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Positroid strata of the complex Grassmannian can be realized as augmentation varieties of Legendrians called positroid links. We prove that the partial order on strata induced by Zariski closure also has a symplectic interpretation, given by exact Lagrangian cobordism.

1. Introduction

Positroid varieties are irreducible subvarieties of the complex Grassmannian that were first introduced in the study of total positivity [Lusztig 1998; Postnikov 2006; Rietsch 2006], and Poisson geometry [Brown et al. 2006]. Open positroid varieties provide a stratification of the complex Grassmannian, and they can be enumerated by a handful of different combinatorial objects.

Positroid varieties are known to admit cluster structures, which have also been found on the coordinate rings of many algebraic varieties arising in representation theory, including double Bruhat cells [Fomin and Zelevinsky 2002], double Bott–Samelson cells [Shen and Weng 2021], positroid strata [Galashin and Lam 2023], and certain Richardson strata [Casals et al. 2022; Galashin et al. 2022; 2023]. Geometrically, this allows one to think of such varieties as the result of gluing algebraic tori along birational mutation maps, and their coordinate rings carry bases whose structure constants are positive integers counting tropical curves [Fock and Goncharov 2009; Gross et al. 2018].

Legendrian links in $\mathbb{R}^3$ are smooth links that are everywhere tangent to the plane field $\ker(dz - y dx)$ which is called the standard contact structure of $\mathbb{R}^3$. Their interpolating objects are exact Lagrangian cobordisms in the symplectization of $\mathbb{R}^3$. Legendrian links and exact Lagrangian cobordisms between them can be studied via the general framework of symplectic field theory [Eliashberg et al. 2000] which aims to use counts of pseudoholomorphic curves to define invariants of contact manifolds and the symplectic cobordisms between them. One such invariant is the Chekanov–Eliashberg differential graded algebra associated to a Legendrian link Λ,

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whose homology is a Legendrian isotopy invariant [Chekanov 2002; Eliashberg et al. 2000].

Under favorable circumstances, the space of augmentations of the Chekanov–Eliashberg dg-algebra forms an algebraic variety $\mathrm{Aug}(\Lambda)$. Exact Lagrangian fillings (cobordisms from the empty set to Λ) induce augmentations.

Augmentations have been shown to be related to simple microlocal sheaves associated to Λ [Shende et al. 2017; 2019; Ng et al. 2020], leading to the idea that augmentation varieties should have cluster structures, with torus charts corresponding to embedded exact Lagrangian fillings of Λ and mutations arising from Lagrangian surgery [Polterovich 1991]. So far this idea has been explored mainly for Legendrian links in the standard contact $\mathbb{R}^3$; see [Casals and Weng 2024]. This bridge between contact geometry and cluster algebras has been fruitful in both directions, having been instrumental in resolving long-standing conjectures on the abundance of Lagrangian fillings of Legendrian links [Casals and Gao 2022] and on the existence of cluster structures on spaces of interest in representation theory [Casals et al. 2022].

We explore this idea further, predicating that when augmentation varieties have compactifications stratified by augmentation varieties of smaller dimension, then the Legendrian submanifolds corresponding to adjacent strata should be related by exact Lagrangian cobordisms. We establish this in the simplest class of examples: positroid strata of complex Grassmannians [Knutson et al. 2013]. It is known that all positroid strata are isomorphic to the moduli space of simple microlocal sheaves of certain Legendrian links Λ in $\mathbb{R}^3$ with framings (marked points) [Shende et al. 2019] and to augmentation varieties of Λ [Casals et al. 2021; 2020]. The top positroid stratum was one of the key motivating examples for the development of cluster algebras [Fomin and Zelevinsky 2002; Scott 2006], while cluster structures on strata of lower dimension were described more recently [Galashin and Lam 2023].

1.1. *Result.* The positroid strata $\Pi^\circ \subset \mathrm{Gr}(k,n)$ of complex Grassmannians are disjoint Zariski locally closed sets; see Definition 4.5. There is a distinguished class of Legendrian links Λ_{Π° in the standard contact $\mathbb{R}^3$, referred to as positroid links whose augmentation varieties are related to the strata by an algebraic isomorphism

$$\Pi^\circ \cong \mathrm{Aug}(\Lambda_{\Pi^\circ}) \times (\mathbb{C}^*)^{N(\Pi^\circ)}, \tag{1}$$

where $N(\Pi^\circ) \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer depending on Π°; see Section 5 for a precise statement. The positroid link Λ_{Π° is the Legendrian (-1)-closure of a braid on k-strands associated to Π°, known as a juggling braid (see Definition 5.4). The positroid strata can be enumerated by bounded affine permutations (see Section 3) and for each pair of integers $1 \leq k < n$, the set of positroid strata of $\mathrm{Gr}(k,n)$ is partially ordered by declaring $\Pi_f^\circ \leq \Pi_g^\circ$ if and only if $\Pi_f^\circ \subset \mathrm{cl}(\Pi_g^\circ)$. Our main result is the following.

Theorem 1.1 (Theorems 6.1 and 6.3). *Given two comparable positroid strata $\Pi_f^\circ \leq \Pi_g^\circ$ in $\mathrm{Gr}(k,n)$, their associated Legendrian links $\Lambda_{\Pi_f^\circ}$ and $\Lambda_{\Pi_g^\circ}$ are related by an exact Lagrangian cobordism from $\Lambda_{\Pi_f^\circ}$ to $\Lambda_{\Pi_g^\circ}$ whose Euler characteristic is*

$$\dim(\Pi_g^\circ) - \dim(\Pi_f^\circ) + \#\,\mathrm{Fix}(g) - \#\,\mathrm{Fix}(f),$$

where $\#\,\mathrm{Fix}$ *denotes the number of fixed points* (*see Definition 3.9*).

Remark 1.2. (1) We defer experts to Theorem 5.8 and Remark 6.2 for a discussion on marked points placed on the positroid links.

(2) Two positroid links being exact Lagrangian cobordant does not imply that the corresponding positroid strata are comparable in the partial order; see Example 7.2 and Remark 7.3.

Note that even for small k and n, complex Grassmannians have many positroid strata, and their partial order is quite complicated; see Example 4.9 and Figure 5. The exact Lagrangian cobordism in Theorem 1.1 is constructed by pinching contractible Reeb chords, which is a well-known technique in contact geometry. We establish that any chain connecting Π_f° and Π_g° in the partial order produces a sequence of pinch moves.

If $\Pi_f^\circ < \Pi_g^\circ$ then from Theorem 1.1 there is an exact Lagrangian cobordism from $\Lambda_{\Pi_f^\circ}$ to $\Lambda_{\Pi_g^\circ}$ consisting of pinch moves. Let r be the number of such moves. From [Pan 2017; Gao et al. 2024] it follows that there is an open embedding relating the augmentation varieties of the ends:

$$\mathrm{Aug}(\Lambda_{\Pi_f^\circ}) \times (\mathbb{C}^*)^r \hookrightarrow \mathrm{Aug}(\Lambda_{\Pi_g^\circ}).$$

This means that if the bounded affine permutations f and g are related by r affine transpositions, under the identification between positroid strata and augmentation varieties in (1) we get an open embedding

$$\Pi_f^\circ \times (\mathbb{C}^*)^{r+N(\Pi_g^\circ)} \hookrightarrow \Pi_g^\circ \times (\mathbb{C}^*)^{N(\Pi_f^\circ)}.$$

As pointed out to us by a referee it is an interesting question whether such embeddings admit a description purely in terms of algebraic combinatorics, i.e., without using the connection with the topology of Legendrians.

Outline. In Section 2 we provide the necessary background on Legendrian links and exact Lagrangian cobordisms. In Section 3 we provide the relevant definitions and properties of bounded affine permutations. We recall the definition of positroid strata of complex Grassmannians in Section 4. In Section 5 we describe positroid links via juggling braids coming from bounded affine permutations. Our main theorem Theorem 1.1 is proven in Section 6. In Section 7 we discuss examples.

2. Background on contact geometry

We briefly review some basic facts on Legendrian links and exact Lagrangian cobordisms. See [Etnyre and Ng 2022] for a more thorough introduction, and [Geiges 2008] for background on contact geometry.

2.1. ***Legendrian links.*** The *standard contact structure* on $\mathbb{R}^3$ is the plane field $\xi := \ker(dz - y\,dx)$. A smooth embedding of circles $\Lambda \subset \mathbb{R}^3$ is called a *Legendrian link* if $T_x\Lambda \subset \xi_x$ for all $x \in \Lambda$. Two Legendrian links are *Legendrian isotopic* if they are smoothly isotopic through Legendrian links. The maps $\pi_L(x, y, z) = (x, y)$ and $\pi_F(x, y, z) = (x, z)$ are called the *Lagrangian projection* and *front projection*, respectively. The Lagrangian projection of a Legendrian link is an immersed curve with zero oriented area. The front projection of a Legendrian link does not have any vertical tangencies but instead has cusp and crossing singularities. Conversely, any immersed disjoint union of circles with cusp and crossing singularities and no vertical tangencies lifts uniquely to a Legendrian link in $\mathbb{R}^3$; see Figure 1 for an example of both projections.

The *Thurston–Bennequin number* $\mathrm{tb}(\Lambda) \in \mathbb{Z}$ of a Legendrian link $\Lambda \subset \mathbb{R}^3$ measures how much the contact structure ξ rotates along Λ, and is defined as the linking number of Λ and its push-off in any direction transverse to ξ. This number is easily computed from a front projection as

$$\begin{aligned}\mathrm{tb}(\Lambda) = \#\text{positive crossings of } \pi_F(\Lambda) - \#\text{negative crossings of } \pi_F(\Lambda) \\ - \#\text{right cusps of } \pi_F(\Lambda).\end{aligned}$$

A *Reeb chord* of Λ is a trajectory of the vector field ∂_z that begins and ends on Λ. Note that the Lagrangian projection induces a bijection between Reeb chords and double points of $\pi_L(\Lambda)$. A Reeb chord of Λ is *contractible* if there exists a smooth homotopy of Λ through Legendrian immersions (such that the Lagrangian projections only have transverse double points throughout the homotopy) that shrinks the length of the Reeb chord to zero; see [Ekholm et al. 2016, Definition 6.13].

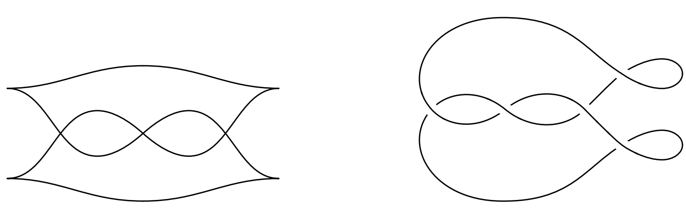

Figure 1. Front (left) and Lagrangian (right) projections of a Legendrian trefoil.

2.2. ***Exact Lagrangian cobordisms.*** The *symplectization* of the standard contact $\mathbb{R}^3$ is defined as $\mathbb{R} \times \mathbb{R}^3$ equipped with the closed nondegenerate 2-form $\omega = d\lambda$ where $\lambda = e^t(dz - y\,dx)$. A *Lagrangian cobordism* from a Legendrian link Λ_- to a Legendrian link Λ_+ is a smooth embedding of a surface $L \subset \mathbb{R} \times \mathbb{R}^3$ such that $\omega|_{TL} = 0$ and such that L is a cylinder over $\Lambda_\pm$ at infinity but is otherwise compact, i.e., there is some $T > 0$ for which $L \cap [-T, T]$ is compact,

$$\mathcal{E}_-(L) := L \cap ((-\infty, -T) \times \mathbb{R}^3) = (-\infty, -T) \times \Lambda_-,$$
$$\mathcal{E}_+(L) := L \cap ((T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+.$$

A Lagrangian cobordism L is *exact* if there is a smooth function $f : L \to \mathbb{R}$ such that $df = \lambda|_L$ and $f|_{\mathcal{E}_\pm(L)}$ is constant. A *Lagrangian filling* is a Lagrangian cobordism with $\Lambda_- = \varnothing$. Exact Lagrangian cobordisms give a reflexive and transitive relation, but not a symmetric one [Chantraine 2015]. All known examples of exact Lagrangian cobordisms between Legendrians with maximal Thurston–Bennequin numbers arise from

- Legendrian isotopy,
- the unique exact Lagrangian disk filling of an unlinked unknot component with maximal Thurston–Bennequin number, and
- pinching a contractible Reeb chord.

The *pinch move* is a local modification of $\Lambda \subset \mathbb{R}^3$, depicted in Figure 2. A pinch move induces an exact Lagrangian cobordism in the symplectization of $\mathbb{R}^3$ from the knot after a pinch move to the knot before the pinch move. When a pinch move is performed, the number of components of the Legendrian link either increases or decreases by one, so the resulting exact Lagrangian cobordism is topologically a pair of pants, and it is often called a saddle cobordism. See Figure 12 for an example of an exact Lagrangian saddle cobordism between two Legendrians.

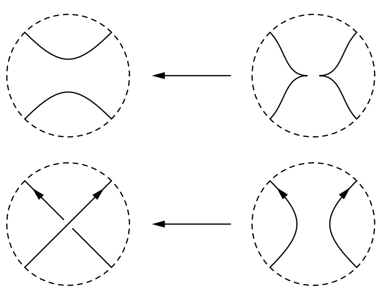

Figure 2. A pinch move in the front (left) and the Lagrangian (right) projection. The arrows show the direction of the induced exact Lagrangian cobordism.

3. Bounded affine permutations

We review bounded affine permutations, the affine analogs of ordinary permutations. See [Knutson et al. 2013] for a more thorough introduction and an interpretation in terms of juggling. Throughout this section let $k, n \in \mathbb{Z}_{\geq 1}$ with $k \leq n$.

Definition 3.1. An *affine permutation of size n* is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(i+n) = f(i) + n$ for all $i \in \mathbb{Z}$. In addition, it is *k-bounded* if

(1) $i \leq f(i) \leq i + n$, and

(2) $\sum_{i=1}^{n} (f(i) - i) = nk$.

Denote the set of k-bounded affine permutations of size n by $\mathrm{Bound}(k, n)$, and a k-bounded affine permutation $f : \mathbb{Z} \to \mathbb{Z}$ by $[f(1), f(2), \ldots, f(n)]$.

Lemma 3.2. *A bijection f is a k-bounded affine permutation of size n if and only if $g := -(-f)^{-1}$ is a k-bounded affine permutation of size n.*

Proof. Let $i \in \mathbb{Z}$ and define $j = -f(i)$. Then $f(i+n) = f(i) + n$ is equivalent to $i + n = -g(-f(i) - n)$, and hence $-g(j) + n = -g(j - n)$. Note that $i \leq f(i) \leq i + n$ is equivalent to $-g((-f)(i)) \leq f(i) \leq -g((-f)(i+n))$. We can rewrite these inequalities as $g(j) \geq j$ and $j \geq g(j-n) = g(j) - n$, which is equivalent to $j \leq g(j) \leq j + n$. Finally, $\sum_{i=1}^{n} (f(i) - i) = nk$ is equivalent to $\sum_{j=1}^{n} (-j + g(j)) = nk$. □

A bounded affine permutation $f \in \mathrm{Bound}(k, n)$ can be visualized in the plane as the set of line segments in $\mathbb{R}^2$ from $(i, 1)$ to $(f(i), 0)$ for all $i \in \mathbb{Z}$; see Figure 3 for an example. Note that once $f(i+n) = f(i) + n$ for all $i \in \mathbb{Z}$, the picture is fully determined by the region in the red dashed box in Figure 3.

Definition 3.3. For a bounded affine permutation $f \in \mathrm{Bound}(k, n)$, a pair $(i, j) \in \{1, \ldots, n\}^2$ is an *affine inversion* if $i < j$ and either $f(i) > f(j)$ or $f(i) < f(j) - n$. The *length* of an affine permutation f, $\ell(f) \in \mathbb{Z}_{\geq 0}$, is the number of affine inversions of f.

Example 3.4. For the bounded affine permutation $f = [3, 5, 6, 4]$ depicted in Figure 3, $(2, 4)$ and $(3, 4)$ are the only affine inversions, and thus $\ell(f) = 2$. These

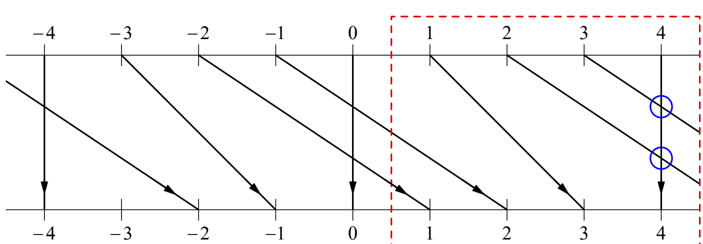

Figure 3. The bounded affine permutation $f = [3, 5, 6, 4] \in \mathrm{Bound}(2, 4)$.

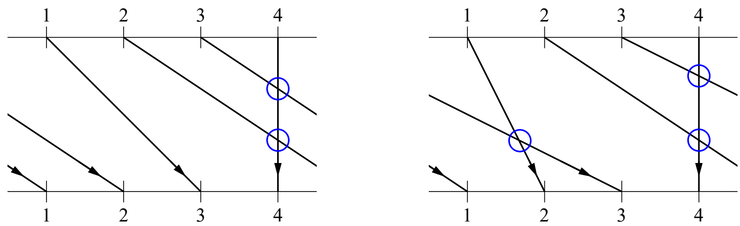

Figure 4. Left: $[3, 5, 6, 4] \in \text{Bound}(2, 4)$. Right: $[2, 5, 7, 4] \in \text{Bound}(2, 4)$.

two affine inversions correspond to the two circles in Figure 3. Similarly, we have $\ell([3, 4, 6, 5]) = 1$ and $\ell([2, 5, 7, 4]) = 3$.

We now equip $\text{Bound}(k, n)$ with a partial order.

Definition 3.5. An affine permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ of size n is a *transposition* if $\sigma(k) = \sigma_i(k)$ for some $i \in \mathbb{Z}$, where

$$\sigma_i(k) := \begin{cases} k+1 & \text{if } k = i \pmod n, \\ k-1 & \text{if } k = i+1 \pmod n, \\ k & \text{if } k \neq i, i+1 \pmod n. \end{cases}$$

Definition 3.6. Let $f, f' \in \text{Bound}(k, n)$. Declare $f \lessdot f'$ if and only if $\ell(f) < \ell(f')$ and there exists an affine transposition σ_i of size n such that $f' = f \circ \sigma_i$ or $f' = \sigma_i \circ f$. Define a relation $<$ on $\text{Bound}(k, n)$ as the transitive closure of the relation $\lessdot$.

It follows from the definition that $(\text{Bound}(k, n), <)$ is a partially ordered set.

Example 3.7. Note that

$$\ell([3, 4, 6, 5]) < \ell([3, 5, 6, 4]) \quad \text{and} \quad [3, 5, 6, 4] = \sigma_4 \circ [3, 4, 6, 5];$$

thus we have $[3, 4, 6, 5] \lessdot [3, 5, 6, 4]$. Similarly

$$\ell([3, 5, 6, 4]) < \ell([2, 5, 7, 4]) \quad \text{and} \quad [2, 5, 7, 4] = \sigma_2 \circ [3, 5, 6, 4],$$

so that $[3, 5, 6, 4] \lessdot [2, 5, 7, 4]$; see Figure 4. Then the induced partial order satisfies $[3, 4, 6, 5] < [2, 5, 7, 4]$. See Figure 5 for the Hasse diagram of the partial order $<$ on $\text{Bound}(2, 4)$.

As in the case of ordinary permutations, one can define cycles of bounded affine permutations.

Lemma 3.8. *Let $f \in \text{Bound}(k, n)$. Then f induces a bijection $\bar{f} : \mathbb{Z}/n \to \mathbb{Z}/n$ defined by $\bar{f}([i]) = [f(i)]$ for all $[i] \in \mathbb{Z}/n$.*

Proof. As f is a bounded affine permutation, for all $i, t \in \mathbb{Z}$, we know $f(i + tn) = f(i) + tn = f(i) \pmod n$. Thus, $\bar{f}$ is well-defined. Moreover, we have a well-defined inverse function $\bar{f}^{-1}$ given by $\bar{f}^{-1}([i]) = [f^{-1}(i)]$. So $\bar{f}$ is a bijection. $\square$

Definition 3.9. A *cycle of length t* of $f \in \text{Bound}(k, n)$ is a tuple $(i_1, \dots, i_t) \in (\mathbb{Z}/n)^t$ up to cyclic permutation such that

$$\bar{f} : i_1 \mapsto i_2 \mapsto \cdots \mapsto i_t \mapsto i_1,$$

where $i_1, \dots, i_t$ are all distinct. A cycle of length 1 is called a *fixed point* of f.

Example 3.10. The affine permutation $f = [3, 5, 6, 4]$ has one cycle of length three being $(1, 3, 2)$ and one cycle of length one being (4).

4. Positroid strata

We collect definitions and known properties of positroid strata of the complex Grassmannian from the literature [Knutson et al. 2013; Galashin and Lam 2021].

4.1. ***Complex Grassmannians.*** Fix $k, n \in \mathbb{Z}_{\geq 1}$ such that $k \leq n$, and write $\text{Mat}(k, n)$ for the set of $k \times n$ matrices with complex entries.

Definition 4.1. The *complex Grassmannian of k-planes* in $\mathbb{C}^n$ is

$$\text{Gr}(k, n) = \{M \in \text{Mat}(k, n) \mid \text{rk}(M) = k\}/\text{row operations}.$$

Complex Grassmannians are smooth projective varieties. A widely used projective embedding of $\text{Gr}(k, n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ is given by Plücker coordinates; see [Harris 1992, Lecture 6].

Definition 4.2. Given $M \in \text{Mat}(k, n)$ with column vectors $M_1, \dots, M_n$ and $1 \leq i_1 < \cdots < i_k \leq n$, we define the *Plücker coordinates* $\Delta_{i_1,\dots,i_k}(M)$ to be

$$\Delta_{i_1,\dots,i_k}(M) = \det[M_{i_1}, M_{i_2}, M_{i_3}, \dots, M_{i_k}].$$

Example 4.3. For $1 \leq i_1 < i_2 \leq 4$, label the $\binom{4}{2} = 6$ homogeneous coordinates of $\mathbb{P}^5$ by Δ_{i_1,i_2}. The corresponding Plücker coordinates on $\text{Gr}(2, 4)$ give a projective embedding of $\text{Gr}(2, 4)$ as a hypersurface in $\mathbb{P}^5$, whose equation is

$$\Delta_{1,3}\Delta_{2,4} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3}.$$

For example, the matrix

$$M = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

has $\Delta_{1,2}(M) = 1$, $\Delta_{1,3}(M) = -1$, $\Delta_{1,4}(M) = 1$, $\Delta_{2,3}(M) = -2$, $\Delta_{2,4}(M) = 1$, and $\Delta_{3,4}(M) = 1$. Note that row operations on M change all Plücker coordinates by a common factor, which is immaterial once one thinks of them as homogeneous coordinates on $\mathbb{P}^5$.

4.2. ***Positroid strata.*** The complex Grassmannian $\mathrm{Gr}(k, n)$ decomposes into disjoint subsets Π°_f labeled by bounded affine permutations $f \in \mathrm{Bound}(k, n)$; see [Knutson et al. 2013]. Any $M \in \mathrm{Mat}(k, n)$ with columns $M_1, \dots, M_n$ extends periodically to a matrix with infinitely many columns, by setting $M_{i+n} = M_i$ for all $i \in \mathbb{Z}$. Define an associated function $f_M : \mathbb{Z} \to \mathbb{Z}$ by

$$f_M(i) = \min\{j \geq i \mid M_i \in \mathrm{Span}(M_{i+1}, \dots, M_j)\}.$$

If $M \in \mathrm{Mat}(k, n)$ has rank k, then $f_M : \mathbb{Z} \to \mathbb{Z}$ is a k-bounded affine permutation of size n that depends only on $[M] \in \mathrm{Gr}(k, n)$.

Example 4.4. The matrix $M \in \mathrm{Mat}(2, 4)$ from Example 4.3 extends periodically to a matrix with infinitely many columns

$$\begin{bmatrix} \cdots & 0 & 1 & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & 1 & 2 & \cdots \\ \cdots & -1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & 0 & 1 & \cdots \end{bmatrix},$$

and the corresponding bounded affine permutation $f_M : \mathbb{Z} \to \mathbb{Z}$ of type $(2, 4)$ is $f_M = [3, 4, 5, 6]$.

Definition 4.5. The *positroid stratum* associated to $f \in \mathrm{Bound}(k, n)$ is defined as

$$\Pi^\circ_f := \{[M] \in \mathrm{Gr}(k, n) \mid f_M = f\}.$$

The adjective positroid comes from the fact that the closure of a stratum is defined by the vanishing of Plücker coordinates $\Delta_{i_1,\dots i_k}$ whose indexing sets $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ form a particular class of matroids [Postnikov 2006]. The term strata refers to the following property.

Theorem 4.6 (Knutson–Lam–Speyer [Knutson et al. 2013, Theorems 5.9 and 5.10]). *Each positroid stratum is locally closed in the Zariski topology, and has closure*

$$\mathrm{cl}(\Pi^\circ_f) = \bigcup_{f' \geq f} \Pi^\circ_{f'}.$$

Definition 4.7 (partial order on positroid strata). Define $\Pi^\circ_1 \leq \Pi^\circ_2$ if and only if $\Pi^\circ_1 \subset \mathrm{cl}(\Pi^\circ_2)$.

It follows immediately that $\leq$ defines a partial order on the set of positroid strata of $\mathrm{Gr}(k, n)$.

Theorem 4.8 [Knutson et al. 2013, Theorem 5.9]. *The codimension of* $\Pi^\circ_f \subset \mathrm{Gr}(k, n)$ *is equal to* $\ell(f)$.

Example 4.9. There are 33 positroid strata $\Pi^\circ_f \subset \mathrm{Gr}(2, 4)$: one of dimension 4, four of dimension 3, ten of dimension 2, twelve of dimension 1, and six of dimension 0. Each dimension corresponds to a row in the Hasse diagram of Figure 5, with the bottom row containing the only top-dimensional stratum.

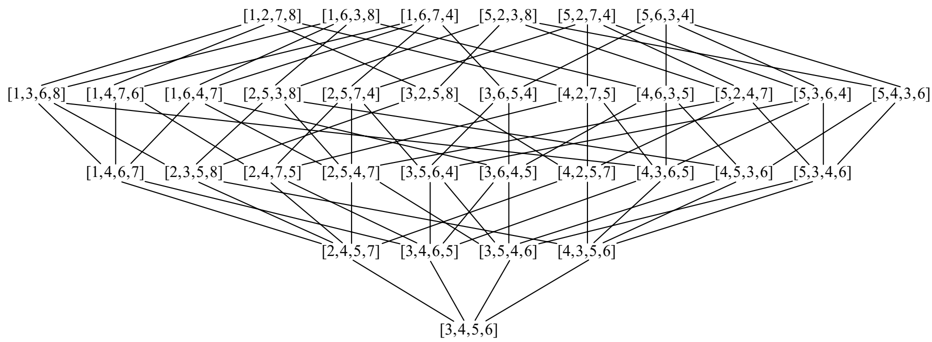

Figure 5. The Hasse diagram of the partial order on Bound(2, 4).

5. Positroid links

We follow Casals, Gorsky, Gorsky and Simental [Casals et al. 2021] and associate a Legendrian link to a bounded affine permutation $f \in \text{Bound}(k, n)$ (see Section 3).

Definition 5.1. Let $f \in \text{Bound}(k, n)$ be a bounded affine permutation. For each $i \in \{1, \dots, n\}$, let

$$A_i(f) := \left\{(x, y) \in \mathbb{R}^2 \mid (2x - f(i) - i)^2 + 4y^2 = (f(i) - i)^2\right\} \cap \{y \geq 0\} \subset \mathbb{R}^2$$

be the upper semicircle of a circle intersecting the x-axis in the points $(i, 0)$ and $(f(i), 0)$. We define the *juggling diagram* associated to f to be the subset $\bigcup_{i=1}^n A_i(f) \subset \mathbb{R}^2$.

Definition 5.2. Let $f \in \text{Bound}(k, n)$ be a bounded affine permutation. After modifying the associated juggling diagram with the moves shown in Figure 6, we obtain a tangle diagram. After enumerating the strands of the tangle diagram from top to bottom we can describe the tangle diagram with a braid word that we denote by $J_k(f)$ and call the *juggling braid* of f.

Remark 5.3. By the definition of a bounded affine permutation, $J_k(f)$ is a positive braid on k strands.

See Figure 7 for examples of juggling diagrams and their corresponding juggling braids.

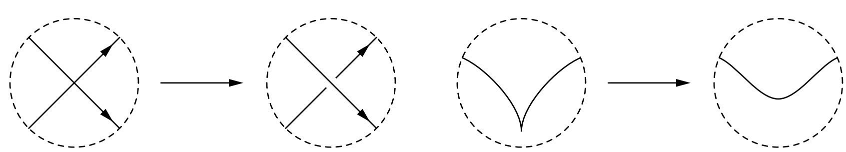

Figure 6. Converting from a juggling diagram to a braid via specified smoothings of cusps and crossings.

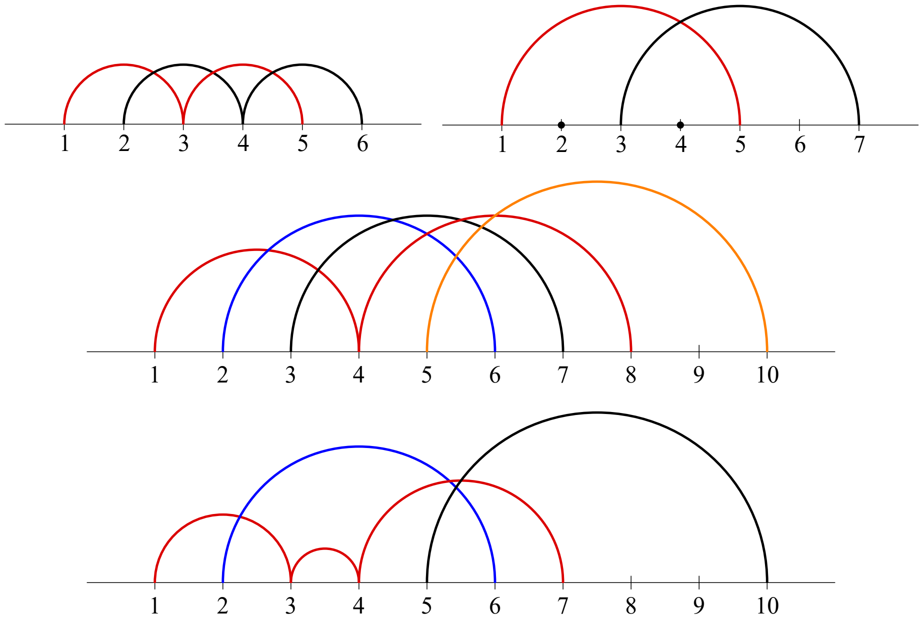

Figure 7. Examples of juggling diagrams and their corresponding juggling braid words. Top left: $J_2([3,4,5,6]) = \sigma_1^3$. Top right: $J_2([5,2,7,4]) = \sigma_1$. Middle: $J_4([4,6,7,8,10]) = \sigma_1\sigma_2\sigma_1\sigma_3(\sigma_2\sigma_1)^2$. Bottom: $J_3([3,6,4,7,10]) = \sigma_1\sigma_2\sigma_1^2$. Note that the dots in the top right indicate fixed points of the bounded affine permutation. The strands are different colors to increase visual clarity.

We will now set some notation. Let $\sigma_1, \dots, \sigma_{k-1}$ denote the Artin generators of the braid group and let Br_k^+ be the submonoid of the braid group generated by non-negative powers of the Artin generators. We let $\Delta_k = (\sigma_1)(\sigma_2\sigma_1)\cdots(\sigma_{k-1}\cdots\sigma_1)$ denote the positive half twist. Let w_0 denote the image of Δ_k in the projection from the braid group to the symmetric group.

Definition 5.4 [Casals et al. 2021, Definition 3.3]. Let $f \in \mathrm{Bound}(k,n)$ be a bounded affine permutation, and let $J_k(f) \in \mathrm{Br}_k^+$ be its associated juggling braid. We define the *positroid link* of f, denoted by Λ_f, to be the Legendrian (-1)-closure (see Figure 8) of the positive braid $J_k(f)\Delta_k \in \mathrm{Br}_k^+$ with the orientation induced by giving all strands of $J_k(f) \in \mathrm{Br}_k^+$ the same orientation.

Remark 5.5. In [Casals et al. 2021], there are other (Legendrian isotopic) descriptions of Λ_f, using other enumerations of positroid strata of the complex Grassmannian such as pairs of permutations (satisfying some properties), Le diagrams, and cyclic rank matrices. For the scope of this article, it suffices to consider juggling braids.

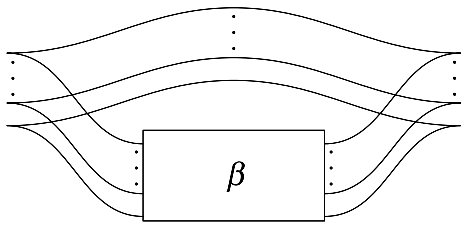

Figure 8. The front diagram of the Legendrian (-1)-closure of the positive braid word $\beta \in \mathrm{Br}_k^+$.

Lemma 5.6. *Let $f \in \mathrm{Bound}(k, n)$. The Thurston–Bennequin number of Λ_f is given by*

$$\mathrm{tb}(\Lambda_f) = |J_k(f)| - \frac{k(k+1)}{2},$$

where $|J_k(f)|$ denotes the length of the braid word $J_k(f) \in \mathrm{Br}_k^+$.

Proof. Recall that the Thurston–Bennequin number of a Legendrian link is given by the writhe minus the number of right cusps of a front diagram; see (2). There are $|J_k(f)| + |\Delta_k|$ positive crossings coming from the crossings in $\beta = J_k(f)\Delta_k$ and k right cusps in the positroid link of f. Note that $|\Delta_k| = \frac{k(k-1)}{2}$. There are $2|\Delta_k| = k(k-1)$ negative crossings coming from the portion of the positroid link of f outside of $\beta = J_k(f)\Delta_k$ in the Legendrian (-1)-closure diagram. The sum of these contributions is

$$\mathrm{tb}(\Lambda_f) = |J_k(f)| + \frac{k(k-1)}{2} - k(k-1) - k = |J_k(f)| - \frac{k(k+1)}{2}. \qquad \square$$

Corollary 5.7. *Let $f \in \mathrm{Bound}(k, n)$. The Thurston–Bennequin number of Λ_f is given by*

$$\mathrm{tb}(\Lambda_f) = \dim \Pi_f^\circ + \#\,\mathrm{Fix}(f) - n,$$

where $\mathrm{Fix}(f) = \{i \in \{1, \dots, n\} \mid f(i) = i\}$.

Proof. The statement of Lemma 3.10 in the first arXiv version of [Casals et al. 2021] states that

$$|J_k(f)| = |w| - |u| + \binom{k}{2} - (n-k) + \#\,\mathrm{Fix}(f),$$

where (u, w) is a pair of permutations called the positroid pair corresponding to the bounded affine permutation f (see [Casals et al. 2021, Definition 2.2] and [Knutson et al. 2013, Proposition 3.15]). It is well-known that $\dim \Pi_f^\circ = |w| - |u|$ (see, e.g., [Knutson et al. 2014, Corollary 3.2]); hence

$$|J_k(f)| = \dim \Pi_f^\circ + \binom{k}{2} - (n-k) + \#\,\mathrm{Fix}(f). \tag{2}$$

Therefore we get

$$\begin{aligned}\operatorname{tb}(\Lambda_f) &= |J_k(f)| - \frac{k(k+1)}{2} \\ &= \dim \Pi_f^\circ + \binom{k}{2} - (n-k) + \#\operatorname{Fix}(f) - \frac{k(k+1)}{2} \\ &= \dim \Pi_f^\circ + \#\operatorname{Fix}(f) - n,\end{aligned}$$

where the first equality is by Lemma 5.6 and the second equality is by (2). □

The main motivation for calling Λ_f a positroid link is the following connection with positroid strata of the complex Grassmannian. To call upon this result, we follow the convention of placing a marked point on each strand in the braid $J_k(f)\Delta_k$, and place each to the right of all crossings in $J_k(f)\Delta_k$ on the respective strand when defining the augmentation variety associated to Λ_f, as in [Casals et al. 2020, Section 2.6].

Theorem 5.8 (Casals–Gorsky–Gorsky–Simental [Casals et al. 2020; 2021]). *Let $f \in \operatorname{Bound}(k, n)$ be a bounded affine permutation, and consider its positroid link Λ_f. Then, there is an algebraic isomorphism*

$$\Pi_f^\circ \cong \operatorname{Aug}(\Lambda_f) \times (\mathbb{C}^*)^{n-\#\operatorname{Fix} f - k}.$$

Proof. We have one marked point in Λ_f for each strand in the braid $J_k(f)\Delta_k$. Then by [Casals et al. 2020, Theorem 2.30] we have $\operatorname{Aug}(\Lambda_f) \cong X_0(J_k(f); w_0)$, where X_0 denotes the *braid variety* as defined in [Casals et al. 2020]. Then, by [Casals et al. 2021, Theorem 1.3] we have

$$\Pi_f^\circ \cong X_0(J_k(f); w_0) \times (\mathbb{C}^*)^{n-\#\operatorname{Fix} f - k},$$

which gives the result. □

Proposition 5.9. *For $f \in \operatorname{Bound}(k, n)$, the number of components of the link Λ_f is given by the number of cycles of f of length at least* 2 (*see Definition 3.9*).

Proof. Consider a cyclic juggling diagram of f which can be obtained from a juggling diagram by first restricting $\bigcup_{i=1}^n A_i(f) \subset \mathbb{R}^2$ to $\{1 \le x \le n\}$ and then extending each arc cyclically. More precisely, we first arrange the juggling diagram of f so that no crossing of $\bigcup_{i=1}^n A_i(f)$ belongs to $\{x \ge n\} \subset \mathbb{R}^2$ by a smooth isotopy of $\bigcup_{i=1}^n A_i(f)$ which leaves the braid word $J_k(f)$ unaffected (up to braid moves); see [Casals et al. 2021, Lemma 2.19]. Then we define the cyclic juggling diagram of f to be the subset

$$\bar{A}(f) := \left(\bigcup_{i=1}^n A_i(f) \cap \{1 \le x \le n\}\right) \cup \left(\bigcup_{\{i \mid f(i) > n\}} A_i^{\text{shift}}(f) \cap \{1 \le x \le n\}\right) \subset \mathbb{R}^2$$

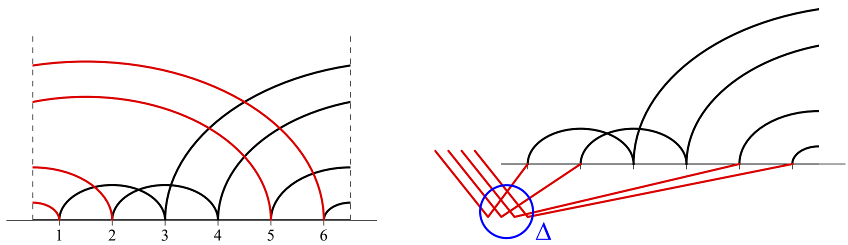

Figure 9. Left: Extending the arcs of [3, 4, 12, 11, 8, 7] cyclically to construct a cyclic juggling diagram. Right: The result of a smooth isotopy from the left diagram, where all the arcs are pulled downwards, demonstrating the half twist obtained from the added arcs of the cyclic juggling diagram.

where

$$A_i^{\text{shift}}(f) := \left\{(x, y) \in \mathbb{R}^2 \mid (2(x+n-1)-f(i)-i)^2+4y^2 = (f(i)-i)^2\right\} \cap \{y \geq 0\} \subset \mathbb{R}^2$$

is the arc $A_i(f)$ shifted to the left by $n-1$. Then, each cycle of f corresponds to a sequence of arcs that closes up onto itself in the cyclic juggling diagram. See Figure 9, left, for an example. We turn the cyclic juggling diagram $\bar{A}(f)$ into a braid by using the smoothing modifications of Figure 6 to obtain a cyclic juggling braid $\bar{J}_k(f)$. We take the (-1)-closure of the $\bar{J}_k(f)$. Then, the resulting link is smoothly isotopic to the (-1)-closure of the juggling braid $\Delta_k J_k(f)$, i.e., the link Λ_f; see Figure 9, right. Thus the number of components of Λ_f is exactly the number of cycles of f. □

6. Construction of the Lagrangian cobordisms

We say that there is a *path* from f to g in $\text{Bound}(k, n)$ if there is a sequence of affine bounded permutations $(h_1, \ldots, h_k)$ (this sequence might be empty) such that

$$f \lessdot h_1 \lessdot \cdots \lessdot h_k \lessdot g.$$

Theorem 6.1. *Given any path from f to g in* $\text{Bound}(k, n)$, *there is an exact Lagrangian cobordism from Λ_g to Λ_f.*

Proof. Recall from Definition 5.2 that any bounded affine permutation f corresponds to a juggling braid $J_k(f)$ which corresponds by Definition 5.4 to a Legendrian Λ_f given by the (-1)-closure of the positive braid $J_k(f)\Delta_k$. There is a convenient Lagrangian projection of Λ_f; see [Casals and Ng 2022, Figure 8]. Since the positive braid $J_k(f)\Delta_k$ contains a positive half twist Δ_k, every crossing in the Lagrangian projection of Λ_f corresponds to a contractible Reeb chord; see [Casals and Ng 2022, Proposition 2.8]. If two affine permutations f and g have the same juggling

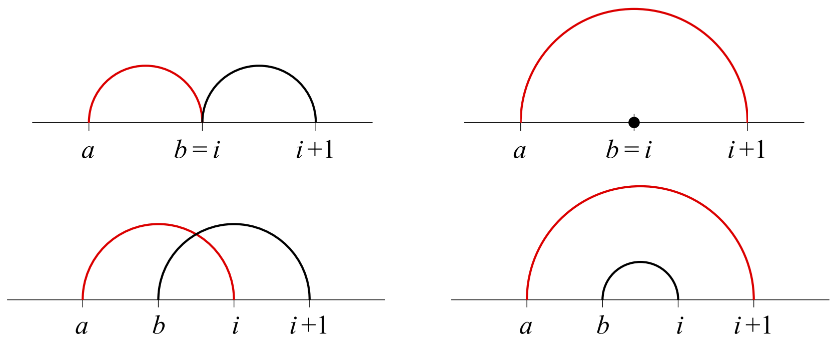

Figure 10. Top: Arcs in the juggling diagrams of f (left) and g (right) when $a < b = i$ where $f \lessdot g$. Bottom: Arcs in the juggling diagrams of f (left) and g (right) when $a < b < i$, where $f \lessdot g$.

braids $J_k(f) = J_k(g)$, then $\Lambda_f = \Lambda_g$ by definition. Suppose now that two affine permutations f and g have juggling braids $J_k(f)$ and $J_k(g)$ such that $J_k(f)$ has one more positive crossing x than $J_k(g)$. By Ng's resolution procedure, the crossing x in the front projection corresponds to a contractible Reeb chord of Λ_f, and we can perform a pinch move at x as in Figure 2 to obtain a Lagrangian saddle cobordism from Λ_g to Λ_f.

Let $f, g \in \mathrm{Bound}(k, n)$ with $f < g$. It suffices to assume $f \lessdot g$, that is, $g = \sigma_i \circ f$ or $g = f \circ \sigma_i$. Recall from Lemma 3.2 that h is a bounded affine permutation if and only if $-(-h)^{-1}$ is. Thus, because $g = f \circ \sigma_i$ is equivalent to $-(-f)^{-1} = \sigma_i \circ (-(-g)^{-1})$, it suffices to consider the case $g = \sigma_i \circ f$.

Namely, assume $g(a) = i + 1$, $g(b) = i$, $f(a) = i$ and $f(b) = i + 1$ for some $a, b, i \in \mathbb{Z}_{\geq 1}$ such that $a < b$. We show that $J_k(f)$ has one more positive crossing than $J_k(g)$ does or $J_k(f) = J_k(g)$. Therefore, there is either an orientable exact Lagrangian saddle cobordism from Λ_g to Λ_f, or the Legendrian links Λ_g and Λ_f are Legendrian isotopic and so are related by a trivial exact Lagrangian cobordism.

Since $g(b) = i$, we know $b \leq i$ so as $a < b$, we have $a < b \leq i$. If $b = i$, the respective juggling diagrams of f and g contain the arcs shown in Figure 10, top. Thus we see that the juggling braids $J_k(f)$ and $J_k(g)$ are equal as braids. If $b < i$, the juggling diagrams of f and g contain the arcs shown in Figure 10, bottom, from which we can immediately conclude that the juggling braid $J_k(f)$ has one more crossing than the juggling braid $J_k(g)$. □

Remark 6.2. In view of Theorem 5.8, a discussion on marked points in the construction of the exact Lagrangian cobordisms in the proof of Theorem 6.1 is warranted. Since both $J_k(f)\Delta_k$ and $J_k(g)\Delta_k$ are k-stranded braids, their Legendrian (-1)-closures are decorated with one marked point per strand of the underlying braid. Any trivial exact Lagrangian cobordism remains trivial when taking marked points into account. Any saddle cobordism induced by a pinch move will involve newly created marked points in order to retain functoriality of the associated Chekanov–Eliashberg dg-algebras with coefficients in $\mathbb{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$; see [Casals and Ng 2022, Section 3.5] and [Gao et al. 2024, Section 2.4]. For our purpose, we will ignore the marked points created by pinch moves by evaluating them to 1.

Theorem 6.3. *Given any path γ from f to g in* Bound(k, n), *the corresponding exact Lagrangian cobordism $L_\gamma(g, f)$ from the proof of Theorem 6.1 satisfies*

$$\chi(L_\gamma(g, f)) = \dim(\Pi_g^\circ) - \dim(\Pi_f^\circ) + \#\operatorname{Fix}(g) - \#\operatorname{Fix}(f),$$

where Π_f° is the open positroid stratum associated to f.

Proof. Let $\gamma = (f_1, \dots, f_t)$ be a sequence of bounded affine permutations such that each pair of adjacent bounded affine permutations are related by an affine transposition, in other words, $f_i \lessdot f_{i-1}$ for all $1 < i \leq t$. So $L_\gamma(f_t, f_1)$ is the corresponding Lagrangian cobordism. For each i, the dimensions of the respective positroid strata differ by 1. Then following the construction, the exact Lagrangian cobordism corresponding to $f_i \lessdot f_{i-1}$ is one of two things:

1. A trivial cobordism, when the change in the juggling diagrams $J_{f_{i-1}}$ to J_{f_i} is the creation of a fixed point. This contributes 1 to $\#\operatorname{Fix}(f_t) - \#\operatorname{Fix}(f_1)$.
2. A saddle cobordism corresponding to a single pinch move, when the change is the removal of a crossing. This contributes 1 to $\chi(L(\gamma))$.

Thus,

$$\dim(\Pi_{f_1}^\circ) - \dim(\Pi_{f_t}^\circ) = \chi(L(\gamma)) + \#\operatorname{Fix}(f_t) - \#\operatorname{Fix}(f_1). \qquad \square$$

Remark 6.4. Theorem 6.3 can also be proved using Corollary 5.7 and the work of Chantraine [2010, Theorem 1.2] which provides the change in the Thurston–Bennequin number for Legendrians related by an exact orientable Lagrangian cobordism.

7. Examples

In Example 7.1 we provide an example of Theorem 1.1 and in Example 7.2 a counterexample to the converse of Theorem 1.1.

Example 7.1. We consider a path $f_1 \lessdot \cdots \lessdot f_7$ in the poset Bound(3, 8) where the bounded affine permutations $f_1, \dots, f_7$ are defined as

$$f_1 = [5, 4, 7, 6, 8, 9, 10, 11], \quad f_2 = [5, 4, 8, 6, 7, 9, 10, 11],$$
$$f_3 = [5, 4, 8, 7, 6, 9, 10, 11], \quad f_4 = [6, 4, 8, 7, 5, 9, 10, 11],$$
$$f_5 = [6, 3, 8, 7, 5, 9, 10, 12], \quad f_6 = [6, 2, 8, 7, 5, 9, 11, 12],$$
$$f_7 = [7, 2, 8, 6, 5, 9, 11, 12].$$

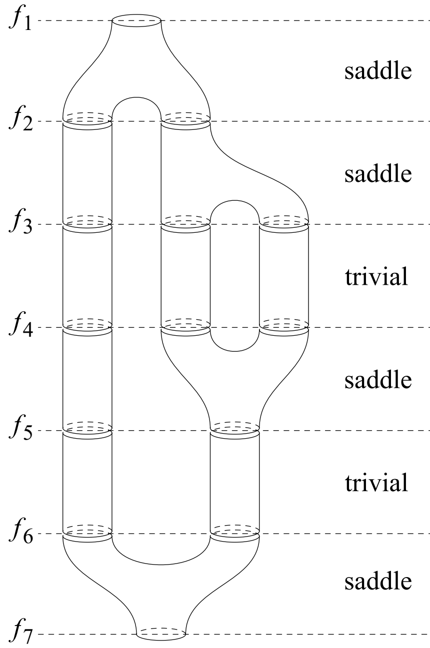

Figure 11. The exact Lagrangian cobordism from Λ_{f_7} to Λ_{f_1} corresponding to $f_1 \lessdot \cdots \lessdot f_7$.

Each f_i corresponds to a positroid stratum of Gr(3, 8) of codimension $i+1$ and a Legendrian link Λ_{f_i}, each of which is the (-1)-closure of the corresponding juggling braid $J_3(f_i)$. The corresponding juggling braids are

$$J_3(f_1) = (\sigma_1\sigma_2)^4\sigma_2\sigma_1\sigma_2, \quad J_3(f_2) = (\sigma_1\sigma_2)^3\sigma_2^2\sigma_1\sigma_2,$$
$$J_3(f_3) = (\sigma_1\sigma_2)^3\sigma_2\sigma_1\sigma_2, \quad J_3(f_4) = (\sigma_1\sigma_2)^3\sigma_2\sigma_1\sigma_2,$$
$$J_3(f_5) = (\sigma_1\sigma_2)^3\sigma_2\sigma_1, \quad J_3(f_6) = (\sigma_1\sigma_2)^3\sigma_2\sigma_1,$$
$$J_3(f_7) = (\sigma_1\sigma_2)^3\sigma_1.$$

We have depicted the corresponding composition of exact Lagrangian cobordisms in Figure 11.

As noted above we have codim $\Pi^\circ_{f_i} = i + 1$. Because we see that $\#\,\mathrm{Fix}(f_1) = 0$ and $\#\,\mathrm{Fix}(f_7) = 2$, Theorem 6.3 gives $\chi(L) = -4$, which correctly predicts that the exact Lagrangian cobordism depicted in Figure 11 has genus 2.

Example 7.2. We now show that the positroid links corresponding to two incomparable positroid strata can still be exact Lagrangian cobordant; this is the converse to Theorem 1.1.

Consider the two bounded affine permutations $f_1, f_2 \in \mathrm{Bound}(2, 6)$ defined by

$$f_1 := [3, 4, 5, 7, 8, 6] \quad \text{and} \quad f_2 := [3, 4, 7, 5, 6, 8].$$

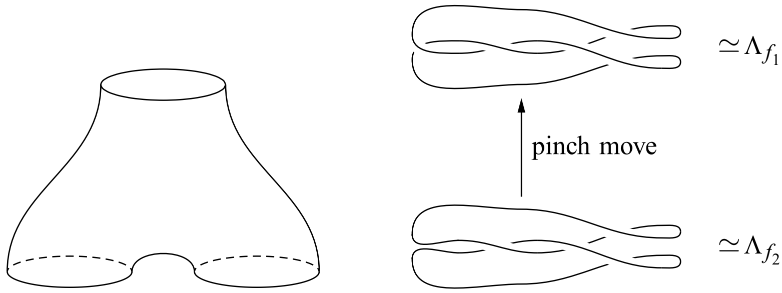

Figure 12. Pinch move giving a saddle cobordism from the Hopf link to the trefoil.

The corresponding juggling braids are $J_2(f_1) = \sigma_1^4$ and $J_2(f_2) = \sigma_1^3$. The two corresponding positroid links are the trefoil and the Hopf link, respectively. The two bounded affine permutations f_1 and f_2 correspond to two different positroid strata in Gr(2, 6) of dimension 6 and are therefore incomparable. However, there is an exact Lagrangian cobordism from Λ_{f_2} to Λ_{f_1} given by a saddle cobordism obtained by performing a pinch move at one of the crossings; see Figure 12.

Remark 7.3. In Example 7.2 we show that the braids σ_1^4 and σ_1^3 may appear as juggling braids of two incomparable bounded affine permutations. They also appear as the juggling braids $J_2(g_1)$ and $J_2(g_2)$, respectively, for $g_1, g_2 \in \text{Bound}(2, 5)$ defined as

$$g_1 := [3, 4, 5, 6, 7] \quad \text{and} \quad g_2 := [4, 3, 5, 6, 7],$$

which *are* comparable. Namely $g_1 \lessdot g_2$ since $g_2 = g_1 \circ \sigma_1$ and $\ell(g_1) = 0 < 1 = \ell(g_2)$.

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