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Integrability and renormalizability for the fully anisotropic SU(2) principal chiral field and its deformations

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Abstract: For the class of $1+1$ dimensional field theories referred to as the non-linear sigma models, there is known to be a deep connection between classical integrability and one-loop renormalizability. In this work, the phenomenon is reviewed on the example of the so-called fully anisotropic SU(2) Principal Chiral Field (PCF). Along the way, we discover a new classically integrable four parameter family of sigma models, which is obtained from the fully anisotropic SU(2) PCF by means of the Poisson-Lie deformation. The theory turns out to be one-loop renormalizable and the system of ODEs describing the flow of the four couplings is derived. Also provided are explicit analytical expressions for the full set of functionally independent first integrals (renormalization group invariants).

Keywords: Integrable Field Theories, Sigma Models

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Contents

1 Introduction

One of the spectacular instances of when ideas from physics and geometry come together is in the study of a class of field theories known as the Non Linear Sigma Models (NLSM). Mathematically, these are defined in terms of maps between two (pseudo-)Riemannian manifolds known as the worldsheet and the target space such that the classical equations of motion take the form of a generalized version of Laplace's equation [1]. In physics, one of the uses of NLSM is as low energy effective field theories with the choice of the target space being dictated by the symmetries of the problem. The first such proposal appeared in a paper of Gell-Mann and Levy [2]. They put forward the following Lagrangian density as an effective field theory of pions:

$$\mathcal{L} = \frac{1}{2}\,\eta^{ij}\,\partial_i\vec{n}\cdot\partial_j\vec{n} \qquad \text{with} \qquad |\vec{n}|^2 = \frac{1}{f^2}\,. \tag{1.1}$$

Here the last equation means that the four component field $\vec{n} = (n_1, n_2, n_3, n_4)$ is constrained to lie on the three dimensional round sphere whose radius coincides with $1/f$. Thus the target space is $\mathbb{S}^3$ equipped with the homogeneous metric while the worldsheet is four dimensional Minkowski spacetime $\mathbb{M}^{1,3}$. The field theory is known as the O(4) sigma model as it possesses O(4) symmetry — the group of isometries of the three-sphere. Ignoring global aspects, one may replace the latter by SU(2) × SU(2) which play the role of the vector and axial symmetries appearing in the 'chiral limit' of QCD. For this reason the model (1.1) is also referred to as the SU(2) principal chiral field.

The O(4) sigma model is rather special in 1 + 1 dimensional spacetime $\mathbb{M}^{1,1}$. In this case, as was pointed out by Polyakov, the Lagrangian (1.1) defines a renormalizable QFT. Following the traditional path-integral quantization, the model should be equipped with a UV cutoff Λ [3]. It was shown to one-loop order that a consistent removal of the UV divergences can be achieved if the bare coupling is given a dependence on the cutoff momentum, described by the RG flow equation [4]

$$\Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}(f^{-2}) = \frac{N-2}{2\pi}\,\hbar + O(\hbar^2)\,. \tag{1.2}$$

Here $\hbar$ stands for the dimensionless Planck constant while $N = 4$ (the computation was performed for the general $\mathrm{O}(N)$ sigma model with target space $\mathbb{S}^{N-1}$). Notice that in the continuous limit $\Lambda \to \infty$ the coupling constant f^2 approaches zero. In turn, the curvature of the sphere to which the fields $n_j(x^0, x^1)$ belong vanishes so that the theory becomes non-interacting. This phenomenon, known as asymptotic freedom, indicates consistency of the quantum field theory. As a result of the work of Polyakov and later Zamolodchikov and Zamolodchikov [5], who proposed the associated scattering theory, it is commonly believed that the $\mathrm{O}(N)$ sigma model in $1+1$ dimensions is a well defined (UV complete) QFT.

The renormalizability of general NLSM in $1+1$ dimensions was discussed in the work of Friedan [6]. He considered the class of theories where the Lagrangian density takes the form

$$\mathcal{L} = \frac{1}{2}\, G_{\mu\nu}(X)\, \eta^{ij} \partial_i X^\mu \partial_j X^\nu \,. \tag{1.3}$$

Here $G_{\mu\nu}(X)$ is the metric written in terms of local coordinates X^μ on the target space. The couplings are encoded in this metric so that the latter is taken to be dependent on the cutoff Λ. Extending the results of Ecker and Honerkamp [7], Friedan computed the RG flow equation to two loops. To the leading order in $\hbar$ it takes the form

$$\partial_\tau G_{\mu\nu} = -\hbar\, R_{\mu\nu} + O(\hbar^2)\,, \qquad\qquad \partial_\tau = -2\pi\Lambda\, \frac{\partial}{\partial\Lambda}\,, \tag{1.4}$$

where $R_{\mu\nu}$ is the Ricci tensor built from the metric. Without the $O(\hbar^2)$ term, (1.4) is usually referred to as the Ricci flow equation [8], which is a partial differential equation for $G_{\mu\nu} = G_{\mu\nu}(X\,|\,\tau)$. It found a remarkable application in mathematics in the proof of the Poincaré conjecture [9, 10].

The question of renormalizability can be addressed within a class of NLSM where the target space metric depends on a finite number of parameters. The simplest example is the $\mathrm{O}(N)$ sigma model whose target manifold belongs to the family of the $(N-1)$ dimensional round spheres, characterized by the radius $1/f$. In this case, the Ricci flow equation boils down to the ordinary differential equation (1.2). Another example is the Principal Chiral Field (PCF), where the target space is the group manifold of a simple Lie group G equipped with the left/right invariant metric. The latter is unique up to homothety and, in local coordinates, is defined by the relation

$$G_{\mu\nu}(X)\, \mathrm{d}X^\mu \mathrm{d}X^\nu = -\frac{1}{e^2}\, \langle \boldsymbol{U}^{-1}\, \mathrm{d}\boldsymbol{U}\,, \boldsymbol{U}^{-1}\, \mathrm{d}\boldsymbol{U}\rangle\,, \tag{1.5}$$

where $\boldsymbol{U} \in G$, e is the homothety parameter and the angular brackets $\langle\cdot,\cdot\rangle$ denote the Killing form in the Lie algebra of G.[1] The Ricci flow (1.4) implies

$$\partial_\tau(e^{-2}) = -\frac{1}{2}\, C_2\, \hbar + O(\hbar^2) \tag{1.6}$$

with C_2 being the value of the quadratic Casimir in the adjoint representation. This equation was essentially obtained in the original work of Polyakov [4], see also [3]. Notice that the SU(2) PCF coincides with the O(4) sigma model. In this case $C_2 = 2$, while (1.6) and (1.2) are the same provided that $e^2 \equiv 2f^2$.

[1] For a classical Lie group we take the Killing form to be the trace over the defining representation.

An example of an NLSM which is renormalizable within a two parametric family is the so-called anisotropic SU(2) PCF. In this case the SU(2) $\times$ SU(2) isometry of the target space is broken down to SU(2) $\times$ U(1) and the manifold is still topologically $\mathbb{S}^3$ but equipped with a certain asymmetric metric. The latter is given by

$$G_{\mu\nu}(X)\,\mathrm{d}X^\mu \mathrm{d}X^\nu = -\frac{1}{e^2}\,\langle \boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U}\,, \mathcal{O}(\boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U})\rangle\,, \tag{1.7}$$

where $\mathcal{O}$ is an operator acting from the Lie algebra $\mathfrak{su}(2)$ to itself depending on the additional deformation parameter r,

$$\mathcal{O}:\ \mathfrak{su}(2)\mapsto\mathfrak{su}(2)\,, \qquad\qquad \mathcal{O}=1+r\,P_3\,, \tag{1.8}$$

and P_3 projects onto the Cartan subalgebra. The Ricci flow equation reduces to a system of ordinary differential equations on e and r:

$$\begin{aligned} -\frac{1}{\hbar}\,\partial_\tau(e^{-2}) &= 1-r \\ -\frac{1}{\hbar}\,\partial_\tau r &= 2\,e^2 r\,(r+1)\,. \end{aligned} \tag{1.9}$$

In the domain $-1<r<0$, similar as with the SU(2) PCF, the theory is asymptotically free and it turns out to be a consistent QFT.

When the τ dependence of the metric, satisfying the Ricci flow equation, is contained in a finite number of parameters, the partial differential equation (1.4) reduces to a system of ordinary ones. From the point of view of physics, this means that the corresponding NLSM depends on a finite number of coupling constants and is one-loop renormalizable within this class. The construction of such solutions is difficult to achieve even when the dimension of the target manifold is low. Among the most impressive early results was the work of Fateev [11], who discovered a three parameter family of metrics solving the Ricci flow equation. The NLSM with this background is a two parameter deformation of the SU(2) PCF, which contains the anisotropic case as a subfamily. A guiding principle for exploring the class of renormalizable NLSM was formulated in the work [12]. It arose from the observation that all the above mentioned models turn out to be classically integrable field theories. It is now believed that there is a deep relation between classical integrability and one-loop renormalizability in $1+1$ dimensional sigma models.

The notion of classical integrability in $1+1$ dimensional field theory requires explanation. Recall that a mechanical system with d degrees of freedom is called integrable (in the Liouville sense) if it possesses d functionally independent Integrals of Motion (IM) in involution. This concept is difficult to extend to a field theory, where the number of degrees of freedom is infinite. A suitable paradigm of integrability in the case of $1+1$ dimensions arose from the works of the Princeton group [13] and was later developed in the papers of Lax [14] and Zakharov and Shabat [15]. A key ingredient is the existence of the so-called Zero Curvature Representation (ZCR) of the Euler-Lagrange equations of the classical field theory:

$$[\partial_i-\boldsymbol{A}_i,\ \partial_j-\boldsymbol{A}_j]=0\,. \tag{1.10}$$

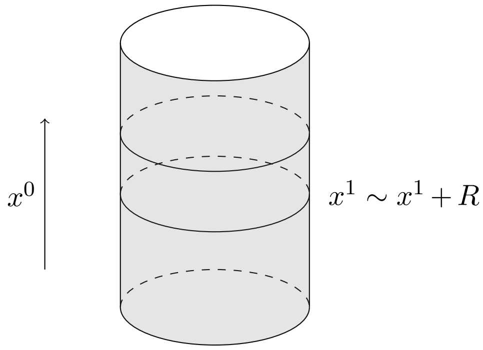

Figure 1. The integration contour for the Wilson loop can be moved freely along the cylinder.

Here $\boldsymbol{A}_i = \boldsymbol{A}_i(x^0, x^1|\lambda)$ is a Lie-algebra valued worldsheet connection which also depends on the auxiliary (spectral) parameter λ. The ZCR implies that the Wilson loops

$$T(\lambda) = \mathrm{Tr}\,\overleftarrow{\mathcal{P}}\exp\left(-\int_{\mathcal{C}} \mathrm{d}x^i\, \boldsymbol{A}_i\right), \tag{1.11}$$

where the trace is taken over some matrix representation of the Lie algebra, are unchanged under continuous deformations of the closed contour $\mathcal{C}$. If suitable boundary conditions are imposed, this can be used to generate IM. For instance, in the case when the worldsheet is the cylinder and the connection is single valued, the contour $\mathcal{C}$ may be chosen to be the equal-time slice at some x^0 as in figure 1. Then, it is easy to see that $T(\lambda)$ does not depend on the choice of x^0, i.e., it is an integral of motion. Due to the dependence on the arbitrary complex variable λ, $T(\lambda)$ constitutes a family of IM. The existence of these may provide a starting point for solving the classical equations of motion by applying the inverse scattering transform [16]. For this reason, we say that a $1+1$ dimensional classical field theory is integrable if it admits the ZCR.[2]

The theme of this paper is the interplay between classical integrability and one-loop renormalizability in sigma models. Its structure is as follows. Section 2 is devoted to a discussion of the so-called fully anisotropic SU(2) PCF, whose target space metric is given by

$$G_{\mu\nu}(X)\,\mathrm{d}X^\mu \mathrm{d}X^\nu = -2\langle \boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U}\,, \mathcal{O}(\boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U})\rangle\,, \qquad \mathcal{O} = I_1\,P_1 + I_2\,P_2 + I_3\,P_3\,. \tag{1.12}$$

Here P_a are projectors onto the basis $\mathtt{t}_a$ of the Lie algebra $\mathfrak{su}(2)$, which is taken to be orthogonal w.r.t. the Killing form. The theory is a two parameter deformation of the SU(2) PCF and it reduces to the latter when $I_1 = I_2 = I_3 = \frac{1}{2}\,e^{-2}$. In addition for the special case $I_1 = I_2$ it becomes the anisotropic SU(2) PCF, whose target space metric was presented above in eq. (1.7). We discuss the classical integrability of the model with metric (1.12). On the other hand, the latter is shown to be a solution of the Ricci flow equation for a certain τ dependence of the couplings $I_a = I_a(\tau)$. The corresponding system of ordinary differential equations is derived and its first integrals are obtained. In section 3 the concept of the Poisson-Lie deformation [17], which preserves integrability, is introduced. We apply it to the fully anisotropic SU(2) PCF and obtain a new classically integrable field theory depending

[2] Such a 'definition' of integrability does not guarantee that the equations of motion can be analytically solved in any sense. Thus, it is a much weaker notion than Liouville integrability in classical mechanics.

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on four parameters. It is argued that the resulting model is one-loop renormalizable. The system of ODEs for the τ dependence of the four couplings is presented and explicit analytical expressions for the renormalization group invariants are provided. The last section is devoted to a discussion. Among other things, it contains the formulae for the renormalization group invariants of the fully anisotropic SU(2) PCF with Wess-Zumino term.

2 Fully anisotropic SU(2) PCF

Following the lecture notes [18], let us gain some intuition about the fully anisotropic SU(2) PCF by considering its classical mechanics counterpart. It is obtained via 'dimensional reduction' where one restricts to field configurations that depend only on the spacetime variable x^0 so that $\boldsymbol{U} = \boldsymbol{U}(x^0)$. Then the Lagrangian density (1.3), (1.12) becomes

$$L = \sum_{a=1}^{3} \frac{I_a\,\omega_a^2}{2}\,, \tag{2.1}$$

where ω_a are defined through the relation

$$\boldsymbol{U}^{-1}\,\dot{\boldsymbol{U}} = -\mathrm{i}\,\sum_{a=1}^{3} \omega_a\,\mathtt{t}_a \tag{2.2}$$

and the dot stands for differentiation w.r.t. the time x^0. Also, the basis for the Lie algebra has been normalized such that

$$\langle \mathtt{t}_a, \mathtt{t}_b \rangle = \frac{1}{2}\,\delta_{ab} \qquad \text{and} \qquad [\mathtt{t}_a, \mathtt{t}_b] = \mathrm{i}\,\epsilon_{abc}\mathtt{t}_c \tag{2.3}$$

with ϵ_{abc} being the Levi-Civita symbol and summation over the repeated index is being assumed. It turns out that the Lagrangian (2.1) describes the free motion of a rigid body where the translational degrees of freedom have been ignored.

Recall that an arbitrary displacement of a rigid body is a composition of a translation and a rotation. For a free moving top, when the net external force is zero, one can without loss of generality consider the case when the centre of mass is at rest. Introduce two right handed coordinate systems called the fixed (laboratory) frame and moving frame, which are defined by the ordered set of unit vectors $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, respectively. The axes of the moving frame coincide with the principal axes of the rigid body w.r.t. the centre of mass. Then the orientation of the body is uniquely specified by a 3×3 special orthogonal matrix which relates the fixed and moving frames as in figure 2. Thus the configuration space of a rigid body with a fixed point coincides with the group manifold of SO(3). The matrix specifying the rotation can be identified with an SU(2) matrix $\boldsymbol{U}$ taken in the adjoint representation. Mathematically this is expressed as

$$\boldsymbol{E}_a \mathtt{t}_a = \boldsymbol{U}\,\boldsymbol{e}_a \mathtt{t}_a\,\boldsymbol{U}^{-1}\,, \tag{2.4}$$

where again summation over $a = 1, 2, 3$ is being assumed. The coefficients ω_a defined in (2.2) coincide with the projections of the instantaneous angular velocity $\boldsymbol{\omega}$ along the principal axes. This can be seen by differentiating both sides of (2.4) w.r.t. time and comparing the result with $\dot{\boldsymbol{e}}_a = \boldsymbol{\omega} \times \boldsymbol{e}_a$.

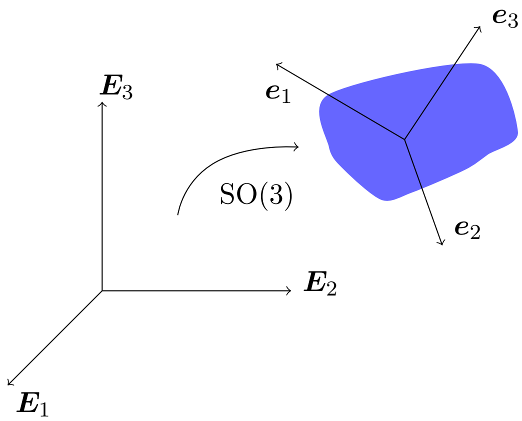

Figure 2. The orientation of the rigid body is uniquely specified by the $3D$ special orthogonal matrix that relates the moving frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ to the fixed frame $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$. The axes of the moving frame are chosen to coincide with the principal axes of inertia.

The classical mechanics system governed by the Lagrangian (2.1) is called the Euler top. The parameters I_a, which were introduced originally as formal couplings in (1.12), coincide with the principal moments of inertia. Notice that the Lagrangian is built from $\boldsymbol{U}^{-1}\dot{\boldsymbol{U}}$ which belongs to the Lie algebra and hence is insensitive to the difference between the groups SU(2) and SO(3).[3]

The Euler top is a textbook example of a Liouville integrable system. The IM that satisfy the conditions of Liouville's theorem are the Hamiltonian H and two more which are built from the angular momentum $\boldsymbol{M}$:

$$H = \sum_{a=1}^{3} \frac{I_a\, \omega_a^2}{2}\,, \qquad\qquad \boldsymbol{M} = \sum_{a=1}^{3} I_a \omega_a\, \boldsymbol{e}_a\,. \tag{2.5}$$

For a free moving body the angular momentum is conserved, i.e., $\dot{\boldsymbol{M}} = 0$. On the other hand, the total time derivative $\dot{\boldsymbol{M}}$ can be written in terms of the canonical Poisson bracket as $\{H, \boldsymbol{M}\}$. Hence, the classical observable $\boldsymbol{M}$ Poisson commutes with the Hamiltonian. This way, the three functionally independent involutive Integrals of Motion may be taken to be

$$H\,, \qquad M_Z \equiv \boldsymbol{M}\cdot\boldsymbol{E}_3 \qquad \text{and} \qquad \boldsymbol{M}^2 = \sum_{a=1}^{3} I_a^2\, \omega_a^2\,. \tag{2.6}$$

It follows from Liouville's theorem that the equations of motion for the Euler top can be integrated in quadratures. The solution is discussed in any standard textbook on classical mechanics see, e.g., [19].

The rigid body with two of the principal moments of inertia equal $I_1 = I_2 \equiv I$ is usually referred to as the symmetric top. In this case the Lagrangian (2.1) possesses invariance w.r.t. rotations about the axis $\boldsymbol{e}_3$. For the symmetric top it is convenient to choose the three

[3] Topologically, the special unitary group SU(2) is the three sphere $\mathbb{S}^3$, while the special orthogonal group SO(3) is the three dimensional real projective space $\mathbb{RP}^3$. The latter coincides with $\mathbb{S}^3$ with antipodal points $\pm\vec{n} \in \mathbb{S}^3$ identified.

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functionally independent, involutive IM to be $\boldsymbol{M}^2, M_Z$ and $M_3 \equiv \boldsymbol{M} \cdot \boldsymbol{e}_3$. Notice that the Hamiltonian is given in terms of these as

$$H = \frac{1}{2I}\,\boldsymbol{M}^2 + \left(\frac{1}{2I_3} - \frac{1}{2I}\right)\,M_3^2 \qquad\qquad (I_1 = I_2 \equiv I)\,. \tag{2.7}$$

The case $I_1 = I_2 = I_3 \equiv I$ is known as the spherical top and the Hamiltonian is proportional to $\boldsymbol{M}^2$. The field theory generalization of the symmetric top is the anisotropic SU(2) PCF (1.3), (1.7), while that of the spherical top is the SU(2) PCF (1.3), (1.5).

Remarkably, the fully anisotropic SU(2) PCF is also an integrable field theory according to the technical definition given in the introduction. Namely, the equations of motion for the model admit the Zero Curvature Representation (1.10). To demonstrate the integrability, it is useful to introduce the currents J_i^a via the formula:

$$\boldsymbol{U}^{-1}\,\partial_i\boldsymbol{U} = -\mathrm{i}\,\sum_{a=1}^{3} J_i^a\,\mathtt{t}_a \qquad\qquad (i = 0, 1)\,. \tag{2.8}$$

Then the Euler-Lagrange equations for the model (1.3), (1.12) can be written as follows:

$$\partial_- J_+^a + \partial_+ J_-^a = \frac{I_b - I_c}{I_a}\,(J_+^b\,J_-^c + J_+^c\,J_-^b)\,, \tag{2.9}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$ while

$$\partial_\pm = \frac{1}{2}\,(\partial_0 \pm \partial_1)\,, \qquad\qquad J_\pm^a = \frac{1}{2}\,(J_0^a \pm J_1^a)\,. \tag{2.10}$$

Note also the kinematic relations (Bianchi identities) which follow directly from the definition (2.8):

$$\partial_- J_+^a - \partial_+ J_-^a = \epsilon_{abc}\,J_+^b J_-^c\,. \tag{2.11}$$

The worldsheet connection for the fully anisotropic SU(2) PCF is rather complicated. For this reason we give it first for the case $I_1 = I_2 = I_3 = \frac{1}{2}\,e^{-2}$ which corresponds to the SU(2) PCF. Then the equations of motion (2.9) simplify greatly since the term in the r.h.s. vanishes. The worldsheet connection $\boldsymbol{A}_\pm$ reads as

$$\boldsymbol{A}_\pm = \frac{\mathrm{i}\,J_\pm^a\,\mathtt{t}_a}{1 \pm \lambda} \qquad\qquad (I_1 = I_2 = I_3) \tag{2.12}$$

and one can easily check that as a consequence of eqs. (2.9) and (2.11),

$$[\partial_+ - \boldsymbol{A}_+,\ \partial_- - \boldsymbol{A}_-] = 0\,. \tag{2.13}$$

This ZCR was first proposed in the work [20] and is valid for the sigma model associated with any simple Lie group G with $\mathrm{i}J_\pm^a\,\mathtt{t}_a$ replaced by $-\boldsymbol{U}^{-1}\,\partial_\pm\boldsymbol{U}$.

The ZCR for the general case with $I_1 \neq I_2 \neq I_3$ was found in [21] and presented in a slightly different form in ref. [22]. In the following, the conventions of the latter paper will be used. To write the result, we swap the two independent combinations of (I_1, I_2, I_3) that enter into the equations of motion for m and ν according to

$$m = \frac{I_2\,(I_1 - I_3)}{I_3\,(I_1 - I_2)}\,, \qquad\qquad \mathrm{cn}^2(\nu, m) = \frac{I_1}{I_2}\,, \tag{2.14}$$

where $\mathrm{cn}(\nu, m)$ is the Jacobi elliptic function with *the parameter* m. Together with sn and dn, it satisfies the relations

$$\mathrm{sn}^2(\nu, m) + \mathrm{cn}^2(\nu, m) = 1\,, \qquad m\,\mathrm{sn}^2(\nu, m) + \mathrm{dn}^2(\nu, m) = 1\,. \tag{2.15}$$

The flat worldsheet connection reads explicitly as

$$\boldsymbol{A}_\pm = \mathrm{i}\sum_{a=1}^{3} w_a(\nu \mp \lambda) J_\pm^a\, \mathtt{t}_a\,, \tag{2.16}$$

where

$$w_1(\lambda) = \frac{\mathrm{sn}(\nu, m)}{\mathrm{sn}(\lambda, m)}\,, \qquad w_2(\lambda) = \frac{\mathrm{sn}(\nu, m)}{\mathrm{cn}(\nu, m)}\frac{\mathrm{cn}(\lambda, m)}{\mathrm{sn}(\lambda, m)}\,, \qquad w_3(\lambda) = \frac{\mathrm{sn}(\nu, m)}{\mathrm{dn}(\nu, m)}\frac{\mathrm{dn}(\lambda, m)}{\mathrm{sn}(\lambda, m)}\,. \tag{2.17}$$

In order to explore the one-loop renormalizability of the fully anisotropic SU(2) PCF, we turn to the analysis of the Ricci flow equation (1.4). It requires one to calculate the Ricci tensor $R_{\mu\nu}$ corresponding to the target space metric $G_{\mu\nu}$ given in (1.12). The computation is straightforward and we do not present it here. Instead, we mention the identity:

$$R_{\mu\nu} = \sum_{a=1}^{3} \frac{(I_a - I_b + I_c)(I_a + I_b - I_c)}{2 I_b I_c} \frac{\partial}{\partial I_a} G_{\mu\nu}\,, \tag{2.18}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Then it follows that the Ricci flow equation is satisfied if the couplings I_a are assigned a τ dependence such that (see also refs. [23, 24])

$$-\frac{1}{\hbar}\partial_\tau(I_a\, I_b) = I_a + I_b - I_c\,, \qquad (a, b, c) = \mathtt{perm}(1, 2, 3)\,. \tag{2.19}$$

This constitutes a set of coupled nonlinear ordinary differential equations describing the flow. Notice that for $I_1 = I_2 = \frac{1}{2e^2}$ and $I_3 = \frac{1+r}{2e^2}$ one recovers the Ricci flow equations for the anisotropic SU(2) PCF (1.9). The latter reduce to the ones for the SU(2) PCF (1.6) with $C_2 = 2$ upon setting $r = 0$.

We found that the system (2.19) possesses two Liouvillian first integrals.[4] They are given by

$$Q_1 = \frac{K(1-m) - (1-p)E(1-m)}{(1-p)E(m) + pK(m)}\,, \qquad Q_2 = \frac{I_1^2\left((p-1)\,E(m) - p\,K(m)\right)^2}{p\,(p-1)\,(p\,m - m + 1)}\,. \tag{2.20}$$

Here $K(m)$ and $E(m)$ stand for the complete elliptic integrals of the first and second kind,

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - m\,\sin^2\theta}}\,, \qquad E(m) = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta\,\sqrt{1 - m\,\sin^2\theta}\,, \tag{2.21}$$

the parameter m is the same as in (2.14), while p coincides with $\mathrm{cn}^2(\nu, m)$ from that formula, i.e.,

$$m = \frac{I_2\,(I_1 - I_3)}{I_3\,(I_1 - I_2)}\,, \qquad p = \frac{I_1}{I_2}\,. \tag{2.22}$$

[4]Liouvillian first integrals are those that are expressed in quadratures in the dependent variables of the differential equation.

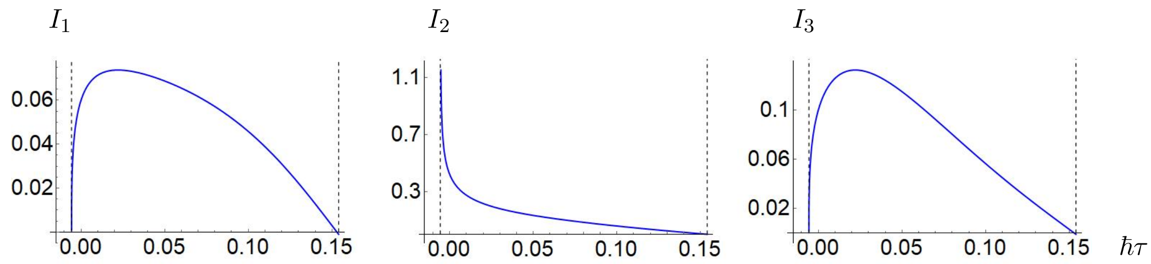

Figure 3. The evolution of I_1, I_2 and I_3 as functions of $\hbar\tau$. The initial conditions at $\tau = 0$ were chosen to be $I_1(0) = 0.06,\ I_2(0) = 0.42,\ I_3(0) = 0.10$. The flow remains real and non-singular in the interval $\tau \in (\tau_{\rm min}, \tau_{\rm max})$ with $\hbar\tau_{\rm min} = -0.006$ and $\hbar\tau_{\rm max} = 0.154$ which is marked by the dashed lines.

The expression (2.20) for the first integrals is one of the original results of this paper.[5] After it was obtained, we discovered that the system of differential equations (2.19) had been introduced, in a slightly different form, in the work of Darboux [26]. Its solution was discussed in refs. [27, 28].

The flow of the couplings I_a as a function of τ can be analyzed numerically. The typical behaviour, for generic initial conditions such that all I_a at $\tau = 0$ are positive and different, is presented in figure 3. One observes from the figure that the solution of (2.19), i.e., the Ricci flow equation, remains real and non-singular only within the finite interval $\tau \in (\tau_{\rm min}, \tau_{\rm max})$. At the end points one of the couplings goes to zero so that the curvature of the target space blows up. As a result, the one-loop approximation is no longer valid and the perturbative analysis is not sufficient to explore whether or not the model can be defined as a consistent (UV complete) QFT.

There exists another three parameter family of deformations of the three dimensional round sphere (1.5). It is the one mentioned in the introduction that was proposed by Fateev in ref. [11]. His metric, depending on (e^2, r, l), can be written as

$$G_{\mu\nu}(X)\,{\rm d}X^\mu {\rm d}X^\nu = -\frac{(1+r)(1+l)}{e^2}\ \frac{\langle \boldsymbol{U}^{-1}\,{\rm d}\boldsymbol{U}\,,\,\mathcal{O}(\boldsymbol{U}^{-1}\,{\rm d}\boldsymbol{U})\rangle}{(1+r)(1+l) - 4rl\,\big(\langle \boldsymbol{U}\mathtt{t}_3\boldsymbol{U}^{-1}, \mathtt{t}_3\rangle\big)^2}\,. \tag{2.23}$$

Here the operator $\mathcal{O}$, acting on the Lie algebra, is given by

$$\mathcal{O} = 1 + r\,P_3 + l\,{\rm Ad}_{\boldsymbol{U}} \circ P_3 \circ {\rm Ad}_{\boldsymbol{U}}^{-1}\,, \tag{2.24}$$

where ${\rm Ad}_{\boldsymbol{U}}$ stands for the adjoint action of the group:

$${\rm Ad}_{\boldsymbol{U}}\,\mathtt{x} = \boldsymbol{U}^{-1}\mathtt{x}\,\boldsymbol{U}\,, \qquad\qquad \mathtt{x} \in \mathfrak{su}(2)\,. \tag{2.25}$$

The Ricci flow equation (1.4) leads to the system of ordinary differential equations for the

[5] A set of first integrals of the system (2.19), similar to (2.20), have also appeared in the recent work [25]. Their results and the ones of our study were achieved independently of each other.

three parameters:

$$
\begin{aligned}
-\frac{1}{\hbar}\,\partial_\tau l &= \frac{2\,e^2\,l\,(1+l+r)}{1+r} \\
-\frac{1}{\hbar}\,\partial_\tau r &= \frac{2\,e^2\,r\,(1+l+r)}{1+l} \\
-\frac{1}{\hbar}\,\partial_\tau (e^{-2}) &= \frac{(1-l-r)(1+l+r)}{(1+l)(1+r)}\,.
\end{aligned}
\tag{2.26}
$$

Notice that for $l=0$ the metric (2.23) becomes the one for the anisotropic SU(2) PCF (1.7), while the above system of differential equations reduces to (1.9).

A remarkable feature of (2.26) is that it possesses solutions where $e^2(\tau)$, $r(\tau)$, $l(\tau)$ are real and non-singular on the half infinite line $(-\infty, \tau_{\max})$ with some real $\tau_{\max}$. In particular, this always happens when the couplings r and l are restricted as $-1 < r(\tau), l(\tau) < 0$. Such solutions of the Ricci flow equation, which can be continued to infinite negative τ, are called 'ancient'. That (2.26) admits ancient solutions suggests that the corresponding NLSM is a consistent QFT. The factorized scattering theory for the model was proposed in ref. [11].

The NLSM with metric (2.23) is an integrable classical field theory. The ZCR for the Euler-Lagrange equations was originally obtained in the work [12]. This way, the Fateev model provides an additional example of the link between integrability and one-loop renormalizability in sigma models.

3 Poisson-Lie deformation

The models discussed above illustrate the connection between integrable NLSM and solutions of the Ricci flow equation. This can be used as a guiding principle for constructing new multiparametric families of metrics that satisfy the Ricci flow. Here we will discuss the so-called Poisson-Lie deformation of integrable NLSM. Such a deformation preserves integrability and allows one to obtain new solutions of the Ricci flow equation. We first illustrate the idea by showing that the anisotropic SU(2) PCF can be obtained as the Poisson-Lie deformation of the SU(2) PCF [17]. Then, a new integrable model is constructed by deforming the fully anisotropic SU(2) PCF.

3.1 Poisson-Lie deformation of PCF

To explain the Poisson-Lie deformation, we start from the Hamiltonian formulation of the model. The latter, in the case of the SU(2) PCF, can be described using the currents J_i^a (2.8). It follows from the Lagrangian (1.5), (1.3) that they form a closed Poisson algebra [16]:[6]

$$
\begin{aligned}
\{J_0^a(x),\, J_0^b(y)\} &= \epsilon_{abc}\; J_0^c(x)\,\delta(x-y) \\
\{J_0^a(x),\, J_1^b(y)\} &= \epsilon_{abc}\; J_1^c(x)\,\delta(x-y) - \delta^{ab}\,\delta'(x-y) \\
\{J_1^a(x),\, J_1^b(y)\} &= 0\,.
\end{aligned}
\tag{3.1}
$$

[6] In our discussion of the Poisson-Lie deformation we set $e^2=1$. Since it appears in an overall factor multiplying the Lagrangian, this has no effect on the classical equations of motion.

These are understood to be equal-time relations with $x^0 = y^0$, while $x \equiv x^1$ and $y \equiv y^1$ are the space coordinates (the dependence of the currents on the time variable has been suppressed). The Hamiltonian is obtained by means of the Legendre transform and is given by

$$H = \frac{1}{2} \int \mathrm{d}x \sum_{a=1}^{3} \left(J_0^a\, J_0^a + J_1^a\, J_1^a \right) . \tag{3.2}$$

One can check that the Hamiltonian equations of motion, $\dot{O} = \{H, O\}$, for the currents are equivalent to eqs. (2.9) and (2.11) with $I_1 = I_2 = I_3$, i.e.,

$$\partial_- J_+^a = \frac{1}{2}\, \epsilon_{abc}\, J_+^b J_-^c \,, \qquad\qquad \partial_+ J_-^a = \frac{1}{2}\, \epsilon_{abc}\, J_-^b J_+^c \,. \tag{3.3}$$

The Poisson algebra (3.1) admits a certain deformation which preserves its defining properties, namely, skew-symmetry, the Jacobi and Leibniz identities. The deformed Poisson bracket relations read explicitly as

$$\begin{aligned} \{\tilde{J}_0^a(x),\, \tilde{J}_0^b(y)\} &= \frac{1}{1+r}\, \epsilon_{abc}\, \tilde{J}_0^c(x)\, \delta(x-y) \\ \{\tilde{J}_0^a(x),\, \tilde{J}_1^b(y)\} &= \frac{1}{1+r}\, \epsilon_{abc}\, \tilde{J}_1^c(x)\, \delta(x-y) - \delta^{ab}\, \delta'(x-y) \\ \{\tilde{J}_1^a(x),\, \tilde{J}_1^b(y)\} &= -\frac{r}{1+r}\, \epsilon_{abc}\, \tilde{J}_0^c(x)\, \delta(x-y) \,. \end{aligned} \tag{3.4}$$

Here r plays the role of the deformation parameter and we switch the notation from J_i^a to $\tilde{J}_i^a$ as the above Poisson brackets will be associated with a different classical field theory. Remarkably, with the same form of the Hamiltonian as (3.2), i.e.,

$$\tilde{H} = \frac{1}{2} \int \mathrm{d}x \sum_{a=1}^{3} \left(\tilde{J}_0^a\, \tilde{J}_0^a + \tilde{J}_1^a\, \tilde{J}_1^a \right) \tag{3.5}$$

the equations of motion do not depend on the deformation parameter. Namely, they coincide with (3.3) upon replacing $J_\pm^a$ by $\tilde{J}_\pm^a = \frac{1}{2}(\tilde{J}_0^a \pm \tilde{J}_1^a)$. This means that the Hamiltonian system defined through (3.4) and (3.5) is integrable by construction. The corresponding flat connection entering into the ZCR takes the same form as for the SU(2) PCF (2.12) but written in terms of the currents $\tilde{J}_\pm^a$:

$$\boldsymbol{A}_\pm = \frac{\mathrm{i}\, \tilde{J}_\pm^a\, \mathtt{t}_a}{1 \pm \lambda} \,. \tag{3.6}$$

The obtained classical field theory is called the Poisson-Lie deformation of the SU(2) PCF. The final and technically most involved step of the procedure is to derive the Lagrangian of the deformed model.

It is well known in classical mechanics how to get from the Hamiltonian to the Lagrangian picture. Consider a mechanical system with a finite number of degrees of freedom d. The Poisson brackets are defined on the algebra of functions on the $2d$-dimensional phase space. In local coordinates $(z^1, \ldots, z^{2d})$ they are given by

$$\{f, g\} = \Omega^{AB}\, \frac{\partial f}{\partial z^A}\, \frac{\partial g}{\partial z^B} \,. \tag{3.7}$$

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Since the Poisson brackets are assumed to be non-degenerate, the inverse of the contravariant tensor Ω^{AB} exists and we will denote it as Ω_{AB}. Due to skew-symmetry of the Poisson brackets, the covariant tensor Ω_{AB} is antisymmetric, i.e., it defines a two-form as $\Omega = \Omega_{AB}\, \mathrm{d}z^A \wedge \mathrm{d}z^B$. Moreover, the Jacobi identity implies that the form is closed, $\mathrm{d}\Omega = 0$. This allows one to write Ω as an exact form, $\Omega = \mathrm{d}\alpha$, at least locally. The action is expressed in terms of the one-form α and the Hamiltonian as

$$S = \int (\alpha - H\, \mathrm{d}t) \tag{3.8}$$

with the integral being taken over a path in the phase space parameterized by the time t. According to the Darboux theorem there exists (locally) a set of canonical variables $(q^1, \ldots, q^d, p_1, \ldots p_d)$ such that $\alpha = \sum_{m=1}^{d} p_m\, \mathrm{d}q^m$. Then the Lagrangian associated with the action (3.8) is given by

$$L = \sum_{m=1}^{d} p_m\, \dot{q}^m - H\,. \tag{3.9}$$

This can be interpreted as the Legendre transform of H where the canonical momenta p_m are replaced by $\dot{q}^m$ as the independent variables.

In order to apply the above procedure to the infinite dimensional Hamiltonian structure (3.4), (3.5), it is useful to realize the Poisson algebra in terms of the fields, similar to the canonical variables p_m and q^m in the finite dimensional case. For this reason we introduce local coordinates X^μ on the group manifold and the corresponding canonical momentum densities Π_μ. They obey the Poisson bracket relations

$$\{\Pi_\mu(x), X^\nu(y)\} = \delta^\nu_\mu\, \delta(x-y)\,, \qquad \{X^\mu(x), X^\nu(y)\} = \{\Pi_\mu(x), \Pi_\nu(y)\} = 0\,. \tag{3.10}$$

In the case $r = 0$, when (3.4) becomes the undeformed algebra (3.1), the currents

$$\boldsymbol{K}_i \equiv \tilde{J}^a_i\, \mathtt{t}_a\big|_{r=0} \qquad (i = 0, 1) \tag{3.11}$$

can be expressed in terms of the canonical fields in the following way; first, define the 3×3 matrix $E^a{}_\mu$ through the relation

$$\mathrm{d}\boldsymbol{U}\, \boldsymbol{U}^{-1} = \mathrm{i}\, E^a{}_\mu\, \mathrm{d}X^\mu\, \mathtt{t}_a\,. \tag{3.12}$$

Its inverse will be denoted by $E^{\mu a}$ so that $E^a{}_\mu E^{\mu b} = \delta^{ab}$. Then with the choice

$$\boldsymbol{K}_0 = E^{\mu a}\, \Pi_\mu\, \mathtt{t}_a, \qquad \boldsymbol{K}_1 = E^a{}_\mu\, \partial_1 X^\mu\, \mathtt{t}_a = -\mathrm{i}\, \partial_1 \boldsymbol{U}\, \boldsymbol{U}^{-1} \tag{3.13}$$

one can check via a direct computation that the Poisson algebra (3.1) with J^a_i replaced by the components of $\boldsymbol{K}_i$ is satisfied. In fact, the r.h.s. of the first equation in (3.13) is just $-\mathrm{i}\, \partial_0 \boldsymbol{U}\, \boldsymbol{U}^{-1}$ written in terms of the canonical fields for the PCF.[7]

For general $r \neq 0$ one should first apply the linear transformation

$$\mathcal{I}^a = \frac{1+r}{2\sqrt{r}} \left(\sqrt{r}\, \tilde{J}^a_0 + \mathrm{i}\, \tilde{J}^a_1\right), \qquad \mathcal{J}^a = \frac{1+r}{2\sqrt{r}} \left(\sqrt{r}\, \tilde{J}^a_0 - \mathrm{i}\, \tilde{J}^a_1\right). \tag{3.14}$$

[7] We realise the algebra (3.1) using the 'left' currents $\boldsymbol{K}_i = -\mathrm{i}\, \partial_i \boldsymbol{U}\, \boldsymbol{U}^{-1}$ rather than the 'right' ones $J^a_i \mathtt{t}_a = \mathrm{i}\, \boldsymbol{U}^{-1}\, \partial_i \boldsymbol{U}$ (2.8) for future convenience. The latter obey the same Poisson bracket relations (3.1).

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This brings the closed Poisson algebra (3.4) to the form:

$$\begin{aligned}
\{\mathcal{I}^a(x),\,\mathcal{I}^b(y)\} &= \epsilon_{abc}\,\mathcal{I}^c(x)\,\delta(x-y) - k\,\delta_{ab}\,\delta'(x-y)\\
\{\mathcal{J}^a(x),\,\mathcal{J}^b(y)\} &= \epsilon_{abc}\,\mathcal{J}^c(x)\,\delta(x-y) + k\,\delta_{ab}\,\delta'(x-y)\\
\{\mathcal{I}^a(x),\,\mathcal{J}^b(y)\} &= 0\,,
\end{aligned} \tag{3.15}$$

where

$$k = \mathrm{i}\,\frac{(1+r)^2}{2\sqrt{r}}\,, \tag{3.16}$$

which is a direct sum of two independent so-called SU(2) current algebras. It turns out that the Poisson algebra generated by $\mathcal{I}^a$ and $\mathcal{J}^a$ can be formally realised in terms of the currents $\boldsymbol{K}_i$ (3.13) as well as the group valued field $\boldsymbol{U} \in \mathrm{SU}(2)$. The explicit formula, along with its verification, is contained in ref. [29] and is given by

$$\begin{aligned}
\mathcal{I}^a\,\mathtt{t}_a &= \frac{1}{2}\left(1 - \mathrm{i}\,\mathrm{Ad}_{\boldsymbol{U}}^{-1}\circ R\circ \mathrm{Ad}_{\boldsymbol{U}}\right)\boldsymbol{K}_0 + k\,\boldsymbol{K}_1\\
\mathcal{J}^a\,\mathtt{t}_a &= \frac{1}{2}\left(1 + \mathrm{i}\,\mathrm{Ad}_{\boldsymbol{U}}^{-1}\circ R\circ \mathrm{Ad}_{\boldsymbol{U}}\right)\boldsymbol{K}_0 - k\,\boldsymbol{K}_1\,.
\end{aligned} \tag{3.17}$$

Here $\mathrm{Ad}_{\boldsymbol{U}}$ stands for the adjoint action of the group, see eq. (2.25), while the linear operator $R : \mathfrak{su}(2) \mapsto \mathfrak{su}(2)$ is defined via its action on the generators as

$$R(\mathtt{t}_1) = \mathtt{t}_2\,, \qquad R(\mathtt{t}_2) = -\mathtt{t}_1\,, \qquad R(\mathtt{t}_3) = 0\,. \tag{3.18}$$

Formulae (3.14), (3.17) and (3.13) allow one to realize the currents $\tilde{J}_0^a$ and $\tilde{J}_1^a$, satisfying the Poisson bracket relations (3.4), through the canonical fields (3.10). The corresponding expression for the Hamiltonian follows from (3.5). In the basis of canonical variables the construction of the Lagrangian is straightforward and is the field theory analogue of the Lengedre transform (3.9). Applying the procedure, where Π_μ maps to $\dot{X}^\mu = \{\tilde{H}, X^\mu\}$, one arrives at the Lagrangian density

$$\mathcal{L} = -\frac{1+r}{e^2}\,\big\langle \boldsymbol{U}^{-1}\,\partial_+\boldsymbol{U}\,,\mathcal{O}\,(\boldsymbol{U}^{-1}\,\partial_-\boldsymbol{U})\big\rangle \qquad \text{with} \qquad \mathcal{O} = \big(1-\sqrt{r}\,R\big)^{-1}. \tag{3.19}$$

Here the dependence on e^2 was restored and we performed the substitution $e^2 \mapsto (1+r)\,e^2$ to keep with the conventions of section 1.

At first glance, in local coordinates, $\mathcal{L}$ can not be written in the form (1.3). Instead, the latter should be modified as

$$\mathcal{L} = 2\,G_{\mu\nu}(X)\,\partial_+X^\mu\,\partial_-X^\nu - B_{\mu\nu}(X)\,\big(\partial_+X^\mu\,\partial_-X^\nu - \partial_-X^\mu\,\partial_+X^\nu\big)\,. \tag{3.20}$$

Here the last term is not invariant w.r.t. the parity transformation $x^1 \mapsto -x^1$, i.e., $\partial_\pm \mapsto \partial_\mp$ and comes about because the Lagrangian density (3.19) is not either. Models of this type motivate a generalization of the NLSM where the target space is additionally equipped with a two form $B = B_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu$ known as the B-field [30]. It turns out that in the SU(2) case the B-field corresponding to $\mathcal{L}$ (3.19) is a closed form (in fact, exact). As a result, the term proportional to $B_{\mu\nu}$ in (3.20) is a total derivative and has no effect on the Euler-Lagrange equations. This way, for the SU(2) case, the obtained sigma model is equivalently described by (1.3) where

$$G_{\mu\nu}(X)\,\mathrm{d}X^\mu\mathrm{d}X^\nu = -\frac{1+r}{e^2}\,\langle \boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U}\,,\mathcal{O}_{\mathrm{sym}}\,(\boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U})\rangle \qquad \text{with} \qquad \mathcal{O}_{\mathrm{sym}} = \big(1+r-r\,P_3\big)^{-1} \tag{3.21}$$

and $P_3 = 1 + R^2 \in \mathrm{End}(\mathfrak{su}(2))$ stands for the projector on the Cartan subalgebra generated by $\mathtt{t}_3$. This way we arrive at the metric of the anisotropic SU(2) PCF (1.7).

It was discussed in section 1 that the anisotropic SU(2) PCF is a integrable classical field theory. Having established that the model is a Poisson-Lie deformation of the SU(2) PCF, we obtain a way to derive the Zero Curvature Representation for the classical equations of motion. Namely, the flat connection is given by (3.6) where the currents $\tilde{J}^a_\pm = \frac{1}{2}(\tilde{J}^a_0 \pm \tilde{J}^a_1)$ entering therein read as

$$\tilde{J}^a_\pm \mathtt{t}_a = (1+r)\,\mathrm{Ad}_{\boldsymbol{U}} \circ\, \big(1 \pm \sqrt{r}\,R\big)\,\big(\partial_\pm \boldsymbol{U}\,\boldsymbol{U}^{-1}\big)\,. \tag{3.22}$$

Indeed, as it follows from the Euler-Lagrange equations for the model (3.19),

$$\partial_- \tilde{J}^a_+ = \frac{1}{2}\,\epsilon_{abc}\,\tilde{J}^b_+\,\tilde{J}^c_-\,, \qquad\qquad \partial_+ \tilde{J}^a_- = \frac{1}{2}\,\epsilon_{abc}\,\tilde{J}^b_-\,\tilde{J}^c_+\,. \tag{3.23}$$

The following comment is in order here. The anisotropic SU(2) PCF admits an integrable generalization, where $\boldsymbol{U}$ belongs to an arbitrary simple Lie group G. The Lagrangian is still given by (3.19) with R being a certain linear operator which is usually referred to as the Yang-Baxter operator. It acts on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and is required to satisfy a skew symmetry condition and the so-called modified Yang-Baxter equation [31]. A possible choice obeying the two properties is specified using the Cartan-Weyl decomposition of the simple Lie algebra, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{h}$ stands for the Cartan subalgebra and $\mathfrak{n}_\pm$ are the nilpotent ones. Namely, the linear operator R is unambiguously defined through the conditions

$$R(\mathtt{e}_\pm) = \mp\mathrm{i}\,\mathtt{e}_\pm\,, \qquad\qquad R(\mathtt{h}) = 0 \qquad\qquad \big(\forall \mathtt{e}_\pm \in \mathfrak{n}_\pm,\, \forall\,\mathtt{h} \in \mathfrak{h}\big)\,. \tag{3.24}$$

The NLSM (3.19) with R being the Yang-Baxter operator was introduced by Klimčík in ref. [32] who called it the Yang-Baxter sigma model. Written in terms of local coordinates, the Lagrangian takes the form (3.20) where, for general group, the second term $\propto B_{\mu\nu}$ is no longer a total derivative and cannot be ignored. The model is classically integrable and the corresponding flat connection is given by the same formulae (3.6) and (3.22) [17].

The Yang-Baxter sigma model also turns out to be a one-loop renormalizable field theory. The proof is based on the extension of the results of the works [6, 7] to the case of an NLSM equipped with a B-field that was carried out in ref. [30], see also the textbook [33]. The one-loop RG flow equations are modified from (1.4) as

$$\begin{aligned} \partial_\tau G_{\mu\nu} &= -\hbar\left(R_{\mu\nu} - \frac{1}{4}H_\mu^{\ \ \sigma\rho}H_{\sigma\rho\nu}\right) + O(\hbar^2) \\ \partial_\tau B_{\mu\nu} &= -\frac{1}{2}\,\hbar\,\nabla_\sigma H^\sigma_{\ \ \mu\nu} + O(\hbar^2)\,. \end{aligned} \tag{3.25}$$

Here $H_{\mu\nu\lambda}$ are the components of the so-called torsion tensor. It is given by the exterior derivative of the B-field, i.e.,

$$H_{\mu\nu\lambda} = \partial_\mu\,B_{\nu\lambda} + \partial_\nu\,B_{\lambda\mu} + \partial_\lambda\,B_{\mu\nu}\,. \tag{3.26}$$

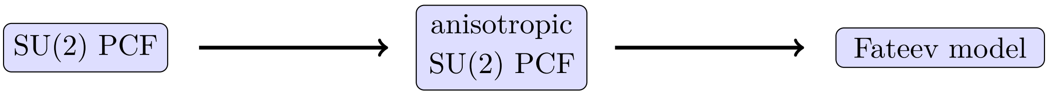

Figure 4. The relation between the various models. The Poisson-Lie deformation is represented by an arrow.

For the model (3.19), (3.20) with $\boldsymbol{U}$ belonging to a simple Lie group, the above equations boil down to a system of ordinary differential equations on e^2 and r. They read as [34]

$$\begin{aligned} -\frac{1}{\hbar}\,\partial_\tau(e^{-2}) &= \frac{1}{2}\,C_2\,(1-r) \\ -\frac{1}{\hbar}\,\partial_\tau r &= C_2\,e^2 r\,(r+1)\,, \end{aligned} \tag{3.27}$$

where, remarkably, the only dependence on the group appears through an overall factor proportional to the value of the quadratic Casimir in the adjoint representation. Note that in the domain $-1 < r < 0$, for which the system (3.27) possesses ancient solutions, the deformation parameter $\sqrt{r}$ entering into the Lagrangian of the Yang-Baxter sigma model (3.19) is an imaginary number. The corresponding target-space metric (3.21) remains real. However, the torsion tensor (3.26), which is non-vanishing outside the SU(2) case, becomes purely imaginary (a related discussion is contained in appendix A of ref. [35]).

We have just discussed that the Poisson-Lie deformation of the PCF yields the Yang-Baxter sigma model. The latter itself can be deformed along the similar line of arguments [17], see also [35] as well as figure 4 for a summary. In the case of $G = \mathrm{SU}(2)$ the obtained theory turns out to be the Fateev model, i.e., the sigma model with target space metric (2.23). For a general simple Lie group G, the two parameter deformation of the PCF was introduced by Klimčík in ref. [17]. The corresponding Lagrangian involves the Yang-Baxter operator R and is given by

$$\mathcal{L} = -\frac{2(1+r)(1+l)}{e^2}\,\langle \boldsymbol{U}^{-1}\,\partial_+\boldsymbol{U}\,,\mathcal{O}\,(\boldsymbol{U}^{-1}\,\partial_-\boldsymbol{U})\rangle \tag{3.28}$$

with

$$\mathcal{O} = \Big(1-\sqrt{l(1+r)}\;R-\sqrt{r(1+l)}\;\mathrm{Ad}_{\boldsymbol{U}}\circ R\circ \mathrm{Ad}_{\boldsymbol{U}}^{-1}\Big)^{-1}\,. \tag{3.29}$$

For $\boldsymbol{U} \in \mathrm{SU}(2)$ the B-field turns out to define a closed two form and has no effect on the equations of motion. It was shown in [36] by an explicit computation that the metric is equivalent to (2.23). For arbitrary simple Lie group G the model (3.28) is classically integrable and the ZCR was found in ref. [37]. One-loop renormalizability was demonstrated in the work [38] using the results of [39]. The differential equations describing the flow of the couplings (e^2, r, l) are

$$\begin{aligned} -\frac{1}{\hbar}\,\partial_\tau(e^{-2}) &= \frac{C_2\,(1-l-r)(l+r+1)}{2\,(1+l)(1+r)} \\ -\frac{1}{\hbar}\,\partial_\tau r &= \frac{C_2\,e^2\,r\,(l+r+1)}{1+l} \\ -\frac{1}{\hbar}\,\partial_\tau l &= \frac{C_2\,e^2\,l\,(l+r+1)}{1+r}\,. \end{aligned} \tag{3.30}$$

They essentially coincide with (2.26) which were derived in Fateev's original paper [11].

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3.2 Poisson-Lie deformation of fully anisotropic SU(2) PCF

Here we obtain a new clasically integrable NLSM as a Poisson-Lie deformation of the fully anisotropic SU(2) PCF. The procedure closely follows that which was explained above on the example of the SU(2) PCF.

The Hamiltonian for the fully anisotropic SU(2) PCF (1.3), (1.12), written in terms of the currents (2.8), is given by

$$H = \frac{1}{2}\sum_{a=1}^{3}\int \mathrm{d}x\, I_a\left(J_0^a J_0^a + J_1^a J_1^a\right), \tag{3.31}$$

while the equal-time Poisson bracket relations for J_i^a read as

$$\begin{aligned}
\{J_0^a(x),\, J_0^b(y)\} &= \frac{I_c}{I_a\, I_b}\,\epsilon_{abc}\; J_0^c(x)\,\delta(x-y)\\
\{J_0^a(x),\, J_1^b(y)\} &= \frac{1}{I_a}\,\epsilon_{abc}\; J_1^c(x)\,\delta(x-y) - \frac{1}{I_a}\,\delta^{ab}\,\delta'(x-y)\\
\{J_1^a(x),\, J_1^b(y)\} &= 0\,.
\end{aligned} \tag{3.32}$$

The above Poisson algebra admits a deformation of the form

$$\begin{aligned}
\{\tilde{J}_0^a(x),\, \tilde{J}_0^b(y)\} &= \frac{I_c-\xi}{I_a\, I_b}\,\epsilon_{abc}\; \tilde{J}_0^c(x)\,\delta(x-y)\\
\{\tilde{J}_0^a(x),\, \tilde{J}_1^b(y)\} &= \frac{I_b-\xi}{I_a\, I_b}\,\epsilon_{abc}\; \tilde{J}_1^c(x)\,\delta(x-y) - \frac{1}{I_a}\,\delta^{ab}\,\delta'(x-y)\\
\{\tilde{J}_1^a(x),\, \tilde{J}_1^b(y)\} &= -\frac{\xi}{I_a\, I_b}\,\epsilon_{abc}\; \tilde{J}_0^c(x)\,\delta(x-y)
\end{aligned} \tag{3.33}$$

depending on the extra parameter ξ. Then, with the Hamiltonian

$$\tilde{H} = \frac{1}{2}\sum_{a=1}^{3}\int \mathrm{d}x\, I_a\left(\tilde{J}_0^a \tilde{J}_0^a + \tilde{J}_1^a \tilde{J}_1^a\right), \tag{3.34}$$

which is formally the same as (3.31) but expressed in terms of the new currents $\tilde{J}_i^a$, the Hamiltonian equations of motion imply

$$\begin{aligned}
\partial_-\tilde{J}_+^a &= \frac{I_a+I_b-I_c}{2I_a}\;\tilde{J}_+^b\,\tilde{J}_-^c - \frac{I_a-I_b+I_c}{2I_a}\;\tilde{J}_+^c\,\tilde{J}_-^b\\
\partial_+\tilde{J}_-^a &= \frac{I_a+I_b-I_c}{2I_a}\;\tilde{J}_-^b\,\tilde{J}_+^c - \frac{I_a-I_b+I_c}{2I_a}\;\tilde{J}_-^c\,\tilde{J}_+^b\,.
\end{aligned} \tag{3.35}$$

Here $(a,b,c) = \mathtt{perm}(1,2,3)$ and summation over repeated indices is not being assumed. The equations (3.35) are equivalent to (2.9), (2.11) up to the replacement $J_i^a \mapsto \tilde{J}_i^a$.

The currents $\tilde{J}_i^a$ obeying the Poisson bracket relations (3.33) can be realized in terms of the fields X^μ and Π_μ subject to the canonical commutation relations (3.10). This is done along the same line of arguments as was discussed in the previous subsection. Namely, one first considers certain linear combinations of $\tilde{J}_i^a$ which obey two independent copies of the classical SU(2) current algebra (3.15) with k being a certain function of the couplings I_a and deformation parameter ξ. Then realizing $\mathcal{I}$ and $\mathcal{J}$ in terms of the canonical variables (see formulae (3.17), (3.13)) and performing the Legendre transform of the Hamiltonian (3.34), one obtains the Lagrangian of the deformed theory. The result of the calculations reads as

$$\mathcal{L} = -4 \Big\langle \boldsymbol{U}^{-1}\partial_-\boldsymbol{U} \, , \, \mathcal{O}_+ \left(\boldsymbol{U}^{-1}\partial_+\boldsymbol{U}\right) \Big\rangle \, , \tag{3.36}$$

where a certain choice of the overall multiplicative factor for the Lagrangian density was made. Here and below we use the notation $\mathcal{O}_\pm$ for the linear operators acting on the Lie algebra $\mathfrak{su}(2)$ given by

$$\mathcal{O}_\pm = \left(\frac{1}{I_1-\xi}\, P_1 + \frac{1}{I_2-\xi}\, P_2 + \frac{1}{I_3-\xi}\, P_3 \pm \sqrt{\gamma}\, \mathrm{Ad}_{\boldsymbol{U}} \circ R \circ \mathrm{Ad}_{\boldsymbol{U}}^{-1}\right)^{-1} \tag{3.37}$$

with

$$\gamma = \frac{\xi}{(I_1-\xi)(I_2-\xi)(I_3-\xi)} \, . \tag{3.38}$$

The Lagrangian density (3.36) is formally not invariant under the parity transformation $x^1 \mapsto -x^1$ (so that $\partial_\pm \mapsto \partial_\mp$). Nevertheless, the theory possesses this symmetry. The reason is because in local coordinates, where $\mathcal{L}$ (3.36) takes the form (3.20), the term $\propto B_{\mu\nu}$ turns out to be a total derivative. Thus one is free to replace $\mathcal{O}_+$ in (3.36) by

$$\mathcal{O}_{\text{sym}} = \frac{1}{2} \left(\mathcal{O}_+ + \mathcal{O}_+^{\mathrm{T}}\right) , \tag{3.39}$$

where the transposition is defined by the condition $\langle \mathtt{x} \, , \mathcal{O}_+ \, \mathtt{y}\rangle = \langle \mathcal{O}_+^{\mathrm{T}}\, \mathtt{x} \, , \mathtt{y}\rangle$ for any $\mathtt{x}, \mathtt{y} \in \mathfrak{su}(2)$. This way, the target space metric for the deformed sigma model can be written as

$$G_{\mu\nu}(X)\, \mathrm{d}X^\mu \mathrm{d}X^\nu = -2 \left\langle \boldsymbol{U}^{-1}\, \mathrm{d}\boldsymbol{U} \, , \mathcal{O}_{\text{sym}}(\boldsymbol{U}^{-1}\, \mathrm{d}\boldsymbol{U})\right\rangle . \tag{3.40}$$

It is worth mentioning that for $I_1 = I_2$ this becomes the Fateev metric (2.23), (2.24) upon the identification of parameters:

$$I_1 = I_2 = \frac{(1+l)^2}{2e^2} \, , \qquad I_3 = \frac{(1+l)\,(1+l+r)}{2e^2} \, , \qquad \xi = \frac{l\,(l+1)}{2e^2} \, . \tag{3.41}$$

By construction the obtained model (3.36) is a classically integrable field theory. The corresponding flat connection takes the same form as for the fully anisotropic SU(2) PCF, i.e.,

$$\boldsymbol{A}_\pm = \mathrm{i} \sum_{a=1}^{3} w_a(\nu \mp \lambda) \tilde{J}_\pm^a \, \mathtt{t}_a \, , \tag{3.42}$$

where the functions $w_a(\lambda)$ are given in (2.14) and (2.17). The formula for the currents $\tilde{J}_\pm^a$ in terms of the SU(2) element $\boldsymbol{U}$ reads as

$$\tilde{J}_\pm^a = \mathrm{i}\, C_a \left\langle \mathcal{O}_\pm \left(\boldsymbol{U}^{-1}\partial_\pm \boldsymbol{U}\right), \mathtt{t}_a \right\rangle , \qquad C_a = \frac{2}{I_a-\xi} \sqrt{\frac{I_b\, I_c}{(I_b-\xi)\,(I_c-\xi)}} \tag{3.43}$$

with $(a,b,c) = \mathtt{perm}(1,2,3)$.

One can check that the metric (3.37)–(3.40) satisfies the Ricci flow equation (1.4). The parameter γ defined in (3.38) turns out to be an RG invariant, i.e.,

$$\frac{1}{\hbar}\, \partial_\tau \gamma = O(\hbar) \, . \tag{3.44}$$

As for the couplings $I_a = I_a(\tau)$, it is convenient to swap these in favour of $\tilde{I}_a$ according to

$$\tilde{I}_a \equiv I_a - \xi\,. \tag{3.45}$$

The latter obey the RG flow equations

$$-\frac{1}{\hbar}\,\partial_\tau(\tilde{I}_a\,\tilde{I}_b) = (1+\gamma\,\tilde{I}_a\,\tilde{I}_b)\,(\tilde{I}_a+\tilde{I}_b-\tilde{I}_c+\gamma\,\tilde{I}_a\,\tilde{I}_b\,\tilde{I}_c)+O(\hbar)\,, \qquad (a,b,c)=\texttt{perm}(1,2,3)\,, \tag{3.46}$$

which may be compared to the underformed case (2.19). The derivation of the above formulae uses the property that the Ricci tensor corresponding to the metric (3.37)–(3.40) can be written as

$$R_{\mu\nu} = \sum_{a=1}^{3} \frac{(\tilde{I}_a-\tilde{I}_b+\tilde{I}_c+\gamma\,\tilde{I}_a\,\tilde{I}_b\,\tilde{I}_c)(\tilde{I}_a+\tilde{I}_b-\tilde{I}_c+\gamma\,\tilde{I}_a\,\tilde{I}_b\,\tilde{I}_c)}{2\tilde{I}_b\,\tilde{I}_c}\left(\frac{\partial G_{\mu\nu}}{\partial \tilde{I}_a}\right)_\gamma\,, \tag{3.47}$$

which generalizes the relation (2.18).

For $\gamma = 0$ the two first integrals of (3.46) coincide with Q_1, Q_2 (2.20)- (2.22) with $I_a \mapsto \tilde{I}_a$. We found that these RG invariants admit a deformation to arbitrary γ. The explicit expressions involve, apart from

$$\tilde{m} = \frac{\tilde{I}_2\,(\tilde{I}_1-\tilde{I}_3)}{\tilde{I}_3\,(\tilde{I}_1-\tilde{I}_2)}\,, \qquad \tilde{p} = \frac{\tilde{I}_1}{\tilde{I}_2}\,, \tag{3.48}$$

also m (2.22), which enters into the functions w_a that appear in the flat connection (3.42). In terms of the parameters $\tilde{I}_a$, it is given by

$$m = \frac{\tilde{I}_2\,(\tilde{I}_1-\tilde{I}_3)\,(1+\gamma\,\tilde{I}_1\,\tilde{I}_3)}{\tilde{I}_3\,(\tilde{I}_1-\tilde{I}_2)\,(1+\gamma\,\tilde{I}_1\,\tilde{I}_2)}\,. \tag{3.49}$$

The two first integrals of the system (3.46) read as

$$\begin{aligned} Q_1^{(\gamma)} &= \frac{K(1-m)-(1-\tilde{p})\,\tilde{m}\,\Pi(1-\tilde{m},1-m)}{(1-\tilde{p})(1-\tilde{m})\,\Pi(\tilde{m},m)+\tilde{p}\,K(m)} \\ Q_2^{(\gamma)} &= \frac{\tilde{m}-m}{\gamma}\,\frac{\big((\tilde{p}-1)(1-\tilde{m})\,\Pi(\tilde{m},m)-\tilde{p}\,K(m)\big)^2}{(\tilde{p}-1)^2\,\tilde{m}\,(1-\tilde{m})}\,, \end{aligned} \tag{3.50}$$

where $\Pi(\tilde{m},m)$ is the complete elliptic integral of the third kind:

$$\Pi(\tilde{m},m) = \int_0^1 \frac{\mathrm{d}t}{(1-\tilde{m}\,t^2)\sqrt{(1-t^2)(1-m\,t^2)}}\,. \tag{3.51}$$

It is straightforward to check that $Q_1^{(0)} = Q_1$, while $\lim_{\gamma\to 0} Q_2^{(\gamma)} = Q_2$.

4 Summary and discussion

In this work we explored the interplay between integrability and one-loop renormalizability for NLSM in 1 + 1 dimensional spacetime. Our main example was the fully anisotropic SU(2) PCF. On the one hand, it was explained that this is a clasically integrable field theory and the Zero Curvature Representation for its equations of motion was reviewed. On the other, the corresponding target space metric satisfies the Ricci flow equation (1.4) so that the fully anisotropic SU(2) PCF is one-loop renormalizable within a three dimensional space of couplings. The system of ODEs describing the flow was derived and its full set of first integrals was obtained, independently from [27, 28].

Another main result is the construction of a classically integrable NLSM depending on four parameters whose Lagrangian density is given by (3.36)–(3.38). It was found by applying a Poisson-Lie deformation to the fully anisotropic SU(2) PCF. The corresponding target space metric turned out to provide a new solution to the Ricci flow equation. The first integrals to the system of ODEs (3.44) and (3.46), which describe the flow of the four couplings, were derived in the course of this work and are given in (3.50).

The class of theories that we discussed admit a modification such that they remain one-loop renormalizable. This is achieved by adding the so-called Wess-Zumino term to the action. The Lagrangian takes the form (3.20) with the B-field no longer being exact. This implies that the target space, together with the Riemannian metric $G_{\mu\nu}$, is equipped with the affine connection, where the torsion $H = \mathrm{d}B$ is non-vanishing [30]. In the case of SU(2), the 3-form H is proportional to the volume form for the group and can be written as

$$H \equiv \mathrm{d}B = \frac{\mathtt{k}}{24\pi} \Big\langle [\boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U} \wedge \boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U}] \wedge \boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U} \Big\rangle . \tag{4.1}$$

Here $\mathtt{k}$ is an additional parameter of the model. In the classical theory there is no contraint on the values it may take, however, upon quantization it is required to be an RG invariant and, furthermore, must be an integer [40]. For the case of the fully anisotropic SU(2) PCF with Wess-Zumino term, the one-loop RG flow equations (3.25) imply the system of ODEs for the couplings:

$$\begin{aligned} -\frac{1}{\hbar}\,\partial_\tau(I_a\,I_b) &= I_a + I_b - I_c - \frac{\mathtt{k}^2}{64\pi^2 I_c} + O(\hbar)\,, \qquad\qquad (a,b,c) = \mathtt{perm}(1,2,3) \\ \frac{1}{\hbar}\,\partial_\tau \mathtt{k} &= 0\,. \end{aligned} \tag{4.2}$$

It possesses two Liouvillian first integrals, which are a simple generalization of (2.20) and in terms of p and m (2.22) take the form

$$Q_1 = \frac{K(1-m) - (1-p)E(1-m)}{(1-p)E(m) + pK(m)}\,, \tag{4.3}$$

$$Q_2 = \frac{I_1^2\,\big((p-1)E(m) - pK(m)\big)^2}{p\,(p-1)(pm - m + 1)} + \frac{\mathtt{k}^2}{64\pi^2}\,\frac{K(m)}{p-1}\,\big((p-1)E(m) - pK(m)\big)\,. \tag{4.4}$$

A complete analysis of the behaviour of the solutions to (4.2) has not been carried out yet. Moreover, the classical integrability of the model has not been established and the Zero Curvature Representation, if it exists, remains unknown to us. These would be interesting questions to pursue in future work. They can also be addressed for the Poisson-Lie deformed theory.

Our work was mainly focused on sigma models associated with the Lie group SU(2). Nevertheless, we expect it to be possible to generalize the Poisson-Lie deformed theory constructed here to the case of higher rank Lie groups. One way to approach the problem

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uses the results of ref. [41]. In that paper, a classically integrable NLSM is introduced, which is a two parameter deformation of the PCF for Lie group SL(N). For $N = 2$ it coincides with the fully anisotropic SU(2) PCF (upon an appropriate choice of reality conditions on the fields and parameters). We expect that this sigma model may also be deformed along the line of arguments presented in section 3. Another possibility for constructing integrable deformations, based on the formalism of the so-called affine Gaudin model, is mentioned in the perspectives section of ref. [41].

Finally, classically integrable multiparametric families of sigma models are of interest to string theory. In particular, the possibility of an integrable elliptic deformation of strings on $\mathrm{Ad}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ was investigated in the recent paper [42].

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