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Non-planar corrections in ABJM theory from quantum M2 branes

Simone Giombi[a] **Stefan A. Kurlyand**[b] **and Arkady A. Tseytlin**[b,1]

[a]*Department of Physics, Princeton University, Princeton, NJ 08544, U.S.A.*

[b]*Theoretical Physics Group, Blackett Laboratory, Imperial College London, SW7 2AZ, U.K.*

E-mail: sgiombi@princeton.edu, s.kurlyand23@imperial.ac.uk, a.tseytlin@imperial.ac.uk

Abstract: The quantization of semiclassical strings in AdS spacetimes yields predictions for the strong-coupling behaviour of the scaling dimensions of the corresponding operators in the planar limit of the dual gauge theory. Finding non-planar corrections requires computing string loops (corresponding to torus and higher genus surfaces), which is a challenging task. It turns out that in the case of the $U_k(N) \times U_{-k}(N)$ ABJM theory there is an alternative approach: one may semiclassically quantize M2 branes in $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ which are wrapped around the 11d circle of radius $1/k = \lambda/N$. Such M2 branes are the M-theory generalization of the strings in $\text{AdS}_4 \times \text{CP}^3$. In this work, we show that by expanding in large M2 brane tension $T_2 \sim \sqrt{kN}$ for fixed k, followed by an expansion in large k, we can predict the large λ asymptotics of the non-planar corrections to the dimensions of the dual ABJM operators. As a specific example, we consider the M2 brane configuration that generalizes the long folded string with large spin in AdS_4, and compute the 1-loop correction to its energy. This calculation allows us to determine non-planar corrections to the universal scaling function or cusp anomalous dimension. We extend our analysis to the semiclassical M2 branes that generalize the "short" and "long" circular strings with two equal angular momenta in CP^3. The "short" M2 brane corresponds to a dual operator whose dimension at strong coupling scales as $\Delta \sim \lambda^{1/4} + \ldots$, and we derive the leading non-planar correction to it.

Keywords: AdS-CFT Correspondence, $1/N$ Expansion, M-Theory, P-Branes

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Contents

1 Introduction and summary

One of the challenging problems in superconformal quantum field theories like $\mathcal{N}=4$ SYM and ABJM [1] ones, which admit a large N expansion, is to compute the conformal dimensions Δ of primary operators as functions of the 't Hooft coupling λ and N. In general,

$$\Delta(\lambda, N) = \Delta_0(\lambda) + \frac{1}{N^2}\Delta_1(\lambda) + \frac{1}{N^4}\Delta_2(\lambda) + \dots \, . \tag{1.1}$$

Here the planar part $\Delta_0(\lambda)$ is controlled by integrability, and, expanded at large λ, it can be matched to the large tension expansion of string energies in the dual string theory (see, e.g., [2–4]). Little, however, is known about the explicit form of the non-planar correction Δ_1, Δ_2, ... In the $\mathcal{N}=4$ SYM theory the first non-planar correction to the cusp anomalous dimension $f(\lambda, N)$ appearing in the large spin expansion of the dimension of an operator like $\mathcal{O} = \mathrm{tr}(\Phi D_+^S \Phi)$

$$\Delta\big|_{S\gg 1} = S + f(\lambda, N)\log S + \dots \tag{1.2}$$

starts at 4-loop order in the weak coupling expansion [5] (see also [6, 7])

$$f(\lambda, N) = \frac{1}{(2\pi)^2}\left[\lambda - \frac{1}{48}\lambda^2 + \frac{11}{11520}\lambda^3 - \left(c_4 + \frac{d_4}{N^2}\right)\lambda^4 + \mathcal{O}(\lambda^5)\right], \tag{1.3}$$

$$c_4 = \frac{73}{20160\times 64} + \frac{1}{8(2\pi)^6}\zeta^2(3)\, , \qquad d_4 = \frac{31}{5040\times 64} + \frac{9}{4(2\pi)^6}\zeta^2(3)\, . \tag{1.4}$$

The $\frac{\lambda^4}{N^2}$ term appears to be universal — it is the same for any matter content [8] as it originates from the quartic Casimir of SU(N). This suggests that in all anomalous dimensions computed at weak coupling the $1/N^2$ correction should first appear at 4 loops, i.e. Δ_1 in (1.1) should be given by

$$\Delta_1\big|_{\lambda\ll 1} = d_4\lambda^4 + d_6\lambda^6 + \dots \, . \tag{1.5}$$

Indeed, similar non-planar behaviour is found for the anomalous dimensions of twist-2 operators with general Lorentz spin [9, 10][1] and also for the Konishi operator [11, 12] where $d_4 \sim \zeta(5)$ (see also [13]).

Less is known about non-planar corrections in the case of the ABJM theory.[2] Given a close analogy with the $\mathcal{N} = 4$ SYM theory, it is natural to expect that here the first non-planar correction should also appear at 4-loop order as in (1.5).

One may conjecture that it may be possible to compute $\Delta_1(\lambda)$ in (1.5) to all orders utilizing somehow the integrability of the planar theory (cf., e.g., [18–20]). If one could do this, then expanding the exact expression for $\Delta_1(\lambda)$ at strong coupling, $\Delta_1\big|_{\lambda\gg 1} \sim \lambda^p + \dots$, one would then determine the power p of the leading term. It could then be compared to the dual string theory side where finding the leading non-planar correction requires computing string 1-loop (torus) correction to string energies, a complicated open problem.

Remarkably, as we shall demonstrate below, there is a way to find the strong-coupling asymptotics of the non-planar corrections $\Delta_1(\lambda), \Delta_2(\lambda), \dots$ in the case of the ABJM model using its duality to M-theory or theory of quantum M2 branes. It turns out that a semiclassical M2 brane quantization in $AdS_4 \times S^7/\mathbb{Z}_k$ captures the leading order $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms at each order in the string coupling $g_s^2 \sim \frac{1}{N^2}$ expansion.

In particular, we will show that for the ABJM cusp anomalous dimension the strong-coupling scaling of the leading non-planar correction is $\frac{\lambda^2}{N^2}$. In general, the prediction for the structure of the large λ expansion of the $1/N^{2s}$ coefficients in (1.2) is

$$f(\lambda, N) = f_0(\lambda) + \frac{1}{N^2}\mathrm{f}_1(\lambda) + \frac{1}{N^4}\mathrm{f}_2(\lambda) + \dots \, , \qquad f_0\big|_{\lambda\gg 1} = \sqrt{2\lambda} + \mathrm{f}_0(\lambda) \, , \tag{1.6}$$

$$\mathrm{f}_s(\lambda)\big|_{\lambda\gg 1} = \lambda^{2r}\left(a_{1s} + \frac{1}{\sqrt{\lambda}}a_{2s} + \dots\right) , \qquad s = 0, 1, 2, \dots \, . \tag{1.7}$$

The few leading coefficients in the strong-coupling expansion of the planar part $f_0(\lambda)$ can be found (as in the $AdS_5 \times S^5$case [21–25]) by quantizing the long folded spinning string in $AdS_4 \times CP^3$ [26–29]. The coefficients a_{1s} of the leading non-planar contributions will be computed below from the 1-loop 3d world-volume correction to the energy of a semiclassical M2 brane spinning in AdS_4 and wrapped on the 11d circle in $S^7/\mathbb{Z}_k$ of radius $\frac{1}{k} = \frac{\lambda}{N}$. The subleading a_{2s} coefficients may be determined from the 2-loop M2 brane correction, etc.

[1] According to [10] at large spin at weak coupling we should expect the anomalous dimension depending on spin as $\Delta_1(S) = \lambda^4\big(d_4 \log S + e_1 + \frac{1}{S}e_2 + \dots\big)$, where e_1, e_2, like d_4, are given by combinations of ζ-values. We thank V. Velizhanin for a comment on this expansion and informing us that the coefficient of the $\frac{1}{S}\log S$ term happens to be zero.

[2] Study of non-planar corrections at leading order at weak coupling in the ABJM theory was initiated in [14]. In [15] the 2-loop correction to the cusp anomaly was found not to contain a non-planar part, but in sect 4.1 [16] (cf. also [17]) the opposite was claimed. We thank M. Lagares for pointing this out.

1.1 Semiclassical expansion for M2 brane in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

While the M2 brane action [30, 31] is formally non-renormalizable, the semiclassical expansion of the corresponding path integral near a "minimal-volume" solution with a non-degenerate induced 3d metric is well defined (at least at the 1-loop order [31–36] where there is no logarithmic UV divergences in a 3d theory). Recent work [37–42] provided a convincing evidence that the semiclassical quantization of the M2 brane is indeed consistent. It was demonstrated that 1-loop M2 brane corrections in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ and $\mathrm{AdS}_7 \times S^4$ match the dual gauge theory (localization) predictions for several "supersymmetric" observables — defect anomaly, $\frac{1}{2}$ BPS Wilson loop and instanton contributions to the supersymmetric partition function (superconformal index) in the 3d ABJM and also 6d (2,0) theory.

This provides a motivation to apply similar semiclassical M2 brane quantization approach also to "non-supersymmetric" observables like non-planar corrections to ABJM anomalous dimensions that are not controlled by integrability or localization.

Let us briefly review some basic relations and notation that we will use below. The $U_k(N) \times U_{-k}(N)$ ABJM theory expanded at large N for fixed k is dual to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background with the metric and 3-form given by

$$ds^2_{11} = \mathrm{L}^2\left(\frac{1}{4}ds^2_{\mathrm{AdS}_4} + ds^2_{S^7/\mathbb{Z}_k}\right), \qquad \mathrm{L} = (2^5\pi^2 Nk)^{1/6}\ell_P\,, \tag{1.8}$$

$$ds^2_{\mathrm{AdS}_4} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\alpha^2 + \cos^2\alpha\, d\beta^2), \tag{1.9}$$

$$ds^2_{S^7/\mathbb{Z}_k} = ds^2_{\mathrm{CP}^3} + \frac{1}{k^2}(d\varphi + k\mathrm{A})^2\,, \qquad \varphi \equiv \varphi + 2\pi\,, \tag{1.10}$$

$$ds^2_{\mathrm{CP}^3} = dz^a d\bar z_a - \bar z_a z^b dz^a d\bar z_b\,, \qquad \mathrm{A} = \frac{1}{2i}(\bar z_a dz^a - z^a d\bar z_a)\,, \qquad \bar z_a z^a = 1, \quad a = 1,\ldots,4\,, \tag{1.11}$$

$$C_3 = -\frac{3}{8}\mathrm{L}^3\cosh\rho\,\sinh^2\rho\,\sin\alpha\, dt\wedge d\rho\wedge d\beta\ \ . \tag{1.12}$$

Taking also k large with $\lambda \equiv \frac{N}{k}$ fixed corresponds to the 't Hooft expansion of the 3d gauge theory in which it is dual to the perturbative type IIA string theory in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ with the coupling g_s and the effective dimensionless string tension T defined with respect to the radius of the AdS_4 part given by (we set $\ell_P = \sqrt{\alpha'}$ as in appendix A in [43])

$$ds^2_{10} = L^2\left(\frac{1}{4}ds^2_{\mathrm{AdS}_4} + ds^2_{\mathrm{CP}^3}\right), \qquad L = g_\mathrm{s}^{1/3}\,\mathrm{L}\,, \tag{1.13}$$

$$g_\mathrm{s} = \left(\frac{\mathrm{L}}{k\,\ell_P}\right)^{3/2} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N}, \qquad \lambda = \frac{N}{k}, \qquad \mathrm{T} = \frac{\frac{1}{4}L^2}{2\pi\alpha'} = \sqrt{\frac{\lambda}{2}} = \frac{\sqrt{\bar\lambda}}{2\pi}, \qquad \bar\lambda \equiv 2\pi^2\lambda\,, \tag{1.14}$$

$$\frac{1}{k^2} = \frac{\lambda^2}{N^2} = \frac{g_\mathrm{s}^2}{8\pi\,\mathrm{T}}\ . \tag{1.15}$$

The M-theory expansion corresponds to $\frac{\mathrm{L}}{\ell_P} \gg 1$ or large N for fixed k, i.e. the expansion in the large effective dimensionless M2 brane tension

$$\mathrm{T}_2 \equiv \mathrm{L}^3 T_2 = \frac{1}{\pi}\sqrt{2Nk}\,, \qquad T_2 = \frac{1}{(2\pi)^2\ell_P^3}\ . \tag{1.16}$$

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Here T_2 is defined with respect to the radius of $S^7/\mathbb{Z}_k$ part so that it is related to the string tension in (1.14) as (note also that in general $\frac{1}{k}\frac{\mathrm{L}^3}{\ell_P^3} = \frac{L^2}{\alpha'}$)

$$\mathrm{T} = \frac{1}{4}\frac{2\pi}{k}\mathrm{T}_2 \, . \tag{1.17}$$

The observables that can be computed in the semiclassical M2 brane expansion can be written as

$$F = \mathrm{T}_2 F_0(k) + F_1(k) + (\mathrm{T}_2)^{-1} F_2(k) + \dots \, , \qquad \mathrm{T}_2 \gg 1 \, . \tag{1.18}$$

This corresponds to the large N expansion for fixed k. Expanding (1.18) further at large k, it may be rewritten as a large N expansion for fixed $\lambda = \frac{N}{k}$ or string-theory expansion in $g_{\rm s}$ for fixed $\mathrm{T} = \sqrt{\frac{\lambda}{2}}$.

Below we will assume that dimensions of ABJM operators with large quantum numbers that, in the planar expansion, are dual to semiclassical strings in $\mathrm{AdS}_4 \times \mathrm{CP}^3$, may be computed as AdS_4 energies of semiclassical M2 branes in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ that are wrapped on the 11d circle φ in (1.10). They will thus have topology $\Sigma^2 \times S^1$, i.e. will generalize the corresponding string solutions reducing to them upon the "double dimensional reduction" [44].

Given a M2 brane solution with a non-degenerate induced 3d metric it is straightforward to expand the corresponding path integral at large T_2 (using, e.g., a static gauge as in [37–39]). The M2 brane action is

$$S = S_B + S_F \, , \qquad S_B = S_{\rm V} + S_{\rm WZ}, \qquad S_{\rm V} = -T_2 \int d^3\xi \, \sqrt{-g} \, , \tag{1.19}$$

$$S_{\rm WZ} = -T_2 \int d^3\xi \, \frac{1}{3!}\epsilon^{ijk} C_{MNK}(X) \, \partial_i X^M \partial_j X^N \partial_k X^K, \tag{1.20}$$

$$S_F = T_2 \int d^3\xi \left[\sqrt{-g} \, g^{ij} \partial_i X^M \, \bar\theta \, \Gamma_M \hat D_j \theta - \frac{1}{2}\epsilon^{ijk} \partial_i X^M \partial_j X^N \, \bar\theta \, \Gamma_{MN} \hat D_k \theta + \dots \right] , \tag{1.21}$$

$$g_{ij} = \partial_i X^M \partial_j X^N G_{MN}(X), \qquad G_{MN} = E^A_M E^A_N, \qquad \Gamma_M = E^A_M(X)\Gamma_A \, , \tag{1.22}$$

$$\hat D_i = \partial_i X^M \hat D_M, \qquad \hat D_M = \partial_M + \frac{1}{4}\Gamma_{AB}\Omega^{AB}_M - \frac{1}{288}(\Gamma^{PNKL}{}_M - 8\Gamma^{PNK}\delta^L_M) F_{PNKL} \, . \tag{1.23}$$

The leading classical and 1-loop contributions to the (euclidean) M2 brane partition function may be written as

$$\mathrm{Z}_{\rm M2} = \int [dX \, d\theta] \, e^{-S[X,\theta]} = \mathcal{Z}_1 \, e^{-\mathrm{T}_2 \bar S_{\rm cl}} \left[1 + \mathcal{O}(\mathrm{T}_2^{-1}) \right] , \tag{1.24}$$

$$\mathcal{Z}_1 = e^{-\Gamma_1} \, , \qquad \Gamma_1 = \frac{1}{2}\sum_i \nu_i \log\det \mathcal{O}_i \, , \tag{1.25}$$

where $\bar S_{\rm cl}$ and fluctuation operators $\mathcal{O}_i$ may depend on the parameter k or the inverse radius of the 11d circle in the 11d metric (1.8), (1.10) and other parameters of a given classical solution (like rotation frequencies, winding numbers, etc.). Then $F = -\log \mathrm{Z}_{\rm M2}$ will have the form given in (1.18).

It is important to note that we are expanding near just one M2 brane saddle (i.e. we are not summing over 3d topologies). Interpreted from the string theory point of view, this

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world-volume loop expansion already captures contributions of all higher string loops (as well as the dependence on the string tension). Indeed, the classical M2 brane action encodes the dependence on the string coupling g_s (cf. (1.15)) via its dependence on the parameter k of the 11d background metric (1.8), (1.10) in which the M2 brane is embedded.

Since the membrane is assumed to have $\Sigma^2 \times S^1$ topology, in the static gauge the S^1 direction may be identified with 11d direction φ in (1.10). Then expanding all 3d world volume fields in Fourier modes in the S^1 coordinate the M2 brane action may be written as a 2d action for the "massless" 2d fields, representing the corresponding type IIA string action in $AdS_4 \times CP^3$, interacting with an infinite set ("KK tower") of 2d fields with masses $m_l^2 = \frac{l^2}{R_{11}^2} = l^2 k^2 = l^2 \frac{8\pi T}{g_s^2}$ $(l = 1, 2, \dots)$ depending on the string coupling.

These massive 2d fields decouple only in the strict $g_s \to 0$ limit, while in general their contributions will encode the string loop corrections to (1.25) or (1.18). Integrating them out one would get an effective non-local action for the "massless" (string or $l = 0$ level) modes. That would realise the idea of having an effective string action on 2-sphere supplemented by "handle operator" contributions that account for the usual sum over the 2d topologies (cf. [45–47]).

1.2 Example: $1/N$ expansion of $\frac{1}{2}$ BPS Wilson loop

To illustrate the above discussion let us review the case of the $1/N$ expansion of the $\frac{1}{2}$ BPS Wilson loop expectation value in the ABJM theory that can be reproduced by the semiclassical M2 brane computation, as was demonstrated in [38]. The exact (at large N and for $k > 2$) result found from localization matrix model on the gauge theory side is [48]

$$\langle W \rangle = \frac{1}{2\,\sin\frac{2\pi}{k}} \, \frac{\text{Ai}\left[(\frac{\pi^2}{2}k)^{1/3}\left(N - \frac{k}{24} - \frac{7}{3k}\right)\right]}{\text{Ai}\left[(\frac{\pi^2}{2}k)^{1/3}\left(N - \frac{k}{24} - \frac{1}{3k}\right)\right]} \,. \tag{1.26}$$

Expanded in large N at fixed k and then further in large $k = \frac{N}{\lambda}$ this may be written as

$$\begin{aligned} \langle W \rangle &= \frac{1}{2\sin\frac{2\pi}{k}}\; e^{\pi\sqrt{\frac{2N}{k}}}\left[1 - \frac{\pi\,(k^2+32)}{24\sqrt{2}\,k^{3/2}}\,\frac{1}{\sqrt{N}} + \mathcal{O}\left(\frac{1}{N}\right)\right] \\ &= \frac{1}{2\sin\frac{2\pi\lambda}{N}}\; e^{\pi\sqrt{2\lambda}}\left[1 - \frac{\pi}{24\sqrt{2}}\,\frac{1}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{N}\right)\right]. \end{aligned} \tag{1.27}$$

The first expansion here may be written as a semiclassical expansion for large effective M2 brane tension T_2 in (1.16)

$$\langle W \rangle = \frac{1}{2\sin\frac{2\pi}{k}}\, e^{\frac{\pi^2}{k}T_2}\left[1 - \frac{k^2+32}{24k}(T_2)^{-1} + \mathcal{O}((T_2)^{-2})\right], \tag{1.28}$$

so that it takes the form of (1.24) or, equivalently, $\log\langle W\rangle$ takes the form of (1.18)

$$\log\langle W \rangle = \frac{\pi^2}{k}T_2 - \log\left(2\sin\frac{2\pi}{k}\right) - \frac{k^2+32}{24k}(T_2)^{-1} + \mathcal{O}((T_2)^{-2}) \,. \tag{1.29}$$

The exponential factor in (1.28) comes from the value of the action of the M2 brane wrapped on $AdS_2 \times S^1$ (ending on a circle at the boundary of AdS_4). The prefactor $\frac{1}{2\sin\frac{2\pi}{k}}$ was reproduced in [38] as the 1-loop contribution (1.25) of the corresponding 3d fluctuations. The subleading $(\mathrm{T}_2)^{-1}$ term in (1.28) should originate from the 2-loop M2 brane contribution, etc.

Expressed in terms of the string tension and the string coupling in (1.14) the prefactor in (1.28) may be written as [49] (cf. (1.15))

$$\begin{aligned}\langle W\rangle &= \frac{1}{2\sin\Big(\sqrt{\frac{\pi}{2}}\,\frac{g_{\rm s}}{\sqrt{\mathrm{T}}}\Big)}\, e^{2\pi\,\mathrm{T}}\Big[1+O(\mathrm{T}^{-1})\Big] \\ &= \frac{\sqrt{\mathrm{T}}}{\sqrt{2\pi}g_{\rm s}}\, e^{2\pi\,T}\bigg[1+\frac{\pi}{12}\frac{g_{\rm s}^2}{\mathrm{T}}+\frac{7\pi^2}{1440}\Big(\frac{g_{\rm s}^2}{\mathrm{T}}\Big)^2+\dots\bigg]\Big[1+O(\mathrm{T}^{-1})\Big]\,. \end{aligned} \tag{1.30}$$

Thus the large k expansion of the 1-loop M2 brane factor $\frac{1}{2\sin\frac{2\pi}{k}}$ captures the leading large string tension (or large λ) corrections at each order in $g_{\rm s}^2$, while the 2-loop and higher M2 brane corrections determine the subleading in $\mathrm{T}^{-1}\sim\frac{1}{\sqrt{\lambda}}$ terms at each order in $g_{\rm s}^2$. Equivalently, (1.28) or (1.30) implies that

$$\langle W\rangle = \frac{\sqrt{\mathrm{T}}}{g_{\rm s}}e^{2\pi\mathrm{T}}\bigg[\Big(\mathrm{c}_{00}+\frac{\mathrm{c}_{10}}{\mathrm{T}}+\dots\Big)+\frac{g_{\rm s}^2}{\mathrm{T}}\Big(\mathrm{c}_{01}+\frac{\mathrm{c}_{11}}{\mathrm{T}}+\dots\Big)+\Big(\frac{g_{\rm s}^2}{\mathrm{T}}\Big)^2\Big(\mathrm{c}_{02}+\frac{\mathrm{c}_{12}}{\mathrm{T}}+\dots\Big)+\dots\bigg], \tag{1.31}$$

where c_{0r} $(r=0,1,2,\dots)$ are determined by the 1-loop M2 brane contribution, c_{1r} — by the 2-loop M2 brane contribution, etc. From the perturbative $\mathrm{AdS}_4\times\mathrm{CP}^3$ string theory perspective, the c_{0r} coefficients represent the leading in $\mathrm{T}^{-1}\sim\alpha'$ terms at each order in the string loop (genus) expansion, i.e. c_{00} is the 1-loop (in string world sheet sense) coefficient on the disk, c_{01} is its counterpart on the disk with one handle, etc.

1.3 Cusp anomalous dimension from fast-spinning M2 brane in AdS_4

The same pattern of non-planar corrections should apply also to other observables like anomalous dimensions, and, in particular, to the cusp anomalous dimension. Namely, the semiclassical quantization of the M2 brane generalization of the long spinning folded string in AdS_4 should lead to the following expansion for $f(k,\mathrm{T}_2)$ in the M2 brane energy or dimension (1.2) (cf. (1.18))

$$f(k,\mathrm{T}_2)=\frac{\pi}{k}\mathrm{T}_2+q_0(k)+q_1(k)(\mathrm{T}_2)^{-1}+q_2(k)(\mathrm{T}_2)^{-2}+\dots\,, \tag{1.32}$$

$$q_r(k)=k^r\Big(p_r^{(0)}+\frac{p_r^{(1)}}{k^2}+\frac{p_r^{(2)}}{k^4}+\dots\Big)\,,\qquad r=0,1,2,\dots\,. \tag{1.33}$$

Since $\mathrm{T}_2\sim k\sqrt{\lambda}$ (cf. (1.16)) the specific large k asymptotics of $q_r(k)$ in (1.33) is the one required to match the inverse string tension ($\frac{1}{\sqrt{\lambda}}$) expansion in the strict tree-level (planar) string theory limit or to match the structure of the expected $1/N^2$ expansion in the 't Hooft limit on the gauge theory side (cf. (1.6), (1.7)).

Thus the condition of matching the string theory expansion like (1.31) fixes the structure of the large k terms in the coefficient functions in the general expression for the semiclassical expansion in (1.18). This requirement assumes that the "double dimensional reduction" relation between the M2 theory and string theory observed at the classical action level extends also to the quantum level. This is implied by the structure of the M2 action as an effective 2d action containing massive KK modes in S^1 direction which should decouple in the $k \to \infty$ limit (assuming that the theory turns out to be well defined in the UV). As in the Wilson loop case reviewed above, we will explicitly verify this at the 1-loop $q_0(k)$ level below.

In particular, the leading large λ asymptotics of each term in the $1/N^2$ expansion, i.e. the coefficients a_{1s} in (1.7), should be the same as the coefficients $p_0^{(s)}$ in the large k expansion of the 1-loop M2 brane function $q_0(k)$, i.e.

$$q_0(k) = p_0^{(0)} + \bar{q}_0(k)\,, \qquad \bar{q}_0(k) = \frac{p_0^{(1)}}{k^2} + \frac{p_0^{(2)}}{k^4} + \dots\,, \qquad a_{1s} = p_0^{(s)}\,, \qquad s = 1, 2, \dots\,. \tag{1.34}$$

As we will find below (cf. (1.15))

$$p_0^{(0)} = -\frac{5\log 2}{2\pi}\,, \qquad \bar{q}_0 = \frac{2\pi}{3k^2} + \frac{2\pi^3}{45k^4} + \dots = \frac{g_s^2}{12\mathrm{T}} + \frac{\pi g_s^2}{1440\mathrm{T}^2} + \dots = \frac{2\pi\lambda^2}{3N^2} + \frac{2\pi^3\lambda^4}{45N^4} + \dots\,. \tag{1.35}$$

At the same time, the string world-sheet loop corrections that represent the $\frac{1}{\sqrt{\lambda}}$ expansion of the planar function f_0 in (1.6), (1.7) come from the leading large k term in the $q_r(k)$ functions (with coefficient $p_r^{(0)}$) in (1.32) or explicitly (cf. (1.16))

$$\mathrm{f}_0(\lambda)\big|_{\lambda\gg 1} = a_{10} + \frac{1}{\sqrt{\lambda}}a_{20} + \frac{1}{(\sqrt{\lambda})^2}a_{30} + \dots\,, \qquad a_{r+1,0} = \left(\frac{\pi}{\sqrt{2}}\right)^r p_r^{(0)}\,, \qquad r = 0, 1, 2, \dots\,. \tag{1.36}$$

To recall, by direct perturbative large tension expansion for a long folded spinning string in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ one finds that at the 1-loop [26–28] and 2-loop [29] orders

$$f_0(\lambda) = \sqrt{2\lambda} + \mathrm{f}_0(\lambda) = \sqrt{2\lambda} - \frac{5\log 2}{2\pi} - \left(\frac{K}{4\pi^2} + \frac{1}{24}\right)\frac{1}{\sqrt{2\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)\,, \tag{1.37}$$

where K is the Catalan's constant. Expressed in terms of the "renormalized tension" containing a $\log 2$ correction [50]

$$h(\lambda)\big|_{\lambda\gg 1} = \sqrt{\frac{\lambda}{2}} - \frac{\log 2}{2\pi} - \frac{1}{48\sqrt{2\lambda}} + \dots\,, \tag{1.38}$$

eq. (1.37) takes the same form as in the $\mathrm{AdS}_5 \times S^5$(i.e. $\mathcal{N} = 4$ SYM) case

$$f_0(\lambda) = 2h(\lambda) - \frac{3\log 2}{2\pi} - \frac{K}{8\pi^2}h^{-1}(\lambda) + \mathcal{O}(h^{-2}(\lambda))\,. \tag{1.39}$$

In general, the ABJM cusp anomalous dimension is expressed in terms of the SYM one as [51]:[3] $f_0(\lambda) = \frac{1}{2}\, f_{0\mathrm{SYM}}(\lambda_{\mathrm{SYM}})$ where one is to replace $\frac{\sqrt{\lambda_{\mathrm{SYM}}}}{4\pi} \to h(\lambda)$. According to the conjecture of [54] the exact expression for $h(\lambda)$ is determined by the relation $\lambda = \frac{\sinh 2\pi h(\lambda)}{2\pi}\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\sinh^2 2\pi h(\lambda)\right)$, implying that at strong coupling

$$h(\lambda) = \frac{1}{\sqrt{2}}\sqrt{\lambda - \frac{1}{24}} - \frac{\log 2}{2\pi} + \mathcal{O}\left(e^{-2\pi\sqrt{2\lambda}}\right)\,, \qquad \lambda \gg 1\,. \tag{1.40}$$

[3]This follows from the equivalence of the BES [2] equations in the $\mathcal{N} = 4$ SYM and ABJM cases and the fact that $h(\lambda)$ (which is not renormalized in the SYM case, $h_{\mathrm{SYM}} = \frac{\sqrt{\lambda_{\mathrm{SYM}}}}{4\pi}$) but should be non-trivial [52, 53] to correctly interpolate between the weak and strong coupling regimes in the ABJM magnon dispersion relation $\epsilon = \frac{1}{2}\sqrt{1 + 16\, h^2(\lambda)\sin^2\frac{p}{2}}$.

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The shift $\lambda \to \lambda - \frac{1}{24}$ may be related to the redefinition $N \to N - \frac{1}{24}(k - k^{-1})$ that follows [55] from the presence of the $R^4 \wedge C_3$ term in the M-theory effective action. This shift of N modifies the relation between L and N in (1.8) and thus the expressions for g_s and T in (1.14).[4] In particular, one gets $\lambda = \frac{N}{k} \to \lambda - \frac{1}{24}(1 - k^{-2}) = \lambda - \frac{1}{24} + \frac{\lambda^2}{24N^2}$, where the $1/N^2$ correction may be ignored in the string tree level (planar) approximation. In general, one gets $f(\lambda) = \sqrt{2\lambda} + \ldots \to \sqrt{2\lambda - \frac{1}{12}(1 - \frac{1}{k^2})} + \ldots = \sqrt{2\lambda - \frac{1}{12}} + \frac{1}{12k^2}\big(\sqrt{2\lambda - \frac{1}{12}}\,\big)^{-1} + \ldots$. The resulting corrections to the coefficients of the $\frac{1}{k^{2n}}$ terms in the M2 brane 1-loop contribution in (1.35) are thus subleading at large λ and will be ignored below. They will become relevant once the 2-loop $q_1(k)$ term in (1.32), (1.33) is taken into account.

1.4 Non-planar corrections from semiclassical M2 branes with two spins in $S^7/\mathbb{Z}_k$

One may apply the same strategy of semiclassically quantizing M2 brane solutions to find the leading strong-coupling asymptotics of the non-planar corrections to the dimensions of other dual ABJM operators. The starting point, like in the familiar $AdS_5 \times S^5$case (for a review see, e.g, [56, 57]), is a classical string solution in $AdS_4 \times CP^3$ that is dual to a particular ABJM operator with large quantum numbers. One is then to find its generalization to an M2 brane in $AdS_4 \times S^7/\mathbb{Z}_k$ which is wrapped also on the 11d circle φ in $S^7/\mathbb{Z}_k$ (1.10) (so that upon the double dimensional reduction or in the $k \to \infty$ limit it reduces to a particular string solution for which one may be able to identify the dual gauge-theory operator).[5]

Considering first the string theory case, let $\mathcal{J}_r$ be a collection of parameters (frequencies, etc.) of a classical solution that are fixed in the large tension expansion. Then the global AdS energy should have the following large tension expansion

$$E = \sqrt{\bar\lambda}\,\mathcal{E}_0(\mathcal{J}_r) + \mathcal{E}_1(\mathcal{J}_r) + \frac{1}{\sqrt{\bar\lambda}}\mathcal{E}_2(\mathcal{J}_r) + \ldots\,, \qquad J_r = \sqrt{\bar\lambda}\,\mathcal{J}_r\,, \tag{1.41}$$

$$\bar\lambda \equiv 2\pi^2\lambda\,, \qquad T = \frac{\sqrt{\lambda}}{\sqrt{2}} = \frac{\sqrt{\bar\lambda}}{2\pi}\,. \tag{1.42}$$

To stress the analogy with the $AdS_5 \times S^5$case we introduced as in (1.14) the rescaled coupling $\bar\lambda$ (used also in [50]). In (1.41) $\mathcal{E}_0$ is the classical contribution, $\mathcal{E}_1$ is the 1-loop world-sheet correction, etc. One can then expand $\mathcal{E}_\ell$ in the limit of small or large $\mathcal{J}_r$, express E in terms of the spins $J_r = \sqrt{\bar\lambda}\mathcal{J}_r$ and compare to the dimensions of the dual gauge theory operators. The 1-loop corrections to energies of two-spin solutions in $AdS_4 \times CP^3$ were discussed in [26, 27, 50, 69, 70].

Since here $\sqrt{\bar\lambda} \gg 1$ while $\mathcal{J}_r$ are fixed, one has $J_r \gg 1$ but one may hope that it may be possible also to capture the strong-coupling behaviour of dimensions of "short" operators with finite values of spins (see [70–73]).

Starting with an M2 brane solution in $AdS_4 \times S^7/\mathbb{Z}_k$ that generalizes a spinning string solution in $AdS_4 \times CP^3$ the analog of the large tension expansion in (1.41) will

[4] Explicitly, we get $g_s = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N}\left(1 - \frac{1}{24\lambda} + \frac{\lambda}{24N^2}\right)^{1/4}$, $\quad \mathrm{T} = \frac{L^2}{8\pi\alpha'} = \sqrt{\frac{\lambda}{2}}\left(1 - \frac{1}{24\lambda} + \frac{\lambda}{24N^2}\right)^{1/2}$.

[5] Classical rotating membrane solutions in flat and AdS spaces were previously discussed, e.g., in [58–68].

be (cf. (1.18), (1.32))[6]

$$E = \frac{\pi}{k} \mathrm{T}_2\, \mathcal{E}_0(\mathcal{J}_r) + \hat{\mathcal{E}}_1(\mathcal{J}_r, k) + (\mathrm{T}_2)^{-1} \hat{\mathcal{E}}_2(\mathcal{J}_r, k) + \dots , \qquad \mathrm{T}_2 = \frac{k}{\pi^2}\sqrt{\lambda} \, . \tag{1.43}$$

In this expansion T_2 is assumed to be large while k (the parameter of the 11d background) and $\mathcal{J}_r$ (the parameters of the classical M2 brane solution) are fixed. To relate this to the small g_s expansion in type IIA string theory in $AdS_4 \times CP^3$ or to the large N expansion in the dual ABJM gauge theory we should then expand $\hat{\mathcal{E}}_r$ in large k for fixed $\mathcal{J}_r$ as in (1.33)

$$\hat{\mathcal{E}}_1(\mathcal{J}_r, k) = \mathcal{E}_1(\mathcal{J}_r) + \frac{1}{k^2}\mathcal{G}_{11}(\mathcal{J}_r) + \frac{1}{k^4}\mathcal{G}_{12}(\mathcal{J}_r) + \dots , \tag{1.44}$$

$$\hat{\mathcal{E}}_2(\mathcal{J}_r, k) = \frac{k}{\pi}\Big[\mathcal{E}_2(\mathcal{J}_r) + \frac{1}{k^2}\mathcal{G}_{21}(\mathcal{J}_r) + \frac{1}{k^4}\mathcal{G}_{22}(\mathcal{J}_r) + \dots\Big], \quad \dots . \tag{1.45}$$

The strong-coupling limit of the leading non-planar correction is thus represented by the $\mathcal{G}_{11}$ term in the 1-loop M2 brane contribution $\hat{\mathcal{E}}_1$.

One may then consider the large or small $\mathcal{J}_r$ limits and finally express the resulting expressions in terms of the quantum numbers $J_r = \sqrt{\lambda}\mathcal{J}_r$ to get predictions for the corresponding gauge theory anomalous dimensions. The above order of limits corresponds to operators for which their quantum numbers do not grow with N, i.e. are fixed in the large N limit so that $N^{-1} J_r \sim \frac{1}{\sqrt{k}}\mathcal{J}_r \ll 1$.

Below we will consider the M2 brane solutions that generalize the "short" (or "slow", $\mathcal{J}_r \ll 1$) and "long" (or "fast", $\mathcal{J}_r \gg 1$) circular string solutions with two equal angular momenta $J_1 = J_2 \equiv J$ in $CP^3 \subset S^7/\mathbb{Z}_k$. These are direct analogs of the string solutions in $AdS_5 \times S^5$for which 1-loop corrections to energies were discussed in [71, 74–76].

The "long" $J_1 = J_2$ string solution in $AdS_4 \times CP^3$ that has the classical energy $E_0 = \sqrt{4J^2 + \lambda}$ was already studied in [69]. Here we will find also its "short" counterpart with $E_0 = \sqrt{2\sqrt{\lambda}J}$. The energies of the corresponding M2 branes wrapped on the 11d circle are given by the same expressions. The dual operators having these quantum numbers should be built out of the 4 bi-fundamental scalars of the ABJM theory as $\mathcal{O} = \mathrm{tr}\big[(Y^1 Y_2^\dagger)^{J_1} (Y^3 Y_4^\dagger)^{J_2}\big] + \dots$.

We will first compute the 1-loop string corrections to the above classical energies. In particular, for the "short" string solution we will find

$$E_{\rm str} = 2\sqrt{\sqrt{\lambda}J} + \frac{1}{2} + \frac{1}{2}\frac{J^{1/2}}{\lambda^{1/4}} - \frac{9}{4}\zeta(3)\frac{J^{3/2}}{\lambda^{3/4}} + \mathcal{O}\left(\frac{J^2}{\lambda}\right) \, . \tag{1.46}$$

This represents a prediction for the subleading strong-coupling corrections to the dimension $\Delta(J)$ of the corresponding dual "short" operator that has "flat-space" scaling $\Delta \sim \sqrt[4]{\lambda}\sqrt{J}$ [21] at leading order in strong coupling. The energy (1.46) has a similar structure as the small-spin expansion for the 1-loop corrected energy of a short folded (S, J) string spinning in AdS_4 and also having orbital momentum in CP^3 that was found in [70][7]

$$E_{\rm str}(S, J) = \sqrt{2\sqrt{\lambda}S} - \frac{1}{2} + \frac{1}{4}\frac{(2S)^{1/2}}{\lambda^{1/4}}\Big[S^{-1}J(J+1) + \frac{3}{2}S - 1\Big] + \dots . \tag{1.47}$$

[6]Here $\frac{\pi}{k}\mathrm{T}_2 = \frac{1}{2}\frac{2\pi}{k}\mathrm{T}_2$ contains in addition to the factor $\frac{2\pi}{k}$ of the length of the 11d circle in (1.10) on which the M2 brane is wrapped an extra $\frac{1}{2}$ due to the scale of the AdS_4 factor in (1.8), cf. (1.16).

[7]For a similar expression in the $AdS_5 \times S^5$case see [70, 77, 78] (see also [79]).

It would be interesting to match (1.46) to the integrability (quantum spectral curve) strong-coupling predictions for the dimensions of the corresponding states.[8] These were found previously for a few ($S = 2,\ J = 1$) [54] and ($S = 1,\ J = 1, 2, 4$) [82] operators of the form $\text{tr}\big[D_+^S(Y^1Y_4^\dagger)^J\big]$ in the sl_2 sector of the ABJM theory (see also [83, 84]).[9]

We will then generalize the string 1-loop computations to the M2 brane ones getting predictions for the non-planar corrections to the dimensions of the above $J_1 = J_2$ operators at strong coupling. For the "short" M2 brane solution we will find the following $1/k^2$ correction to (1.46)

$$
\begin{aligned}
E_{\rm M2} =& 2\sqrt{\sqrt{\bar\lambda}J} + \frac{1}{2} + \frac{1}{2}\bar\lambda^{-1/4}J^{1/2} - \frac{9}{4}\zeta(3)\bar\lambda^{-3/4}J^{3/2} + \mathcal{O}(\bar\lambda^{-1}J^2) \\
&+ \frac{1}{k^2}\Big[\zeta(2)\big(-4\bar\lambda^{3/4}J^{-3/2} + 8\bar\lambda^{1/4}J^{-1/2}\big) + \mathcal{O}(\bar\lambda^{-1/4}J^{1/2})\Big] + \mathcal{O}\left(\frac{1}{k^4}\right) .
\end{aligned}
\tag{1.48}
$$

From the string theory point of view the membrane correction term $\sim \frac{1}{k^2} = \frac{g_s^2}{4\sqrt{\bar\lambda}}$ represents the leading large tension asymptotics of the string 1-loop (torus) contribution.

On the dual ABJM gauge theory side (1.48) should be understood as the expansion first in $1/N^2$ and then in large λ for fixed quantum number J. The $\frac{1}{k^2} = \frac{\lambda^2}{N^2} = \frac{\bar\lambda^2}{(2\pi^2)^2N^2}$ term in (1.48) then represents a prediction for the leading non-planar correction to the dimension of the corresponding "short" operator.

In the "long" M2 brane case we will find

$$
\begin{aligned}
E_{\rm M2} =& 2J + \frac{1}{4}\bar\lambda J^{-1}(1 - 2\log 2\,\bar\lambda^{-1/2} + \ldots) + \frac{1}{2}c_1\bar\lambda J^{-2}(1 + \ldots) + \ldots \\
&+ \frac{1}{k^2}\,\zeta(2)\big(-8\bar\lambda^{-1/2}J - 2\bar\lambda^{1/2}J^{-1} + \frac{3}{16}\bar\lambda^{3/2}J^{-3} + \ldots\big) + \mathcal{O}\left(\frac{1}{k^4}\right) .
\end{aligned}
\tag{1.49}
$$

Here $c_1 \approx -0.336$ and the $\frac{1}{k^2} = \frac{\bar\lambda^2}{(2\pi^2)^2N^2}$ term represents a prediction for the strong-coupling limit of the leading non-planar correction to the dimension of the corresponding operator with the large spin J.

The rest of this paper is organized as follows. In section 2 we will present the 1-loop M2 brane computation that generalizes the leading strong coupling $\text{AdS}_4 \times \text{CP}^3$ string theory contribution to the ABJM cusp anomaly to the non-planar level.

In section 3 we will consider the M2 brane generalizations of the "long" and "short" circular string solutions with equal spins in CP^3 and compute the corresponding 1-loop corrections to the AdS_4 energies in the large k expansion.

Some open problems will be mentioned in section 4. There are also several appendices containing some details of the computations. In appendix D we will make some comments on non-planar corrections to the multi-wound Wilson loop and the Brehmstrahlung function in the ABJM theory related to the discussion in section 4.

[8]For a recent exposition of the strong-coupling QSC results in the $\text{AdS}_5 \times S^5$case see [80, 81].

[9]The BMN operator which is the vacuum in the sl_2 sector corresponds to $\text{tr}(Y^1Y_4^\dagger)^J$ in the representation $[J, 0, J]$ of SU(4). For early discussions of integrability of the ABJM theory see [52, 85, 86].

2 1-loop correction to energy of M2 brane spinning in AdS_4

In this section we will compute the 1-loop correction to the partition function (1.24) expanded near the classical M2 brane solution in $AdS_4 \times S^7/\mathbb{Z}_k$ that generalizes the infinitely long rotating folded string [21, 22, 26] in AdS_4. This will determine the function $q_0(k)$ in (1.32)–(1.35), i.e. the leading large λ corrections at each order in the $1/N^2$ expansion in the cusp anomalous dimension in the ABJM theory.

In terms of the $AdS_4 \times S^7/\mathbb{Z}_k$ coordinates in (1.9), (1.10) the relevant large-spin (infinitely long) membrane solution is

$$t = \kappa\,\xi^0, \qquad \rho = \kappa\,\xi^1, \qquad \alpha = 0\,, \qquad \beta = \kappa\,\xi^0, \qquad \varphi = \xi^2\,, \tag{2.1}$$

with CP^3 coordinates in (1.11) being trivial and κ being a constant parameter. Here ξ^i $(i = 0, 1, 2)$ are the membrane world-volume coordinates with $\xi^2 \in (0, 2\pi)$. One of 4 segments of the folded closed string is represented by $\xi^1 = (0, \frac{\pi}{2})$. We will consider the limit $\kappa \to \infty$ in which the rescaled $\xi^1 \to \kappa\xi^1$ can be decompactified. The corresponding classical AdS_4 energy and the spin satisfy ($\mathcal{S} = \frac{S}{\sqrt{\lambda}} \gg 1$)

$$E_0 - S = \frac{1}{4}\frac{(2\pi)^2}{k}\mathrm{T}_2\,\kappa = \sqrt{\lambda}\,\kappa = \sqrt{2\lambda}\,\log S\,, \qquad \kappa = \frac{1}{\pi}\log \mathcal{S} \gg 1\ . \tag{2.2}$$

The dependence on the parameter κ can be scaled away by redefining the coordinates ξ^0, ξ^1; it will then appear only as an overall factor in the log of the quantum partition function or in the quantum correction to the energy. It is useful also to perform the Euclidean continuation $\xi^0 \to i\xi^0$. The resulting induced 3d metric is flat (cf. (1.8))

$$g_{ij} = \frac{1}{4}\mathrm{L}^2\bar{g}_{ij}, \qquad \bar{g}_{ij} = \left(1, 1, \frac{4}{k^2}\right)\ . \tag{2.3}$$

We will expand all 3d fluctuation fields in Fourier modes in $\xi^2 = \varphi$ getting an effective 2d field theory on $\mathbb{R}^2$ with $l = 0$ sector representing the modes of the type IIA string on $AdS_4 \times CP^3$ and the $l \neq 0$ tower being the genuine membrane modes. The derivation of the corresponding fluctuation operators in (1.25) is very similar to the case of the $AdS_2 \times S^1$ M2 brane representing the circular Wilson loop that was discussed in [38, 87].

We will fix the static gauge setting to zero the fluctuations of t, ρ and φ so that the non-zero bosonic fluctuations will be those of $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta - \xi^0$ and of the 6 real CP^3 coordinates. Extracting the overall factor $\frac{1}{4}\mathrm{L}^2$ the resulting fluctuation operators will contain the "free" part

$$-\bar{g}^{ij}\partial_i\partial_j = -\left(\partial_0^2 + \partial_1^2 + \frac{1}{4}k^2\partial_2^2\right) \quad \to \quad p^2 + \frac{1}{4}k^2l^2\,, \qquad p^2 = p_0^2 + p_1^2\,, \qquad l = 0, \pm1, \pm2, \dots\,, \tag{2.4}$$

plus effective mass terms. Here p_α are the momenta in the non-compact ξ^0 and ξ^1 directions and l is the mode number in the circular ξ^2 direction. One finds that the 6 real CP^3 fluctuations have masses

$$m_l^2 = \frac{1}{4}kl(kl + 2) \qquad \text{(6 modes)}, \tag{2.5}$$

where the linear in kl term comes from the mixing between the constant $d\varphi$ term and $k\mathrm{A}$ (which is quadratic in fluctuations) in the $S^7/\mathbb{Z}_k$ metric in (1.10). The $l = 0$ modes in (2.5) are 6 massless excitations in the corresponding folded string spectrum in $AdS_4 \times CP^3$. Note that if $k=1$ we get 6 tachyonic modes with $l=-1$ indicating an instability of the membrane wrapped on a big circle of S^7.[10] As we are interested in the large $k=\frac{N}{\lambda}$ expansion, below we will assume that $k>1$ but will return to the $k=1$ case at the end of this section.

Expanding the volume $\int\sqrt{g}$ part (1.19) of the membrane action we get for the quadratic Lagrangian for the 2 remaining bosonic 3d fluctuations $\tilde{\alpha}$ and $\tilde{\beta}$ (scaling out the $\frac{1}{4}\mathrm{L}^2$ factor in (2.3))

$$L_{\rm V}=\frac{1}{2}\Big[\sinh^2\rho\,\cosh^2\rho\,\partial^i\tilde{\beta}\partial_i\tilde{\beta}+\sinh^2\rho\,(\partial^i\tilde{\alpha}\partial_i\tilde{\alpha}+\tilde{\alpha}^2)\Big]\,, \tag{2.6}$$

where $\rho=\xi^1$. After the 3d field redefinition $(\tilde{\beta},\tilde{\alpha})\to(u,v)$

$$\tilde{\beta}=(\sinh\rho\,\cosh\rho)^{-1}u\,,\qquad\qquad \tilde{\alpha}=(\sinh\rho)^{-1}v\,, \tag{2.7}$$

and integrating by parts we get (as in the static-gauge analysis in the $AdS_5\times S^5$case [22])

$$L_{\rm V}=\frac{1}{2}(\partial^i u\partial_i u+4u^2+\partial^i v\partial_i v+2v^2)\ . \tag{2.8}$$

In addition, there is a contribution coming from the WZ term in the membrane action (1.20) with C_3 given by (1.12). Using (2.1) it leads to the mixing term $L_{\rm WZ}\sim v\partial_2 u$ so that in total we get

$$L(u,v)=L_{\rm V}+L_{\rm WZ}=\frac{1}{2}(\partial^i u\partial_i u+4u^2+\partial^i v\partial_i v+2v^2)-3v\partial_2 u\ . \tag{2.9}$$

Expanding in modes in ξ^2 we have $\partial_2\to\frac{1}{2}ikl$ and thus diagonalizing (2.9) find two towers of 2d scalars with the following masses (which are positive for any kl)

$$m^2_{l,+}=3+\frac{1}{4}k^2l^2+\sqrt{1+\frac{9}{4}k^2l^2}\,,\qquad\qquad m^2_{l,-}=3+\frac{1}{4}k^2l^2-\sqrt{1+\frac{9}{4}k^2l^2}\ . \tag{2.10}$$

For $l=0$ we reproduce the values of masses (4 and 2) of the two AdS_4 fluctuations in the corresponding string theory limit.[11]

Finding the quadratic fermionic Lagrangian from (1.21) is very similar to the $AdS_2\times S^1$ membrane case [38] and one gets 8 fermionic towers in flat 2d space with masses

$$m_l=\frac{1}{2}kl\pm1\quad(3{+}3\text{ modes})\,,\qquad\qquad m_l=\frac{1}{2}kl\quad(2\text{ modes})\ . \tag{2.11}$$

For $l=0$ this reproduces the spectrum of the fermionic fluctuations for the infinite folded string in $AdS_4\times CP^3$ [26, 27].

[10] The classical spinning membrane solution in $AdS_4\times S^7$ that corresponds to a folded spinning string in AdS_4 was discussed also in [67] and this reference had comments on its instability by analogy with a string wrapped on a circle in the sphere.

[11] Note that the mass of the fluctuation of the coordinate α transverse to the AdS_3 subspace where the string moves is the same 2 as in the case of the AdS_5 string solution (where there are two such modes). The only mass that changes is that of the fluctuation of β as the string is rotating in this direction. In general, the mass of such mode is $4+R^{(2)}$ where $R^{(2)}$ is the curvature of induced metric. For comparison, in the case of the $AdS_2\times S^1$ membrane in [38] the mass terms of the corresponding fluctuations in (2.9) were both equal to 2 (due to the shift of mass term 4 by the scalar curvature $R^{(2)}=-2$ of AdS_3) and then $m^2_{l,\pm}=2+\frac{1}{4}k^2l^2\pm\frac{3}{2}kl$. As l takes both positive and negative values this is equivalent to having 2 modes with $m_l^2=\frac{1}{4}(kl-2)(kl-4)$.

Integrating out 8+8 towers of fluctuations in $\mathbb{R}^2$ and summing over l we then get the 1-loop partition function $\mathcal{Z}_1$ in (1.25). Since all fluctuation operators have constant coefficients, $\log \mathcal{Z}_1$ will be proportional to the 3d volume containing the κ^2 factor from rescaling of ξ^0 and ξ^1 (cf. (2.1)). The 1-loop correction to the world-volume energy will then scale as κ^2. Since $t = \kappa\xi^0$ the corresponding correction to the AdS$_4$ energy will scale as $\kappa = \frac{1}{\pi}\log S$ leading to the expression for the 1-loop term q_0 in the scaling function $f(k, \mathrm{T}_2)$ in (1.32). Explicitly, we find

$$\Gamma_1 = -\log \mathcal{Z}_1 = \frac{1}{2} V q_0, \qquad V = \kappa^2 \int d\xi^0 d\xi^1, \qquad E_1 = \pi q_0 \kappa = q_0 \log S\,, \tag{2.12}$$

$$q_0 = \int \frac{d^2p}{(2\pi)^2}\Big[Q_0(p^2) + 2\sum_{l=1}^{\infty} Q_l(p^2)\Big] = p_{00} + \bar{q}_0(k)\,, \tag{2.13}$$

$$\begin{aligned} Q_l(p^2) = {}& \log\Big[p^2 + 3 + \frac{1}{4}k^2l^2 + \sqrt{1+\frac{9}{4}k^2l^2}\Big] + \log\Big[p^2 + 3 + \frac{1}{4}k^2l^2 - \sqrt{1+\frac{9}{4}k^2l^2}\Big] \\ & + 3\log\Big[p^2 + \frac{1}{4}(kl)^2 + \frac{1}{2}kl\Big] + 3\log\Big[p^2 + \frac{1}{4}(kl)^2 - \frac{1}{2}kl\Big] \\ & - 3\log\Big[p^2 + \Big(1+\frac{1}{2}kl\Big)^2\Big] - 3\log\Big[p^2 + \Big(1-\frac{1}{2}kl\Big)^2\Big] - 2\log\Big[p^2 + \Big(\frac{1}{2}kl\Big)^2\Big]\,. \end{aligned} \tag{2.14}$$

Here in (2.13) we followed (1.34) and separated the k-independent contribution p_{00} to q_0 coming from the $l = 0$ (string-theory) part Q_0 of the integrand. Computing the 2d momentum integral gives

$$p_{00} = \int \frac{d^2p}{(2\pi)^2}\, Q_0(p^2) = \int_0^\infty \frac{dp^2}{4\pi}\Big[\log(p^2{+}4) + \log(p^2{+}2) + 4\log p^2 - 6\log(p^2{+}1)\Big] = -\frac{5\log 2}{2\pi}\,, \tag{2.15}$$

thus reproducing the value of the 1-loop correction to the cusp anomaly in string theory in AdS$_4 \times$ CP^3 given in (1.37).

The integral of Q_l with $l > 0$ giving $\bar{q}_0(k)$ in (2.13) is also UV finite (as one can check explicitly by doing the integral over p^2 between 0 and Λ and taking the limit $\Lambda \to \infty$)

$$\begin{aligned} \bar{Q}_l \equiv {}& \int_0^\infty \frac{dp^2}{2\pi}\, Q_l(p^2) \\ = {}& -\frac{1}{8\pi}\Big[-3(kl{-}2)(kl{-}4)\log(kl{-}2) - 3(kl{+}2)(kl{+}4)\log(kl{+}2) + (kl)^2\log[(kl)^2] \\ & + [(kl)^2{+}12]\log\big[(kl)^4 - 12(kl)^2 + 128\big] + 2\sqrt{9(kl)^2+4}\log\frac{(kl)^2+12+2\sqrt{9(kl)^2+4}}{(kl)^2+12-2\sqrt{9(kl)^2+4}}\Big]\,. \end{aligned} \tag{2.16}$$

Expanding in large k then gives

$$\bar{Q}_l = \frac{4}{\pi(kl)^2} + \frac{4}{\pi(kl)^4} - \frac{1616}{15\pi(kl)^6} - \frac{38944}{35\pi(kl)^8} - \frac{447488}{105\pi(kl)^{10}} + \frac{2227200}{77\pi(kl)^{12}} + \dots\,. \tag{2.17}$$

The remaining sum over M2 modes in (2.13) thus converges, leading to

$$\bar{q}_0 = \sum_{l=1}^{\infty}\bar{Q}_l = \frac{2\pi}{3k^2} + \frac{2\pi^3}{45k^4} - \frac{1616\pi^5}{14175k^6} - \frac{19472\pi^7}{165375k^8} - \frac{447488\pi^9}{9823275k^{10}} + \frac{20519936\pi^{11}}{655539885k^{12}} + \dots\,. \tag{2.18}$$

This determines the non-planar coefficients p_{0r} in $\bar{q}_0$ in (1.34) in terms of the values of $\zeta(2m) = \sum_{l=1}^{\infty} \frac{1}{l^{2m}}$, thus reproducing (1.35).

In the above derivation of (2.16) we assumed that $k > 1$ when (2.16) is real. Analytically continuing $\bar{Q}_l$ to $k = 1$ we get an imaginary part[12]

$$\bar{q}_0\big|_{k=1} = -0.663 + 1.125i \ . \tag{2.19}$$

As already noted below eq. (2.5), this reflects an instability of the membrane that rotates only in AdS_4 and is wrapped on a circle inside S^7 which is contractable.

3 M2 branes rotating in $S^7/\mathbb{Z}_k$

Let us now provide an illustration of the strategy described in section 1.4 and consider 1-loop corrections to the two membrane solutions that generalize the "short" and "long" circular string solutions with two angular momenta $J_1 = J_2$ in $\mathbb{R}_t \times \mathrm{CP}^3$ part of $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$.[13] We shall first describe these string solutions in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ (with the "long" one previously found in [69]) and then generalize them to M2 brane solutions in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. The M2 branes will be located at the center of AdS_4 with $t = \xi^0$, wrapped on the 11d circle φ in (1.10) and rotating in CP^3. We shall then compute the 1-loop corrections to the energies of these "short" and "long" M2 brane solutions and study their expansions in spins and 11d radius $\frac{1}{k}$.

3.1 Classical solutions

It will be useful to use the explicit parametrization of $S^7/\mathbb{Z}_k$ in terms of the 7 angles choosing the 4 complex coordinates subject to $z_a \bar{z}_a = 1$ in as

$$z_1 = \cos\chi \cos\frac{\theta_1}{2} \exp\left[i\left(\frac{\varphi}{k} + \frac{\psi+\phi_1}{2}\right)\right], \quad z_2 = \cos\chi \sin\frac{\theta_1}{2} \exp\left[i\left(\frac{\varphi}{k} + \frac{\psi-\phi_1}{2}\right)\right],$$
$$z_3 = \sin\chi \cos\frac{\theta_2}{2} \exp\left[i\left(\frac{\varphi}{k} - \frac{\psi-\phi_2}{2}\right)\right], \quad z_4 = \sin\chi \sin\frac{\theta_2}{2} \exp\left[i\left(\frac{\varphi}{k} - \frac{\psi+\phi_2}{2}\right)\right], \tag{3.1}$$

[12] The imaginary part is equal to 9/8 and arises from the $l = 1$ term. The real part is obtained by evaluating the sum numerically.

[13] These are direct counterparts of the string solutions in $\mathrm{AdS}_5 \times S^5$describing a rigid circular string rotating in two orthogonal planes in S^5 with $J_1 = J_2 = \sqrt{\lambda}\mathcal{J}$ having two branches: "long" one with $\mathcal{J} \geq \frac{1}{2}$ and "short" one with $\mathcal{J} \leq \frac{1}{2}$ [88] (see also [74, 89]). While the radius of the "long" string is fixed to be that of S^5 so it is never small and admits a "fast-string" expansion $\mathcal{J} = \frac{J}{\sqrt{\lambda}} \gg 1$, the "short" one may have an arbitrarily small radius and spin and thus has a "slow-string" limit $\mathcal{J} \ll 1$ when it probes the near-flat region of S^5. These are among the simplest rigid string solutions with explicitly known spectrum of small fluctuations. For the "long" branch the 1-loop corrections to the energy were computed in the large $\mathcal{J}$ expansion [74–76], with "non-analytic" terms found in [90, 91]. The 1-loop correction to the energy of the "short" solution was found in [71–73]. Note that in addition to the circular solution there is also a folded string solution with 2 spins in S^5 that has less energy for given values of spins [92]. The study of these simplest solutions played an important role in establishing the integrability approach to the spectrum of strings in $\mathrm{AdS}_5 \times S^5$.

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so that the $S^7/\mathbb{Z}_k$ metric in (1.11) takes the form

$$\begin{aligned} ds^2_{S^7/\mathbb{Z}_k} &= ds^2_{\mathrm{CP}^3} + \frac{1}{k^2}(d\varphi + k\mathrm{A})^2\,, \\ \mathrm{A} &= \frac{1}{2}\Big[\cos(2\chi)d\psi + \sin^2\chi\cos\theta_2 d\phi_2 + \cos^2\chi\cos\theta_1 d\phi_1\Big]\,, \\ ds^2_{\mathrm{CP}^3} &= d\chi^2 + \cos^2\chi\sin^2\chi\Big(d\psi + \frac{1}{2}\cos\theta_1 d\phi_1 - \frac{1}{2}\cos\theta_2 d\phi_2\Big)^2 \\ &\quad + \frac{1}{4}\cos^2\chi\Big(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\Big) + \frac{1}{4}\sin^2\chi\Big(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\Big)\,. \end{aligned} \tag{3.2}$$

Here $\chi \in [0, \pi/2)$, $\varphi \in [0, 2\pi)$, $\psi \in [0, 2\pi)$, $\theta_i \in [0, \pi)$, $\phi_i \in [0, 2\pi)$.

3.1.1 String solutions

Starting with the bosonic part of the string action in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ we shall fix the conformal gauge. Then the relevant $\mathbb{R}_t \times \mathrm{CP}^3$ part of the action may be written in terms of that of the CP^3 sigma model as (cf. (1.11), (1.14))

$$S_{\rm str} = -2\mathrm{T}\int d^2\xi\,\Big[-\frac{1}{4}(\partial_\alpha t)^2 + |D_\alpha z^a|^2 - \Lambda(\xi)(|z^a|^2 - 1)\Big]\,, \qquad \mathrm{T} = \frac{\sqrt{\lambda}}{2\pi}\,. \tag{3.3}$$

Here $\alpha = (0, 1)$, $\Lambda(\xi)$ is a Lagrange multiplier imposing the $\bar{z}_a z_a = 1$ constraint on 4 complex coordinates z_a. D_α is a U(1) covariant derivative containing an auxiliary gauge field A_α

$$D_\alpha z^a = \partial_\alpha z^a - iA_\alpha z^a\,, \qquad z^a \to e^{i\epsilon}z^a\,, \qquad A_\alpha \to A_\alpha + \partial_\alpha\epsilon\,, \qquad \epsilon = \epsilon(\xi)\,. \tag{3.4}$$

The equations of motion that follow from (3.3) are

$$\partial_\alpha\partial^\alpha t = 0\,, \qquad D_\alpha D^\alpha z^a = -\Lambda z^a\,, \qquad \Lambda = |D_\alpha z^a|^2\,, \qquad \bar{z}_a z_a = 1\,, \tag{3.5}$$

$$A_\alpha = \frac{1}{2i}(\bar{z}_a\partial_\alpha z^a - z^a\partial_\alpha\bar{z}_a)\,, \qquad |D_\alpha z^a|^2 = \eta^{\alpha\beta}[\partial_\alpha\bar{z}_a\partial_\beta z^a - (\bar{z}_a\partial_\alpha z^a)(z^b\partial_\beta\bar{z}_b)]\,. \tag{3.6}$$

We thus get the expressions that correspond to the metric (1.11) (with A_α related to the 1-form A and z^a being the embedding coordinates of S^7). In addition, we have the conformal gauge constraints ($g_{\alpha\beta}$ is the induced metric)

$$g_{00} + g_{11} = 0\,, \qquad g_{01} = 0\,, \qquad g_{\alpha\beta} = -\frac{1}{4}\partial_\alpha t\partial_\beta t + (D_{(\alpha}z^a)^\dagger D_{\beta)}z^a\,. \tag{3.7}$$

The action (3.3) is invariant under the global SU(4) symmetry. We may choose its Cartan generators as

$$H_1 = \frac{i}{2}\,\mathrm{diag}(1, -1, 0, 0)\,, \qquad H_2 = \frac{i}{2}\,\mathrm{diag}(0, 0, 1, -1)\,, \qquad H_3 = \frac{i}{2}\,\mathrm{diag}(1, 1, -1, -1)\,, \tag{3.8}$$

which correspond to the Killing vector fields ∂_{ϕ_1}, ∂_{ϕ_2} and ∂_ψ of (3.2) respectively. The associated conserved charges or angular momenta and the AdS_4 energy then are

$$J_r = 2\mathrm{T}\int_0^{2\pi} d\xi_1\,\Big[(D_0 z)^\dagger H_r z - z^\dagger H_r D_0 z\Big]\,, \qquad E_0 = \mathrm{T}\int_0^{2\pi} d\xi_1\,\partial_0 t\,. \tag{3.9}$$

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We shall consider a class of "rigid" string solutions for which (cf. (3.1); $a = 1,\ldots,4$)

$$t = \kappa\,\xi^0\,, \qquad z_a = r_a\; e^{i\gamma_a(\xi)}\,, \qquad \gamma_a = w_a\xi^0 + m_a\xi^1\,, \tag{3.10}$$

where r_a, w_a and m_a are constant "radii", frequencies and winding numbers. Fixing the U(1) gauge symmetry of (3.3) by the $A_0 = 0$ condition,[14] the equations of motion (3.5) together with the Virasoro constraints (3.7) reduce to the system of algebraic equations on the parameters in (3.10)

$$A_0 = 0, \qquad A_1 = \sum_a r_a^2 m_a\,, \qquad -w_a^2 + (m_a - A_1)^2 = \Lambda\,, \qquad \sum_a r_a^2 w_a = 0\,, \tag{3.11}$$

$$\sum_a w_a(m_a - A_1)r_a^2 = 0\,, \qquad \frac{1}{4}\kappa^2 = \sum_a r_a^2(m_a - A_1)^2 + \sum_a r_a^2 w_a^2\,, \qquad \sum_a r_a^2 = 1\,. \tag{3.12}$$

Evaluated on (3.10) the angular momenta in (3.9) may be written as

$$J_1 = 4\pi\mathrm{T}(w_1 r_1^2 - w_2 r_2^2), \qquad J_2 = 4\pi\mathrm{T}(w_3 r_3^2 - w_4 r_4^2), \qquad J_3 = 4\pi\mathrm{T}(w_1 r_1^2 + w_2 r_2^2 - w_3 r_3^2 - w_4 r_4^2). \tag{3.13}$$

We shall consider two special solutions of (3.10)–(3.12) for which $J_1 = J_2$, $J_3 = 0$.

The first is the "short" one

$$r_1 = r_2 \equiv a, \qquad r_3 = r_4 = \sqrt{\frac{1}{2} - a^2}, \qquad m_1 = m_2 = -m_3 = -m_4 \equiv \frac{1}{2}m, \tag{3.14}$$

$$w_1 = -w_2 = 2m\left(\frac{1}{2} - a^2\right), \qquad w_3 = -w_4 = 2ma^2, \qquad \kappa^2 = 32m^2a^2\left(\frac{1}{2} - a^2\right),$$
$$A_0 = 0, \qquad A_1 = 2m\left(a^2 - \frac{1}{4}\right), \qquad g_{\alpha\beta} = \mathrm{c}^2\eta_{\alpha\beta}, \qquad \mathrm{c}^2 = \frac{1}{8}\kappa^2, \qquad \Lambda = 0. \tag{3.15}$$

Explicitly, for z^a in (3.10) we get

$$z^1 = a\; e^{i[m(1-2a^2)\xi^0 + \frac{1}{2}m\xi^1]}, \qquad z^2 = a e^{i[-m(1-2a^2)\xi^0 + \frac{1}{2}m\xi^1]},$$
$$z^3 = \sqrt{\frac{1}{2} - a^2}\; e^{i[2ma^2\xi^0 - \frac{1}{2}m\xi^1]}, \qquad z^4 = \sqrt{\frac{1}{2} - a^2}\; e^{i[-2ma^2\xi^0 - \frac{1}{2}m\xi^1]}, \tag{3.16}$$

or, equivalently, in terms of the CP^3 angles in (3.1), (3.2)

$$\cos\chi_0 = \sqrt{2}a, \quad \theta_1 = \theta_2 = \frac{\pi}{2}, \quad \psi = m\xi^1, \quad \phi_1 = 4m\left(\frac{1}{2} - a^2\right)\xi^0, \quad \phi_2 = 4ma^2\xi^0\,. \tag{3.17}$$

Here $0 \le a \le \frac{1}{\sqrt{2}}$ and m is the winding number that takes integer values.[15] The corresponding charges are

$$J_3 = 0, \qquad J_1 = J_2 \equiv J = \sqrt{\lambda}\mathcal{J}, \qquad \mathcal{J} = 8ma^2\left(\frac{1}{2} - a^2\right) = \frac{1}{4}m^{-1}\kappa^2,$$
$$E_0 = \sqrt{\lambda}\kappa = \sqrt{4m\sqrt{\lambda}J}. \tag{3.18}$$

[14] Note that under the gauge transformation (3.4) the phases $\gamma_a(\xi)$ in (3.10) are all shifted by $\epsilon(\xi)$. Thus a solution found in the $A_0 = 0$ gauge that may have $\sum_a \gamma_a \neq 0$ may be transformed into a gauge where $\sum_a \gamma_a = 0$. Note also that the charges (3.9) are invariant under the gauge transformation (3.4).

[15] As $\xi^1 \in (0, 2\pi)$ one could think that m should take only even values. Note, however, that under $\xi^1 \to \xi^1 + 2\pi$ we get $z^a \to e^{\pm im\pi}z^a$ which for any integer m is just an overall phase of z^a which is a trivial symmetry of CP^3 (global part of the U(1) gauge symmetry).

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Here the spin is bounded, i.e. $0 \le \mathcal{J} \le \frac{1}{2}m$ or $0 \le J \le \frac{1}{2}m\sqrt{\bar\lambda}$, with the maximum at $a = \frac{1}{2}$ and the minimum at $a = 0$ or $a = \frac{1}{\sqrt{2}}$. Note that like for the analogous solution in $AdS_5 \times S^5$ [74] the relation between the energy and spin is the same as for the corresponding solution in flat space (i.e. for a circular string rotating in 2 orthogonal planes in $\mathbb{R}^4$).

To see that for $a \to 0$ the solution reduces to its flat-space analog one is to do a U(1) gauge transformation $z_a \to e^{i\frac{m}{2}\xi^1} z_a$ that sets the $a \to 0$ value of A_1 (equal to $-\frac{1}{2}m$ in (3.14)) to zero.[16] Then

$$a \to 0: \qquad z^1 \to a\, e^{im(\xi^0+\xi^1)}\,, \qquad z^2 \to a\, e^{im(-\xi^0+\xi^1)}\,, \qquad z^3 \to \frac{1}{\sqrt{2}}\,, \qquad z^4 \to \frac{1}{\sqrt{2}}\,. \tag{3.19}$$

The second special solution of (3.10)–(3.12) (that was already found in [69]) is a "long" one for which $\mathcal{J}$ and thus J is not bounded. Here (cf. (3.11), (3.13)–(3.18))

$$r_1{=}r_2{=}r_3{=}r_4{=}\frac{1}{2}, \qquad m_1{=}m_2{=}-m_3{=}-m_4{\equiv}\frac{1}{2}m, \qquad w_1{=}-w_2{=}w_3{=}-w_4{=}\mathcal{J}, \tag{3.20}$$

$$A_0{=}A_1{=}0, \qquad \kappa^2{=}4\mathcal{J}^2{+}m^2, \qquad g_{\alpha\beta}{=}\mathrm{c}^2\eta_{\alpha\beta}, \qquad \mathrm{c}^2{=}\frac{1}{4}m^2, \qquad \Lambda{=}-\mathcal{J}^2{+}\frac{1}{4}m^2, \tag{3.21}$$

$$z_1{=}\frac{1}{2}e^{i(\mathcal{J}\xi^0+\frac{1}{2}m\xi^1)}, \quad z_2{=}\frac{1}{2}e^{i(-\mathcal{J}\xi^0+\frac{1}{2}m\xi^1)}, \quad z_3{=}\frac{1}{2}e^{i(\mathcal{J}\xi^0-\frac{1}{2}m\xi^1)}, \quad z_4{=}\frac{1}{2}e^{-i(\mathcal{J}\xi^0+\frac{1}{2}m\xi^1)}, \tag{3.22}$$

$$\chi{=}\frac{\pi}{4}, \qquad \theta_1{=}\theta_2{=}\frac{\pi}{2}, \qquad \psi{=}m\xi^1, \qquad \phi_1{=}\phi_2{=}2\mathcal{J}\xi^0, \tag{3.23}$$

$$J_3{=}0, \qquad J_1{=}J_2{=}J{=}\sqrt{\bar\lambda}\mathcal{J}, \qquad E_0{=}\sqrt{\bar\lambda}\kappa{=}\sqrt{\bar\lambda}\sqrt{4\mathcal{J}^2{+}m^2}{=}\sqrt{4J^2{+}m^2\bar\lambda}\,. \tag{3.24}$$

Like for the similar $AdS_5 \times S^5$solution [74] here the energy expanded at large $\mathcal{J}$ has a familiar "fast-string" form

$$E_0 = 2J + \frac{m^2\bar\lambda}{4J} - \frac{m^4\bar\lambda^2}{64J^3} + \dots\,. \tag{3.25}$$

Note that the two solutions (3.14) and (3.20) coincide in the special case of $a = \frac{1}{2}$ and $\mathcal{J} = \frac{1}{2}m$ when in both cases $E = m\sqrt{2\bar\lambda}$ and z^a in (3.16), (3.22) have $\pm\xi^0 \pm \xi^1$ as their phases.[17]

3.1.2 M2 brane solutions

Let us now discuss how to "uplift" the above string solutions to the M2 brane solutions in $AdS_4 \times S^7/\mathbb{Z}_k$ so that the brane wrapped on 11d angle φ and rotating in CP^3.

As is well known, the "double dimensional reduction" relates the M2 brane action in 11d supergravity background (1.19)–(1.23) to the type IIA string action in the corresponding

[16] Note that since $\xi^1 \in [0, 2\pi)$ one cannot in general transform a constant A_1 component of the potential in (3.11), (3.14) to zero if $\frac{1}{2\pi}\int d\xi^1\, A_1$ is not a (half) integer.

[17] For comparison, in the $AdS_5 \times S^5$case [74] the "short" solution written in S^5 embedding coordinates is $t = \kappa\xi^0$, $X_1 + iX_2 = a\, e^{im(\xi^0+\xi^1)}$, $X_3 + iX_4 = a\, e^{im(\xi^0-\xi^1)}$, $X_3 + iX_4 = \sqrt{1-2a^2}$ with $\kappa^2 = 4m^2a^2 = 4\mathcal{J}$, $J_1 = J_2 = \sqrt{\lambda}\mathcal{J}$, $E = \sqrt{4m\sqrt{\lambda}J}$. For the "long" solution $X_1 + iX_2 = \frac{1}{\sqrt{2}}\, e^{i(\mathcal{J}\xi^0+m\xi^1)}$, $X_3 + iX_4 = \frac{1}{\sqrt{2}}\, e^{i(\mathcal{J}\xi^0-m\xi^1)}$, $X_3 + iX_4 = 0$, $E = \sqrt{4J^2 + m^2\lambda}$.

10d background [44, 93, 94]. Namely, with a 10+1 split of the target space coordinates and a 2+1 split of the world volume coordinates one assumes that

$$X^M = (X^\mu, \varphi)\,, \quad \xi^i = (\xi^\alpha, \xi^2)\,, \qquad \varphi = \xi^2\,, \quad \partial_2 X^\mu = 0\,, \quad \partial_\varphi G_{MN} = 0\,, \quad \partial_\varphi C_{MNP} = 0\,, \tag{3.26}$$

and to get the string action keeps only the zero mode in the Fourier expansion of the M2 brane fields in ξ^2. In the present case of the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background (1.8)–(1.12), (3.2) where φ is the isometric coordinate of the $\mathrm{U}(1)_k$ fiber of $S^7/\mathbb{Z}_k$ the conditions $\partial_\varphi G_{MN} = \partial_\varphi C_{MNP} = 0$ are indeed satisfied.

Considering an M2 brane located at the center of AdS_4 and moving in $S^7/\mathbb{Z}_k$ the bosonic part of its action may be written like in (3.3) in terms of coordinates z^a of $\mathbb{C}^4/\mathbb{Z}_k$ with the additional constraint $z^a \bar{z}_a = 1$ imposed by a Lagrange multiplier:

$$S = -\mathrm{T}_2 \int d^3\xi \, \sqrt{-\det g_{ij}} \left[1 - \frac{1}{2}\Lambda(\xi)(\bar{z}_a z^a - 1)\right], \qquad g_{ij} = -\frac{1}{4}\partial_i t \partial_j t + \partial_{(i}\bar{z}_a \partial_{j)} z^a\,. \tag{3.27}$$

Here $z^a \equiv e^{\frac{2\pi i}{k}} z^a$ or, equivalently, given by (3.1). The effective tension T_2 was defined in (1.16). The corresponding equations of motion are

$$\nabla^2 t = 0\,, \qquad \nabla^2 z^a = -\Lambda z^a\,, \qquad \nabla^2 = \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g} g^{ij}\partial_j)\,, \qquad \bar{z}_a z^a = 1\,. \tag{3.28}$$

It is straightforward to check that they are satisfied by $t = \kappa \xi^0$ and

$$z^a(\xi^i) = e^{\frac{i}{k}\xi^2}\, z^a(\xi^\alpha)\,, \qquad \xi^2 \in [0, 2\pi)\,, \tag{3.29}$$

where $z^a(\xi^\alpha)$ solve the equations (3.5), (3.7) for a string in $\mathbb{R} \times \mathrm{CP}^3$. The induced 3d metric g_{ij} can be written as

$$g_{ij} = \begin{pmatrix} g_{\alpha\beta} + A_\alpha A_\beta & \frac{1}{k}A_\alpha \\ \frac{1}{k}A_\beta & \frac{1}{k^2} \end{pmatrix}, \qquad g_{\alpha\beta} = \mathrm{c}^2 \eta_{\alpha\beta}\,, \tag{3.30}$$

where A_α and $g_{\alpha\beta}$ are given by (3.6) and (3.7) respectively.

As in the string case, the action (3.27) is invariant under the global SU(4) symmetry and time t translations. In particular, for $z^a(\xi^\alpha)$ satisfying the Virasoro constraints (3.7), the expressions for the conserved charges can be written as in (3.9)

$$J_r = \frac{1}{k}\mathrm{T}_2 \int_0^{2\pi} d\xi^1 \int_0^{2\pi} d\xi^2 \Big[(\partial_0 z)^\dagger H_r z - z^\dagger H_r \partial_0 z\Big]\,, \qquad E_0 = \frac{1}{2k}\mathrm{T}_2 \int_0^{2\pi} d\xi^1 \int_0^{2\pi} d\xi^2 \; \partial_0 t\,. \tag{3.31}$$

These coincide with the corresponding string charges (3.9) as $2\mathrm{T} = \frac{2\pi}{k}\mathrm{T}_2$ (see (1.17)).

Thus the M2 brane counterparts of the "short" and "long" string solutions are represented by (3.29) with $z^a(\xi^\alpha)$ given by (3.16) and (3.22) respectively and the same values of spins and energies as in (3.18) and (3.24). For these solutions both $g_{\alpha\beta}$ and A_α are constant (see (3.15), (3.21)) so that g_{ij} in (3.30) is also constant

$$ds_3^2 = g_{ij}\, d\xi^i d\xi^j = \mathrm{c}^2\,[-(d\xi^0)^2 + (d\xi^1)^2] + \frac{1}{k^2}(d\xi^2 + kA_1 d\xi^1)^2\,. \tag{3.32}$$

Note that while for the "long" solution in (3.21) one has $\mathrm{c}^2 = \frac{1}{4}m^2$ and $A_\alpha = 0$ so that the 3d metric is diagonal, for the "short" one in (3.15) $\mathrm{c}^2 = \frac{1}{8}\kappa^2$ and $A_0 = 0$ but $A_1 = 2m(a^2 - \frac{1}{4})$ is non-zero (and, as already mentioned above, cannot be, in general, eliminated by a redefinition of ξ^2). As a result, in the "short" case g_{ij} in (3.32) represents a non-trivial torus in the (ξ^1, ξ^2) directions

$$ds_3^2 = -\mathrm{c}^2(d\xi^0)^2 + \frac{1}{k^2}|d\xi^2 + \tau d\xi^1|^2, \qquad \tau = \tau_1 + i\tau_2, \qquad \mathrm{c} = \sqrt{\frac{m}{2}\mathcal{J}}, \tag{3.33}$$

$$\tau_1 = k\,A_1 = 2km\left(a^2 - \frac{1}{4}\right) = -\frac{1}{2}km\sqrt{1-2\mathcal{J}}, \qquad \tau_2 = k\,\mathrm{c} = 2kma\sqrt{\frac{1}{2} - a^2} = k\sqrt{\frac{m}{2}\mathcal{J}}. \tag{3.34}$$

For the "long" solution one may also write the diagonal metric in the form (3.33) where

$$\mathrm{c} = \frac{1}{2}m\,, \qquad \tau = i\tau_2\,, \qquad \tau_2 = k\,\mathrm{c} = \frac{1}{2}km\ . \tag{3.35}$$

3.2 1-loop correction to the energy

Our aim will be to compute the 1-loop corrections to the energies of the "short" and "long" M2 brane solutions. The first step is to find the corresponding quadratic fluctuation action that follows from (1.19)–(1.23). This is can be done, e.g., in the static gauge as in [37–39] (see also a discussion in appendix A). Like in the case of the long folded M2 brane solution in section 2 the induced metric (3.32) is constant (cf. (2.3)) as are the derivatives of the background 3d fields so that the fluctuation Lagrangian has constant coefficients and the spectrum of fluctuation frequencies is straightforward to find.

In particular, the 8 bosonic fluctuations propagating in the induced 3-metric (3.32), (3.33) will be described by a coupled quadratic 2-derivative action with constant coefficients. For a single 3d scalar field $X(\xi)$ with mass M the corresponding Klein-Gordon operator will be (cf. (2.4))

$$(-g^{ij}\partial_i\partial_j + M^2)X \quad \to \quad \mathrm{c}^{-2}\big[\partial_0^2 - (\partial_1 - kA_1\partial_2)^2 - k^2\mathrm{c}^2\partial_2^2 + \mathrm{c}^2M^2\big]X\ . \tag{3.36}$$

Expanding in Fourier modes in ξ^i as

$$X(\xi) = \int \frac{d\omega}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \tilde{X}_{nl}(\omega)\ e^{i(\omega\xi^0 + n\xi^1 + l\xi^2)}\,, \tag{3.37}$$

the frequencies $\omega(n,l)$ corresponding to (3.36) may be written as (cf. (3.33))

$$\omega^2(n,l) = |n - \tau\, l|^2 + \mathrm{c}^2M^2 = (n - \tau_1 l)^2 + (\tau_2 l)^2 + \mathrm{c}^2M^2\ . \tag{3.38}$$

Assuming that one can diagonalize the 8×8 matrices for the bosonic and fermionic characteristic frequencies one will then get the 1-loop correction to the AdS_4 energy given by

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\Omega(n,l)\,, \qquad \Omega(n,l) = \sum_B \omega_{\rm B}(n,l) - \sum_F \omega_{\rm F}(n,l)\,, \tag{3.39}$$

where $\Omega(n,l)$ depends on the parameters of a solution, i.e. $\mathcal{J}$ and m. The sum over l can be split as in (2.13) into the $l=0$ (string) contribution and that of the rest of the M2 brane $l\neq 0$ ("KK") modes, i.e.

$$E_1 = E_{1,\text{str}} + E_{1,\text{kk}}\,, \qquad E_{1,\text{str}} = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Omega(n,0)\,, \qquad E_{1,\text{kk}} = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\sum_{l\neq 0}\Omega(n,l)\,. \tag{3.40}$$

In practice, finding the explicit expressions for the frequencies $\omega_{\text{B}}(n,l)$ and $\omega_{\text{F}}(n,l)$ and thus $\Omega(n,l)$ is hard due to non-trivial mixing of the transverse fluctuations (cf. appendix C). One can use instead an equivalent representation for E_1 in terms of the 1-loop partition function (cf. (2.12) and a discussion in [73])[18]

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dw}{2\pi}\,\log\frac{\text{D}_{\text{B}}(w^2,\tau,\mathcal{J})}{\text{D}_{\text{F}}(w^2,\tau,\mathcal{J})}\,. \tag{3.41}$$

Here $\text{D}_{\text{B,F}}$ are the determinants of the quadratic fluctuation matrices for the bosons and fermions obtained after expanding in the Fourier modes as in (3.37) and $w=i\omega$.

3.2.1 "Short" M2 brane

Below we will consider the case of the minimal winding number $m=1$ (corresponding to the state with minimal energy for given spins). Let us first discuss the contribution of the string ($l=0$) modes. For the 8=1+3+2+2 bosonic modes one finds the following expressions for the characteristic frequencies (see appendix B)

$$\begin{aligned}
&l=0: \qquad \omega^2 = n^2\,, \qquad\quad \omega^2 = n^2 + 4\mathcal{J} \quad (3\ \text{modes})\,,\\
&\omega^2 = n^2 + 2 - 3\mathcal{J} \pm \sqrt{\mathcal{J}^2 + 4n^2 - 4\mathcal{J}n^2} + 2\sqrt{(1-2\mathcal{J})(1+n^2-\mathcal{J}\pm\sqrt{\mathcal{J}^2+4n^2-4\mathcal{J}n^2})}\,,\\
&\omega^2 = n^2 + 2 - 3\mathcal{J} \pm \sqrt{\mathcal{J}^2 + 4n^2 - 4\mathcal{J}n^2} - 2\sqrt{(1-2\mathcal{J})(1+n^2-\mathcal{J}\pm\sqrt{\mathcal{J}^2+4n^2-4\mathcal{J}n^2})}\,.
\end{aligned} \tag{3.42}$$

The $8=2\times 2+2\times 2$ fermionic $l=0$ frequencies are

$$\begin{aligned}
l=0: \qquad &\omega^2 = 1 + n^2 + \mathcal{J} \pm 2\sqrt{\mathcal{J} + n^2 - \mathcal{J}n^2} &&(2\ \text{modes})\,,\\
&\omega^2 = 1 + n^2 - \mathcal{J} \pm 2\sqrt{(1-2\mathcal{J})\,(\mathcal{J}+n^2)} &&(2\ \text{modes})\,,
\end{aligned} \tag{3.43}$$

where $\mathcal{J} = 8a^2(\frac{1}{2}-a^2)$, $\kappa^2 = 4\mathcal{J}$ (see (3.18) with $m=1$). Separating the special $n=0,1,2$ modes, $E_{1,\text{str}}$ in (3.40) may be written as

$$E_{1,\text{str}} = \frac{1}{2\sqrt{\mathcal{J}}}\left[\frac{1}{2}\Omega(0,0) + \Omega(1,0) + \Omega(2,0) + \sum_{n=3}^{\infty}\Omega(n,0)\right], \tag{3.44}$$

[18] In general, in the $l=0$ string case the contributions of some low-n modes may require special treatment.

where $\Omega(n,0)$ is the total contribution of the bosonic and fermionic modes as in (3.39). Expanding in small $\mathcal{J}$ we get

$$\begin{aligned}\Omega(0,0) &= -4+6\sqrt{\mathcal{J}}+2\mathcal{J}+\mathcal{J}^2+\mathcal{O}(\mathcal{J}^{5/2})\,,\\ \Omega(1,0) &= 2-2\sqrt{\mathcal{J}}+\frac{1}{2}\mathcal{J}-\frac{211}{32}\mathcal{J}^2+\mathcal{O}(\mathcal{J}^{5/2})\,,\\ \Omega(2,0) &= -\frac{1}{3}\mathcal{J}+\frac{131}{216}\mathcal{J}^2+\mathcal{O}(\mathcal{J}^{5/2})\,,\end{aligned} \tag{3.45}$$

$$\frac{1}{2}\Omega(0,0)+\Omega(1,0)+\Omega(2,0) = \sqrt{\mathcal{J}}+\frac{7}{6}\mathcal{J}-\frac{4741}{864}\mathcal{J}^2+\mathcal{O}(\mathcal{J}^{5/2}), \tag{3.46}$$

$$\sum_{n=3}^{\infty}\Omega(n,0) = q_1\mathcal{J}+q_2\mathcal{J}^2+\mathcal{O}(\mathcal{J}^{5/2})\,,$$

$$\begin{aligned} q_1 &= -\sum_{n=3}^{\infty}\frac{2}{n(n^2-1)} = -\frac{1}{6}\,,\\ q_2 &= \sum_{n=3}^{\infty}\frac{23n^4-29n^2+10}{2n^3\,(n^2-1)^3} = \frac{4741}{864}-\frac{9}{2}\zeta(3)\ .\end{aligned} \tag{3.47}$$

As a result,

$$E_{1,\rm str} = \frac{1}{2}+\frac{1}{2}\sqrt{\mathcal{J}}-\frac{9}{4}\zeta(3)\mathcal{J}^{3/2}+\mathcal{O}\left(\mathcal{J}^2\right) = \frac{1}{2}+\frac{1}{2}\frac{\sqrt{J}}{\bar{\lambda}^{1/4}}-\frac{9}{4}\zeta(3)\frac{J^{3/2}}{\bar{\lambda}^{3/4}}+\mathcal{O}\left(\frac{J^2}{\bar{\lambda}}\right), \tag{3.48}$$

Combined with the classical contribution in (3.18) this gives

$$E_{\rm str} = 2\sqrt{\sqrt{\bar{\lambda}}J}+\frac{1}{2}+\frac{1}{2}\frac{J^{1/2}}{\bar{\lambda}^{1/4}}-\frac{9}{4}\zeta(3)\frac{J^{3/2}}{\bar{\lambda}^{3/4}}+\mathcal{O}\left(\frac{J^2}{\bar{\lambda}}\right), \tag{3.49}$$

which has similar structure as the corresponding expression in the $AdS_5 \times S^5$case [71].

Let us now consider the $l \neq 0$ (membrane-mode) contribution $E_{1,\rm kk}$ to the 1-loop energy in (3.40). We will be interested in its expansion first at large k and then in small $\mathcal{J}$. If the small $\mathcal{J}$ limit is taken before the large k one directly in $\Omega(n,l)$, i.e. before summing over n,l, this leads to inconsistencies, since the frequency lattice in (3.38) becomes degenerate as $\tau_2 \sim \mathcal{J} \to 0$ (cf. (3.34)).[19] Thus it is important that the large k limit is to be taken before the small $\mathcal{J}$ one.[20] This implies that

$$k \gg 1, \quad \mathcal{J} \ll 1\ : \qquad\qquad \tau_2 = \frac{1}{\sqrt{2}}k\sqrt{\mathcal{J}} \gg 1\ . \tag{3.50}$$

Below we shall use the integral representation (3.41) for the 1-loop energy and treat τ and $\mathcal{J}$ as independent parameters, assuming that $\tau_2 \gg 1$ and $\mathcal{J} \ll 1$. We will replace τ with its explicit value in (3.34) at the end of the calculation.

[19] One can draw some analogy with what one finds for the non-holomorphic Eisenstein series $E(s,\tau)$ as a function of τ. If one considers its Fourier expansion, it can be seen that the series has a regular behaviour for $\tau_2 \to \infty$, while for $\tau_2 \to 0$ it is divergent because of the asymptotics of the modified Bessel function $K_\nu(x) \sim x^{-\nu}$ near zero.

[20] As discussed in section 1.4 this is consistent with the standard 't Hooft large N expansion on the gauge theory side where one should first take N large and then consider limits of small or large λ and small or large $\frac{J}{\sqrt{\lambda}}$.

Let us expand the integrand in (3.41) as

$$\log \frac{\mathrm{D_B}(w^2,\tau,\mathcal{J})}{\mathrm{D_F}(w^2,\tau,\mathcal{J})}\Big|_{\mathcal{J}\to 0} = \mathcal{Q}_0(w^2,\tau) + \sqrt{\mathcal{J}}\,\mathcal{Q}_1(w^2,\tau) + \mathcal{J}\,\mathcal{Q}_2(w^2,\tau) + \dots \tag{3.51}$$

Using the expressions in appendix B one finds that the expressions for the determinants $\mathrm{D_B}$ and $\mathrm{D_F}$ depend on τ only through p^2 and q defined as

$$p^2 \equiv |n-\tau l|^2 = q^2 + (\tau_2 l)^2\,, \qquad\qquad q \equiv n - \tau_1 l\ . \tag{3.52}$$

Explicitly, we get[21]

$$\mathcal{Q}_0 = \log \frac{(p^2+w^2)^5\left[p^6+3p^4w^2+p^2\left(-8q^2+3w^4+8w^2-16\right)-8q^2\left(w^2-4\right)+w^2\left(w^2+4\right)^2\right]}{\left[p^4+2p^2w^2-2q^2+(w^2+1)^2\right]^4}, \tag{3.53}$$

$$\mathcal{Q}_1 = \frac{8\sqrt{2}q\sqrt{p^2-q^2}\left[4p^4+p^2\left(-6q^2+14w^2-17\right)-6q^2\left(w^2-4\right)+10w^4+23w^2+4\right]}{\left[p^4+2p^2w^2-2q^2+(w^2+1)^2\right]\left[p^6+3p^4w^2+p^2(-8q^2+3w^4+8w^2-16)-8q^2(w^2-4)+w^2(w^2+4)^2\right]}. \tag{3.54}$$

It is also straightforward to find $\mathcal{Q}_2$ but its expression is somewhat long so we will not present it here. Let us define

$$\mathcal{E}_r(\tau) = \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_0^\infty \frac{dw}{2\pi}\ \mathcal{Q}_r(w^2,p^2,q)\ . \tag{3.55}$$

Then combining (3.41), (3.51), (3.55) we get

$$E_1 = \frac{1}{\kappa}\sum_{r=0}^{\infty}\mathcal{E}_r(\tau)\,\mathcal{J}^{r/2} = \frac{1}{2}\sum_{r=0}^{\infty}\mathcal{E}_r(\tau)\,\mathcal{J}^{(r-1)/2}\,, \tag{3.56}$$

where we used that according to (3.18) $\kappa = 2\sqrt{\mathcal{J}}$.

To evaluate $\mathcal{E}_r(\tau)$ in (3.55) we may assume that $\mathcal{Q}_r$ with an even r is an even function of q while $\mathcal{Q}_r$ with an odd r is an odd function of q (we checked this property for low values of $r = 0, 1, 2$ that we will consider below). Then the sum of odd $\mathcal{Q}_r$ over (n, l) in (3.55) is zero, since the terms with (n, l) and $(-n, -l)$ contribute with an opposite sign. Thus we may consider only $\mathcal{E}_r(\tau)$ with an even r. One can further use that since the dependence of $\mathcal{Q}_r$ on τ is only via p^2 and q, they are periodic functions of τ_1 and thus can be expressed in Fourier series as

$$\mathcal{E}_r(\tau_1+1,\tau_2) = \mathcal{E}_r(\tau_1,\tau_2)\,, \qquad\qquad \mathcal{E}_r = \sum_{s=-\infty}^{\infty} e_r^{(s)}(\tau_2)\ e^{2\pi i s\tau_1}\,, \tag{3.57}$$

$$e_r^{(s)}(\tau_2) = \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_0^\infty \frac{dw}{2\pi}\int_0^1 d\tau_1'\, e^{-2\pi i s\tau_1'}\ \mathcal{Q}_r(w^2,p'^2,q')\,, \tag{3.58}$$

where p' and q' are assumed to depend on τ_1'.

[21] To get the expansion in terms of $\mathcal{J}$, we have assumed that $a < \frac{1}{2}$ and used (3.18) to express a in terms of $\mathcal{J}$, i.e. $a^2 = \frac{1}{4} - \frac{1}{4}\sqrt{1-2\mathcal{J}}$.

Let us first consider the $l = 0$ term in the sum (which should be the string-theory contribution already discussed above). In this case from (3.52) we have $q^2 = p^2 = n^2$, i.e. do not depend on τ and thus the only non-vanishing term in (3.58) is the one with $s = 0$, i.e.[22]

$$(\mathcal{E}_{1,\text{str}})_r = e_r^{(0)}(0) = \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{dw}{2\pi}\, \mathcal{Q}_r(w^2, n^2, n)\ . \tag{3.59}$$

The remaining sum over n and $l \neq 0$ in (3.58) can be written as:

$$\begin{aligned}
&2\sum_{l=1}^{\infty}\sum_{n=-\infty}^{\infty}\int_0^\infty \frac{dw}{2\pi}\int_0^1 d\tau_1'\, e^{-2\pi i s\tau_1'}\, \mathcal{Q}_r \\
&\qquad = 2\sum_{l=1}^{\infty}\sum_{n \bmod l}\int_0^\infty \frac{dw}{2\pi}\int_{-\infty}^{\infty} d\tau_1\, e^{-2\pi i s\tau_1'}\mathcal{Q}_r(w^2, |n-\tau' l|^2, n-\tau_1' l) \\
&\qquad = 2\sum_{l=1}^{\infty}\sum_{n \bmod l} e^{-2\pi i sn/l}\int_0^\infty \frac{dw}{2\pi}\int_{-\infty}^{\infty} d\tau_1'\, e^{-2\pi i s\tau_1'}\mathcal{Q}_r(w^2, (\tau_1'^2+\tau_2^2)l^2, \tau_1' l)\ .
\end{aligned} \tag{3.60}$$

Here we used the assumption that $\mathcal{Q}_r$ is an even function of q and also the properties of the sum and the periodicity of the integral over τ_1', and finally shifted $\tau_1' \to \tau_1' + n/l$.

The integral over τ_1' is hard to evaluate explicitly even for $\mathcal{Q}_0$ in (3.53). To proceed, we shall focus on the large k expansion, i.e. assume as in (3.50) that $\tau_2 \gg 1$. Rescaling the integration variables $w = \tau_2 y$ and $\tau_1' = \tau_2 x$, we get for (3.60)

$$2\tau_2^2\sum_{l=1}^{\infty}\sum_{n \bmod l} e^{-2\pi i sn/l}\int_0^\infty \frac{dy}{2\pi}\int_{-\infty}^{\infty} dx\ e^{-2\pi i s\tau_2 x}\ \mathcal{Q}_r\big(\tau_2^2(x^2+1)l^2, \tau_2 x l, \tau_2^2 y^2\big)\ . \tag{3.61}$$

We first note that if $\mathcal{Q}_r$ is an integrable function of x, the integral over x vanishes in the limit of $\tau_2 \to \infty$ if $s \neq 0$ due to the Riemann-Lebesgue lemma. This suggests that for $\tau_2 \gg 1$ the contribution $e_r^{(s\neq 0)}$ in (3.58) will be suppressed relative to $e_r^{(0)}$ in (3.59).[23]

If the same is true for all terms in (3.61), i.e. the terms with $s \neq 0$ are exponentially suppressed, then the leading-order terms in the expression for $\mathcal{E}_r(\tau)$ in (3.57) can be written as $(\mathcal{E}_{1,\text{str}})_r$ in (3.59) plus $(\mathcal{E}_{1,\text{ kk}})_r$, i.e.

$$\mathcal{E}_r(\tau) = (\mathcal{E}_{1,\text{str}})_r + (\mathcal{E}_{1,\text{kk}})_r\,, \qquad (\mathcal{E}_{1,\text{kk}})_r = 2\tau_2^2\sum_{l=1}^{\infty} l\int_0^\infty \frac{dy}{2\pi}\int_{-\infty}^{\infty} dx\Big(\frac{\hat{\mathcal{Q}}_r^{(4)}}{\tau_2^4} + \frac{\hat{\mathcal{Q}}_r^{(6)}}{\tau_2^6} + \ldots\Big) + \ldots\ . \tag{3.62}$$

Here we have assumed that $\mathcal{Q}_r$ admits the large τ_2 expansion of the form $\sum_{m=4}^{\infty}\hat{\mathcal{Q}}_r^{(m)}\,\tau_2^{-m}$, where $\hat{\mathcal{Q}}_r^{(m)} = \hat{\mathcal{Q}}_r^{(m)}(\tau_2^2(x^2+1)l^2, \tau_2 x l, \tau_2^2 y^2)$ as in (3.61). We have checked explicitly that this is true for $r = 0$ and 2.

[22] As was already alluded to above, the representation of the 1-loop correction to energy (3.39) in terms of the integral in (3.41) is valid when the integrand in (3.41) does not have branch points on the real axis as a function of w. As this is not true in the special cases when $l = 0$ and $n = 0, \pm 1, \pm 2$, we are to use instead the representation (3.45) for these contributions. For the other values of n the results following from (3.59) and (3.46) coincide.

[23] For instance, in the case of non-holomorphic Eisenstein series, such terms are exponentially suppressed. This also happens for the integral $\int_{-\infty}^{\infty} dx\ e^{-2\pi i s\tau_2 x}\ \frac{f(x)}{(x^2+x_0^2)^\ell} \sim e^{-2\pi s\tau_2|x_0|}$, $\tau_2 \gg 1$, where $f(x)$ is a polynomial with degree less than $\ell \in \mathbb{Z}$ and has no poles at $x_0 \in \mathbb{R}$.

We then arrive at the following expressions for $(\mathcal{E}_{1,\mathrm{kk}})_0$ and $(\mathcal{E}_{1,\mathrm{kk}})_2$:

$$(\mathcal{E}_{1,\mathrm{kk}})_0 = -\frac{4\zeta(2)}{\tau_2^2} - \frac{152\zeta(6)}{15\tau_2^6} + \dots , \qquad (\mathcal{E}_{1,\mathrm{kk}})_2 = \frac{8\zeta(2)}{\tau_2^2} + \frac{10\zeta(4)}{\tau_2^4} + \dots . \tag{3.63}$$

Using (3.50), i.e. that $\tau_2^2 = \frac{1}{2}k^2\mathcal{J}$, and plugging (3.63) into the expansion in (3.56) we conclude that the membrane-mode contribution to the 1-loop energy of the "short" solution can be written as

$$E_{1,\mathrm{kk}} = \mathcal{J}^{-1/2}\Big(- \frac{4\zeta(2)}{k^2\mathcal{J}} + \dots \Big) + \mathcal{J}^{1/2}\Big(\frac{8\zeta(2)}{k^2\mathcal{J}} + \dots \Big) + \dots + \mathcal{O}\left(\frac{1}{k^4}\right) . \tag{3.64}$$

Combining this with the classical part of the energy in (3.18) and the string ($l=0$) 1-loop contribution in (3.48) or (1.46) we then get the following prediction for the 1-loop corrected "short" M2 brane energy

$$\begin{aligned} E_{\mathrm{M2}} =& 2\sqrt{\sqrt{\bar\lambda}J} + \frac{1}{2} + \frac{1}{2}\bar\lambda^{-1/4}J^{1/2} - \frac{9}{4}\zeta(3)\bar\lambda^{-3/4}J^{3/2} + \mathcal{O}\Big(\bar\lambda^{-1}J^2\Big) \\ &+ \frac{1}{k^2}\Big[\zeta(2)\Big(- 4\bar\lambda^{3/4}J^{-3/2} + 8\bar\lambda^{1/4}J^{-1/2}\Big) + \mathcal{O}\Big(\bar\lambda^{-1/4}J^{1/2}\Big)\Big] + \mathcal{O}\left(\frac{1}{k^4}\right) . \end{aligned} \tag{3.65}$$

Note that like in the fast-spinning M2 brane case considered in section 2 (cf. (2.17), (2.18)), here the leading $\frac{1}{k^2}$ correction is also proportional to $\zeta(2) = \frac{\pi^2}{6}$. On the dual ABJM gauge theory side (3.65) $\bar\lambda = 2\pi^2\lambda$ and $\frac{1}{k^2} = \frac{\lambda^2}{N^2}$ (see (1.15)) is a prediction for the leading non-planar correction to the dimension of the corresponding "short" operator.[24]

3.2.2 "Long" M2 brane

Let us now consider a similar computation of 1-loop correction to the energy of the $m=1$ "long" M2 brane solution that generalizes the string solution (3.20)–(3.24). Here one has diagonal induced 3-metric as in (2.3) (cf. (3.33))

$$A_\alpha = 0 , \qquad \kappa^2 = 4\mathcal{J}^2 + 1 , \qquad ds_3^2 = \frac{1}{4}\left[-(d\xi^0)^2 + (d\xi^1)^2 + \frac{4}{k^2}(d\xi^2)^2\right] . \tag{3.66}$$

The characteristic frequency polynomials for the "long" solution are given in appendix C.

[24] As was mentioned above (cf. (3.19)), the "short" string or "short" M2 brane solution has a direct analog in $\mathbb{R}^{1,9}\times S^1$ flat space. There the circular M2 brane is rotating with $J_1 = J_2$ in two orthogonal planes in $\mathbb{R}^4 \subset \mathbb{R}^{1,9}$ and is wrapped on S^1 of radius R_{11}. To take the flat space limit we need to identify the radius of S^1 as $R_{11} = \frac{\mathrm{L}}{k}$ that will be fixed in the large $\mathrm{L} \sim L$ limit along with the parameters κ and a of the solution in (3.19) (cf. (3.14)). To get the energy and spin in the flat space limit and relate to string theory we need to rescale $E \to \frac{1}{2}LE$ so that E and J will have canonical mass dimensions (1 and 0). Then using that $\sqrt{\bar\lambda} = \frac{L^2}{4\alpha'}$ (cf. (1.14), (1.15)) we conclude that all string corrections to the classical term $E = 2\sqrt{\alpha'^{-1}J}$ in the first line of (3.65) vanish, in agreement with the fact that the free superstring spectrum in flat space is not deformed by α' corrections. At the same time, the $R_{11}^2 \sim g_s^2$ dependent corrections in the second line of (3.65) survive, with the leading one proportional to $\zeta(2)\alpha'^{-1/2}g_s^2J^{-3/2}$, i.e. we get $E = 2\sqrt{\alpha'^{-1}J}\big[1 - \frac{1}{2}\zeta(2)g_s^2J^{-2} + \dots\big]$. This can be checked 1-loop computation of the energy of the "short" M2 solution directly in the flat space case and may be related to the expectation that masses of massive superstring states may received 1-loop (and higher order) corrections (cf. [95–99]). Note that a non-zero 1-loop correction to the energy of a different $J_1 = J_2$ supermembrane solution in flat space (where the membrane was rotating in 2 planes with the "radii" being periodic functions of ξ^1 and ξ^2 but was not wrapped on S^1) was found also in [35].

Like in the "short" case let us first consider the string theory $l=0$ contribution. We find as in [69] that there are $8 = 1+3+2\times 2$ bosonic

$$
\begin{aligned}
l=0: \qquad & \omega^2 = n^2+4\mathcal{J}^2-1\,, \qquad \omega^2 = n^2+4\mathcal{J}^2+1 && (3 \text{ modes}), \\
& \omega^2 = n^2+2\mathcal{J}^2 \pm \sqrt{4\mathcal{J}^4+(4\mathcal{J}^2+1)n^2} && (2\times 2 \text{ modes}), \qquad (3.67)
\end{aligned}
$$

and $8 = 2\times 2+4$ fermionic fluctuation frequencies

$$
\begin{aligned}
l=0: \qquad & \omega^2 = n^2+5\mathcal{J}^2+\frac{1}{4} \pm \sqrt{(4\mathcal{J}^2+1)(4\mathcal{J}^2+n^2)} && (2\times 2 \text{ modes}), \\
& \omega^2 = n^2+\mathcal{J}^2+\frac{1}{4} && (4 \text{ modes}). \qquad (3.68)
\end{aligned}
$$

These frequencies agree with (3.42), (3.43) for $\mathcal{J}=\frac{1}{2}$ when the "short" and "long" solutions become equivalent. Note that in contrast to what happens in the $AdS_5\times S^5$case (where the $m=1$ solution is unstable [88]) these frequencies are always real, i.e. the $J_1=J_2$ solution in $AdS_4\times CP^3$ is stable for any $\mathcal{J}$.

The 1-loop energy string energy $E_{1,\text{str}}$ is given again by the general expressions in (3.39), (3.40). Here we will be interested in its expansion in the large spin limit $\mathcal{J}\gg 1$. Like in the $AdS_5\times S^5$case [90, 91, 100], in addition to the "analytic" contributions (with even powers of $\mathcal{J}^{-1}$) discussed already in [69], there are also "non-analytic" terms (with odd powers of $\mathcal{J}^{-1}$), i.e. for large $\mathcal{J}$

$$
E_{1,\text{str}} = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Omega(n,0;\mathcal{J}) = E_1^{\text{an}}+E_1^{\text{non}}+\mathcal{O}(e^{-\mathcal{J}})\;. \qquad (3.69)
$$

To sum up the series over n we apply the Abel-Plana summation formula (with a slight modification due to an additional branch cut coming from the bosonic modes).[25] As a result (here $\kappa = \sqrt{4\mathcal{J}^2+1}$)

$$
\begin{aligned}
E_1^{\text{an}} &= \frac{i}{\kappa}\int_0^1 ds\cot(\pi s)\left[\sqrt{4\mathcal{J}^2+1+(s-i\sqrt{1-s^2})^2}-\sqrt{4\mathcal{J}^2+1+(s+i\sqrt{1-s^2})^2}\right] \qquad (3.70)\\
&= \frac{1}{2\mathcal{J}^2}\left[\frac{1}{4}+\sum_{n=1}^{\infty}\left(n\sqrt{n^2-1}-n^2+\frac{1}{2}\right)\right]-\frac{1}{8\mathcal{J}^4}\left[\frac{3}{16}+\sum_{n=1}^{\infty}\left(\frac{3}{8}-n^4+n\sqrt{n^2-1}\left(\frac{1}{2}+n^2\right)\right)\right]\\
&\quad +\mathcal{O}\left(\frac{1}{\mathcal{J}^6}\right),\\
E_1^{\text{non}} &= \frac{1}{\kappa}\Bigg[2\int_0^\infty ds\sqrt{2\mathcal{J}^2+s^2+\sqrt{4\mathcal{J}^4+s^2(4\mathcal{J}^2+1)}}+2\int_1^\infty ds\sqrt{2\mathcal{J}^2+s^2-\sqrt{4\mathcal{J}^4+s^2(4\mathcal{J}^2+1)}}\\
&\quad +\int_0^\infty dt\left(3\sqrt{s^2+4\mathcal{J}^2+1}+\sqrt{s^2+4\mathcal{J}^2-1}-4\sqrt{s^2+\mathcal{J}^2+\frac{1}{4}}-4\sqrt{s^2+4\mathcal{J}^2}\right)\Bigg]\,. \qquad (3.71)
\end{aligned}
$$

[25] To get E_1^{non} in [101] an alternative method using the Sommerfeld-Watson transform was applied.

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The sums that appear in (3.70) converge, and, in particular, the coefficient of the leading $\frac{1}{\mathcal{J}^2}$ term is the same as in [69][26]

$$c_1 \equiv \frac{1}{4} + \sum_{n=1}^{\infty} \left(n\sqrt{n^2-1} - n^2 + \frac{1}{2} \right) \approx -0.336 \, . \tag{3.72}$$

Evaluating the integrals in (3.71) and expanding for $\mathcal{J}$, we get $1/\mathcal{J}$ and $1/\mathcal{J}^3$ contributions with $\log 2$ coefficients. Then the combined result for (3.69) is

$$E_{1,\rm str}\Big|_{\mathcal{J}\to\infty} = -\frac{\log 2}{2\mathcal{J}} + \frac{c_1}{2\mathcal{J}^2} + \frac{\log 2}{16\mathcal{J}^3} + \mathcal{O}\left(\frac{1}{\mathcal{J}^4}\right) , \tag{3.73}$$

This is to be added to the large $\mathcal{J}$ expansion of the classical energy in (3.24)

$$E_0 = \bar{\lambda}^{1/2}\sqrt{4\mathcal{J}^2+1} = \bar{\lambda}^{1/2}\left[2\mathcal{J} + \frac{1}{4\mathcal{J}} - \frac{1}{64\mathcal{J}^3} + \mathcal{O}\left(\frac{1}{\mathcal{J}^5}\right)\right] . \tag{3.74}$$

As a result, the 1-loop string energy can be put into the form

$$E_{\rm str}\Big|_{\mathcal{J}\to\infty} = 2J + \frac{\bar{h}^2(\bar{\lambda})}{4J} + c_1\frac{\bar{\lambda}}{2J^2} - \frac{\bar{h}^4(\bar{\lambda})}{64J^3} + \dots , \tag{3.75}$$

where $J = \bar{\lambda}^{1/2}\mathcal{J}$ and $\bar{h}(\bar{\lambda}) \equiv 2\pi h(\lambda) = \bar{\lambda}^{1/2} - \log 2 + \dots$ with $h(\lambda)$ that appeared in (1.38), (1.40). A similar result that the replacement $\sqrt{\bar{\lambda}} \to \bar{h}(\bar{\lambda})$ happens only for the coefficients of the "odd" $1/J^{2r+1}$ terms in the expansion of the energy (that are then directly related to those in the $\mathrm{AdS}_5 \times S^5$case) was found in [50] for a circular rotating string with spins S and J stretched in both AdS_4 and CP^3.

Let us now turn to the $l \neq 0$ membrane mode contribution $E_{1,\rm kk}$ in (3.40). Using the integral representation (3.41) we get

$$E_{1,\rm kk} = \frac{1}{\kappa}\sum_{n=-\infty}^{\infty}\sum_{l\neq 0}\int_0^{\infty}\frac{dw}{2\pi}\,\mathcal{E}(w^2,n^2,(\tau_2 l)^2,\mathcal{J}) , \qquad \mathcal{E} = \log\frac{\mathrm{D_B}(w^2,\tau_2,\mathcal{J})}{\mathrm{D_F}(w^2,\tau_2,\mathcal{J})} , \qquad \tau_2 \equiv \frac{1}{2}k \, . \tag{3.76}$$

While the induced metric in (3.66) is diagonal, to keep the analogy with the "short" M2 case (cf. (3.50), (3.51)) we introduced as in (3.35) the coefficient $\tau_2 = \frac{1}{2}k$ that will be again large in the $k \gg 1$ limit we are interested in.

Here we should first expand in large τ_2 and then in large $\mathcal{J}$. As follows from the explicit form of the determinants $\mathrm{D_B}$ and $\mathrm{D_F}$ in appendix C the integrand $\mathcal{E}$ in (3.76) turns out to be an even function of both n and l. To compute (3.76) in the large τ_2 limit we may try to follow the same strategy as in the "short" M2 brane case discussed above. For that we may formally introduce a parameter τ_1 (to be taken to zero at the end) shifting $n \to n - \tau_1 l$ as in (3.51), (3.52). Then we can take $l \neq 0$ and consider first the sum over n following the same steps as in (3.58)–(3.62). Like in the string case, one may split the sum into the integral part and finite series and the expectation is that the contribution of the latter is

[26]The $\frac{1}{\mathcal{J}^2}$ correction is essentially the same as in the $\mathrm{AdS}_5 \times S^5$case [76] (in general, with the winding numbers related by $m \to \frac{1}{2}m$).

exponentially will be suppressed when $k \gg 1$ and $l \neq 0$. This suggests that like in (3.62) the sum over n can be effectively replaced by an integral

$$\tau_2 \gg 1 : \qquad E_{1,\rm kk} \approx \frac{1}{\kappa}\sum_{l=1}^{\infty}\int_0^{\infty}\frac{dw}{\pi}\int_{-\infty}^{\infty} dn\, \mathcal{E}(w^2, n^2, (\tau_2 l)^2, \mathcal{J}) \ . \tag{3.77}$$

Assuming that $\mathcal{J}/\tau_2 \ll 1$ we may rescale the integration variables in (3.77) as $w = \tau_2 y$, $n = \tau_2 x$ (cf. (3.61))

$$E_{1,\rm kk} = \frac{\tau_2^2}{\pi\kappa}\sum_{l=1}^{\infty}\int_0^{\infty} dy \int_{-\infty}^{\infty} dx \left[\frac{\mathcal{E}^{(2)}(\mathcal{J})}{\tau_2^2} + \frac{\mathcal{E}^{(4)}(\mathcal{J})}{\tau_2^4} + \dots\right], \tag{3.78}$$

where we can further expand the integrand at large $\mathcal{J}$. As a result (cf. (3.64))

$$\begin{aligned} E_{1,\rm kk} =& \frac{\zeta(2)}{\tau_2^2}\Big(- 2\mathcal{J} - \frac{1}{2\mathcal{J}} + \frac{3}{64\mathcal{J}^3} + \dots\Big) + \frac{\zeta(4)}{\tau_2^4}\Big(\frac{21}{4}\mathcal{J}^3 + \dots\Big) + \mathcal{O}\Big(\frac{1}{\tau_2^6}\Big) \\ =& \frac{4\zeta(2)}{k^2}\Big(- 2\mathcal{J} - \frac{1}{2\mathcal{J}} + \frac{3}{64\mathcal{J}^3} + \dots\Big) + \frac{16\zeta(4)}{k^4}\Big(\frac{21}{4}\mathcal{J}^3 + \dots\Big) + \mathcal{O}\Big(\frac{1}{k^6}\Big) \ . \end{aligned} \tag{3.79}$$

Combining this with the string part (3.75) we get (cf. (3.65))

$$\begin{aligned} E_{\rm M2} =& 2J + \frac{\bar\lambda}{4J}(1 - 2\log 2\, \bar\lambda^{-1/2} + \dots) + c_1\frac{\bar\lambda}{2J^2}(1 + \dots) + \dots \\ &+ \frac{1}{k^2}\,\zeta(2)\Big(- 8\bar\lambda^{-1/2}J - 2\frac{\bar\lambda^{1/2}}{J} + \frac{3\bar\lambda^{3/2}}{16J^3} + \dots\Big) + \mathcal{O}\Big(\frac{1}{k^4}\Big) \ . \end{aligned} \tag{3.80}$$

Here the $\frac{1}{k^2} = \frac{\bar\lambda^2}{(2\pi^2)^2 N^2}$ term represents the prediction for the strong-coupling limit of the leading non-planar correction to the dimension of the corresponding operator with the large spin J.

4 Concluding remarks

In this paper we discussed the $AdS_4 \times S^7/\mathbb{Z}_k$ M2 brane counterparts of the computations of 1-loop corrections to energies of the three string solutions in $AdS_4 \times CP^3$: "long" folded string with large spin in AdS_4 and "short" and "long" circular strings with equal angular momenta $J_1 = J_2$ in CP^3. As a result, we obtained predictions for the leading non-planar corrections to scaling dimensions of the corresponding dual ABJM operators at strong coupling.

In all cases the $1/N^2$ term is proportional to $\zeta(2) = \frac{\pi^2}{6}$. This is related to the fact that $\frac{1}{k} = \frac{\lambda}{N}$ is the radius of the 11d circle φ which is identified with the cylindrical M2 brane dimension ξ^2 so that the dependence on the corresponding Fourier mode number l is via kl. As a result, the coefficient of the $\frac{1}{k^2}$ term is proportional to $\sum_{l=1}^{\infty} l^{-2} = \zeta(2)$.

There are several obvious generalizations. One may consider the M2 brane analog of the folded spinning string in AdS_4 with an extra orbital momentum J in CP^3. Taking the limit when $S \gg J \gg \sqrt{\lambda} \gg 1$ with $\frac{\sqrt{\lambda}}{J}\ln\frac{S}{J}$=fixed determines the generalized cusp anomaly or scaling function. In the $AdS_5 \times S^5$string case this solution was studied in [102, 103]. In this limit the resulting string fluctuation Lagrangian has constant coefficients and thus finding the quantum corrections to the classical energy is straightforward.[27] For the string in $AdS_4 \times CP^3$ the 1-loop correction to the energy of such (S, J) solution was already found in [26, 27] and a generalization to the M2 brane case in $AdS_4 \times S^7/\mathbb{Z}_k$ should not be a problem.

One can also consider a M2 brane analog of another (S, J) string solution where the string is wrapped on a circle in both AdS_4 and CP^3 part (here $S = mJ$ where m is a wrapping number). This is the direct analog of the solution in $AdS_5 \times S^5$studied in [76, 89, 104]. The 1-loop correction to the energy of this circular (S, J) string in $AdS_4 \times CP^3$ was computed in [50] and a generalization to the M2 brane case should be again straightforward. Expanding in small S/J one may relate [73, 105] the leading term in the string energy (or in the dimension of the dual operator) to the so called slope function which, in the planar limit, is known exactly from the integrability [54, 106, 107].

The slope function turns out to be very similar to the Brehmstrahlung function that can be found from localization (via circular BPS WL connection) and, in the planar limit, from the integrability (see [108–110]). Assuming the analogy between the slope function and the Brehmstrahlung function continues also in the ABJM case, ref. [54] suggested a conjecture for the $h(\lambda)$ function that enters the ABJM magnon dispersion relation (1.40), and it passed all tests so far.[28] It would be interesting to use the above M2 brane approach to find a prediction for non-planar corrections to the slope function at strong coupling, and then to compare it to the known expression for the Brehmstrahlung function $B(\lambda, N)$ in the ABJM theory (see [111–118]). One may also consider a direct M2 brane computation of non-planar corrections to the Brehmstrahlung function following the approach of [38] and generalizing to the case of non-trivial wrapping number w. One may then get the Brehmstrahlung function by taking a derivative over w of the large N expansion of the log of the Wilson loop expectation value (see a discussion in appendix D).

At a more conceptual level, it would be remarkable to find a way to do similar computations of non-planar corrections in the type IIB $AdS_5 \times S^5$superstring dual to $\mathcal{N} = 4$ SYM theory. While we utilized the fact that the type IIA string theory has an uplift to M-theory, allowing to apply the semiclassical M2 brane approach, there is no obvious analog of this procedure in the type IIB string theory. At the same time, the exact localization results for the expectation values of the $\frac{1}{2}$ BPS Wilson loops in SYM and ABJM theories exhibit very similar structure when expanded in $1/N$ [38, 119]. Expressing $\langle W \rangle$ in terms of the string coupling g_s and the string tension T in the ABJM theory we have (see (1.30))

$$\begin{aligned} \langle W \rangle_{\text{ABJM}} &= \frac{\sqrt{\mathrm{T}}}{\sqrt{2\pi} g_{\text{s}}} e^{2\pi T} \left[1 + \frac{\pi}{12} \frac{g_{\text{s}}^2}{\mathrm{T}} + \frac{7\pi^2}{1440} \left(\frac{g_{\text{s}}^2}{\mathrm{T}} \right)^2 + \dots \right], \quad && \mathrm{T} = \sqrt{\frac{\lambda}{2}}, \\ g_{\text{s}} &= \frac{\sqrt{\pi}}{N} (2\lambda)^{5/4}, && \lambda = \frac{N}{k}. \end{aligned} \tag{4.1}$$

In the case of $\mathcal{N} = 4$ SYM theory, expressing $\langle W \rangle$ in (D.2) in terms of the corresponding

[27] Moreover, in the $AdS_5 \times S^5$case the (S, J) solution in this limit is related by an analytic continuation to the circular 2-spin solution in S^5, implying a relation between the fluctuation frequencies [102].

[28] In [54] the comparison was made between the structure of the integral representation for the $\frac{1}{6}$ BPS WL and the ABJM slope function found there.

g_s and T gives

$$\langle W\rangle_{\mathrm{SYM}} = \frac{\sqrt{\mathrm{T}}}{2\pi g_\mathrm{s}}\, e^{2\pi T}\left[1+\frac{\pi}{12}\frac{g_\mathrm{s}^2}{\mathrm{T}}+\frac{\pi^2}{288}\left(\frac{g_\mathrm{s}^2}{\mathrm{T}}\right)^2+\dots\right], \qquad \mathrm{T}=\frac{\sqrt{\lambda}}{2\pi}, \quad g_\mathrm{s}=\frac{g_{\mathrm{YM}}^2}{4\pi}, \quad \lambda = g_{\mathrm{YM}}^2 N. \tag{4.2}$$

Remarkably, the two expansions in (4.1) and (4.2) have the same universal form, and, moreover, the leading 1-loop g_s^2 string correction terms happen to have the same coefficients [119].

Surprisingly, the same coefficient of the $\frac{g_\mathrm{s}^2}{\mathrm{T}}$ term is found also for the leading non-planar correction to the ABJM cusp anomalous dimension $f(\lambda, N)$ in (1.2), (1.32) coming from the 1-loop M2 brane contribution we computed in section 2. Including also the leading string contributions, we get from (1.34) and (1.35)

$$\begin{aligned} f_{\mathrm{ABJM}}(\mathrm{T}, g_\mathrm{s}) &= \frac{1}{\pi}\left[2\pi\mathrm{T} - \frac{5}{2}\log 2 + \mathcal{O}(\mathrm{T}^{-1}) \ + \frac{\pi}{12}\frac{g_\mathrm{s}^2}{\mathrm{T}} + \frac{\pi^2}{1440}\left(\frac{g_\mathrm{s}^2}{\mathrm{T}}\right)^2 + \dots\right] \\ &= \frac{1}{\pi}\left[\pi\sqrt{2\lambda} - \frac{5}{2}\log 2 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{2\pi^2}{3}\frac{\lambda^2}{N^2} + \frac{2\pi^3}{45}\frac{\lambda^4}{N^4} + \dots\right]. \end{aligned} \tag{4.3}$$

If we make a bold conjecture that the coincidence of the order g_s^2 string 1-loop coefficients observed in the Wilson loop expressions in (4.1) and (4.2) should extend also to the cusp anomaly, we may then make a prediction that in the SYM theory the analog of (4.3) should read

$$\begin{aligned} f_{\mathrm{SYM}}(\mathrm{T}, g_\mathrm{s}) &= \frac{1}{\pi}\left[2\pi\mathrm{T} - 3\log 2 + \mathcal{O}(\mathrm{T}^{-1}) + \frac{\pi}{12}\frac{g_\mathrm{s}^2}{\mathrm{T}} + \gamma_1\pi^2\left(\frac{g_\mathrm{s}^2}{\mathrm{T}}\right)^2 + \dots\right] \\ &= \frac{1}{\pi}\left[\sqrt{\lambda} - 3\log 2 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{1}{12}\frac{\lambda^{3/2}}{N^2} + \frac{\gamma_1}{36}\frac{\lambda^3}{N^4} + \dots\right]. \end{aligned} \tag{4.4}$$

Here the $1/N^2$ term should be representing the strong coupling limit of the leading non-planar correction and we introduce γ_1 as a coefficient of the subleading non-planar term. It would be very interesting to confirm this prediction that the leading non-planar correction to the SYM cusp anomalous dimension that scales as λ^4 at weak coupling (see (1.3)) should scale as $\lambda^{3/2}$ at strong coupling.

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A Quadratic fluctuation action

To find the 1-loop correction to the energy one needs to expand the M2 brane action (1.19)–(1.23) in the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background near a given classical solution. The spectrum of bosonic fluctuations will in general contain 8 physical ("transverse") modes and 3 unphysical ("longitudinal") modes. The latter can be eliminated by imposing a static gauge. Alternatively, one can just isolate the fluctuations in the normal directions to the surface (see, e.g, [120–123] for similar discussions).

Viewing the membrane as a surface in 11d spacetime, one can define an orthonormal basis e_i on the membrane world volume (here $i, j = 0, 1, 2$ and A, B are tangent-space 11d indices)

$$\langle e_i, e_j \rangle = e_i^A e_j^B \eta_{AB} = \eta_{ij} \ . \tag{A.1}$$

For a pair of tangent vector fields X, Y and the Levi-Civita connection ∇ in the target space one can define the connection ∇^{T} on the brane (corresponding to the induced metric g_{ij}) and the extrinsic curvature K as $\nabla_X Y = \nabla^{\mathrm{T}}_X Y + \mathrm{K}(X, Y)$. For a vector X in the normal bundle we also define: $\nabla_X \mathrm{X} = -\mathrm{A}_{\mathrm{X}}(X) + \nabla^{\perp}_X \mathrm{X}$ where $\nabla^{\perp}$ is the connection on the normal bundle and $\mathrm{A}_{\mathrm{X}}(X)$ is the Weingarten operator, related to the extrinsic curvature as $\langle \mathrm{A}_{\mathrm{X}}(X), Y \rangle = \langle \mathrm{K}(X, Y), \mathrm{X} \rangle$. The bosonic equations of motion for the M2 brane following from (1.19), (1.20) can be written as (here $\mathrm{K}_{ij} = \mathrm{K}(e_i, e_j)$)

$$\eta^{ij} (\mathrm{K}_{ij})_A + \frac{1}{3!} \epsilon^{ijk} F_{ABCD} E^B(e_i) E^C(e_j) E^D(e_k) = 0 \,, \tag{A.2}$$

where E^A is a basis of the target space 1-forms. The quadratic fluctuation part of the bosonic M2 brane action for the fluctuations X in the normal directions is then ($d^3V = d^3\xi \sqrt{-g}$)

$$\begin{aligned} S_{B,2} = &-T_2 \int d^3V \Big[\eta^{ij} \langle \nabla^{\perp}_{e_i} \mathrm{X}, \nabla^{\perp}_{e_j} \mathrm{X} \rangle + \eta^{ij} \langle R(\mathrm{X}, e_i) \mathrm{X}, e_j \rangle - \langle \mathrm{K}^{ij}, \mathrm{X} \rangle \langle \mathrm{K}_{ij}, \mathrm{X} \rangle + \big(\eta^{ij} \langle \mathrm{K}_{ij}, \mathrm{X} \rangle \big)^2 \Big] \\ &+ \frac{1}{3!} T_2 \int d^3V \ \epsilon^{ijk} \Big[(3 F_{DABC} \mathrm{X}^D (\nabla_{e_i} \mathrm{X}^A) e_j^B e_k^C + (\nabla_L F_{ABCD}) \mathrm{X}^L \mathrm{X}^D E^A(e_i) E^B(e_j) E^C(e_k) \Big] \,, \end{aligned} \tag{A.3}$$

where R is the Riemann curvature.[29] Using an orthonormal basis n_p ($p = 1, \ldots, 8$) in the normal bundle we get

$$S_{B,2} = -T_2 \int d^3V \Big[\eta^{ij} (\nabla^{\perp}_{e_i} \mathrm{X})^p (\nabla^{\perp}_{e_j} \mathrm{X})_p + \mathrm{X}^p M_{pq} \mathrm{X}^q \Big] + (F_4\text{-terms}) \,, \tag{A.4}$$

$$M_{pq} = \langle R(\mathrm{n}_p, e_i) \mathrm{n}_q, e_j \rangle \eta^{ij} - \langle \mathrm{K}^{ij}, \mathrm{n}_p \rangle \langle \mathrm{K}_{ij}, \mathrm{n}_q \rangle + \langle \mathrm{tr}\,\mathrm{K}, n_p \rangle \langle \mathrm{tr}\,\mathrm{K}, \mathrm{n}_q \rangle \,, \quad \mathrm{tr}\,\mathrm{K} = \eta^{ij} \mathrm{K}_{ij} \ . \tag{A.5}$$

The quadratic fermionic part of the M2 brane action in (1.21) can be written also as (see, e.g., [39, 87])

$$S_{F,2} = T_2 \int d^3V \eta^{ij} \bar{\theta} (1 - \Gamma) \rho_i \mathcal{D}_{e_j} \theta, \tag{A.6}$$

$$\rho_i = E^A(e_i) \Gamma_A, \quad \{\rho_i, \rho_j\} = 2\eta_{ij} \mathbb{1}_{32}, \quad \Gamma = \frac{1}{3!} \epsilon^{ijk} \rho_i \rho_j \rho_k, \quad \Gamma^2 = 1, \quad \rho^i \Gamma = \Gamma \rho^i = \frac{1}{2} \epsilon^{ijk} \rho_{jk}. \tag{A.7}$$

[29] When a membrane is not coupled to C_3 in (1.20), (A.2) it becomes the equation for a minimal surface in the target space, i.e. $\eta^{ij} \mathrm{K}_{ij} = 0$ and then (2.5) follows from a known expression for the second variation of the minimal volume action (see, e.g., [124]).

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To lowest order the κ-symmetry acts as $\delta\theta = (1+\Gamma)\kappa$. A convenient choice of the κ-symmetry gauge is $(1+\Gamma)\theta = 0$.

For an M2 brane with non-trivial dynamics only in the $S^7/\mathbb{Z}_k$ part of 11d space one can use the induced metric (3.30) in local coordinates in (3.29) to define the orthonormal frame on the world volume as ($\partial_i = \partial/\partial\xi^i$)

$$e_i = \left(\mathrm{c}^{-1}(\partial_\alpha - kA_\alpha\partial_2),\ k\partial_2\right) . \tag{A.8}$$

Then C_3 in (1.12) does not contribute to the membrane equations of motion (A.2) which are equivalent to[30] $\eta^{ij}\mathrm{K}_{ij} = \eta^{ij}\mathrm{K}(e_i, e_j) = 0$. Then (A.4) may be written as:

$$S_{B,2} = -T_2 \int d^3V \Big[\eta^{ij}\langle\nabla^\perp_{e_i}\mathrm{X}, \nabla^\perp_{e_j}\mathrm{X}\rangle + \eta^{ij}\langle R(\mathrm{X}, e_i)\mathrm{X}, e_j\rangle - \langle\mathrm{K}^{ij}, \mathrm{X}\rangle\langle\mathrm{K}_{ij}, \mathrm{X}\rangle\Big] . \tag{A.9}$$

Using an orthonormal basis n_p in the normal bundle we have

$$NN = \mathrm{X}^p\mathrm{n}_p\,, \qquad \langle\mathrm{n}_p, \mathrm{n}_q\rangle = \delta_{pq}\,, \qquad \langle\mathrm{K}_{ij}, \mathrm{n}_p\rangle = \langle\nabla_{e_i}e_j, \mathrm{n}_p\rangle\,, \tag{A.10}$$

$$\nabla^\perp_{e_i}\mathrm{X} = \nabla^\perp_{e_i}(\mathrm{n}_p\mathrm{X}^p) = \mathrm{n}_p(\partial_{e_i}\mathrm{X}^p) + \mathrm{n}_q\Omega^q{}_p(e_i)\mathrm{X}^p\,, \qquad \Omega_{qp}(e_i) = \langle\mathrm{n}_q, \nabla_{e_i}\mathrm{n}_p\rangle . \tag{A.11}$$

The fermionic part (1.12) is determined by the operator:

$$\not{\mathcal{D}} = \rho^i\mathcal{D}_{e_i} = \rho^i\Big[\nabla_{e_i} + \frac{1}{12}E^A(e_i)(\Gamma_A\mathrm{F}_4 - 3\mathrm{F}_{4A})\Big]\,, \qquad \nabla_{e_i} = \partial_{e_i} + \frac{1}{4}\Omega^{AB}(e_i)\Gamma_{AB}\,, \tag{A.12}$$

$$\mathrm{F}_4 \equiv \frac{1}{4!}F_{ABCD}\Gamma^{ABCD}\,, \qquad \mathrm{F}_{4A} \equiv \frac{1}{3!}F_{ABCD}\Gamma^{BCD} . \tag{A.13}$$

Here Ω^{AB} is the spin connection on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, $F_4 = dC_3$ is proportional to the volume form of AdS_4 and E^A is a coframe in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. Using the orthonormal frame (e_i, n_p) one may split the Γ^A-matrices as (ρ_i, γ_p).

For the metric in explicit coordinates in (3.2) we may determine (e_i, n_p) in terms of the local coordinate basis in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ as follows. For the "short" membrane solution corresponding to (3.14), (3.29) we get (we set $m = 1$)

$$e_0 = \mathrm{c}^{-1}\left(\kappa\partial_t + (2-4a^2)\partial_{\phi_1} + 4a^2\partial_{\phi_2}\right), \qquad e_1 = \mathrm{c}^{-1}\left(\partial_\psi - k\left(2a^2 - \frac{1}{2}\right)\partial_\varphi\right), \qquad e_2 = k\partial_\varphi,$$

$$\mathrm{n}_i = \partial_{\eta^i}, \qquad \mathrm{n}_4 = \partial_\chi, \qquad \mathrm{n}_5 = \sqrt{2}a^{-1}\partial_{\theta_1}, \qquad \mathrm{n}_6 = \sqrt{2}\left(\frac{1}{2} - a^2\right)^{-1/2}\partial_{\theta_2},$$

$$\mathrm{n}_7 = 2(\partial_{\phi_1} - \partial_{\phi_2}), \qquad \mathrm{n}_8 = 2\partial_t + 2\sqrt{1-2a^2}a^{-1}\partial_{\phi_1} + 4a(1-2a^2)^{-1/2}\partial_{\phi_2}, \tag{A.14}$$

where $\kappa = 4\sqrt{2}a\sqrt{\frac{1}{2} - a^2}$ and $\mathrm{c} = 2a\sqrt{\frac{1}{2} - a^2}$.[31] For the "long" membrane solution with $m = 1$ corresponding to (3.23), (3.26) we get

$$e_0 = \mathrm{c}^{-1}(\kappa\partial_t + 2\mathcal{J}(\partial_{\phi_1} + \partial_{\phi_2}))\,, \qquad e_1 = \mathrm{c}^{-1}\partial_\psi\,, \qquad e_2 = k\partial_\varphi\,,$$

$$\mathrm{n}_i = \partial_{\eta^i}\,, \qquad \mathrm{n}_4 = \partial_\chi\,, \qquad \mathrm{n}_5 = 2\sqrt{2}\partial_{\theta_1}\,, \qquad \mathrm{n}_6 = 2\sqrt{2}\partial_{\theta_2}\,,$$

$$\mathrm{n}_7 = 2(\partial_{\phi_1} - \partial_{\phi_2})\,, \qquad \mathrm{n}_8 = 4\mathcal{J}\partial_t + 2\kappa(\partial_{\phi_1} + \partial_{\phi_2}))\,, \tag{A.15}$$

where $\kappa = \sqrt{4\mathcal{J}^2 + 1}$ and $\mathrm{c} = \frac{1}{2}$.

[30] To make the connection with (A.3) explicit, one may view the M2 brane world volume as a 3-surface $\mathcal{M}$ in $\mathbb{R} \times \mathbb{C}^4$ and use that $\mathcal{M} \subset \mathbb{R} \times S^7 \subset \mathbb{R} \times \mathbb{C}^4$.

[31] Here $\mathrm{n}_i = \partial_{\eta_i}$ $(i = 1, 2, 3)$ correspond to the normal directions in AdS_4 where η^i are the "Cartesian" part of coordinates in AdS_4, i.e. $ds^2_{\mathrm{AdS}_4} = -\frac{(1+\eta^2)^2}{(1-\eta^2)^2}dt^2 + \frac{4}{(1-\eta^2)^2}d\eta^i d\eta^i$.

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In both cases, the non-zero part of the fermionic operator (A.12) takes the form:

$$\not{\mathcal{D}} = \rho^i \mathcal{D}_{e_i} = \rho^i \nabla^{\perp}_{e_i} - \frac{3}{4}(\mathrm{n}_8)^t \gamma_8\gamma_1\gamma_2\gamma_3 \,, \qquad \nabla^{\perp}_{e_i} = \partial_{e_i} + \frac{1}{4}\Omega^{pq}(e_i)\gamma_{pq} \,. \tag{A.16}$$

We note that the part with $\rho^i\rho^j\Omega_{jp}(e_i)$ is zero since

$$\rho^i\rho^j\Omega_{jp}(e_i) = \rho^i\rho^j\langle e_j, \nabla_{e_i}\mathrm{n}_p\rangle = -\rho^i\rho^j\langle \nabla_{e_i}e_j, \mathrm{n}_p\rangle = -\frac{1}{2}\{\rho^i,\rho^j\}\langle \mathrm{K}_{ij}, n_p\rangle = -(\mathrm{tr}\ \mathrm{K})_p = 0\,, \tag{A.17}$$

and also that the projection of the connection ∇^{T} on the M2 brane vanishes because the induced metric is flat.

B Fluctuation frequency polynomials for "short" M2 brane

The determinant of the 8×8 bosonic fluctuation operator (with $m=1$) in the Fourier representation (3.37) that appears in (3.41) may be written as (cf. (3.38), (3.34), (3.52))

$$\mathrm{D_B} = (-\omega^2 + 4\mathcal{J} + p^2)^3\, P_{\mathrm{B}}(\omega, n, l, \tau, \mathcal{J})\,, \tag{B.1}$$

$$p^2 = (n-\tau_1 l)^2 + (\tau_2 l)^2\,, \qquad q = n - \tau_1 l\,, \qquad \tau_1 = -\frac{1}{2}k\sqrt{1-2\mathcal{J}}, \qquad \tau_2 = \frac{1}{\sqrt{2}}k\sqrt{\mathcal{J}}\ . \tag{B.2}$$

Here P_{B} is an order 10 polynomial in ω

$$\begin{aligned}
P_{\mathrm{B}} =& \omega^{10} + \omega^8(12\mathcal{J} - 5p^2 - 8) \\
&+ 2\omega^6\Big[18\mathcal{J}^2 - 4\mathcal{J}(5p^2-2q^2+6) - 4\sqrt{2-4\mathcal{J}}\sqrt{\mathcal{J}}q\sqrt{p^2-q^2} + 5p^4 + 12p^2 - 4q^2 + 8\Big] \\
&+ \omega^4\Big[24\sqrt{2-4\mathcal{J}}\mathcal{J}^{3/2}q\sqrt{p^2-q^2} + 32\mathcal{J}^3 - 4\mathcal{J}^2(11p^2+12q^2+4) + 8\sqrt{\mathcal{J}}\sqrt{2-4\mathcal{J}}(3p^2-4)q\sqrt{p^2-q^2} \\
&\qquad + 8\mathcal{J}[6p^4 + p^2(7-6q^2) + 11q^2] - 2[5p^6 + 12p^4 + p^2(8-12q^2) + 16q^2]\Big] \\
&+ \omega^2\Big[-16\sqrt{2-4\mathcal{J}}\mathcal{J}^{3/2}q\sqrt{p^2-q^2}(3p^2+2q^2-2) - 16\sqrt{2-4\mathcal{J}}\mathcal{J}^{5/2}q\sqrt{p^2-q^2} + 32\mathcal{J}^3\left(q^2-2p^2\right) \\
&\qquad + 4\mathcal{J}^2[-9p^4 + 8p^2\left(2q^2+3\right) + 4q^2\left(4q^2-5\right)] - 8\sqrt{2-4\mathcal{J}}\sqrt{\mathcal{J}}p^2\left(3p^2-8\right)q\sqrt{p^2-q^2} \\
&\qquad - 8\mathcal{J}[3p^6 - 6p^4\left(q^2+1\right) + p^2\left(22q^2+4\right) + 2q^2\left(q^2-2\right)] + 5p^8 + 8p^6 - 8p^4\left(3q^2+2\right) + 64p^2q^2\Big] \\
&- (6\mathcal{J} + p^2 - 4)\Big[16\sqrt{2}\mathcal{J}^{5/2}q\left(4q^2-3p^2\right)\sqrt{p^2-q^2} + 8\sqrt{2}\mathcal{J}^{3/2}q\left(2p^4+3p^2-4q^2\right)\sqrt{p^2-q^2} \\
&\qquad\qquad + 16\mathcal{J}^2\left(p^4-5p^2q^2+4q^4\right) - 8\sqrt{2}\sqrt{\mathcal{J}}p^4q\sqrt{p^2-q^2} \\
&\qquad\qquad - 2\mathcal{J}\left[5p^6 + p^4\left(4-8q^2\right) - 12p^2q^2 + 8q^4\right] + p^8 + 4p^6 - 8p^4q^2\Big]\ .
\end{aligned} \tag{B.3}$$

For the fermionic fluctuations we find

$$\mathrm{D_F} = [P_{\mathrm{F}}(\omega, n, l, \tau, \mathcal{J})]^2\,, \tag{B.4}$$

where $P_{\rm F}$ is 4-th order polynomial in ω^2

$$
\begin{aligned}
&P_{\rm F}= \\
&\omega^8-4\left(p^2+1\right)\omega^6+\omega^4\Big[6J^2+4J\left(p^2+2q^2-2\right)-4\sqrt{2-4J}\sqrt{J}q\sqrt{p^2-q^2}+6p^4+8p^2-4q^2+6\Big] \\
&+\omega^2\Big[16\sqrt{2-4J}J^{3/2}q\sqrt{p^2-q^2}-16J^3+4J^2\left(3p^2-8q^2-3\right)+8\sqrt{J}\sqrt{2-4J}\left(p^2+1\right)q\sqrt{p^2-q^2} \\
&\qquad -8J[p^4+p^2\left(2q^2+1\right)-2]-4\left(p^2+1\right)\left(p^4-2q^2+1\right)\Big] \qquad\qquad (B.5)\\
&-8\sqrt{2-4J}J^{3/2}q\sqrt{p^2-q^2}\left(3p^2+q^2-4\right)-60\sqrt{2-4J}J^{5/2}q\sqrt{p^2-q^2}+9J^4-12J^3\left(5p^2-10q^2+2\right) \\
&+2J^2\Big[-5p^4+4p^2\left(5q^2+7\right)+8q^4-62q^2+11\Big]+4\sqrt{J}\sqrt{2-4J}q\left(-p^4+2q^2-1\right)\sqrt{p^2-q^2} \\
&+4J\Big[p^6+p^4\left(2q^2+1\right)-p^2\left(4q^2+3\right)-5q^4+10q^2-2\Big]+\left(p^4-2q^2+1\right)^2 .
\end{aligned}
$$

C Fluctuation frequency polynomials for "lonng" M2 brane

Let us start with the bosonic fluctuations. We will specify to the case of the minimal winding $m=1$. The determinant of the 8×8 fluctuation operator in the Fourier representation (3.37) that appears in (3.41) may be written as (cf. (3.38))

$$
{\rm D_B} = (-\omega^2+4\mathcal{J}^2+n^2+{\rm q}^2+1)^3\, P_{\rm B}(\omega,n,{\rm q},\mathcal{J})\,, \qquad {\rm q}\equiv\tau_2 l=\frac{1}{2}kl\,, \qquad (C.1)
$$

where $P_{\rm B}$ given by

$$
P_{\rm B}=\det\begin{pmatrix}
n^2+{\rm q}^2-\omega^2-1 & 0 & 0 & 2i\mathcal{J}\omega & 0\\
0 & 4\mathcal{J}^2+n^2+{\rm q}^2-\omega^2 & 0 & -\frac{i(n+{\rm q})}{\sqrt{2}} & -\frac{1}{2}i\sqrt{8\mathcal{J}^2+2}(n+{\rm q})\\
0 & 0 & 4\mathcal{J}^2+n^2+{\rm q}^2-\omega^2 & -\frac{i(n-{\rm q})}{\sqrt{2}} & \frac{1}{2}i\sqrt{8\mathcal{J}^2+2}(n-{\rm q})\\
-2i\mathcal{J}\omega & \frac{i(n+{\rm q})}{\sqrt{2}} & \frac{i(n-{\rm q})}{\sqrt{2}} & n^2+{\rm q}^2-\omega^2 & 0\\
0 & \frac{1}{2}i\sqrt{8\mathcal{J}^2+2}(n+{\rm q}) & -\frac{1}{2}i\sqrt{8\mathcal{J}^2+2}(n-{\rm q}) & 0 & n^2+{\rm q}^2-\omega^2
\end{pmatrix} \qquad (C.2)
$$

It is a polynomial of order 5 in ω^2 with the explicit form being

$$
\begin{aligned}
&P_{\rm B}= \\
&\omega^{10}+\omega^8(-12J^2-5n^2-5{\rm q}^2+1)+\omega^6[48J^4+8J^2(5n^2+5{\rm q}^2-1)+2(n^2+{\rm q}^2)(5n^2+5{\rm q}^2-3)] \\
&-2\omega^4[32J^6+8J^4(5n^2+5{\rm q}^2-1)+8J^2(n^2+{\rm q}^2)(3n^2+3{\rm q}^2-2)+(n^2+{\rm q}^2)^2(5n^2+5{\rm q}^2-6)+n^2+{\rm q}^2)] \\
&+\omega^2\Big[32J^4(n^2+{\rm q}^2-1)(n^2+{\rm q}^2)+8J^2[3n^6+n^4(9{\rm q}^2-4)+n^2(9{\rm q}^4-10{\rm q}^2+1)+3{\rm q}^6-4{\rm q}^4+{\rm q}^2] \\
&\qquad +5n^8+10n^6(2{\rm q}^2-1)+5n^4(6{\rm q}^4-6q{\rm q}^2+1)+n^2(20{\rm q}^6-30{\rm q}^4+6{\rm q}^2)+5{\rm q}^4({\rm q}^2-1)^2\Big] \\
&-(n^2+{\rm q}^2-1)\Big[n^6(4J^2+4{\rm q}^2-2)+n^4[4J^2(3{\rm q}^2-1)+6{\rm q}^4-6{\rm q}^2+1] \\
&\qquad\qquad +2n^2{\rm q}^2[6J^2({\rm q}^2-2)+2{\rm q}^4-3{\rm q}^2-1]+{\rm q}^4({\rm q}^2-1)(4J^2+{\rm q}^2-1)+n^8\Big] . \qquad (C.3)
\end{aligned}
$$

The characteristic frequencies are solutions of ${\rm D_B}=0$. The 3 decoupled modes with

$$
\omega=\sqrt{4\mathcal{J}^2+n^2+{\rm q}^2+1}\,. \qquad (C.4)
$$

correspond to the transverse fluctuations of the M2 brane in the AdS_4 directions.

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Similarly, for the fermionic fluctuations we find

$$\mathrm{D_F} = \left[P_\mathrm{F}(\omega, n, \mathrm{q}, \mathcal{J})\right]^2 , \tag{C.5}$$

where P_F is 4-th order polynomial in ω^2

$$\begin{aligned} &P_\mathrm{F} = \\ &\omega^8+\omega^6(-12J^2-4n^2-4\mathrm{q}^2-1)+\omega^4\left[30J^4+J^2(32n^2+32\mathrm{q}^2+5)+6(n^2+\mathrm{q}^2)^2+2n^2+2\mathrm{q}^2+\frac{3}{8}\right] \\ &-\omega^2(4J^2+4n^2+4\mathrm{q}^2+1)\left[7J^4+6J^2(n^2+\mathrm{q}^2)+(n^2+\mathrm{q}^2)^2+\frac{1}{16}\right] \\ &+\mathrm{q}^8+4\mathrm{q}^6(2J^2+n^2)+\mathrm{q}^4\left[22J^4+2J^2(24n^2+1)+6n^4-\frac{1}{8}\right] \\ &+\frac{1}{4}\mathrm{q}^2\left[96J^6+16J^4(11n^2+1)+J^2(96n^4+24n^2-2)+16n^6+3n^2\right] \\ &+\frac{1}{256}(4J^2+4n^2+1)^2(12J^2+4n^2-1)^2 \end{aligned} \tag{C.6}$$

Thus each ω that is solves $\mathrm{D_F} = 0$ has degeneracy two.

D Non-planar corrections to ABJM Brehmstrahlung function

In the case of the $\mathcal{N} = 4$ SU(N) SYM theory the Brehmstrahlung function may be found from the exact localization result [125] for the expectation value of the $\frac{1}{2}$ BPS circular Wilson loop as [108]

$$B_\mathrm{SYM} = \frac{1}{2\pi^2}\lambda\frac{\partial}{\partial\lambda}\log\langle W\rangle_\mathrm{SYM} , \tag{D.1}$$

$$\langle W\rangle_\mathrm{SYM} = N^{-1}e^{\frac{\lambda}{8N^2}(N-1)}L^1_{N-1}\left(-\frac{\lambda}{4N}\right) = \frac{2N}{\sqrt{\lambda}}I_1(\sqrt{\lambda})\left[1+\frac{1}{96}\frac{\lambda^{3/2}}{N^2}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}+\dots\right], \tag{D.2}$$

$$\begin{aligned} B_\mathrm{SYM}(\lambda,N) &= B^{(\infty)}_\mathrm{SYM}(\lambda)+\frac{1}{128\pi^2}\frac{\lambda^{3/2}}{N^2}+\dots , \\ B^{(\infty)}_\mathrm{SYM}(\lambda) &= \frac{\sqrt{\lambda}}{4\pi^2}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} = \frac{\sqrt{\lambda}}{4\pi^2}-\frac{3}{8\pi^2}+\dots . \end{aligned} \tag{D.3}$$

To get the Brehmstrahlung function one may use the original definition as a derivative over the angle of a small cusp or one may start with the expression for the $\frac{1}{2}$ BPS Wilson loop wrapped w times on the circle and then [111]

$$B(\lambda,N) = \frac{1}{4\pi^2}\frac{\partial}{\partial \mathrm{w}}\log\langle W\rangle\Big|_{\mathrm{w}=1} . \tag{D.4}$$

In the $\mathcal{N} = 4$ SYM case this leads to the same expression as in (D.2) since the dependence on w can be incorporated into $\langle W\rangle$ in (D.2) by $\sqrt{\lambda} \to \mathrm{w}\sqrt{\lambda}$.

Ref. [111] has shown that (D.4) applies also in the ABJM case for the Brehmstrahlung function given in terms of the $\frac{1}{6}$ BPS Wilson loop defined on a small cusp. One may also find the Brehmstrahlung function corresponding to either $\frac{1}{2}$ or $\frac{1}{6}$ BPS Wilson loops by using a generalization of the identity [108] that expresses $B(\lambda, N)$ as a derivative of the logarithm of the latitude Wilson loop with respect to the small latitude angle [112, 115, 126, 127].[32] In the planar ($N = \infty$) limit one finds the following strong coupling expansion for the Brehmstrahlung function corresponding to the $\frac{1}{2}$ BPS Wilson loop [115]

$$B^{(\infty)}_{\text{ABJM}} = \frac{1}{2\pi}\sqrt{\frac{\lambda}{2}} - \frac{1}{4\pi^2} - \frac{1}{96\pi}\frac{1}{\sqrt{2\lambda}} + \dots , \tag{D.5}$$

which matches the string theory prediction at the two leading orders [79, 113]. Finding non-planar corrections in this approach is hard as the exact localization result is not known for a non-trivial cusp angle. An alternative approach was suggested in [117, 118].

One may conjecture that in general the multi-wrapped Wilson loop expectation value is the same as the one for the loop in the w-fundamental representation. The corresponding localization result was found in [48] and is a simple generalization of the expression given above in (1.26)

$$\langle W\rangle_{\text{ABJM}} = \frac{1}{2\,\sin\frac{2\pi \text{w}}{k}}\frac{\text{Ai}\Big[(\frac{\pi^2}{2}k)^{1/3}\Big(N-\frac{k}{24}-\frac{1}{3k}-\frac{2\text{w}}{k}\Big)\Big]}{\text{Ai}\Big[(\frac{\pi^2}{2}k)^{1/3}\Big(N-\frac{k}{24}-\frac{1}{3k}\Big)\Big]}\,. \tag{D.6}$$

We have checked explicitly that using (D.6) in (D.4) one gets the result for the corresponding Brehmstrahlung function which is equivalent to the one found using other definitions of $B(\lambda, N)$ in the $\frac{1}{2}$ BPS Wilson loop case [115, 116, 118] (see eq. (7.14) in [118]).

The expansion of (D.6) at large N for fixed k is similar to the one in (1.27)

$$\langle W\rangle_{\text{ABJM}} = \frac{1}{2\sin\frac{2\pi\text{w}}{k}}\,e^{\pi\text{w}\sqrt{\frac{2N}{k}}}\left[1-\frac{\pi\text{w}\,(k^2+24\text{w}+8)}{24\sqrt{2}\,k^{3/2}}\frac{1}{\sqrt{N}}+O\left(\frac{1}{N}\right)\right]. \tag{D.7}$$

Using this in (D.4) and expanding in large k we then get

$$\begin{aligned} B_{\text{ABJM}} &= \frac{1}{4\pi}\sqrt{\frac{2N}{k}} - \frac{1}{2\pi k}\cot\frac{2\pi}{k} + \dots = \frac{1}{2\pi}\sqrt{\frac{\lambda}{2}} - \frac{1}{4\pi^2} + \dots + \frac{1}{3k^2} + \frac{4\pi^2}{45k^4} + \frac{32\pi^4}{945k^6} + \dots \\ &= B^{(\infty)}_{\text{ABJM}} + \frac{\lambda^2}{3N^2} + \frac{4\pi^2\lambda^4}{45N^4} + \frac{32\pi^4\lambda^6}{945N^6} + \dots\,. \end{aligned} \tag{D.8}$$

It is interesting to note that the coefficients of the first two leading non-planar corrections here are the same (up to an overall 2π factor) as in the cusp anomaly function in (1.35), (2.18).

It would be important to explain the dependence of (D.7) on w from the semiclassical M2 brane point of view. One possible approach is to generalize the discussion in [38] to the case when the minimal surface is wrapped w times on the boundary circle. While the dependence of exponential in (D.7) on w then follows simply from the value of the classical

[32] For $\frac{1}{2}$ BPS Wilson loop this identity was proposed and proved perturbatively in [126], and for the corresponding Brehmstrahlung function it was first introduced and then proved exactly in [127]. In the $\frac{1}{6}$ BPS Wilson loop case a similar identity for the Brehmstrahlung function was proved in [112] and further elaborated on in [115]. For a review of the Brehmstrahlung function in the ABJM theory see the contribution of L. Bianchi in [128] and also [129].

action, it is not clear how a particular w-dependence of the $\frac{1}{\sin}$ prefactor in (D.6), (D.7) may come out of the 1-loop M2 brane contribution generalizing w = 1 one in [38].

Somewhat surprisingly, the dependence of the tree-level $e^{\pi \mathrm{w}\sqrt{\frac{2N}{k}}}$ and 1-loop $\frac{1}{2\sin\frac{2\pi \mathrm{w}}{k}}$ prefactors in (D.7) on w is actually the same as in the case when the M2 brane is wrapped w times not on the AdS_4 boundary circle but on the 11d circle φ. In this case we have effectively $\varphi \to \mathrm{w}\varphi$ and thus the radius $1/k$ in (1.10) is rescaled to w/k. This leads to $\frac{2\pi}{k} \to \frac{2\pi \mathrm{w}}{k}$ in the M2 brane 1-loop correction. The factor of w in the exponent in (D.7) then also has an obvious origin: the classical M2 brane action is proportional to the length of the 11d circle, i.e. $\frac{2\pi \mathrm{w}}{k}$, with the additional dependence on N and k coming from the effective M2 brane tension factor T_2 in (1.16). However, the w-dependence of the subleading terms in (D.7) (that should originate from the two and higher loop M2 brane corrections) does not appear to have a similar simple explanation.

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