,0) & \dots\\
\mathrm{gr}_\psi(U) &= (-2;0,0) & \lambda_w &= (0;0,0).
\end{aligned}$$

4.4. **The winding number and total weight.** The algebra $\mathcal{A}_-^0$ has two other gradings: the *length* and the *multiplicity at the chord* ρ_4. The length was defined in Section 3.1; recall in particular that $|U| = 4$. We will think of the multiplicity at ρ_4 as the winding number, and denote it by wn. Specifically, define $\mathrm{wn}(\rho_1) = \mathrm{wn}(\rho_2) = \mathrm{wn}(\rho_3) = 0$ and $\mathrm{wn}(\rho_4) = \mathrm{wn}(U) = 1$, and extend wn to all of $\mathcal{A}_-^0$ by $\mathrm{wn}(ab) = \mathrm{wn}(a) + \mathrm{wn}(b)$. Equivalently, there is a homomorphism $G' \to \mathbb{Z}$ by $(m;a,b,c,d) \mapsto d$ and wn is the composition of gr' with this projection. In particular, wn sends λ_w to 1.

Sometimes, it is convenient to combine the above gradings. In Section 5, we will formulate a uniqueness result for the algebra, phrased in terms of a grading

$$\gamma = \mathrm{gr} \times \mathrm{wn} \tag{4.24}$$

with values in the group $\Gamma = G \times \mathbb{Z}$.

4.5. **A mod-2 grading.** We conclude with the rather dull mod-2 grading:

Lemma 4.25. *The map $\epsilon\colon G(\mathbb{T}) \to \mathbb{Z}/2\mathbb{Z}$, defined by $\epsilon(m;a,b) \equiv m + \frac{a-b}{2} + ab \pmod 2$ is a homomorphism. Further, the map ϵ sends λ to 1 and the grading $\mathrm{gr}_\psi(a)$ of every homogeneous algebra element a to 0.*

Proof. To verify that ϵ is a homomorphism, observe that

$$\begin{aligned}\epsilon(m;a,b)+\epsilon(n;c,d) &= m+n+\frac{a-b+c-d}{2}+ab+cd\\ &\equiv m+n+ad-bc+\frac{a+c-(b+d)}{2}+(a+c)(b+d)\\ &= \epsilon(m+n+ad-bc;a+c,b+d).\end{aligned}$$

To verify the statement about the gradings of algebra elements, it suffices to compute $\epsilon(\mathrm{gr}_\psi(\rho_1)) = \epsilon(\mathrm{gr}_\psi(\rho_2)) = \epsilon(\mathrm{gr}_\psi(\rho_3)) = \epsilon(\mathrm{gr}_\psi(\rho_4)) = \epsilon(\mathrm{gr}_\psi(U)) = 0$. Similarly, $\epsilon(\lambda) = 1 + \frac{0-0}{2} + 0 = 1$. ☐

Remark 4.26. A homomorphism $G(\mathbb{T}) \to \mathbb{Z}/2\mathbb{Z}$ sending λ to 0 is an element of $H^1(\mathbb{T};\mathbb{Z}/2\mathbb{Z})$. The homomorphisms sending λ to 1 are obtained from ϵ by adding one of these four maps.

Note that since λ is a commutator in G', there is no homomorphism $G' \to \mathbb{Z}/2\mathbb{Z}$ sending λ to 1.

Remark 4.27. For the case of $\widehat{HF}$, modulo-2 gradings on the bordered algebras and modules have been constructed previously by Petkova [Pet18] and Hom-Lidman-Watson [HLW17], for surfaces of arbitrary genus.

5. Abstract approach to existence and uniqueness of the torus algebra

The goal of this section is to provide a more algebraic characterization of the weighted algebra $\mathcal{A}_-$.

5.1. A_∞ deformations and Hochschild cohomology.

In this section, we show that A_∞ deformations of an associative algebra are controlled by Hochschild cohomology. This material is well-known, but we recall it here for the reader's convenience. See, for instance, [Sei15, Section 3a], [LP11], [She15, Section 2.3] and the references therein. In particular, the group-graded context of [She15, Section 2.3] is close to the setting of interest here (cf. Section 6).

Fix a commutative ring $\Bbbk$; for us, $\Bbbk$ will be a finite direct sum of copies of $\mathbb{F}_2$ or $\mathbb{F}_2[U]$. Unless otherwise specified tensor products are over $\Bbbk$.

By an A_n-*algebra* we mean a projective $\Bbbk$-module A together with $\Bbbk$-linear maps $\{\mu_i\colon A^{\otimes i} \to A\}_{i=2}^n$ satisfying those A_∞-algebra relations with at most $n+1$ inputs. (So, in this section, we are only considering A_n-algebras with trivial differential.) A *homomorphism* of A_n-algebras $f\colon \mathcal{A}\to\mathcal{B}$ consists of maps $\{f_i\colon A^{\otimes i}\to B\}_{i=1}^{n-1}$, satisfying the relations for an A_∞-algebra homomorphism with at most n inputs.

Fix an associative algebra A and an augmentation $\epsilon\colon A\to\Bbbk$. Let $A_+ = \ker(\epsilon)$ be the augmentation ideal and let $\Pi\colon A\to A_+$ denote the projection to the augmentation ideal.

We are interested in A_∞-algebra structures on A so that $\mu_1 = 0$, μ_2 is the given multiplication on A, and the operations are strictly unital and $\Bbbk$-multilinear. (Unitality of the A_∞-algebra can be formulated as the condition that the operations μ_i for $i > 2$ satisfy $\mu_i = \mu_i \circ (\overbrace{\Pi\otimes\cdots\otimes\Pi}^{i})$.) We will define such structures inductively. The obstructions to extending, and different choices of extensions, will be given in terms of the Hochschild cochains.

Definition 5.1. *The* (reduced) bar complex *of an augmented associative algebra A over $\Bbbk$ is given by*

$$\mathrm{Bar}(A) = A\otimes A \longleftarrow A\otimes A_+\otimes A \longleftarrow A\otimes A_+\otimes A_+\otimes A \longleftarrow \cdots$$

where the differential $A\otimes A_+^{\otimes n}\otimes A \to A\otimes A_+^{\otimes(n-1)}\otimes A$ is specified by

$$a_0\otimes\cdots\otimes a_{n+1} \mapsto \sum_{i=0}^{n} a_0\otimes\cdots\otimes a_i a_{i+1}\otimes a_{i+2}\otimes\cdots\otimes a_{n+1}. \tag{5.2}$$

This is a chain complex of (A,A)-bimodules, and in fact is a resolution of A.

The Hochschild cochain complex *of A is given by*

$$HC^*(A) = \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(\mathrm{Bar}(A), A). \tag{5.3}$$

The grading is chosen so that $\mathrm{Hom}(A\otimes A, A)$ *lies in grading* 0*.*

Note that we can absorb the A factors on the left and right into the Hom, to get

$$HC^n(A) = \mathrm{Hom}_{\Bbbk\otimes\Bbbk}(A_+^{\otimes n}, A). \tag{5.4}$$

Let δ denote the differential on this model for the Hochschild complex.

Remark 5.5. In the terminology of [LOT15], $HC^*(A) = \mathrm{Mor}({}^A[\mathbb{I}]_A, {}^A[\mathbb{I}]_A)$, the complex of type DA bimodule morphisms from the identity DA bimodule of A to itself.

Definition 5.6. *Given an associative algebra A (with no differential), an A_n-algebra structure $\mathcal{A}$ on A with μ_2 the given multiplication on A (and $\mu_1 = 0$) is an A_n* deformation of A*.*

Proposition 5.7. *Let A be an associative algebra and let $\mathcal{A}$ be an A_{n-1}-algebra that is a deformation of A. Then there is a Hochschild cochain obstruction class $\mathfrak{O}_n \in HC^{n+1}(A)$ so that:*

(A$\mathfrak{O}$-1) $\mathfrak{O}_n$ is a cocycle.
(A$\mathfrak{O}$-2) $\mathfrak{O}_n$ is a coboundary if and only if there is an operation μ_n making $\mathcal{A}$ into an A_n-algebra; indeed, the operation $\mu_n \in HC^n(A)$ is a cochain with $\delta(\mu_n) = \mathfrak{O}_n$.
(A$\mathfrak{O}$-3) If μ_n and μ'_n are cochains with $\delta(\mu_n) = \delta(\mu'_n) = \mathfrak{O}_n$ then $\mu_n - \mu'_n$ is itself a coboundary if and only if there is an A_n-homomorphism f between the corresponding structures with $f_1 = \mathbb{I}$ and $f_j = 0$ for $1 < j < n-1$.
(A$\mathfrak{O}$-4) If $\mathcal{A}$ and $\mathcal{A}'$ are two A_n deformations with $\mu_i = \mu'_i$ for all $i < n$, and $\mu_n - \mu'_n$ is a coboundary, then their respective obstruction cocycles $\mathfrak{O}_{n+1}, \mathfrak{O}'_{n+1} \in HC^{n+2}(A)$ are cohomologous.

There are analogous statements for maps. In particular, given A_n deformations $\mathcal{A}$ and $\mathcal{A}'$ of A and an A_{n-1}-homomorphism $f\colon \mathcal{A} \to \mathcal{A}'$ so that $f_1\colon A \to A$ is the identity map, there is an obstruction class $\mathfrak{F}_n \in HC^n(A)$ so that:

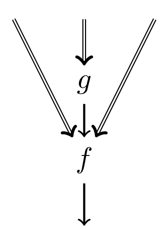
(A$\mathfrak{F}$-1) $\mathfrak{F}_n$ is a cocycle.
(A$\mathfrak{F}$-2) $\mathfrak{F}_n$ is a coboundary if and only if there is an A_n-homomorphism extending f.

Proof. Given a map $f\colon A_+^{\otimes i} \to A$, we will also let f denote the extension $f\circ\Pi^{\otimes i}\colon A^{\otimes i} \to A$. To define the obstruction cocycle $\mathfrak{O}_n$, we use a composition map

$$\mathrm{Hom}(A_+^{\otimes i}, A)\otimes\mathrm{Hom}(A_+^{\otimes j}, A) \to \mathrm{Hom}(A_+^{\otimes i+j-1}, A),$$

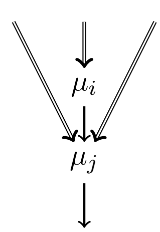
which we denote by $\star$, defined by

$$f_i \star g_j = \sum_{\ell=0}^{i-j+1} f_i(\mathbb{I}_{A_+^{\otimes \ell}} \otimes \circ g_j \otimes \mathbb{I}_{A_+^{\otimes(i-\ell-1)}}) = \tag{5.8}$$



(This uses the extension of f, via pre-composition with Π.) Let

$$\mathfrak{O}_n = \sum_{\substack{i,j\geq 3\\ i+j=n+2}} \mu_i \star \mu_j = \sum_{\substack{i,j\geq 3\\ i+j=n+2}} \tag{5.9}$$



The A_∞ relation with $n+1$ inputs is the condition that

$$\delta\mu_n = \sum_{\substack{i,j\geq 3\\ i+j=n+2}} \mu_i \star \mu_j; \tag{5.10}$$

i.e., $\delta\mu_n = \mathfrak{O}_n$. Property (A$\mathfrak{O}$-2) follows.

To verify that $\mathfrak{O}_n$ is a cocycle, it helps to have the following easily verified identity: for any $f_i\colon A_+^{\otimes i} \to A$ and $g_j\colon A_+^{\otimes j} \to A$,

$$\delta(f_i \star g_j) = (\delta f_i) \star g_j + f_i \star (\delta g_j) + \mu_2(f_i, g_j) + \mu_2(g_j, f_i). \tag{5.11}$$

Using this identity and Equation (5.10), we see that

$$\begin{aligned}\delta\mathfrak{O}_n &= \sum_{\substack{i,k\geq 3\\ i+k=n+2}} (\delta\mu_i)\star\mu_k + \mu_i \star (\delta\mu_k) + \mu_2(\mu_i,\mu_k) + \mu_2(\mu_k,\mu_i)\\ &= \left(\sum_{\substack{i,j,k\geq 3\\ i+j+k=n+4}} (\mu_i\star\mu_j)\star\mu_k\right) + \left(\sum_{\substack{i,j,k\geq 3\\ i+j+k=n+4}} \mu_i\star(\mu_j\star\mu_k)\right).\end{aligned}$$

Note that $\star$ is not associative; rather,

$$(a\star b)\star c + a\star(b\star c) = a\circ(\mathbb{I}\otimes b\otimes\mathbb{I}\otimes c\otimes\mathbb{I})\circ\Delta^5 + a\circ(\mathbb{I}\otimes c\otimes\mathbb{I}\otimes b\otimes\mathbb{I})\circ\Delta^5, \tag{5.12}$$

where

$$\Delta^m\colon \mathcal{T}^*(A_+) \to \overbrace{(\mathcal{T}^*(A_+)\otimes\cdots\otimes\mathcal{T}^*(A_+))}^{m} \tag{5.13}$$

denotes the comultiplication map on $\mathcal{T}^*(A_+)$ applied $m-1$ times. Property (A$\mathfrak{O}$-1) follows.

To verify Property (A$\mathfrak{O}$-3), we argue as follows. Let $\mathcal{A}$ and $\mathcal{A}'$ be the A_n algebras such that $\mu_i = \mu_i'$ for all $i<n$, but with possibly different μ_n and μ_n'. The A_∞ relation with n inputs for a map $f\colon\mathcal{A}\to\mathcal{A}'$ with $f_1=\mathbb{I}$ and $f_j = 0$ for $1\leq j<n-1$ is precisely the condition that

$$\delta(f_{n-1}) = \mu_n - \mu_n'.$$

To verify Property (A$\mathfrak{O}$-4), observe that the hypothesis that $\mu_i=\mu_i'$ for all $i<n$ ensures that

$$\mathfrak{O}_{n+1} - \mathfrak{O}_{n+1}' = \mu_n\star\mu_3 + \mu_3\star\mu_n + \mu_n'\star\mu_3 + \mu_3\star\mu_n'.$$

Our hypotheses also give us a c_{n-1} with

$$\delta c_{n-1} = \mu_n - \mu_n'.$$

Now, using Equation (5.11) (and using the fact that $\delta\mu_3 = 0$), we see that

$$\mathfrak{O}_{n+1} - \mathfrak{O}_{n+1}' = \delta(c_{n-1}\star\mu_3 + \mu_3\star c_{n-1}).$$

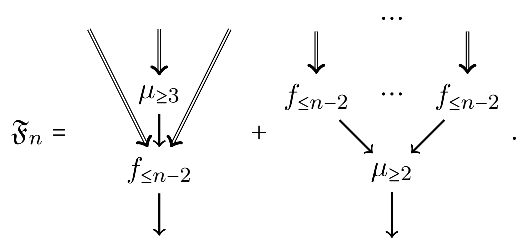
We now turn to the statements for maps. Fix A_n deformations $\mathcal{A}$ and $\mathcal{A}'$ of A and an A_{n-1}-homomorphism $f\colon\mathcal{A}\to\mathcal{A}'$ with f_1 the identity map. To define the obstruction cocycle $\mathfrak{F}_n$, we introduce some notation. Extend $\{f_i\colon A_+^{\otimes i}\to A\}_{i\leq n-2}$ to a map $F_n\colon A^{\otimes n}\to\mathcal{T}^*A$ by summing over all ways of parenthesizing $A^{\otimes n}$ into strings of $\leq n-2$ elements and applying the appropriate f_i to each string. For example, for $n=4$,

$$\begin{aligned}&F_4(a_1\otimes a_2\otimes a_3\otimes a_4)\\ &= f_1(a_1)\otimes f_1(a_2)\otimes f_1(a_3)\otimes f_1(a_4) + f_2(a_1\otimes a_2)\otimes f_1(a_3)\otimes f_1(a_4) + f_1(a_1)\otimes f_2(a_2\otimes a_3)\otimes f_1(a_4)\\ &\qquad + f_1(a_1)\otimes f_1(a_2)\otimes f_2(a_3\otimes a_4) + f_2(a_1\otimes a_2)\otimes f_2(a_3\otimes a_4).\end{aligned}$$

Define the obstruction class by

$$\mathfrak{F}_n = \Big(\sum_{\substack{j\geq 3\\ i+j=n+1}} f_i \star \mu_j \Big) + \Big(\mu \circ F_n\Big) \colon A_+^{\otimes n} \to A, \tag{5.14}$$

or graphically by



The condition on a map $f_{n-1}\colon A_+^{\otimes n-1} \to A$ that

$$\delta f_{n-1} = \mathfrak{F}_n$$

is precisely the A_∞ relation with n inputs for the map $\{f_i\}_{i=1}^{n-1}$, so Part (A$\mathfrak{F}$-2) holds.

To verify that $\mathfrak{F}$ is a cocycle, we introduce a little more notation. Given $g_j\colon A_+^{\otimes j} \to A$, let $\widetilde{g_j}\colon \mathcal{T}^*(A_+) \to \mathcal{T}^*(A)$ be the map defined by

$$\widetilde{g_j}(a_1 \otimes \cdots \otimes a_i) = \sum_{m=0}^{i-j} a_1 \otimes \cdots \otimes a_m \otimes g_j(a_{m+1}, \dots, a_{m+j}) \otimes a_{m+j+1} \otimes \cdots \otimes a_i. \tag{5.15}$$

In particular, $f \star g = f \circ \widetilde{g_j}$. (Again, recall that we are abusing notation so $f = f \circ \Pi^{\otimes n}$.) There is a differential $D\colon \mathrm{Hom}(\mathcal{T}^*(A), \mathcal{T}^*(A)) \to \mathrm{Hom}(\mathcal{T}^*(A), \mathcal{T}^*(A))$ defined by

$$D(\Phi) = \widetilde{\mu_2} \circ \Phi + \Phi \circ \widetilde{\mu_2}. \tag{5.16}$$

The fact that μ_2 is associative implies that $D^2 = 0$.

Given maps $\{\phi_i \in \mathrm{Hom}_{\Bbbk\otimes\Bbbk}(A_+^{\otimes j}, A)\}_{j=1}^m$ with $\phi_1 = \mathbb{I}$, let $\widetilde{\widetilde{\phi}}\colon \mathcal{T}^*(A_+) \to \mathcal{T}^*(A)$ denote the map defined by

$$\widetilde{\widetilde{\phi}}(a_1, \dots, a_t) = \sum_{i_1+\cdots+i_k=t} \phi_{i_1}(a_1, \dots, a_{i_1}) \otimes \phi_{i_2}(a_{i_1+1}, \dots, a_{i_1+i_2}) \otimes \cdots \otimes \phi_{i_k}(a_{m-i_k+1}, \dots a_t). \tag{5.17}$$

Given maps $\{f_i \in \mathrm{Hom}(A^{\otimes i}, A)\}_{i=1}^m$, we will let $f = \sum_{i=1}^m f_i\colon \mathcal{T}^*(A) \to A$, and similarly let $f_{\leq n} = \sum_{i=1}^n f_i$, and so forth. Conversely, let $(g)_m$ denote the m-input component of g. Observe that

$$\delta((f \circ \widetilde{\widetilde{\phi}})_m) = (\delta(f) \circ \widetilde{\widetilde{\phi}})_{m+1} + (f \circ D\widetilde{\widetilde{\phi}})_{m+1} + \sum_{\substack{i,j\\ i+j=m+1, i>1}} \mu_2(\phi_i, (f \circ \widetilde{\widetilde{\phi}})_j) + \mu_2((f \circ \widetilde{\widetilde{\phi}})_j, \phi_i). \tag{5.18}$$

In this notation,

$$\mathfrak{F}_n = f_{\leq n-2} \star \mu_{\geq 3} + \mu_{\geq 2} \circ \widetilde{\widetilde{f_{\leq n-2}}} = f_{\leq n-2} \star \mu_{\geq 3} + \mu_{\geq 3} \circ \widetilde{\widetilde{f_{\leq n-2}}} + \sum_{\substack{i>1,\ j>1\\ i+j=n+1}} \mu_2(f_i, f_j).$$

By hypothesis, we have maps $\{f_i\colon A^{\otimes i} \to A\}_{i=1}^{n-2}$, that satisfy the A_∞ relation with k inputs for all $k \leq n-1$. As noted earlier, the A_∞ relation on these maps with k inputs can be formulated as

$$\delta(f_{k-1}) = f_{\leq k-2} \star \mu_{\geq 3} + \mu_{\geq 2} \circ \widetilde{\widetilde{f_{\leq k-2}}} = \left(\sum_{\substack{i+j=k+1\\ j\geq 3}} f_i \star \mu_j + \mu_j \circ \widetilde{\widetilde{f}}_{\leq k} \right) + \sum_{\substack{i>1, j>1\\ i+j=k}} \mu_2(f_i, f_j),$$

which we abbreviate

$$\delta(f_{k-1}) = (f_{\leq k-2} \star \mu_{\geq 3})_k + (\mu_{\geq 3} \circ \widetilde{\widetilde{f_{\leq k-2}}})_k + \mu_2(f_{>1}, f_{>1})_k. \tag{5.19}$$

We will also use a reformulation of the A_n-algebra homomorphism relations, stated in terms of the map D from Equation (5.16). Abusing notation, given a tensor product $g^1 \otimes \cdots \otimes g^k$ of maps $\mathcal{T}^*(A) \to A$ (such as $\widetilde{\widetilde{f_{\leq n-1}}}$ or $D(\widetilde{\widetilde{f_{\leq n-1}}})$), let $(g^1 \otimes \cdots \otimes g^k)_{\leq n}$ denote the restriction that no individual g^i has more than n inputs. Then the reformulation is

$$\big(D(\widetilde{\widetilde{f_{\leq n-1}}})\big)_{\leq n} = \big(\widetilde{\widetilde{f_{\leq n-2}}} \circ \widetilde{\mu}_{\geq 3} + \widetilde{\mu_{\geq 3}} \circ \widetilde{\widetilde{f_{\leq n-2}}}\big)_{\leq n} \tag{5.20}$$

Another identity we shall use is that

$$\delta(\mu_2(f,g)) = \mu_2(\delta(f),g) + \mu_2(f,\delta(g)).$$

With these preliminaries in hand, we compute

$$\begin{aligned}\delta\big((f_{\leq n-2} \star \mu_{\geq 3})_n\big) &= \big(\delta(f_{\leq n-2}) \star \mu_{\geq 3} + f_{\leq n-3} \star \delta(\mu_{\geq 3}) + \mu_2(f_{>1}, \mu_{\geq 3}) + \mu_2(\mu_{\geq 3}, f_{>1})\big)_{n+1}\\ &= \big(\mu_{\geq 3} \circ \widetilde{\widetilde{f_{\leq n-3}}} \circ \widetilde{\mu_{\geq 3}} + \mu_2(f_{>1} \star \mu_{\geq 3}, f_{>1}) + \mu_2(f_{>1}, f_{>1} \star \mu_{\geq 3})\\ &\qquad + \mu_2(f_{>1}, \mu_{\geq 3}) + \mu_2(\mu_{\geq 3}, f_{>1})\big)_{n+1}\\ &= \big(\mu_{\geq 3} \circ \widetilde{\widetilde{f_{\leq n-3}}} \circ \widetilde{\mu_{\geq 3}} + \mu_2(f \star \mu_{\geq 3}, f_{>1}) + \mu_2(f_{>1}, f \star \mu_{\geq 3})\big)_{n+1}.\end{aligned} \tag{5.21}$$

In going from the first to the second line above, we are using a cancellation of $(f_{\leq n-1} \star \mu_{\geq 3}) \star \mu_{\geq 3}$ against $f_{\leq n-2} \star (\mu_{\geq 3} \star \mu_{\geq 3})$, which uses Equation (5.12).

Next, we compute

$$\begin{aligned}\delta\big((\mu_{\geq 3} \circ \widetilde{\widetilde{f_{\leq n-2}}})_n\big) &= \big(\delta(\mu_{\geq 3}) \circ \widetilde{\widetilde{f_{\leq n-2}}} + \mu_{\geq 3} \circ D(\widetilde{\widetilde{f_{\leq n-2}}}) + \mu_2(\mu_{\geq 3} \circ \widetilde{\widetilde{f}}, f_{>1}) + \mu_2(f_{>1}, \mu_{\geq 3} \circ \widetilde{\widetilde{f}})\big)_{n+1}\\ &= \big(\mu_{\geq 3} \circ \widetilde{\widetilde{f_{\leq n-3}}} \circ \widetilde{\mu_{\geq 3}} + \mu_2(\mu_{\geq 3} \circ \widetilde{\widetilde{f}}, f_{>1}) + \mu_2(f_{>1}, \mu_{\geq 3} \circ \widetilde{\widetilde{f}})\big)_{n+1},\end{aligned} \tag{5.22}$$

and also

$$\delta\big(\mu_2(f_{>1}, f_{>1})_n\big) = \mu_2(\delta f_{>1}, f_{>1})_{n+1} + \mu_2(f_{>1}, \delta f_{>1})_{n+1} = \mu_2(\delta f, f_{>1})_{n+1} + \mu_2(f_{>1}, \delta f)_{n+1}. \tag{5.23}$$

Adding Equations (5.21), (5.22), and (5.23), and using associativity of μ_2, and once again using Equation (5.19), we see that

$$\begin{aligned}\delta(\mathfrak{F}_n) &= \mu_2(\delta f + f \star \mu_{\geq 3} + \mu_{\geq 3} \circ \widetilde{\widetilde{f}}, f_{>1})_{n+1} + \mu_2(f_{>1} \delta f + f \star \mu_{\geq 3} + \mu_{\geq 3} \circ \widetilde{\widetilde{f}})_{n+1}\\ &= \mu_2(\delta f + f \star \mu_{\geq 3} + \mu_{\geq 3} \circ \widetilde{\widetilde{f}} + \mu_2(f_{>1}, f_{>1}), f_{>1})_{n+1} + \mu_2(f_{>1}, \delta f + f \star \mu_{\geq 3} + \mu_{\geq 3} \circ \widetilde{\widetilde{f}}\\ &\qquad + \mu_2(f_{>1}, f_{>1}))_{n+1}\\ &= 0,\end{aligned}$$

verifying Property (A$\mathfrak{F}$-1). ☐

Corollary 5.24. *Let $\mathcal{A}$ be an A_n-structure on the associative algebra A. If $HH^{m+2}(A) = 0$ for all $m \geq n$ then $\mathcal{A}$ extends to an A_∞-algebra structure on A. If in addition $HH^{m+1}(A) = 0$ for all $m \geq n$ then this extension is unique up to isomorphism.*

Proof. Suppose that $HH^{m+2}(A) = 0$ for all $m \geq n$. By Properties (A$\mathfrak{O}$-1) we can inductively find the requisite sequence of elements $\mu_k \in HC^k$ with $\delta(\mu_k) = \mathfrak{O}_k$ for all $k \geq m+1$, giving an extension of $\mathcal{A}$ to an A_∞ algebra.

Suppose that $HH^{m+1}(A) = 0$ for all $m \geq n$, and let $\mathcal{A}$ and $\mathcal{A}'$ be two A_∞ deformations of A that agree as A_n algebras. Choose $f_1 = \mathbb{I}$ and $f_k = 0$ for all $k = 1, \dots, n$. By hypothesis, these are the components of an A_n-homomorphism from $\mathcal{A}$ to $\mathcal{A}'$. For the inductive step, suppose that we have components $\{f_i \colon A^{\otimes i} \to A\}_{i=1}^k$ for $n < k$ of an A_k homomorphism, the obstruction $\mathfrak{F}_k$ to extending it to an A_{k+1}-homomorphism lies in $HH^k(A)$ by Properties (A$\mathfrak{F}$-1) and (A$\mathfrak{F}$-2). This map is an isomorphism since f_1 is invertible. ☐

We will actually be interested in deforming a (G,λ)-graded associative algebra to a (G,λ)-graded A_∞-algebra (so that μ_n has grading λ^{n-2}). (For G abelian, this case was studied by Sheridan [She15, Section 2.3].) We will assume further that the distinguished central element $\lambda \in G$ has infinite order.

In this setting, define a grading on the bar complex $\mathrm{Bar}(A)$ by viewing the n^{th} term as $A \otimes (A_+[1])^{\otimes n} \otimes A$, i.e., $\mathrm{gr}(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = \lambda^n \mathrm{gr}(a_0)\cdots\mathrm{gr}(a_{n+1})$. Then, the differential has grading λ^{-1}. If we are interested in deformations which preserve this grading—and we are—then we are interested in the subcomplex $HC^*_G(A) \subset HC^*(A)$ of morphisms which respect the grading by $G/\langle\lambda\rangle$, i.e., maps $f\colon \mathrm{Bar}(A) \to A[1]$ so that $\mathrm{gr}_{A[1]}(f(x)) = \lambda^k \mathrm{gr}_{\mathrm{Bar}(A)}(x)$, or equivalently $\mathrm{gr}_A(f(x)) = \lambda^{k-1}\mathrm{gr}_{\mathrm{Bar}(A)}(x)$, for some $k \in \mathbb{Z}$. In addition to the grading by n, this complex has an obvious $\mathbb{Z}$-grading (by k), and the differential decreases this grading by 1. A graded A_∞ operation lies in grading -1.

Let $HC^{i,j}_G(A)$ denote the part $HC^*_G(A)$ consisting of $(\Bbbk\otimes\Bbbk)$-module maps $A_+[1]^{\otimes i} \to A[1]$ of grading j. Explicitly, if $f \in HC^{i,j}_G(A)$, then

$$\mathrm{gr}(f(x_1,\dots,x_i)) = \lambda^{i+j-1}\mathrm{gr}(x_1)\cdots\mathrm{gr}(x_i).$$

The differential has the property that

$$\delta\colon HC^{r,d}_G(A) \to HC^{r+1,d-1}_G(A).$$

Proposition 5.7 has the following (G,λ)-graded analogue:

Proposition 5.25. *Let A be a (G,λ)-graded associatve algebra and let $\mathcal{A}$ be a (G,λ)-graded A_{n-1} deformation of A. Then, the bigradings of the obstruction classes are given by $\mathfrak{O}_n \in HC^{n+1,-2}_G(A)$ and $\mathfrak{F}_n \in HC^{n,-1}_G(A)$, and an operation μ_n defining a (G,λ)-graded A_n-deformation lies in $HC^{n,-1}_G(A)$. Moreover, $\mathfrak{O}_n$ is the obstruction to extending the (G,λ)-graded A_{n-1} deformation to a (G,λ)-graded A_n deformation. If $\mathcal{A}$ and $\mathcal{A}'$ are two (G,λ)-graded A_n deformations of A, and f is a (G,λ)-graded homomorphism of the underlying A_{n-1} deformations, then $\mathfrak{F}_n$ is the obstruction to extending f to a (G,λ)-graded A_n homomorphism.*

Proof. This is a straightforward adaptation of the proof of Proposition 5.7. ☐

Corollary 5.26. *Let A be a (G,λ)-graded associative algebra and $\mathcal{A}$ a (G,λ)-graded A_{n-1} deformation of A. If $HH^{m+1,-2}_G(A) = 0$ for all $m \geq n$ then $\mathcal{A}$ extends to a (G,λ)-graded A_∞-algebra structure on A. If $HH^{m,-1}_G(A) = 0$ for all $m \geq n$ then any two (G,λ)-graded A_∞-algebra structures on A extending $\mathcal{A}$ are A_∞-isomorphic.*

Proof. This follows from Proposition 5.25 exactly as Corollary 5.24 follows from Proposition 5.7. ☐

Remark 5.27. The discussion in our previous paper [LOT15, Section 2.5.3.] gives a grading on the space of morphisms from $\mathrm{Bar}(A)$ to A by $G \times_{G\times G} G$, which is the set of conjugacy classes in G. The definition of $HC^{*,*}_G(A)$ then restricts to morphisms lying over the conjugacy class $\{\lambda^n\}$ for $n \in \mathbb{Z}$.

5.2. **The cobar complex of the torus algebra.** Let $\Bbbk$ be a finite direct sum of copies of $\mathbb{F}_2$, A be an augmented associative $\Bbbk$-algebra and A_+ be the augmentation ideal. Note that the dual space $\mathrm{Hom}(A_+,\mathbb{F}_2)$ is a $\Bbbk$-bimodule.

Definition 5.28. *The* reduced cobar algebra $\mathrm{Cob}(A)$ *is the dual chain complex to* $\mathrm{Bar}(A)$*, that is, the direct sum over n of the dual of $(A_+)^{\otimes n}$. The multiplication on* $\mathrm{Cob}(A)$ *is the transpose of the comultiplication Δ on* $\mathrm{Bar}(A)$*, which in turn is defined by*

$$\Delta(a_1\otimes\cdots\otimes a_n) = \sum_{i=0}^{n}(a_1\otimes\cdots\otimes a_i)\otimes(a_{i+1}\otimes\cdots\otimes a_n) \in \mathrm{Bar}(A)\otimes\mathrm{Bar}(A).$$

The differential on $\mathrm{Cob}(A)$ *is the transpose of the map*

$$\sum_{m,n\geq 0}\mathbb{I}^{\otimes m}\otimes\mu_2\otimes\mathbb{I}^{\otimes n}\colon \mathrm{Bar}(A)\to\mathrm{Bar}(A).$$

Under modest finiteness assumptions, we can describe $\mathrm{Cob}(A)$ more explicitly. The easiest case is if A is finite-dimensional over $\mathbb{F}_2$, in which case $\mathrm{Cob}(A)$ is the tensor algebra on A_+^*, with differential given by

$$\delta^{\mathrm{Cob}}(a_1^* \otimes \cdots \otimes a_n^*) = \sum_{i=1}^{n} a_1^* \otimes \cdots \otimes \mu_2^*(a_i^*) \otimes \cdots \otimes a_n^*,$$

where μ_2^* is the dual of the multiplication μ_2 on $\mathcal{A}$. (Here and below, undecorated tensor products are over $\Bbbk$.) We use the convention that

$$(a_n \otimes \cdots \otimes a_1)^* = a_1^* \otimes \cdots \otimes a_n^*.$$

For example, when $A = \mathcal{A}_-^{0,\mathrm{as}}$, we have that $\mu_2^*(\rho_{12}^*) = \rho_2^* \otimes \rho_1^*$.

More generally, suppose that A is filtered by subspaces $F_0 A \subset F_1 A \subset \cdots \subset A$ with $A = \bigcup_i F_i A$, and so that each $F_i A$ is finite-dimensional, $\Bbbk \subset F_0 A$, and $\mu_2(F_i A, F_j A) \subset F_{i+j} A$. There is an induced filtration on $\mathrm{Bar}(A)$, which we also denote F_i, and

$$\mathrm{Cob}(A)_n = (A_+^{\otimes n})^* \cong \varprojlim (F_i(A_+^{\otimes n}))^*$$

$$\mathrm{Cob}(A) = \bigoplus_{n=1}^{\infty} \mathrm{Cob}(A)_n.$$

Explicitly, given an $\mathbb{F}_2$-basis $\{e_i\}$ for A_+, let $A_+^\dagger$ be the subspace of A_+^* spanned by the dual basis elements e_i^*. Then $\mathrm{Cob}(A)_n$ is the completion of $(A_+^\dagger)^{\otimes n}$ with respect to the (descending) filtration dual to F_i.

Turning to the example of interest to us, the algebra $\mathcal{A}_-^{0,\mathrm{as}}$ from Section 3.1 has a filtration by the winding number grading, wn, so that each filtration level is finite-dimensional. Hence, the cobar algebra $\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})$ is the completion of the tensor algebra on $(\mathcal{A}_-^{0,\mathrm{as}})_+^\dagger$ with respect to this filtration.

The algebra $\mathcal{A}_-^{0,\mathrm{as}}$ also has a grading by $\Gamma = G \times \mathbb{Z}$, with grading given by $\gamma = \mathrm{gr} \times \mathrm{wn}$, as defined in Section 4. This induces a Γ-grading on $\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})$ by the formula

$$\gamma^{\mathrm{Cob}}(a_1^* \otimes \cdots \otimes a_n^*) = \lambda^{-n} \gamma(a_1)^{-1} \cdots \gamma(a_n)^{-1}. \tag{5.29}$$

(Since the cobar algebra is the completion of the tensor algebra on $(\mathcal{A}_-^{0,\mathrm{as}})_+^\dagger$, not every element is a finite linear combination of homogeneous elements.)

There is an automorphism $\alpha \colon \Gamma \to \Gamma$ defined by

$$\alpha((j; a, b) \times i) = (j + 2i; -a, -b) \times (-i). \tag{5.30}$$

The following relationship between $\mathcal{A}_-^{0,\mathrm{as}}$ and its cobar algebra can be seen as a kind of Koszul self-duality.

Lemma 5.31. *There is a quasi-isomorphism of Γ-graded differential algebras*

$$\phi \colon (\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}}), \gamma^{\mathrm{Cob}}) \to (\mathcal{A}_-^{0,\mathrm{as}}, \alpha \circ \gamma)$$

specified by $\phi(\iota_0) = \iota_1$, $\phi(\iota_1) = \iota_0$, $\phi(\rho_i^*) = [\rho_i]$ for $i = 1, \dots, 4$, and $\phi(a^*) = 0$ if $|a| > 1$.

(When thinking of $\mathcal{A}_-^{0,\mathrm{as}}$ as a bimodule quasi-isomorphic to $\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})$, we will write elements of $\mathcal{A}_-^{0,\mathrm{as}}$ inside brackets.)

Proof. By construction, ϕ is a ring homomorphism. Direct computation shows that if $\rho_i \cdots \rho_\ell \neq 0$ then

$$\lambda^{-\ell+i-1} \cdot \gamma(\rho_i)^{-1} \cdots \gamma(\rho_\ell)^{-1} = \alpha(\gamma(\rho_i) \cdots \gamma(\rho_\ell)).$$

It follows that ϕ respects the grading.

Consider the homomorphism $j \colon \mathcal{A}_-^{0,\mathrm{as}} \to \mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})$ specified by $j(\iota_0) = \iota_1$, $j(\iota_1) = \iota_0$, $j(\rho_i) = \rho_i^*$ for $i = 1, \dots, 4$. Clearly,

$$\phi \circ j = \mathbb{I}. \tag{5.32}$$

The image of j is spanned by elements of the form $\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_\ell^*$. Any element of $\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})$ can be written as a series in elements in the image of j and further elements of the form

$$\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_\ell^* \otimes a_1^* \otimes \cdots \otimes a_m^*, \tag{5.33}$$

where the a_i are basic algebra elements and a_1 is the first element in the sequence with length greater than one or for which $a_1 = \rho_{\ell-1}$. Given such an element, let $k = 0$ if the element is in the image of j and $k = \ell - i + 1$ otherwise, and consider the homotopy operator

(5.34)

$$H(\overbrace{\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_\ell^*}^{k} \otimes a_1^* \otimes \cdots \otimes a_m^*) = \begin{cases} \rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_{\ell-1}^* \otimes (a_1 \cdot \rho_\ell)^* \otimes a_2^* \otimes \cdots \otimes a_m^* & \text{if } k > 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Since the total winding number of the output of H and the input of H are the same, H extends (continuously) to all of $\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})$.

We claim that the following formula holds:

$$\delta^{\mathrm{Cob}} \circ H + H \circ \delta^{\mathrm{Cob}} = \mathbb{I} + j \circ \phi. \tag{5.35}$$

It suffices to verify Equation (5.35) for each element $\xi = \rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_\ell^* \otimes a_1^* \otimes \cdots \otimes a_m^*$. There are three cases:

- If $\xi \in \mathrm{Im}(j)$, $H(\xi) = 0$, $\delta^{\mathrm{Cob}}(\xi) = 0$, and $j \circ \phi(\xi) = \xi$, so Equation (5.35) is immediate.
- Suppose that $k > 0$ and $a_1 \cdot \rho_\ell = 0$; or alternatively, that $k = 0$. Then, $H(\xi) = 0$, $\phi(\xi) = 0$ and the only non-zero term in $H(\delta^{\mathrm{Cob}}(\xi))$ is ξ itself, corresponding to the factorization $a_1 = \rho_{\ell+1} \cdot a_1'$ for some a_1'.
- Suppose that $k > 0$ and $a_1 \cdot \rho_\ell \neq 0$. In that case, ξ is the term in $\delta^{\mathrm{Cob}} \circ H$ corresponding to the factorization of $a_1 \cdot \rho_\ell$ as a_1 times ρ_ℓ. All other terms in $\delta^{\mathrm{Cob}} \circ H$ cancel corresponding terms in $H \circ \delta^{\mathrm{Cob}}$.

Together, Equations (5.32) and (5.35) ensure that ϕ is a quasi-isomorphism, as claimed. ☐

The cobar algebra is of interest to us because of its relationship with the Hochschild complex. Suppose that A is an augmented associative $\Bbbk$-algebra (such as $\mathcal{A}_-^{0,\mathrm{as}}$). Assume that A is endowed with an increasing filtration so that each F_iA finite-dimensional. The tensor product $A \otimes_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$ inherits a decreasing filtration from the filtration on $\mathrm{Cob}(A)$ (which does not use the filtration on the A-factor). Let $A \mathbin{\widehat{\otimes}}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$ denote the completion with respect to this filtration on each $A \otimes_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)_n$. Equivalently, $A \mathbin{\widehat{\otimes}}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$ is the direct sum over n of the completion of $A \otimes_{\Bbbk\otimes\Bbbk} (A_+^\dagger)^{\otimes n}$. Given a (potentially infinite) basis $\{e_i\}$ for A_+, define a differential on $A \mathbin{\widehat{\otimes}}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$ by

$$\partial(b \otimes \xi) = b \otimes \left(\delta^{\mathrm{Cob}}(\xi)\right) + \sum_i e_i b \otimes \left(\xi \otimes e_i^*\right) + b e_i \otimes \left(e_i^* \otimes \xi\right) \tag{5.36}$$

and then extending linearly to the completion (which contains infinite sums of elements of the form $b \otimes [\xi]$). The assumption that each F_iA is finite-dimensional implies this gives a well-defined map. (The last two terms come from partly dualizing the operation $\mu_2 \colon A \otimes A \to A$ to maps $A \to A \otimes A^*$ and $A \to A^* \otimes A$.)

Lemma 5.37. *If A is a filtered algebra, and each F_iA is finite-dimensional, there is an isomorphism of chain complexes*

$$A \mathbin{\widehat{\otimes}}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A) \cong HC^*(A) \tag{5.38}$$

where on the left side we use the differential from Equation (5.36).

Proof. This is straightforward from the definitions. ☐

In the application to $\mathcal{A}_-^{0,\mathrm{as}}$, we actually want to extend scalars from $\mathbb{F}_2$ to $\mathbb{F}_2[U]$. So, given an algebra A as above, let $A[U] = A \otimes_{\mathbb{F}_2} \mathbb{F}_2[U]$. Consider $A[U] \otimes_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$; note that the U-variable appears only on the $A[U]$-factor. The increasing filtration on A induces a decreasing filtration on $A[U] \widehat{\otimes}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)_n$ and we again have a completed tensor product $A[U] \widehat{\otimes}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$ and, given a basis $\{e_i\}$ for A_+, a differential induced by Equation (5.36). We have the following analogue of Lemma 5.37, reformulating the Hochschild cochain complex of $A[U]$ over $\Bbbk[U]$:

Lemma 5.39. *If A is a filtered algebra, and each $F_i A$ is finite-dimensional, there is an isomorphism of chain complexes*

$$A[U] \widehat{\otimes}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A) \cong HC^*(A[U]). \tag{5.40}$$

On the left side of Equation (5.40), the differential is given by Equation (5.36). On the right side of Equation (5.40), U is viewed as an element of the ground ring.

Proof. Again, this is straightforward from the definitions. $\square$

Example 5.41. We consider the torus algebra. Since (as proved below) $\mu_3 = 0$, it follows that the obstruction class from Proposition 5.7 satisfies $\mathfrak{O}_4 = 0$. (See Eq. (5.9).) There is a map $\mu_4\colon (\mathcal{A}_-^{0,\mathrm{as}})^{\otimes 4} \to \mathcal{A}_-^{0,\mathrm{as}}[U]$ constructed in Section 3.1, whose non-trivial operations are

$$\mu_4(\rho_4,\rho_3,\rho_2,\rho_1\cdot a) = Ua, \qquad \mu_4(b\cdot\rho_4,\rho_3,\rho_2,\rho_1) = Ub,$$

and the additional operations obtained by cyclically permuting $\rho_4,\rho_3,\rho_2,\rho_1$. The 5-input A_∞-relation (which holds by a very easy special case of Theorem 3.19) is equivalent to the statement that $\delta\mu_4 = 0$; i.e., $\mu_4 \in HC^*(A)$ is a Hochschild cocycle.

Under the isomorphism from Equation (5.40), this cocycle μ_4 corresponds to the element of $A[U] \widehat{\otimes}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$ specified by

$$U \otimes (\rho_1^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_4^*) + U\rho_2 \otimes (\rho_{12}^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_4^*) + U\rho_{23} \otimes (\rho_{123}^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_4^*) + \cdots$$
$$+ U\rho_3 \otimes (\rho_1^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_{34}^*) + U\rho_{23} \otimes (\rho_1^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_{234}^*) + \cdots + \cdots$$

(where the last $\cdots$ corresponds to cyclically permuting the set $1,2,3,4$). For this sum to make sense, we need the completed tensor product $A[U] \widehat{\otimes}_{\Bbbk\otimes\Bbbk} \mathrm{Cob}(A)$.

We use Lemma 5.31 to obtain a useful small model for the Hochschild complex (compare [LOT11]), which we describe after introducing some notation.

Definition 5.42. *The* small model Hochschild complex *C^* is defined as follows. As a vector space, C^* is generated by $a \otimes [b]$ with $a \in \mathcal{A}_-^{0,\mathrm{as}}$ and $[b] \in \mathcal{A}_-^{0,\mathrm{as}}$ are basic elements with the property that $i \cdot a \cdot j = a$ and $[j' \cdot b \cdot i'] = [b]$, for some idempotents $i, j \in \{\iota_0, \iota_1\}$ and complementary idempotents i', j'. (That is, if $i = \iota_0$ then $i' = \iota_1$.) We endow C^* with the following further structure:*

- *a $\mathbb{Z}$-grading, the* length grading *specified by $|a \otimes [b]| = |b|$ (when b is a basic algebra element).*
- *a $G \times \mathbb{Z}$-grading, specified by*

$$\gamma(a \otimes [b]) = \lambda \cdot \gamma(a) \cdot \alpha(\gamma(b)).$$

- *a differential*

$$\partial(a \otimes [b]) = \sum_{i=1}^{4} \left(\rho_i \cdot a \otimes [b \cdot \rho_i] + a \cdot \rho_i \otimes [\rho_i \cdot b]\right). \tag{5.43}$$

We let $C^*_\Gamma \subset C^*$ be the portion in grading $0 \times \mathbb{Z} \subset G \times \mathbb{Z}$, i.e., generated by $a \otimes [b]$ with the property that

$$\gamma(a) \cdot \alpha(\gamma(b)) = \lambda^{k-1},$$

for some integer k, called the homological grading *of $a \otimes [b]$. Let $C^{n,k}_\Gamma \subset C^{n,k}_\Gamma$ denote the portion with length grading n and homological grading k. The differential sends $C^{n,k}$ to $C^{n+1,k-1}$.*

For example, $\rho_{2341}\otimes[\rho_{1234}]\in C_\Gamma^{4,-1}$. (The element $U\iota_1\otimes[\rho_{1234}]$ also lies in this bigrading.)

To see that C^* is a chain complex, note that for any $i,j\in\{1,\dots,4\}$, at least one of $\rho_i\rho_j$ and $\rho_j\rho_i$ vanishes.

Proposition 5.44. *The chain complex C_Γ^* is quasi-isomorphic to the complex $HC_\Gamma^*(\mathcal{A}_-^{0,\mathrm{as}}[U])$; in particular $H^{n,k}(C_\Gamma)\cong HH^{n,k}(\mathcal{A}_-^{0,\mathrm{as}}[U])$.*

Proof. Recall that elements of $HC_\Gamma^{n,k}(\mathcal{A}_-^{0,\mathrm{as}})\subset\mathcal{A}_-^{0,\mathrm{as}}\otimes_{\Bbbk\otimes\Bbbk}(\mathcal{A}_-^{0,\mathrm{as}})_+^{\otimes n}$ are series with terms of the form $a_0\otimes(a_1^*\otimes\cdots\otimes a_n^*)$, where the a_i are all basic algebra elements, with the property that the right idempotent of a_1 (which is the left idempotent of a_1^*) agrees with the right idempotent of a_0, and the left idempotent of a_n (which is the right idempotent of a_n^*) agrees with the left idempotent of a_0. The Γ-grading is computed by $\lambda\cdot\gamma(a_0)\cdot\gamma^{\mathrm{Cob}}(a_1^*\otimes\cdots\otimes a_n^*)=\gamma(a_0)\cdot\lambda^{1-n}(\gamma(a_1)^{-1}\cdots\gamma(a_n)^{-1})$.

Consider the map $HC^*(\mathcal{A}_-^{0,\mathrm{as}})\to C^*$ induced by

$$\mathbb{I}\otimes\phi\colon\mathcal{A}_-^{0,\mathrm{as}}\otimes_{\Bbbk\otimes\Bbbk}\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})\to\mathcal{A}_-^{0,\mathrm{as}}\otimes_{\Bbbk\otimes\Bbbk}\mathcal{A}_-^{0,\mathrm{as}},$$

where ϕ is the map from Lemma 5.31. Since ϕ sends any element of $\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}})_n$ with filtration greater than $n/4+1$ to 0, $\mathbb{I}\otimes\phi$ indeed induces a map from $A[U]\widehat{\otimes}_{\Bbbk\otimes\Bbbk}\mathrm{Cob}(A)\cong HC^*(A[U])$ to C^*.

Comparing the differential on $HC^*(\mathcal{A}_-^{0,\mathrm{as}})$ from Equation (5.36) (with basic algebra elements as the basis) with Equation (5.43), we see that $\mathbb{I}\otimes\phi$ is a chain map.

Observe that $HC_\Gamma^{*,*}(\mathcal{A}_-^{0,\mathrm{as}})$ is a direct summand of $HC^*(\mathcal{A}_-^{0,\mathrm{as}})$, $C_\Gamma^{*,*}$ is a direct summand of C^*, and $\mathbb{I}\otimes\phi$ takes $HC_\Gamma^{*,*}(\mathcal{A}_-^{0,\mathrm{as}})$ to $C_\Gamma^{*,*}$.

If we filter $HC_\Gamma^{*,*}(\mathcal{A}_-^{0,\mathrm{as}})$ by the sum of the lengths of the input elements and filter $C_\Gamma^{*,*}$ by the length grading from Definition 5.42 then $\mathbb{I}\otimes\phi$ is a filtered chain map. The induced map at the E_1-page of the associated spectral sequence is $\mathbb{I}\otimes\phi_*$ where ϕ_* is the isomorphism $H_*(\mathrm{Cob}(\mathcal{A}_-^{0,\mathrm{as}}))\to\mathcal{A}_-^{0,\mathrm{as}}$ from Lemma 5.31 (or, rather, its restriction to the Γ-graded part). It follows that $\mathbb{I}\otimes\phi$ is a quasi-isomorphism from the completion of $HC_\Gamma^{*,*}(\mathcal{A}_-^{0,\mathrm{as}})$ to the completion of $C_\Gamma^{*,*}$. However, for each fixed Γ-grading on $HC_\Gamma^{*,*}(\mathcal{A}_-^{0,\mathrm{as}})$, for any element $b\otimes[\xi]\in HC_\Gamma^{n,k}(\mathcal{A}_-^{0,\mathrm{as}})$ there is a bound on the difference between the length of b and four times the winding number of ξ. Hence, for each pair of integers (n,k), $HC_\Gamma^{n,k}(\mathcal{A}_-^{0,\mathrm{as}})$ is already complete with respect to the length filtration on $\mathcal{A}_-^{0,\mathrm{as}}$. Similarly, for $C_\Gamma^{*,*}$, there are finitely many elements in each grading (see the proof of Proposition 5.46 below) so $C_\Gamma^{*,*}$ is also already complete. Hence, $\mathbb{I}\otimes\phi$ is a quasi-isomorphism $HC_\Gamma^*(\mathcal{A}_-^{0,\mathrm{as}}[U])\to C_\Gamma^{*,*}$, as desired. ☐

5.3. Uniqueness of $\mathcal{A}_-^0$.

Theorem 5.45. *Up to isomorphism, there is a unique A_∞ deformation of $\mathcal{A}_-^{0,\mathrm{as}}$ over $\mathbb{F}_2[U]$ satisfying the following conditions:*

1. *The deformation is $\Gamma=G\times\mathbb{Z}$-graded, where the gradings of the chords ρ_i is defined by $\gamma(\rho_i)=\mathrm{gr}(\rho_i)\times\mathrm{wn}(\rho_i)$. (The gradings gr and wn are defined in Section 4.)*
2. *The operations satisfy $\mu_4(\rho_4,\rho_3,\rho_2,\rho_1)=U\iota_1$ and $\mu_4(\rho_3,\rho_2,\rho_1,\rho_4)=U\iota_0$.*

The conditions of the theorem immediately imply that $\gamma(U)=(-2;0,0)\times 1$. Also, the relation $\mu_4(\rho_4,\rho_3,\rho_2,\rho_1)=U\iota_1$ implies the relation $\mu_4(\rho_3,\rho_2,\rho_1,\rho_4)=U\iota_0$, by considering the 5-input A_∞-relations.

A key step in the proof is a computation of (part of) the Hochschild cohomology of $\mathcal{A}_-^{0,\mathrm{as}}$:

Proposition 5.46. *The graded Hochschild cohomology* $HH_\Gamma^{*,*}(\mathcal{A}_-^{0,\mathrm{as}}[U])$ *of* $\mathcal{A}_-^{0,\mathrm{as}}[U]$ *over* $\Bbbk[U]$ *satisfies*

$$HH_\Gamma^{n,-1}(\mathcal{A}_-^{0,\mathrm{as}}[U]) = \begin{cases} \mathbb{F}_2 & n = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$HH_\Gamma^{n,-2}(\mathcal{A}_-^{0,\mathrm{as}}[U]) = \begin{cases} \mathbb{F}_2 & n = 5 \\ 0 & \text{otherwise} \end{cases}$$

Moreover, suppose $\xi \in HC_\Gamma^{4,-1}(\mathcal{A}_-^{0,\mathrm{as}}[U])$ *is a cycle and* $\xi(\rho_4 \otimes \rho_3 \otimes \rho_2 \otimes \rho_1) = U$*. Then* ξ *represents a generator of* $HH_\Gamma^{4,-1}(\mathcal{A}_-^{0,\mathrm{as}}[U])$*.*

Proof. Proposition 5.44 supplies a smaller quasi-isomorphic model for this complex C, where

$$C_\Gamma^{n,k} \subset \mathcal{A}_-^{0,\mathrm{as}} \otimes_{\Bbbk\otimes\Bbbk} \mathcal{A}_-^{0,\mathrm{as}},$$

is generated by elements of the form $a \otimes [b]$ with $a, b \in \mathcal{A}_-^{0,\mathrm{as}}$ for which:

(HC-1) the right idempotent of b is complementary to the left idempotent of a and the left idempotent of b is complementary to the right idempotent of a.

(HC-2) the gradings satisfy

$$\gamma(a \otimes [b]) = \lambda \cdot \gamma(a) \cdot \alpha(\gamma(b)) = \lambda^k,$$

where α is as in Equation (5.30) and $|b| = n$.

(We will typically suppress the $\otimes$ symbol from $a \otimes [b]$.)

The above conditions ensure that any such element $a[b]$ must be one of:

(C-1) the following elements $a \otimes [b]$

$$\rho_1[\rho_1], \qquad \rho_{123}[\rho_{123}], \qquad \iota_0[\iota_1], \qquad \iota_1[\iota_0];$$

(C-2) any of the elements obtained by multiplying the above b by some further element b' with $|b'| = 4s$, and while also multiplying a by some further element a' with $|a'| = 4s$;

(C-3) any element obtained by adding some $i \in \mathbb{Z}/4\mathbb{Z}$ to all the indices in any of the above obtained elements.

(In particular, the elements $\rho_{1234}[\rho_{4123}]$, $\rho_{1234}[\rho_{2341}]$, and $U[\rho_{1234}]$ are all obtained from $a[b] = \iota_1[\iota_0]$ by multiplying both a and b by length four algebra elements.)

In a little more detail, suppose that $a[b] = U^n\rho_{i,\dots,j+1}[\rho_{\ell,\dots,m+1}]$ is such an element; so that $|a| = j - i + 4n$ and $|b| = \ell - m$. Then, the $G/\langle\lambda\rangle$ factor of Condition (HC-2) ensures that $|a| \equiv |b|$ mod 4; furthermore, if $|a| \not\equiv 0 \pmod 4$, then $i \equiv \ell \pmod 4$ and $j \equiv m \pmod 4$. Condition (HC-1) now excludes the possibility that $|a| \equiv 2 \pmod 4$. Finally, the $\mathbb{Z}$ factor of Condition (HC-2) now ensures that $\mathrm{wn}(a) = \mathrm{wn}(b)$ and so $|a| = |b|$. The above classification follows.

Direct computation of the homological grading k gives:

$$k(\rho_1[\rho_1]) = 0, \qquad k(\rho_{123}[\rho_{123}]) = 0, \qquad k(\iota_0[\iota_1]) = k(\iota_1[\iota_0]) = 1,$$

and the usual symmetry obtained by adding $i \in \mathbb{Z}/4\mathbb{Z}$ to all the subscripts in the first two equations above. All other homological gradings are determined by the property that

$$k(a \cdot a'[b \cdot b']) = k(a[b]) - 2s,$$

if $|a'| = |b'| = 4s$. In particular,

$$k(\rho_{1234}[\rho_{4123}]) = k(U[\rho_{1234}]) = k(\rho_{1234}[\rho_{2341}]) = -1.$$

It follows that $HH^{n,-1} = 0$ unless $n = 4$, and $HC^{n,-2} = HH^{n,-2} = 0$ unless $n = 5, 7$.

We now compute the differentials of elements involving terms with $k = -1$ and $k = -2$:

$$\begin{aligned}
\partial(\rho_{123}[\rho_{123}]) &= \rho_{1234}[\rho_{4123}] + \rho_{4123}[\rho_{1234}] \\
\partial(\rho_{1234}[\rho_{4123}]) &= \rho_{41234}[\rho_{41234}] \\
\partial(\rho_{4123}[\rho_{1234}]) &= \rho_{41234}[\rho_{41234}] \\
\partial(U[\rho_{1234}]) &= U\rho_4[\rho_{41234}] + U\rho_1[\rho_{12341}] \\
\partial(\rho_{12341}[\rho_{12341}]) &= 0 \\
\partial(U\rho_1[\rho_{1234123}]) &= 0 \\
\partial(\rho_{1234123}[\rho_{1234123}]) &= \rho_{12341234}[\rho_{41234123}] + \rho_{41234123}[\rho_{12341234}] \\
\partial(U\rho_{123}[\rho_{1234123}]) &= U\rho_{1234}[\rho_{41234123}] + U\rho_{4123}[\rho_{12341234}]
\end{aligned}$$

All other such terms are obtained by adding $i \in \mathbb{Z}/4\mathbb{Z}$ to all of the indices in any of these expressions.

For $n = 4$, there are two kinds of cycles,

$$\rho_{1234}[\rho_{4123}] + \rho_{4123}[\rho_{1234}] = \partial(\rho_{123}[\rho_{123}])$$

(there are four cycles of this form), and

$$U[\rho_{1234}] + U[\rho_{2341}] + U[\rho_{3412}] + U[\rho_{4123}] \tag{5.47}$$

(there is a unique cycle of this form). This proves the claim about $HH_\Gamma^{n,-1}$.

Turning to $HH_\Gamma^{n,-2}$, for $n = 5$, the single homology class is

$$U\rho_1[\rho_{12341}] \sim U\rho_2[\rho_{23412}] \sim U\rho_3[\rho_{34123}] \sim U\rho_4[\rho_{41234}].$$

Finally, for $n = 7$ there are no cycles at all. □

Proof of Theorem 5.45. Throughout this proof, by "deformation" we mean "Γ-graded deformation".

It is immediate from Proposition 5.25 and Proposition 5.46 that there is no nontrivial A_3 deformation of $\mathcal{A}_-^{0,\mathrm{as}}$. Thus, taking $\mu_3 = 0$, $[\mathfrak{O}_4] = 0$ since taking $\mu_4 = 0$ defines an A_4 algebra. (In fact, $\mathfrak{O}_4$ vanishes as a chain.) Thus, the choices of A_4 deformation of $\mathcal{A}_-^{0,\mathrm{as}}$ correspond to $HH_\Gamma^{4,-1}(\mathcal{A}_-^{0,\mathrm{as}}) \cong \mathbb{F}_2$. Thus, there is a single nontrivial A_4 deformation. Moreover, from the description of the generator of $HH_\Gamma^{4,-1}(\mathcal{A}_-^{0,\mathrm{as}}) \cong \mathbb{F}_2$ in Proposition 5.46, this deformation satisfies and is characterized by Property (2) of the statement of Theorem 5.45. Now, again by Proposition 5.46, $HH_\Gamma^{m+1,-2}(\mathcal{A}_-^{0,\mathrm{as}}) = HH_\Gamma^{m,-1}(\mathcal{A}_-^{0,\mathrm{as}}) = 0$ for all $m > 4$, so by Corollary 5.26, this deformation extends uniquely to an A_∞ deformation of $\mathcal{A}_-^{0,\mathrm{as}}$. □

Remark 5.48. The non-trivial deformation from Theorem 5.45 appears in bordered Floer homology. For example, consider the A_∞ module for the solid torus, as in [LOT23, Figure 38]. This has a single generator a with $m_2(a,\iota_1) = a$, and actions $m_3(a,\rho_2,\rho_1) = a$, $m_3(a,\rho_4,\rho_3) = U\cdot a$. Composing these two actions gives a non-zero term in the A_∞ relation with input sequence $(a,\rho_4,\rho_3,\rho_2,\rho_1)$. Since $m_1 = 0$, the only possible term that can cancel this sequence is $m_2(a,\mu_4(\rho_4,\rho_3,\rho_2,\rho_1))$, forcing $\mu_4(\rho_4,\rho_3,\rho_2,\rho_1)$ as in the theorem. This is a formalization of the more geometric observation: composing the holomorphic disks giving the m_3 operations, we obtain a one-dimensional moduli space whose other end consists of a curve that covers T^2 once, with boundary asymptotics as given by $\rho_4,\rho_3,\rho_2,\rho_1$; compare also Section 3.5; and see [LOT23, Figure 7]. (This deformation also appears from the wrapped Fukaya category, as discussed in Section 6.)

5.4. **Weighted algebras and Hochschild cohomology.**

Next, we discuss deforming A_∞-algebras into weighted algebras. This is similar to the discussion in [Sei15, Section 3b] and [She15, Section 2.4]; again, the group-graded setting of [She15, Section 2.4] is particularly relevant.

Fix an augmented A_∞-algebra $\mathcal{A}^0 = (A,\{\mu_m\})$ over $\Bbbk$ with underlying vector space A and augmentation ideal $A_+ \subset A$. By a *weighted deformation* of $\mathcal{A}^0$ we mean a weighted A_∞-algebra $(A,\{\mu_m^k\})$

with the same underlying vector space as $\mathcal{A}^0$ and whose weight-zero operations are the same as for $\mathcal{A}^0$: i.e., $\mu_m^0 = \mu_m$ for all $m \geq 0$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both weighted deformations of the same undeformed A_∞ algebra. A *homomorphism of deformations* from $\mathcal{A}$ to $\mathcal{B}$ is a sequence of maps $f^\bullet = \{f^W : \mathcal{T}^*(A_+) \to B\}_{W=0}^{\infty}$ satisfying the weighted A_∞ homomorphism relations

$$\sum_{a+b=W} f^a \circ (\mathbb{I} \otimes \mu^b \otimes \mathbb{I}) \circ \Delta^3 + \sum_{a+w_1+\cdots+w_m=W} \mu^a \circ (f^{w_1} \otimes \cdots \otimes f^{w_m}) \circ \Delta^m = 0 \tag{5.49}$$

for each $W \geq 0$. In words, the second sum expresses the sum of all ways of parenthesizing the tensor product into disjoint bundles and applying some f^v to each bundle, and then channeling the outputs into a μ^a so that the total weight of the f's plus the weight a is W.

Like A_∞ deformations, we will build weighted deformations step-by-step. By a *W-truncated weighted A_∞-algebra* we mean a vector space A and operations $\mu_m^w : A^{\otimes m} \to A$ for $m \geq 0$ and $0 \leq w \leq W$, $(m,w) \neq (0,0)$, satisfying the weighted A_∞-algebra relations up to weight W (i.e., the ones only involving the operations defined). A *W-truncated weighted deformation* of an A_∞-algebra $\mathcal{A}^0$ is a W-truncated weighted A_∞-algebra $\mathcal{A}^W$ whose undeformed (unweighted) A_∞-algebra is $\mathcal{A}^0$.

Let $\mathcal{A}^W$ and $\mathcal{B}^W$ be W-truncated weighted deformations of $\mathcal{A}^0$. By a *homomorphism of W-truncated weighted deformations* from $\mathcal{A}^W$ to $\mathcal{B}^W$ we mean maps $f_m^w : A^{\otimes m} \to A$ for $0 \leq w \leq W$ and all $m \geq 0$ with $(m,w) \neq (0,0)$, such that:

- $f_1^0 = \mathbb{I}$,
- $f_m^0 = 0$ for $m \neq 1$, and
- the f_m^w satisfy the weighted A_∞-algebra homomorphism relations in Equation (5.49) up to weight W.

(The first two conditions specify that f^0 is the identity map of A_∞-algebras.) An *isomorphism* is an invertible homomorphism; by the proof of Lemma 2.8, every homomorphism of W-truncated weighted deformations is an isomorphism.

Definition 5.50. *Let $\mathcal{A}^0$ be an augmented A_∞-algebra over $\mathbb{k}$, with underlying vector space A and augmentation ideal $A_+ \subset A$.*

Let ${}^{\mathcal{A}^0}[\mathbb{I}]_{\mathcal{A}^0}$ denote the identity type DA bimodule over $\mathcal{A}^0$ (see [LOT15]). The Hochschild cochain complex *of $\mathcal{A}^0$ is given by*

$$HC^*(\mathcal{A}^0) = \mathrm{Mor}({}^{\mathcal{A}^0}[\mathbb{I}]_{\mathcal{A}^0}, {}^{\mathcal{A}^0}[\mathbb{I}]_{\mathcal{A}^0})$$

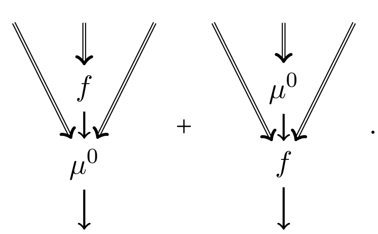
of strictly unital type DA bimodule morphisms from ${}^{\mathcal{A}^0}[\mathbb{I}]_{\mathcal{A}^0}$ to itself. $HH(\mathcal{A}^0)$ is the homology of this complex.

Explicitly, as a vector space,

$$HC^*(\mathcal{A}^0) = \prod_{n=0}^{\infty} \mathrm{Hom}_{\mathbb{k}\otimes\mathbb{k}}(\mathbb{k} \otimes (A_+)^{\otimes n}, A \otimes \mathbb{k}) = \prod_{n=0}^{\infty} \mathrm{Hom}_{\mathbb{k}\otimes\mathbb{k}}((A_+)^{\otimes n}, A).$$

The differential is given as follows. Recall the operation $\star$ from Equation (5.8). Let $\mu^0 = \sum_i \mu_i^0 \in \prod_{i=1}^{\infty} \mathrm{Hom}(A_+^{\otimes i}, A_+)$. Then the differential of f is given by

$$\delta(f) = \mu^0 \star f + f \star \mu^0 =$$



Unlike the associative case (Definition 5.1), in the A_∞-setting the Hochschild cohomology is not graded unless $\mathcal{A}^0$ is graded.

Proposition 5.51. *Let $\mathcal{A}^0$ be an A_∞-algebra and $\mathcal{A}^{W-1}$ a $(W-1)$-truncated weighted deformation of $\mathcal{A}^0$. Then there is a Hochschild cochain obstruction class $\mathfrak{O}^W \in HC^*(\mathcal{A}^0)$ so that:*

($\infty\mathfrak{O}$-1) *$\mathfrak{O}^W$ is a cocycle.*
($\infty\mathfrak{O}$-2) *$\mathfrak{O}^W$ is a coboundary if and only if there are operations $\mu^W = \{\mu^W_m\}_{m=0}^\infty$ making $\mathcal{A}^W$ into a W-truncated weighted A_∞ algebra; indeed, the operation μ^W is a cochain with $\delta(\mu^W) = \mathfrak{O}^W$.*
($\infty\mathfrak{O}$-3) *If μ^W and $\overline{\mu}^W$ are cochains with $\delta(\mu^W) = \delta(\overline{\mu}^W) = \mathfrak{O}^W$, then $\mu^W - \overline{\mu}^W$ is itself a coboundary if and only if there is a homomorphism f of W-truncated deformations between the structures induced by μ^W and $\overline{\mu}^W$ with $f^w = 0$ for all $0 < w < W$.*
($\infty\mathfrak{O}$-4) *Suppose $\mathcal{A}$ and $\overline{\mathcal{A}}$ are two W-truncated deformations, $W > 0$, with $\mu^w = \overline{\mu}^w$ for all $w < W$. If $\mu^W - \overline{\mu}^W$ is a coboundary then their respective obstruction cocycles $\mathfrak{O}^{W+1}$ and $\overline{\mathfrak{O}}^{W+1}$ are cohomologous.*

There are analogous statements for maps. In particular given W-truncated deformations $\mathcal{A}$ and $\overline{\mathcal{A}}$ of $\mathcal{A}^0$, for some $W > 0$, and a homomorphism $f^{\leq W-1}\colon \mathcal{A}^{\leq W-1} \to \overline{\mathcal{A}}^{\leq W-1}$ of $W-1$-truncated weighted deformations, there is an obstruction class $\mathfrak{F}^W \in HC^*(\mathcal{A}^0)$ so that:

($\infty\mathfrak{F}$-1) *$\mathfrak{F}^W$ is a cocycle.*
($\infty\mathfrak{F}$-2) *$\mathfrak{F}^W$ is a coboundary if and only if there is a homomorphism $\mathcal{A}^W \to \overline{\mathcal{A}}^W$ of W-truncated weighted deformations extending f.*

Proof. Let $\mu^v_*\colon \mathcal{T}^* A_+ \to A_+$ be the direct sum of the maps $\mu^v_n\colon A_+^{\otimes n} \to A_+$.

We think of the operation $\star$ from Equation (5.8) as follows. Given $f\colon \mathcal{T}^*(A_+) \to A_+$ and $g\colon \mathcal{T}^*(A_+) \to A_+$,

$$f \star g = g \circ (\mathbb{I} \otimes f \otimes \mathbb{I}) \circ \Delta^3,$$

where Δ^3 is as in Equation (5.13). We extend the operation $\star$ to sequences of maps $f^\bullet = \{f^W\colon \mathcal{T}^*(A_+) \to A_+\}_{W=0}^\infty$ and $g^\bullet = \{g^W\colon \mathcal{T}^*(A_+) \to A^+\}_{W=0}^\infty$, letting $f^\bullet \star g^\bullet = \{(f\star g)^W\colon \mathcal{T}^*(A_+) \to A_+\}_{W=0}^\infty$ be the sequence of maps whose components are given by

$$(f^\bullet \star g^\bullet)^W = \sum_{a+b=W} f^a \circ (\mathbb{I} \otimes g^b \otimes \mathbb{I}) \circ \Delta^3.$$

Given a sequence of maps $f^\bullet = \{f^W\colon \mathcal{T}^*(A_+) \to A_+\}_{W=0}^\infty$, let $f^{\bullet \geq 1}$ denote the sequence of maps $\phi^\bullet = \{\phi^W\colon \mathcal{T}^*(A_+) \to A_+\}_{W=0}^\infty$ with $\phi^0 = 0$ and $\phi^W = f^W$ for all $W > 0$.

The obstruction class $\mathfrak{O}^W$ is defined by

$$\mathfrak{O}^W = (\mu^{\bullet \geq 1} \star \mu^{\bullet \geq 1})^W;$$

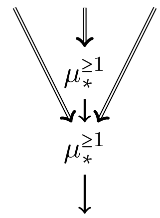
i.e., $\mathfrak{O}^W\colon \mathcal{T}^*(A_+) \to A_+$ is the map

$$\mathfrak{O}^W = \sum_{\substack{a+b=W \\ 1 \leq a \leq W-1}} \mu^a \star \mu^b,$$

whose components $\mathfrak{O}^W_n\colon \overbrace{A_+ \otimes \cdots \otimes A_+}^{n} \to A_+$ are given by

$$\mathfrak{O}^W_n = \sum_{\substack{a+b=W \\ 1 \leq a \leq W-1}} \sum_{\substack{i+j-1=n \\ 1 \leq i \leq n+1}} \mu^a_i \star \mu^b_j.$$

Graphically,

$\mathfrak{O}^W =$ 

.

Given $f^\bullet = \{f^w\colon \mathcal{T}^*(A_+) \to A_+\}_{w=0}^\infty$, we define a sequence of maps $\overline{\overline{f}}^\bullet = \{\overline{\overline{f}}^W\colon \mathcal{T}^*(A_+) \to \mathcal{T}^*(A_+)\}_{W=0}^\infty$ whose components $\overline{\overline{f}}^W$ are defined by

$$\overline{\overline{f}}^W = \sum_{m=1}^\infty \sum_{w_1+\cdots+w_m=W} (f^{w_1} \otimes \cdots \otimes f^{w_m}) \circ \Delta^m. \tag{5.52}$$

Given $g^\bullet = \{g^w\colon \mathcal{T}^*(A_+) \to A_+\}_{w=0}^\infty$ and $\phi^\bullet = \{\phi^w\colon \mathcal{T}^*(A_+) \to \mathcal{T}^*(A_+)\}_{w=0}^\infty$, we can define their weighted composition $g^\bullet \circ \phi^\bullet = \{(g^\bullet \circ \phi^\bullet)^w\colon \mathcal{T}^*(A_+) \to A_+\}_{w=0}^\infty$ by

$$(g^\bullet \circ \phi^\bullet)^W = \sum_{a+b=W} g^a \circ \phi^b.$$

In this notation, the W^{th} weighted A_∞-homomorphism relation is

$$(f^\bullet \star \mu^\bullet)^W + (\mu^\bullet \circ \overline{\overline{f}}^\bullet)^W = 0$$

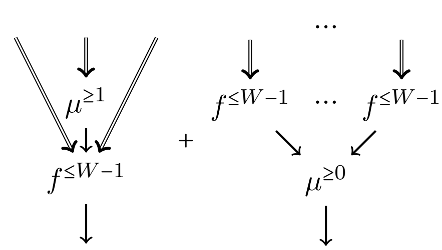
Now, suppose that f is only a $(W-1)$-truncated homomorphism, with components $\{f^v\colon \mathcal{T}^* A_+ \to A_+\}_{v=0}^{W-1}$. The obstruction class $\mathfrak{F}^W$ is defined to be

$$\mathfrak{F}^W = (f^\bullet \star \mu^{\bullet \geq 1})^W + (\mu^\bullet \circ \overline{\overline{f}}^\bullet)^W;$$

which, in turn, is shorthand for

$$\mathfrak{F}^W = \sum_{\substack{a+b=W\\ 0\leq a\leq W-1}} f^a \star \mu^b + \sum_{\substack{a+b=W\\ 0\leq a\leq W}} \mu^a \circ F^b.$$

We represent this equation graphically by

$\mathfrak{F}^W =$ 

.

(We write here $f^{\leq W-1}$ to bear in mind that there are no terms f^i with $i \geq W$.)

Having defined $\mathfrak{O}^W$ and $\mathfrak{F}^W$, we now check they satisfy the requisite properties.

The weight W A_∞ relation for μ^w can be written

$$\delta\mu^W = (\mu^{\bullet \geq 1} \star \mu^{\bullet \geq 1})^W = \mathfrak{O}^W, \tag{5.53}$$

which is Property ($\infty\mathfrak{O}$-2).

We verify Property ($\infty\mathfrak{O}$-1) after introducing some notation.

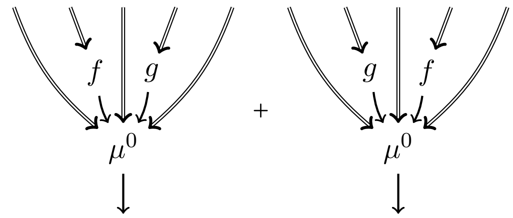
Equation (5.11) generalizes, as follows. Given $f\colon \mathcal{T}^*(A_+) \to A_+$ and $g\colon \mathcal{T}^*(A_+) \to A_+$, let

$$\eta(f, g)\colon \mathcal{T}^*(A_+) \to A_+$$

be the map

$$\eta(f,g) = \mu^0 \circ (\mathbb{I} \otimes f \otimes \mathbb{I} \otimes g \otimes \mathbb{I}) \circ \Delta^5 + \mu^0 \circ (\mathbb{I} \otimes g \otimes \mathbb{I} \otimes f \otimes \mathbb{I}) \circ \Delta^5. \tag{5.54}$$

Graphically

$\eta(f,g) =$ 

(So, for an associative algebra, $\eta(f,g) = \mu_2(f,g) + \mu_2(g,f)$.) Equation (5.11) readily generalizes to

$$\delta(f \star g) = (\delta f) \star g + f \star (\delta g) + \eta(f,g). \tag{5.55}$$

Extend η to sequences of maps $f^\bullet$ and $g^\bullet$ as usual, letting $\eta^\bullet(f^\bullet, g^\bullet) = \{\eta^W(f^\bullet, g^\bullet)\colon \mathcal{T}^*(A_+) \to A_+\}_{W=0}^\infty$ be the sequence of maps whose W^{th} component is given by

$$\eta^W(f^\bullet, g^\bullet) = \sum_{a+b=W} \eta(f^a, g^b).$$

Equation (5.55) generalizes to

$$\delta(f^\bullet \star g^\bullet) = (\delta f^\bullet) \star g^\bullet + f^\bullet \star (\delta g^\bullet) + \eta^\bullet(f^\bullet, g^\bullet). \tag{5.56}$$

To verify Property ($\infty\mathfrak{O}$-1), observe that

$$\begin{aligned}
\delta \mathfrak{O}^W &= \delta(\mu^{\bullet \geq 1} \star \mu^{\bullet \geq 1})^W = (\delta(\mu^{\bullet \geq 1}) \star \mu^{\bullet \geq 1})^W + (\mu^{\bullet \geq 1} \star \delta(\mu^{\bullet \geq 1}))^W + \eta^\bullet(\mu^{\bullet \geq 1}, \mu^{\bullet \geq 1}) \\
&= ((\mu^{\bullet \geq 1} \star \mu^{\bullet \geq 1}) \star \mu^{\bullet \geq 1})^W + (\mu^{\bullet \geq 1} \star (\mu^{\bullet \geq 1} \star \mu^{\bullet \geq 1}))^W \\
&= 0
\end{aligned}$$

(Here, the last step uses the analogue of Equation (5.12).)

Let $\mathcal{A}$ and $\overline{\mathcal{A}}$ be the W-truncated deformations with weighted operations μ^w and $\overline{\mu}^w$ respectively. The weight W A_∞ relation for a W-weighted map $f\colon \mathcal{A} \to \overline{\mathcal{A}}$ with $f^0 = \mathbb{I}$ and $f^w = 0$ for all $0 < w < W$ is precisely the condition

$$\delta(f^W) = \mu^W - \overline{\mu}^W;$$

and this is Property ($\infty\mathfrak{O}$-3).

To verify Property ($\infty\mathfrak{O}$-4), observe that if $\mathcal{A}$ and $\overline{\mathcal{A}}$ have $\mu^w = \overline{\mu}^w$ for all $w < W$, and

$$\mu^W - \overline{\mu}^W = \delta c$$

for some $c \in HC^*(\mathcal{A}^0)$, then

$$\begin{aligned}
\mathfrak{O}^{W+1} - \overline{\mathfrak{O}}^{W+1} &= \mu^W \star \mu^1 + \mu^1 \star \mu^W - \overline{\mu}^W \star \mu^1 - \mu^1 \star \overline{\mu}^W \\
&= (\mu^W - \overline{\mu}^W) \star \mu^1 + \mu^1 \star (\mu^W - \overline{\mu}^W) \\
&= \delta(c \star \mu^1 + \mu^1 \star c).
\end{aligned}$$

(The last line uses Equation (5.55), $\eta(\mu^W - \overline{\mu}^W, \mu^1) + \eta(\mu^1, \mu^W - \overline{\mu}^W) = 0$, and the A_∞ relation that guarantees that $\delta\mu^1 = 0$.)

Next, we consider the case for maps. Fix two W-truncated deformations $\mathcal{A}$ and $\overline{\mathcal{A}}$ of $\mathcal{A}^0$, and a homomorphism of the underlying $(W-1)$-truncated deformations

$$f^{\leq W-1}\colon \mathcal{A}^{\leq W-1} \to \overline{\mathcal{A}}^{\leq W-1},$$

with $f^0_1\colon A \to A$ the identity map and $f^0_n = 0$ for $n > 1$. We wish to extend this to a homomorphism of W-truncated deformations by introducing a new component $f^W\colon \mathcal{T}^*(A_+) \to A_+$. The weight W A_∞ relation has the form

$$\delta f^W = \mathfrak{F}^W;$$

so Property (∞𝔉-2) follows.

To verify Property (∞𝔉-1), we introduce some more notation.

Given $\phi\colon \mathcal{T}^*(A_+) \to A_+$, let $\overline{\phi}\colon \mathcal{T}^*(A_+) \to \mathcal{T}^*(A_+)$ be the induced map

$$\overline{\phi} = (\mathbb{I}_{\mathcal{T}^*(A_+)} \otimes \phi \otimes \mathbb{I}_{\mathcal{T}^*(A_+)}) \circ \Delta^3.$$

There is a differential

$$D\colon \mathrm{Hom}(\mathcal{T}^*(A_+), \mathcal{T}^*(A_+)) \to \mathrm{Hom}(\mathcal{T}^*(A_+), \mathcal{T}^*(A_+))$$

defined by

$$D(\Phi) = \overline{\mu^0} \circ \Phi + \Phi \circ \overline{\mu^0}.$$

Equation (5.18) has the following analogue:

$$\delta(f^\bullet \circ \overline{\overline{\phi}}^\bullet) = (\delta(f^\bullet) \circ \overline{\overline{\phi}}^\bullet) + (f^\bullet \circ D\overline{\overline{\phi}}^\bullet) + (\mu^0 \star f^\bullet) \circ \overline{\overline{\phi}}^\bullet + \mu^0 \star (f^\bullet \circ \overline{\overline{\phi}}^\bullet). \tag{5.57}$$

(The last term is a $\star$ of a single map $\mu^0 \in \mathrm{Mor}(\mathcal{T}^*(A_+), A_+)$ with a sequence of maps $g^\bullet = f^\bullet \circ \overline{\overline{\phi}}^\bullet$. This is to be interpreted as a sequence of maps whose W^{th} component is $\mu^0 \star g^W$.) Equation (5.57) follows from the following identities:

$$\begin{aligned}
\delta(f^\bullet \circ \overline{\overline{\phi}}^\bullet) &= \mu^0 \star (f^\bullet \circ \overline{\overline{\phi}}^\bullet) + (f^\bullet \circ \overline{\overline{\phi}}^\bullet \circ \overline{\mu}^0)\\
f^\bullet \circ D\overline{\overline{\phi}}^\bullet &= (f^\bullet \circ \overline{\mu}^0 \circ \overline{\overline{\phi}}^\bullet) + (f^\bullet \circ \overline{\overline{\phi}}^\bullet \circ \overline{\mu}^0)\\
(\delta f^\bullet) \circ \overline{\overline{\phi}}^\bullet &= (\mu^0 \star f^\bullet) \circ \overline{\overline{\phi}}^\bullet + (f^\bullet \star \mu^0) \circ \overline{\overline{\phi}}^\bullet.
\end{aligned}$$

The maps $f^\bullet = \{f^w\colon \mathcal{T}^*(A_+) \to A_+\}_{w=0}^\infty$ form the components of a weighted A_∞ homomorphism if

$$\mu^{\bullet\geq 0} \circ \overline{\overline{f}}^\bullet + \overline{\overline{f}}^\bullet \circ \mu^{\bullet\geq 0} = 0. \tag{5.58}$$

When $f^\bullet$ is merely a homomorphism of W-truncated deformations, i.e. $f^\bullet = \{f^w\colon \mathcal{T}^*(A_+) \to A_+\}_{w=0}^W$, we require that Equation (5.58) holds for the components indexed by $w = 0, \dots, W$.

The w^{th} component of Equation (5.58) has two alternative formulations. We begin with the analgue of Equation (5.19), which states that if $f^\bullet$ is a homomorphism of W-truncated deformations, then for each $w = 0, \dots, W$,

$$\delta f^w = (f^\bullet \star \overline{\mu^{\bullet\geq 1}})^w + (\mu^{\bullet\geq 1} \circ \overline{\overline{f}}^\bullet)^w + \mu^0 \circ \overline{\overline{f}}^w + \mu^0 \star f^w; \tag{5.59}$$

i.e.,

$$\delta f^w = \sum_{\substack{a+b=w\\ b\geq 1}} \left(f^a \star \mu^b + \mu^b \circ \overline{\overline{f}}^b\right) + \sum_{w_1+\dots+w_m=w} \mu^0 \circ (f^{w_1} \otimes \dots \otimes f^{w_m}) \circ \Delta^m + \mu^0 \star f^w.$$

Equation (5.59) follows immediately from the homomorphism relation

$$\mu^0 \circ (\overline{\overline{f}}^w) + (f^w \star \mu^0) = (f^\bullet \star \mu^{\bullet\geq 1} + \mu^{\bullet\geq 1} \star f^\bullet)^w,$$

together with the definition

$$\delta(f^w) = \mu^0 \star f^w + f^w \star \mu^0.$$

Since $f^0 = \mathbb{I}$, the terms in $\mu^0 \circ \overline{\overline{f}}^w$ that involve the f^w component cancel against the terms in $\mu^0 \star f^w$; thus, we can rewrite Equation (5.59) as:

$$\delta f^w = (f^\bullet \star \mu^{\bullet\geq 1})^w + (\mu^{\bullet\geq 1} \circ \overline{\overline{f}}^\bullet)^w + \mu^0 \circ (\overline{\overline{f}}^{\bullet<w})^w. \tag{5.60}$$

Here, $\mu^0 \circ (\overline{\overline{f}}^{\bullet<w})^w$ is the map which, given an element of $\mathcal{T}^*(A_+)$, splits the element into m tensor factors, applies f^{w_i} to the i^{th} tensor factor, for all choices of $(w_1,\dots,w_m)$ with $0 \le w_i \le w-1$ and $\sum_{i=1}^m w_i = m$, and then applies μ^0_m to the outputs.

The analogue of Equation (5.20) is

$$D(\overline{\overline{f}})^W = (\overline{\mu}^{\bullet\ge1} \circ \overline{\overline{f}})^W + (\overline{\overline{f}} \circ \overline{\mu}^{\bullet\ge1})^W. \tag{5.61}$$

Using the A_∞ homomorphism relation for all $w < W$, we get the following version:

$$\begin{aligned} D(\overline{\overline{f}}^{\bullet<W})^W &= (\overline{\mu^{\bullet\ge1}} \circ \overline{\overline{f}}^{\bullet})^W + (\overline{\overline{f}} \circ \overline{\mu^{\bullet\ge1}})^W \\ &\quad + \overline{(\mu^0 \circ \overline{\overline{f^{\bullet<W}}})^W} + \overline{(\mu^{\bullet\ge1} \circ \overline{\overline{f^{\bullet<W}}})^W} - \overline{(f^{\bullet<W} \star \mu^{\bullet\ge1})^W} \end{aligned} \tag{5.62}$$

With this notational background in place, we turn to the verification of Property ($\infty\mathfrak{F}$-1). By hypothesis, Equations (5.60) holds. In particular, although we cannot assume that Equation (5.59) holds for $w = W$, we do have that

$$(\delta f^\bullet \star \mu^{\bullet\ge1})^W = \left(\left((f^\bullet \star \mu^{\bullet\ge1}) + (\mu^{\bullet\ge1} \circ \overline{\overline{f}}^{\bullet}) + (\mu^0 \circ \overline{\overline{f^\bullet}}) + (\mu^0 \star f^\bullet)\right) \star \mu^{\bullet\ge1}\right)^W, \tag{5.63}$$

using Equation (5.60) in the components $w = 0,\dots,W-1$. Thus, by Equations (5.55) and (5.63),

$$\begin{aligned} \delta(f^\bullet \star \mu^{\bullet\ge1})^W &= ((\delta f^\bullet) \star \mu^{\bullet\ge1})^W + (f^\bullet \star \delta(\mu^{\bullet\ge1}))^W + \eta^W(f^\bullet, \mu^{\bullet\ge1}) \\ &= ((f^\bullet \star \mu^{\bullet\ge1}) \star \mu^{\bullet\ge1})^W + ((\mu^{\bullet\ge1} \circ \overline{\overline{f}}^{\bullet}) \star \mu^{\bullet\ge1})^W + ((\mu^0 \circ \overline{\overline{f^\bullet}} + \mu^0 \star f^\bullet) \star \mu^{\bullet\ge1})^W \\ &\quad + (f^\bullet \star (\mu^{\bullet\ge1} \star \mu^{\bullet\ge1}))^W + \eta^W(f^\bullet, \mu^{\bullet\ge1}) \\ &= ((\mu^{\bullet\ge1} \circ \overline{\overline{f}}^{\bullet}) \star \mu^{\bullet\ge1})^W + ((\mu^0 \circ \overline{\overline{f^\bullet}}) \star \mu^{\bullet\ge1})^W + ((\mu^0 \star f^\bullet) \star \mu^{\bullet\ge1})^W + \eta^W(f^\bullet, \mu^{\bullet\ge1}). \end{aligned} \tag{5.64}$$

Note that we cancelled above the terms $(f^\bullet \star \mu^{\bullet\ge1}) \star \mu^{\bullet\ge1}$ and $f^\bullet \star (\mu^{\bullet\ge1} \star \mu^{\bullet\ge1})$ as in Equation (5.12).

Similarly, using Equation (5.61) for $w < W$, we see that

$$(\mu^{\bullet\ge1} \circ D\overline{\overline{f}}^{\bullet})^W = \left(\mu^{\bullet\ge1} \circ (\overline{\mu^{\bullet\ge1}} \circ \overline{\overline{f}}^{\bullet}) + \mu^{\bullet\ge1} \circ (\overline{\overline{f}}^{\bullet} \circ \overline{\mu^{\bullet\ge1}})\right)^W \tag{5.65}$$

Thus, by Equations (5.57), (5.65), and (5.53)

$$\begin{aligned} \delta(\mu^{\bullet\ge1} \circ \overline{\overline{f}}^{\bullet})^W &= \left(\delta(\mu^{\bullet\ge1}) \circ \overline{\overline{f}}^{\bullet} + \mu^{\bullet\ge1} \circ (D\overline{\overline{f}}^{\bullet}) + (\mu^0 \star \mu^{\bullet\ge1}) \circ \overline{\overline{f}}^{\bullet} + \mu^0 \star (\overline{\mu^{\bullet\ge1}} \circ \overline{\overline{f}}^{\bullet})\right)^W \\ &= \Big((\mu^{\bullet\ge1} \star \mu^{\bullet\ge1}) \circ \overline{\overline{f}}^{\bullet} + \mu^{\bullet\ge1} \circ (\overline{\mu^{\bullet\ge1}} \circ \overline{\overline{f}}^{\bullet}) + \mu^{\bullet\ge1} \circ (\overline{\overline{f}}^{\bullet} \circ \overline{\mu^{\bullet\ge1}}) \\ &\quad + (\mu^0 \circ (\overline{\mu^{\bullet\ge1}} \circ \overline{\overline{f}}^{\bullet}) + \mu^0 \star (\mu^{\bullet\ge1} \circ \overline{\overline{f}}^{\bullet})\Big)^W \\ &= \left(\mu^{\bullet\ge1} \circ (\overline{\overline{f}}^{\bullet} \circ \overline{\mu^{\bullet\ge1}}) + (\mu^0 \circ (\overline{\mu^{\bullet\ge1}} \circ \overline{\overline{f}}^{\bullet}) + \mu^0 \star (\mu^{\bullet\ge1} \circ \overline{\overline{f}}^{\bullet})\right)^W. \end{aligned} \tag{5.66}$$

Above, we have used that $\mu^{\bullet\ge1} \star \mu^{\bullet\ge1} = \mu^{\bullet\ge1} \circ \overline{\mu}^{\bullet\ge1}$.

By Equation (5.57), and using the identities $\delta(\mu^0) = 0$ and $\mu^0 \star \mu^0 = 0$, we see that

$$\begin{aligned} \delta(\mu^0 \circ (\overline{\overline{f^{\bullet<W}}})^W) &= (\delta\mu^0) \circ \overline{\overline{f^{\bullet<W}}}^W + \mu^0 \circ (D\overline{\overline{f^{\bullet<W}}})^W + (\mu^0 \star \mu^0) \circ \overline{\overline{f^{\bullet<W}}}^W + \mu^0 \star (\mu^0 \circ \overline{\overline{f^{\bullet<W}}})^W \\ &= \mu^0 \circ (D\overline{\overline{f^{\bullet<W}}})^W + \mu^0 \star (\mu^0 \circ \overline{\overline{f^{\bullet<W}}})^W. \end{aligned} \tag{5.67}$$

Applying μ^0 to Equation (5.62), we find that

$$\begin{aligned} \mu^0 \circ D(\overline{\overline{f^{\bullet<W}}})^W &= \mu^0 \circ (\overline{\mu^{\bullet\ge1}} \circ \overline{\overline{f}}^{\bullet})^W + \mu^0 \circ (\overline{\overline{f}}^{\bullet} \circ \overline{\mu^{\bullet\ge1}})^W \\ &\quad + \mu^0 \star (\mu^0 \circ \overline{\overline{f^{\bullet<W}}})^W + \mu^0 \star (\mu^{\bullet\ge1} \circ \overline{\overline{f^{\bullet<W}}})^W + \mu^0 \star (f^{\bullet<W} \star \mu^{\bullet\ge1})^W \end{aligned} \tag{5.68}$$

Adding up Equations (5.64), (5.66),(5.67), and (5.68), together with the following case of Equation (5.12)

$$\mu^0 \star (f^{\bullet<W} \star \mu^{\bullet\geq 1})) + (\mu^0 \star f^{\bullet<W}) \star \mu^{\bullet\geq 1} + \eta^W(f^\bullet, \mu^{\bullet\geq 1}) = 0,$$

we find that

$$\delta(\mathfrak{F}^W) = \delta(f^\bullet \star \mu^{\bullet\geq 1} + \mu^{\bullet\geq 1} \circ \overline{\overline{f}}^\bullet + \mu^0 \circ \overline{\overline{f^{\bullet<W}}})^W = 0,$$

verifying Property ($\infty\mathfrak{F}$-1). ☐

We turn next to the graded case. Fix a group Γ and central elements λ_d and λ_w. Assume further that λ_d and λ_w generate a subgroup isomorphic to $\mathbb{Z}^2$. (We will be working here with $\Gamma = G \times \mathbb{Z}$, where $\mathbb{Z}$ is winding number grading, as in Section 4.4.)

If $\mathcal{A}^0$ is a (Γ, λ_d)-graded A_∞ algebra then we can consider the complex $HC_\Gamma^{W,*}(\mathcal{A}^0)$ generated by elements of $\mathrm{Mor}([\mathbb{I}], [\mathbb{I}])$ which shift the grading by $\lambda_d^\ell \lambda_w^W$ for some ℓ.

More explicitly, the Γ-grading on $HC^*(\mathcal{A}^0)$ is specified by

$$\gamma((a_1 \otimes \cdots \otimes a_n) \mapsto b) := \lambda_d^{1-n} \gamma(b) \cdot \gamma(a_n)^{-1} \cdots \gamma(a_1)^{-1}.$$

Let $HC_\Gamma^{W,\ell}$ be the portion with

$$\gamma((a_1 \otimes \cdots \otimes a_n) \mapsto b) = \lambda_w^W \lambda_d^\ell.$$

Then δ maps $HC_\Gamma^{W,\ell}$ to $HC_\Gamma^{W,\ell-1}$.

With these remarks in place, we have the following graded version of Proposition 5.51 (compare Proposition 5.25, as well as [She15, Section 2.4]):

Proposition 5.69. *Given a $(\Gamma, \lambda_d, \lambda_w)$-graded $(W-1)$-truncated, weighted A_∞ deformation $\mathcal{A}^{W-1}$ of $\mathcal{A}^0$, the obstruction class $\mathfrak{O}^W \in HC_\Gamma^{W,-2}(\mathcal{A}^0)$ is the obstruction to extending $\mathcal{A}^{W-1}$ to a W-truncated, weighted A_∞ deformation of $\mathcal{A}^0$. Given two $(\Gamma, \lambda_d, \lambda_w)$-graded W-truncated deformations $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{A}^0$ and a $(\Gamma, \lambda_d, \lambda_w)$-graded homomorphism f between their underlying $(W-1)$-truncated parts, the class $\mathfrak{F}^W \in HH_\Gamma^{W,-1}(\mathcal{A}^0)$ is the obstruction to extending f to a $(\Gamma, \lambda_d, \lambda_w)$-graded W-truncated homomorphism.*

Corollary 5.70. *Let $\mathcal{A}^0$ be a (Γ, λ_d)-graded A_∞-algebra and $\mathcal{A}^W$ a $(\Gamma, \lambda_d, \lambda_w)$-graded W-truncated weighted deformation of $\mathcal{A}^0$. If $HH_\Gamma^{w,-2}(\mathcal{A}^0) = 0$ for all $w > W$ then $\mathcal{A}^W$ extends to a $(\Gamma, \lambda_d, \lambda_w)$-graded weighted A_∞-algebra structure on A. If $HH_\Gamma^{w,-1}(\mathcal{A}^0) = 0$ for all $w > W$ then any two $(\Gamma, \lambda_d, \lambda_w)$-graded weighted A_∞-algebra structures on A extending $\mathcal{A}^W$ are isomorphic.*

Proof. This follows readily from Proposition 5.69; cf. the proof of Corollary 5.24. ☐

5.5. **Uniqueness of $\mathcal{A}_-$.** In this section, we view the ground ring for $\mathcal{A}_-^0$ as $\Bbbk = \mathbb{F}_2 \oplus \mathbb{F}_2$, not $\Bbbk[U]$. So, our augmentation is a map $\mathcal{A}_-^0 \to \Bbbk$, and there is a corresponding augmentation ideal.

Theorem 5.71. *Up to isomorphism, there is a unique weighted deformation $\mathcal{A}_-$ of $\mathcal{A}_-^0$ such that:*

(1) $\mathcal{A}_-$ is $\Gamma = G \times \mathbb{Z}$-graded and
(2) $\mu_0^1 = \rho_{1234} + \rho_{2341} + \rho_{3412} + \rho_{4123}$.

It follows that the distinguished central elements in $G \times \mathbb{Z}$ are

$$\lambda_d = \lambda := (1; 0, 0) \times 0 \tag{5.72}$$

$$\lambda_w = (0; 0, 0) \times 1 \tag{5.73}$$

(as in Section 4.2).

Like Theorem 5.45, Theorem 5.71 follows from a computation of certain Hochschild cohomology groups. Before giving that computation, we adapt the material from Section 5.2 to $\mathcal{A}_-^0$.

Under mild assumptions on an augmented A_∞ algebra $\mathcal{A}^0$, we can construct its cobar algebra $\mathrm{Cob}(\mathcal{A}^0)$, as follows. Given an augmented A_∞-algebra $\mathcal{A}^0 = (A, \{\mu_n\colon A_+^{\otimes n} \to A\}_{n=1}^\infty)$, let $\mathrm{Cob}(\mathcal{A}^0)$ be the component-wise dual of the bar complex,

$$\mathrm{Cob}(\mathcal{A}^0) = \bigoplus_{n=0}^{\infty} \mathrm{Hom}(A_+^{\otimes n}, \mathbb{F}_2)$$

with multiplication induced by the comultiplication on the bar complex. Let

$$(\mu^0)^*\colon A_+^* \to \mathrm{Hom}\Big(\bigoplus_{n=1}^{\infty} A_+^{\otimes n}, \mathbb{F}_2\Big) \cong \prod_{n=1}^{\infty} \big(A_+^{\otimes n}\big)^*$$

be dual to the operations $\mu_n^0\colon A_+^{\otimes n} \to A_+$. Call $\mathcal{A}^0$ *bounded enough for cobar* if the image of $(\mu^0)^*$ lies in

$$\mathrm{Cob}(\mathcal{A}^0) = \bigoplus_{n=1}^{\infty} \mathrm{Hom}(A_+^{\otimes n}, \mathbb{F}_2) \subset \prod_{n=1}^{\infty} \mathrm{Hom}(A_+^{\otimes n}, \mathbb{F}_2).$$

Under this assumption, the differential on the bar complex dualizes to a differential on $\mathrm{Cob}(\mathcal{A}^0)$.

As in Section 5.2, we will make the differential more explicit with the help of a filtration. Assume that A is filtered by subspaces $F_0A \subset F_1A \subset \cdots \subset A$ with $A = \bigcup_i F_iA$, and so that each F_iA is finite-dimensional, $\Bbbk \subset F_0A$, and $\mu_n(F_{i_1}A, F_{i_2}A, \cdots, F_{i_n}A) \subset F_{i_1+\cdots+i_n}A$. Then the n-input part of $\mathrm{Cob}(\mathcal{A}^0)$ is given by $\mathrm{Cob}(\mathcal{A}^0)_n = \varprojlim(F_i(A_+^{\otimes n}))^*$. Pick a basis $\{e_i\}$ for A_+ and define $A_+^\dagger \subset (A_+)^*$ to be the subspace spanned by the elements e_i^*. Then $\mathrm{Cob}(\mathcal{A}^0)_n$ is the completion of $(A_+^\dagger)^{\otimes n}$. The differential on $\mathrm{Cob}(\mathcal{A}^0)$ is induced by the formula

$$\delta^{\mathrm{Cob}}(a_1^* \otimes \cdots \otimes a_k^*) = \sum_{i=1}^{k} a_1^* \otimes \cdots \otimes (\mu^0)^*(a_i^*) \otimes \cdots \otimes a_k^*.$$

There is also a grading on $\mathrm{Cob}(\mathcal{A}^0)$ as defined in Equation (5.29), though $\mathrm{Cob}(\mathcal{A}^0)$ is not the direct sum of its graded pieces.

In the application to the torus algebra, a suitable filtration F_i is given, for instance, by the winding number wn. We will always take the elements $U^m\rho_{i,\dots,j}$ and $U^m\iota_i$ as the basis $\{e_i\}$. Call elements of this basis $\mathbb{F}_2$-*basic elements*.

Example 5.74. For $\mathrm{Cob}(\mathcal{A}_-^0)$, the differential of U^* is given by

$$\delta^{\mathrm{Cob}}(U^*) = \rho_1^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_4^* + \rho_2^* \otimes \rho_3^* \otimes \rho_4^* \otimes \rho_1^* + \rho_3^* \otimes \rho_4^* \otimes \rho_1^* \otimes \rho_2^* + \rho_4^* \otimes \rho_1^* \otimes \rho_2^* \otimes \rho_3^*.$$

The differential of $(U\rho_1)^*$ is given by

$$\delta^{\mathrm{Cob}}\big((U\rho_1)^*\big) = U^* \otimes \rho_1^* + \rho_1^* \otimes U^* + \rho_3^* \otimes \rho_4^* \otimes \rho_1^* \otimes \rho_{12}^* + \rho_{41}^* \otimes \rho_1^* \otimes \rho_2^* \otimes \rho_3^*.$$

The differential of $(U^2)^*$ is given by

$$\begin{aligned}\delta^{\mathrm{Cob}}\big((U^2)^*\big) = U^* \otimes U^* &+ (U\rho_1)^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_4^* + \rho_1^* \otimes (U\rho_2)^* \otimes \rho_3^* \otimes \rho_4^* + \rho_1^* \otimes \rho_2^* \otimes (U\rho_3)^* \otimes \rho_4^* \\ &+ \rho_1^* \otimes \rho_2^* \otimes \rho_3^* \otimes (U\rho_4)^* + (\text{cyclic permutations of indices in these terms}) \\ &+ \rho_{41}^* \otimes \rho_1^* \otimes \rho_2^* \otimes \rho_{23}^* \otimes \rho_3^* \otimes \rho_4^* + (\text{cyclic permutations of these 6 chords}) \\ &+ \rho_{12}^* \otimes \rho_2^* \otimes \rho_3^* \otimes \rho_{34}^* \otimes \rho_4^* \otimes \rho_1^* + (\text{cyclic permutations of these 6 chords}).\end{aligned}$$

The following lemma will quickly lead to a proof that $\mathcal{A}_-^0$ is bounded enough for cobar:

Lemma 5.75. *Let $\mathcal{A}_-^0$ be a Γ-graded deformation of $\mathcal{A}_-^{0,\mathrm{as}}$. Given $h = (j;\mathfrak{a},\mathfrak{b}) \times m \in \Gamma$, there is an upper bound on n for which there is a sequence $(a_1,\dots,a_n)$ of $\mathbb{F}_2$-basic elements in the augmentation ideal A_+ of $\mathcal{A}_-^{0,\mathrm{as}}[U]$, with the following properties:*

- *(c-1) The tensor product $(a_n \otimes \cdots \otimes a_1) \neq 0 \in A_+^{\otimes n}$, or equivalently $(a_1^* \otimes \cdots \otimes a_n^*) \neq 0$ in $(A_+^*)^{\otimes n}$.*
- *(c-2) The gradings satisfy $\lambda^n\gamma(a_n)\cdots\gamma(a_1) = h^{-1}$, or equivalently $\gamma^{\mathrm{Cob}}(a_1^* \otimes \cdots \otimes a_n^*) = h$.*

Proof. Recall that a $\mathbb{F}_2$-basic element a_i of A_+ is of the form $U^{\ell_i}\rho^i$, where $\ell_i \geq 0$, or $U^{\ell}\iota_k$ where $\ell > 0$.

Let a be a basic algebra element. For $i = 1, \dots, 4$, let $\mathrm{wn}_i(a)$ denote the multiplicity of an element of $\mathcal{A}^0_-$ at ρ_i; i.e., for a chord ρ, $\mathrm{gr}'(\rho) = (-1/2; \mathrm{wn}_1(\rho), \mathrm{wn}_2(\rho), \mathrm{wn}_3(\rho), \mathrm{wn}_4(\rho))$, and wn_4 was the winding number grading wn of Section 4.4. We extend this to $(a_1^* \otimes \cdots \otimes a_n^*)$ by

$$\mathrm{wn}_i(a_1^* \otimes \cdots \otimes a_n^*) = -\sum_{j=1}^{n} \mathrm{wn}_i(a_j).$$

Perusing the gradings from Section 4.2, we see that if $\mathrm{gr}(\rho) = (j; \alpha, \beta)$, then $\alpha + \beta = \mathrm{wn}_2(\rho) - \mathrm{wn}_4(\rho)$. Thus, $\mathfrak{a} + \mathfrak{b} = -\mathrm{wn}_2 + \mathrm{wn}_4$. Since $\mathrm{wn}_4 = m$, wn_2 is now determined by $(j; \mathfrak{a}, \mathfrak{b})$.

Suppose that x of the a_i have the form $U^{\ell_i}\iota_k$, and let $y = n - x$. Property (c-1) ensures that

$$-\mathrm{wn}_2 - \mathrm{wn}_4 \geq \lfloor y/2 \rfloor, \tag{5.76}$$

for if some element a_i^* has $\mathrm{wn}_2(a_i^*) + \mathrm{wn}_4(a_i^*) = 0$ but $\mathrm{wn}_1(a_i^*) + \mathrm{wn}_3(a_i^*) \neq 0$, then the next $j > i$ so that $a_j^* \neq (U^{n_j}\iota_k)^*$ has the property that $\mathrm{wn}_2(a_j^*) + \mathrm{wn}_4(a_j^*) \neq 0$.

Now, there are only finitely many sequences $b_1^* \otimes \cdots \otimes b_y^*$ with the following properties:

- The length of the sequence is fixed. (It is y, as above.)
- Each b_i is a Reeb element
- $\mathrm{wn}_4(b_1^* \otimes \cdots \otimes b_y^*)$ is fixed. (It is m, as in the statement.)

Thus, for all sequences as above, there is a constant c with the property that the Maslov component of $\mathrm{gr}(b_1)^{-1} \cdots \mathrm{gr}(b_y)^{-1}$ is bounded below by c.

Next, $\gamma^{\mathrm{Cob}}((U^{\ell}\iota_k)^*) = \lambda^{2\ell}$. It follows that the Maslov component j of $\lambda^{-n}\,\mathrm{gr}(a_1)^{-1} \cdots \mathrm{gr}(a_n)^{-1}$ satisfies

$$j \geq x + c. \tag{5.77}$$

The upper bounds from Equation (5.76) and (5.77) give the desired upper bound in terms of h on n for which there is a sequence $(a_1, \dots, a_n)$ with Properties (c-1) and (c-2): $n \leq j - c + 2\mathfrak{a} + 2\mathfrak{b} - 4m + 1$. ☐

Corollary 5.78. *Any Γ-graded deformation of $\mathcal{A}^{0,\mathrm{as}}_-$ is bounded enough for cobar.*

Proof. The Γ-grading hypothesis ensures that

$$\gamma^{\mathrm{Cob}}((\mu_n^0)^*(b^*)) = \lambda^{-1} \cdot \gamma^{\mathrm{Cob}}(b^*).$$

So, this is immediate from Lemma 5.75. ☐

Remark 5.79. For the particular Γ-graded deformation $\mathcal{A}^0_-$ of $\mathcal{A}^{0,\mathrm{as}}_-$ constructed geometrically in Section 3, Lemma 3.23 (applied to operations with $w = 0$) immediately implies that $\mathcal{A}^0_-$ is bounded enough for cobar.

Lemma 5.31 identifies the cobar algebra of $\mathcal{A}^{0,\mathrm{as}}_-$. We promote this to an identification of the cobar algebra of $\mathcal{A}^0_-$, as follows. Consider $\mathcal{A}' = \mathcal{A}^{0,\mathrm{as}}_-[h]/(h^2)$, equipped with the differential ∂' that vanishes on $\mathcal{A}^{0,\mathrm{as}}_- \subset \mathcal{A}'$ and satisfies

$$\partial' h = \rho_{1234} + \rho_{2341} + \rho_{3412} + \rho_{4123}.$$

A Γ-grading on $\mathcal{A}'$ is specified by

$$\gamma'(h) = (-1; 0, 0) \times 1$$

and the condition that the natural inclusion map $i\colon \mathcal{A}^{0,\mathrm{as}}_- \to \mathcal{A}'$ (with image the elements without factors of h) satisfies

$$\gamma'(i([a])) = \alpha(\gamma(a)),$$

where α is as in Equation (5.30).

Lemma 5.80. *There is a quasi-isomorphism of Γ-graded algebras $\phi'\colon \mathrm{Cob}(\mathcal{A}_-^0) \to \mathcal{A}'$ satisfying $\phi'(\iota_0) = \iota_1$, $\phi'(\iota_1) = \iota_0$, $\phi'(\rho_i^*) = [\rho_i]$ for $i = 1, \dots, 4$, $\phi'(U^*) = [h]$, and $\phi'(a^*) = 0$ for all $\mathbb{F}_2$-basic elements a with $a \neq U$ and $|a| > 1$.*

(Here, by quasi-isomorphism, we mean a ring homomorphism that induces an isomorphism from each graded part of the homology of $\mathrm{Cob}(\mathcal{A}_-^0)$ to the corresponding graded part of $\mathcal{A}'$; recall that $\mathrm{Cob}(\mathcal{A}_-^0)$ is not the direct sum of its homogeneous pieces.)

Proof. The map ϕ' extends continuously to $A_+^* \subset \mathrm{Cob}(\mathcal{A}_-^0)$ because all elements with sufficiently large wn_4 are in the kernel. It is well-defined on $\mathrm{Cob}(\mathcal{A}_-^0)$ because we defined $\mathrm{Cob}(\mathcal{A}_-^0)$ using a direct sum instead of a direct product. The fact that ϕ' is a chain map and respects the gradings is straightforward.

As a chain complex, $\mathrm{Cob}(\mathcal{A}_-^0)$ is generated by elements of the form

$$\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_j^* \otimes a_1^* \otimes \cdots \otimes a_m^*,$$

$m \geq 0$, where a_1 has length greater than one, or $a_1 = \rho_{j-1}$. (Recall that the length of U is defined to be 4.)

Consider the chain map

$$j'\colon \mathcal{A}' \to \mathrm{Cob}(\mathcal{A}_-^0)$$

defined by $j'(\iota_0) = \iota_1$, $j'(\iota_1) = \iota_0$ and

$$\begin{aligned} j'([h]) &= U^* \\ j'([\overbrace{\rho_i \dots \rho_j}^{k}]) &= \rho_i^* \otimes \cdots \otimes \rho_j^* && k \geq 1 \\ j'([h\overbrace{\rho_i \dots \rho_j}^{k}]) &= \rho_i^* \otimes \cdots \otimes \rho_j^* \otimes U^* + \rho_i^* \otimes \cdots \otimes \rho_{j-1}^* \otimes (\rho_{j-1}\rho_j)^* \otimes \rho_j^* \otimes \rho_{j+1}^* \otimes \rho_{j+2}^* && k \geq 1. \end{aligned}$$

(The map j' is a chain map, but it is not a ring homomorphism.) We will prove that j' and ϕ' are homotopy inverses. It is easy to see that $\phi' \circ j' = \mathbb{I}$ as chain maps, so we focus on the reverse direction.

Consider a pure generator $b_1^* \otimes \cdots \otimes b_m^*$, and suppose that there is some $j \in 1, \dots, m$ so that U divides b_j. Let ℓ be the minimum i so that U divides b_j. We then associate to the pure generator the following integer:

$$v = \sum_{i=\ell}^{m} |b_i|.$$

In cases where there is no j so that U divides b_j, let $v = 0$. Let $\mathcal{F}^{v_0} \subset \mathrm{Cob}(\mathcal{A}_-^0)$ denote the subset generated by all generators with $v \leq v_0$. Clearly, $\mathcal{F}^{v_0}$ is a subcomplex of $\mathrm{Cob}(\mathcal{A}_-^0)$. The image of j' is contained in $\mathcal{F}^4$. (Not every element of $\mathrm{Cob}(\mathcal{A}_-^0)$ lies in some $\mathcal{F}^v$, but $\mathrm{Cob}_n(\mathcal{A}_-^0)$ is the completion of $\bigcup_v(\mathcal{F}^v \cap \mathrm{Cob}_n(\mathcal{A}_-^0))$ with respect to the winding number filtration.)

Consider the map $H'\colon \mathrm{Cob}(\mathcal{A}_-^0) \to \mathrm{Cob}(\mathcal{A}_-^0)$ defined by

$$H'(b_1^* \otimes \cdots \otimes b_m^*) = \begin{cases} 0 & \text{if } \forall j,\ U \text{ does not divide } b_j. \\ 0 & \text{if } b_\ell \neq U \text{ or } b_\ell = U \text{ and } \ell = m. \\ b_1^* \otimes \cdots \otimes b_{\ell-1}^* \otimes (b_{\ell+1} \cdot U)^* \otimes b_{\ell+2}^* \otimes \cdots \otimes b_m^* & \text{if } b_\ell = U \text{ and } \ell < m. \end{cases}$$

and extending continuously to the completion.

Obviously, $H'(\mathcal{F}^{v_0}) \subset \mathcal{F}^{v_0}$. Moreover, if $v > 4$, then we claim that

$$(\mathbb{I} + \delta^{\mathrm{Cob}} \circ H' + H' \circ \delta^{\mathrm{Cob}})(\mathcal{F}^v) \subset \mathcal{F}^{v-1}. \tag{5.81}$$

We check this for each element of the form

$$x = b_1^* \otimes \cdots \otimes b_m^*,$$

as follows.

- If there U does not divide any of the b_j then $x \in \mathcal{F}^0$ and $H'(x) = H'(\delta^{\mathrm{Cob}}(x)) = 0$, so $(\mathbb{I} + \delta^{\mathrm{Cob}} \circ H' + H' \circ \delta^{\mathrm{Cob}})(x) = x$ still lies in $\mathcal{F}^0 \subset \mathcal{F}^4$.
- Suppose that $b_\ell \neq U$. In this case, $H'(x) = 0$, and $H' \circ \delta^{\mathrm{Cob}}(x) = x$; in particular,
$$\mathbb{I} + \delta^{\mathrm{Cob}} \circ H' + H' \circ \delta^{\mathrm{Cob}} = 0.$$
- Suppose that $b_\ell = U$ and $m > \ell$. Terms in $\delta^{\mathrm{Cob}} \circ H'$ arising from $\delta^{\mathrm{Cob}}(b_k^*)$ for $k > \ell + 1$ cancel with corresponding terms in $H' \circ \delta^{\mathrm{Cob}}$. Terms in $\delta^{\mathrm{Cob}} \circ H'$ arising from $\delta^{\mathrm{Cob}}((b_{\ell+1} \cdot U)^*)$ are of the following types:
  - those (using the part of δ^{Cob} adjoint to μ_2^*) that cancel against the terms in $H' \circ \delta^{\mathrm{Cob}}$ coming from $\delta^{\mathrm{Cob}}(b_{\ell+1})$;
  - the term x;
  - terms arising from μ_i^* with $i > 2$; these automatically lie in $\mathcal{F}^{v-4}$; and
  - additional terms corresponding to factorizations of $b_{\ell+1} \cdot U = a_1 \cdot a_2$ (which dualize to $a_2^* \otimes a_1^*$), where U does not divide a_2. These lie in $\mathcal{F}^{v-1}$.

On the other hand, on $\mathcal{F}^4$, H' vanishes, so $\delta^{\mathrm{Cob}} \circ H' + H' \circ \delta^{\mathrm{Cob}} = 0$.

Define
$$G = \lim_{n \to \infty} (\mathbb{I} + \delta^{\mathrm{Cob}} \circ H' + H' \circ \delta^{\mathrm{Cob}})^n \colon \mathrm{Cob}(\mathcal{A}_-^0) \to \mathcal{F}^4.$$
To see this limit exists, note that by Lemma 5.75, for any particular grading and winding number filtration level, every basis element lies in $\mathcal{F}^v$ for some bounded v. Since $\mathbb{I} + \delta^{\mathrm{Cob}} \circ H' + H' \circ \delta^{\mathrm{Cob}}$ decreases the filtration level until we get to $\mathcal{F}^4$, where it is the identity, G is a well-defined continuous map. For the same reason, the map G is homotopic to the identity map. So, $\mathrm{Cob}(\mathcal{A}_-^0)$ is quasi-isomorphic to its subcomplex $\mathcal{F}^4$.

Next, let H be the homotopy from the proof of Lemma 5.31. Consider the homotopy operator $H'' \colon \mathcal{F}^4 \to \mathcal{F}^4$ defined by
$$H''(x) = \begin{cases} H(x) & \text{if } x \in \mathcal{F}^0 \\ H(x_0) \otimes U^* & \text{if } x = x_0 \otimes U^* \text{ with } x_0 \in \mathcal{F}^0 \end{cases}$$
and extending continuously to all of $\mathcal{F}^4$. (In particular, $H''(U) = 0$.) We claim that
$$\big(\delta^{\mathrm{Cob}} \circ H'' + H'' \circ \delta^{\mathrm{Cob}} + \mathbb{I}\big)|_{\mathcal{F}^4} = j' \circ \phi'. \tag{5.82}$$
Specifically, for $x_0 \in \mathcal{F}^0$ of the form $x_0 = a_1^* \otimes \cdots \otimes a_n^*$ with $|a_i| \neq 1$ for some i or $a_{i+1} \cdot a_i \neq 0$, the proof of Equation (5.35) proves that $(\delta^{\mathrm{Cob}} \circ H'' + H'' \circ \delta^{\mathrm{Cob}})(x) = x$. This argument readily adapts also to $x \in \mathcal{F}^4$ of the form $x = x_0 \otimes U^*$ with x_0 as above. For the remaining generators x of the form $\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_j^*$ or $\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_j^* \otimes U^*$ for which H'' vanishes, Equation (5.82) is a straightforward verification. Equation (5.82) finishes the proof that ϕ' and j' are inverse quasi-isomorphisms of chain complexes. ☐

Our interest in the cobar algebra comes from its relation to the Hochschild complex. (Compare Equation (5.40).) First,
$$\mathcal{A}^0 \mathbin{\widehat{\otimes}}_{\Bbbk \otimes \Bbbk} \mathrm{Cob}(\mathcal{A}^0) \cong HC(\mathcal{A}^0), \tag{5.83}$$
where $\widehat{\otimes}$ denotes the completed tensor product with respect to the filtration on $\bigoplus_n \mathcal{A}^0 \otimes \mathrm{Hom}(A_+^{\otimes n}, \mathbb{F}_2)$ induced by the length filtration on $A_+^{\otimes n}$. (This completion also completes the direct sum over n in the definition of $\mathrm{Cob}(\mathcal{A}^0)$ to a direct product.) Note that for each grading on $\mathcal{A}^0 \mathbin{\widehat{\otimes}} \mathrm{Cob}(\mathcal{A}^0)$, by Lemma 5.75, it is equivalent to complete by the length filtration on the first, $\mathcal{A}^0$, factor.

Define a differential on $\mathcal{A}^0 \mathbin{\widehat{\otimes}} \mathrm{Cob}(\mathcal{A}^0)$ by setting, for $a \in \mathcal{A}^0$ and $x^* \in \mathrm{Cob}(\mathcal{A}^0)$,
$$\partial(a \otimes x^*) = a \otimes \delta^{\mathrm{Cob}}(x^*) + \sum_{\mu_n^0(b_1,\dots,b_k,a,c_1,\dots,c_\ell)=d} d \otimes c_\ell^* \otimes \cdots \otimes c_1^* \otimes x^* \otimes b_k^* \otimes \cdots \otimes b_1^* \tag{5.84}$$

(with the b_i and c_i in the sum being basic algebra elements) and extending continuously. The fact that this sum converges follows from the fact that each filtration level on $\mathcal{A}^0$ is finite-dimensional. With respect to this differential, Equation (5.83) is an isomorphism of chain complexes.

Lemma 5.80 leads to a smaller model of the Hochschild complex, as follows. Define an analogue $\mathfrak{C}^*$ of the complex C^* from Definition 5.42. As a vector space, $\mathfrak{C}^* = \mathcal{A}^0_- \widehat{\otimes}_{\Bbbk\otimes\Bbbk} \mathcal{A}'$, where $\widehat{\otimes}$ indicates we are completing with respect to the winding number filtration on $\mathcal{A}'$. (Again by Lemma 5.75, in each grading this is equivalent to completing with respect to either the winding number filtration or the length filtration on $\mathcal{A}^0_-$.) Write an element $a \otimes b \in \mathcal{A}^0_- \otimes \mathcal{A}'$ of $\mathfrak{C}^*$ as $a[b]$. The fact that the tensor product is over $\Bbbk \otimes \Bbbk$ means that the right idempotent of b is complementary to the left idempotent of a, and the left idempotent of b is complementary to the right idempotent of a. An arbitrary element of $\mathfrak{C}^*$ is a possibly infinite linear combination of elements of the form $a[b]$, with finitely many terms where b has winding number $< k$.

The Γ-grading on $\mathfrak{C}^*$ is specified by $\gamma(a[b]) = \lambda \cdot \gamma(a) \cdot \gamma'(b)$. Define the differential on basic elements by

$$
\begin{aligned}
\partial(a[b]) &= a[\partial b] + \sum_{i=1}^{4} (\rho_i \cdot a[b \cdot \rho_i] + a \cdot \rho_i[\rho_i \cdot b]) \\
&+ \sum_{i=1}^{4} \big(\mu_4(\rho_{i+3}, \rho_{i+2}, \rho_{i+1}, a)[b \cdot \rho_{i+1,i+2,i+3}] + \mu_4(\rho_{i+2}, \rho_{i+1}, a, \rho_{i-1})[\rho_{i-1} \cdot b \cdot \rho_{i+1,i+2}] \\
&\quad + \mu_4(\rho_{i+1}, a, \rho_{i-1}, \rho_{i-2})[\rho_{i-2,i-1} \cdot b \cdot \rho_{i+1}] + \mu_4(a, \rho_{i-1}, \rho_{i-2}, \rho_{i-3})[\rho_{i-3,i-2,i-1} \cdot b] \big)
\end{aligned} \tag{5.85}
$$

and extend continuously.

For example,

$$\partial(\rho_{1234}[\iota_1]) = \rho_{12341}[\rho_1] + \rho_{41234}[\rho_4] + U\rho_{123}[\rho_{123}] + U\rho_{234}[\rho_{234}] \tag{5.86}$$

$$\partial(\iota_0[\iota_1]) = \rho_1[\rho_1] + \rho_2[\rho_2] + \rho_3[\rho_3] + \rho_4[\rho_4] = \partial(\iota_1[\iota_0]) \tag{5.87}$$

$$\partial(\rho_1[\rho_1]) = U[\rho_{1234}] + U[\rho_{3412}] + U[\rho_{4123}] + U[\rho_{2341}]. \tag{5.88}$$

Observe that the differential commutes with the $\mathbb{F}_2[U]$-module structure.

Lemma 5.89. *This differential makes $\mathfrak{C}^*$ into a chain complex.*

Proof. Write

$$
\begin{aligned}
\partial_1(a[b]) &= a[\partial b] \\
\partial_2(a[b]) &= \sum_{i=1}^{4} (\rho_i \cdot a[b \cdot \rho_i] + a \cdot \rho_i[\rho_i \cdot b]) \\
\partial_4(a[b]) &= \sum_{i=1}^{4} \mu_4(\rho_{i+3}, \rho_{i+2}, \rho_{i+1}, a)[b \cdot \rho_{i+1,i+2,i+3}] + \mu_4(\rho_{i+2}, \rho_{i+1}, a, \rho_{i-1})[\rho_{i-1} \cdot b \cdot \rho_{i+1,i+2}] \\
&\quad + \mu_4(\rho_{i+1}, a, \rho_{i-1}, \rho_{i-2})[\rho_{i-2,i-1} \cdot b \cdot \rho_{i+1}] + \mu_4(a, \rho_{i-1}, \rho_{i-2}, \rho_{i-3})[\rho_{i-3,i-2,i-1} \cdot b].
\end{aligned}
$$

It is immediate that

$$\partial_1^2 = \partial_1\partial_2 + \partial_2\partial_1 = \partial_1\partial_4 + \partial_4\partial_1 = \partial_2^2 = 0.$$

To see that $\partial_2\partial_4 + \partial_4\partial_2 = 0$ is a case check. The most interesting case is when a has length 1, say, $a = \rho_1$. In that case, we have

$$\begin{aligned}
\partial_2(\partial_4(\rho_1[b])) &= \partial_2\big(\mu_4(\rho_4,\rho_3,\rho_2,\rho_1)[b\cdot\rho_{234}] + \mu_4(\rho_3,\rho_2,\rho_1,\rho_4)[\rho_4\cdot b\cdot\rho_{23}] \\
&\quad + \mu_4(\rho_2,\rho_1,\rho_4,\rho_3)[\rho_{34}\cdot b\cdot\rho_2] + \mu_4(\rho_1,\rho_4,\rho_3,\rho_2)[\rho_{234}\cdot b]\big) \\
&= U\rho_1[b\cdot\rho_{2341}] + U\rho_2[\rho_2\cdot b\cdot\rho_{234}] + U\rho_4[\rho_4\cdot b\cdot\rho_{234}] \\
&\quad + U\rho_3[\rho_{34}\cdot b\cdot\rho_{23}] + U\rho_4[\rho_4\cdot b\cdot\rho_{234}] + U\rho_2[\rho_{234}\cdot b\cdot\rho_2] + U\rho_3[\rho_{34}\cdot b\cdot\rho_{23}] \\
&\quad + U\rho_1[\rho_{1234}\cdot b] + U\rho_2[\rho_{234}\cdot b\cdot\rho_2] + U\rho_4[\rho_{234}\cdot b\cdot\rho_4] \\
&= U\rho_1[\rho_{1234}\cdot b] + U\rho_1[b\cdot\rho_{2341}] + U\rho_2[\rho_2\cdot b\cdot\rho_{234}] + U\rho_4[\rho_{234}\cdot b\cdot\rho_4] \\
\partial_4(\partial_2(\rho_1[b])) &= \partial_4\big(\rho_{12}[\rho_2\cdot b] + \rho_{41}[b\cdot\rho_4]\big) \\
&= \mu_4(\rho_{12},\rho_1,\rho_4,\rho_3)[\rho_{3412}\cdot b] + \mu_4(\rho_4,\rho_3,\rho_2,\rho_{12})[\rho_2\cdot b\cdot\rho_{234}] \\
&\quad + \mu_4(\rho_{41},\rho_4,\rho_3,\rho_2)[\rho_{234}\cdot b\cdot\rho_4] + \mu_4(\rho_3,\rho_2,\rho_1,\rho_{41})[b\cdot\rho_{4123}] \\
&= U\rho_1[\rho_{3412}\cdot b] + U\rho_2[\rho_2\cdot b\cdot\rho_{234}] + U\rho_4[\rho_{234}\cdot b\cdot\rho_4] + U\rho_1[b\cdot\rho_{4123}].
\end{aligned}$$

So,

$$\partial_2(\partial_4(\rho_1[b])) + \partial_4(\partial_2(\rho_1[b])) = U\rho_1[(\rho_{1234}+\rho_{3412})\cdot b] + U\rho_1[b\cdot(\rho_{2341}+\rho_{4123})] = 0.$$

Finally, we check that $\partial_4^2 = 0$. Suppose $a'[\rho_{i,\dots,j}\cdot b\cdot\rho_{k,\dots,\ell}]$ appears in $\partial_4^2(a[b])$. Then there is a composition of two μ_4-operations taking $\rho_\ell\otimes\cdots\otimes\rho_k\otimes a\otimes\rho_j\otimes\cdots\otimes\rho_i$ to a'. It follows that the first operation must be left-extended or right-extended, and have a (which must have length >1) as the left-most or right-most term, respectively. If the output of the first operation is not the right-most (respectively left-most) term of the second operation then the product in brackets vanishes: if we write $a = \rho_{m,\dots,n}$ then in the left-extended case, say, the result of the first operation is $\rho_{m,\dots,n-1}[\rho_{n-3,n-2,n-1}b]$ and if $\rho_{m,\dots,n-1}$ is not the last input to the next operation then the coefficient of b has a term $\rho_{n-2}\rho_{n-3,n-2,n-1} = 0$. In the remaining case, the pair of operations must be either

$$\mu_4(\mu_4(\rho_{k+2},\rho_{k+1},\rho_k,a),\rho_{\ell+2},\rho_{\ell+1},\rho_\ell) \qquad \text{or} \qquad \mu_4(\rho_{k+2},\rho_{k+1},\rho_k,\mu_4(a,\rho_{\ell_2},\rho_{\ell+1},\rho_\ell)).$$

These two cases cancel with each other. ☐

For integers W,ℓ, let $\mathfrak{C}_\Gamma^{W,\ell}\subset\mathfrak{C}^*$ denote the subspace spanned by elements $a[b]$ with

$$\gamma(a[b]) = \lambda_d\cdot\gamma(a)\cdot\gamma'(b) = \lambda_w^W\lambda_d^\ell, \tag{5.90}$$

and

$$\mathfrak{C}_\Gamma = \bigoplus_{W,\ell}\mathfrak{C}^{W,\ell}.$$

The following is an analogue of Proposition 5.44:

Proposition 5.91. *The chain complex $\mathfrak{C}_\Gamma^*$ is quasi-isomorphic to the complex $HC_\Gamma^*(\mathcal{A}_-^0)$; in particular $H^{W,k}(\mathfrak{C}_\Gamma)\cong HH_\Gamma^{W,k}(\mathcal{A}_-^0)$.*

Proof. Consider the map

$$\mathbb{I}\otimes\phi'\colon HC^*(\mathcal{A}_-^0) = \mathcal{A}_-^0\,\widehat{\otimes}\,\mathrm{Cob}(\mathcal{A}_-^0)\to\mathcal{A}_-^0\otimes_{\Bbbk\otimes\Bbbk}\mathcal{A}'$$

where ϕ' is as in Lemma 5.80. That is, define

$$(\mathbb{I}\otimes\phi')(\textstyle\sum a_i\otimes x_i^*) = \sum a_i[\phi'(x_i^*)].$$

Since ϕ' preserves the winding number, this induces a well-defined map of completed tensor products.

It is immediate from the definitions that this respects the Γ-gradings.

To see that $\mathbb{I}\otimes\phi'$ is a chain map, write the differential on $\mathfrak{C}^*$ as $\partial_1+\partial_2+\partial_4$ as in Lemma 5.89. Given an element $a\otimes x^*\in HC^*(\mathcal{A}^0)$, it is straightforward to check that $(\mathbb{I}\otimes\phi')(a\otimes\delta^{\mathrm{Cob}}(x^*)) = \partial_1(a[x])$: both sides are only non-zero if $x^* = U^*$. Similarly, terms of the form $d\otimes c^*\otimes x^*$ (respectively $d\otimes x^*\otimes b^*$) in the differential on $HC^*(\mathcal{A}^0)$ coming from operations $\mu_2^0(a,c) = d$ (respectively

$\mu_2^0(b,a) = d$) (Equation (5.84)) are mapped to zero by $\mathbb{I}\otimes\phi'$ unless c (respectively b) has length 1, in which case these terms correspond to ∂_2. Similarly, the only terms in the differential on $HC^*(\mathcal{A}^0)$ coming from μ_4^0-operations involving a which are not mapped to zero by $\mathbb{I}\otimes\phi'$ have the other three chords of length 1, and these terms correspond to ∂_4. Finally, by Lemma 3.24, every non-zero μ_n^0 with $n > 4$ has at least two inputs with length > 1, and hence the corresponding terms are mapped to zero by $\mathbb{I}\otimes\phi'$.

To see that $\mathbb{I}\otimes\phi'$ is a quasi-isomorphism, filter both $HC^*_\Gamma(\mathcal{A}^0_-)$ and $\mathfrak{C}^*_\Gamma$ by the total length of the element of $\mathcal{A}^0_-$. Both are complete with respect to this filtration. By Lemma 5.80, the map $\mathbb{I}\otimes\phi'$ induces an isomorphism of E^1-pages of the resulting spectral sequences, hence is a quasi-isomorphism. □

Proposition 5.92. *The Hochschild cohomology groups $HH^{W,k}_\Gamma(\mathcal{A}^0_-)$, $W > 0$, have*

$$HH^{W,-1}_\Gamma(\mathcal{A}^0_-) = \begin{cases} (\mathbb{F}_2)^2 & W = 1 \\ 0 & \textit{otherwise} \end{cases}$$

and $HH^{W,-2}_\Gamma(\mathcal{A}^0_-)$ is entirely supported in weight (W) grading 1. Moreover, one can choose a basis for $HH^{1,-1}_\Gamma(\mathcal{A}^0_-)$ so that one basis element sends $1\in\Bbbk$ to $\rho_{1234}+\rho_{2341}+\rho_{3412}+\rho_{4123}$ and the other sends $1\in\Bbbk$ to $U = U(\iota_0+\iota_1)$.

Proof. By Proposition 5.91, we can use $\mathfrak{C}^*_\Gamma$ to compute the Γ-graded Hochschild cohomology. In turn, this complex is generated by the following elements $a[b]$:

($\mathfrak{C}$-1) the elements
$$a[b] \in \{\rho_1[\rho_1], \rho_{123}[\rho_{123}], \iota_0[\iota_1], \iota_1[\iota_0]\},$$

($\mathfrak{C}$-2) any of the elements obtained by multiplying the above b by some further element b' with $|b'| = 4s_1$, and also multiplying a by some further element a' with $|a'| = 4s_2$. Here, a' might contain factors of U, and b' might contain (at most one factor of) h; with the understanding that $|U| = |h| = 4$.

($\mathfrak{C}$-3) any element obtained by adding some $i\in\mathbb{Z}/4\mathbb{Z}$ to all the indices in any of the above elements.

The proof is straightforward. Observe that in Case ($\mathfrak{C}$-2), s_1 and s_2 can be distinct. (Compared with Case (C-2), we now have more flexibility in the gradings.)

Next, we compute the (W,k) bigradings. The elements $\rho_1[\rho_1]$ and $\rho_{123}[\rho_{123}]$ have $(W,k) = (0,0)$, as do the additional six elements obtained by cyclically permuting indices; moreover, $\iota_0[\iota_1]$ and $\iota_1[\iota_0]$ have $(W,k) = (0,1)$. All other bigradings are determined by the relations

$$W(a\cdot a'[b\cdot b']) = W(a[b]) + s_1 - s_2 \qquad\text{and}\qquad k(a\cdot a'[b\cdot b']) = k(a[b]) - 2s_1$$

where $|a'| = 4s_1$ and $|b'| = 4s_2$, provided b' is not divisible by h; and

$$W(a[b\cdot h]) = W(a[b]) - 1 \qquad\text{and}\qquad k(a[b\cdot h]) = k(a[b]) + 1.$$

Thus, provided that $W > 0$, there are no non-zero elements with $k = 0$ and the following elements with $k = -1$ or $k = -2$:

- $k = -1$: $\rho_{1234}[\iota_1]$, $\rho_{2341}[\iota_0]$, $\rho_{3412}[\iota_1]$, $\rho_{4123}[\iota_0]$, $U\iota_0[\iota_1]$ and $U\iota_1[\iota_0]$.
- $k = -2$: $\rho_{1234123}[\rho_{123}]$, $U\rho_{123}[\rho_{123}]$, $\rho_{12341}[\rho_1]$, $U\rho_1[\rho_1]$, $\rho_{12341234}[h]$, $U\rho_{1234}[h]$, $U^2\iota_0[h\iota_1]$ and $U^2\iota_1[h\iota_0]$ and 18 more obtained by shifting the indices of all but the last two terms.

From the computation above, all of these elements have weight $W = 1$, proving the claim about $HH^{W,-2}_\Gamma(\mathcal{A}^0_-)$.

Differentials of terms with $k = -1$ are specified by Equations (5.86) and (5.87) (along with the usual symmetry of adding $i\in\mathbb{Z}/4\mathbb{Z}$ to all the subscripts in Equation (5.86)). It follows that $H^{>0,-1}(\mathfrak{C}) \cong (\mathbb{F}_2)^2$, generated by $\rho_{1234}[\iota_1]+\rho_{2341}[\iota_0]+\rho_{3412}[\iota_1]+\rho_{4123}[\iota_0]$ and $U\iota_0[\iota_1]+U\iota_1[\iota_0]$, as desired. □

Proof of Theorem 5.71. Throughout this proof, by deformation we will mean a Γ-graded deformation.

Since the trivial deformation ($\mu_n^w = 0$ for all $w > 0$) defines a weighted A_∞-algebra, the class $\mathfrak{O}^1$ must vanish. Thus, by Proposition 5.51, the isomorphism classes of 1-truncated weighted deformations correspond to $HH_\Gamma^{1,-1}(\mathcal{A}_-^0)$. By Proposition 5.92, this group is isomorphic to $(\mathbb{F}_2)^2$, and there is a unique generator satisfying Condition (2) of the theorem. By Corollary 5.70 and the fact that $HH_\Gamma^{W,-2} = HH_\Gamma^{W,-1} = 0$ for $W > 1$ from Proposition 5.92, this deformation extends uniquely to a weighted A_∞-algebra structure. □

Finally, we note that Theorems 5.45 and 5.71 also hold with the refined grading, by $G(T^2)$:

Corollary 5.93. *Up to isomorphism, there is a unique A_∞-deformation of $\mathcal{A}_-^{0,\mathrm{as}}$ over $\mathbb{F}_2[U]$ satisfying the following conditions:*

(1) The deformation is $G(T^2) \times \mathbb{Z}$-graded, where the gradings of the chords ρ_i is defined by $\gamma(\rho_i) = \mathrm{gr}_\psi(\rho_i) \times \mathrm{wn}(\rho_i)$.
(2) The operations satisfy $\mu_4(\rho_4, \rho_3, \rho_2, \rho_1) = U\iota_1$ and $\mu_4(\rho_3, \rho_2, \rho_1, \rho_4) = U\iota_0$.

Similarly, up to isomorphism, there is a unique weighted deformation $\mathcal{A}_-$ of $\mathcal{A}_-^0$ such that:

(1) $\mathcal{A}_-$ is $G(T^2) \times \mathbb{Z}$-graded and
(2) $\mu_0^1 = \rho_{1234} + \rho_{2341} + \rho_{3412} + \rho_{4123}$.

Proof. For the unweighted case, by Lemma 4.21, any such deformation of $\mathcal{A}_-^{0,\mathrm{as}}$ induces a $G \times \mathbb{Z}$-graded deformation. So, the result follows from Theorem 5.45. The weighted statement is similar, but using Theorem 5.71. □

Remark 5.94. The non-trivial weighted deformation from Theorem 5.71 is the one which appears in bordered Floer homology. Weighted actions on the modules are counts of rigid pseudo-holomorphic curves, where the weights signify the total number of Reeb orbits on the curve. In one-dimensional families, these Reeb orbits can wander off on the α-side, limiting to Reeb orbits either on boundary degenerations or curves at east infinity. (See [LOT23, Section 3].) The term μ_0^1 above comes from counting certain rigid curves at east infinity; see for example [LOT23, Figure 8].

6. The torus algebra and the wrapped Fukaya category

Let $\mathcal{A}_-^0$ denote the undeformed A_∞-algebra of $\mathcal{A}_-$. In this section we give an alternate interpretation of $\mathcal{A}_-^0$ in terms of the wrapped Fukaya category of the punctured torus.

Recall [AS10, Aur14] that the *wrapped Fukaya category* $\mathcal{W}(M)$ of a symplectic manifold (M, ω) with convex, conical ends has objects Lagrangians $L \subset (M, \omega)$ which are conical at infinity. Typically, these Lagrangians are also equipped with brane data—gradings and pin-structures—but we will not use brane data in this section. We will also later restrict to simply-connected Lagrangians. To define the morphism spaces, one chooses a Hamiltonian function H so that on the conical end $[1, \infty) \times Z$, $H(r, x) = r^2$. If ϕ^1 denotes the time-1 flow of H then $\mathrm{Hom}(L_0, L_1) = CF(\phi^1(L_0), L_1)$. The A_∞-composition map

$$\mathrm{Hom}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}(L_{n-1}, L_n) \to \mathrm{Hom}(L_0, L_n)$$

counts holomorphic polygons with Maslov index $2 - n$ with boundary on

$$\left(\phi^n(L_0), \phi^{n-1}(L_1), \phi^{n-2}(L_2), \dots, \phi^1(L_{n-1}), L_n\right)$$

(in counterclockwise order) to obtain an element of $CF(\phi^n(L_0), L_n)$ and then uses a rescaling trick and a continuation map to map $CF(\phi^n(L_0), L_n)$ to $CF(\phi^1(L_0), L_n) = \mathrm{Hom}(L_0, L_n)$ via a quasi-isomorphism.

(We have followed Auroux's exposition; Abouzaid-Seidel's original definition is more algebraic. Note that we are not using composition order, but rather the order of "morphisms" which is more natural for an algebra. See Figure 16 for our convention on the orientations of polygons.)

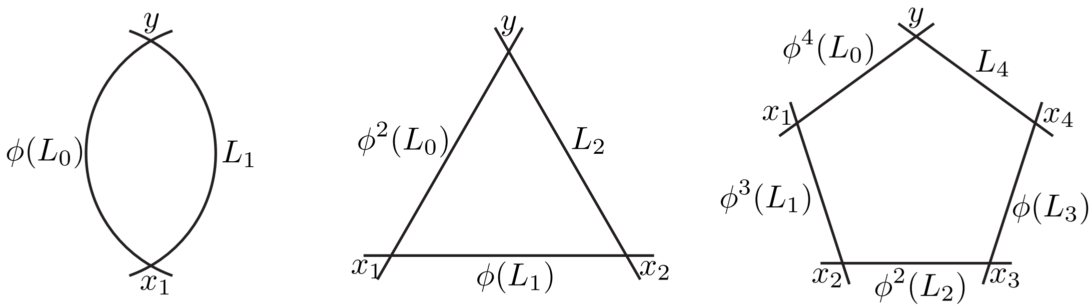


FIGURE 16. **Orientations of polygons.** Left: the model for a bigon contributing a term y in $\mu_1(x)$. Center: the model for a triangle contributing a term y in $\mu_2(x_1, x_2)$. Right: the model for a pentagon contributing a term y in $\mu_4(x_1, x_2, x_3, x_4)$. This convention is consistent with Auroux's [Aur14] but is the opposite convention from [OSz04].

We are interested in the case that $M = T^2 \setminus \{p\}$ is a punctured torus, where the holomorphic curve counts are combinatorial (by the Riemann mapping theorem) and the rescaling map $CF(\phi^n(L_0), L_n) \to CF(\phi^1(L_0), L_n) = \mathrm{Hom}(L_0, L_n)$ is induced by an obvious surface diffeomorphism.

Let α_1 and α_2 be the arcs from Figure 1, where the puncture p is $\alpha_1 \cap \alpha_2$. The following is a special case of a result of Auroux [Aur14, Theorem 4.7]:

Theorem 6.1. *[Aur10a, Aur10b, Aur14] The wrapped Fukaya category of the punctured torus, $\mathcal{W}(T^2 \setminus \{p\})$, is quasi-equivalent to* $\mathrm{End}_{\mathcal{W}}(\alpha_1 \oplus \alpha_2)$.

We consider a relative version of the wrapped Fukaya category of $T^2 \setminus \{p\}$ by fixing a point $z \in T^2 \setminus \{p\}$ not in the conical end, considering Lagrangians disjoint from z, and a Hamiltonian function H which vanishes in a neighborhood of z. We can then enhance the Floer complexes $CF(\phi^1(L_0), L_n)$ to be modules over $\mathbb{F}_2[U]$ by counting holomorphic disks $u: \mathbb{D}^2 \to T^2 \setminus \{p\}$ with weight $U^{n_z(u)}$, where $n_z(u)$ is the multiplicity of u at z. The polygon counts used to define the higher compositions on the wrapped Fukaya category inherit a similar weight. Let $\mathcal{W}_z(T^2 \setminus \{p\})$ denote this relative Fukaya category. (For a more general discussion of relative Fukaya categories, see [She15].)

With these definitions in hand, the goal of this section is to prove:

Theorem 6.2. *There is an A_∞-quasi-isomorphism* $\mathcal{A}_-^0 \simeq \mathrm{End}_{\mathcal{W}_z}(\alpha_1 \oplus \alpha_2)$.

Corollary 6.3. *There is an (ungraded) A_∞-quasi-isomorphism* $\mathcal{A}_-^0|_{U=1} \simeq \mathrm{End}_{\mathcal{W}}(\alpha_1 \oplus \alpha_2) \simeq \mathcal{W}(T^2 \setminus \{p\})$.

(In fact, the proof of Theorem 6.2 also specifies a grading on $\mathcal{A}_-^0|_{U=1}$ so that this becomes a graded quasi-isomorphism.)

We will deduce Theorem 6.2 from the uniqueness theorem for $\mathcal{A}_-^0$, Theorem 5.45. Since gradings play an essential role in the uniqueness theorem, we introduce a corresponding notion of gradings on the wrapped Fukaya category in Section 6.1, before returning to the model computations needed to prove Theorem 6.2 in Section 6.2.

6.1. **Gradings on the wrapped Fukaya category.** In this section, we define a grading on the wrapped Fukaya category using (a variant of) the notions of anchored Lagrangians [FOOO10, She15]. Fix a symplectic manifold (M, ω) with $\pi_2(M) = 0$; for instance, $(M, \omega) = (T^2 \setminus \{p\}, \mathrm{Area})$. Let $\mathcal{L}ag(M)$ denote the space of Lagrangian subspaces of TM. The space $\mathcal{L}ag(M)$ fibers over M, with fiber diffeomorphic to the Grassmanian $\mathcal{L}ag$ of Lagrangian planes in $\mathbb{R}^{2n}$. Any Lagrangian submanifold $L \subset M$ has a canonical lift $\tilde{L} \subset \mathcal{L}ag(M)$.

Recall that $\mathcal{L}ag$ is path connected and $\pi_1(\mathcal{L}ag) \cong \mathbb{Z}$; the isomorphism is given by the Maslov index. Thus, since $\pi_2(M) = 0$, for basepoints $b \in M$ and $\tilde{b} \in \mathcal{L}ag(M)$, we have an exact sequence

$$0 \to \mathbb{Z} = \pi_1(\mathcal{L}ag) \to \pi_1(\mathcal{L}ag(M), \tilde{b}) \to \pi_1(M, b) \to 0.$$

Lemma 6.4. *The action of $\pi_1(M,b)$ on $\pi_1(\mathcal{L}ag)$ is trivial.*

Proof. Recall that the action of $\gamma \in \pi_1(M,b)$ on $\eta \in \pi_1(\mathcal{L}ag(T_bM), \tilde{b})$ is given by applying the homotopy lifting property to the diagram

$$\begin{array}{ccc} [0,1] \times \{0\} & \xrightarrow{\ \eta\ } & \mathcal{L}ag(M) \\ \big\downarrow & \nearrow & \big\downarrow \\ [0,1] \times [0,1] & \xrightarrow[(s,t)\mapsto\gamma(t)]{} & M. \end{array}$$

and restricting the dashed arrow to $[0,1]\times\{1\}$. (We may be negligent about basepoints since $\pi_1(\mathcal{L}ag)$ is abelian.) Since $\pi_1(\mathcal{L}ag) \cong \mathbb{Z}$ and γ acts by a group automorphism on $\pi_1(\mathcal{L}ag)$, the induced element $\gamma \cdot \eta$ has the form $\pm\eta$. So, it suffices to show that for each γ there is some nontrivial η with $\gamma \cdot \eta = \eta$.

Fix an almost complex structure J on M compatible with ω, making TM into a Hermitian vector bundle. Consequently, TM inherits an action of S^1 which takes Lagrangian subspaces to Lagrangian subspaces. Hence, we have an action $\cdot : S^1 \times \mathcal{L}ag(M) \to \mathcal{L}ag(M)$. Observe that the image of $\tilde{b}$ under the action of S^1 is $n = \dim(M)/2$ times a generator of $\pi_1(\mathcal{L}ag(T_bM), \tilde{b}) \cong \mathbb{Z}$. Let η be this loop in $\pi_1(\mathcal{L}ag(T_bM), \tilde{b})$. Then we can take the dashed arrow to be the map $(s,t) \mapsto e^{2\pi i t} \cdot \gamma(s)$, and the restriction to $\{1\} \times [0,1]$ is again η. ☐

Consequently, $\pi_1(\mathcal{L}ag(M), \tilde{b})$ is a $\mathbb{Z}$ central extension of $\pi_1(M,b)$. Let $\lambda \in \pi_1(\mathcal{L}ag(M), \tilde{b})$ be the image of the positive generator of $\pi_1(\mathcal{L}ag)$ (the one with $\mu = 1$ or, equivalently, the direction induced by an almost complex structure; in the one-dimensional case, this is counterclockwise rotation).

Given a pair of connected Lagrangians $L, L' \subset M$, let $S(L,L')$ be the set of homotopy classes of maps $([0,1], \{0\}, \{1\}) \to (\mathcal{L}ag(M), \tilde{L}, \tilde{L}')$, i.e., the homotopy classes of paths from $\tilde{L}$ to $\tilde{L}'$. The group $\pi_1(\mathcal{L}ag) = \mathbb{Z}$ acts on $S(L,L')$ by concatenating with loops in the fiber. In particular, we can view λ as an element of $S(L,L)$.

The following is a simple extension of Definition 4.18 to A_∞-categories (rather than algebras):

Definition 6.5. *Given a groupoid G with a central element λ_s or λ (Definition 4.17) and an A_∞-category $\mathscr{C}$, with composition operations denoted μ_n, a* grading of $\mathscr{C}$ by G *consists of:*

- *For each object $L \in \mathrm{Ob}(\mathscr{C})$, an object $s(L) \in \mathrm{Ob}(G)$.*
- *For each pair of objects L, L' of $\mathscr{C}$, a decomposition*

$$\mathrm{Hom}(L,L') = \bigoplus_{\gamma \in \mathrm{Hom}_G(s(L), s(L'))} \mathrm{Hom}(L,L';\gamma).$$

An element $x \in \mathrm{Hom}_G(s(L), s(L');\gamma)$ is called homogeneous, *and we write* $\mathrm{gr}(x) = \gamma$.

These data are required to satisfy the condition that for homogeneous elements $x_i \in \mathrm{Hom}(L_i, L_{i+1})$, $i = 1, \dots, n$, the element $\mu_n(x_1, x_2, \dots, x_n)$ is homogeneous and

$$\mathrm{gr}\big(\mu_n(x_1, x_2, \dots, x_n)\big) = \lambda^{n-2}\,\mathrm{gr}(x_1)\,\mathrm{gr}(x_2)\cdots\mathrm{gr}(x_n). \tag{6.6}$$

If the Lagrangian submanifolds L are all simply-connected then we can form a groupoid $\mathcal{G}$ whose objects are Lagrangians in M, whose morphism sets are $\mathrm{Hom}(L,L') = S(L,L')$, and whose composition $\mathrm{Hom}(L,L') \times \mathrm{Hom}(L',L'') \to \mathrm{Hom}(L,L'')$ sends (γ,γ') to the concatenation $\gamma * \eta * \gamma'$, where η is a path in L' from $\gamma(1)$ to $\gamma'(0)$. This concatenation is well-defined because L' is simply-connected.

For any Lagrangian L, $S(L,L) \cong \pi_1(\mathcal{L}ag(M))$ which, in particular, contains an element λ_L, and these elements λ are central in the sense described in Definition 6.5.

The grading on the Fukaya category by $\mathcal{G}$ is given as follows. Consider the following two kinds of paths in the Lagrangian Grassmanian:

- Given Lagrangians L and L' with $L \pitchfork L'$ and $x \in L \cap L'$ let $\gamma_x^{L\to L'}$ be the standard path in $\mathcal{L}ag(T_xM)$ from T_xL to T_xL', corresponding to moving in the negative direction from T_xL and remaining transverse to T_xL for all positive time. ("Negative" means opposite the direction induced by an almost complex structure, that is, negative means moving from Jv towards v; compare [RS93]. In the one-dimensional case, this is clockwise in the plane.)
- Given $m,n \in \mathbb{R}$ and $x \in \phi^m(L)$ let $\eta_x^{\phi^m(L)\to\phi^n(L)}$ be the path in $\mathcal{L}ag(M)$ from $T_x\phi^m(L)$ to $T_{\phi^{n-m}(x)}\phi^n(L)$ induced by the Hamiltonian isotopy of L. (That is, the Hamiltonian induces a path of Lagrangians, hence a path in the Lagrangian Grassmanian bundle.)

Then, given Lagrangians L_0 and L_1 with $\phi^1(L_0) \pitchfork L_1$ and an element $x \in \phi^1(L_0)\cap L_1$ the element $\mathrm{gr}(x) \in S(L_0,L_1)$ is the concatenation

$$\mathrm{gr}(x) = \eta_{\phi^{-1}(x)}^{\phi^0(L_0)\to\phi^1(L_0)} * \gamma_x^{\phi^1(L_0)\to L_1} \tag{6.7}$$

of the path from $T_{\phi^{-1}(x)}L_0$ to $T_x\phi^1(L_0)$ induced by the Hamiltonian isotopy and the positive path in T_xM from $T_x\phi^1(L_0)$ to T_xL_1.

It is sometimes convenient to view all the elements $\mathrm{gr}(x)$ as having the same endpoints, by defining some additional paths:

- Given $x,y \in L$, let $\epsilon_{x\to y}^L$ be a path in $\tilde{L}$ from T_xL to T_yL.

Then if we choose a point $q_0 \in L_0$ and $q_1 \in L_1$ we can also write

$$\mathrm{gr}(x) = \epsilon_{q_0\to\phi^{-1}(x)}^{L_0} * \eta_{\phi^{-1}(x)}^{\phi^0(L_0)\to\phi^1(L_0)} * \gamma_x^{\phi^1(L_0)\to L_1} * \epsilon_{x\to q_1}^{L_1}. \tag{6.8}$$

Proposition 6.9. *Equation (6.7) or (6.8) defines a grading of the wrapped Fukaya category $\mathcal{W}(M)$ of simply-connected Lagrangians in M by the groupoid $\mathcal{G}$.*

Proof. We need to check the compatibility condition (6.6). The idea is that if x_0 appears as a term in $\mu_n(x_1,\cdots,x_n)$ then there is a holomorphic disk connecting $\phi^{n-1}(x_1)$, $\phi^{n-2}(x_2),\dots,x_n$ and x_0 with Maslov index $2-n$. We will use the following lemma, which relates the gradings on points in $\phi^n(L_n)\cap L_0$ and $\phi^1(L_n)\cap L_0$, so the product of the gradings of the x_i is equal to λ^{n-2} times the grading of x_0.

Lemma 6.10. *Let $x \in \phi^1(L_0)\cap L_n$ and let x' be a point in $\phi^n(L_0)\cap L_n$ which corresponds to x under dilation. Let $q_0 \in L_0$ and $q_n \in L_n$ be any points. Then we have the following equality in $S(L_0,L_n)$:*

$$\epsilon_{q_0\to\phi^{-1}(x)}^{L_0} * \eta_{\phi^{-1}(x)}^{\phi^0(L)\to\phi^1(L)} * \gamma_x^{\phi^1(L_0)\to L_n} * \epsilon_{x\to q_n}^{L_n} = \epsilon_{q_0\to\phi^{-n}(x')}^{L_0} * \eta_{\phi^{-n}(x')}^{\phi^0(L)\to\phi^n(L)} * \gamma_{x'}^{\phi^n(L_0)\to L_n} * \epsilon_{x'\to q_n}^{L_n}.$$

Proof. The statement is clearly independent of the choice of H (defining ϕ), q_0, and q_n and of compactly supported Hamiltonian isotopies of L_0 and L_n. So, let $q_0 = q_n \in L_0 \pitchfork L_n$ outside the conical end and choose H to vanish in a neighborhood of q_0. Then the two loops are homotopic to

$$\epsilon_{q_0\to x}^{\phi^1(L_0)} * \gamma_x^{\phi^1(L_0)\to L_n} * \epsilon_{x\to q_0}^{L_n} \qquad \text{and} \qquad \epsilon_{q_0\to x'}^{\phi^n(L_0)} * \gamma_{x'}^{\phi^n(L_0)\to L_n} * \epsilon_{x'\to q_0}^{L_n}.$$

So, we want to show that the loop

$$\begin{aligned}\epsilon_{q_0\to x}^{\phi^1(L_0)} * \gamma_x^{\phi^1(L_0)\to L_n} * \epsilon_{x\to q_0}^{L_n} * \epsilon_{q_0\to x'}^{L_n} * \left(\gamma_{x'}^{\phi^n(L_0)\to L_n}\right)^{-1} * \epsilon_{x'\to q_0}^{\phi^n(L_0)} \\ = \epsilon_{q_0\to x}^{\phi^1(L_0)} * \gamma_x^{\phi^1(L_0)\to L_n} \epsilon_{x\to x'}^{L_n} * \left(\gamma_{x'}^{\phi^n(L_0)\to L_n}\right)^{-1} * \epsilon_{x'\to q_0}^{\phi^n(L_0)}\end{aligned}$$

is nullhomotopic. Applying the homotopy given by dilation (with varying dilation parameter) to the last three terms, this loop becomes

$$\epsilon^{\phi^1(L_0)}_{q_0\to x} * \gamma^{\phi^1(L_0)\to L_n}_x * \epsilon^{L_n}_{x\to x} * \left(\gamma^{\phi^1(L_0)\to L_n}_x\right)^{-1} * \epsilon^{\phi^1(L_0)}_{x\to q_0}$$

which is nullhomotopic, as desired. ☐

Returning to the proof of Proposition 6.9, for $i = 1,\dots,n$ let $x_i \in \phi^1(L_{i-1}) \cap L_i$ and let $x_0 \in \phi^1(L_0) \cap L_n$, and suppose that x_0 appears as a term in $\mu_n(x_1,\cdots,x_n)$. Since this composition map counts holomorphic disks with Maslov index $2-n$, we have

$$\begin{aligned}(6.11)\quad &\epsilon^{\phi^n(L_0)}_{x_0'\to\phi^{n-1}(x_1)} * \gamma^{\phi^n(L_0)\to\phi^{n-1}(L_1)}_{\phi^{n-1}(x_1)} * \epsilon^{\phi^{n-1}(L_1)}_{\phi^{n-1}(x_1)\to\phi^{n-2}(x_2)} * \gamma^{\phi^{n-1}(L_1)\to\phi^{n-2}(L_2)}_{\phi^{n-2}(x_2)} * \epsilon^{\phi^{n-2}(L_2)}_{\phi^{n-2}(x_2)\to\phi^{n-3}(x_3)}\\ &\qquad * \cdots * \gamma^{\phi^1(L_{n-1})\to L_n}_{x_n} * \epsilon^{L_n}_{x_n\to x_0'} * (\gamma^{\phi^n(L_0)\to L_n}_{x_0'})^{-1} = \lambda^{2-n}.\end{aligned}$$

See, for instance, [Aur14, Definition 1.8 and Formula (2.5)], and note in particular that the inverse of the negative path $\gamma^{\phi^n(L_0)\to L_n}_{x_0}$ at x_0 from $\phi^n(L_0)$ to L_n is the positive path at x_0 from L_n to L_0.

On the other hand,

$$\begin{aligned}&\mathrm{gr}(x_1)\,\mathrm{gr}(x_2)\cdots\mathrm{gr}(x_n)\,\mathrm{gr}(x_0)^{-1}\\ &\quad= \eta^{\phi^0(L_0)\to\phi^1(L_0)}_{\phi^{-1}(x_1)} * \gamma^{\phi^1(L_0)\to L_1}_{x_1} * \epsilon^{L_1}_{x_1\to\phi^{-1}(x_2)} * \eta^{\phi^0(L_1)\to\phi^1(L_1)}_{\phi^{-1}(x_2)} * \gamma^{\phi^1(L_1)\to L_2}_{x_2} * \epsilon^{L_2}_{x_2\to\phi^{-1}(x_3)}\\ &\qquad * \cdots * \eta^{\phi^0(L_{n-1})\to\phi^1(L_{n-1})}_{\phi^{-1}(x_n)} * \gamma^{\phi^1(L_{n-1})\to L_n}_{x_n} * \epsilon^{L_n}_{x_n\to x_0} * \left(\eta^{\phi^0(L_0)\to\phi^1(L_0)}_{\phi^{-1}(x_0)} * \gamma^{\phi^1(L_0)\to L_n}_{x_0}\right)^{-1}\\ &\quad= \eta^{\phi^0(L_0)\to\phi^1(L_0)}_{\phi^{-1}(x_1)} * \gamma^{\phi^1(L_0)\to L_1}_{x_1} * \epsilon^{L_1}_{x_1\to\phi^{-1}(x_2)} * \eta^{\phi^0(L_1)\to\phi^1(L_1)}_{\phi^{-1}(x_2)} * \gamma^{\phi^1(L_1)\to L_2}_{x_2} * \epsilon^{L_2}_{x_2\to\phi^{-1}(x_3)}\\ &\qquad * \cdots * \eta^{\phi^0(L_{n-1})\to\phi^1(L_{n-1})}_{\phi^{-1}(x_n)} * \gamma^{\phi^1(L_{n-1})\to L_n}_{x_n} * \epsilon^{L_n}_{x_n\to x_0'} * \left(\eta^{\phi^0(L_0)\to\phi^n(L_0)}_{\phi^{-n}(x_0')} * \gamma^{\phi^n(L_0)\to L_n}_{x_0'}\right)^{-1}\end{aligned}$$

where the second equality uses Lemma 6.10. Since Hamiltonian isotopies take standard paths associated to Lagrangian intersections to standard paths associated to Lagrangian intersections and paths in Lagrangians to paths in Lagrangians, we have, for instance,

$$\gamma^{\phi^1(L_0)\to L_1}_{x_1} * \epsilon^{L_1}_{x_1\to\phi^{-1}(x_2)} * \eta^{\phi^0(L_1)\to\phi^1(L_1)}_{\phi^{-1}(x_2)} = \eta^{\phi^1(L_0)\to\phi^2(L_0)}_{x_1} * \gamma^{\phi^2(L_0)\to\phi(L_1)}_{\phi(x_1)} * \epsilon^{\phi(L_1)}_{\phi(x_1)\to x_2}.$$

Consequently, commuting the copies of η to the left gives

$$\begin{aligned}\mathrm{gr}(x_1)\cdots\mathrm{gr}(x_n)\,\mathrm{gr}(x_0)^{-1} &= \eta^{\phi^0(L_0)\to\phi^n(L_0)}_{\phi^{-n}(x_0)} * \gamma^{\phi^n(L_0)\to\phi^{n-1}(L_1)}_{\phi^{n-1}(x_1)} * \epsilon^{\phi^{n-1}(L_1)}_{\phi^{n-1}(x_1)\to\phi^{n-2}(x_2)}\\ &\quad * \cdots * \gamma^{\phi^1(L_{n-1})\to L_n}_{x_n} * \epsilon^{L_n}_{x_n\to x_0} * \left(\gamma^{\phi^n(L_0)\to L_n}_{x_0}\right)^{-1} * \left(\eta^{\phi^0(L_0)\to\phi^n(L_0)}_{\phi^{-n}(x_0)}\right)^{-1}.\end{aligned}$$

By Formula (6.11), since λ is central, this is equal to $\lambda^{2-n} \in S(L_0, L_0)$, as desired. ☐

We can reduce this groupoid grading to a grading by a group, as in Section 4.3.2, as follows. Fix a basepoint $\tilde{b} \in \mathcal{L}ag(M)$. For each Lagrangian L choose a point $\tilde{b}_L \in \tilde{L}$ and a path η_L in $\mathcal{L}ag(M)$ from $\tilde{b}$ to $\tilde{b}_L$, i.e., an *anchor* for L. Let $G = \pi_1(\mathcal{L}ag(M), \tilde{b})$. For each pair L_0, L_1 there is an induced identification $S(L_0, L_1) \cong G$ which sends a path γ from $\tilde{L}_0$ to $\tilde{L}_1$ to the concatenation $\eta_{L_0} * \nu_0 * \gamma * \bar{\nu}_1 * \eta_{L_1}^{-1}$ where ν_i is a path in $\tilde{L}_i$ from $\tilde{b}_{L_i}$ to $\gamma(i)$. Since L_i is simply connected, this is independent of the choice of paths ν_0 and ν_1. Further, this construction defines a homomorphism of groupoids from $\mathcal{G}$ to G sending λ to λ. Hence, it induces a grading of $\mathcal{W}(M)$ by G.

We are interested in the case that $M = T^2 \setminus \{p\}$ and the Lagrangians are α_1 and α_2. In this case, the grading group $\pi_1(\mathcal{L}ag(T^2 \setminus \{p\}), \tilde{b})$ is a $\mathbb{Z}$ central extension of the free group F_2. Indeed, the circle bundle

$$\mathcal{L}ag \to \mathcal{L}ag(T^2 \setminus \{p\}) \to T^2 \setminus \{p\}$$

is trivial, since any surface bundle over a punctured surface is, so we have $\pi_1(\mathcal{L}ag(T^2 \setminus \{p\}), \tilde{b}) \cong \mathbb{Z} \times F_2$. This isomorphism is not canonical; to fix one, let $a = S^1 \times \{q\}$ and $b = \{q\} \times S^1$ in $S^1 \times S^1 = T^2$, viewed as oriented loops so that $a\cdot b=1$. The curves a,b are Lagrangian, so have canonical lifts $\tilde{a},\tilde{b}\subset \mathcal{L}ag(T^2\setminus\{p\})$. Then $\{\lambda,\tilde{a},\tilde{b}\}$ is a set of generators for $\pi_1(\mathcal{L}ag(T^2\setminus\{p\},\tilde{b}))$, giving an isomorphism $\pi_1(\mathcal{L}ag(T^2\setminus\{p\},\tilde{b}))\cong\mathbb{Z}\times F_2$.

If we impose the relation $\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}=\lambda^{-2}$ we obtain the group G. Hence, we have a surjection $\pi_1(\mathcal{L}ag(T^2\setminus\{p\}),\tilde{b})\to G$, and we can consider the Fukaya category of $T^2\setminus\{p\}$ graded by G.

We will also exploit one further *winding number grading*, coming from the fact that we are considering a relative Fukaya category. Fix a path γ_z from z to p so that:

- γ_z approaches $\{p\}$ from the region labeled ρ_4 in Figure 1.
- γ_z is disjoint from $\alpha_1\cup\alpha_2$.

(See, e.g., Figure 18.) Choose the Hamiltonian H so that ϕ^1 has a single fixed point ι_0 on α_1 and a single fixed point ι_1 on α_2, and H is constant in a neighborhood of z. Given a point $\rho\in\phi^1(\alpha_i)\cap\alpha_j$, let $\mathrm{wn}(\rho)$ be the algebraic intersection number of the path in $\phi^1(\alpha_i)$ from ι_i to ρ with γ_z. More generally, given $\rho\in\phi^m(\alpha_i)\cap\phi^n(\alpha_j)$, $m>n$, let η_ρ (respectively ν_ρ) be the path in $\phi^m(\alpha_i)$ from ι_i to ρ (respectively in $\phi^n(\alpha_j)$ from ι_j to ρ), and set

$$\mathrm{wn}(\rho)=\gamma_z\cdot(\eta_\rho*\overline{\nu_\rho}),$$

the difference of algebraic intersection numbers. Finally, define $\mathrm{wn}(U)=1$.

Lemma 6.12. *If $\rho\in\phi^m(\alpha_i)\cap\phi^n(\alpha_j)$ and ρ' is the corresponding point in $\phi^{m-n}(\alpha_i)\cap\alpha_j$ then $\mathrm{wn}(\rho)=\mathrm{wn}(\rho')$. Similarly, if $\rho\in\phi^n(\alpha_i)\cap\alpha_j$ and ρ' is the point in $\phi^1(\alpha_i)\cap\alpha_j$ which corresponds to ρ under rescaling then $\mathrm{wn}(\rho)=\mathrm{wn}(\rho')$. Finally, the composition maps in $\mathrm{End}_{\mathcal{W}_z}(\alpha_1\oplus\alpha_2)$ preserve wn.*

Proof. The first statement follows from the observation that the loops $(\eta_\rho*\overline{\nu_\rho})$ and $(\eta_{\rho'}*\overline{\nu'_\rho})$ are isotopic in $T^2\setminus\{p,z\}$. The second statement is clear. The third statement follows from the fact that the intersection number of the boundary of a polygon with γ_z is the same as the multiplicity with which the polygon covers z. ☐

6.2. **Model computations and the proof of Theorem 6.2.** In this section, we prove Theorem 6.2, i.e., that $\mathrm{End}_{\mathcal{W}_z}(\alpha_1\oplus\alpha_2)$ is quasi-isomorphic to $\mathcal{A}^0_-$. In fact, we will choose perturbations defining $\mathrm{End}_{\mathcal{W}_z}(\alpha_1\oplus\alpha_2)$ so that the two algebras are isomorphic.

The identification of an $\mathbb{F}_2$-basis for $\mathcal{A}^0_-$ and $\mathrm{End}_{\mathcal{W}_z}(\alpha_1\oplus\alpha_2)$ is shown in Figure 17. In words, let $M=T^2\setminus\{p\}$, fix a cylindrical neighborhood V of the puncture, and choose α_1 and α_2 so that, in cylindrical polar coordinates around the puncture, $\alpha_1\cap\partial V=\{(0,0),(0,\pi)\}$ and $\alpha_2\cap\partial V=\{(0,\pi/2),(0,3\pi/2)\}$. Choose a perturbation $\tilde{\alpha}_i$ of α_i so that the two intervals in $\tilde{\alpha}_i\cap V$ are slightly counter-clockwise of the $\alpha_i\cap V$ and so that $\tilde{\alpha}_i$ intersects α_i in a single point.

Let $q^0_{0,0}=\alpha_1\cap\tilde{\alpha}_1$ and $q^0_{1,1}=\alpha_2\cap\tilde{\alpha}_2$, and let $\alpha^k_i=\phi^k(\tilde{\alpha}_i)$. Then, still using cylindrical polar coordinates, we have

$$\begin{aligned}
\alpha_1\cap\alpha^1_1&=\{(n-\epsilon,0)\mid n=1,2,\dots\}\cup\{(n-\epsilon,\pi)\mid n=1,2,\dots\}\cup\{q^0_{0,0}\}\\
\alpha_2\cap\alpha^1_2&=\{(n-\epsilon,\pi/2)\mid n=1,2,\dots\}\cup\{(n-\epsilon,3\pi/2)\mid n=1,2,\dots\}\cup\{q^0_{1,1}\}\\
\alpha_1\cap\alpha^1_2&=\{(n+1/2,0)\mid n=0,1,\dots\}\cup\{(n+1/2,\pi)\mid n=0,1,\dots\}\\
\alpha_2\cap\alpha^1_1&=\{(n+1/2,\pi/2)\mid n=0,1,\dots\}\cup\{(n+1/2,3\pi/2)\mid n=0,1,\dots\},
\end{aligned}$$

where the ϵs are small, unimportant positive real numbers.

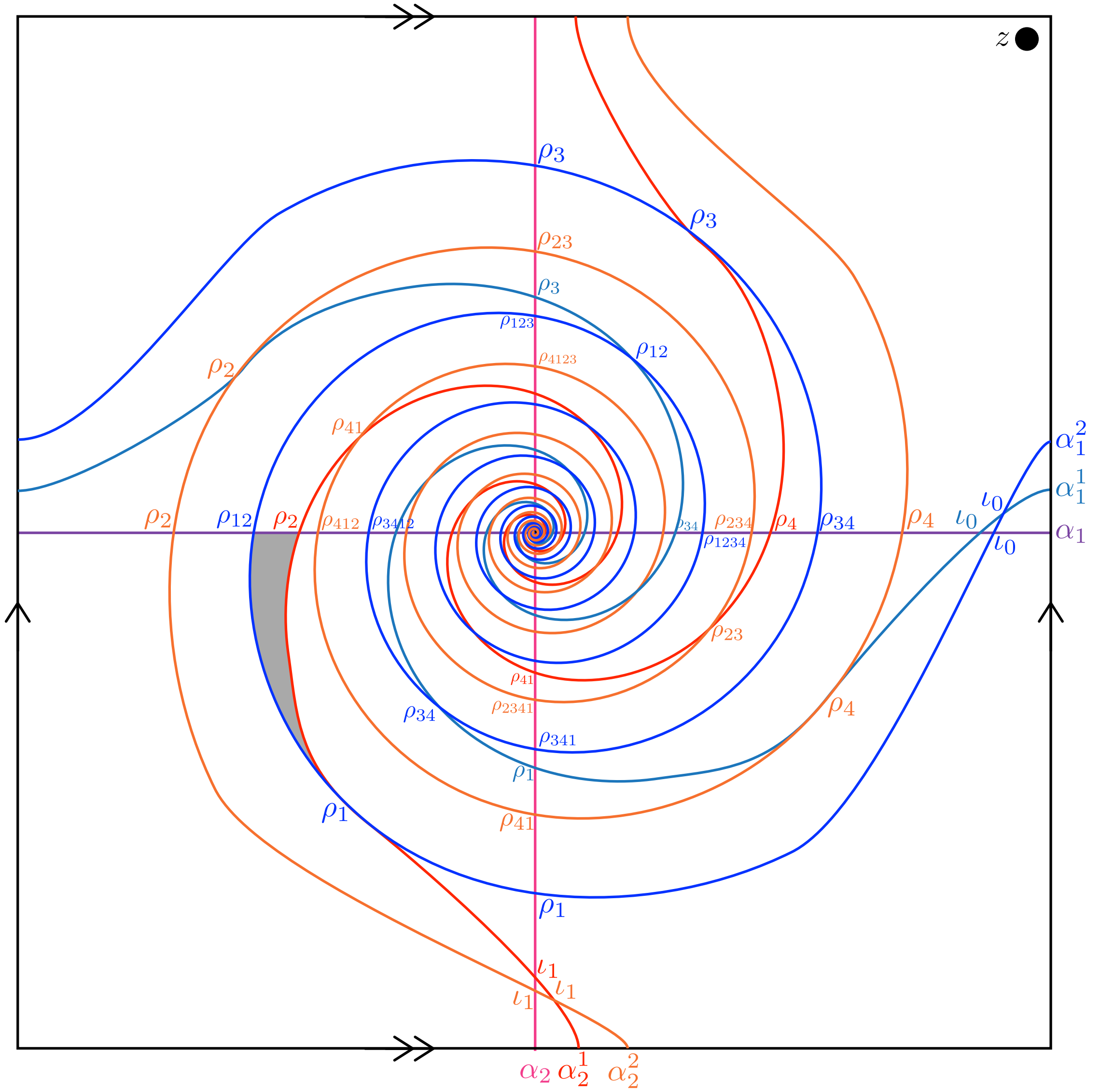


FIGURE 17. **Wrapped Fukaya category of the torus.** The puncture is in the middle of the picture. The shaded triangle corresponds to the operation $\mu_2(\rho_1, \rho_2) = \rho_{12}$. Here, we have perturbed the α-curves slightly to resolve the triple intersections at the fixed points ι_0 and ι_1 (which are fixed points of ϕ).

The identification between these generators and $\mathcal{A}_-^0$ is given by

$$q^0_{0,0} \leftrightarrow \iota_0 \qquad\qquad q^0_{1,1} \leftrightarrow \iota_1$$

$$(n-\epsilon, 0) \leftrightarrow \begin{cases} \rho_{34}\rho_{1234}^{(n-1)/2} & n \text{ odd} \\ \rho_{1234}^{n/2} & n \text{ even} \end{cases} \qquad (n-\epsilon, \pi) \leftrightarrow \begin{cases} \rho_{12}\rho_{3412}^{(n-1)/2} & n \text{ odd} \\ \rho_{3412}^{n/2} & n \text{ even} \end{cases}$$

$$(n-\epsilon, \pi/2) \leftrightarrow \begin{cases} \rho_{23}\rho_{4123}^{(n-1)/2} & n \text{ odd} \\ \rho_{4123}^{n/2} & n \text{ even} \end{cases} \qquad (n-\epsilon, 3\pi/2) \leftrightarrow \begin{cases} \rho_{41}\rho_{2341}^{(n-1)/2} & n \text{ odd} \\ \rho_{2341}^{n/2} & n \text{ even} \end{cases}$$

$$(n+1/2, 0) \leftrightarrow \begin{cases} \rho_4(\rho_{1234})^{n/2} & n \text{ even} \\ \rho_{234}(\rho_{2341})^{(n-1)/2} & n \text{ odd} \end{cases} \qquad (n+1/2, \pi) \leftrightarrow \begin{cases} \rho_2(\rho_{3412})^{n/2} & n \text{ even} \\ \rho_{412}(\rho_{3412})^{(n-1)/2} & n \text{ odd} \end{cases}$$

$$(n+1/2, \pi/2) \leftrightarrow \begin{cases} \rho_3(\rho_{4123})^{n/2} & n \text{ even} \\ \rho_{123}(\rho_{4123})^{(n-1)/2} & n \text{ odd} \end{cases} \qquad (n+1/2, 3\pi/2) \leftrightarrow \begin{cases} \rho_1(\rho_{2341})^{n/2} & n \text{ even} \\ \rho_{341}(\rho_{2341})^{(n-1)/2} & n \text{ odd.} \end{cases}$$

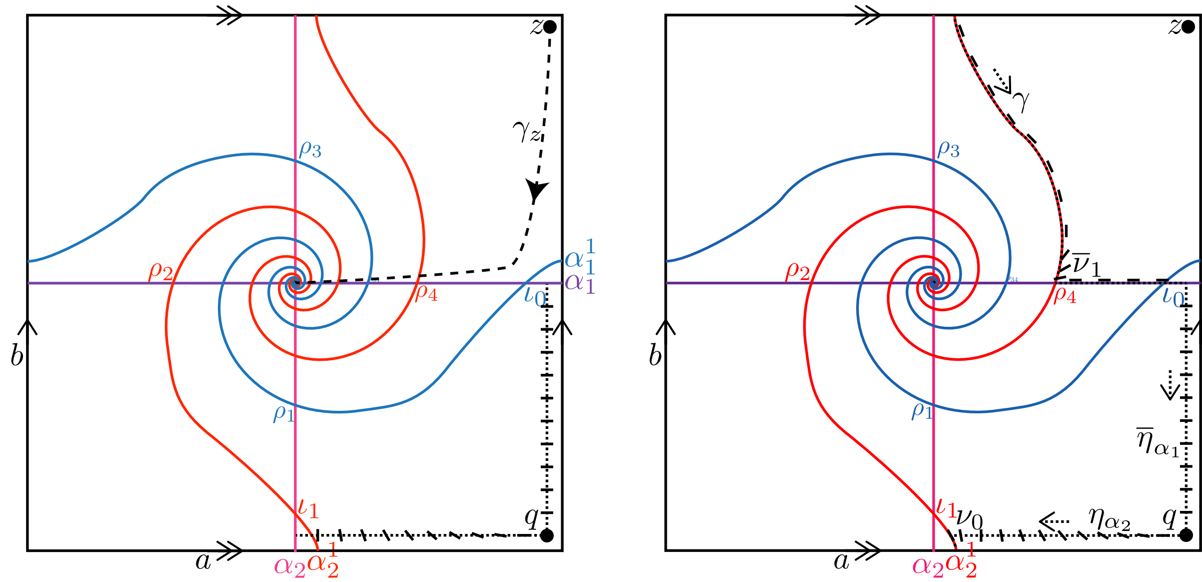


FIGURE 18. **Identifying the gradings on the Fukaya category of the torus.** Left: the base paths to the two Lagrangians used to identify the gradings (dotted). Right: the loop giving the grading of ρ_4. The short black segments indicate the lifts of the paths to the Lagrangian Grassmannian bundle. The paths ν_0 and $\overline{\nu}_1$ correspond to the planes turning at the end; more precisely, these happen at the intersection points where γ ends, but we have merged them slightly with η_{α_2} and γ to make the drawing legible.

We will henceforth abuse notation and use ρ_I to refer to both the element of $\mathcal{A}_{-}^{0}$ and the corresponding element of $\operatorname{End}_{\mathcal{W}_z}(\alpha_1 \oplus \alpha_2)$.

Lemma 6.13. *The above identification of basis elements intertwines the winding number gradings. Further, there is a homomorphism from $\mathbb{Z} \times F_2$ to $G(T^2)$ so that if we let gr_F denote the induced $G(T^2)$-valued grading on the Fukaya category then the identification of basis elements intertwines the grading $\mathrm{gr}_F - (2\,\mathrm{wn}; 0, 0)$ on the Fukaya category and the grading gr_ψ from Section 4.3.2 on $\mathcal{A}_{-}$.*

Proof. It is clear that the identification respects the winding number grading; see Figure 18 on the left for the curve γ_z.

To specify the grading on the Fukaya category, choose anchors for the two Lagrangians as shown in Figure 18. There is then a loop of Lagrangian subspaces of the tangent space to T^2 associated to each intersection point; an example is shown on the right in Figure 18. Trivialize the Lagrangian Grassmanian bundle using the section of lines at slope $\pi/4$ in the figure, so to extract an integer from the Maslov (fiber) component of the grading, one counts with sign how many times the loop of Lagrangian subspaces passes through slope $\pi/4$.

We will construct the homomorphism $\mathbb{Z} \times F_2$ to $G(T^2)$ below. To see that it intertwines the gradings of generators, it suffices to show that it intertwines the gradings of ρ_1, ρ_2, ρ_3, ρ_4, and U.

It is straightforward to compute that the $\mathbb{Z} \times F_2$-valued grading $\widetilde{\mathrm{gr}}_F$ is given by

$$\widetilde{\mathrm{gr}}_F(\rho_1) = (0; \mathbb{I}) \qquad \widetilde{\mathrm{gr}}_F(\rho_2) = (-1; a^{-1})$$

$$\widetilde{\mathrm{gr}}_F(\rho_3) = (0; b * a) \qquad \widetilde{\mathrm{gr}}_F(\rho_4) = (-1; b^{-1}).$$

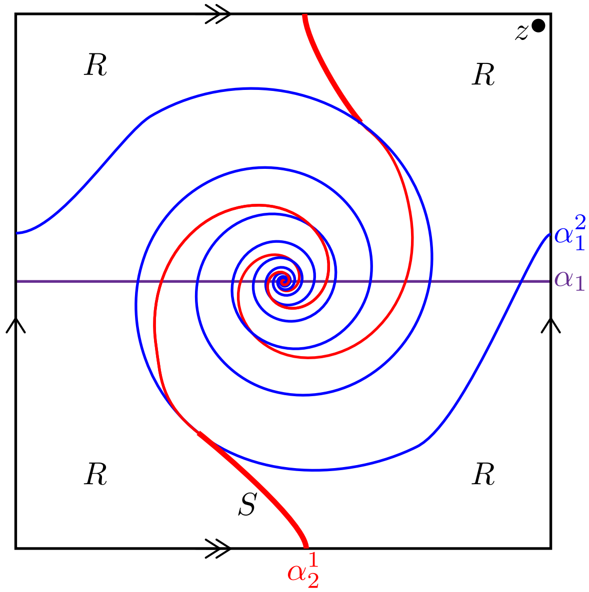


FIGURE 19. **No triangles cover z.** The segment S on α_j discussed in the proof of Lemma 6.14 is in bold, and the region R is indicated.

(The final computation is illustrated on the right in Figure 18.) By construction, we have $\widetilde{\mathrm{gr}}_F(U) = (0;0)$. The winding number grading is given by

$$\mathrm{wn}(\rho_1) = \mathrm{wn}(\rho_2) = \mathrm{wn}(\rho_3) = 0 \qquad \mathrm{wn}(\rho_4) = \mathrm{wn}(U) = 1.$$

Consider the homomorphism from $\mathbb{Z} \times F_2$ to $G(T^2)$ defined by

$$(1;0) \mapsto \lambda = (1;0,0) \qquad (0;a) \mapsto (-1/2;-1,0) \qquad (0;b) \mapsto (-1/2;0,1).$$

The image of $\widetilde{\mathrm{gr}}_F$ under this map is

$$\mathrm{gr}_F(\rho_1) = (0;0,0) \qquad \mathrm{gr}_F(\rho_2) = (-1;0,0)\cdot(1/2;1,0) = (-1/2;1,0)$$
$$\mathrm{gr}_F(\rho_3) = (-1/2;0,1)\cdot(-1/2;-1,0) = (0;-1,1) \qquad \mathrm{gr}_F(\rho_4) = (-1;0,0)\cdot(1/2;0,-1) = (-1/2;0,-1)$$

To identify this with the grading on $\mathcal{A}_-$ we subtract two times the winding number grading from the Maslov component of the grading, to get

$$\mathrm{gr}(\rho_1) = (0;0,0) \qquad \mathrm{gr}(\rho_2) = (-1/2;1,0)$$
$$\mathrm{gr}(\rho_3) = (0;-1,1) \qquad \mathrm{gr}(\rho_4) = (-5/2;0,-1) \qquad \mathrm{gr}(U) = (-2;0,0).$$

This agrees with the grading gr_ψ in Section 4.3.2, as claimed. □

Lemma 6.14. *The above identification of basis elements intertwines the operations μ_2.*

Proof. This is similar to the computation for T^*S^1 [Aur14]. For any of the operations $\rho_I\rho_J = \rho_{I\cup J}$ there is a unique small triangle contained in the conical end of $T^2 \smallsetminus \{p\}$ connecting $\phi^1(\rho_I)$, ρ_J, and the image of $\rho_{I\cup J}$ under rescaling. This is straightforward to verify, for instance, by considering lifts of the α-arcs to the universal cover of the conical end of $T^2 \smallsetminus \{p\}$. We may take the lift of each component of $\alpha_i \cap [0,\infty)\times Z$ to be a straight line at slope 0, and $\phi^1(\alpha_j)$ and $\phi^2(\alpha_i)$ to be straight lines at slopes π and 2π. The choice of a lift of $\alpha_i \cap [0,\infty)\times Z$ and the points $\rho_{I\cup J}$ and ρ_J fixes the other two lifts, and the three lines cut out a triangle. We just need to verify that the third vertex of the triangle is at the point $\phi^1(\rho_I)$. Writing $\widetilde{\phi}^1(x,y) = (x, y+\pi x)$ for the induced action on $[0,\infty)\times Z$, this is a straightforward computation. Triangles realizing the operations $\iota_i\rho_J=\rho_J$ and $\rho_J\iota_i=\rho_J$ can be found similarly.

Further, we show that there is no immersed triangle in $T^2\smallsetminus\{p\}$ with boundary on α_i, $\phi^1(\alpha_j)$ and $\phi^2(\alpha_k)$ covers the region R containing z. If all of the α_i that appear are the same, then R is a non-trivial annulus that cannot be part of a disk. Otherwise, one α_i appears one time; suppose it is $\phi^1(\alpha_2)$, as in Figure 19, and consider the resulting segment S, indicated there, which has R on both sides. The boundary of such a triangle would then have to traverse S an even number of times (since R is on both sides of the segment). Since the boundary of the triangle is immersed, in fact S must be traversed 0 times. Then the purported triangle contains a non-nullhomotopic loop, the preimage of the loop in R passing through S once, which is a contradiction. Hence, terms of the form $U^m\rho_K$ for $m>0$ do not appear in $\rho_I\rho_J$. So, it follows from Lemma 6.13 that there are no more terms in $\rho_I\rho_J$. □

Lemma 6.15. *The operation* μ_4 *on* $\mathrm{End}_{\mathcal{W}_z}(\alpha_1\oplus\alpha_2)$ *satisfies*

$$\mu_4(\rho_4,\rho_3,\rho_2,\rho_1)=U\iota_1$$
$$\mu_4(\rho_3,\rho_2,\rho_1,\rho_4)=U\iota_0.$$

Proof. By inspection, there is a holomorphic pentagon contributing each of these terms. See Figure 20. □

Proof of Theorem 6.2. This is immediate from Corollary 5.93 and Lemmas 6.13, 6.14, and 6.15. □

Remark 6.16. While $\mathcal{A}_-^0$ has a grading by $\mathbb{Z}\times F_2$, $\mathcal{A}_-$ does not: with respect to this grading, the element $\mu_0^1=\rho_{1234}+\rho_{2341}+\rho_{3412}+\rho_{4123}$ does not have grading a central element of $\mathbb{Z}\times F_2$ (as required) and, in fact, is not even homogeneous. After quotienting by the relation $\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}=\lambda^{-2}$, the grading of μ_0^1 becomes well-defined and central.

7. Signs

In this section, we lift the discussion above from $\mathbb{F}_2$-coefficients to $\mathbb{Z}$-coefficients. Specifically, we lift the weighted algebra $\mathcal{A}_-$ to a weighted algebra over $\mathbb{Z}$, denoted $\mathcal{A}_{-;\mathbb{Z}}$, and extend the uniqueness theorems, Theorems 5.45 and 5.71, to $\mathbb{Z}$-coefficients. Note that for the case of $\widehat{HF}$, there is also work of Knowles-Petkova lifting not just the bordered algebra but also the modules over it (defined via nice diagrams) to $\mathbb{Z}$-coefficients [KP21].

7.1. Weighted algebras over $\mathbb{Z}$.

So far, we have discussed group-graded algebras with characteristic 2. We explain how to extend this to arbitrary characteristic.

In order to define A_∞-algebras with characteristic $\neq 2$, the algebra needs to be equipped with a $\mathbb{Z}/2\mathbb{Z}$-valued grading, which we denote by $\|\cdot\|$ here. This grading is needed to formulate the Leibniz rule:

$$d(a\cdot b)=(da)\cdot b+(-1)^{\|a\|}a\cdot(db)$$

or, more generally, the n-input A_∞ relation

$$\sum_{n=r+s+t}(-1)^{r+st}\mu_{r+1+t}\circ(\mathbb{I}^{\otimes r}\otimes\mu_s\otimes\mathbb{I}^{\otimes t})=0. \tag{7.1}$$

In interpreting the above expression, we follow conventions of Keller [Kel01], according to which

$$(f\otimes g)(x\otimes y)=(-1)^{\|g\|\|x\|}f(x)\otimes g(y). \tag{7.2}$$

The A_∞-algebra homomorphism relation on $\{f_n\colon A_+^{\otimes n}\to B_+\}$ takes the form

$$\sum_{n=r+s+t}(-1)^{r+st}f_{r+1+t}\circ(\mathbb{I}^{\otimes r}\otimes\mu_s\otimes\mathbb{I}^{\otimes t})=\sum_{n=i_1+\dots i_m}(-1)^{\sigma}\mu_m\circ(f_{i_1}\otimes\cdots\otimes f_{i_m}),$$

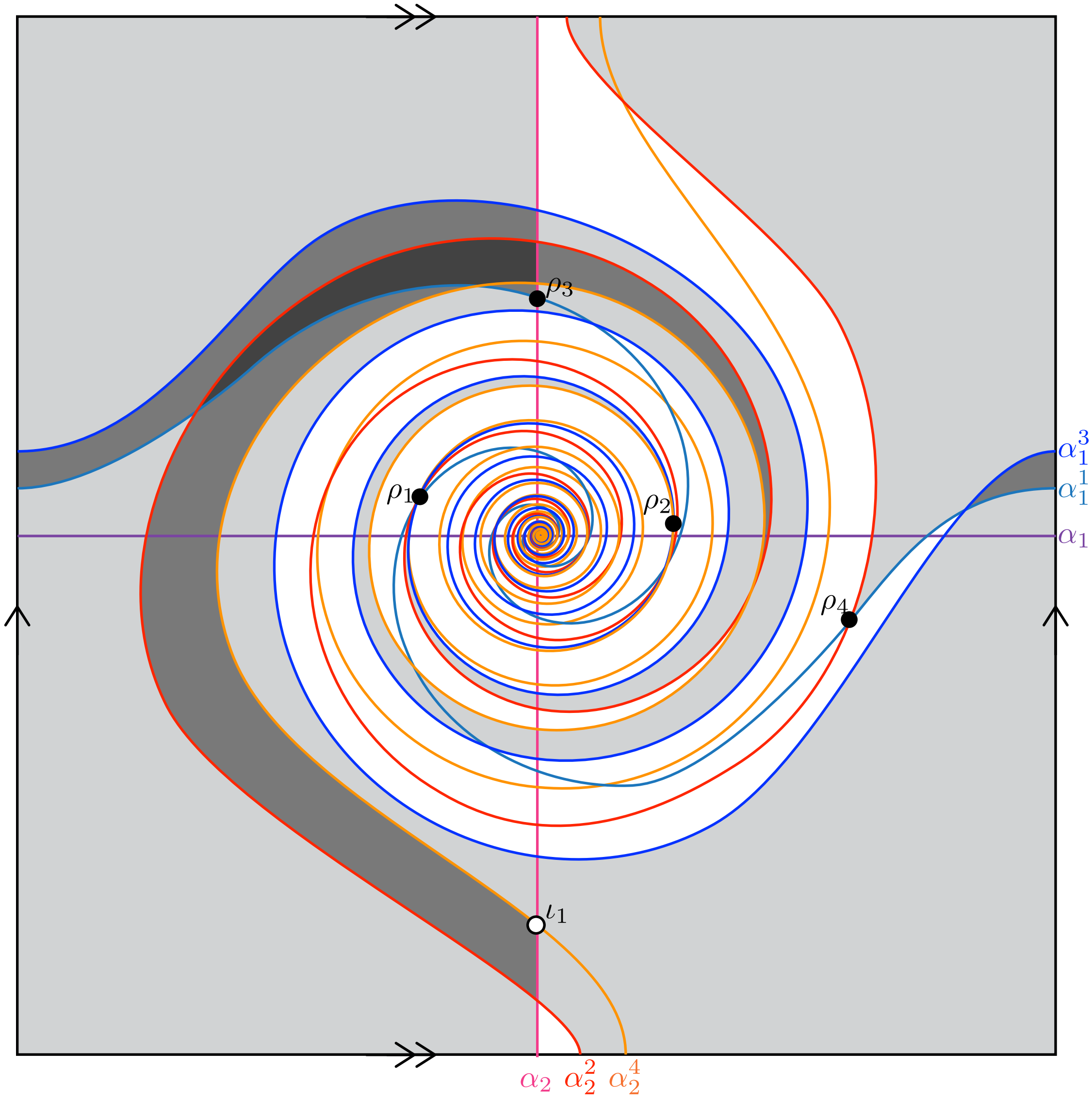


FIGURE 20. **The μ_4-operation.** The shaded region is the image of the pentagon contributing $\mu_4(\rho_2,\rho_1,\rho_4,\rho_3) = U\iota_1$. Lightly shaded regions are covered once, medium-shading twice, and darkly-shaded regions are covered three times.

where $\sigma = \sigma(i_1,\dots,i_m)$ is given by

$$\sigma = \sum_{k=1}^{m-1}(m-k)(i_k-1). \tag{7.3}$$

The A_∞ relation has the obvious weighted generalization (for each n, w)

$$\sum_{\substack{n=r+s+t\\ w=u+v}}(-1)^{r+st}\mu^u_{r+1+t}\circ(\mathbb{I}^{\otimes r}\otimes\mu^v_s\otimes\mathbb{I}^{\otimes t}) = 0. \tag{7.4}$$

(That is, the weight is treated as having even grading, so does not contribute to the sign.) Similarly, the a weighted A_∞ homomorphism relation (for each n and w) is given by

$$\sum_{\substack{n=r+s+t\\w=w_1+w_2}}(-1)^{r+st}f^{w_1}_{r+1+t}\circ(\mathbb{I}^{\otimes r}\otimes\mu^{w_2}_s\otimes\mathbb{I}^{\otimes t}) = \sum_{\substack{n=i_1+\dots i_m\\w=w_0+w_1+\dots+w_m}}(-1)^{\sigma}\mu^{w_0}_m\circ(f^{w_1}_{i_1}\otimes\dots\otimes f^{w_m}_{i_m}), \tag{7.5}$$

with σ as in Equation (7.3).

Another convention we adhere to is the following: if (C_*,∂) and (C'_*,∂') are $\mathbb{Z}/2\mathbb{Z}$-graded complexes, then $\mathrm{Mor}(C_*,C'_*)$ is also $\mathbb{Z}/2\mathbb{Z}$-graded, with differential given by

$$\partial(f)=\partial\circ f-(-1)^{\|f\|}f\circ\partial. \tag{7.6}$$

7.2. **Construction of the bordered algebras over $\mathbb{Z}$.** Define $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}$ exactly as in Section 3.1, with the understanding that the path algebra is taken over $\mathbb{Z}$, rather than $\mathbb{F}_2$. For the purposes of signs, we equip this algebra with the $\mathbb{Z}/2\mathbb{Z}$ grading with the property that all of $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}$ is supported in degree 0. In other words, use the grading ϵ from Section 4.5.

Define the operations on $\mathcal{A}_{-;\mathbb{Z}}$ as in Definition 3.10, by counting tiling patterns; every tiling pattern counts positively. Similarly, no signs appear in the definition of μ^1_0 from Equation (3.11). The operations μ^w_n are non-zero only when n is even. This simplifies the sign computations: the sign appearing in the weighted A_∞ relation (Equation (7.4)) is $(-1)^r$. Additionally, since $\mathcal{A}_{-;\mathbb{Z}}$ is supported in $\mathbb{Z}/2\mathbb{Z}$-grading 0, the sign in Equation (7.2) drops out.

A more careful look at the proof of Theorem 3.19 gives the following generalization:

Theorem 7.7. *The operations above give $\mathcal{A}_{-;\mathbb{Z}}$ the structure of a weighted A_∞-algebra over $\mathbb{Z}$.*

Proof. We must check that the cancelling terms in the proof of Theorem 3.19 indeed come with opposite signs.

We leave the case with ≤ 3 inputs or ≤ 2 inputs and weight ≥ 1 to the reader.

We consider the other cancellations in the order they are verified in Theorem 3.19, now with sign. As in that proof, we can consider input sequences of basic algebra elements, with at most one element which is an idempotent (i.e., not Reeb-like); and the idempotent comes either as the first or last element in the sequence. When the idempotent is the first element, we have terms of type $(2{*}{-})$ and $({*}2{+})$. Evidently, the term of type $(2{*}{-})$ has $r=1$ (and $t=0$), so it contributes to the A_∞ relation with sign -1; while the term of type $({*}2{+})$ has $r=0$ (and $s=2$), so it contributes to the A_∞ relation with sign $+1$. Thus these terms cancel with sign. Similarly, when the idempotent comes at the end, we obtain cancellation of terms of type $(2{*}{+})$, which counts with sign $+1$ (as it has $r=0$ and even s), against terms of type $({*}2{-})$, which counts with sign -1 (as it has odd r and $t=0$).

Next, we consider the cancellation of terms as in Lemma 3.15. For the factorization of $\rho^i=\sigma_1\cdot\sigma_2$, we have a term of type $({*}2{*})$ in the notation of the proof of Theorem 3.19; or $\mathfrak{P}$, in the notation of Lemma 3.15. The μ_2 node has $r=i-1$. Consider the corresponding element of $\mathfrak{S}_1\cup\mathfrak{T}\cup\mathfrak{L}\cup\mathfrak{R}$ under the correspondence from Lemma 3.15. If the term is of type $\mathfrak{T}$, the term has $r=i$. If the terms is of type $\mathfrak{P}\cup\mathfrak{L}\cup\mathfrak{R}$ we have a composition of two terms; and the higher one can have ρ_{i+1} as its first input, in which case $r=i$; or it can have ρ_i as its last input, in which case $r=i-s$, where the second term has s inputs. In any of these cases, by Lemma 3.7, the composite contributes with a sign of $(-1)^i$.

For the remaining cases—the cancellations coming from the bijection M from the proof of Theorem 3.19–all cancellations are between a term of type + (i.e., $({*}0{+})$, $(2{*}{+})$, $({*}2{+})$, and $[{*}{*}{+}]$, where each $*\in\{L,C,R\}$) and a term of type $-$. By construction, terms of type + contribute $+1$, and terms of type $-$ contribute -1, so the cancellation holds with signs, as well. ∎

7.3. **Section 5.1 revisited.** There are two equivalent formulations of the A_∞ relations, and we will find it to convenient to move between them. One formulation is stated in Equation (7.1). There is an alternative, somewhat simpler formulation, as follows.

To set notation, let $A[1]$ denote A with the following grading shift: $A[1]_g = A_{g\cdot\lambda^{-1}}$. In particular, if A is $\mathbb{Z}$-graded, then $A[1]_n = A_{n-1}$. Thus, the identity map can be viewed as a map $\sigma\colon A \to A[1]$, of degree $+1$. (Note that we follow homological conventions, rather than the cohomological conventions of [Kel02].)

We now view the A_∞ operations as maps $\boldsymbol{\mu}_k\colon A_+[1]^{\otimes k} \to A[1]$ of degree -1. With this normalization, the A_∞ relation takes the simpler form:

$$\sum_{n=r+s+t} \boldsymbol{\mu}_{r+1+t} \circ (\mathbb{I}^{\otimes r} \otimes \boldsymbol{\mu}_s \otimes \mathbb{I}^{\otimes t}) = 0. \tag{7.8}$$

(Our $\boldsymbol{\mu}_n$ are the maps b_n from [GJ90, Kel02].) Formula (7.8) is equivalent to Formula (7.1), as follows. The operations $\boldsymbol{\mu}_n$ and μ_n are related by the following commutative diagram:

$$\begin{array}{ccc} (A[1])^{\otimes n} & \xrightarrow{\boldsymbol{\mu}_n} & A[1] \\ \uparrow{\scriptstyle \sigma^{\otimes n}} & & \uparrow{\scriptstyle \sigma} \\ A^{\otimes n} & \xrightarrow{\mu_n} & A. \end{array}$$

Then, the relation $0 = (\boldsymbol{\mu}\circ\boldsymbol{\mu})_n \circ \sigma^{\otimes n}$ gives

$$\begin{aligned} 0 &= \sum_{\{r,s|r+1+s=n\}} \boldsymbol{\mu}_{r+1+s} \circ (\mathbb{I}_{A[1]^{\otimes r}} \otimes \boldsymbol{\mu}_s \otimes \mathbb{I}_{A[1]^{\otimes t}}) \circ (\sigma^{\otimes r} \otimes \sigma^{\otimes s} \otimes \sigma^{\otimes t}) \\ &= \sum_{\{r,s|r+1+s=n\}} (-1)^{\|\boldsymbol{\mu}_s\|\cdot\|\sigma^{\otimes r}\|} \boldsymbol{\mu}_{r+1+s} \circ (\sigma^{\otimes(s+1)} \otimes \mathbb{I}_{A[1]^{\otimes t}}) \circ (\mathbb{I}_{A_+^{\otimes r}} \otimes (\sigma^{-1} \circ \boldsymbol{\mu}_s \circ \sigma^{\otimes s}) \otimes \sigma^{\otimes t}) \\ &= \sum_{\{r,s|r+1+s=n\}} (-1)^{\|\boldsymbol{\mu}_s\|\cdot\|\sigma^{\otimes r}\|} \boldsymbol{\mu}_{r+1+s} \circ (\sigma^{\otimes(s+1)} \otimes \mathbb{I}_{A[1]^{\otimes t}}) \circ (\mathbb{I}_{A_+^{\otimes r}} \otimes \mu_s \otimes \sigma^{\otimes t}) \\ &= \sum_{\{r,s|r+1+s=n\}} (-1)^{\|\boldsymbol{\mu}_s\|\cdot\|\sigma^{\otimes r}\| + \|\mu_s\|\cdot\|\sigma^{\otimes t}\|} \boldsymbol{\mu}_{r+1+s} \circ (\sigma^{\otimes(s+1+t)}) \circ (\mathbb{I}_{A_+^{\otimes r}} \otimes \mu_s \otimes \mathbb{I}_{A_+^{\otimes t}}) \\ &= \sum_{\{r,s|r+1+s=n\}} (-1)^{r+st} \mu_{r+1+s}(\mathbb{I}_{A_+^{\otimes r}} \otimes \mu_s \otimes \mathbb{I}_{A_+^{\otimes t}}), \end{aligned}$$

since $\mu_n = \sigma^{-1} \circ \boldsymbol{\mu}_n \circ \sigma^{\otimes n}$, $\|\boldsymbol{\mu}_s\| = -1$, $\|\sigma^{\otimes \ell}\| = -\ell$, so $\|\mu_s\| = s-2$. We prefer to use the normalization $\boldsymbol{\mu}_n$ in the proofs below, since the formulas are a little simpler.

We also normalize the A_∞-homomorphism relation as follows. For us, an A_∞-homomorphism is a collection of maps $\{\boldsymbol{f}_n\colon A_+[1]^n \to B_+[1]\}$ of degree 0, satisfying

$$\sum_{n=r+s+t} \boldsymbol{f}_{r+1+t} \circ (\mathbb{I}^{\otimes r} \otimes \boldsymbol{\mu}_s \otimes \mathbb{I}^{\otimes t}) = \sum_{n=i_1+\dots i_m} \boldsymbol{\mu}_m(\boldsymbol{f}_{i_1} \otimes \cdots \otimes \boldsymbol{f}_{i_m}). \tag{7.9}$$

The bar complex is defined by

$$\mathrm{Bar}(A) = \bigoplus_{n=0}^{\infty} A \otimes (A_+[1])^{\otimes n} \otimes A,$$

with differential

$$\partial\colon A \otimes (A_+[1])^{\otimes n} \otimes A \to A \otimes (A_+[1])^{\otimes(n-1)} \otimes A$$

defined by

$$\sum_{r=0}^{n} \mathbb{I}_{A^{\otimes r}[r-1]} \otimes \boldsymbol{\mu}_2 \otimes \mathbb{I}_{A^{\otimes(n-1-r)}[n-2-r]}.$$

(Compare Equation (5.2).) We can define the Hochschild complex as in Definition 5.1, by

$$HC^*(A) = \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(\mathrm{Bar}(A), A),$$

with its induced differential.

Equivalently, the analogue of Equation (5.4) gives

$$HC^n(A) = \mathrm{Hom}_{\Bbbk\otimes\Bbbk}((A_+[1])^{\otimes n}, A),$$

with differential

$$\delta \boldsymbol{f}_n = \boldsymbol{\mu}_2(\mathbb{I}_{A_+[1]} \otimes \boldsymbol{f}_n) - (-1)^{\|\boldsymbol{f}_n\|}\boldsymbol{f}_n \circ \left(\sum_{r=0}^{n-1} \mathbb{I}_{A_+[1]^{\otimes r}} \otimes \boldsymbol{\mu}_2 \otimes \mathbb{I}_{A_+[1]}^{n-r}\right) + \boldsymbol{\mu}_2 \circ (\boldsymbol{f}_n \otimes \mathbb{I}_{A_+[1]}). \tag{7.10}$$

(The sign comes from Equation (7.6).)

The isomorphism of chain complexes

$$\Phi\colon \bigoplus_n \mathrm{Hom}_{\Bbbk\otimes\Bbbk}(A_+^{\otimes n}, A) \to \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(\mathrm{Bar}(A), A)$$

is defined by

$$\Phi(\boldsymbol{f}) = (\boldsymbol{\mu}_2 \circ (\mathbb{I}_A \otimes \boldsymbol{\mu}_2)) \circ (\mathbb{I}_A \otimes \boldsymbol{f} \otimes \mathbb{I}_A).$$

In particular, if an A_n algebra has only $\boldsymbol{\mu}_2$ and $\boldsymbol{\mu}_n$, the A_∞ relation with $n+1$ inputs is

$$\delta\boldsymbol{\mu}_n = 0.$$

From its construction the Hochschild complex inherits a $\mathbb{Z}/2\mathbb{Z}$ grading.

Proposition 7.11. *The statement of Proposition 5.7 holds in arbitrary characteristic.*

Proof. Signs are incorporated into the definition of $\star$, as follows:

$$\boldsymbol{f}_i \star \boldsymbol{g}_m = \boldsymbol{f}_i \circ \left(\sum_{i=r+1+t} \mathbb{I}^{\otimes r} \otimes \boldsymbol{g}_m \otimes \mathbb{I}^{\otimes t}\right).$$

Thus,

$$\delta \boldsymbol{f}_n = \boldsymbol{\mu}_2 \star \boldsymbol{f}_n - (-1)^{\|\boldsymbol{f}_n\|}\boldsymbol{f}_n \star \boldsymbol{\mu}_2.$$

With these sign conventions, define

$$\mathfrak{O}_n = -\sum_{\substack{i,j\geq 3\\ i+j=n+2}} \boldsymbol{\mu}_i \star \boldsymbol{\mu}_j.$$

With this notation, using Equation (7.10) and the fact that $\|\boldsymbol{\mu}_n\| = 1$, the A_∞ relation with $n+1$ inputs (Equation (7.8)) becomes

$$\delta\boldsymbol{\mu}_n + \sum_{\substack{i,j\geq 3\\ i+j=n+2}} \boldsymbol{\mu}_i \star \boldsymbol{\mu}_j = 0$$

(compare Equation (5.10)). Thus, Property (A$\mathfrak{O}$-2) holds in arbitrary characteristic.

To verify Property (A$\mathfrak{O}$-1), we use the following refinement of Equation (5.11):

$$\delta(\boldsymbol{f}_i \star \boldsymbol{g}_j) = (\delta\boldsymbol{f}_i) \star \boldsymbol{g}_j + (-1)^{\|\boldsymbol{f}_i\|}\boldsymbol{f}_i \star (\delta\boldsymbol{g}_j) - \boldsymbol{\mu}_2(\boldsymbol{f}_i, \boldsymbol{g}_j) - (-1)^{\|\boldsymbol{f}_i\|\|\boldsymbol{g}_j\|}\boldsymbol{\mu}_2(\boldsymbol{g}_j, \boldsymbol{f}_i). \tag{7.12}$$

The last two terms, for example, cancel against terms in $(\delta\boldsymbol{f}_i) \star \boldsymbol{g}_j$, as follows. Note that $(\delta\boldsymbol{f}_i)$ includes a term $\boldsymbol{\mu}_2 \circ (\mathbb{I} \otimes \boldsymbol{f}_i)$. When computing $\big(\boldsymbol{\mu}_2 \circ (\mathbb{I} \otimes \boldsymbol{f}_i)\big) \star \boldsymbol{g}_j$, the term with $r = 0$ is given by

$$(\mathbb{I} \otimes \boldsymbol{f}_i) \circ (\boldsymbol{g}_j \otimes \mathbb{I}^{\otimes i}) = (-1)^{\|\boldsymbol{f}_i\|\|\boldsymbol{g}_j\|}\boldsymbol{g}_j \otimes \boldsymbol{f}_i.$$

Also, $(\delta\boldsymbol{f}_i)$ includes another term, $\boldsymbol{\mu}_2(\boldsymbol{f}_i \otimes \mathbb{I})$. The $r = i$, $t = 0$ term in $\big(\boldsymbol{\mu}_2 \circ (\boldsymbol{f}_i \otimes \mathbb{I})\big) \star \boldsymbol{g}_j$ is

$$\boldsymbol{\mu}_2 \circ (\boldsymbol{f}_i \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \boldsymbol{g}_j) = \boldsymbol{\mu}_2(\boldsymbol{f}_i, \boldsymbol{g}_j).$$

Thus, since $\|\boldsymbol{\mu}_i\| = \|\boldsymbol{\mu}_j\| = 1$, we have that

$$\delta(\mathfrak{O}_n) = -\sum_{\substack{i,j\geq 3\\ i+j=n+2}} \Big[(\delta\boldsymbol{\mu}_i)\star\boldsymbol{\mu}_j - \boldsymbol{\mu}_i\star(\delta\boldsymbol{\mu}_j)\Big]$$
$$= \sum_{\substack{i,j,k\geq 3\\ i+j+k=n+2}} \Big[(\boldsymbol{\mu}_i\star\boldsymbol{\mu}_j)\star\boldsymbol{\mu}_k - \boldsymbol{\mu}_i\star(\boldsymbol{\mu}_j\star\boldsymbol{\mu}_k)\Big].$$

To complete the verification of Property (A$\mathfrak{O}$-1), we use the following signed version of Equation (5.12):

$$(a\star b)\star c - a\star(b\star c) = a\circ(\mathbb{I}\otimes b\otimes\mathbb{I}\otimes c\otimes\mathbb{I})\circ\Delta^5 + (-1)^{\|b\|\|c\|}a\circ(\mathbb{I}\otimes c\otimes\mathbb{I}\otimes b\otimes\mathbb{I})\circ\Delta^5. \tag{7.13}$$

To verify Property (A$\mathfrak{O}$-3), observe that the existence of a homomorphism $\boldsymbol{f}\colon\mathcal{A}\to\mathcal{A}'$ is equivalent to the existence of c_{n-1} with $\delta(c_{n-1}) = \mu_n - \mu_n'$. The proof of Property (A$\mathfrak{O}$-4) from before applies similarly.

Turning to the statements about maps, we define $\tilde{\boldsymbol{f}}$ as in Equation (5.15), D as in Equation (5.16), and $\widetilde{\widetilde{\phi}}$ as in Equation (5.17).

We will assume that the components of ϕ_i all have $\mathbb{Z}/2\mathbb{Z}$ grading equal to zero. In particular, under the assumption that $\|\phi_i\| = 0$, Equation (5.18) is replaced by

$$\delta((\boldsymbol{f}\circ\widetilde{\widetilde{\phi}})_m) = (\delta(\boldsymbol{f})\circ\widetilde{\widetilde{\phi}})_{m+1} + (-1)^{\|f\|}(\boldsymbol{f}\circ D\widetilde{\widetilde{\phi}})_{m+1} - \sum_{\substack{i,j\\ i+j=m+1, i>1}}\Big(\boldsymbol{\mu}_2(\phi_i,(\boldsymbol{f}\circ\widetilde{\widetilde{\phi}})_j) + \boldsymbol{\mu}_2((\boldsymbol{f}\circ\widetilde{\widetilde{\phi}})_j,\phi_i)\Big).$$

Summing the maps $\boldsymbol{\mu}_k\colon A_+[1]^{\otimes k}\to A$ over all $k\leq n$ gives a map

$$\boldsymbol{\mu}_{\leq n}\colon\bigoplus_{k<n} A_+[1]^{\otimes k}\to A.$$

Let

$$\mathfrak{F}_n = \Big(\sum_{\substack{j\geq 3\\ i+j=n+1}}\boldsymbol{f}_i\star\boldsymbol{\mu}_j\Big) - \Big(\boldsymbol{\mu}_{\leq n}\circ\widetilde{\widetilde{\boldsymbol{f}}}\Big)_n\colon A_+[1]^{\otimes n}\to A,$$

where $(\cdot)_n$ denotes the restriction of a map from $\mathcal{T}^*A_+[1]$ to the summand $A_+[1]^{\otimes n}$. Moreover, since the components of $\boldsymbol{f}$ all have degree zero, the A_{n-1}-homomorphism relation is equivalent to

$$\delta\boldsymbol{f}_{n-1} - (\boldsymbol{f}_{\leq n-2}\star\boldsymbol{\mu}_{\geq 3})_n + (\boldsymbol{\mu}_{\geq 2}\circ\widetilde{\widetilde{\boldsymbol{f}}})_n = 0,$$

i.e.,

$$\delta\boldsymbol{f}_{n-1} = \mathfrak{F}_n.$$

Property (A$\mathfrak{F}$-2) follows in arbitrary characteristic.

Analogous to Equation (5.19), we can formulate this A_∞ relation for maps as

$$\delta(\boldsymbol{f}_{n-1}) - (\boldsymbol{f}_{\leq n-2}\star\boldsymbol{\mu}_{\geq 3})_n + (\boldsymbol{\mu}_{\geq 3}\circ\widetilde{\widetilde{\boldsymbol{f}_{\leq n-2}}})_n - (\boldsymbol{\mu}_2(\boldsymbol{f}_{>1},\boldsymbol{f}_{>1}))_n = 0. \tag{7.14}$$

If we define

$$D(\Phi) = \widetilde{\boldsymbol{\mu}_2}\circ\Phi - (-1)^{\|\Phi\|}\Phi\circ\widetilde{\boldsymbol{\mu}_2},$$

then the A_∞ relation for maps can be reformulated (analogous to Equation (5.20)) as

$$D(\widetilde{\widetilde{\boldsymbol{f}_{\leq n-1}}}) - \Big(\widetilde{\widetilde{\boldsymbol{f}_{\leq n-2}}}\circ\widetilde{\boldsymbol{\mu}_{\geq 3}} + \widetilde{\boldsymbol{\mu}_{\geq 3}}\circ\widetilde{\widetilde{\boldsymbol{f}_{\leq n-2}}}\Big)_n = 0 \tag{7.15}$$

Analogous to Equation (5.21), we find that

$$\begin{aligned}
\delta\Big((\boldsymbol{f}_{\le n-2} \star \boldsymbol{\mu}_{\ge 3})_n\Big) &= \Big(\delta(\boldsymbol{f}_{\le n-2}) \star \boldsymbol{\mu}_{\ge 3} + \boldsymbol{f}_{\le n-3} \star \delta(\boldsymbol{\mu}_{\ge 3}) - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{\mu}_{\ge 3}) - \boldsymbol{\mu}_2(\boldsymbol{\mu}_{\ge 3}, \boldsymbol{f}_{>1})\Big)_{n+1} \\
&= \Big((\boldsymbol{f}_{\le n-3} \star \boldsymbol{\mu}_{\ge 3}) \star \boldsymbol{\mu}_{\ge 3} - (\boldsymbol{\mu}_{\ge 3} \circ \overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-3}}}}) \circ \widetilde{\boldsymbol{\mu}_{\ge 3}} - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1} \star \boldsymbol{\mu}_{\ge 3}, \boldsymbol{f}_{>1}) \\
&\quad - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{f}_{>1} \star \boldsymbol{\mu}_{\ge 3}) - \boldsymbol{f}_{\le n-3} \star (\boldsymbol{\mu}_{\ge 3} \star \boldsymbol{\mu}_{\ge 3}) - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{\mu}_{\ge 3}) - \boldsymbol{\mu}_2(\boldsymbol{\mu}_{\ge 3}, \boldsymbol{f}_{>1})\Big)_{n+1} \\
&= \Big(-(\boldsymbol{\mu}_{\ge 3} \circ \overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-3}}}}) \circ \widetilde{\boldsymbol{\mu}_{\ge 3}} - \boldsymbol{\mu}_2(\boldsymbol{f} \star \boldsymbol{\mu}_{\ge 3}, \boldsymbol{f}_{>1}) - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{f} \star \boldsymbol{\mu}_{\ge 3})\Big)_{n+1}.
\end{aligned}$$

Analogous to Equation (5.22):

$$\begin{aligned}
\delta\Big((\boldsymbol{\mu}_{\ge 3} \circ \overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-2}}}})_n\Big) &= \Big(\delta(\boldsymbol{\mu}_{\ge 3}) \circ \overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-2}}}} - \boldsymbol{\mu}_{\ge 3} \circ D(\overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-2}}}}) - \boldsymbol{\mu}_2(\boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}}, \boldsymbol{f}_{>1}) - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}})\Big)_{n+1} \\
&= \Big(-(\boldsymbol{\mu}_{\ge 3} \star \boldsymbol{\mu}_{\ge 3}) \circ \overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-2}}}} + \boldsymbol{\mu}_{\ge 3} \circ (\widetilde{\boldsymbol{\mu}_{\ge 3}} \circ \overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-2}}}}) \\
&\quad - \boldsymbol{\mu}_{\ge 3} \circ (\overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-3}}}} \circ \widetilde{\boldsymbol{\mu}_{\ge 3}}) - \boldsymbol{\mu}_2(\boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}}, \boldsymbol{f}_{>1}) - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}})\Big)_{n+1} \\
&= \Big(-\boldsymbol{\mu}_{\ge 3} \circ (\overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-3}}}} \circ \widetilde{\boldsymbol{\mu}_{\ge 3}}) - \boldsymbol{\mu}_2(\boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}}, \boldsymbol{f}_{>1}) - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}})\Big)_{n+1}
\end{aligned}$$

and analogous to Equation (5.23):

$$\delta\Big(\boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{f}_{>1})_n\Big) = -\boldsymbol{\mu}_2(\delta \boldsymbol{f}_{>1}, \boldsymbol{f}_{>1})_{n+1} - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \delta \boldsymbol{f}_{>1})_{n+1} = -\boldsymbol{\mu}_2(\delta \boldsymbol{f}, \boldsymbol{f}_{>1})_{n+1} - \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \delta \boldsymbol{f})_{n+1}.$$

Adding these expressions up, we find that

$$\begin{aligned}
\delta(\mathfrak{F}_n) &= \delta\Big((\boldsymbol{f}_{\le k-2} \star \boldsymbol{\mu}_{\ge 3})_n\Big) - \delta\Big((\boldsymbol{\mu}_{\ge 3} \circ \overline{\widetilde{\widetilde{\boldsymbol{f}_{\le n-2}}}})_n\Big) - \delta\Big((\boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{f}_{>1}))_n\Big) \\
&= -\Big(\boldsymbol{\mu}_2(\boldsymbol{f}_{\le n-2} \star \boldsymbol{\mu}_{\ge 3} - \boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}} - \delta \boldsymbol{f}_{\le n-2}, \boldsymbol{f}_{>1}) + \boldsymbol{\mu}_2(\boldsymbol{f}_{>1}, \boldsymbol{f}_{\le n-2} \star \boldsymbol{\mu}_{\ge 3} - \boldsymbol{\mu}_{\ge 3} \circ \widetilde{\widetilde{\boldsymbol{f}}} - \delta \boldsymbol{f}_{\le n-2})\Big)_{n+1} \\
&= 0.
\end{aligned}$$

Property (A$\mathfrak{F}$-1) follows in arbitrary characteristic. ☐

7.4. **Uniqueness of $\mathcal{A}^0_{-;\mathbb{Z}}$.** The following analogue of Theorem 5.45 holds:

Theorem 7.16. *Up to isomorphism, there is a unique A_∞-deformation of $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}$ over $\mathbb{Z}[U]$ satisfying the following conditions:*

(1) The deformation is $\Gamma = G \times \mathbb{Z}$-graded, where the gradings of the chords ρ_i is defined by $\gamma(\rho_i) = \mathrm{gr}(\rho_i) \times \mathrm{wn}(\rho_i)$. (The gradings gr and wn are defined in Section 4.)
(2) The operations satisfy $\mu_4(\rho_4, \rho_3, \rho_2, \rho_1) = U\iota_1$ and $\mu_4(\rho_3, \rho_2, \rho_1, \rho_4) = U\iota_0$.

This follows from a Hochchild homology computation, which is facilitated by a signed analogue of Lemma 5.31, which we state after a few remarks. Recall that the cobar algebra is obtained by dualizing the tensor algebra over $A_+[1]$. We will use the trivial $\mathbb{Z}/2\mathbb{Z}$ grading on the torus algebra (cf. Section 4.5). The cobar algebra inherits a $\mathbb{Z}/2\mathbb{Z}$ grading from this description. Concretely, the elements ρ_i^* in $\mathrm{Cob}(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}})$ all have odd $\mathbb{Z}/2\mathbb{Z}$ grading.

Lemma 7.17. *There is a quasi-isomorphism of Γ-graded algebras (over $\mathbb{Z}$)*

$$\phi\colon (\mathrm{Cob}(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}), \gamma^{\mathrm{Cob}}) \to (\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}, \alpha \circ \gamma)$$

specified by $\phi(\iota_0) = \iota_1$, $\phi(\iota_1) = \iota_0$, $\phi(\rho_i^*) = [\rho_i]$ for $i = 1, \dots, 4$, and $\phi(a^*) = 0$ if $|a| > 1$. The map ϕ shifts $\mathbb{Z}/2\mathbb{Z}$ gradings: it identifies $\mathrm{Cob}(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}})$ with the above $\mathbb{Z}/2\mathbb{Z}$ grading with the algebra $(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}, \alpha \circ \gamma)$, with a $\mathbb{Z}/2\mathbb{Z}$ grading which is the mod-2 reduction of the length grading.

Proof. The proof of Lemma 5.31 can be adapted, noting that the element $\rho_i^* \otimes \rho_{i+1}^* \cdots \otimes \rho_{\ell-1} \otimes \rho_\ell^* \in \mathrm{Cob}(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}})$ has $\mathbb{Z}/2\mathbb{Z}$ grading specified by $\ell - i + 1$, which coincides with the length of $\phi(\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_{\ell-1}^* \otimes \rho_\ell^*)$.

Equation (5.35) (now in characteristic different than 2) follows, using the following sign refinement of the homotopy operator from Equation (5.34):

$$H(\overbrace{\rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_\ell^*}^{k} \otimes a_1^* \otimes \cdots \otimes a_m^*) = \begin{cases} (-1)^{\ell-i} \rho_i^* \otimes \rho_{i+1}^* \otimes \cdots \otimes \rho_{\ell-1}^* \otimes (a_1 \cdot \rho_\ell)^* \otimes a_2^* \otimes \cdots \otimes a_m^* & k > 0 \\ 0 & k = 0. \end{cases}$$

This completes the proof. ☐

We have the following sign refinement of Proposition 5.46:

Proposition 7.18. *The graded Hochschild cohomology $HH^{*,*}_\Gamma(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} \otimes \mathbb{Z}[U])$ of $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[U]$ over $\Bbbk \otimes \mathbb{Z}[U]$ satisfies*

$$HH^{n,-1}_\Gamma(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} \otimes \mathbb{Z}[U]) = \begin{cases} \mathbb{Z} & n = 4 \\ 0 & \textit{otherwise} \end{cases}$$

$$HH^{n,-2}_\Gamma(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} \otimes \mathbb{Z}[U]) = \begin{cases} \mathbb{Z} & n = 5 \\ 0 & \textit{otherwise} \end{cases}$$

Moreover, suppose $\xi \in HC^{4,-1}_\Gamma(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} \otimes \mathbb{Z}[U])$ is a cycle and $\xi(\rho_4 \otimes \rho_3 \otimes \rho_2 \otimes \rho_1) = U$. Then ξ represents a generator of $HH^{4,-1}_\Gamma(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} \otimes \mathbb{Z}[U])$.

Proof. As in the proof of Proposition 5.46, we will perform the Hochschild cohomology computations with the help of a small model of the cobar algebra.

Specifically, define $C^*_{\mathbb{Z}}$ to be the sign-refined analogue of Definition 5.42, defined using $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}$ in place of $\mathcal{A}^{0,\mathrm{as}}_{-}$. That is, $C^*_{\mathbb{Z}}$ is the free $\mathbb{Z}$-module generated by elements of the form $a \otimes [b]$ where $a \in \mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}$ and $[b] \in \mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}$ are basic elements with the property that $i \cdot a \cdot j = a$ and $[j \cdot b \cdot i] = [b]$, for some idempotents $i, j \in \{\iota_0, \iota_1\}$. The differential is defined by the following analogue of Equation (5.43):

$$\partial(a \otimes [b]) = \sum_{i=1}^{4} \Big((-1)^{|b|} \rho_i \cdot a \otimes [b \cdot \rho_i] + a \cdot \rho_i \otimes [\rho_i \cdot b] \Big).$$

Define $C^*_{\Gamma;\mathbb{Z}} \subset C^*$ to be the portion in grading $0 \times \mathbb{Z} \subset G \times \mathbb{Z}$. The analogue of Proposition 5.44 gives a quasi-isomorphism

$$H^{n,k}(C_{\Gamma;\mathbb{Z}}) \cong HH^{n,k}(\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} \otimes \mathbb{Z}[U]).$$

(This proof uses Lemma 7.17 in place of Lemma 5.31, which was used to prove Proposition 5.44.)

With these remarks in place, the proof of Proposition 5.46 applies, with minor modifications. The vanishing of homology for grading reasons follows exactly as before. In the present case, the differentials appearing in the computation of that proposition involving elements with $k = -1$ and -2 have signs in them. For example:

$$\partial(\rho_{123}[\rho_{123}]) = \rho_{1234}[\rho_{4123}] - \rho_{4123}[\rho_{1234}]$$
$$\partial(U[\rho_{1234}]) = U\rho_1[\rho_{12341}] + U\rho_4[\rho_{41234}].$$

The homology class $HH^{4,-1}_\Gamma$ is represented by the cycle

$$U[\rho_{1234}] - U[\rho_{2341}] + U[\rho_{3412}] - U[\rho_{4123}]$$

The homology class $HH^{5,-1}$ is represented by

$$U\rho_1[\rho_{12341}] \sim U\rho_2[\rho_{23412}] \sim U\rho_3[\rho_{34123}] \sim U\rho_4[\rho_{41234}].$$ ☐

Proof of Theorem 7.16. This proof is the same as the proof of Theorem 5.45, replacing the use of Proposition 5.7 (which required characteristic 2) with its analogue, Proposition 7.11, and replacing the characteristic 2 Hochschild cohomology computation of Proposition 5.46 with the more general Proposition 7.18. ☐

7.5. **Weighted algebras and Hochschild cohomology revisited.** We explain how to put signs into the discussion of Section 5.4.

Let $\Bbbk$ be an algebra that is free as a $\mathbb{Z}$-module. Fix an augmented A_∞-algebra $\mathcal{A}^0 = (A, \{\boldsymbol{\mu}_m\})$ over $\Bbbk$ with underlying $\mathbb{Z}$-module A and augmentation ideal $A_+ \subset A$. By a *weighted deformation* of $\mathcal{A}^0$ we mean a weighted A_∞-algebra $(A, \{\boldsymbol{\mu}^k_m\})$ with the same underlying vector space as $\mathcal{A}^0$ and whose weight-zero operations are the same as for $\mathcal{A}^0$: $\boldsymbol{\mu}^0_m = \boldsymbol{\mu}_m$ for all $m \geq 0$. If $\mathcal{A}$ and $\mathcal{B}$ are both weighted deformations of the same undeformed A_∞-algebra, a *homomorphism of deformations* from $\mathcal{A}$ to $\mathcal{B}$ is a sequence of $\mathbb{Z}/2\mathbb{Z}$-grading-preserving maps $\boldsymbol{f}^\bullet = \{\boldsymbol{f}^W : \mathcal{T}^*(A_+[1]) \to A_+[1]\}_{W=0}^\infty$ satisfying the weighted A_∞ relation

$$\sum_{a+b=W} \boldsymbol{f}^a \circ (\mathbb{I} \otimes \boldsymbol{\mu}^b \otimes \mathbb{I}) \circ \Delta^3 - \sum_{w_1+\cdots+w_m=W} (\boldsymbol{f}^{w_1} \otimes \cdots \otimes \boldsymbol{f}^{w_m}) \circ \Delta^m = 0$$

for each $W \geq 0$.

In this case, the Hochschild complex of $\mathcal{A}^0$ is given, as a $\mathbb{Z}$-module, by

$$HC^*(\mathcal{A}^0) = \prod_{n=0}^\infty \mathrm{Hom}_{\Bbbk\otimes\Bbbk}(\Bbbk \otimes (A_+)^{\otimes n}, A \otimes \Bbbk) = \prod_{n=0}^\infty \mathrm{Hom}_{\Bbbk\otimes\Bbbk}((A_+)^{\otimes n}, A),$$

with differential specified by

$$\delta(\boldsymbol{f}) = \boldsymbol{\mu}^0 \star \boldsymbol{f} - (-1)^{\|\boldsymbol{f}\|} \boldsymbol{f} \star \boldsymbol{\mu}^0,$$

with $\star$ as in Proposition 7.11.

Proposition 7.19. *The statement of Proposition 5.51 holds in arbitrary characteristic.*

Proof. In place of Equation (5.53), the signed weight W A_∞ relation takes the form

$$\delta\boldsymbol{\mu}^W + (\boldsymbol{\mu}^{\bullet\geq 1} \star \boldsymbol{\mu}^{\bullet\geq 1})^W = 0.$$

Letting

$$\mathfrak{O}^W = -(\boldsymbol{\mu}^{\bullet\geq 1} \star \boldsymbol{\mu}^{\bullet\geq 1})^W,$$

the A_∞ relation is equivalent to

$$\delta\boldsymbol{\mu}^W = \mathfrak{O}^W.$$

Hence, Property ($\infty\mathfrak{O}$-2) follows in arbitrary characteristic.

We verify Property ($\infty\mathfrak{O}$-1) as follows. We modify the definition of η: if $\|\boldsymbol{f}^\bullet\| = \|\boldsymbol{g}^\bullet\| = 1$, then let

$$\eta^W(\boldsymbol{f}^\bullet, \boldsymbol{g}^\bullet) = \boldsymbol{\mu}^0 \circ (\mathbb{I} \otimes \boldsymbol{f}^\bullet \otimes \mathbb{I} \otimes \boldsymbol{g}^\bullet \otimes \mathbb{I}) \circ \Delta^5 - \boldsymbol{\mu}^0 \circ (\mathbb{I} \otimes \boldsymbol{g}^\bullet \otimes \mathbb{I} \otimes \boldsymbol{f}^\bullet \otimes \mathbb{I}) \circ \Delta^5.$$

Equation (5.56) generalizes to

$$\delta(f^\bullet \star g^\bullet) = (\delta f^\bullet) \star g^\bullet + (-1)^{\|f^\bullet\|} f^\bullet \star (\delta g^\bullet) - \eta^\bullet(f^\bullet, g^\bullet) \tag{7.20}$$

(cf. Equation (7.12)). (We are using here that $\|\boldsymbol{f}^\bullet\| = \|\boldsymbol{g}^\bullet\| = 1$.)

To verify Property ($\infty\mathfrak{O}$-1), observe that

$$\begin{aligned}\delta\mathfrak{O}^W = -\delta(\boldsymbol{\mu}^{\bullet\geq 1} \star \boldsymbol{\mu}^{\bullet\geq 1})^W &= -\big(\delta(\boldsymbol{\mu}^{\bullet\geq 1}) \star \boldsymbol{\mu}^{\bullet\geq 1}\big)^W + \big(\boldsymbol{\mu}^{\bullet\geq 1} \star \delta(\boldsymbol{\mu}^{\bullet\geq 1})\big)^W + \eta^\bullet(\boldsymbol{\mu}^{\bullet\geq 1}, \boldsymbol{\mu}^{\bullet\geq 1}) \\ &= -\big((\boldsymbol{\mu}^{\bullet\geq 1} \star \boldsymbol{\mu}^{\bullet\geq 1}) \star \boldsymbol{\mu}^{\bullet\geq 1}\big)^W + \big(\boldsymbol{\mu}^{\bullet\geq 1} \star (\boldsymbol{\mu}^{\bullet\geq 1} \star \boldsymbol{\mu}^{\bullet\geq 1})\big)^W \\ &= 0,\end{aligned}$$

in view of Equation (7.13).

Defining $\overline{\overline{\boldsymbol{f}}}^{\bullet}$ as in Equation (5.52), we find that the weight W A_∞-homomorphism relation (Equation (7.5)) takes the form

$$(\boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet})^W - (\boldsymbol{\mu}^{\bullet} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet})^W = 0$$

(cf. Equation (7.9)). Analogous to Equation (5.60), we can formulate this as

$$\delta \boldsymbol{f}^W = (\boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet\geq 1})^W - (\boldsymbol{\mu}^{\bullet\geq 1} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet})^W - \boldsymbol{\mu}^0 \circ (\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W.$$

Equivalently, if we let

$$\begin{aligned}\mathfrak{F}^W &= (\boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet\geq 1})^W - (\boldsymbol{\mu}^{\bullet\geq 1} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet})^W - \boldsymbol{\mu}^0 \circ (\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W, \\ &= (\boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet\geq 1})^W - (\boldsymbol{\mu}^{\bullet\geq 1} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet})^W - \boldsymbol{\mu}^0 \circ (\overline{\overline{\boldsymbol{f}^{\bullet}}})^W + \boldsymbol{\mu}^0 \star \boldsymbol{f}^{\bullet}\end{aligned}$$

then the A_∞-homomorphism relation takes the form

$$\delta \boldsymbol{f}^W = \mathfrak{F}^W.$$

Property (∞$\mathfrak{F}$-2) follows.

To verify Property (∞$\mathfrak{F}$-1), we use the following signed version of Equation (5.61):

$$D(\overline{\overline{\boldsymbol{f}}}^{\bullet})^w + (\overline{\boldsymbol{\mu}^{\bullet\geq 1}} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet})^w - (\overline{\overline{\boldsymbol{f}}}^{\bullet} \circ \overline{\boldsymbol{\mu}^{\bullet\geq 1}})^w = 0$$

(cf. Equation (7.15)) where

$$D(\Phi) = \overline{\boldsymbol{\mu}^0} \circ \Phi - (-1)^{\|\Phi\|} \Phi \circ \overline{\boldsymbol{\mu}^0}.$$

In place of Equation (5.63), we have

$$(\delta \boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet\geq 1})^W = \Big(\big((\boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet\geq 1}) - (\boldsymbol{\mu}^{\bullet\geq 1} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet}) - (\boldsymbol{\mu}^0 \circ \overline{\overline{\boldsymbol{f}^{\bullet<W}}})\big) \star \boldsymbol{\mu}^{\bullet\geq 1}\Big)^W.$$

In place of Equation (5.57), we have:

$$\delta(\boldsymbol{f}^{\bullet} \circ \overline{\overline{\phi}}^{\bullet}) = (\delta(\boldsymbol{f}^{\bullet}) \circ \overline{\overline{\phi}}^{\bullet}) + (-1)^{\|\boldsymbol{f}^{\bullet}\|}(\boldsymbol{f}^{\bullet} \circ D\overline{\overline{\phi}}^{\bullet}) - (\boldsymbol{\mu}^0 \star \boldsymbol{f}^{\bullet}) \circ \overline{\overline{\phi}}^{\bullet} + \boldsymbol{\mu}^0 \star (\boldsymbol{f}^{\bullet} \circ \overline{\overline{\phi}}^{\bullet})$$

(cf. Equation (7.14)). In place of Equation (5.62), if we know that the weight $< W$ A_∞-homomorphism relations hold, we have

$$\begin{aligned}&D(\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W + (\overline{\boldsymbol{\mu}^{\bullet\geq 1}} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet})^W - (\overline{\overline{\boldsymbol{f}}}^{\bullet} \circ \overline{\boldsymbol{\mu}^{\bullet\geq 1}})^W \\ &\qquad = \overline{(\boldsymbol{\mu}^0 \circ \overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W} + \overline{(\boldsymbol{\mu}^{\bullet\geq 1} \circ \overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W} - \overline{(\boldsymbol{f}^{\bullet<W} \star \boldsymbol{\mu}^{\bullet\geq 1})^W}\end{aligned}$$

In place of Equation (5.64):

$$\begin{aligned}\delta(\boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet\geq 1})^W &= ((\delta \boldsymbol{f}^{\bullet}) \star \boldsymbol{\mu}^{\bullet\geq 1})^W + (\boldsymbol{f}^{\bullet} \star \delta(\boldsymbol{\mu}^{\bullet\geq 1}))^W - \eta^W(\boldsymbol{f}^{\bullet}, \boldsymbol{\mu}^{\bullet\geq 1}) \\ &= \Big((\boldsymbol{f}^{\bullet} \star \boldsymbol{\mu}^{\bullet\geq 1}) \star \boldsymbol{\mu}^{\bullet\geq 1} - (\boldsymbol{\mu}^{\bullet\geq 1} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet}) \star \boldsymbol{\mu}^{\bullet\geq 1} - (\boldsymbol{\mu}^0 \circ \overline{\overline{\boldsymbol{f}}}^{\bullet}) \star \boldsymbol{\mu}^{\bullet\geq 1} + (\boldsymbol{\mu}^0 \star \boldsymbol{f}^{\bullet}) \star \boldsymbol{\mu}^{\bullet\geq 1}\Big)^W \\ &\qquad - \big(\boldsymbol{f}^{\bullet} \star (\boldsymbol{\mu}^{\bullet\geq 1} \star \boldsymbol{\mu}^{\bullet\geq 1})\big)^W - \eta^W(\boldsymbol{f}^{\bullet}, \boldsymbol{\mu}^{\bullet\geq 1}) \\ &= -((\boldsymbol{\mu}^{\bullet\geq 1} \circ \overline{\overline{\boldsymbol{f}}}^{\bullet}) \star \boldsymbol{\mu}^{\bullet\geq 1})^W - ((\boldsymbol{\mu}^0 \circ \overline{\overline{\boldsymbol{f}}}^{\bullet}) \star \boldsymbol{\mu}^{\bullet\geq 1})^W + \big((\boldsymbol{\mu}^0 \star \boldsymbol{f}^{\bullet}) \star \boldsymbol{\mu}^{\bullet\geq 1}\big)^W \\ &\qquad - \eta^W(\boldsymbol{f}^{\bullet}, \boldsymbol{\mu}^{\bullet\geq 1}).\end{aligned}$$

In place of Equation (5.66):

$$\begin{aligned}\delta(\boldsymbol{\mu}^{\bullet\geq 1}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet})^W &= \Big(\delta(\boldsymbol{\mu}^{\bullet\geq 1})\circ\overline{\overline{\boldsymbol{f}}}^{\bullet} - \boldsymbol{\mu}^{\bullet\geq 1}\circ(D\overline{\overline{\boldsymbol{f}}}^{\bullet}) - (\boldsymbol{\mu}^0\star\boldsymbol{\mu}^{\bullet\geq 1})\circ\overline{\overline{\boldsymbol{f}}}^{\bullet} + \boldsymbol{\mu}^0\star(\boldsymbol{\mu}^{\bullet\geq 1}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet})\Big)^W\\ &= \Big(-(\boldsymbol{\mu}^{\bullet\geq 1}\star\boldsymbol{\mu}^{\bullet\geq 1})\circ\overline{\overline{\boldsymbol{f}}}^{\bullet} + \boldsymbol{\mu}^{\bullet\geq 1}\circ(\overline{\boldsymbol{\mu}^{\bullet\geq 1}}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet}) - \boldsymbol{\mu}^{\bullet\geq 1}\circ(\overline{\overline{\boldsymbol{f}}}^{\bullet}\circ\overline{\boldsymbol{\mu}^{\bullet\geq 1}})\\ &\qquad - (\boldsymbol{\mu}^0\circ(\overline{\boldsymbol{\mu}^{\bullet\geq 1}}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet}) + \boldsymbol{\mu}^0\star(\boldsymbol{\mu}^{\bullet\geq 1}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet})\Big)^W\\ &= \Big(-\boldsymbol{\mu}^{\bullet\geq 1}\circ(\overline{\overline{\boldsymbol{f}}}^{\bullet}\circ\overline{\boldsymbol{\mu}^{\bullet\geq 1}}) - \boldsymbol{\mu}^0\circ(\overline{\boldsymbol{\mu}^{\bullet\geq 1}}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet}) + \boldsymbol{\mu}^0\star(\boldsymbol{\mu}^{\bullet\geq 1}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet})\Big)^W\end{aligned}$$

In place of Equation (5.67):

$$\begin{aligned}\delta(\boldsymbol{\mu}^0\circ(\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W) &= (\delta\boldsymbol{\mu}^0)\circ(\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W - \boldsymbol{\mu}^0\circ(D\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W - (\boldsymbol{\mu}^0\star\boldsymbol{\mu}^0)\circ(\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W\\ &\quad + \boldsymbol{\mu}^0\star(\boldsymbol{\mu}^0\circ\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W\\ &= -\boldsymbol{\mu}^0\circ(D\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W + \boldsymbol{\mu}^0\star(\boldsymbol{\mu}^0\circ\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W.\end{aligned}$$

In place of Equation (5.68):

$$\begin{aligned}\boldsymbol{\mu}^0\circ D(\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W &= -\boldsymbol{\mu}^0\circ(\overline{\boldsymbol{\mu}^{\bullet\geq 1}}\circ\overline{\overline{\boldsymbol{f}}}^{\bullet})^W + \boldsymbol{\mu}^0\circ(\overline{\overline{\boldsymbol{f}}}^{\bullet}\circ\overline{\boldsymbol{\mu}^{\bullet\geq 1}})^W\\ &\quad + \boldsymbol{\mu}^0\star(\boldsymbol{\mu}^0\circ\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W + \boldsymbol{\mu}^0\star(\boldsymbol{\mu}^{\bullet\geq 1}\circ\overline{\overline{\boldsymbol{f}^{\bullet<W}}})^W - \boldsymbol{\mu}^0\star(\boldsymbol{f}^{\bullet<W}\star\boldsymbol{\mu}^{\bullet\geq 1})^W.\end{aligned}$$

Finally, using

$$(\boldsymbol{\mu}^0\star\boldsymbol{f}^{\bullet})\star\boldsymbol{\mu}^{\bullet\geq 1} - \boldsymbol{\mu}^0\star(\boldsymbol{f}^{\bullet}\star\boldsymbol{\mu}^{\bullet\geq 1}) - \eta^W(\boldsymbol{f}^{\bullet},\boldsymbol{\mu}^{\bullet\geq 1}) = 0$$

(a special case of Equation (7.13)), it now follows that $\delta(\mathfrak{F}^W) = 0$, that is, that Property ($\infty\mathfrak{F}$-1) holds in arbitrary characteristic. ☐

7.6. **Uniqueness of $\mathcal{A}_{-;\mathbb{Z}}$.** In this section, we view the ground ring for $\mathcal{A}^0_{-;\mathbb{Z}}$ as $\Bbbk = \mathbb{Z}\oplus\mathbb{Z}$, rather than $\mathbb{Z}[U]$; so our augmentation is a map $\mathcal{A}^0_{-;\mathbb{Z}}\to\Bbbk$, and there is a corresponding augmentation ideal.

Theorem 7.21. *Up to isomorphism, there is a unique weighted deformation $\mathcal{A}_{-;\mathbb{Z}}$ of $\mathcal{A}^0_{-;\mathbb{Z}}$ such that:*

(1) $\mathcal{A}_{-;\mathbb{Z}}$ is $\Gamma = G\times\mathbb{Z}$-graded and
(2) $\mu^1_0 = \rho_{1234} + \rho_{2341} + \rho_{3412} + \rho_{4123}$.

We sketch the modifications needed to make to the discussion in Section 5.5 to hold over $\mathbb{Z}$. Endow $\mathcal{A}'_{\mathbb{Z}} = \mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}[h]/(h^2)$, with the $\mathbb{Z}/2\mathbb{Z}$-grading which coincides with the mod 2 reduction of the length grading on $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}\subset\mathcal{A}'_{\mathbb{Z}}$, and so that $\|h\|$ is odd. With this understood, we have the following analogue of Lemma 5.80:

Lemma 7.22. *There is a quasi-isomorphism of $\Gamma\times\mathbb{Z}/2\mathbb{Z}$-graded algebras $\phi'\colon \mathrm{Cob}(\mathcal{A}^0_{-;\mathbb{Z}})\to\mathcal{A}'_{\mathbb{Z}}$, determined by $\phi'(\iota_0) = \iota_1$, $\phi'(\iota_1) = \iota_0$, $\phi'(\rho_i^*) = [\rho_i]$ for $i = 1,\dots,4$, $\phi'(U^*) = [h]$, and $\phi'(a^*) = 0$ if $a\neq U$ and length $|a| > 1$. This map is $\mathbb{Z}/2\mathbb{Z}$-grading preserving, using the induced $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathrm{Cob}(\mathcal{A}^0_{-;\mathbb{Z}})$, corresponding to a length grading on $\mathcal{A}'_{\mathbb{Z}}$ (with the understanding that $|h|$ is odd).*

Proof. The proof of Lemma 5.80 applies, with signs added as in the proof of Lemma 7.17. ☐

The small model of the Hochschild complex is defined as before, as $\mathfrak{C}^*_{\mathbb{Z}} = \mathcal{A}^0_{-;\mathbb{Z}} \widehat{\otimes}_{\Bbbk\otimes\Bbbk} \mathcal{A}'_{\mathbb{Z}}$ as before. Signs are now inserted into the differential, as follows:

$$\begin{aligned}\partial(a[b]) &= a[\partial b] + \sum_{i=1}^{4}\Big((-1)^{|b|}\rho_i \cdot a[b\cdot\rho_i] + a\cdot\rho_i[\rho_i\cdot b]\Big)\\ &\quad+\sum_{i=1}^{4}\big((-1)^{|b|}\mu_4(\rho_{i+3},\rho_{i+2},\rho_{i+1},a)[b\cdot\rho_{i+1,i+2,i+3}] + \mu_4(\rho_{i+2},\rho_{i+1},a,\rho_{i-1})[\rho_{i-1}\cdot b\cdot\rho_{i+1,i+2}]\\ &\qquad + (-1)^{|b|}\mu_4(\rho_{i+1},a,\rho_{i-1},\rho_{i-2})[\rho_{i-2,i-1}\cdot b\cdot\rho_{i+1}] + \mu_4(a,\rho_{i-1},\rho_{i-2},\rho_{i-3},)[\rho_{i-3,i-2,i-1}\cdot b].\end{aligned}$$

For example, Equation (5.88) is replaced by

$$\partial(\rho_1[\rho_1]) = -U[\rho_{1234}] + U[\rho_{2341}] - U[\rho_{3412}] + U[\rho_{4123}].$$

Lemma 5.89 has the following analogue:

Lemma 7.23. *This differential makes $\mathfrak{C}^*_{\mathbb{Z}}$ into a chain complex.*

Proof. The proof follows along the lines of Lemma 5.89. Again, we break up the differential into its components ∂_i for $i = 1,2,4$, according to which μ_i action contributes. It is immediate that $\partial_1^2 = 0$. The identity $\partial_1\circ\partial_2 + \partial_2\circ\partial_1$ follows quickly from the fact that $|h| \equiv 1 \pmod 2$, as does $\partial_1\partial_4 + \partial_4\partial_1 = 0$.

Again, verifying $\partial_2\partial_4 + \partial_4\partial_2 = 0$ is a case check. For example, as in the proof of Lemma 5.89, only now keeping track of signs, we find that

$$\partial_2\partial_4(\rho_1[b]) + \partial_4\partial_2(\rho_1[b]) = U\rho_1[(\rho_{1234}+\rho_{3412})\cdot b] - U\rho_1[b\cdot(\rho_{4123}+\rho_{2341})] = 0.$$

The cancellation in $\partial_4^2 = 0$ is also straightforward, and is left to the reader. ☐

Specializing gradings, we have $\mathfrak{C}^{W,\ell}_{\Gamma;\mathbb{Z}} \subset \mathfrak{C}^*_{\mathbb{Z}}$ (analogous to $\mathfrak{C}^{W,\ell}_{\Gamma} \subset \mathfrak{C}^*$; see Equation (5.90)). We now have the following analogue of Proposition 5.91:

Proposition 7.24. *The chain complex $\mathfrak{C}^*_{\Gamma;\mathbb{Z}}$ is quasi-isomorphic to the complex $HC^*_\Gamma(\mathcal{A}^0_{-;\mathbb{Z}})$; in particular $H^{w,k}(\mathfrak{C}_{\Gamma;\mathbb{Z}}) \cong HH^{w,k}_\Gamma(\mathcal{A}^0_{-;\mathbb{Z}})$.*

Proof. The proof of Proposition 5.91 applies with minor changes. ☐

Proposition 5.92 now has the following analogue:

Proposition 7.25. *The Hochschild cohomology groups $HH^{w,k}_\Gamma(\mathcal{A}^0_{-;\mathbb{Z}})$, $w > 0$, have*

$$HH^{w,-1}_\Gamma(\mathcal{A}^0_{-;\mathbb{Z}}) = \begin{cases}\mathbb{Z}^2 & w = 1\\ 0 & \text{otherwise}\end{cases}$$

and $HH^{w,-2}_\Gamma(\mathcal{A}^0_{-;\mathbb{Z}})$ is entirely supported in weight (w) grading 1. Moreover, one can choose a basis for $HH^{1,-1}_\Gamma(\mathcal{A}^0_{-;\mathbb{Z}})$ so that one basis element sends $1\in\Bbbk$ to $\rho_{1234}+\rho_{2341}+\rho_{3412}+\rho_{4123}$ and the other sends $1\in\Bbbk$ to $U = U(\iota_0+\iota_1)$.

Proof. This follows as in the proof of Proposition 5.92. Again, the generators of the homology are $\rho_{1234}[\iota_1] + \rho_{2341}[\iota_0] + \rho_{3412}[\iota_1] + \rho_{4123}[\iota_0]$ and $U\iota_0[\iota_1] + U\iota_1[\iota_0]$. ☐

Proof of Theorem 7.21. As in the case of Theorem 5.71, this follows from the above Hochschild computation (Proposition 7.25) and deformation theory (Proposition 7.19). ☐

7.7. **Application to the Fukaya category.** Sheridan showed that the anchored Fukaya category can be defined with $\mathbb{Z}$, instead of $\mathbb{F}_2$, coefficients [She15]. In particular, his construction makes $\mathrm{End}_{\mathcal{W}_z}(\alpha_1 \oplus \alpha_2; \mathbb{Z})$ into an A_∞-algebra over $\mathbb{Z}[U]$. For his, or any other, way of lifting the wrapped Fukaya category of the torus to $\mathbb{Z}$-coefficients, we have:

Theorem 7.26. *There is an A_∞-quasi-isomorphism $\mathcal{A}^0_{-;\mathbb{Z}} \simeq \mathrm{End}_{\mathcal{W}_z}(\alpha_1 \oplus \alpha_2; \mathbb{Z})$.*

Proof. This follows from Theorem 7.16 and the observation that for any way of assigning signs in the multiplication on $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}} = (\mathrm{End}_{\mathcal{W}_z}(\alpha_1 \oplus \alpha_2), \mu_2)$ there is an isomorphism to $\mathcal{A}^{0,\mathrm{as}}_{-;\mathbb{Z}}$, and similarly if one negates the equations in Theorem 7.16 (2) above one obtains an isomorphic A_∞-algebra (by sending U to $-U$). (Note that the A_∞-relation with inputs $(\rho_4, \rho_3, \rho_2, \rho_1, \rho_4)$ ensures that both equations in Theorem 7.16 (2) have the same sign.) ☐

References

[AB22] Haniya Azam and Christian Blanchet, *Topological Fukaya categories of surfaces*, J. Pure Appl. Algebra **226** (2022), no. 6, Paper No. 106941, 30.

[Abo08] Mohammed Abouzaid, *On the Fukaya categories of higher genus surfaces*, Adv. Math. **217** (2008), no. 3, 1192–1235.

[AS10] Mohammed Abouzaid and Paul Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), no. 2, 627–718.

[AS20] Denis Auroux and Ivan Smith, *Fukaya categories of surfaces, spherical objects, and mapping class groups*, 2020, arXiv:2006.09689.

[Aur10a] Denis Auroux, *Fukaya categories and bordered Heegaard-Floer homology*, Proceedings of the International Congress of Mathematicians. Volume II (New Delhi), Hindustan Book Agency, 2010, pp. 917–941.

[Aur10b] ———, *Fukaya categories of symmetric products and bordered Heegaard-Floer homology*, J. Gökova Geom. Topol. GGT **4** (2010), 1–54, arXiv:1001.4323.

[Aur14] ———, *A beginner's introduction to Fukaya categories*, Contact and symplectic topology, Bolyai Soc. Math. Stud., vol. 26, János Bolyai Math. Soc., Budapest, 2014, pp. 85–136.

[CS20] Benjamin Cooper and Peter Samuelson, *The Hall algebras of surfaces I*, J. Inst. Math. Jussieu **19** (2020), no. 3, 971–1028.

[DHKK14] G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich, *Dynamical systems and categories*, The influence of Solomon Lefschetz in geometry and topology, Contemp. Math., vol. 621, Amer. Math. Soc., Providence, RI, 2014, pp. 133–170.

[Efi12] Alexander I. Efimov, *Homological mirror symmetry for curves of higher genus*, Adv. Math. **230** (2012), no. 2, 493–530.

[FOOO09] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[FOOO10] ———, *Anchored Lagrangian submanifolds and their Floer theory*, Mirror symmetry and tropical geometry, Contemp. Math., vol. 527, Amer. Math. Soc., Providence, RI, 2010, pp. 15–54.

[GJ90] Ezra Getzler and John D. S. Jones, *A_∞-algebras and the cyclic bar complex*, Illinois J. Math. **34** (1990), no. 2, 256–283.

[HLW17] Jennifer Hom, Tye Lidman, and Liam Watson, *The Alexander module, Seifert forms, and categorification*, J. Topol. **10** (2017), no. 1, 22–100.

[HRW24] Jonathan Hanselman, Jacob Rasmussen, and Liam Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, J. Amer. Math. Soc. **37** (2024), no. 2, 391–498.

[Kel01] Bernhard Keller, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl. **3** (2001), no. 1, 1–35, arXiv:math.RA/9910179.

[Kel02] Bernhard Keller, *A-infinity algebras in representation theory*, Representations of algebra. Vol. I, II, Beijing Norm. Univ. Press, Beijing, 2002, pp. 74–86.

[KP21] Douglas Knowles and Ina Petkova, *Bordered Heegaard Floer homology with integral coefficients over the torus algebra*, 2021, arXiv:2104.15120.

[KS02] Mikhail Khovanov and Paul Seidel, *Quivers, Floer cohomology, and braid group actions*, J. Amer. Math. Soc. **15** (2002), no. 1, 203–271, arxiv:math.QA/0006056.

[LOT11] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston, *Heegaard Floer homology as morphism spaces*, Quantum Topol. **2** (2011), no. 4, 381–449, arXiv:1005.1248.

[LOT13] ———, *A faithful linear-categorical action of the mapping class group of a surface with boundary*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 4, 1279–1307.

[LOT15] ______, *Bimodules in bordered Heegaard Floer homology*, Geom. Topol. **19** (2015), no. 2, 525–724, arXiv:1003.0598.

[LOT18] ______, *Bordered Heegaard Floer homology*, Mem. Amer. Math. Soc. **254** (2018), no. 1216, viii+279, arXiv:0810.0687.

[LOT20] ______, *Diagonals and A-infinity tensor products*, 2020, arXiv:2009.05222.

[LOT23] ______, *Bordered HF^- for three-manifolds with torus boundary*, 2023, arXiv:2305.07754.

[LP11] Yankı Lekili and Timothy Perutz, *Fukaya categories of the torus and Dehn surgery*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 20, 8106–8113.

[LP17] Yankı Lekili and Alexander Polishchuk, *Arithmetic mirror symmetry for genus 1 curves with n marked points*, Selecta Math. (N.S.) **23** (2017), no. 3, 1851–1907.

[OSz04] Peter S. Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158, arXiv:math.SG/0101206.

[Pet18] Ina Petkova, *The decategorification of bordered Heegaard Floer homology*, J. Symplectic Geom. **16** (2018), no. 1, 227–277.

[PS19] James Pascaleff and Nicolò Sibilla, *Topological Fukaya category and mirror symmetry for punctured surfaces*, Compos. Math. **155** (2019), no. 3, 599–644.

[RS93] Joel Robbin and Dietmar Salamon, *The Maslov index for paths*, Topology **32** (1993), no. 4, 827–844.

[Sei11] Paul Seidel, *Homological mirror symmetry for the genus two curve*, J. Algebraic Geom. **20** (2011), no. 4, 727–769.

[Sei15] ______, *Homological mirror symmetry for the quartic surface*, Mem. Amer. Math. Soc. **236** (2015), no. 1116, vi+129, arXiv:math/0310414v4.

[She15] Nick Sheridan, *Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space*, Invent. Math. **199** (2015), no. 1, 1–186, arXiv:1111.0632.

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