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Quantum Field Theory and Statistical Systems

Scaling limit of the ground state Bethe roots for the inhomogeneous XXZ spin - $\frac{1}{2}$ chain

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ABSTRACT

It is known that for the Heisenberg XXZ spin - $\frac{1}{2}$ chain in the critical regime, the scaling limit of the vacuum Bethe roots yields an infinite set of numbers that coincide with the energy spectrum of the quantum mechanical 3D anharmonic oscillator. The discovery of this curious relation, among others, gave rise to the approach referred to as the ODE/IQFT (or ODE/IM) correspondence. Here we consider a multiparametric generalization of the Heisenberg spin chain, which is associated with the inhomogeneous six-vertex model. When quasi-periodic boundary conditions are imposed the lattice system may be explored within the Bethe Ansatz technique. We argue that for the critical spin chain, with a properly formulated scaling limit, the scaled Bethe roots for the ground state are described by second order differential equations, which are multi-parametric generalizations of the Schrödinger equation for the anharmonic oscillator.

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1. Introduction and summary

Since the seminal work of Polyakov [1], it is believed that the isotropic and homogeneous scaling behaviour of a critical statistical system can be adequately described within the framework of Conformal Field Theory (CFT). While this fundamental insight remains unproven in general, there exists an enormous amount of supporting evidence. Much of it comes from the study of 2D classical statistical models. In many cases, such models admit an equivalent description in terms of 1D quantum spin chains. Then the problem of identification of the underlying CFT can be approached by studying the scaling limit of the spectrum of a spin chain Hamiltonian [2]. This provides a convenient way for a numerical investigation of the critical behaviour. For the so-called integrable statistical systems, a detailed analytical study is also possible. By integrable, what is usually meant is that the model can be explored via the Bethe Ansatz approach or some variant thereof [3].

In the case of integrable systems at criticality, apart from the conformal structure of the underlying continuous theory, it is natural to investigate also the integrable structure of the corresponding CFT [4–6]. The latter is encoded in the scaling limit of the Bethe roots – the solutions of the Bethe Ansatz equations. The most powerful approach for studying the integrable structure of CFT or, more generally, QFT is the so-called ODE/IM correspondence [7–10]. The abbreviation IM stands for Integrable Model. For the most interesting examples these are 2D Integrable Quantum Field Theories (IQFT) and, as such, we prefer to use the term ODE/IQFT correspondence rather than ODE/IM.

One of the simplest manifestations of the ODE/IQFT correspondence, which is directly relevant to this work, occurs in the scaling limit of the XXZ spin - $\frac{1}{2}$ chain. The latter is the archetypical example of an integrable model and, in fact, the Bethe Ansatz approach was originally introduced for its special (isotropic) case [11]. The Hamiltonian of the spin-$\frac{1}{2}$ XXZ chain of length N is given by

$$\mathbb{H}_{\rm XXZ} = -\sum_{m=1}^{N} \left(\sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} + \Delta\, \sigma^z_m \sigma^z_{m+1} \right) . \tag{1.1}$$

With quasi-periodic boundary conditions imposed [14],

$$\sigma^x_{N+m} \pm \mathrm{i}\, \sigma^y_{N+m} = \omega^{\pm 2} \left(\sigma^x_m \pm \mathrm{i}\, \sigma^y_m \right), \qquad \sigma^z_{N+m} = \sigma^z_m , \tag{1.2}$$

$\mathbb{H}_{\rm XXZ}$ commutes with the z - component of the total spin operator and the eigenstates are labelled by the (half-)integer numbers S^z. The energy spectrum is described in terms of solutions of the Bethe Ansatz equations:

$$\left(\frac{1 + q^{+1}\, \zeta_j}{1 + q^{-1}\, \zeta_j} \right)^N = -\omega^2\, q^{2S^z} \prod_{i=1}^{N/2 - S^z} \frac{\zeta_i - q^{+2}\, \zeta_j}{\zeta_i - q^{-2}\, \zeta_j} , \tag{1.3}$$

where q parameterizes the anisotropy as $\Delta = \frac{1}{2}(q + q^{-1})$. It turns out that the system is critical when $-1 \le \Delta < 1$ [12–14] or, equivalently, q is a unimodular number, i.e.,

$$q = \mathrm{e}^{\mathrm{i}\gamma} \,. \tag{1.4}$$

In this case, for an even number of sites, the ground state of $\mathbb{H}_{\mathrm{XXZ}}$, i.e., the state with the lowest energy, is non-degenerate and occurs in the sector $S^z = 0$. It was proven in the work of Yang and Yang [15] that for $\gamma \in (0, \pi]$ and periodic boundary conditions ($\omega = 1$), the solution set to the Bethe Ansatz equations corresponding to the ground state has ζ_j all real, distinct positive numbers. The proof may be extended to quasi-periodic boundary conditions when

$$\omega = \mathrm{e}^{\mathrm{i}\pi \mathtt{k}} \tag{1.5}$$

with sufficiently small real $\mathtt{k}$, namely,

$$|\mathtt{k}| < \frac{\gamma}{2\pi} \,. \tag{1.6}$$

The vacuum Bethe roots can be naturally ordered as

$$0 < \zeta_1 < \zeta_2 < \ldots < \zeta_{N/2} \,. \tag{1.7}$$

In the limit when the number of sites N goes to infinity, most of the roots become densely distributed such that $(\zeta_j - \zeta_{j+1})/\zeta_j \sim 1/N$. However, at the edges of the distribution the Bethe roots develop a certain scaling behaviour. In particular, keeping j fixed as $N \to \infty$ the following limits exist

$$\lim_{\substack{N\to\infty \\ j-\text{fixed}}} N^{2(1-\gamma/\pi)} \zeta_j \tag{1.8}$$

and form an infinite set of non-vanishing numbers. These turn out to admit a remarkable description in terms of the Schrödinger equation for the 3D anharmonic oscillator [9],

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{l(l+1)}{x^2} + x^{2\alpha} - E \right) \Psi = 0 \,. \tag{1.9}$$

Namely, the differential equation possesses only a discrete component of the spectrum and the corresponding energy levels E_j, coincide with the limits (1.8) up to an overall factor provided the parameters of the ODE are taken to be

$$\alpha = \frac{\pi}{\gamma} - 1 \,, \qquad l + \frac{1}{2} = \frac{\pi \mathtt{k}}{\gamma} \,. \tag{1.10}$$

A similar relation between the scaling behaviour of the Bethe roots of not only the ground state, but also the class of so-called low energy excited states, holds true (see, e.g., ref. [16]). Moreover such a phenomenon, i.e., the ODE/IQFT correspondence, has been observed for a number of other integrable, critical statistical models as well [17–21].

The XXZ spin chain belongs to the integrability class of a 2D classical statistical system known as the six-vertex model. As was shown by Baxter in the work [22], the latter admits a multiparametric generalization, which is solvable within the Bethe Ansatz approach. In this case, the equations (1.3) are modified to

$$\prod_{J=1}^{N} \frac{\eta_J + q^{+1}\,\zeta_j}{\eta_J + q^{-1}\,\zeta_j} = -\omega^2\, q^{2S^z} \prod_{i=1}^{N/2-S^z} \frac{\zeta_i - q^{+2}\,\zeta_j}{\zeta_i - q^{-2}\,\zeta_j} \qquad \left(j = 1, 2, \ldots, \tfrac{N}{2} - S^z \right) \tag{1.11}$$

and involve the complex parameters $\{\eta_J\}_{J=1}^{N}$, the so-called inhomogeneities. The corresponding statistical system is known as the inhomogeneous six-vertex model. With the parameters q and η_J obeying certain conditions, the lattice system develops critical behaviour. We'll assume that the number of sites N is divisible by r and the inhomogeneities obey the r-site periodicity condition

$$\eta_{J+r} = \eta_J \qquad \left(J = 1, 2, \ldots, N - r \,;\; N/r \in \mathbb{Z} \right). \tag{1.12}$$

In the scaling limit, $N \to \infty$ while the integer r is kept fixed. This ensures the presence of translational invariance for the continuous theory.

The critical behaviour of the r - site periodic inhomogeneous six-vertex model can be explored by studying the spectrum of the Hamiltonian $\mathbb{H}$ of a certain spin chain. For the case $r = 1$ it coincides, up to an overall multiplicative factor and additive constant, with $\mathbb{H}_{\mathrm{XXZ}}$ (1.1). The explicit form of $\mathbb{H}$ with $r \geq 2$ is rather cumbersome (see, e.g., [23,24] and the work [25], where the same notation as in this paper is used). What is important is that $\mathbb{H}$ is given by a sum over terms that describe the interactions of up to $r + 1$ adjacent $\frac{1}{2}$ spins, i.e., the corresponding Hamiltonian density is local. The quasi-periodic boundary conditions (1.2) are still assumed and, similar to the homogeneous case, the Hamiltonians commute with the z - projection of the total spin operator.

A numerical investigation shows that the spin chain governed by the Hamiltonian $\mathbb{H}$ exhibits critical behaviour when the anisotropy parameter q is a unimodular number, $q = \mathrm{e}^{\mathrm{i}\gamma}$. It turns out that the phase can be restricted to the domain $\gamma \in (0, \pi)$ and the critical behaviour is described differently as $\frac{\pi}{r} A < \gamma < \frac{\pi}{r} (A + 1)$. Inside each segment labelled by the integer A we will parameterize γ by real positive n such that

$$\gamma = \frac{\pi}{r} A + \frac{\pi}{n + r} \qquad (A = 0, 1, \ldots, r - 1 \,,\; n > 0) \,. \tag{1.13}$$

Mapping out the critical surfaces in the space of the parameters $\big(\gamma,\{\eta_\ell\}\big)$ is a complicated problem that, apart from some special cases, has not been fully achieved. A useful starting point is when the inhomogeneities are given by

$$\eta_\ell=(-1)^r\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}\,(2\ell-1)}\qquad\qquad (\ell=1,\ldots,r) \tag{1.14}$$

for which the model possesses $\mathcal{Z}_r$ invariance (see, e.g., sec. 7 in ref. [25]). In this case the description of the scaling limit for the ground state is especially simple. For N divisible by $2r$ and $S^z=0$ the corresponding Bethe roots are split into r groups of equal size. The Bethe roots within each group lie on the rays

$$\arg(\zeta)=-\frac{\pi}{r}\,A,\ \frac{\pi}{r}\,(2-A),\ \ldots,\ \frac{\pi}{r}\,\big(2\,(r-1)-A\big)\qquad (\mathrm{mod}\,2\pi)\ . \tag{1.15}$$

These can be labelled by the integer $a=1,2,\ldots,r$ and we denote by $\zeta_m^{(a)}$ the roots lying on the a^{th} ray ordered by their absolute value. Then the real positive numbers $\big|\zeta_m^{(a)}\big|^r$ with $m=1,2,\ldots,N/(2r)$ are the same for any a. They turn out to coincide with the solution to (1.3) with $S^z=0$ and the substitutions $q\mapsto \mathrm{e}^{\frac{\mathrm{i}\pi r}{n+r}}$, $N\mapsto N/r$ which corresponds to the ground state of the XXZ spin chain. It follows that the limits

$$E_m^{(a)}=\lim_{\substack{N\to\infty\\ m-\text{fixed}}}\left(\frac{N}{rN_0}\right)^{\frac{2n}{r(n+r)}}\zeta_m^{(a)} \tag{1.16}$$

exist and are non-vanishing. Here for later convenience, an extra n dependent factor has been introduced with

$$N_0=\frac{\sqrt{\pi}\,\Gamma(1+\frac{r}{2n})}{r\Gamma(\frac{3}{2}+\frac{r}{2n})}\ . \tag{1.17}$$

The numbers $E_m^{(a)}$ would be expressed in terms of the spectrum of the ODE (1.9). For the purpose of this work, it is convenient to re-write the differential equation using the variables $y=\frac{2}{r}\log(x)+\frac{2}{n+r}\log(\frac{r}{2})$ and $\psi=x^{-\frac{1}{2}}\,\Psi$, so that it becomes[1]

$$\Big[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-1)^A\,E^r\,\mathrm{e}^{ry}\Big]\,\psi=0 \tag{1.18}$$

with

$$p=\frac{n+r}{2}\ \mathtt{k}\ . \tag{1.19}$$

The above differential equation admits the Jost solution $\psi_p(y)$, which is uniquely defined through the asymptotic

$$\psi_p(y)=\mathrm{e}^{py}\qquad\text{as}\qquad y\to-\infty\qquad\qquad \big(\Re e(p)\geq 0\big)\ . \tag{1.20}$$

For generic E the solution grows unboundedly as $y\to+\infty$. However, at the special values $E=E_m^{(a)}$ (1.16), one has

$$\psi_p(y)\Big|_{E=E_m^{(a)}}\to 0\qquad\text{as}\qquad y\to+\infty\ . \tag{1.21}$$

In this work we consider the spin chain, where the $\mathcal{Z}_r$ symmetry is "softly" broken for finite system size. This is carried out by treating the inhomogeneities η_ℓ as N - dependent bare couplings of the spin chain Hamiltonian which tend to the values (1.14) as $N\to\infty$. The quantities

$$\mathfrak{a}_s=\frac{1}{s}\left(\frac{N}{rN_0}\right)^{d_s}\ \frac{1}{r}\ \sum_{\ell=1}^{r}(\eta_\ell)^{-s} \tag{1.22}$$

will be taken to be independent of N, i.e., they play the rôle of the RG invariants. With their proper specialization (which, among other things, involves fixing the exponent $d_s>0$) the Bethe roots possess a scaling behaviour similar to that as for the $\mathcal{Z}_r$ invariant case. In particular, for the ground state in the sector with $S^z=0$, the roots may still be split into r groups with roughly equal phases. Then there exist the limits (1.16), where the integer m labels the roots according to their absolute values. We will argue that $E_m^{(a)}$ can be described as above, but with the differential equation (1.18) replaced by the ODE of the form

$$\Big[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-1)^A\,E^r\,\mathrm{e}^{ry}-\sum_{(\mu,j)\in\Xi_{r,A}}c_{\mu,j}\,E^\mu\,\mathrm{e}^{\left((A\mu-rj)\,\frac{n+r}{r}+\mu\right)y}\Big]\,\psi=0\ . \tag{1.23}$$

[1] With some abuse of notation we use the same symbol E in eqs. (1.9) and (1.18). In fact, E from (1.9) coincides with the combination $(-1)^A\,(2/r)^{\frac{2n}{n+r}}\,E^r$ with E from (1.18). Also, $\alpha=\frac{n}{r}$ and $l+\frac{1}{2}=\frac{2p}{r}$.

The coefficients $c_{\mu,j}$ are related to the values of the RG invariants $\mathfrak{a}_s$ from eq. (1.22), while the summation indices μ and j run over a certain integer set. The latter is given by[2]

$$\Xi_{r,A} = \big\{(\mu,j)\ :\ \ \tfrac{rj}{A} < \mu < \tfrac{r}{A+1}\,(j+1)\ \ \&\ \ j\geq 0\big\} \qquad \text{for} \qquad A=1,2,\dots,r-2\,, \tag{1.24}$$

while for $A=0$,

$$\Xi_{r,0} = \big\{(\mu,j)\ :\ \ \mu=1,2,\dots,r-1\ \ \&\ \ j=0\big\}\,. \tag{1.25}$$

As $0<A<r-1$ we expect that it is possible to organize the scaling limit in such a way that the differential equation (1.23) appears for any given values of the coefficients $c_{\mu,j}$ (some restrictions on the domain of n may be required). In particular, the specification of the RG invariants such that there is only one non-vanishing term occurring in the sum in (1.23) can be found in Appendix F for $r=3,4,\dots,10$. When $A=0$ only the coefficient $c_{\mu,0}$ with $\mu\geq\frac{r}{2}$ can be chosen at will, while the others turn out to be not independent. The case $A=r-1$ was already discussed in ref. [21]. Its peculiar feature is that all of the exponents d_s in (1.22) turn out to be zero. The scaling of the Bethe roots for the ground state is still described by means of the differential equation of the form (1.23) with

$$\Xi_{r,r-1} = \big\{(\mu,j)\ :\ \ \mu=j+1\ \ \&\ \ j=0,1,\dots,r-2\big\} \qquad\qquad (A=r-1)\,. \tag{1.26}$$

2. The case odd r and $A=\frac{r-1}{2}$

An important source of intuition in the study of the spin - $\frac{1}{2}$ Heisenberg chain comes from the XX spin chain — the special case when the anisotropy parameter Δ in (1.1) is set to zero or, equivalently, $q=\mathrm{i}$. By means of the Jordan-Wigner transformation, the lattice model can then be mapped to a system of free fermions, which dramatically simplifies the analysis. A similar phenomenon occurs for the spin chain associated with the r-site periodic inhomogeneous six-vertex model for any odd r,

$$r=3,5,7,\dots\,, \tag{2.1}$$

when the integer A in formula (1.13) for the argument of q is chosen to be

$$A=\frac{r-1}{2}\,. \tag{2.2}$$

In this case

$$q=\mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}\pi(r-n)}{2r(n+r)}} \tag{2.3}$$

so that as $n=r$ one has $q=\mathrm{i}$. Below we discuss the scaling limit of the ground state for the spin chain in the regime (2.2) (r odd) using the "free fermion" point as a launch pad.

2.1. Scaling limit of the Bethe roots for the free fermion point

At $q=\mathrm{i}$ the Bethe Ansatz equations (1.11) for the r-site periodic model (1.12) become

$$\left(\prod_{\ell=1}^{r}\frac{\eta_\ell+\mathrm{i}\zeta_j}{\eta_\ell-\mathrm{i}\zeta_j}\right)^{\frac{N}{r}} = -\mathrm{e}^{\mathrm{i}\pi(2\mathtt{k}+S^z)}\,. \tag{2.4}$$

Taking the logarithm of both sides, one obtains

$$\frac{N}{\mathrm{i}\pi r}\,\log\left(\prod_{\ell=1}^{r}\frac{1+\mathrm{i}\zeta_j/\eta_\ell}{1-\mathrm{i}\zeta_j/\eta_\ell}\right) = 2n_j-1+2\mathtt{k}+S^z \tag{2.5}$$

with n_j being certain integers. For the vacuum state of the $\mathcal{Z}_r$ invariant case, where the inhomogeneities are given by (1.14), the Bethe roots are split into the groups $\zeta_m^{(a)}$ with $a=1,\dots,r$. Then it turns out that with a proper choice of the branch of the logarithm,

$$\frac{N}{\mathrm{i}\pi r}\,\log\left(\prod_{\ell=1}^{r}\frac{1+\mathrm{i}\zeta_m^{(a)}/\eta_\ell}{1-\mathrm{i}\zeta_m^{(a)}/\eta_\ell}\right) = 2m-1+2\mathtt{k} \qquad\qquad \big(m=1,2,\dots,N/(2r)\big)\,, \tag{2.6}$$

where S^z has been set to zero. One may expect that the above formula remains true when the inhomogeneities η_ℓ differ slightly from the values $(-1)^r\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)}$.

[2] The number of admissible pairs (μ,j) (1.24) coincides with the total number of solutions of the Diophantine equations $A\mu-rj=L$ subject to the constraints $1\leq\mu<r-L\ \&\ j\geq0$ for $L=1,2,\dots$.

To perform the scaling limit, we write ζ in the form

$$\zeta = \left(\frac{\pi}{2N}\right)^{\frac{1}{r}} E \tag{2.7}$$

and keep E fixed as $N \to \infty$. Further, the inhomogeneities are assigned an N - dependence in such a way that the quantities

$$\mathfrak{a}_{2j+1} = \frac{1}{2j+1}\left(\frac{2N}{\pi}\right)^{1-\frac{2j+1}{r}} \frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-2j-1} \qquad \left(j = 1, \dots, \tfrac{r-1}{2}\right) \tag{2.8}$$

remain independent of the length of the spin chain. For $j = A = \frac{r-1}{2}$, the exponent of the N - dependent factor in the r.h.s. vanishes and, without loss of generality, $\mathfrak{a}_r$ can be set to be any positive number. We take its value to be the same as for the $\mathcal{Z}_r$ invariant case (1.14), i.e., $\mathfrak{a}_r = \frac{1}{r}$. This way, in the scaling limit, the Bethe Ansatz equations (2.6) simplify to

$$\frac{(-1)^A}{r} E^r + \sum_{j=0}^{A-1}(-1)^j\, \mathfrak{a}_{2j+1}\, E^{2j+1} = 2m - 1 + 2\mathtt{k} \qquad (m = 1, 2, \dots)\,. \tag{2.9}$$

For given m, the above algebraic equation possesses exactly r complex solutions, which we denote as $E_m^{(a)}$. Assuming that the absolute value of $\mathfrak{a}_{2j+1}$ is sufficiently small, the roots may be specified by the condition that

$$E_m^{(a)} \sim \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)}\,\big((2m-1+2\mathtt{k})\,r\big)^{\frac{1}{r}} \qquad \text{for} \qquad m \gg 1 \tag{2.10}$$

and $E_m^{(a)}$ are split into r groups of roots which have approximately the same phase

$$\arg\left(E_m^{(a)}\right) \approx \frac{\pi}{r}\,\big(2a - A\big) \qquad (a = 1, \dots, r)\,. \tag{2.11}$$

As such, the scaling limit of the Bethe roots for the ground state is described by the formula

$$E_m^{(a)} = \lim_{\substack{N\to\infty\\ m-\text{fixed}}} \left(\frac{2N}{\pi}\right)^{\frac{1}{r}} \zeta_m^{(a)}\,. \tag{2.12}$$

This provides an illustration of the scaling relation (1.16) discussed in the Introduction.

It is clear how to come up with a differential equation, whose spectrum is given by $E_m^{(a)}$. One just takes (1.18) with $n = r$ and replaces $(-1)^A E^r$ by $r\lambda(E)$, where

$$\lambda(E) = \frac{(-1)^A}{r} E^r + \sum_{j=0}^{\frac{r-3}{2}}(-1)^j\, \mathfrak{a}_{2j+1}\, E^{2j+1}\,. \tag{2.13}$$

The ODE becomes

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{2ry} - r\lambda(E)\,\mathrm{e}^{ry}\Big]\,\psi = 0\,, \tag{2.14}$$

which is a form of the confluent hypergeometric equation. The solution defined via the asymptotic condition (1.20) reads explicitly as

$$\psi_p(y) = \mathrm{e}^{py}\,\exp\Big(-\tfrac{1}{r}\,\mathrm{e}^{ry}\Big)\,{}_1F_1\Big(\tfrac{1}{2} + \tfrac{p}{r} - \tfrac{\lambda}{2}, 1 + \tfrac{2p}{r}, \tfrac{2}{r}\,\mathrm{e}^{ry}\Big)\,. \tag{2.15}$$

For $y \to +\infty$, it develops the asymptotic behaviour

$$\psi_p(y) \asymp (2r)^{\frac{p}{r}+\frac{1}{2}}\,\frac{\Gamma(1+\frac{p}{r})}{2\sqrt{\pi}}\,D_+\left(\frac{r}{2}\right)^{\frac{1}{2}\lambda} \exp\Big(\tfrac{1}{r}\,\mathrm{e}^{ry} - \big(\lambda + 1\big)\,\tfrac{ry}{2} + o(1)\Big)\,, \tag{2.16}$$

where

$$D_+ = \frac{\Gamma(\frac{1}{2} + \frac{p}{r})}{\Gamma\big(\frac{1}{2} + \frac{p}{r} - \frac{1}{2}\lambda(E)\big)}\,. \tag{2.17}$$

The Jost solution $\psi_p(y)$ vanishes at large y when E is a zero of $D_+ = D_+(E)$. Hence, the spectrum is defined through the equation $\lambda(E) = 2m - 1 + \frac{2p}{r}$ with $m = 1, 2, 3, \dots$. This coincides with (2.9), provided $\frac{p}{r} = \mathtt{k}$ as in (1.19) with $n = r$.

2.2. Differential equation

For the case $q = \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}\pi(r-n)}{2r(n+r)}}$ with arbitrary $n > 0$ we propose that the differential equation describing the scaling limit of the Bethe roots for the ground state is given by

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A\, E^r\, \mathrm{e}^{ry} - \sum_{j=0}^{A-1} c_{2j+1}\, E^{2j+1}\, \mathrm{e}^{\left(\frac{n+r}{2} - \frac{n-r}{2r}(2j+1)\right)y}\Big]\,\psi = 0 \qquad \big(A = \tfrac{r-1}{2}\big)\,. \tag{2.18}$$

When all the coefficients c_{2j+1} vanish, the ODE becomes that for the $\mathcal{Z}_r$ invariant case (1.18). Also, as $n = r$, it reduces to the equation (2.14) where $\lambda(E)$ is given by (2.13), provided c_{2j+1} are set to $(-1)^j\, r\, \mathfrak{a}_{2j+1}$. In general, these coefficients would be some functions of the RG invariants of the form (1.22).

The ODE may be advocated for via the same types of arguments that were originally developed in ref. [9] for the Schrödinger equation for the anharmonic oscillator (which is equivalent to (1.18)). They are based on the invariance of (2.18) w.r.t. the transformation

$$\hat{\Omega}\ :\ y \mapsto y + \frac{2\pi\mathrm{i}}{n+r}\,, \qquad E \mapsto q^{-2}\, E \tag{2.19}$$

that holds true not only for $c_{2j+1} = 0$, but any values of these coefficients. Using this symmetry, one can derive the set of functional relations for the connection coefficients of the ODE. The most fundamental one is

$$\mathrm{e}^{\frac{2\pi\mathrm{i}p}{n+r}}\, D_+(qE)\, D_-(q^{-1}E) - \mathrm{e}^{-\frac{2\pi\mathrm{i}p}{n+r}}\, D_+(q^{-1}E)\, D_-(qE) = 2\mathrm{i}\sin\big(\tfrac{2\pi p}{n+r}\big)\,, \tag{2.20}$$

which is obeyed by the spectral determinants $D_\pm(E)$. In order to define the latter, apart from $\psi_p(y)$ (1.20), one should consider other solutions of the differential equation, $\psi_{-p}(y)$ and $\chi(y)$.

For large negative y, the potential in (2.18) approaches p^2. As a result, the ODE possesses a solution satisfying the asymptotic condition

$$\psi_{-p}(y) \to \mathrm{e}^{-py} \qquad \text{as} \qquad y \to -\infty \qquad \big(\Re e(p) \le 0\big)\,. \tag{2.21}$$

It turns out that the analytic continuation of $\psi_{-p}(y)$ to the full complex p - plane results in a meromorphic function of p. This allows one to specify the solution $\psi_{-p}(y)$ for any complex p, except for the discrete set of real, positive values $p = \frac{1}{2}(n+r),\, (n+r),\, \frac{3}{2}(n+r), \ldots$. Similarly $\psi_p(y)$, defined in the Introduction in formula (1.20), makes sense for any $p \ne -\frac{1}{2}(n+r),\, -(n+r), \ldots$. The Wronskian $W[\psi_{-p}, \psi_p] \equiv \psi_{-p}\,\partial_y\psi_p - \psi_p\,\partial_y\psi_{-p}$ is given by

$$W[\psi_{-p}, \psi_p] = 2p \tag{2.22}$$

so that the two functions form a basis in the space of solutions of the ODE (2.18) if $\frac{2p}{n+r} \notin \mathbb{Z}$.

The solution $\chi(y)$ is unambiguously defined through its asymptotic behaviour

$$\chi(y) \asymp \exp\Big(-\tfrac{n+r}{4}\, y - \tfrac{2}{n+r}\, \mathrm{e}^{\frac{n+r}{2}y} + o(1)\Big) \qquad \text{as} \qquad y \to +\infty \qquad (n > r)\,. \tag{2.23}$$

Note that, for now, we make the technical assumption $n > r$, while the case $0 < n < r$ will be discussed below. The spectral determinants are expressed via the Wronskians of $\psi_{\pm p}(y)$ and $\chi(y)$ as

$$D_\pm(E) = \frac{\sqrt{\pi}}{\Gamma(1 \pm \frac{2p}{n+r})}\, (n+r)^{-\frac{1}{2} \mp \frac{2p}{n+r}}\, W[\chi, \psi_{\pm p}]\,. \tag{2.24}$$

They are entire functions of E and the overall multiplicative factor has been chosen in such a way that

$$D_\pm(0) = 1\,. \tag{2.25}$$

For the proof of the relation (2.20) one should expand χ in the basis of solutions $\psi_{\pm p}$. It follows from (2.24) that

$$\chi(y) = \frac{1}{\sqrt{\pi(n+r)}}\Big(\Gamma\big(-\tfrac{2p}{n+r}\big)\,(n+r)^{-\frac{2p}{n+r}}\, D_-(E)\,\psi_{+p}(y) + \Gamma\big(\tfrac{2p}{n+r}\big)\,(n+r)^{\frac{2p}{n+r}}\, D_+(E)\,\psi_{-p}(y)\Big)\,. \tag{2.26}$$

Then one applies the symmetry transformation $\hat{\Omega}$ (2.19) to both sides of this equation and computes the Wronskian of $\chi(y)$ and $\hat{\Omega}\chi(y)$. Taking into account that $W[\hat{\Omega}\chi, \chi] = 2\mathrm{i}$, which is a consequence of (2.23), one easily arrives at (2.20).

To make a link to the lattice system, recall that the inhomogeneous six-vertex model possesses a commuting family of operators, a prominent member of which is the Baxter Q - operator [26]. In fact, there are two of them [5,6]. We will use the notation $\mathbb{A}_\pm(\zeta)$ borrowed from ref. [25] and the reader may consult sec. 3 of that work for their construction. Important properties of $\mathbb{A}_\pm(\zeta)$ are that they commute amongst themselves for different values of the spectral parameter,

$$\big[\mathbb{A}_\pm(\zeta_1),\, \mathbb{A}_\pm(\zeta_2)\big] = \big[\mathbb{A}_\pm(\zeta_1),\, \mathbb{A}_\mp(\zeta_2)\big] = 0\,, \tag{2.27}$$

and satisfy the quantum Wronskian relation

$$q^{+2\mathbb{P}}\ \mathbb{A}_+\big(q^{+1}\zeta\big)\,\mathbb{A}_-\big(q^{-1}\zeta\big)-q^{-2\mathbb{P}}\ \mathbb{A}_-\big(q^{+1}\zeta\big)\,\mathbb{A}_+\big(q^{-1}\zeta\big)=\big(q^{+2\mathbb{P}}-q^{-2\mathbb{P}}\big)\,f(\zeta)\,. \tag{2.28}$$

Here

$$f(\zeta)=\prod_{J=1}^{N}\big(1+\zeta/\eta_J\big)\,, \tag{2.29}$$

while

$$q^{2\mathbb{P}}=\mathrm{e}^{\mathrm{i}\pi\mathrm{k}}\,q^{\mathbb{S}^z} \tag{2.30}$$

with $\mathbb{S}^z$ being the z - projection of the total spin operator. In addition, in the sector with given value of $S^z\geq 0$, the eigenvalues of $\mathbb{A}_+(\zeta)$ are polynomials in ζ of order $N/2-S^z$. The above immediately implies that the zeroes of these polynomials coincide with the roots ζ_j of the Bethe Ansatz equations (1.11). The eigenvalues of $\mathbb{A}_-(\zeta)$ are likewise polynomials in ζ of order $N/2+S^z$, whose zeroes obey the equations similar to (1.11). The relation (2.28) holds true for arbitrary inhomogeneities, though we focus on the case where η_J obey the r - site periodicity condition (1.14).

The Hamiltonian $\mathbb{H}$ associated with the inhomogeneous six-vertex model belongs to the commuting family, i.e.,

$$\big[\mathbb{H},\mathbb{A}_\pm(\zeta)\big]=0\,. \tag{2.31}$$

Let $A_\pm(\zeta)$ be the eigenvalues of $\mathbb{A}_\pm(\zeta)$ corresponding to the ground state of the spin chain. In the $\mathcal{Z}_r$ invariant case the scaling relation for the Bethe roots (1.16) can be equivalently written as

$$\lim_{N\to\infty}A_\pm\Big(\big(N/(rN_0)\big)^{-\frac{2n}{r(n+r)}}\,E\Big)=D_\pm(E)\qquad\qquad (n>r) \tag{2.32}$$

with the constant N_0 defined in (1.17) and $D_\pm(E)$ are the spectral determinants for the ODE (1.18). In the scaling limit, the quantum Wronskian relation for $A_\pm(\zeta)$, which follows from (2.28), becomes the functional relation (2.20). Motivated by the analysis of the free fermion point, we propose to extend (2.32) to the case when $D_\pm$ are the spectral determinants for the ODE (2.18). In taking the limit, the inhomogeneities become N - dependent such that the values of

$$\mathfrak{a}_{2j+1}=\frac{1}{2j+1}\left(\frac{N}{rN_0}\right)^{1-\frac{2j+1}{r}}\ \frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-2j-1}\qquad\big(j=0,\ldots,\tfrac{r-3}{2}\big) \tag{2.33}$$

are kept fixed. Like in the free fermion case, $\mathfrak{a}_r$ can be specified to be $\frac{1}{r}$ or, equivalently,[3]

$$\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-r}=(-1)^{r-1}\,. \tag{2.34}$$

As for the quantities $\mathfrak{a}_s$ (1.22) with s even, we set them to be zero:

$$\sum_{\ell=1}^{r}(\eta_\ell)^{-2j}=0\qquad\qquad\big(j=1,\ldots,\tfrac{r-1}{2}\big)\,. \tag{2.35}$$

This can be thought of as taking the exponent d_s in (1.22) to be $+\infty$ for even s.

Our numerical work in support of the relation (2.32) went along the following line. First of all we note that for given values of $\mathfrak{a}_{2j+1}$, formulae (2.33)-(2.35) may be regarded as a system of equations for the determination of the inhomogeneities. It is easy to show that $(\eta_\ell)^{-1}$ are zeroes of a single polynomial equation of order r, whose coefficients are expressed in terms of $\mathfrak{a}_{2j+1}$. This defines the set $\{\eta_\ell\}_{\ell=1}^r$ up to permutations, which has no effect on the Bethe Ansatz equations. The roots corresponding to the ground state of the Hamiltonian $\mathbb{H}$ can be obtained by continuously deforming the solution to the Bethe Ansatz equations from the $\mathcal{Z}_r$ invariant case with $\mathfrak{a}_{2j+1}=0$ to the given values of the RG invariants. Then one may consider the sums

$$h_s^{(N)}=s^{-1}\sum_{j=1}^{\frac{N}{2}}\big(\zeta_j\big)^{-s}\qquad\qquad (s=1,2,\ldots)\,. \tag{2.36}$$

Taking into account that

$$A_+(\zeta)=\prod_{j=1}^{\frac{N}{2}}\left(1-\frac{\zeta}{\zeta_j}\right), \tag{2.37}$$

[3] In this section, r is always taken to be odd so that the sign factor in the r.h.s. of (2.34) is one. Nevertheless, keeping it there makes the formula applicable for any integer r, which will be explored later.

the convergence of the l.h.s. of (2.32) implies the existence of the limits

$$h_s^{(\infty)} = \lim_{N\to\infty} \left(\frac{rN_0}{N}\right)^{\frac{2sn}{r(n+r)}} h_s^{(N)} . \tag{2.38}$$

For $n > r$ this was indeed observed numerically. The spectral determinant $D_+(E)$ occurring in the scaling limit of $A_+(\zeta)$ is an entire function of E which satisfies the normalization condition $D_+(0) = 1$. Hence the series

$$\log D_+(E) = -\sum_{s=1}^{\infty} J_s\, E^s \tag{2.39}$$

has a finite radius of convergence. This way, the scaling relation (2.32) is equivalent to the infinite set of so-called sum rules:

$$h_s^{(\infty)} = J_s \qquad\qquad (s = 1, 2, \ldots) . \tag{2.40}$$

They can be used to provide numerical support for the ODE/IQFT correspondence as was originally done in ref. [19] in the study of the scaling limit of the Fateev-Zamolodchikov spin chain [27–29].

It is possible to calculate the first few coefficients J_s analytically by applying perturbation theory in E to the ODE (2.18). From the results of the work [30] one can show that

$$\begin{aligned} J_1 &= c_1\rho_1\ f_1(h,g_0) \\ J_2 &= \big(c_1\rho_1\big)^2\ f_2(h,g_0) \\ J_3 &= \big(c_1\rho_1\big)^3\ f_3(h,g_0) + c_3\rho_3\, f_1(h,g_1)\ . \end{aligned} \tag{2.41}$$

The functions f_s are given in the Appendix A, while

$$\rho_{2j+1} = \frac{(n+r)^{2g_j-2}}{\Gamma^2(1-g_j)} \tag{2.42}$$

and

$$h = \frac{p}{n+r}\ , \qquad g_j = \frac{1}{2} - \frac{(2j+1)(n-r)}{2r(n+r)}\ . \tag{2.43}$$

Since $h = \frac{\mathtt{k}}{2}$, see (1.19), formula (2.40) yields a prediction for the dependence of $h_s^{(\infty)}$ on the twist parameter. This can be checked even without the explicit knowledge of the relation between the RG invariants $\mathfrak{a}_{2j+1}$ and the coefficients c_{2j+1} of the differential equation. Note that the expressions for J_s for larger s are not available in analytical form. Nevertheless, it may be verified that the $\mathtt{k}$ - dependence of both sides of (2.40) agrees for $s \geq 4$ by computing J_s via a numerical integration of the ODE (2.18).

Through the study of the cases $r = 3, 5, 7$ we found that the relation between the coefficients of the differential equation and the RG invariants takes the general form

$$c_{2j+1} = C_j^{(j)}\, \mathfrak{a}_{2j+1} + \sum_{k=2}^{A-j} \sum_{\substack{j_1+\ldots+j_k=j+(k-1)A \\ 0\leq j_1,\ldots,j_k\leq A-1}} C_{j_1\cdots j_k}^{(j)}\ \mathfrak{a}_{2j_1+1}\cdots \mathfrak{a}_{2j_k+1}\ . \tag{2.44}$$

Here, for given r, $C_{j_1\cdots j_k}^{(j)}$ are some n - dependent constants. In the case $r = n$, all $C_{j_1\cdots j_k}^{(j)}$ with $k \geq 2$ vanish, while $C_j^{(j)} = (-1)^j r$. In what follows, we present a conjectured formula that expresses $\mathfrak{a}_{2j+1}$ in terms of c_{2j+1}, i.e., the inversion of (2.44). However, this first requires a discussion of the case $0 < n < r$.

2.3. Relation between the RG invariants and the coefficients of the ODE

For $0 < n \leq r$ the asymptotic condition (2.23), which is used to specify the subdominant as $y \to +\infty$ solution of the differential equation, is not literally applicable. The function $\chi(y)$ can be instead defined by means of its WKB asymptotic, and the spectral determinants are then calculated, as before, by means of formula (2.24). This turns out to be most efficient for the numerical computation of $D_\pm(E)$. However, for our purposes, it would be useful to introduce them for generic $n > 0$ via analytic continuation. Assuming $n > r$, we first derive the large - E asymptotic behaviour of $D_+(E)$. The latter has zeroes at $E_m^{(a)}$ defined through the condition (1.21):

$$D_+\big(E_m^{(a)}\big) = 0 . \tag{2.45}$$

These accumulate along the rays

$$\arg(E) = -\frac{\pi}{r}\, A,\ \frac{\pi}{r}\,(2-A),\ \ldots,\ \frac{\pi}{r}\,\big(2\,(r-1)-A\big) \qquad\quad (\mathrm{mod}\, 2\pi) . \tag{2.46}$$

The rays are the Stokes lines that divide the complex E - plane into r wedges, in which the large - E asymptotic behaviour of the spectral determinant is described differently. Inside each wedge, labelled by $a = 1, 2, \ldots, r$, we can parameterize E in terms of θ as[4]

$$E = (-1)^{r-1}\, e^{\frac{i\pi}{r}(2a-1-A)}\, e^{\frac{2n\theta}{r(n+r)}} \tag{2.47}$$

with

$$\Big|\Im m(\theta)\Big| < \frac{\pi(n+r)}{2n}\,. \tag{2.48}$$

Then the standard WKB analysis yields

$$\log D_+ \asymp \frac{N_0\, e^{\theta}}{\cos(\frac{\pi r}{2n})} - \sum_{j=0}^{A-1} \frac{D_{2j+1}}{\sin\big(\frac{\pi(2j+1)(n-r)}{2nr}\big)}\, e^{\frac{(2j+1)}{r}\big(\theta + i\pi(2a-1-A)\big)} - \frac{2np\,\theta}{r(n+r)} + \log C_p + o(1) \tag{2.49}$$

as $\Re e(\theta) \to +\infty$. Here

$$D_{2j+1} = \frac{\sqrt{\pi}\,\Gamma\big(\frac{1}{2} - \frac{n-r}{2nr}(2j+1)\big)}{2n\Gamma\big(1 - \frac{n-r}{2nr}(2j+1)\big)} \sum_{k=1}^{A} e^{-\frac{i\pi}{2}(r+1)(k-1)}\, \frac{\Gamma\big(m-1+\frac{n-r}{2nr}(2j+1)\big)}{m!\,\Gamma\big(\frac{n-r}{2nr}(2j+1)\big)}$$
$$\times \sum_{\substack{j_1+\ldots+j_k = j+(k-1)A \\ j_1,\ldots,j_k \geq 0}} c_{2j_1+1}\cdots c_{2j_k+1} \tag{2.50}$$

and

$$C_p = \sqrt{\frac{r}{n+r}}\; r^{\frac{2p}{r}}\, (n+r)^{-\frac{2p}{n+r}}\, \frac{\Gamma(1+\frac{2p}{r})}{\Gamma(1+\frac{2p}{n+r})}\,. \tag{2.51}$$

For $0 < n < r$ the spectral determinant can be defined to be the entire function of E, whose zeroes coincide with $E_m^{(a)}$ (2.45) and which obeys the asymptotic formula (2.49). This specifies $D_+(E)$ for any $n > 0$ except for a discrete set of values when the coefficients in the expansion (2.49) possess simple poles:

$$n = \frac{r(2j+1)}{2j+1+2wr} \qquad \text{with} \qquad j = 0, \ldots, A-1;\ w = 0, 1, 2, \ldots\ . \tag{2.52}$$

The spectral determinant $D_-(E)$ may be introduced analogously. Together, $D_\pm(E)$ satisfy the quantum Wronskian relation (2.20) for any $n > 0$ except (2.52).[5] We will exclude such exceptional n from our analysis to avoid making the discussion too technical (see, e.g., ref. [39] for a treatment of the cases $r = 1, 2$).

The scaling limit of the individual Bethe roots,

$$\lim_{\substack{N\to\infty \\ m-\text{fixed}}} \left(\frac{N}{rN_0}\right)^{\frac{2n}{r(n+r)}} \zeta_m^{(a)}\,, \tag{2.53}$$

exists for any positive n. However as $0 < n \leq r$, it turns out there are issues with the existence of the limit (2.38). To properly define $h_s^{(\infty)}$ certain subtractions need to be made so that the large - N limit can be taken. Namely, introduce the "regularized" version of the sum $h_s^{(N)}$:

$$h_s^{(N,\text{reg})} = s^{-1}\sum_{m=1}^{M}(\zeta_m)^{-s} + \frac{(-1)^{s-1}N}{2sr\cos(s\gamma)}\sum_{\ell=1}^{r}(\eta_\ell)^{-s}\,, \tag{2.54}$$

where M stands for the total number of Bethe roots and in the case of the ground state $M = \frac{N}{2}$. Without going into details, we mention that the form of the counterterm is motivated by the Bethe Ansatz equations (1.11) with the inhomogeneities subject to the r - site periodicity condition $\eta_{J+r} = \eta_J$. Taking into account the definition of $\mathfrak{a}_s$ (1.22), the above formula can be re-written as

$$h_s^{(N,\text{reg})} = s^{-1}\sum_{m=1}^{M}(\zeta_m)^{-s} + \frac{(-1)^{s-1}\, rN_0\mathfrak{a}_s}{2\cos(s\gamma)}\left(\frac{N}{rN_0}\right)^{1-d_s}\,. \tag{2.55}$$

This will be used to define $h_s^{(N,\text{reg})}$ for any value of $A = 0, 1, 2, \ldots, r-2$. In the case we are currently considering, where r is odd and $A = \frac{r-1}{2}$, all $\mathfrak{a}_{2j}$ with $j = 1, 2, \ldots, A$ are set to zero, so that

---

[4] For odd r, which we consider in this section, one can neglect the factor $(-1)^{r-1}$. It is included in eq. (2.47) since it will be applied for any r, $A = 0, 1, \ldots, r-2$ and to make the notation consistent with that of the work [31], devoted to the case $A = 0$. The remark also carries over to eq. (2.62), below.

[5] If we require the spectral determinants to be entire functions of E, the quantum Wronskian relation (2.20) must be modified when n takes the values (2.52). The simplest illustration is provided by the case $r = n$, where $D_+(E)$ is given by (2.17), while $D_-(E)$ is obtained from the former via the substitution $p \mapsto -p$.

$$h_{2j}^{(N,\mathrm{reg})} = h_{2j}^{(N)} \,. \tag{2.56}$$

At the same time, the values of $\mathfrak{a}_{2j+1}$ with $j = 0, 1, \ldots, A-1$ are generic complex numbers, which are held fixed in taking the scaling limit, and the exponent d_{2j+1} in (2.55) is given by

$$d_{2j+1} = 1 - \frac{2j+1}{r} \qquad\qquad (j = 0, 1, \ldots, A-1)\,. \tag{2.57}$$

Hence,

$$\left(\frac{rN_0}{N}\right)^{\frac{2(2j+1)n}{r(n+r)}} h_{2j+1}^{(N,\mathrm{reg})} = \left(\frac{rN_0}{N}\right)^{\frac{2(2j+1)n}{r(n+r)}} h_{2j+1}^{(N)} + \frac{(-1)^j\, rN_0\, \mathfrak{a}_{2j+1}}{2\sin\left(\frac{\pi(2j+1)(n-r)}{2r(n+r)}\right)} \left(\frac{rN_0}{N}\right)^{\frac{(2j+1)(n-r)}{r(n+r)}} \,. \tag{2.58}$$

For $n > r$ the second term in the r.h.s. tends to zero as $N \to \infty$ and can be neglected. However, as $0 < n < r$ it grows and cancels the divergent behaviour of the first term, ensuring the existence of the limit

$$h_s^{(\infty)} = \lim_{N\to\infty} \left(\frac{rN_0}{N}\right)^{\frac{2sn}{r(n+r)}} h_s^{(N,\mathrm{reg})} \,. \tag{2.59}$$

Note that even when $n > r$, the last formula is more efficient than (2.38) for numerical purposes since the inclusion of the counterterm greatly improves the convergence. For instance, we observed

$$\left(\frac{rN_0}{N}\right)^{\frac{2n}{r(n+r)}} h_1^{(N,\mathrm{reg})} = h_1^{(\infty)} + O\big(N^{-2}\big) \qquad (n > r-2)\,, \tag{2.60}$$

while for $0 < n < r-2$ the correction term goes to zero as $N \to \infty$ with a certain exponent which depends on n.

Closely related to the divergent behaviour of $h_s^{(N)}$ for $0 < n \leq r$ is that some of the coefficients in the expansion of the spectral determinant (2.39) possess a simple pole at $n = r$. It is worth discussing this in some detail since it turns out to give a hint of how to obtain the relations between the RG invariants and the coefficients of the differential equation. Since $E_m^{(a)}$ are zeroes of $D_+(E)$, it follows that J_s can be written in the form

$$J_s = \frac{1}{s} \sum_{m=1}^{\Lambda} \sum_{a=1}^{r} \big(E_m^{(a)}\big)^{-s} + \Xi_s(\Lambda)\,. \tag{2.61}$$

Here the remainder $\Xi_s(\Lambda) \to 0$ as the cutoff $\Lambda \to \infty$ for $n > r$. Obtaining the large - Λ behaviour of $\Xi_s(\Lambda)$ requires knowledge of the asymptotics of the zeroes $E_m^{(a)}$ as $m \gg 1$. The latter can be extracted from the ODE (2.18) using the WKB approximation.

The asymptotic formula (2.49) for $D_\pm(E)$ holds true inside the wedge in the complex E - plane described by eqs. (2.47), (2.48) and can not be applied along its boundary. At the boundary of the a^{th} and $(a+1)^{\mathrm{th}}$ (mod r) wedges, E can be parameterized by real positive θ as

$$E = (-1)^{r-1}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)}\, \mathrm{e}^{\frac{2n\theta}{r(n+r)}} \qquad\qquad (\theta > 0)\,. \tag{2.62}$$

Along this ray, the asymptotic behaviour of $D_+(E)$ is given by

$$D_+ \asymp 2C_p\, \mathrm{e}^{-\frac{2np\theta}{r(n+r)}}\, \mathrm{e}^{-\chi_a(\theta)} \cos\big(\phi_a(\theta)\big) \qquad\qquad (\theta \to +\infty)\,, \tag{2.63}$$

where

$$\chi_a(\theta) = N_0 \tan\left(\tfrac{\pi r}{2n}\right) \mathrm{e}^{\theta} + \sum_{j=0}^{A-1} D_{2j+1}\, \cot\left(\tfrac{\pi(2j+1)(n-r)}{2nr}\right) \mathrm{e}^{\frac{(2j+1)}{r}\left(\theta + \mathrm{i}\pi(2a-A)\right)} + o(1) \tag{2.64}$$

and

$$\phi_a(\theta) = N_0 \mathrm{e}^{\theta} + \sum_{j=0}^{A-1} \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)(2j+1)}\, D_{2j+1}\, \mathrm{e}^{\frac{2j+1}{r}\theta} - \frac{\pi p}{r} + o(1)\,. \tag{2.65}$$

In view of eq. (2.62) we write the zeroes $E_m^{(a)}$ in the form

$$E_m^{(a)} = (-1)^{r-1}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)}\, \exp\left(\tfrac{2n}{r(n+r)}\, \theta_m^{(a)}\right)\,. \tag{2.66}$$

Then $\theta_m^{(a)}$ solves the equation

$$\phi_a\big(\theta_m^{(a)}\big) = \pi\big(m - \tfrac{1}{2}\big) \qquad\quad (m = 1, 2, \ldots)\,, \tag{2.67}$$

which is similar to the usual Bohr-Sommerfeld quantization condition. It is convenient to re-write the latter in a different way. Introduce X and Y such that

$$X=-\mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)}\,\mathrm{e}^{\frac{1}{r}\theta_m^{(a)}}\,,\qquad Y=-\mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)}\left(\frac{\pi}{N_0}\left(m-\frac{1}{2}+\frac{p}{r}\right)\right)^{\frac{1}{r}} \tag{2.68}$$

as well as

$$G_{r-2j-1}=(-1)^j\ \frac{D_{2j+1}}{N_0}\ . \tag{2.69}$$

Formula (2.67), upon dropping the term $o(1)$ in (2.65), then takes the form

$$X^r+\sum_{m=1}^{A}G_{2m}X^{r-2m}=Y^r\ , \tag{2.70}$$

which can be considered as a polynomial equation to determine X given Y. Since m is assumed to be large it follows from the definition (2.68) that $|Y|\gg 1$. Moreover, eq. (2.70) has r roots and we should focus on the one, $X(Y)$, such that

$$X(Y)=Y+o(1)\qquad \text{as}\qquad Y\to\infty\ . \tag{2.71}$$

The Lagrange formula [32,33] (see also Appendix B) gives this root as a series in inverse powers of Y:

$$X(Y)=Y+\sum_{\substack{j=0\\ 2j+1\neq 0\ \mathrm{mod}\, r}}^{\infty}R_{2j+1}\ Y^{-2j-1} \tag{2.72}$$

with

$$R_{2j+1}=\frac{1}{r}\sum_{\substack{\alpha_1+2\alpha_2+\ldots+(j+1)\alpha_{j+1}=j+1\\ \alpha_1,\ldots,\alpha_{j+1}\geq 0}}\frac{(-1)^{\alpha_1+\ldots+\alpha_{j+1}}}{\alpha_1!\alpha_2!\ldots\alpha_{j+1}!}\ \frac{\Gamma(\alpha_1+\ldots+\alpha_{j+1}-\frac{2j+1}{r})}{\Gamma(1-\frac{2j+1}{r})}\ G_2^{\alpha_1}G_4^{\alpha_2}\ldots G_{2j+2}^{\alpha_{j+1}}\ . \tag{2.73}$$

In the case at hand, as it follows from (2.69) and (2.50), G_{2m} is given by

$$\begin{aligned}G_{2m}=(-1)^{A-m}\ &\frac{\Gamma\big(\frac{3}{2}+\frac{r}{2n}\big)\Gamma\big(\frac{1}{2}-\frac{(r-2m)(n-r)}{2nr}\big)}{\Gamma\big(\frac{r}{2n}\big)\Gamma\big(1-\frac{(r-2m)(n-r)}{2nr}\big)}\ \sum_{k=1}^{m}(-1)^{(A+1)(k-1)}\ \frac{\Gamma\big(k-1+\frac{n-r}{2nr}(r-2m)\big)}{m!\,\Gamma\big(\frac{n-r}{2nr}(r-2m)\big)}\\ &\times\sum_{\substack{j_1+\ldots+j_k=kA-m\\ j_1,\ldots,j_k\geq 0}}c_{2j_1+1}\cdots c_{2j_k+1}\ .\end{aligned} \tag{2.74}$$

Combining the above with (2.66) yields the asymptotic formula for the zeroes:

$$E_m^{(a)}\asymp \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)}\ |Y_m|^{\frac{2n}{n+r}}\ \left(1+\sum_{k=1}^{A}R_{2k-1}\ \mathrm{e}^{-\frac{\mathrm{i}\pi}{r}(4a+1)k}\ |Y_m|^{-2k}+O\big(m^{-1}\big)\right)^{\frac{2n}{n+r}}\qquad (m\gg 1)\ , \tag{2.75}$$

where

$$|Y_m|=\left(\frac{\pi}{N_0}\left(m-\frac{1}{2}+\frac{p}{r}\right)\right)^{\frac{1}{r}}\ . \tag{2.76}$$

The remainder $\Xi_s(\Lambda)$ in (2.61) involves a sum over negative integer powers of $E_m^{(a)}$. They are expressed through $Q_{2k-1}(z)$, which are the coefficients in the formal expansion

$$\Bigg(1+\sum_{\substack{k=1\\ 2k-1\neq 0\ \mathrm{mod}\, r}}^{\infty}R_{2k-1}\ Y^{-2k}\Bigg)^{z}=1+\sum_{k=1}^{\infty}Q_{2k-1}(z)\ Y^{-2k}\ . \tag{2.77}$$

Then

$$\big(E_m^{(a)}\big)^{-s}\asymp \mathrm{e}^{-\frac{\mathrm{i}\pi}{r}(2a-2-A)s}\ |Y_m|^{-s\nu}\ \left(1+\sum_{k=1}^{A}Q_{2k-1}(-s\nu)\ \mathrm{e}^{-\frac{\mathrm{i}\pi}{r}(4a+1)k}\ |Y_m|^{-2k}+O\big(m^{-1}\big)\right), \tag{2.78}$$

where we use

$$\nu=\frac{2n}{n+r}\ . \tag{2.79}$$

Taking s to be odd and performing the sum over a leads to

$$\sum_{a=1}^{r}\big(E_m^{(a)}\big)^{-s}\asymp r\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(s-1)}\ Q_{r-s-1}(-s\nu)\ |Y_m|^{-r+s(1-\nu)}+O\Big(m^{-2+(1-\nu)\frac{s}{r}}\Big)\qquad (s=2j+1)\ . \tag{2.80}$$

It follows that the large Λ behaviour for the remainder in (2.61) is given by

$$\Xi_{2j+1}(\Lambda)=\frac{(-1)^j\, rN_0}{\left(\frac{\pi(2j+1)(n-r)}{r(n+r)}\right)}\ \frac{Q_{r-2j-2}\big(-(2j+1)\nu\big)}{2j+1}\ \left(\frac{N_0}{\pi\Lambda}\right)^{\frac{(2j+1)(n-r)}{r(n+r)}}+O\Big(\Lambda^{-1-\frac{(2j+1)(n-r)}{r(n+r)}}\Big)\,. \tag{2.81}$$

Let's compare the residues of (2.81) and (2.58) at the pole $n=r$. It leads to the relation

$$\mathfrak{a}_{2j+1}=\frac{1}{2j+1}\ Q_{r-2j-2}\big(-(2j+1)\nu\big)\ , \tag{2.82}$$

or, explicitly, taking into account the definition (2.77),

$$\mathfrak{a}_{2j+1}=\frac{1}{2j+1}\sum_{k=1}^{A-j}(-1)^k\ \frac{\Gamma\big((2j+1)\nu+k\big)}{k!\Gamma\big((2j+1)\nu\big)}\sum_{\substack{j_1+j_2\ldots+j_k=A-j-k\\ j_1,\ldots,j_k\geq 0}} R_{2j_1+1}\ldots R_{2j_k+1}\ , \tag{2.83}$$

where R_{2j+1} is given by eqs. (2.73) and (2.74). Of course, the above is expected to be valid only for $n=r$. Nevertheless, we found from the numerical work that it holds true for any $n>0$. The relation can be inverted, whereupon it takes the form (2.44). The coefficients $C^{(j)}_{j_1\cdots j_k}$ are rather cumbersome in general and the simplest of them is given by

$$C^{(j)}_j=(-1)^j\ \frac{r\Gamma\big(\frac{r}{2n}\big)\Gamma\big(1-\frac{(2j+1)(n-r)}{2rn}\big)}{\Gamma\big(\frac{1}{2}+\frac{r}{2n}\big)\Gamma\big(\frac{1}{2}-\frac{(2j+1)(n-r)}{2rn}\big)}\ . \tag{2.84}$$

Formula (2.44) is written out explicitly in the Appendix C for the cases $r=3,5,7$. Numerical work confirms its validity.

2.4. Some remarks about the scaling limit

In our discussion we focused on describing the large - N behaviour of the Bethe roots in the vicinity of $\zeta=0$. A similar analysis can be performed for the Bethe roots located near $\zeta=\infty$. In this case, (1.16) is replaced by

$$\bar{E}^{(a)}_m=\lim_{\substack{N\to\infty\\ m-\text{fixed}}}\left(\frac{N}{rN_0}\right)^{\frac{2n}{r(n+r)}}\left(\zeta^{(a)}_{M_a-m}\right)^{-1}, \tag{2.85}$$

where for the ground state in the sector $S^z=0$ all M_a are equal and coincide with $N/(2r)$. If the scaling limit is taken with $\mathfrak{a}_{2j+1}$ (2.33) kept fixed and the conditions (2.34), (2.35) are imposed on the inhomogeneities, it turns out that $\bar{E}^{(a)}_m$ are described by the differential equation corresponding to the $\mathcal{Z}_r$ invariant case similar to (1.18). This can be observed by plotting the Bethe roots in the complex β - plane with $\beta=-\frac{1}{2}\log(\zeta)$, which is done in the left panel of Fig. 1 for the case $r=3$. One sees that for the roots on the right side of the distribution, for which $|\zeta_j|\ll 1$, there is significant deviation from the lines $\Im m(\beta)=\pm\frac{\pi}{6},\ \frac{\pi}{2}$ so that their scaling behaviour is described via the differential equation (2.18) with $r=3$. The Bethe roots depicted on the left side of the figure have $|\zeta_j|\gg 1$ and are visibly located along the lines with $\Im m(\beta)=\pm\frac{\pi}{6},\ \frac{\pi}{2}$. Their scaling limit turns out to be same as that of the Bethe roots for the $\mathcal{Z}_3$ invariant case.

Keeping in mind the invariance of the Bethe Ansatz equations (1.11) w.r.t. the simultaneous inversion $\zeta_j\mapsto\zeta_j^{-1}$, $\eta_\ell\mapsto(\eta_\ell)^{-1}$ and $\omega\mapsto\omega^{-1}$ ($\mathtt{k}\mapsto-\mathtt{k}$), it is clear how to organize the scaling limit such that the scaling behaviour of the roots near $\zeta=0$ and $\zeta=\infty$ is reversed. One should treat

$$\bar{\mathfrak{a}}_s=\frac{1}{s}\left(\frac{N}{rN_0}\right)^{\bar{d}_s}\ \frac{1}{r}\ \sum_{\ell=1}^{r}(\eta_\ell)^s \tag{2.86}$$

with $s=2j+1$ and $\bar{d}_{2j+1}=1-\frac{2j+1}{r}$ as the RG invariants and impose the additional restrictions on the inhomogeneities:

$$\sum_{\ell=1}^{r}(\eta_\ell)^{2j}=0\qquad\qquad (j=1,2,\ldots,A) \tag{2.87}$$

and

$$\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^r=(-1)^{r-1}\,. \tag{2.88}$$

The numbers $\bar{E}^{(a)}_m$ (2.85) would then be described by the "barred" version of the ODE (2.18):

$$\Big[-\partial^2_{\bar{y}}+\bar{p}^2+\mathrm{e}^{(n+r)\bar{y}}-(-1)^A\,\bar{E}^r\,\mathrm{e}^{r\bar{y}}-\sum_{j=0}^{A-1}\bar{c}_{2j+1}\,\bar{E}^{2j+1}\,\mathrm{e}^{\left(\frac{n+r}{2}-\frac{n-r}{2r}(2j+1)\right)\bar{y}}\Big]\,\bar{\psi}=0\qquad \big(A=\tfrac{r-1}{2}\big)\,. \tag{2.89}$$

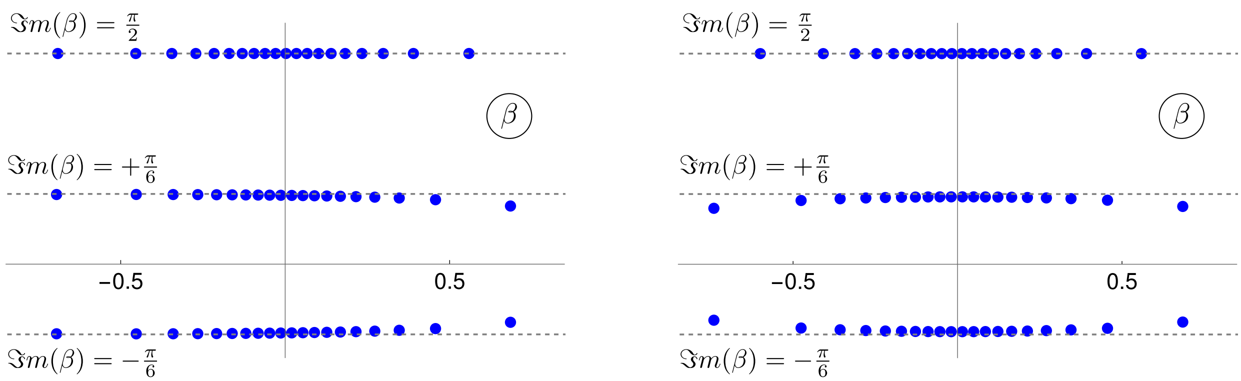

Fig. 1. Depicted are plots of the Bethe roots in the complex plane $\beta = -\frac{1}{2}\log(\zeta)$ for $r = 3$, $N = 120$, $n = 5$ and $\Bbbk = 0.05$. For the left panel, the inhomogeneities are taken to satisfy the conditions (2.33) - (2.35) with $\mathfrak{a}_1 = 0.4$. The right panel corresponds to the case when the spin chain possesses $\mathcal{CP}$ and $\mathcal{T}$ symmetry. The inhomogeneities are specified according to (2.96) - (2.98) with $\mathfrak{b}_1 = 0.4$.

Here $\bar{p} = -p = -\frac{n+r}{2}\ \Bbbk$, while formula (2.44) with the replacements $(\mathfrak{a}_{2j+1}, c_{2j+1}) \mapsto (\bar{\mathfrak{a}}_{2j+1}, \bar{c}_{2j+1})$ gives the relation between the RG invariants and the coefficients of the differential equation.

In view of applications to QFT, of special interest are the spin chains admitting extra global symmetries — charge conjugation $\mathcal{C}$, parity $\mathcal{P}$ and time reversal $\mathcal{T}$. If the inhomogeneities obey the condition

$$\eta_\ell = \big(\eta_{r+1-\ell}\big)^{-1} \qquad (\ell = 1, 2, \ldots, r) \tag{2.90}$$

then the system possesses $\mathcal{CP}$ invariance, i.e.,

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\,\mathbb{H}\,\hat{\mathcal{C}}\hat{\mathcal{P}} = \mathbb{H}\,. \tag{2.91}$$

For the action of the involutory operators $\hat{\mathcal{C}}$ and $\hat{\mathcal{P}}$ on the space of states of the spin chain we refer the reader to sec. 4 of the paper [25]. In the latter, it is also described how the $\mathcal{CP}$ transformation acts on the Baxter Q - operators:

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\,\mathbb{A}_\pm(\zeta)\,\hat{\mathcal{C}}\hat{\mathcal{P}} = \zeta^{\frac{N}{2}\mp\mathbb{S}^z}\ \mathbb{A}_\mp\big(\zeta^{-1}\big)\,\big[\mathbb{A}_\pm^{(\infty)}\big]^{-1}, \tag{2.92}$$

where

$$\mathbb{A}_\pm^{(\infty)} = \lim_{\zeta\to\infty} \zeta^{-\frac{N}{2}\pm\mathbb{S}^z}\,\mathbb{A}_\pm(\zeta)\,. \tag{2.93}$$

Note that $\hat{\mathcal{C}}\hat{\mathcal{P}}$ intertwines the sectors with S^z and $-S^z$. If, in addition to (2.90), the inhomogeneities obey the reality condition

$$\eta_\ell^* = \eta_\ell^{-1}\,, \tag{2.94}$$

then the spin chain possesses time reversal symmetry $\mathcal{T}$ (again see ref. [25] for details). This way, the requirements (2.90) and (2.94) will guarantee that the field theory underlying the critical behaviour of the spin chain is $\mathcal{CP}$ and $\mathcal{T}$ invariant. Also note that, without loss of generality, the inhomogeneities can be chosen in such a way to satisfy the condition

$$\prod_{\ell=1}^{r} \eta_\ell = 1 \tag{2.95}$$

even for the spin chain, where no other constraints on η_ℓ are being assumed.

For the $\mathcal{Z}_r$ invariant case, $\eta_\ell = (-1)^r\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)}$ and the conditions (2.90), (2.94) and (2.95) are satisfied. For the model, which possesses $\mathcal{CP}$ and $\mathcal{T}$ symmetries but $\mathcal{Z}_r$ invariance is broken, the inhomogeneities can be parameterized through the phases δ_ℓ as

$$\eta_\ell = (-1)^r\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)+\mathrm{i}\delta_\ell} \tag{2.96}$$

with

$$\delta_\ell = \delta_\ell^* = -\delta_{r+1-\ell}\,, \qquad \delta_1 + \ldots + \delta_r = 0\,. \tag{2.97}$$

In the case when r is odd and $A = \frac{r-1}{2}$, one can consider the limit with $\delta_\ell \to 0$ as $N \to \infty$ such that

$$\delta_\ell = -\delta_{r+1-\ell} = 2\sum_{j=0}^{A-1} \sin\big(\tfrac{\pi}{r}(2\ell-1)(2j+1)\big)\ \mathfrak{b}_{2j+1}\left(\frac{rN_0}{N}\right)^{1-\frac{2j+1}{r}} \qquad (\ell = 1, \ldots, A+1) \tag{2.98}$$

keeping the real numbers $\mathfrak{b}_{2j+1}$ fixed. The Bethe roots for the ground state of the spin chain with $N = 120$ and $\mathfrak{b}_1 = 0.4$ are depicted in the right panel of Fig. 1, which may be compared with the plot in the left panel. Our numerical work for the cases $r = 3, 5, 7$ shows that the Bethe roots still possess the scaling behaviour (1.16) and (2.85) with $E_m^{(a)}$ and $\bar{E}_m^{(a)}$ described in terms of the differential equations (2.18) and (2.89), respectively. If we assume that the scaling limit exists then formula (2.92) implies that the coefficients c_{2j+1} must coincide with their barred counterparts:

$$\bar{c}_{2j+1} = c_{2j+1} \qquad (j = 0, \dots, A-1)\,. \tag{2.99}$$

The relation between $\mathfrak{b}_{2j+1}$ and c_{2j+1} turns out to take the form

$$c_{2j+1} = C_j^{(j)}\, \mathfrak{b}_{2j+1} + \dots\,, \tag{2.100}$$

where the ellipses involve a sum over monomials in $\mathfrak{b}_{2j'+1}$ with $j' < j$, each of which is quadratic or of higher order in the RG invariants. The coefficient for the linear term coincides with $C_j^{(j)}$ from (2.44) and is given by (2.84). It is worthing noting that though the $N \to \infty$ limit (2.59) involving the regularized sum over the Bethe roots $h_s^{(N,\mathrm{reg})}$ still exists and is described by (2.40), the rate of convergence is considerably slower. Moreover, the definition of $h_s^{(N,\mathrm{reg})}$ requires different subtractions, depending on the RG invariant $\mathfrak{b}_{2j+1}$ but not on $\mathtt{k}$, in order to ensure the existence of the limit for any $n > 0$. In particular, when the scaling limit was defined as in sec. 2.2, $h_{2j}^{(N,\mathrm{reg})} = h_{2j}^{(N)}$, see (2.56), while for the case (2.98) this is no longer true.

Thus we see that there is freedom in the way in which one takes the scaling limit — different prescriptions give rise to the ODEs of the same form (2.18) and (2.89) which describe the scaling of the Bethe roots for the vacua. The coefficients of the differential equations depend on the set of RG invariants that specify the scaling procedure. Note that the number of non-vanishing RG invariants may not necessarily coincide with the number of coefficients c_{2j+1}. For instance, we discussed the scaling limit, where $\sum_{\ell=1}^{r}(\eta_\ell)^{-2j}$ were set to zero for $j = 1, 2, \dots, A$. As was mentioned before, this formally corresponds to taking the exponent d_{2j} in (1.22) to be $+\infty$. However, if the condition is relaxed and one considers the limit with non-vanishing RG invariants $\mathfrak{a}_{2j}$ but d_{2j} sufficiently large positive numbers, the ODEs describing the scaling limit remain the same. In particular for the cases $r = 3, 5$ we found that if the exponent $d_{2j} > \frac{1}{r}$, the presence of the non-zero RG invariants $\mathfrak{a}_{2j}$ does not essentially affect the scaling behaviour.

2.5. Low energy spectrum in the scaling limit

In this paper, we focus on describing the scaling of the Bethe roots for the ground state of the spin chain, which may be regarded as a rather formal mathematical problem. From the physical point of view the most interesting question would be the study of the field theory underlying the critical behaviour of the lattice system. While this goes beyond the scope of our work, here we nevertheless give a brief discussion of the spectrum of the Hamiltonian $\mathbb{H}$ in the scaling limit in the regime with $\frac{\pi}{2}(1-\frac{1}{r}) < \gamma < \frac{\pi}{2}(1+\frac{1}{r})$ and r odd. The analysis is mostly informed by a study of the free fermion point with $\gamma = \frac{\pi}{2}$.

2.5.1. Low energy spectrum for the $\mathcal{Z}_r$ invariant spin chain

For a given solution of the Bethe Ansatz equations (1.11), with the inhomogeneities obeying the r - site periodicity condition (1.12), the corresponding eigenvalue of the Hamiltonian $\mathbb{H}$ reads as

$$\mathcal{E} = 2\mathrm{i} \sum_{\ell=1}^{r} \sum_{m=1}^{N/2-S^z} \left(\frac{1}{1+\zeta_m q^{-1}/\eta_\ell} - \frac{1}{1+\zeta_m q/\eta_\ell} \right). \tag{2.101}$$

The Hamiltonian commutes with the r - site translation operator $\mathbb{K}$, whose eigenvalue $\mathcal{K}$ is computed from the Bethe roots as (see, e.g., sec. 6 of ref. [25] for details)

$$\mathcal{K} = \mathrm{e}^{\mathrm{i}r\pi\mathtt{k}}\, q^{-r(\frac{N}{2}-S^z)} \prod_{\ell=1}^{r} \prod_{m=1}^{N/2-S^z} \frac{\zeta_m + \eta_\ell\, q^{+1}}{\zeta_m + \eta_\ell\, q^{-1}}\,. \tag{2.102}$$

The lowest energy state in the sector $S^z = 0$ is translationally invariant, i.e., $\mathcal{K}_{\mathrm{GS}} = 1$. As was mentioned in the Introduction, in the $\mathcal{Z}_r$ invariant case with

$$\eta_\ell = (-1)^r\, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)} \qquad (\ell = 1, \dots, r) \tag{2.103}$$

the corresponding Bethe roots are simply related to those for the ground state of the XXZ spin chain. Using well known facts about the latter, it is straightforward to deduce the large - N behaviour for the ground state energy:

$$\mathcal{E}_{\mathrm{GS}}^{(0)} = N e_\infty + \frac{2\pi r v_{\mathrm{F}}}{N} \left(\frac{1}{2}(n+r)\mathtt{k}^2 - \frac{r}{12} \right) + o(N^{-1})\,, \tag{2.104}$$

where the specific bulk energy e_∞ and the Fermi velocity v_{F} are given by

$$e_\infty = -\frac{2v_{\mathrm{F}}}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\sinh\left(\frac{rt}{n}\right)}{\sinh\left(\frac{(n+r)t}{n}\right)\cosh(t)}\,, \qquad v_{\mathrm{F}} = \frac{r(n+r)}{n}\,. \tag{2.105}$$

The superscript "(0)" has been introduced in order to emphasize that the above formulae are valid for the $\mathcal{Z}_r$ invariant spin chain. Eq. (2.104) leads one to expect that in the scaling limit $\mathbb{H}^{(0)}$ can be written as

$$\mathbb{H}^{(0)} \asymp N e_\infty + \frac{2\pi r v_{\rm F}}{N}\, \hat{H}_{\rm CFT} + o(N^{-1})\,. \tag{2.106}$$

Here $\hat{H}_{\rm CFT}$ stands for the Hamiltonian of the underlying CFT,

$$\hat{H}_{\rm CFT} = \int_0^{2\pi} \frac{{\rm d}x}{2\pi}\,\big(T + \overline{T}\,\big)\,, \tag{2.107}$$

with T and $\overline{T}$ being the chiral components ($\partial\overline{T} = \bar{\partial}T = 0$) of the canonically normalized energy-momentum tensor.

For the XXZ spin chain (1.1), corresponding to $r = 1$, the low energy states for any value of the anisotropy parameter $-1 < \Delta < 1$ can be classified in the same way as for the free fermion point with $\Delta = 0$. Namely, one should choose the basis of eigenstates of the Hamiltonian to be the Bethe states Ψ_N, which are labelled by the solution of the Bethe Ansatz equations. In the scaling limit Ψ_N takes the form of the tensor product of the chiral states

$$\underset{N\to\infty}{\rm slim}\, \Psi_N = |\alpha\rangle \otimes |\bar{\alpha}\rangle\,, \tag{2.108}$$

where the symbol "slim" is used as a reminder that the formula applies for the class of low energy states only. Also, strictly speaking, the existence of the limit requires that the Bethe state Ψ_N be properly normalized (for details see ref. [16]). As was discussed in [10] (see also [34]), in a given sector with fixed S^z the chiral state $|\alpha\rangle$ can be labelled by two sets of non-negative integers

$$1 \le n_1^\pm < n_2^\pm < \ldots < n_{M^\pm}^\pm\,. \tag{2.109}$$

Similarly $|\bar{\alpha}\rangle$ is labelled by

$$1 \le \bar{n}_1^\pm < \bar{n}_2^\pm < \ldots < \bar{n}_{\bar{M}^\pm}^\pm \tag{2.110}$$

and the difference $\bar{M}^+ - \bar{M}^-$ should coincide with $M^- - M^+$. The latter can be identified with the so-called winding number:

$$\mathtt{w} = M^- - M^+ = \bar{M}^+ - \bar{M}^-\,. \tag{2.111}$$

Formula (2.108) implies that the eigenvalues of the CFT Hamiltonian $\hat{H}_{\rm CFT}$ split into two terms, $I_1 + \bar{I}_1$, where I_1 ($\bar{I}_1$) stands for the contribution of the chiral state $|\alpha\rangle$ ($|\bar{\alpha}\rangle$). In the case of the scaling limit of the XXZ spin chain,

$$I_1 = \frac{p^2}{n+1} - \frac{1}{24} + \mathtt{L}\,, \qquad \bar{I}_1 = \frac{\bar{p}^2}{n+1} - \frac{1}{24} + \bar{\mathtt{L}} \qquad (r=1)\,. \tag{2.112}$$

Here

$$p = \frac{S^z}{2} + \frac{n+1}{2}\,(\mathtt{k}+\mathtt{w})\,, \qquad \bar{p} = \frac{S^z}{2} - \frac{n+1}{2}\,(\mathtt{k}+\mathtt{w})\,, \tag{2.113}$$

while the non-negative integers $\mathtt{L}$, $\bar{\mathtt{L}}$ are the conformal levels. In terms of the sets (2.109), (2.110) they are given by

$$\begin{aligned} \mathtt{L} &= \sum_{j=1}^{M^-}\big(n_j^- - \tfrac12\big) + \sum_{j=1}^{M^+}\big(n_j^+ - \tfrac12\big) - \sum_{j=1}^{|\mathtt{w}|}\big(j - \tfrac12\big) \\ \bar{\mathtt{L}} &= \sum_{j=1}^{\bar{M}^-}\big(\bar{n}_j^- - \tfrac12\big) + \sum_{j=1}^{\bar{M}^+}\big(\bar{n}_j^+ - \tfrac12\big) - \sum_{j=1}^{|\mathtt{w}|}\big(j - \tfrac12\big) \end{aligned}\,. \tag{2.114}$$

As for the eigenvalue of the lattice translation operator for the low energy state Ψ_N, one has

$$\mathcal{K} = (-1)^{\mathtt{w}} \exp\bigg(\frac{2\pi {\rm i} r}{N}\,\big(I_1 - \bar{I}_1\big)\bigg) \qquad (r=1)\,. \tag{2.115}$$

In Appendix D we briefly discuss the spectrum of the Baxter Q - operator $\mathbb{A}_+(\zeta)$ (2.27) in the scaling limit for the $\mathcal{Z}_r$ invariant spin chain at the free fermion point $\gamma = \frac{\pi}{2}$. For a low energy Bethe state in the sector with given S^z, the chiral states $|\alpha\rangle$ and $|\bar{\alpha}\rangle$ appearing in the r.h.s. of (2.108) can be labelled by the positive integers

$$1 \le n_{1,a}^\pm < n_{2,a}^\pm < \ldots < n_{M_a^\pm,a}^\pm \tag{2.116}$$

and

$$1 \le \bar{n}_{1,a}^\pm < \bar{n}_{2,a}^\pm < \ldots < \bar{n}_{\bar{M}_a^\pm,a}^\pm\,, \tag{2.117}$$

respectively, where $a = 1, 2, \ldots, r$. The number of elements in the sets $\{n^{\pm}_{j,a}\}$ and $\{\bar{n}^{\pm}_{j,a}\}$, i.e., $M^{\pm}_a$ and $\bar{M}^{\pm}_a$, are subject to the constraint

$$\sum_{a=1}^{r}\big(M_a^- - M_a^+\big) + \sum_{a=1}^{r}\big(\bar{M}_a^- - \bar{M}_a^+\big) = 0\,. \tag{2.118}$$

For $r > 1$, the relation (2.111) is replaced by

$$\frac{1}{r}\sum_{a=1}^{r}\big(M_a^- - M_a^+\big) = \frac{1}{r}\sum_{a=1}^{r}\big(\bar{M}_a^+ - \bar{M}_a^-\big) = \mathtt{w} + \frac{\mathtt{m}}{r}\,, \tag{2.119}$$

which, together with the winding number $\mathtt{w} \in \mathbb{Z}$, involves

$$\mathtt{m}\ :\qquad \mathtt{m} = 0, 1, 2, \ldots, r-1\,. \tag{2.120}$$

Recall that N is always assumed to be divisible by the odd integer r. For this reason, in taking the limit (2.108) the size of the lattice should be increased by $2r$ in order to remain in the sector with given S^z. As such, one can introduce an extra quantum number

$$\mathtt{s} = \frac{N}{2} - S^z \qquad (\mathrm{mod}\ r)\,, \tag{2.121}$$

which takes the values

$$\mathtt{s} = 0, 1, 2, \ldots, r-1\,. \tag{2.122}$$

It is expected that, away from the free fermion point with $\frac{\pi}{2}(1-\frac{1}{r}) < \gamma < \frac{\pi}{2}(1+\frac{1}{r})$, the chiral states can be characterized in the same way and, in particular, are labelled by $S^z, \mathtt{w} \in \mathbb{Z}$ and $\mathtt{m}, \mathtt{s} = 0, 1, \ldots, r-1$.

In order to describe the eigenvalues of the conformal Hamiltonian (2.106), one should distinguish between the following four cases:

$$\begin{aligned} &\text{(i)}\ \ \tfrac{\mathtt{s}+S^z}{r} - S^z \text{ even and } \mathtt{k} \in (-\tfrac{1}{2}, 0)\,, && \text{(ii)}\ \ \tfrac{\mathtt{s}+S^z}{r} - S^z \text{ even and } \mathtt{k} \in (0, \tfrac{1}{2}) \\ &\text{(iii)}\ \ \tfrac{\mathtt{s}+S^z}{r} - S^z \text{ odd and } \mathtt{k} \in (-\tfrac{1}{2}, 0)\,, && \text{(iv)}\ \ \tfrac{\mathtt{s}+S^z}{r} - S^z \text{ odd and } \mathtt{k} \in (0, \tfrac{1}{2}) \end{aligned} \tag{2.123}$$

(note that $\frac{\mathtt{s}+S^z}{r} - S^z$ is always an integer, as follows from the definition (2.121) and that r is odd). Also, introduce the integers $\tilde{\mathtt{m}}$ and $\tilde{\mathtt{s}}$ as

$$\tilde{\mathtt{m}} = \begin{cases} |r - \mathtt{s} - \mathtt{m}| & \text{for cases (i) \& (iv)} \\ |\mathtt{s} - \mathtt{m}| & \text{for cases (ii) \& (iii)} \end{cases}\,, \qquad \tilde{\mathtt{s}} = \begin{cases} +\mathtt{s} & \text{for case (i)} \\ -\mathtt{s} & \text{for case (ii)} \\ r - \mathtt{s} & \text{for case (iii)} \\ \mathtt{s} - r & \text{for case (iv)} \end{cases}\,. \tag{2.124}$$

Based on an analysis of the free fermion point and numerical work we come to

Conjecture: The eigenvalues of $\hat{H}_{\rm CFT}$ (2.106) are given by $I_1 + \bar{I}_1$ with

$$I_1 = \frac{p^2}{n+r} - \frac{r}{24} + \frac{\mathtt{m}(r-\mathtt{m})}{2r} + \mathtt{L}\,, \qquad \bar{I}_1 = \frac{\bar{p}^2}{n+r} - \frac{r}{24} + \frac{\tilde{\mathtt{m}}(r-\tilde{\mathtt{m}})}{2r} + \bar{\mathtt{L}}\,, \tag{2.125}$$

in the cases (i) and (iv) and

$$I_1 = \frac{p^2}{n+r} - \frac{r}{24} + \frac{\tilde{\mathtt{m}}(r-\tilde{\mathtt{m}})}{2r} + \mathtt{L}\,, \qquad \bar{I}_1 = \frac{\bar{p}^2}{n+r} - \frac{r}{24} + \frac{\mathtt{m}(r-\mathtt{m})}{2r} + \bar{\mathtt{L}}\,, \tag{2.126}$$

in the cases (ii) and (iii). Here p and $\bar{p}$ are expressed in terms of the quantum numbers S^z, $\mathtt{w}$, $\mathtt{m}$ and $\mathtt{s}$ according to

$$p + \bar{p} = S^z\,, \qquad \frac{p - \bar{p}}{n+r} = \mathtt{k} + \mathtt{w} + \frac{\mathtt{m}}{r} + \frac{\tilde{\mathtt{s}}}{2r}\,, \tag{2.127}$$

while $\mathtt{L}$, $\bar{\mathtt{L}}$ are non-negative integers. The eigenvalues of the r - site lattice translation operator for the low energy states are given by

$$\mathcal{K} = (-1)^{\frac{r-1}{2}(\frac{N}{2} - S^z) + \mathtt{m} + r\mathtt{w}}\ \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\tilde{\mathtt{s}}}\ \exp\left(\frac{2\pi\mathrm{i}r}{N}\ \big(I_1 - \bar{I}_1\big)\right). \tag{2.128}$$

2.5.2. Some comments on the low energy spectrum away from the $\mathcal{Z}_r$ invariant point

Let $\mathcal{E}_{\rm vac}$ (vacuum energy) be the lowest energy of the spin chain in the sector with given S^z and $N/2 - S^z$ is divisible by r, i.e., the quantum number $\mathtt{s}$ (2.121) is equal to zero. Note that by the ground state energy, $\mathcal{E}_{\rm GS}$, what is meant is the vacuum energy for $S^z = 0$. We studied the large - N behaviour of $\mathcal{E}_{\rm vac}$, where the inhomogeneities were taken to satisfy the conditions (2.33)-(2.35) with $\mathfrak{a}_{2j+1}$ being treated as fixed, N - independent parameters. It was found that the extensive part does not depend on the values of the RG invariants and, of course, S^z so that the specific bulk energy e_∞ is still given by eq. (2.105). However, the difference between $\mathcal{E}_{\rm vac}$ and $\mathcal{E}^{(0)}_{\rm vac}$ (the vacuum energy for the $\mathcal{Z}_r$ invariant spin chain) goes at large N as

$$\mathcal{E}_{\rm vac}-\mathcal{E}_{\rm vac}^{(0)}=\frac{2\pi r v_{\rm F}}{N}\,f(\mathfrak{a}_1,\ldots,\mathfrak{a}_{r-2})+o(N^{-1})\ , \tag{2.129}$$

where f is a certain polynomial in $\mathfrak{a}_{2j+1}$ whose coefficients depend on n. Note that f is independent of the value of S^z as well as the twist parameter $\mathtt{k}$. This motivated us to separate out the factor $2\pi r v_{\rm F}$ in (2.129), where the Fermi velocity is defined by the same formula as for the $\mathcal{Z}_r$ invariant case, i.e., (2.105). For $r=3$,

$$f=-C\,\mathfrak{a}_1^3 \tag{2.130}$$

with C being a positive n - dependent coefficient which vanishes as n^{-1} for $n\to\infty$ and diverges $\sim n^{-2}$ as $n\to 0$. Also, at the free fermion point $C|_{n=3}=\frac{1}{2}$.

It turns out that the form of the leading behaviour (2.129) is not universal in that it depends on the way in which one performs the scaling limit. For example, as was discussed, the limit can be organized differently such that the inhomogeneities obey (2.90), (2.94) and the spin chain Hamiltonian possesses both $\mathcal{CP}$ and $\mathcal{T}$ symmetries for any number of lattice sites. If $r=3$, there would be a single real valued RG invariant $\mathfrak{b}_1$, which is defined by eqs. (2.96) - (2.98). Then, we observed that

$$\mathcal{E}_{\rm vac}-\mathcal{E}_{\rm vac}^{(0)}=-\frac{C_1}{N^{\frac{1}{3}}}\,\mathfrak{b}_1^2-\frac{2\pi r v_{\rm F}}{N}\,C_2\,\mathfrak{b}_1^3+o(N^{-1})\qquad\qquad (r=3)\ , \tag{2.131}$$

where C_1, C_2 are some positive constants depending on n.

Regardless of the scheme in which the scaling limit is taken, the operator $N(\mathbb{H}-\mathcal{E}_{\rm GS})$, when restricted to the class of low energy states, is expected to possess the $N\to\infty$ limit. The low energy states can be chosen to be the Bethe states, which are specified by solutions to the Bethe Ansatz equations. It seems to be possible to unambiguously and continuously deform the corresponding Bethe roots away from the $\mathcal{Z}_r$ invariant case, at least in some domain of the values of the RG invariants. A preliminary numerical study shows that the spectrum of the operator

$$\hat{H}\equiv\frac{1}{2}\,(n+r)\,\mathtt{k}^2-\frac{r}{12}+\frac{1}{2\pi r v_{\rm F}}\ \underset{N\to\infty}{\rm slim}\ N\big(\mathbb{H}-\mathcal{E}_{\rm GS}\big) \tag{2.132}$$

depends on the RG invariants. For $n=r$, i.e., at the free fermion point, this can also be confirmed via a straightforward extension of the analysis presented in Appendix D. A detailed investigation of the spectrum of $\hat{H}$ lies beyond the scope of the current paper.

3. Basic constraints on ODEs for spin chain with anisotropy $0<\gamma<\pi$

The quantum Wronskian relation(s), obeyed by the Q - operators, first appeared in the series of works [4–6] and has since then become a central notion in integrability in Quantum Field Theory and Statistical Mechanics. In the case of the inhomogeneous six-vertex model this would be the relation (2.28) satisfied by $\mathbb{A}_\pm(\zeta)$ and it holds true for any values of the anisotropy parameter q and inhomogeneities η_J. It is expected that, with a properly defined scaling limit, the eigenvalues of $\mathbb{A}_\pm(\zeta)$ corresponding to the low energy states would become the functions $D_\pm(E)$, which obey the functional relation inherited from the quantum Wronskian relation. For the ground state of the spin chain this would be (2.20) with $p=\frac{n+r}{2}\,\mathtt{k}$. In developing the ODE/IQFT correspondence the main step is to identify $D_\pm(E)$ with the spectral determinants of a certain $2^{\rm nd}$ order ODE. Unfortunately, we do not know of a systematic way to do this at the moment. While the previous section was devoted to the case with r odd and $q={\rm e}^{{\rm i}\gamma}$, where $\gamma\in\big(\frac{\pi}{2}(1-\frac{1}{r}),\frac{\pi}{2}(1+\frac{1}{r})\big)$, it contains the essential hints for obtaining the differential equation appearing in the scaling limit of the spin chain for the other domains

$$\gamma\in\big(\tfrac{\pi}{r}A,\tfrac{\pi}{r}(A+1)\big) \tag{3.1}$$

labelled by the integer $A=0,1,\ldots,r-1$. Among them is that the class of possible ODEs is greatly reduced by the requirement that the differential equation be invariant w.r.t. the transformation

$$\hat{\Omega}\ :\ y\mapsto y+\frac{2\pi{\rm i}}{n+r}\,,\qquad\qquad E\mapsto q^{-2}\,E\,, \tag{3.2}$$

which is used to derive the quantum Wronskian relation for the spectral determinants. Our proposal is that the scaling limit can be defined in such a way that the ground state of the spin chain is described in terms of the differential equation (1.23), i.e.,

$$\Big[-\partial_y^2+p^2+{\rm e}^{(n+r)y}-(-1)^A\,E^r\,{\rm e}^{ry}-\delta U(y)\Big]\,\psi=0 \tag{3.3}$$

with

$$\delta U(y)=\sum_{(\mu,j)\in\Xi_{r,A}}c_{\mu,j}\,E^\mu\,{\rm e}^{\left((A\mu-rj)\,\frac{n+r}{r}+\mu\right)y} \tag{3.4}$$

and its barred counterpart. It is easy to check that the ODE is indeed invariant w.r.t. the transformation $\hat{\Omega}$ with $q={\rm e}^{\frac{{\rm i}\pi}{r}A+\frac{{\rm i}\pi}{n+r}}$. Formally this holds true for integer j and arbitrary values of μ. However, in order for the spectral determinant to be an entire function of E, μ should be a non-negative integer. To specify the ODE completely one needs to describe the set $\Xi_{r,A}$, which occurs in the sum in (3.4). For the case r odd and $A=\frac{r-1}{2}$, this has already been carried out in the previous section,

$$\Xi_{r,\frac{r-1}{2}} = \big\{(\mu, j)\ :\ \ \mu = 2j+1\ \ \&\ \ j = 0,\dots,\tfrac{r-3}{2}\big\} \qquad\qquad (r\ \text{odd})\,. \tag{3.5}$$

Then the differential equation (3.3) becomes (2.18) with $c_{2j+1} \equiv c_{2j+1,j}$

3.1. The case $A = r - 1$

The ODE/IQFT correspondence in the regime $\pi(1-\frac{1}{r}) < \gamma < \pi$ was proposed in ref. [21]. Here we give a brief summary of the results of that work relevant to our current discussion.

For $A = r - 1$ the integers μ and j are specified as

$$\mu = j+1 \qquad \text{with} \qquad j = 0,\dots,A-1\,. \tag{3.6}$$

Then the differential equation (3.3) takes the form

$$\bigg[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} + (-E)^r\,\mathrm{e}^{ry} - \sum_{\mu=1}^{r-1} c_\mu\, E^\mu\, \mathrm{e}^{ry+\frac{n}{r}(r-\mu)y}\bigg]\,\psi = 0\,. \tag{3.7}$$

Performing the change of variables

$$z = -E^{-1}\,\mathrm{e}^{\frac{n}{r}y}\,, \qquad \Psi = \mathrm{e}^{\frac{n}{2r}y}\,\psi \tag{3.8}$$

one arrives at

$$\Big[-\partial_z^2 + z^{-2}\Big(\big(\tfrac{rp}{n}\big)^2 - \tfrac{1}{4}\Big) + \kappa^2 z^{-2+\xi r}\,P_r(z)\Big]\,\Psi = 0\,. \tag{3.9}$$

Here

$$\kappa^2 = (-1)^r\left(\frac{rp}{n}\right)^2 E^{\frac{r(n+r)}{n}}\,, \qquad\qquad \xi = \frac{r}{n}\,, \tag{3.10}$$

while $P_r(z)$ is a polynomial of degree r given by

$$P_r(z) = z^r - \sum_{\mu=1}^{r-1}(-1)^\mu\, c_\mu\, z^{r-\mu} + 1\,. \tag{3.11}$$

If one assumes that all the coefficients c_μ are sufficiently small, then the roots of this polynomial would be close to $\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a+1)}$ and hence simple. This way, $P_r(z)$ can be written as

$$P_r(z) = \prod_{a=1}^{r}(z - z_a) \qquad \text{with} \qquad \prod_{a=1}^{r}(-z_a) = 1\,, \tag{3.12}$$

where z_a are distinct.

It was proposed in section 12 of ref. [21] that the differential equation (3.9) (3.12) describes the scaling limit of the Bethe roots for the ground state of the spin chain with $q = -\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(\beta^2-1)}$, where $0 < \beta^2 \equiv \frac{r}{n+r} < 1$. In taking the limit, the inhomogeneities are assumed to be fixed and sufficiently close to their values for the $\mathcal{Z}_r$ invariant case. Also, without loss of generality, the normalization condition was imposed as

$$\prod_{\ell=1}^{r}\eta_\ell = 1\,. \tag{3.13}$$

In our notation, this prescription for the scaling limit corresponds to setting the exponent d_s in the definition of the RG invariants $\mathfrak{a}_s$ (1.22) to be zero, i.e.,

$$\mathfrak{a}_s = \frac{1}{sr}\sum_{\ell=1}^{r}(\eta_\ell)^{-s} \qquad\qquad (s = 1,2,\dots,r-1)\,. \tag{3.14}$$

The relation between the set $\{\mathfrak{a}_s\}_{s=1}^{r-1}$ and the coefficients $\{c_\mu\}_{\mu=1}^{r-1}$ of the differential equation (3.7) is rather complicated. It is described explicitly for the case $r=2$ in section 12.3 in [21].

For $A = r-1$ the full class of ODEs which appear in the scaling limit of the low energy excited states of the spin chain is known [21]. It will not be presented here. We just mention that though the integrable structure underlying the CFT depends on the RG invariants, they have no effect on the conformal structure. In particular, the eigenvalues and degeneracies of the Hamiltonian $\hat{H}$ defined through (2.132) does not depend on $\mathfrak{a}_s$ (3.14) provided that their absolute values are sufficiently small.

3.2. General restrictions on integers μ and j

As was mentioned in the Introduction, our work is devoted to the study of the scaling limit of the spin chain with "softly" broken $\mathcal{Z}_r$ symmetry. At the level of the differential equation (3.3), this can be formulated as the requirement that the term δU (3.4) may be treated as a perturbation. In view of this, we impose the following constraints:

(a) The function

$$\delta U(y) = \sum_{(\mu,j)} c_{\mu,j}\, E^{\mu}\, \mathrm{e}^{\left((A\mu - rj)\,\frac{n+r}{r} + \mu\right) y} \tag{3.15}$$

vanishes as $y \to -\infty$:

$$\lim_{y\to-\infty} \delta U(y) = 0 \qquad \text{for} \quad \forall n > 0\,. \tag{3.16}$$

Then one can still introduce the Jost solutions of the ODE, which are involved in the derivation of the quantum Wronskian relation (2.20), through the asymptotic conditions

$$\psi_{\pm p}(y) \to \mathrm{e}^{\pm p y} \qquad \text{as} \qquad y \to -\infty \qquad \big(\Re e(\pm p) \geq 0\big)\,. \tag{3.17}$$

It is easy to see that (3.16) implies that both the integers μ and j can not be simultaneously zero as well as the inequalities

$$\mu \geq \max\Big(0, \frac{rj}{A}\Big) \qquad \text{for} \qquad A \geq 1 \tag{3.18a}$$

$$\mu \geq 0 \;\&\; j \leq 0 \qquad \text{for} \qquad A = 0\,. \tag{3.18b}$$

Here we also took into account that μ must be non-negative, which is required in order that the spectral determinant admits a power series expansion at $E = 0$.

(b) The other constraint is that $\delta U(y)$ should grow slower than the term $\propto \mathrm{e}^{(n+r)y}$ in eq. (3.3) as $y \to +\infty$, i.e.,

$$\lim_{y\to+\infty} \mathrm{e}^{-(n+r)y}\, \delta U(y) = 0 \qquad \text{for} \quad \forall n > 0\,. \tag{3.19}$$

This implies

$$\mu < \frac{r}{A+1}\,(j+1) \qquad (0 \leq A < r-1)\,. \tag{3.20}$$

It follows from (3.19) that the zeroes of the spectral determinants would accumulate along the same rays as for the $\mathcal{Z}_r$ invariant case. For $A = r-1$ it was discussed in the previous subsection that $\mu = j+1$ and the inequality (3.20) is saturated. In turn, the Stokes lines are considerably more cumbersome to describe (see, e.g., Appendix B in ref. [21]). This makes the case $A = r-1$ somewhat special.

The conditions (3.18) and (3.20) have some immediate consequences. In particular, one has for the integer j that

$$0 \leq j \leq A-1 \qquad (A = 1, \dots, r-2)\,. \tag{3.21}$$

Since we exclude the case when both μ and j are zero, it follows from (3.18a) that μ is a positive integer and, furthermore,

$$1 \leq \mu < \frac{A}{A+1}\, r\,, \qquad (A = 1, \dots, r-2)\,. \tag{3.22}$$

As for $A = 0$, the possible values of the pair (μ, j) are described as

$$\mu = 1, 2, \dots, r-1 \;\&\; j = 0 \qquad (A = 0)\,. \tag{3.23}$$

If $\delta U(y)$ is given by (3.15), where the summation is taken over the pairs (μ, j) satisfying the above conditions, the spectral determinants for the ODE (3.3) can be introduced along the same lines as was done in secs. 2.2 and 2.3.[6] The proof that they satisfy the quantum Wronskian relation (2.20), which is based on the invariance of the differential equation w.r.t. the transformation (3.2), remains unchanged. Further, $D_{\pm}(E)$ are entire functions of E.

4. ODE with single coefficient $c_{\mu,j}$ for $A\mu > rj$ and $A = 1, \dots, r-2$

We now come up against the problem of giving a precise description of the scaling limit of the spin chain that would result in the differential equation belonging to the class (3.3) and (3.4). This involves specifying the exponents d_s entering into the definition

[6] Strictly speaking formula (2.23) used to specify the subdominant at $y \to +\infty$ solution χ, entering into the definition (2.24) of the spectral determinant, is not valid in general. Nevertheless, χ may still be unambiguously introduced via its large - y asymptotic obtained within the WKB approximation.

of the RG invariants $\mathfrak{a}_s$ (1.22) as well as relating their values to the coefficients $c_{\mu,j}$ of the ODE. However, when carried out in full generality, one runs into, besides the technical complexity, also conceptual issues. For this reason, we first focus our attention on formulating the scaling limit such that the corresponding differential equation is given by

$$\Big[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-1)^A\,E^r\,\mathrm{e}^{ry}-c\,E^\mu\,\mathrm{e}^{\left((A\mu-rj)\,\frac{n+r}{r}+\mu\right)y}\Big]\,\psi=0\qquad\quad (c\equiv c_{\mu,j})\,, \tag{4.1}$$

where the pair of integers (μ,j) obeys the conditions

$$\frac{rj}{A}<\mu<\frac{r}{A+1}\,(j+1)\ \&\ j\geq 0\qquad\quad (1\leq A<r-1)\,. \tag{4.2}$$

Notice that we have excluded the case $A\mu=rj$ which, as will be discussed below, requires a separate treatment.

4.1. Bohr-Sommerfeld quantization condition

The experience gained in sec. 2.3 suggests to start with the analysis of the large - E asymptotic behaviour of the spectral determinant. A computation, based on the WKB approximation, leads to the asymptotic formula

$$\begin{aligned}\log D_+\asymp\ &\frac{N_0\mathrm{e}^\theta}{\cos(\frac{\pi r}{2n})}-\sum_{1\leq k\leq\frac{r}{2M}}\mathrm{e}^{(r-2kM)\frac{\theta}{r}}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-1-A)k\mu}\,\big((-1)^{(r-1)\mu}\,c\big)^k\\ &\times\frac{\Gamma\big(\frac{k\mu}{r}-\frac{n+r}{2nr}\,(r-2kM)\big)\Gamma\big(k-\frac{1}{2}-\frac{k\mu}{r}+\frac{n+r}{2nr}\,(r-2kM)\big)}{2n\sqrt{\pi}\,k!}-\frac{2np\,\theta}{r(n+r)}+\log C_p+o(1)\,.\end{aligned} \tag{4.3}$$

Here N_0 and C_p are given by (1.17) and (2.51), respectively, while

$$M=(j+1)\,r-(A+1)\,\mu>0 \tag{4.4}$$

is a positive integer (which follows from (4.2)). Again, θ parameterizes E in the wedge labelled by the integer a as

$$E=(-1)^{r-1}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-1-A)}\,\mathrm{e}^{\frac{2n\theta}{r(n+r)}} \tag{4.5}$$

and the asymptotic formula is valid for $\Re e(\theta)\to+\infty$ with $\big|\Im m(\theta)\big|<\frac{\pi(n+r)}{2n}$.

One can also derive the Bohr-Sommerfeld quantization condition, from which the location of the zeroes $E_m^{(a)}$ of the spectral determinant $D_+(E)$ for $m\gg 1$ may be deduced. Writing the zeroes in terms of $\theta_m^{(a)}$ according to (2.66), one finds that

$$X=\mathrm{e}^{-\frac{\mathrm{i}\pi\mu}{2rM}(2a-A)}\,\mathrm{e}^{\frac{1}{r}\theta_m^{(a)}} \tag{4.6}$$

with $a=1,2,\ldots,r$ and $m\gg 1$ solves the equation

$$X^r+\sum_{k=1}^{[\frac{r}{2M}]}G_{2kM}\,X^{r-2kM}+o(1)=Y^r\,. \tag{4.7}$$

Here $[\ldots]$ stands for the integer part and the notation

$$Y=\mathrm{e}^{-\frac{\mathrm{i}\pi\mu}{2rM}(2a-A)}\left(\frac{\pi}{N_0}\,\Big(m-\frac{1}{2}+\frac{p}{r}\Big)\right)^{\frac{1}{r}}\,,\qquad G_{2kM}=\mathfrak{g}_{2k}\,\Big((-1)^{(r-1)\mu}\,c\Big)^k \tag{4.8}$$

as well as

$$\mathfrak{g}_{2k}=\frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\Gamma(\frac{r}{2n})}\ \frac{\Gamma\big(\frac{k}{r}\,L+\frac{1}{2n}\,(r-2kM)\big)}{k!\,\Gamma\big(\frac{3}{2}-k+\frac{k}{r}\,L+\frac{1}{2n}\,(r-2kM)\big)} \tag{4.9}$$

is being used. Also, together with the positive integer M (4.4), we introduce

$$L=A\mu-rj>0\,, \tag{4.10}$$

which is a positive integer due to the condition (4.2). Notice that the argument of the Γ - function occurring in the numerator of the second fraction in $\mathfrak{g}_{2k}$ is always positive since, as follows from the upper limit in the sum in (4.7), $k\leq\frac{r}{2M}$. It vanishes when this bound is saturated and $L=0$, which is only possible if both A and r are divisible by $2K$ and

$$j=\frac{A}{2K}\,(2K-1)\,,\qquad\qquad \mu=\frac{r}{2K}\,(2K-1)\qquad\qquad (K=1,2,\ldots)\,. \tag{4.11}$$

Then, the last term in the sum in (4.7) would be proportional to $\log(X/\mathrm{const})$. This, among other things, makes the case $L=0$ somewhat special and is one of the reasons why it will be excluded from the discussion of this section.

If the positive integer M entering into the quantization condition (4.7) exceeds $r/2$, the sum vanishes and hence, as $m \to +\infty$,

$$E_m^{(a)} = (-1)^{r-1}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)} \left(\frac{\pi}{N_0}\left(m-\frac{1}{2}+\frac{p}{r}\right)\right)^{\frac{2n}{r(n+r)}} \left(1+o(m^{-1})\right) \qquad (M > r/2)\,. \tag{4.12}$$

If $0 < M \le r/2$, i.e.,

$$\frac{r}{A+1}\left(j+\frac{1}{2}\right) \le \mu < \frac{r}{A+1}\,(j+1)\,, \tag{4.13}$$

the sum in (4.7) is not trivial and the leading large - m asymptotic formula for $E_m^{(a)}$ reads as

$$E_m^{(a)} = (-1)^{r-1}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)} \left(\frac{\pi}{N_0}\left(m-\frac{1}{2}+\frac{p}{r}\right)\right)^{\frac{2n}{r(n+r)}} \left(1+O\left(m^{-\frac{2M}{r}}\right)\right) \qquad (0 < M \le r/2)\,. \tag{4.14}$$

In both cases the zeroes accumulate along the same Stokes lines as in the $\mathcal{Z}_r$ invariant spin chain. However, the subleading term in (4.14), which evaluates to a complex number, decays considerably slower than $o(m^{-1})$ from eq. (4.12) so that the zeroes approach to the rays less rapidly when the value of the integer μ is restricted to the interval (4.13). This, in turn, impacts the convergence of the series

$$J_s = \frac{1}{s}\sum_{m=1}^{\infty}\sum_{a=1}^{r}\left(E_m^{(a)}\right)^{-s}, \tag{4.15}$$

which would require, in general, the introduction of counterterms. Similar to what was discussed in sec. 2.3, one can make use of this to try to deduce the relation between the RG invariants (1.22) and the coefficient c in (4.1).

4.2. Specification of the scaling limit

Let's first consider the case when the pair (μ, j) satisfies the inequality

$$\mu \ge \frac{r}{A+1}\left(j+\frac{1}{2}\right) \tag{4.16}$$

in addition to (4.2). Numerical work shows that it is possible to organize the scaling limit in such a way that the scaled Bethe roots for the ground state of the spin chain are described by the ODE of the form (4.1). The exponent d_s, which appears in the definition of the RG invariants,

$$\mathfrak{a}_s = \frac{1}{s}\left(\frac{N}{rN_0}\right)^{d_s}\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-s}, \tag{4.17}$$

should be taken to be

$$d_s = \frac{2M}{r}\, i_s\,, \tag{4.18}$$

where i_s denotes the smallest positive integer such that

$$s = \mu\, i_s \pmod r \tag{4.19}$$

and, as before, $M = (j+1)\,r - (A+1)\,\mu$. If s is not divisible by μ modulo r, i.e., the integer i_s does not exist, the corresponding RG invariant can be set to zero:

$$\mathfrak{a}_s = 0 \qquad \text{if} \qquad s \nmid \mu \pmod r\,. \tag{4.20}$$

In addition, we assume the normalization condition for the inhomogeneities

$$\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-r} = (-1)^{r-1}\,. \tag{4.21}$$

The inequality (4.16) guarantees that there are RG invariants $\mathfrak{a}_s$ for which $0 < d_s \le 1$. In this case one can use the trick discussed in sec. 2.3 to deduce the relation

$$\mathfrak{a}_s = (-1)^{s-(j+1)i_s+\mathcal{N}_s}\, Z_s\, c^{i_s}\,. \tag{4.22}$$

Here the integer $\mathcal{N}_s$ is defined by the condition

$$s = \mu\, i_s + r\,\mathcal{N}_s\,. \tag{4.23}$$

The coefficients Z_s are built from $\mathfrak{g}_{2k}$ (4.9) as

$$Z_s = \frac{1}{s}\sum_{m=1}^{i_s}(-1)^m\,\frac{\Gamma(s\nu+m)}{m!\Gamma(s\nu)}\sum_{\substack{k_1,\ldots,k_m\geq 1\\ k_1+k_2+\ldots+k_m=i_s}}\mathfrak{R}_{k_1}\ldots\mathfrak{R}_{k_m} \tag{4.24}$$

with

$$\mathfrak{R}_k = \frac{1}{r}\sum_{\substack{\alpha_1+2\alpha_2+\ldots+k\alpha_k=k\\ \alpha_1,\ldots,\alpha_k\geq 0}}\frac{(-1)^{\alpha_1+\ldots+\alpha_k}}{\alpha_1!\alpha_2!\ldots\alpha_k!}\,\frac{\Gamma(\alpha_1+\ldots+\alpha_k-\frac{2kM-1}{r})}{\Gamma(1-\frac{2kM-1}{r})}\;\mathfrak{g}_2^{\alpha_1}\mathfrak{g}_4^{\alpha_2}\ldots\mathfrak{g}_{2k}^{\alpha_k} \tag{4.25}$$

and

$$\nu = \frac{2n}{n+r}\,. \tag{4.26}$$

It needs to be emphasized that the argument used to obtain (4.22) can not be literally applied to the case when the exponent d_s for the RG invariant $\mathfrak{a}_s$ exceeds one. Nevertheless, we found that under certain conditions which will be clarified in the examples below, the scaling limit of the Bethe roots is described by the differential equation (4.1) if the RG invariants are specified as in (4.20), (4.22) - (4.26) for any s. Moreover, numerical work shows that the values of some of the RG invariants $\mathfrak{a}_s$, where s is divisible by μ modulo r and $d_s > 1$ have no effect on the scaling limit and can be set to zero. In particular, for $r \leq 10$ this was always observed if the corresponding exponent $d_s \geq 2$. Unfortunately, we do not have at hand a full description of the minimal set of non-trivial RG invariants that works for generic values of r and $A = 1,\ldots,r-2$.[7] For this reason, we list them in the tables in Appendix F, along with the corresponding exponents d_s for all the cases with $r = 3,4,\ldots,10$, which were deduced from a numerical study.

4.2.1. *Example: $r = 5,\ A = 1$ and $(\mu, j) = (2, 0)$*

In this case we found that there are three non-trivial RG invariants $\mathfrak{a}_2$, $\mathfrak{a}_4$ and $\mathfrak{a}_1$, whose corresponding exponents are given by $d_2 = \frac{2}{5}$, $d_4 = \frac{4}{5}$ and $d_1 = \frac{6}{5}$, respectively. A specialization of formulae (4.22) - (4.26) yields

$$\mathfrak{a}_2 = -\tfrac{\nu}{5}\,\mathfrak{g}_2\,c \tag{4.27a}$$

$$\mathfrak{a}_4 = \tfrac{\nu}{5}\left(\mathfrak{g}_4 + \tfrac{1}{10}\,(4\nu-1)\,\mathfrak{g}_2^2\right)c^2$$

$$\mathfrak{a}_1 = -\tfrac{\nu}{5}\left(\mathfrak{g}_6 + \tfrac{1}{5}\,(\nu+1)\,\mathfrak{g}_2\mathfrak{g}_4 + \tfrac{1}{150}\,(\nu+1)(\nu-4)\,\mathfrak{g}_2^3\right)c^3\,, \tag{4.27b}$$

where $\nu = \frac{2n}{n+5}$ and the functions $\mathfrak{g}_{2k}$ are defined in eq. (4.9) with $M = 1$ and $L = 2$. Then, according to the above discussion, the ODE appearing in the scaling limit is expected to be

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+5)y} + E^5\,\mathrm{e}^{5y} - c\,E^2\,\mathrm{e}^{\frac{2}{5}(n+10)\,y}\right]\psi = 0\,, \tag{4.28}$$

where recall that $p = \frac{1}{2}\,(n+5)\,\mathtt{k}$, see (1.19). One may observe that both $\mathfrak{g}_2$ and $\mathfrak{g}_4$ are non-singular functions of n for any $n > 0$. In contrast $\mathfrak{g}_6$, which reads explicitly as

$$\mathfrak{g}_6 = \frac{\Gamma(\frac{3}{2}+\frac{5}{2n})}{\Gamma(\frac{5}{2n})}\,\frac{\Gamma(\frac{6}{5}-\frac{1}{2n})}{6\,\Gamma(-\frac{3}{10}-\frac{1}{2n})}\,, \tag{4.29}$$

possesses simple poles when $n = \frac{5}{2(1+5k)}$ with $k = 1,2,3,\ldots$, accumulating at $n = 0$. As a result, the proportionality coefficient $Z_1^{(\mu=2)} = -\mathfrak{a}_1/c^3$ also contains poles at the same values of n, see eq. (4.27b). Moreover, $Z_1^{(\mu=2)}$ has zeroes with the largest one located at $n = \frac{5}{2}$. From the numerical work, we were able to confirm that the scaling limit of the Bethe roots for the ground state is governed by the ODE (4.28) for $n > \frac{5}{2}$. As for the domain $0 < n \leq \frac{5}{2}$, it turns out to be difficult to explore for reasons to be discussed below.

4.2.2. *Example: $r = 5,\ A = 1$ and $(\mu, j) = (1, 0)$*

So far, we have taken μ and j to obey the inequality (4.16) in addition to (4.2). For $\mu < \frac{r}{A+1}\,(j+\frac{1}{2})$ or, equivalently, the integer M (4.4) exceeds $\frac{r}{2}$, it follows from formula (4.18) that all the exponents corresponding to the non-trivial RG invariants are greater than one. The numerical study shows that the relations obtained assuming μ satisfies both $\frac{r}{A+1}\,(j+\frac{1}{2}) \leq \mu$ and $\frac{rj}{A} < \mu < \frac{r}{A+1}\,(j+1)$ remain valid if the former condition is dropped. This can be illustrated on the example $r = 5$, $A = 1$ and $(\mu, j) = (1, 0)$. Then, there is only one non-trivial RG invariant $\mathfrak{a}_1$ with corresponding exponent $d_1 = \frac{6}{5} > 1$. Formula (4.22), specialized to the case at hand, reads as

$$\mathfrak{a}_1 = Z_1^{(\mu=1)}\,c \qquad \text{with} \qquad Z_1^{(\mu=1)} = \frac{\Gamma(\frac{1}{2}+\frac{5}{2n})}{5\,\Gamma(\frac{5}{2n})}\,\frac{\Gamma(\frac{1}{5}-\frac{1}{2n})}{\Gamma(\frac{7}{10}-\frac{1}{2n})}\,. \tag{4.30}$$

[7] Here and below by "non-trivial RG invariants" we mean those $\mathfrak{a}_s$, whose values affect the scaling limit of the Bethe roots.

Notice that the factor $Z_1^{(\mu=1)}$ remains finite and non-vanishing as $n > \frac{5}{2}$. In this domain, it was numerically verified that the corresponding differential equation is given by

$$\left[-\partial_y^2 + p^2 + e^{(n+5)y} + E^5\, e^{5y} - c\, E\, e^{\frac{1}{5}(n+10)\,y} \right] \psi = 0\,. \tag{4.31}$$

4.2.3. Example: combining the cases $\mu = 1$ and $\mu = 2$ for $r = 5,\ A = 1,\ j = 0$

We found it rather surprising that in the prescription for taking the scaling limit, relations (4.22)-(4.26) hold true for the RG invariants $\mathfrak{a}_s$ with $d_s > 1$ at least for sufficiently large n. For this reason, we performed an additional numerical study with a slightly different setup. The exponents d_s for all the non-trivial RG invariants (4.17) were taken as in (4.18). Furthermore, the value of the RG invariants $\mathfrak{a}_s$ with $0 < d_s \le 1$ were fixed according to (4.22), while those with $d_s > 1$ were kept as free parameters. It was observed that for sufficiently large n the scaling limit of the Bethe roots for the ground state is described in terms of the differential equation of the form

$$\left[-\partial_y^2 + p^2 + e^{(n+r)y} - (-1)^A\, E^r\, e^{ry} - c\, E^\mu\, e^{\left((A\mu - rj)\,\frac{n+r}{r} + \mu\right) y} - \delta\tilde{U}(y) \right] \psi = 0\,, \tag{4.32}$$

where

$$\delta\tilde{U}(y) = \sum_{s\in\Sigma} \tilde{c}_s\, E^s\, e^{\left((As - rj)\,\frac{n+r}{r} + s\right) y} \tag{4.33}$$

and

$$\Sigma = \left\{ s:\ \ s = \mu\, i\ (\mathrm{mod}\, r)\ \ \text{with}\ i = \left[\tfrac{r}{2M}\right] + 1, \ldots, r-1 \quad \&\quad \tfrac{rj}{A} \le s < \tfrac{r}{A+1}\,(j + \tfrac{1}{2}) \right\}. \tag{4.34}$$

The appearance of such an ODE may be expected since it belongs to the class (3.3), (3.4), while the presence of the term $\delta\tilde{U}$ does not affect the Bohr-Sommerfeld quantization condition (4.7), which was derived for the case with $\delta\tilde{U} = 0$. The coefficients $\tilde{c}_s$ depend on c as well as the values of $\mathfrak{a}_s$ with $d_s > 1$. They can be obtained numerically, for instance, from the study of the first few sum rules, along the lines of sec. 2.2.

As an example, consider again the case with $r = 5,\ A = 1,\ (\mu, j) = (2, 0)$. The RG invariants $\mathfrak{a}_2$ and $\mathfrak{a}_4$, with corresponding exponents $d_2 = \frac{2}{5}$ and $d_4 = \frac{4}{5}$, are taken as in (4.27a), while the value of

$$\mathfrak{a}_1 = \left(\frac{N}{5N_0}\right)^{\frac{6}{5}} \frac{1}{5} \sum_{\ell=1}^{5} (\eta_\ell)^{-1} \tag{4.35}$$

is kept free now. Then, the scaling limit of the Bethe roots for the ground state would be described by the ODE

$$\left[-\partial_y^2 + p^2 + e^{(n+5)y} + E^5\, e^{5y} - c\, E^2\, e^{\frac{2}{5}(n+10)\,y} - \tilde{c}_1\, E\, e^{\frac{1}{5}(n+10)y} \right] \psi = 0\,. \tag{4.36}$$

Among other things, the differential equation implies

$$\left(\frac{N}{5N_0}\right)^{-\frac{2n}{5(n+5)}} h_1^{(N,\mathrm{reg})} = \tilde{c}_1\, \frac{(n+5)^{\frac{2}{n+5} - \frac{8}{5}}}{\Gamma^2(\frac{4}{5} - \frac{1}{n+5})}\, f_1\big(\tfrac{\mathtt{k}}{2}, \tfrac{1}{5} + \tfrac{1}{n+5}\big) + o(1) \qquad (N \to \infty)\,, \tag{4.37}$$

where

$$h_1^{(N,\mathrm{reg})} = \sum_{m=1}^{N/2} (\zeta_m)^{-1} + \frac{N}{2r\cos(\frac{\pi}{r} + \frac{\pi}{n+r})} \sum_{\ell=1}^{r} (\eta_\ell)^{-1} \tag{4.38}$$

with $r = 5$ and the function f_1 is defined in formula (A.1) in Appendix A. The dependence of the l.h.s. on c, the twist parameter $\mathtt{k}$ and the RG invariant $\mathfrak{a}_1$ can be investigated numerically. One finds that

$$\tilde{c}_1 = P_0\, c^3 + P_3\, \mathfrak{a}_1 \qquad\qquad \text{for} \qquad\qquad n > \tfrac{5}{2} \tag{4.39}$$

with certain n dependent constants P_0 and P_3. Since $\tilde{c}_1$ must vanish as $\mathfrak{a}_1 = -c^3 Z_1^{(\mu=2)}$, one obtains

$$P_0/P_3 = \tfrac{\nu}{5}\left(\mathfrak{g}_6 + \tfrac{1}{5}\,(\nu+1)\,\mathfrak{g}_2\mathfrak{g}_4 + \tfrac{1}{150}\,(\nu+1)(\nu-4)\,\mathfrak{g}_2^3\right). \tag{4.40}$$

Also notice that when $c = 0$, the RG invariants $\mathfrak{a}_2 = \mathfrak{a}_4 = 0$ and one gets back the case with $\mu = 1$ discussed above. This yields the relation $P_3 = 1/Z_1^{(\mu=1)}$ or, explicitly,

$$P_3 = \frac{5\,\Gamma(\frac{5}{2n})}{\Gamma(\frac{1}{2} + \frac{5}{2n})}\, \frac{\Gamma(\frac{7}{10} - \frac{1}{2n})}{\Gamma(\frac{1}{5} - \frac{1}{2n})}\,. \tag{4.41}$$

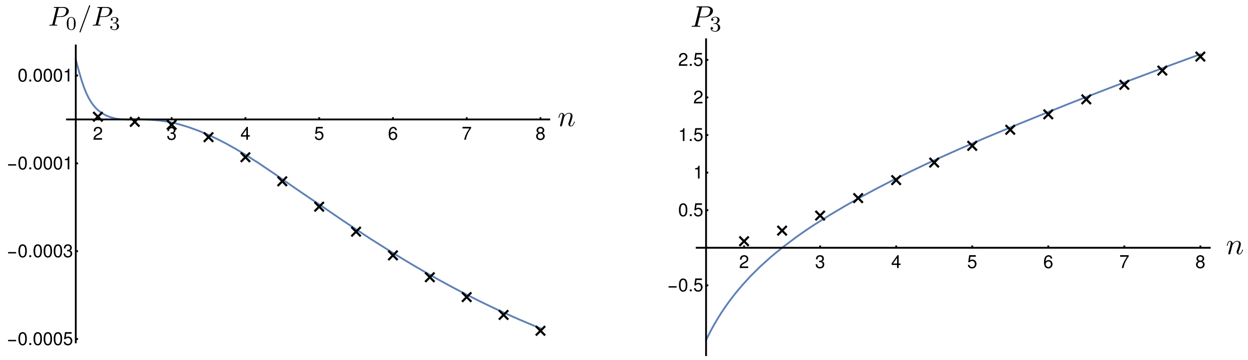

Fig. 2. The crosses represent numerical data for the ratio P_0/P_3 (left panel) and P_3 (right panel) obtained through the study of the sum rule for the Bethe roots (4.37) - (4.39). The scaling limit is taken for the spin chain with $r=5$, where the RG invariants are specified as in eqs. (4.17) and (4.18) with $A=1$, $j=0$ and $\mu=2$. The values of $\mathfrak{a}_2$ and $\mathfrak{a}_4$ were fixed according to (4.27a), $\mathfrak{a}_3$ was set to zero, while the computation was performed for various values of $\mathfrak{a}_1$. The solid line is a plot of the analytic formula given by (4.40) and (4.41) for the left and right panels, respectively. Note that there is a loss of accuracy as $n \lesssim \frac{5}{2}$.

In Fig. 2, numerical data for P_0 and P_3 obtained from the study of the sum rule (4.37) with $\tilde{c}_1$ as in (4.39) is compared against the predictions (4.40), (4.41). One observes that there is apparent deviation between the numerical results and the analytic expressions as n becomes smaller than $\frac{5}{2}$. This is due to the slow decay of the remainder term $o(1)$ in eq. (4.37). A simple fit of the form $b_1 + b_2\,N^{-\delta}$ of the numerical data, which worked well for large n, yields a small value of the exponent δ and, in fact, becomes highly unstable. As such, we were not able to determine with confidence the limiting value of the l.h.s. of (4.37) as $n < \frac{5}{2}$. In all likelihood, this means that its large - N behaviour can not be adequately described as $b_1 + b_2\,N^{-\delta}$ containing a single small exponent δ.

4.2.4. *Example:* $r=7$, $A=1$ *and* $(\mu, j) = (1, 0)$

To understand better the problems that occurred for the cases with $r=5$ considered above, it is instructive to turn to another example, where $r=7$, $A=1$ and $(\mu, j) = (1, 0)$. Then there is only a single non-trivial RG invariant

$$\mathfrak{a}_1 = \left(\frac{N}{rN_0}\right)^{d_1} \frac{1}{r} \sum_{\ell=1}^{r} (\eta_\ell)^{-1} \tag{4.42}$$

with $r=7$ and $d_1 = \frac{10}{7}$. The scaling limit of the Bethe roots for the ground state is described by the ODE

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+7)y} + E^7\,\mathrm{e}^{7y} - c\,E\,\mathrm{e}^{\frac{1}{7}(n+14)\,y}\right]\psi = 0\,, \tag{4.43}$$

where c, therein, is related to the value of $\mathfrak{a}_1$ as

$$\mathfrak{a}_1 = \frac{\Gamma(\frac{1}{2}+\frac{7}{2n})}{7\,\Gamma(\frac{7}{2n})}\,\frac{\Gamma(\frac{1}{7}-\frac{3}{2n})}{\Gamma(\frac{9}{14}-\frac{3}{2n})}\,c\,. \tag{4.44}$$

The proportionality coefficient is free of singularities and zeroes as $n > \frac{21}{2}$. It turns out that when the position of the first pole in the $\mathfrak{a}_s$ — c relation occurs at sufficiently large values of n, one can obtain more accurate numerical data in its vicinity. In particular, we found with confidence that

$$\left(\frac{N}{7N_0}\right)^{-\frac{2n}{7(n+7)}} h_1^{(N,\mathrm{reg})} = c\,\frac{(n+7)^{\frac{2}{n+7}-\frac{12}{7}}}{\Gamma^2(\frac{6}{7}-\frac{1}{n+7})}\,f_1\big(\tfrac{\mathrm{k}}{2},\tfrac{1}{7}+\tfrac{1}{n+7}\big) + o(1) \qquad \big(n > \tfrac{21}{2}\big)\,, \tag{4.45}$$

where

$$o(1) = \begin{cases} O\big(c\,N^{\frac{6}{7}-\frac{4n}{49}}\big) & \text{for} \quad \frac{21}{2} < n < 35 \\ O\big(c\,N^{-2}\big) & \text{for} \quad n > 35 \end{cases}\,. \tag{4.46}$$

It follows from the relation (4.44) that when n approaches $\frac{21}{2}$ from above with the RG invariant $\mathfrak{a}_1$ kept fixed, c must tend to zero. Then, the differential equation (4.43) becomes the one corresponding to the $\mathcal{Z}_7$ invariant case at $n = \frac{21}{2}$. We also performed a numerical study in the domain $0 < n \le \frac{21}{2}$. It was found that if $\mathfrak{a}_1$ is still specified as in (4.42) with $d_1 = \frac{10}{7}$, the scaling limit of the Bethe roots does not depend on the value of this RG invariant and is the same as for $\mathfrak{a}_1 = 0$. However, it impacts the rate of convergence, in particular,

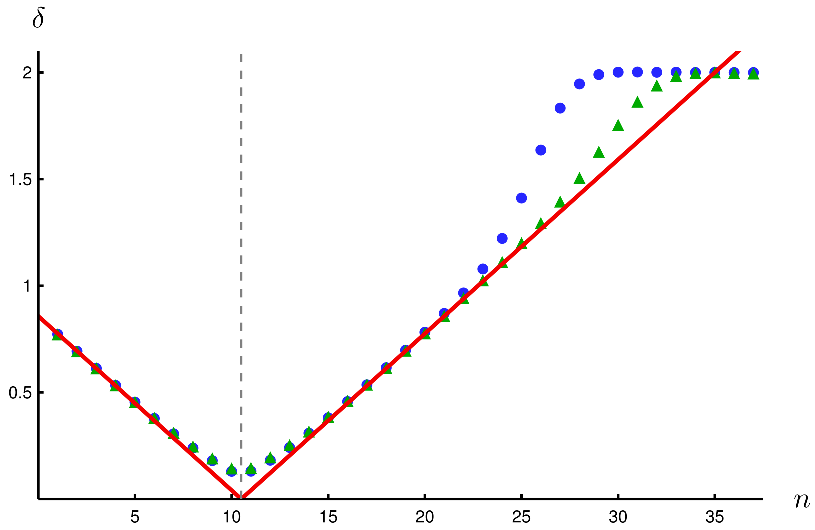

Fig. 3. The figure corresponds to the case $r=7$, $A=1$ and $(\mu,j)=(1,0)$. In order to numerically analyze the correction terms in eqs. (4.45) and (4.47) the quantity $(\frac{N}{7N_0})^{-\frac{2n}{7(n+7)}}\, h_1^{(N,\mathrm{reg})}$ was computed for increasing N. This was fitted with $h_1^{(\infty)}+\mathrm{const}\times N^{-\delta}$, where $h_1^{(\infty)}$ stands for the known limiting value, i.e., given in (4.45) for $n>\frac{21}{2}$ and zero for $n<\frac{21}{2}$. The results for δ are represented by the blue dots. The green triangles depict data for δ that was obtained by applying the similar procedure to $(\frac{N}{7N_0})^{-\frac{2n}{7(n+7)}}\,(h_1^{(N,\mathrm{reg})}|_{\mathtt{k}=0.02}-h_1^{(N,\mathrm{reg})}|_{\mathtt{k}=0.04})$ instead. The solid red line is a plot of $\delta=|\frac{6}{7}-\frac{4n}{49}|$. Note that near $n\approx 10.5$ the value of the exponent δ becomes small and the simple fit with a single exponent seems to be unreliable. For $n>35$ one observes $\delta=2$ with good accuracy. Again, in the vicinity $n\lesssim 35$ the fit with a single exponent looks inadequate, which may explain the significant spread in the numerical data obtained via the fitting procedures.

$$\left(\frac{N}{7N_0}\right)^{-\frac{2n}{7(n+7)}} h_1^{(N,\mathrm{reg})}=O\big(c\,N^{-\frac{6}{7}+\frac{4n}{49}}\big)\qquad\big(0<n<\tfrac{21}{2}\big)\,. \tag{4.47}$$

In Fig. 3 the numerical data in support of eqs. (4.46) and (4.47) is presented.

We explored the possibility of achieving a non-trivial scaling limit by making the exponent in (4.42) n - dependent such that $0<d_1<\frac{10}{7}$. It was found that if d_1 is replaced by $\tilde d_1$ where

$$\tilde d_1=\frac{4}{7}+\frac{4n}{49}\qquad\big(0<n<\tfrac{21}{2}\big) \tag{4.48}$$

then the limits

$$h_i^{(\infty)}=\lim_{N\to\infty}\left(\frac{N}{7N_0}\right)^{-\frac{2in}{7(n+7)}} h_i^{(N,\mathrm{reg})}\qquad(i=1,2,\ldots) \tag{4.49}$$

exist and are non-vanishing for generic values of the RG invariant $\mathfrak{a}_1$ in (4.42). This suggests that the Bethe roots for the ground state exhibit the scaling behaviour.

4.2.5. The case $\frac{rj}{A}<\mu<\frac{r}{A+1}\,(j+\frac{1}{2})$

For $A=1$, and $(\mu,j)=(1,0)$ it is possible to give a uniform description for any $r\ge 5$, which incorporates the case $r=7$ just discussed. With regards to this, it should be pointed out that $\mathfrak{a}_1$ has exponent $d_1=\frac{2}{r}(r-2)$, while formula (4.18) implies that for all the other RG invariants $d_s>2$. As was already mentioned, their values do not affect the scaling limit and they can be set to zero. This way, the scaling of the Bethe roots depends only on the single RG invariant $\mathfrak{a}_1$ (4.42). Similar to what was discussed for $r=7$ we confirmed that, in order to achieve a non-trivial scaling limit, the exponent d_1 should be taken as

$$d_1=\begin{cases}\frac{2}{r}\,(r-2) & \text{for}\quad n>\frac{1}{2}\,r\,(r-4)\\ \frac{4}{r^2}\,(n+r) & \text{for}\quad 0<n<\frac{1}{2}\,r\,(r-4)\end{cases}. \tag{4.50}$$

For $n>\frac{1}{2}\,r\,(r-4)$ the scaled Bethe roots are described by the differential equation

$$\Big[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}+E^r\,\mathrm{e}^{ry}-c\,E\,\mathrm{e}^{\frac{1}{r}(n+2r)\,y}\Big]\,\psi=0\,. \tag{4.51}$$

We believe that formulae (4.50) and (4.51) are applicable for $A=1$, $(\mu,j)=(1,0)$ and any $r\ge 5$, despite that for $r=5$ they are difficult to confirm numerically in the domain $0<n<\frac{5}{2}$.

If one accepts that all the RG invariants with $d_s \geq 2$ may be ignored, an interesting prediction is reached for the case of generic values of r, A and j with μ restricted by the condition

$$\frac{rj}{A} < \mu < \frac{r}{A+1}\left(j+\frac{1}{2}\right). \tag{4.52}$$

The latter implies that the only non-trivial RG invariant is $\mathfrak{a}_\mu$ with the corresponding exponent $d_\mu = \frac{2}{r}\big((j+1)r-(A+1)\,\mu\big)$ such that $1 < d_\mu < 2$. It is expected that a scaling limit, which is different than the one for the $\mathcal{Z}_r$ invariant case, can be achieved provided

$$d_\mu = \begin{cases} \frac{2M}{r} & \text{for} \quad n > n_{\min} \\ \frac{4Ln}{r^2} + \frac{2}{r}(r-M) & \text{for} \quad 0 < n < n_{\min} \end{cases}, \tag{4.53}$$

where

$$n_{\min} = \frac{r}{L}\,(M - \tfrac{r}{2}) \qquad (L > 0) \tag{4.54}$$

and we use the integers M and L defined in eqs. (4.4) and (4.10), respectively. For $n > n_{\min}$, the differential equation appearing in the scaling limit would be (4.1). If in the domain $0 < n < n_{\min}$ the exponent d_μ is not assigned an n dependence but kept fixed as $d_\mu = \frac{2M}{r}$, the scaled Bethe roots would be the same as for the $\mathcal{Z}_r$ invariant case. When the exponent is modified according to the second line of (4.53) we observed in many examples that the Bethe roots exhibit non-trivial scaling behaviour. Describing the corresponding ODE for $0 < n < n_{\min}$ is beyond the scope of this work.

It should be emphasized that formula (4.54) does not make sense for $L = 0$. This is again a signal that the case $A\mu = rj$ requires special attention.

Let us give a short summary of this section. For sufficiently large $n > n_{\min}$, the scaling limit can be organized such that the scaled Bethe roots are described by the differential equation (4.1), (4.2). The specialization of the RG invariants is given by eqs. (4.17), (4.20) with exponents (4.18). The values of the non-vanishing $\mathfrak{a}_s$ are related to the parameters of the differential equation as in (4.22) - (4.26). The values of some of the RG invariants with $d_s > 1$ turn out to have no effect on the scaling limit and can be taken to be zero. For given A and (μ, j) obeying the inequalities (4.2), the minimal sets of non-trivial $\mathfrak{a}_s$, along with their exponents, are listed in the tables contained in Appendix F for $3 \leq r \leq 10$. Therein, we also quote the corresponding values of the lower bound $n_{\min}$.

5. The case $A\mu = rj$ with $A = 1, \ldots, r-2$

A formal specification (ignoring the constraint (4.2)) of the ODE (4.1) to the case at hand yields

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A\, E^r\, \mathrm{e}^{ry} - c\, E^\mu\, \mathrm{e}^{\mu y}\Big]\,\psi = 0\,. \tag{5.1}$$

In view of the condition $A\mu = rj$, μ is not an arbitrary integer from the segment $[1, r-1]$. Instead, if H is the greatest common divisor of r and A,

$$H = \gcd(r, A)\,, \tag{5.2}$$

then

$$\mu = \frac{rJ}{H} \qquad (J = 1, \ldots, H-1)\,. \tag{5.3}$$

It turns out that the case $A\mu = rj$ with $A = 1, \ldots, r-2$ falls outside of the scope of the discussion in the previous section and the ODE (5.1) does not actually appear in the scaling limit of the spin chain (except when both r, A are even and $\mu = \frac{r}{2}$).

5.1. Special features of the scaling limit

Let's first consider the case when the positive integer $\mu < \frac{r}{2}$ or, equivalently, $J < \frac{H}{2}$. Similarly to what was discussed in sec. 4.2.5 one may expect that all of the RG invariants have no effect on the scaling limit apart from $\mathfrak{a}_\mu$, whose corresponding exponent $d_\mu = \frac{2}{H}\,(H-J)$ is such that $1 < d_\mu < 2$. In fact, a numerical study shows that the scaling behaviour does not depend on the value of $\mathfrak{a}_\mu$ as well, and the scaled Bethe roots coincide with those for the $\mathcal{Z}_r$ invariant case.[8]

For r even, $\mu = \frac{r}{2}$ is among the admissible values (5.3). Then it is sufficient to keep only one RG invariant $\mathfrak{a}_{\frac{r}{2}}$ to be non-vanishing with exponent $d_{\frac{r}{2}} = 1$. However formulae (4.24)-(4.26) for the coefficient $Z_{\frac{r}{2}}$, which relates the value of $\mathfrak{a}_{\frac{r}{2}}$ with the parameter c in the differential equation, give infinity. This suggests to modify the definition of the RG invariant as

[8] A non-trivial scaling behaviour for the Bethe roots for the ground state can be achieved by swapping $d_\mu = \frac{2}{H}\,(H-J)$ for $\tilde{d}_\mu = \frac{2J}{H} < 1$ in the definition of the RG invariant $\mathfrak{a}_\mu$ (4.17). This follows from the relation between the case $A\mu = rj > 0$ and $A = 0$ described in sec. 5.2 together with the results of sec. 7.

$$\mathfrak{a}_{\frac{r}{2}} \equiv \frac{2}{r^2}\,\frac{1}{\log\big(\frac{N}{rN_0}\big)}\,\bigg(\frac{N}{rN_0}\bigg)\,\sum_{\ell=1}^{r}(\eta_\ell)^{-\frac{r}{2}}\ . \tag{5.4}$$

With such a setup, the investigation of the scaling behaviour becomes difficult to carry out numerically. Even $N\sim 5000$ turns out to be insufficient to reliably observe the existence of a scaling limit. Nevertheless, in Appendix E we present a numerical scheme, which dramatically improves convergence and allows one to confirm that the differential equation occurring in the scaling limit is given by (5.1) with $\mu=\frac{r}{2}$. The parameter c is related to the value of the RG invariant as

$$\mathfrak{a}_{\frac{r}{2}} = (-1)^{\frac{1}{2}(r-A+2)}\,\frac{2}{r\sqrt{\pi}}\,\frac{\Gamma(\frac{1}{2}+\frac{r}{2n})}{\Gamma(1+\frac{r}{2n})}\,c\ . \tag{5.5}$$

Note that the argument of the logarithm in (5.4) involves the constant rN_0. It can be replaced by any positive number without affecting the scaling limit.

In the case $\mu>\frac{r}{2}$ the integer J in formula (5.3) is restricted as $\frac{H}{2}<J\leq H-1$. It turns out that the minimal set of non-trivial RG invariants consists of

$$\mathfrak{a}_s\ :\quad s=\tfrac{rJ}{H}\,i_s\quad(\mathrm{mod}\ r)\,,\qquad\qquad i_s=1,2,\ldots,\big[\tfrac{H}{2(H-J)}\big]\,. \tag{5.6}$$

The corresponding exponents are given by

$$d_s=\frac{2}{H}\,(H-J)\,i_s \tag{5.7}$$

and are all less than or equal to one. A numerical study shows that the Bethe roots for the ground state develop a scaling behaviour, which differs from the one that occurs in the $\mathcal{Z}_r$ invariant case. However, the scaling limit is not described by the ODE (5.1). For instance, when $r=9$, $A=3$ ($H=3$) and $\mu=6$ there is only one non-trivial RG invariant $\mathfrak{a}_6$ with exponent $d_6=\frac{2}{3}$. The differential equation yields the prediction that the limits (2.59) with $s=1,2,\ldots,8$ vanish except for

$$h_6^{(\infty)}=\lim_{N\to\infty}\bigg(\frac{9N_0}{N}\bigg)^{\frac{4n}{3(n+9)}}\,h_6^{(N,\mathrm{reg})}\ . \tag{5.8}$$

However, we found that together with $h_6^{(\infty)}$, the limit

$$h_3^{(\infty)}=\lim_{N\to\infty}\bigg(\frac{9N_0}{N}\bigg)^{\frac{2n}{3(n+9)}}\,h_3^{(N,\mathrm{reg})} \tag{5.9}$$

exists and is not zero.

The first idea that may come to mind for describing the scaling limit for $\mu=\frac{rJ}{H}>\frac{r}{2}$ is to replace (5.1) by the more general differential equation of the form

$$\Big[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-1)^A\,E^r\,\mathrm{e}^{ry}-c\,E^{\mu}\,\mathrm{e}^{\mu y}-\sum_i b_i\,E^{\mu_i}\,\mathrm{e}^{\mu_i y}\Big]\,\psi=0\,, \tag{5.10}$$

where the summation runs over the positive integer $i=\big[\frac{H}{2(H-J)}\big]+1,\ \ldots,\ \big[\frac{H-1}{H-J}\big]$ and

$$\mu_i=i\,\mu-(i-1)\,r\qquad\qquad\qquad\big(\mu=\tfrac{rJ}{H}\big)\ . \tag{5.11}$$

Numerical data supports this conjecture, where the values of the RG invariants $\mathfrak{a}_s$ (5.6) are related to c as in eqs. (4.22)-(4.26). However, while the coefficients b_i depend simply on c, they turn out to be complicated functions of $p=\frac{1}{2}(n+r)\,\mathtt{k}$ and n:

$$b_i=c^i\ \hat{b}_i(p,n)\,. \tag{5.12}$$

For instance, the differential equation (5.10), specialized to the case $r=9$, $A=3$ ($H=3$) and $\mu=6$, leads to the prediction

$$h_3^{(\infty)}=b_1\,\frac{(n+9)^{-\frac{2(n+6)}{n+9}}}{\Gamma^2\big(\frac{n+6}{n+9}\big)}\,f_1\big(\tfrac{\mathtt{k}}{2},\tfrac{3}{n+9}\big) \tag{5.13}$$

$$h_6^{(\infty)}=c\,\frac{(n+9)^{-\frac{2(n+3)}{n+9}}}{\Gamma^2\big(\frac{n+3}{n+9}\big)}\,f_1\big(\tfrac{\mathtt{k}}{2},\tfrac{6}{n+9}\big)+b_1^2\,\frac{(n+9)^{-\frac{4(n+6)}{n+9}}}{\Gamma^4\big(\frac{n+6}{n+9}\big)}\,f_2\big(\tfrac{\mathtt{k}}{2},\tfrac{3}{n+9}\big)\,.$$

Eliminating b_1, one obtains

$$h_6^{(\infty)}=c\,\frac{(n+9)^{-\frac{2(n+3)}{n+9}}}{\Gamma^2\big(\frac{n+3}{n+9}\big)}\,f_1\big(\tfrac{\mathtt{k}}{2},\tfrac{6}{n+9}\big)+\Bigg(\frac{h_3^{(\infty)}}{f_1\big(\frac{\mathtt{k}}{2},\frac{3}{n+9}\big)}\Bigg)^2\,f_2\big(\tfrac{\mathtt{k}}{2},\tfrac{3}{n+9}\big)\ . \tag{5.14}$$

Here the functions f_1 and f_2 are given by eqs. (A.1)-(A.4) in the Appendix A, while the $\mathfrak{a}_s$ — c relation reads explicitly as

Table 1
Compared is numerical data obtained for $h_6^{(\infty)}$ in two different ways for $r=9,\ A=3$ and various values of $\mathtt{k}$ and n. In the first case, $h_6^{(\infty)}$ was calculated from the solution of the Bethe Ansatz equations via formulae (5.8), (2.55) and (2.36). In the second, it was found from the r.h.s. of (5.14), where $h_3^{(\infty)}$ was calculated numerically using eq. (5.9). The value of the single non-trivial RG invariant $\mathfrak{a}_6$, with exponent $d_6=\frac{2}{3}$, was fixed according to (5.15), where $c=1$. Computations were performed for spin chains with $N\lesssim 5000$.

n	$\mathtt{k}=0.02$		$\mathtt{k}=0.04$	
	$h_6^{(\infty)}$ from (5.8)	$h_6^{(\infty)}$ from (5.14)	$h_6^{(\infty)}$ from (5.8)	$h_6^{(\infty)}$ from (5.14)
10	0.189683	0.189499	0.182812	0.182641
20	0.096347	0.096342	0.089699	0.089695
30	0.070086	0.070084	0.063443	0.063442
40	0.057235	0.057232	0.050550	0.050548

Table 2
Listed are all possible cases where $A\mu=rj$ with $\mu>\frac{r}{2}$ up to $r=10$. Apart from the integers s and exponents d_s, which specify the RG invariants $\mathfrak{a}_s$ belonging to the set (5.6), the values of the integers μ_i (5.11) entering into the differential equation (5.10) are given in the last column.

		μ	s	d_s	μ_i
$r=6$	$A=3$	4	$\{4\}$	$d_4=\frac{2}{3}$	$\mu_2=2$
$r=8$	$A=4$	6	$\{6,4\}$	$d_6=\frac{1}{2},\ d_4=1$	$\mu_3=2$
$r=9$	$A=3$	6	$\{6\}$	$d_6=\frac{2}{3}$	$\mu_2=3$
	$A=6$	6	$\{6\}$	$d_6=\frac{2}{3}$	$\mu_2=3$
$r=10$	$A=5$	6	$\{6\}$	$d_6=\frac{4}{5}$	$\mu_2=2$
		8	$\{8,6\}$	$d_8=\frac{2}{5},\ d_6=\frac{4}{5}$	$\mu_3=4,\ \mu_4=2$

$$\mathfrak{a}_6=-\frac{\Gamma\big(\frac{3}{2n}\big)\Gamma\big(\frac{3}{2}+\frac{9}{2n}\big)}{(n+9)\Gamma\big(1+\frac{9}{2n}\big)\Gamma\big(\frac{1}{2}+\frac{3}{2n}\big)}\ c\ . \tag{5.15}$$

Numerical data in support of (5.14) is presented in Table 1.

In Table 2 we list the cases where $A\mu=rj$ and $\mu>\frac{r}{2}$ for all possible values of $r\le 10$. Together with the integers s labelling the non-trivial RG invariants and the corresponding exponents d_s, presented are the values of the integers μ_i entering into the differential equation (5.10). Note that the two cases $A=3$ and $A=6$ with $r=9$ lead to the same ODE. In fact, it is easy to check that the two sets of Bethe roots for the ground state are related as: $\{\zeta_j\}|_{A=3}=\{\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\,\zeta_j\}|_{A=6}$.

In sec. 4.1 a subtlety was mentioned that occurs when both r, A are even with μ and j given by (4.11). Setting $K=1$ into the formula $\mu=\frac{r}{2K}(2K-1)$ one gets back the case $\mu=\frac{r}{2}$ discussed above. If $K>1$ then $\mu>\frac{r}{2}$. For r even we expect that the RG invariant $\mathfrak{a}_{\frac{r}{2}}$ should be specified as in (5.4). Moreover, it seems likely that (5.5) is generalized as

$$\mathfrak{a}_{\frac{r}{2}}=(-1)^{\frac{1}{2}(r-A+2)}\ \frac{2\Gamma(K-\frac{1}{2})}{r\pi K!}\ \frac{\Gamma(\frac{1}{2}+\frac{r}{2n})}{\Gamma(1+\frac{r}{2n})}\ c^K\ . \tag{5.16}$$

Unfortunately, we were not able to numerically verify this relation for $K>1$. Notice that the first such case corresponds to $r=8$, $A=4$ and $\mu=6$, see Table 2.

5.2. Relation to $A=0$

It turns out that the cases $A\mu=rj$ with $A=1,2,\ldots,r-2$ and $A=0$ are related. The link already occurs for the Bethe roots corresponding to the ground state of the spin chain at finite lattice size N. It allows one to carry over the discussion of the case $A=0$ contained in sec. 7 which, among other things, gives a relation between the coefficients b_i in the ODE (5.10) with the eigenvalues of the so-called quasi-shift operators.

Recall that the integer μ is used to specify the minimal set of non-trivial RG invariants $\mathfrak{a}_s$. In view of (4.20), when performing the scaling limit, one may take

$$\sum_{\ell=1}^{r}(\eta_\ell)^{-s}=0\qquad\text{for}\qquad s\neq\sigma,2\sigma,\ldots,r\ , \tag{5.17}$$

where $\sigma>1$ stands for the greatest common divisor of μ and r:

$$\sigma=\gcd(r,\mu)\,. \tag{5.18}$$

The condition (5.17) may be fulfilled if one sets

$$\eta_{\ell+\frac{r}{\sigma}} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\sigma}}\, \eta_\ell\ . \tag{5.19}$$

When s is a multiple of σ, i.e., $s=\sigma m$ with $m=1,2,\ldots,r/\sigma-1$, then instead of (5.17) one has

$$\frac{1}{rm}\sum_{\ell=1}^{r/\sigma}(\eta_\ell)^{-\sigma m} = \begin{cases} \left(\frac{rN_0}{N}\right)^{d_{\sigma m}} \mathfrak{a}_{\sigma m} & \text{if} \quad \sigma m \neq \frac{r}{2} \\ \frac{1}{\log(\frac{N}{rN_0})}\left(\frac{rN_0}{N}\right)\mathfrak{a}_{\frac{r}{2}} & \text{if} \quad \sigma m = \frac{r}{2} \end{cases}, \tag{5.20}$$

while the normalization condition (4.21) becomes

$$\frac{\sigma}{r}\sum_{\ell=1}^{r/\sigma}(\eta_\ell)^{-r} = (-1)^{r-1}\,. \tag{5.21}$$

For given values of $\mathfrak{a}_{\sigma m}$ and N, this may be treated as a system of r/σ equations for determining the set $\{\eta_1,\ldots,\eta_{r/\sigma}\}$ and hence, in view of (5.19), all the inhomogeneities.

The $N/2$ Bethe roots for the ground state ($S^z=0$) are split into groups with roughly equal phases, which can be labelled by the integer $a=1,2,\ldots,r$ such that $\arg(\zeta_m^{(a)}) \approx \frac{\pi}{r}\,(2a-2-A) \mod 2\pi$. If (5.19) is obeyed and

$$\zeta_m^{(a+\frac{r}{\sigma})} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\sigma}}\,\zeta_m^{(a)}\,, \tag{5.22}$$

then the algebraic system of $N/2$ Bethe Ansatz equations (1.11) reduces to a system of $\tilde{N}/2$ equations with

$$\tilde{N} = N/\sigma\,. \tag{5.23}$$

Namely,

$$\left(\prod_{\ell=1}^{\tilde{r}}\frac{\tilde{\eta}_\ell+\tilde{q}^{+1}\,\tilde{\zeta}_j}{\tilde{\eta}_\ell+\tilde{q}^{-1}\,\tilde{\zeta}_j}\right)^{\frac{\tilde{N}}{\tilde{r}}} = -\omega^2\prod_{i=1}^{\tilde{N}/2}\frac{\tilde{\zeta}_i-\tilde{q}^{+2}\,\tilde{\zeta}_j}{\tilde{\zeta}_i-\tilde{q}^{-2}\,\tilde{\zeta}_j} \qquad \big(j=1,2,\ldots,\tfrac{\tilde{N}}{2}\big)\,, \tag{5.24}$$

where

$$\{\tilde{\zeta}_j\}_{j=1}^{\tilde{N}/2} = \bigcup_{a=1}^{\tilde{r}}\left\{(-1)^I\,\big(\zeta_j^{(a)}\big)^\sigma\right\}_{j=1}^{N/(2r)} \tag{5.25}$$

and we also use

$$\tilde{r}=r/\sigma\,, \qquad \tilde{q}=(-1)^I\,q^\sigma\,, \qquad \tilde{\eta}_\ell = (-1)^{\sigma-1}\,(\eta_\ell)^\sigma \qquad (\ell=1,2,\ldots,\tilde{r})\,. \tag{5.26}$$

The above formulae involve the sign factor $(-1)^I$ with $I=\frac{A\sigma}{r}$, which is an integer due to the condition $A\mu=rj$.

From the definition (5.26), it follows that

$$\tilde{q} = \exp\left(\frac{\mathrm{i}\pi}{\tilde{n}+\tilde{r}}\right) \qquad \text{with} \qquad \tilde{n}=n/\sigma\,, \tag{5.27}$$

i.e., the argument of $\tilde{q}$ lies in the interval $(0,\frac{\pi}{\tilde{r}})$. Thus the algebraic system (5.24) coincides with the Bethe Ansatz equations for the $\tilde{r}$ - site periodic spin chain with the set of inhomogeneities $\{\tilde{\eta}_\ell\}_{\ell=1}^{\tilde{r}}$ in the regime $\tilde{A}=0$. In view of (5.20), the RG invariants $\tilde{\mathfrak{a}}_s$ are expressed in terms of the original ones as

$$\tilde{\mathfrak{a}}_s = (-1)^{(\sigma-1)s}\,\sigma^{1-d_{\sigma s}}\,\mathfrak{a}_{\sigma s} \qquad \text{and} \qquad \tilde{d}_s = d_{\sigma s}\,. \tag{5.28}$$

It turns out that if $\{\zeta_j\}$ is the solution to the Bethe Ansatz equations (1.11) corresponding to the ground state, then $\tilde{\zeta}_j$ (5.25) are the Bethe roots for the ground state of the spin chain with (N,r,q,η_ℓ) swapped for $(\tilde{N},\tilde{r},\tilde{q},\tilde{\eta}_\ell)$ defined in eqs. (5.23) and (5.26).

6. General differential equation for $A=1,2,\ldots,r-2$

Consider the equation

$$\left[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-1)^A\,E^r\,\mathrm{e}^{ry}-\sum_{(\mu,j)\in\Xi_{r,A}}c_{\mu,j}\,E^\mu\,\mathrm{e}^{\left((A\mu-rj)\,\frac{n+r}{r}+\mu\right)y}\right]\psi=0\,, \tag{6.1}$$

where the sum goes over pairs of non-negative integers belonging to the set

$$\Xi_{r,A} = \left\{(\mu,j)\ :\ \ \frac{rj}{A}<\mu<\frac{r}{A+1}\,(j+1)\ \ \&\ \ j\geq 0\right\}\,. \tag{6.2}$$

We have excluded (μ, j) for which the condition $A\mu = rj$ is satisfied for the reasons discussed in the previous section. It turns out that for any $r \geq 3$ and $A = 1, 2, \dots, r-2$ the number of admissible pairs obeys the condition

$$|\Xi_{r,A}| = |\Xi_{r,r-1-A}| \tag{6.3}$$

and is bounded as[9]

$$[\tfrac{r+1}{4}] \leq |\Xi_{r,A}| \leq [\tfrac{r-1}{2}] \,. \tag{6.4}$$

This way, the amount of terms in the sum in (6.1) is less than the number of functionally independent RG invariants. Thus one arrives at the hypothesis that for sufficiently large n and given $c_{\mu,j}$, it is always possible to organize the scaling limit such that the scaled Bethe roots are described by the ODE (6.1).

The specification of the RG invariants $\mathfrak{a}_s$ and their relation to the coefficients $c_{\mu,j}$ can be deduced, in principle, for any given value of r and $A = 1, 2, \dots, r-2$ similar to how it was done previously. Unfortunately, we did not find a convenient form for these relations that works in general. For odd r, $A = \frac{r-1}{2}$ and $n > 0$ they are given by eqs. (2.83), (2.73) and (2.74). Below we present the explicit formulae for even r and $A = \frac{r}{2} - 1$, $\frac{r}{2}$ as well as arbitrary r with $A = 1, r-2$.

6.1. The case $A = 1$

The definition (6.2) for the set of admissible values of (μ, j), specialized to $A = 1$, yields

$$\Xi_{r,1} = \big\{(\mu, j) \;:\;\; \mu = 1, \dots, \big[\tfrac{r-1}{2}\big] \;\;\&\;\; j = 0\big\} \,, \tag{6.5}$$

so that the ODE (6.1) takes the form

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} + E^r\,\mathrm{e}^{ry} - \sum_{\mu=1}^{[\frac{r-1}{2}]} c_\mu\, E^\mu\, \mathrm{e}^{\frac{n+2r}{r}\mu y}\right]\psi = 0\,. \tag{6.6}$$

One should distinguish between the case when r is an odd or even integer. For the former, a computation shows that the exponents for the RG invariants $\mathfrak{a}_s$ are given by

$$d_s = \frac{2}{r}\,o_s \qquad (r-\text{odd})\,, \tag{6.7}$$

where

$$o_s \equiv \begin{cases} r - 2s & \text{for} \quad s = 1, 2, \dots, \frac{r-1}{2} \\ 2r - 2s & \text{for} \quad s = \frac{r+1}{2}, \frac{r+3}{2}, \dots, r-1 \end{cases} . \tag{6.8}$$

The values of $\mathfrak{a}_s$ are related to the coefficients of the ODE (6.6) as

$$\mathfrak{a}_s = (-1)^{\frac{(r+1)}{2}o_s}\; \frac{1}{s}\sum_{k=1}^{o_s}(-1)^k\,\frac{\Gamma(s\nu + k)}{k!\Gamma(s\nu)} \sum_{\substack{j_1+j_2\ldots+j_k=o_s-k \\ j_1,\ldots,j_k\geq 0}} R_{2j_1+1}\ldots R_{2j_k+1} \tag{6.9}$$

with $\nu = \frac{2n}{n+r}$ and we use the same notation as in (2.73). Namely

$$R_{2j+1} = \frac{1}{r}\sum_{\substack{\alpha_1+2\alpha_2+\ldots+(j+1)\alpha_{j+1}=j+1 \\ \alpha_1,\ldots,\alpha_{j+1}\geq 0}} \frac{(-1)^{\alpha_1+\ldots+\alpha_{j+1}}}{\alpha_1!\alpha_2!\ldots\alpha_{j+1}!}\,\frac{\Gamma(\alpha_1+\ldots+\alpha_{j+1}-\frac{2j+1}{r})}{\Gamma(1-\frac{2j+1}{r})}\,G_2^{\alpha_1}G_4^{\alpha_2}\ldots G_{2j+2}^{\alpha_{j+1}} \tag{6.10}$$

where now

$$G_{2m} = \frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\Gamma(\frac{r}{2n})}\sum_{k=1}^{m}\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(m-k)}\,\frac{\Gamma\big(\frac{k}{2}-(\frac{1}{2r}+\frac{1}{n})m+\frac{r}{2n}\big)}{k!\Gamma\big(\frac{3}{2}-\frac{k}{2}-(\frac{1}{2r}+\frac{1}{n})m+\frac{r}{2n}\big)} \sum_{\substack{\mu_1+\ldots+\mu_k=\frac{1}{2}(kr-m) \\ 1\leq\mu_1,\ldots,\mu_k\leq\frac{r-1}{2}}} c_{\mu_1}\cdots c_{\mu_k}\,. \tag{6.11}$$

Our numerical work shows that the value of the RG invariant $\mathfrak{a}_s$ with $s = \frac{r+1}{2}$, whose exponent is $d_s = 2 - \frac{2}{r}$, has no effect on the scaling limit. Thus it can be set to zero and the number of non-trivial RG invariants is equal to $r-2$.

In the case of even r the analogous relations read as follows. The exponents corresponding to $\mathfrak{a}_s$ are given by

$$d_s = \frac{4}{r}\,e_s \qquad (r-\text{even})\,, \tag{6.12}$$

[9] We checked (6.4) up to $r = 200$. If r is a prime number then $|\Xi_{r,A}| = \frac{r-1}{2}$ for any $A = 1, 2, \dots, r-2$, while as $A = \frac{r}{2} - 1$ and $A = \frac{r}{2}$ (r even) one has $|\Xi_{r,A}| = [\frac{r}{4}]$.

where we use

$$e_s \equiv \begin{cases} \frac{r}{2}-s & \text{for} \quad s=1,\dots \frac{r}{2}-1 \\ r-s & \text{for} \quad s=\frac{r}{2},\dots,r-1 \end{cases} . \tag{6.13}$$

As for the $\mathfrak{a}_s$ — c_μ relation, one has

$$\mathfrak{a}_s = (-1)^{e_s-1}\ \frac{1}{s}\times \begin{cases} \frac{1}{2\mathrm{i}}\ \big(F_s^{(+)}-F_s^{(-)}\big) & \text{for} \quad s=1,\dots,\frac{r}{2}-1 \\ \frac{1}{2}\big(F_s^{(+)}+F_s^{(-)}\big) & \text{for} \quad s=\frac{r}{2},\dots,r-1 \end{cases} . \tag{6.14}$$

Here

$$F_s^{(\pm)} = \sum_{k=1}^{e_s}(-1)^k\ \frac{\Gamma(\frac{s\nu}{2}+k)}{k!\Gamma(\frac{s\nu}{2})} \sum_{\substack{j_1+j_2\dots+j_k=e_s-k \\ j_1,\dots,j_k\geq 0}} R^{(\pm)}_{2j_1+1}\dots R^{(\pm)}_{2j_k+1} \tag{6.15}$$

with

$$R^{(\pm)}_{2j+1} = \frac{2}{r} \sum_{\substack{\alpha_1+2\alpha_2+\ldots+(j+1)\alpha_{j+1}=j+1 \\ \alpha_1,\dots,\alpha_{j+1}\geq 0}} \frac{(-1)^{\alpha_1+\ldots+\alpha_{j+1}}}{\alpha_1!\alpha_2!\dots\alpha_{j+1}!}\ \frac{\Gamma(\alpha_1+\ldots+\alpha_{j+1}-\frac{2}{r}(2j+1))}{\Gamma(1-\frac{2}{r}(2j+1))} \tag{6.16}$$

$$\times \big(G_2^{(\pm)}\big)^{\alpha_1}\big(G_4^{(\pm)}\big)^{\alpha_2}\dots\big(G_{2j+2}^{(\pm)}\big)^{\alpha_{j+1}} \tag{6.17}$$

and

$$G^{(\pm)}_{2m} = \frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\Gamma(\frac{r}{2n})}\ \sum_{k=1}^{m} \mathrm{e}^{\frac{\mathrm{i}\pi}{2}(r\pm 1)k}\ \frac{\Gamma\big(\frac{k}{2}-(\frac{1}{r}+\frac{2}{n})m+\frac{r}{2n}\big)}{k!\Gamma\big(\frac{3}{2}-\frac{k}{2}-(\frac{1}{r}+\frac{2}{n})m+\frac{r}{2n}\big)} \sum_{\substack{\mu_1+\ldots+\mu_k=\frac{r}{2}k-m \\ 1\leq\mu_1,\dots,\mu_k\leq\frac{r}{2}-1}} c_{\mu_1}\cdots c_{\mu_k}\ . \tag{6.18}$$

The following comment is in order. Formula (6.14) yields that $\mathfrak{a}_{r-1}\equiv 0$. Moreover, the RG invariant $\mathfrak{a}_{\frac{r}{2}}$ has exponent $d_{\frac{r}{2}}=2$ and its value does not affect the scaling limit. In fact, for even r the set of $r-3$ RG invariants turns out to not be minimal in general. For example, we found that in the cases with $r\leq 10$ the scaled Bethe roots are also insensitive to the value of $\mathfrak{a}_{\frac{r}{2}+1}$, in spite that the corresponding exponent $d_{\frac{r}{2}+1}<2$. This is similar to what happens for the RG invariant $\mathfrak{a}_{\frac{r-1}{2}}$ in the case of odd r.

Finally, the above formulae for both odd and even r are literally applicable provided

$$n > \frac{1}{2}\,r\,(r-4)\ . \tag{6.19}$$

6.2. *The case even r and $A=\frac{r}{2}-1$*

Since

$$\boldsymbol{\Xi}_{r,\frac{r}{2}-1} = \big\{(\mu,j)\ :\ \ \mu=2j+1\ \ \&\ \ j=0,1,\dots,\big[\tfrac{r}{4}\big]-1\big\}\ , \tag{6.20}$$

the corresponding ODE reads explicitly as

$$\Bigg[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}+(-1)^{\frac{r}{2}}\,E^r\,\mathrm{e}^{ry}-\sum_{j=0}^{[\frac{r}{4}]-1} c_{2j+1}\,E^{2j+1}\,\mathrm{e}^{\left(\frac{n+r}{2}-(2j+1)\,\frac{n}{r}\right)y}\Bigg]\,\psi=0\ . \tag{6.21}$$

Based on an analysis of the differential equation and a numerical study of the scaling limit, we expect that there are only the $[\frac{r}{4}]$ non-trivial RG invariants:

$$\mathfrak{a}_{2j+1} = \frac{1}{2j+1}\left(\frac{N}{rN_0}\right)\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-2j-1} \qquad \big(0\leq j\leq\big[\tfrac{r}{4}\big]-1\big)\ . \tag{6.22}$$

The relation between their values and the coefficients c_{2j+1} is especially simple

$$\mathfrak{a}_{2j+1} = (-1)^j\ \frac{\Gamma(\frac{1}{2}+\frac{r}{2n})}{r\Gamma(\frac{r}{2n})}\ \frac{\Gamma\big(\frac{1}{2}-\frac{2j+1}{r}\big)}{\Gamma\big(1-\frac{2j+1}{r}\big)}\ c_{2j+1}\ . \tag{6.23}$$

The differential equation (6.21) describes the scaling limit of the ground state Bethe roots for any $n>0$.

6.3. The case even r and $A = \frac{r}{2}$

The set $\Xi_{r,\frac{r}{2}}$ coincides with $\Xi_{r,\frac{r}{2}-1}$ (6.20):

$$\Xi_{r,\frac{r}{2}} = \big\{(\mu, j)\ :\quad \mu = 2j+1\ \ \&\ \ j = 0,1,\ldots,\big[\tfrac{r}{4}\big]-1\big\} \tag{6.24}$$

and the corresponding ODE takes the form

$$\Bigg[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^{\frac{r}{2}}\, E^r\, \mathrm{e}^{ry} - \sum_{j=0}^{[\frac{r}{4}]-1} c_{2j+1}\, E^{2j+1}\, \mathrm{e}^{\left(\frac{n+r}{2}+2j+1\right)y}\Bigg]\,\psi = 0\,. \tag{6.25}$$

The specification of the RG invariants and the $\mathfrak{a}_s$ — c_{2j+1} relation looks similar to those for the case $A = 1$ and we'll use the analogous notations. However, now one should distinguish between even and odd $\frac{r}{2}$. For even $\frac{r}{2}$ the exponents for the RG invariants $\mathfrak{a}_s$ are given by

$$d_s = \frac{2}{r}\, o_s \qquad \big(r/2 - \text{even}\big)\,, \tag{6.26}$$

where

$$o_s \equiv \begin{cases} \frac{r}{2} - s & \text{for}\quad s = 1,3,\ldots,\frac{r}{2}-1 \\ r - s & \text{for}\quad s = \frac{r}{2},\, \frac{r}{2}+2,\,\ldots, r-2 \end{cases}. \tag{6.27}$$

The values of $\mathfrak{a}_s$ are related to the coefficients of the ODE (6.25) as

$$\mathfrak{a}_s = (-1)^{[\frac{o_s}{2}]-1}\ \frac{1}{s}\sum_{k=1}^{o_s}(-1)^k\,\frac{\Gamma(s\nu+k)}{k!\Gamma(s\nu)} \sum_{\substack{j_1+j_2\ldots+j_k=o_s-k\\ j_1,\ldots,j_k\geq 0}} R_{2j_1+1}\cdots R_{2j_k+1} \tag{6.28}$$

with $\nu = \frac{2n}{n+r}$ and we use the same notation (6.10) where now

$$G_{2m} = \frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\Gamma(\frac{r}{2n})}\sum_{k=1}^{m}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}rk}\,\frac{\Gamma\big(\frac{k}{2}-\frac{m}{n}+\frac{r}{2n}\big)}{k!\Gamma\big(\frac{3}{2}-\frac{k}{2}-\frac{m}{n}+\frac{r}{2n}\big)} \sum_{\substack{j_1+\ldots+j_k=\frac{r-2}{4}k-\frac{m}{2}\\ 0\leq j_1,\ldots,j_k\leq\frac{r}{4}-1}} c_{2j_1+1}\cdots c_{2j_k+1}\,. \tag{6.29}$$

In the case of odd $\frac{r}{2}$ the exponents are given by

$$d_s = \frac{4}{r}\, e_s \qquad \big(r/2 - \text{odd}\big)\,, \tag{6.30}$$

where

$$e_s \equiv \begin{cases} \frac{1}{4}\,(r-2s) & \text{for}\quad s = 1,\,3,\,\ldots,\,\frac{r}{2}-2 \\ \frac{1}{2}\,(r-s) & \text{for}\quad s = \frac{r}{2}+1,\,\frac{r}{2}+3,\,\ldots,\,r-4 \end{cases}. \tag{6.31}$$

The $\mathfrak{a}_s$ — c_{2j+1} relation reads as

$$\mathfrak{a}_s = (-1)^{e_s-1}\ \frac{1}{s}\times\begin{cases} \frac{1}{2\mathrm{i}}\,\big(F_s^{(-)} - F_s^{(+)}\big) & \text{for}\qquad s = 1,\,3,\,\ldots,\,\frac{r}{2}-2 \\ \frac{1}{2}\,\big(F_s^{(+)} + F_s^{(-)}\big) & \text{for}\qquad s = \frac{r}{2}+1,\,\frac{r}{2}+3,\,\ldots,\,r-4 \end{cases}. \tag{6.32}$$

Here the functions $F_s^{(\pm)}$ are defined by eqs. (6.15) and (6.16) but now

$$G_{2m}^{(\pm)} = \frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\Gamma(\frac{r}{2n})}\sum_{k=1}^{m}\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(\frac{r-2}{2}\pm 1)k}\,\frac{\Gamma\big(\frac{k}{2}-\frac{2m}{n}+\frac{r}{2n}\big)}{k!\Gamma\big(\frac{3}{2}-\frac{k}{2}-\frac{2m}{n}+\frac{r}{2n}\big)} \sum_{\substack{j_1+\ldots+j_k=\frac{r-2}{4}k-m\\ 0\leq j_1,\ldots j_k\leq\frac{r-2}{4}-1}} c_{2j_1+1}\cdots c_{2j_k+1}\,. \tag{6.33}$$

The differential equation (6.25) describes the scaling limit of the ground state Bethe roots for any $n > 0$.

6.4. The case $A = r - 2$

The set of admissible values of (μ, j) (6.2) are given by

$$\Xi_{r,r-2} = \big\{(\mu, j)\ :\quad \mu = j+1\ \ \&\ \ j = 0,1,\ldots,\big[\tfrac{r-1}{2}\big]-1\big\} \tag{6.34}$$

and the ODE becomes

$$\left[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-E)^r\,\mathrm{e}^{ry}-\sum_{\mu=1}^{[\frac{r-1}{2}]}c_\mu\,E^\mu\,\mathrm{e}^{\left(n+r-\frac{2n+r}{r}\,\mu\right)y}\right]\psi=0\,. \tag{6.35}$$

We have found that for any $n>0$ the differential equation describes the scaling limit of the Bethe roots for the ground state if the minimal set of non-trivial RG invariants consists of the $[\frac{r}{2}]$ members

$$\mathfrak{a}_s=\frac{1}{s}\left(\frac{N}{rN_0}\right)^{\frac{2s}{r}}\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-s}\qquad\left(s=1,\ldots,\left[\tfrac{r}{2}\right]\right). \tag{6.36}$$

Their values are related to the coefficients of eq. (6.35) as

$$\mathfrak{a}_s=\frac{1}{s}\sum_{k=1}^{s}(-1)^k\,\frac{\Gamma(s\nu+k)}{k!\Gamma(s\nu)}\sum_{\substack{j_1+j_2\ldots+j_k=s-k\\ j_1,\ldots,j_k\geq 0}}R_{2j_1+1}\ldots R_{2j_k+1}\,. \tag{6.37}$$

Here $\nu=\frac{2n}{n+r}$ and again we use the notation R_{2j+1} (6.10) with

$$G_{2m}=\frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\Gamma(\frac{r}{2n})}\sum_{k=1}^{m}\frac{\Gamma\left(k-(\frac{2}{r}+\frac{1}{n})m+\frac{r}{2n}\right)}{k!\,\Gamma\left(\frac{3}{2}-(\frac{2}{r}+\frac{1}{n})m+\frac{r}{2n}\right)}\sum_{\substack{\mu_1+\ldots+\mu_k=m\\ 1\leq\mu_1,\ldots,\mu_k\leq[\frac{r-1}{2}]}}c_{\mu_1}\cdots c_{\mu_k}\,. \tag{6.38}$$

7. ODEs for $A=0$

As $A=0$ the integer μ may take any value from 1 to $r-1$, while $j=0$ (see (3.23)). Then the ODE (3.3), (3.4) becomes

$$\left[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-E^r\,\mathrm{e}^{ry}-\sum_{\mu=1}^{r-1}c_\mu\,E^\mu\,\mathrm{e}^{\mu y}\right]\psi=0\,. \tag{7.1}$$

This is a more general version of equation (5.10), which describes the scaling limit of the Bethe roots for the ground state when $A\mu=rj>0$. The latter is connected to the case $A=0$ since, as was explained in sec. 5.2, the ground state Bethe roots for $A\mu=rj$ and $A=0$ are simply related. In view of this, one may expect that in performing the scaling limit in the regime $\gamma\in(0,\frac{\pi}{r})$ one should take the RG invariants $\mathfrak{a}_s$ with $\frac{r}{2}\leq s\leq r-1$ to be non-vanishing with the corresponding exponents given by

$$d_s=\tfrac{2}{r}\,(r-s)\qquad\left(s=\left[\tfrac{r+1}{2}\right],\ldots,r-1\right). \tag{7.2}$$

The $\mathfrak{a}_s$ — c_μ relation is obtained along the similar lines as before yielding that

$$\mathfrak{a}_s=(-1)^{s-1}\,\frac{1}{s}\sum_{k=1}^{r-s}(-1)^k\,\frac{\Gamma(s\nu+k)}{k!\Gamma(s\nu)}\sum_{\substack{j_1+j_2\ldots+j_k=r-s-k\\ j_1,\ldots,j_k\geq 0}}R_{2j_1+1}\ldots R_{2j_k+1}\qquad\left(\tfrac{r}{2}<s\leq r-1\right), \tag{7.3}$$

where $\nu=\frac{2n}{n+r}$ and we use the notation R_{2j+1} (6.10) with

$$G_{2m}=\frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\Gamma(\frac{r}{2n})}\sum_{k=1}^{m}\frac{\Gamma\left(\frac{r-2m}{2n}\right)}{k!\,\Gamma\left(\frac{3}{2}-k+\frac{r-2m}{2n}\right)}\sum_{\mu_1+\ldots+\mu_k=rk-m}c_{\mu_1}\cdots c_{\mu_k}\,. \tag{7.4}$$

These formulae are literally applicable for $\frac{r}{2}<s\leq r-1$. If r is even and $s=\frac{r}{2}$ the summand in the r.h.s. of (7.3) at $k=1$ coincides with νR_{r-1}, which contains a divergent term G_r. As was discussed, this signals that the definition of the RG invariant $\mathfrak{a}_{\frac{r}{2}}$ needs to be modified:

$$\mathfrak{a}_{\frac{r}{2}}\equiv\frac{2}{r^2}\,\frac{1}{\log\left(\frac{N}{rN_0}\right)}\left(\frac{N}{rN_0}\right)\sum_{\ell=1}^{r}(\eta_\ell)^{-\frac{r}{2}} \tag{7.5}$$

and, similar to (5.5), we expect that its value is expressed through the coefficients of the ODE as

$$\mathfrak{a}_{\frac{r}{2}}=\frac{2\Gamma(\frac{1}{2}+\frac{r}{2n})}{\sqrt{\pi}r\Gamma(1+\frac{r}{2n})}\sum_{k=1}^{\frac{r}{2}}(-1)^{\frac{r}{2}-k}\,\frac{\Gamma\left(k-\frac{1}{2}\right)}{\sqrt{\pi}k!}\sum_{\mu_1+\ldots+\mu_k=r(k-\frac{1}{2})}c_{\mu_1}\cdots c_{\mu_k}\,. \tag{7.6}$$

In the case under consideration, in taking the scaling limit, the $[\frac{r}{2}]$ non-trivial RG invariants can be treated as the independent parameters. A quick inspection shows that formulae (7.3) - (7.6) involve the $[\frac{r}{2}]$ coefficients c_μ, where $\mu=[\frac{r+1}{2}],[\frac{r+1}{2}]+1,\ldots,r-1$ only. These relations can be inverted to express $c_{[\frac{r+1}{2}]},\ldots,c_{r-1}$ in terms of $\mathfrak{a}_{[\frac{r+1}{2}]},\ldots,\mathfrak{a}_{r-1}$. The remaining coefficients of the ODE c_μ

Table 3
Listed are the numerical values of $B^{(1)}_{22}$ and $B^{(2)}_{2}$ from the relation (7.12), which expresses the coefficients c_1, c_2 in the differential equation (7.10) in terms of b_1, b_2 defined by (7.7), (7.8) with $r = 3$.

n	$B^{(1)}_{22}$	$B^{(2)}_{2}$	n	$B^{(1)}_{22}$	$B^{(2)}_{2}$
0.5	−24.31181	6.335556	5.5	−214.2092	17.56490
1.0	−33.51001	8.000000	6.0	−244.4040	18.57161
1.5	−45.26519	9.248695	6.5	−276.6034	19.57701
2.0	−59.22168	10.37020	7.0	−310.8065	20.58141
2.5	−75.27314	11.43904	7.5	−347.0127	21.58504
3.0	−93.37681	12.48197	8.0	−385.2213	22.58808
3.5	−113.5123	13.51060	8.5	−425.4320	23.59065
4.0	−135.6686	14.53070	9.0	−467.6444	24.59284
4.5	−159.8393	15.54535	9.5	−511.8583	25.59473
5.0	−186.0204	16.55638	10.0	−558.0735	26.59636

with $\mu = 1, \ldots, [\frac{r-1}{2}]$ depend on the RG invariants and the anisotropy parameter $\gamma = \frac{\pi}{n+r}$. Contrary to c_μ with $\mu \geq \frac{r}{2}$, they also turn out to be complicated functions of $p = \frac{1}{2}(n+r)\mathtt{k}$, where $\mathtt{k}$ is the twist parameter. While there are no simple analytical expressions for these c_μ, in principle, they can be determined numerically. One way of doing so is by studying the scaling behaviour of the sums of inverse powers of the Bethe roots, i.e., the sum rules. In practice, however, obtaining c_μ becomes a difficult computational task for $r \geq 6$. In this regard, we found it also useful to consider the products

$$\mathcal{K}^{(\ell)} = \mathrm{e}^{\mathrm{i}\pi\mathtt{k}}\, q^{-\frac{N}{2}} \prod_{m=1}^{M} \frac{\zeta_m + \eta_\ell\, q^{+1}}{\zeta_m + \eta_\ell\, q^{-1}} \qquad (\ell = 1, 2, \ldots, r), \tag{7.7}$$

where $M = N/2 - S^z$ so that in the case of the ground state $M = N/2$. Such products are the eigenvalues of the so-called quasi-shift operators, which are members of the commuting family that includes the spin chain Hamiltonian (for details see, e.g., sec. 6.2 of ref. [25]). Note that $\prod_{\ell=1}^{r} \mathcal{K}^{(\ell)}$ coincides with the eigenvalue of the lattice translation operator (2.102). Having at hand the numerical values of the Bethe roots $\{\zeta_m\}_{m=1}^{N/2}$ corresponding to the ground state and computing (7.7) for increasing numbers of lattice sites, we observed that the following limits exist

$$b_\mu = \lim_{N\to\infty} \left[\left(\frac{N}{rN_0} \right)^{\frac{r-2\mu}{r}} \frac{1}{2\pi \mathrm{i} r} \sum_{\ell=1}^{r} \mathrm{e}^{\frac{\mathrm{i}\pi}{r}\mu(r+1-2\ell)} \log\left(\mathcal{K}^{(\ell)}\right) \right] \tag{7.8}$$

(here the branches of the logarithms are chosen in such a way that $\log(\mathcal{K}^{(\ell)})$ vary continuously with the values of the RG invariants and vanish when all $\mathfrak{a}_s = 0$). Moreover, it was checked for $r = 3, 4, 5$ that a relation of the form

$$c_\mu = \sum_{k=1}^{r-\mu} \sum_{\substack{\mu_1+\ldots+\mu_k=(k-1)r+\mu \\ 1\leq\mu_1,\ldots,\mu_k\leq r-1}} B^{(\mu)}_{\mu_1\ldots\mu_k}\, b_{\mu_1} \cdots b_{\mu_k} \qquad (\mu = 1, \ldots, r-1) \tag{7.9}$$

is satisfied by the $r-1$ numbers b_μ and the coefficients c_μ. Here $B^{(\mu)}_{\mu_1\ldots\mu_k}$ depends on n and r, but not on the twist parameter $\mathtt{k}$.

Let's illustrate the above on the simplest example with $r = 3$. In this case, the value of the RG invariant $\mathfrak{a}_1$ may be set to zero. The coefficient c_2 in the differential equation

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+3)y} - E^3\,\mathrm{e}^{3y} - c_1\, E\,\mathrm{e}^{y} - c_2\, E^2\,\mathrm{e}^{2y} \right] \psi = 0 \tag{7.10}$$

is related to $\mathfrak{a}_2$ as

$$\mathfrak{a}_2 = -\frac{\Gamma(\frac{1}{2}+\frac{3}{2n})}{3\Gamma(\frac{3}{2n})}\, \frac{\Gamma(\frac{1}{2n})}{\Gamma(\frac{1}{2}+\frac{1}{2n})}\, c_2\,. \tag{7.11}$$

Formula (7.9), specialized to the case at hand, becomes

$$c_1 = B^{(1)}_{1}\, b_1 + B^{(1)}_{22}\, (b_2)^2\,, \qquad\qquad c_2 = B^{(2)}_{2}\, b_2\,. \tag{7.12}$$

Numerical values of $B^{(1)}_{22}$ and $B^{(2)}_{2}$ for different n are presented in Table 3. As for $B^{(1)}_{1}$, we found that the analytical expression

$$B^{(1)}_{1} = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2n})}{\Gamma(\frac{1}{2}+\frac{1}{2n})} \tag{7.13}$$

is in agreement with the data within the accuracy of our computations (at least $\sim 10^{-6}$ relative error).

The numerical computation of the limits b_μ (7.8) is straightforward for any r. Hence, knowledge of $B^{(\mu)}_{\mu_1\ldots\mu_k}$ in the relation (7.9) would provide an effective way of determining all $r-1$ coefficients c_μ in the differential equation (7.1). Unfortunately, at the current moment the analytical formula for $B^{(\mu)}_{\mu_1\ldots\mu_k}$ is not known.

The following comment is in order here. In sec. 2.4, devoted to the case $A=\frac{r-1}{2}$ with odd r, it was mentioned that the Bethe roots for a low energy state develop a scaling behaviour not only in the vicinity of $\zeta=0$, but also $\zeta=\infty$. This is manifest in that the limits

$$\bar{E}^{(a)}_m=\lim_{\substack{N\to\infty\\ m-\text{fixed}}}\left(\frac{N}{rN_0}\right)^{\frac{2n}{r(n+r)}}\left(\zeta^{(a)}_{M_a-m}\right)^{-1} \tag{7.14}$$

exist and are non-vanishing (for the ground state $S^z=0$ and $M_a=N/(2r)$). In the case $A=0$, we found that if the scaling limit is performed, where the set of non-trivial RG invariants is taken to be $\{\mathfrak{a}_s\}_{s=[\frac{r+1}{2}]}^{r-1}$ with corresponding exponents $d_s=\frac{2}{r}(r-s)$, then the scaled Bethe roots $\bar{E}^{(a)}_m$ are described in terms of the ODE of the form

$$\left[-\partial_{\bar{y}}^2+\bar{p}^2+\mathrm{e}^{(n+r)\bar{y}}-\bar{E}^r\,\mathrm{e}^{r\bar{y}}-\sum_{\mu=1}^{[\frac{r}{2}]}\bar{c}_\mu\,\bar{E}^\mu\,\mathrm{e}^{\mu\bar{y}}\right]\bar{\psi}=0\,. \tag{7.15}$$

The coefficients $\bar{c}_\mu$, entering therein, depend on the values of the RG invariants, the anisotropy n and $\bar{p}=-\frac{1}{2}(n+r)\,\mathtt{k}$.

For given values of $\mathfrak{a}_s$ and N the relations (4.17), along with the normalization condition (4.21), may be treated as a system of algebraic equations, which determines the set of inhomogeneities $\{\eta_\ell\}_{\ell=1}^r$. For large N it is not difficult to show that

$$\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^r=(-1)^{r-1}+O(N^{-2})\,, \tag{7.16}$$

while

$$\frac{1}{s}\left(\frac{N}{rN_0}\right)^{\bar{d}_s}\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^s=\bar{\mathfrak{a}}_s+O(N^{-2})\qquad(s=1,\ldots,r-1)\,. \tag{7.17}$$

Here

$$\bar{d}_s=\frac{2s}{r} \tag{7.18}$$

and the limiting values $\bar{\mathfrak{a}}_s$ are certain polynomials of the RG invariants $\mathfrak{a}_1,\ldots,\mathfrak{a}_{r-1}$. In particular,

$$\begin{aligned}
\bar{\mathfrak{a}}_1&=(-1)^r\,\mathfrak{a}_{r-1}\\
\bar{\mathfrak{a}}_2&=(-1)^r\,\mathfrak{a}_{r-2}+\frac{r}{2}\,(\mathfrak{a}_{r-1})^2\\
\bar{\mathfrak{a}}_3&=(-1)^r\,\mathfrak{a}_{r-3}+r\,\mathfrak{a}_{r-2}\,\mathfrak{a}_{r-1}+(-1)^r\,\frac{r^2}{3}\,(\mathfrak{a}_{r-1})^3\qquad(r>3)\,.
\end{aligned} \tag{7.19}$$

The scaling limit can be performed with the set of RG invariants chosen to be $\{\bar{\mathfrak{a}}_s\}$. The latter would be defined by formula (7.17) with the term $O(N^{-2})$ ignored, and are kept independent of N. Also, one may impose the normalization condition (7.16), again, without the remainder term. The two schemes based on the different sets of RG invariants lead to the same limiting values $E^{(a)}_m$ and $\bar{E}^{(a)}_m$, provided that $\mathfrak{a}_s$ and $\bar{\mathfrak{a}}_s$ are related as in (7.19). The scaled Bethe roots would be described by the ODEs (7.1) and (7.15). Since the Bethe Ansatz equations (1.11) are invariant w.r.t. the simultaneous inversion $\zeta_j\mapsto\zeta_j^{-1}$, $\eta_\ell\mapsto(\eta_\ell)^{-1}$ and $\omega\mapsto\omega^{-1}$ ($\mathtt{k}\mapsto-\mathtt{k}$), the results for $E^{(a)}_m$ carry over to $\bar{E}^{(a)}_m$ with a change in nomenclature only.

Comment on the $\mathcal{Z}_r$ invariant case with $\eta_\ell=(-1)^r\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)}$

In the work [31], the scaling limit for the low energy states of the $\mathcal{Z}_r$ invariant spin chain in the regime $\gamma\in(0,\frac{\pi}{r})$ was considered. It was found that such states can be labelled by S^z, the winding number $\mathtt{w}\in\mathbb{Z}$, the pair of non-negative integers $(\mathtt{L},\bar{\mathtt{L}})$; and also, for r even, s which may take any real value. The corresponding eigenvalues of the CFT Hamiltonian, defined according to eqs. (2.105), (2.106), are given by

$$\mathcal{E}_{\rm CFT}=\frac{p^2+\bar{p}^2}{n+r}-\frac{r}{12}+\mathtt{L}+\bar{\mathtt{L}}+\begin{cases}\dfrac{s^2}{2n} & \text{for}\quad r\text{ even}\\ 0 & \text{for}\quad r\text{ odd}\end{cases}\,, \tag{7.20}$$

where

$$p=\frac{S^z}{2}+\frac{n+r}{2}\,(\mathtt{k}+\mathtt{w})\,,\qquad\bar{p}=\frac{S^z}{2}-\frac{n+r}{2}\,(\mathtt{k}+\mathtt{w})\,. \tag{7.21}$$

For any low energy Bethe state, the differential equations were proposed, which describe the scaling limit of the Bethe roots. In particular, the ODEs for the so-called "primary" Bethe states, for which the integers $\mathtt{L} = \bar{\mathtt{L}} = 0$, take the form

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - E^r\,\mathrm{e}^{ry} + \sum_{\mu=1}^{[\frac{r}{2}]} (-1)^\mu\, s_\mu\, E^\mu\, \mathrm{e}^{\mu y} \right] \psi = 0 \tag{7.22a}$$

$$\left[-\partial_{\bar{y}}^2 + \bar{p}^2 + \mathrm{e}^{(n+r)\bar{y}} - \bar{E}^r\,\mathrm{e}^{r\bar{y}} - \sum_{\mu=1}^{[\frac{r}{2}]} (-1)^\mu\, \bar{s}_\mu\, \bar{E}^\mu\, \mathrm{e}^{\mu \bar{y}} \right] \bar{\psi} = 0 \tag{7.22b}$$

The coefficients s_μ, $\bar{s}_\mu$ are expressed in terms of the eigenvalues of the quasi-shift operators (7.7):

$$s_\mu = +B_\mu \lim_{N\to\infty} \left[\left(\frac{N}{rN_0}\right)^{\frac{r-2\mu}{r}} \frac{1}{2\pi \mathrm{i} r} \sum_{\ell=1}^{r} \mathrm{e}^{+\frac{\mathrm{i}\pi}{r}\mu(r+1-2\ell)}\, \log\left(\mathcal{K}^{(\ell)}\right) \right], \tag{7.23a}$$

$$\bar{s}_\mu = -B_\mu \lim_{N\to\infty} \left[\left(\frac{N}{rN_0}\right)^{\frac{r-2\mu}{r}} \frac{1}{2\pi \mathrm{i} r} \sum_{\ell=1}^{r} \mathrm{e}^{-\frac{\mathrm{i}\pi}{r}\mu(r+1-2\ell)}\, \log\left(\mathcal{K}^{(\ell)}\right) \right] \tag{7.23b}$$

where

$$B_\mu = (-1)^{(r-1)\mu}\, 2n\, \frac{\sqrt{\pi}\,\Gamma(1+\frac{r-2\mu}{2n})}{\Gamma(\frac{1}{2}+\frac{r-2\mu}{2n})} \qquad \left(\mu = 1,\ldots,[\tfrac{r}{2}]\right). \tag{7.24}$$

Note that when r is even, $s_{\frac{r}{2}} = \bar{s}_{\frac{r}{2}}$ and coincides with s, which appears in formula (7.20).

The pair (7.22) resembles the differential equations (7.1) and (7.15). However, it describes the scaling limit of a class of low energy excited states, not just the ground state. The values of s_μ and $\bar{s}_\mu$ specify the primary Bethe state and are not arbitrary. For odd r they belong to a certain discrete set, which is determined through a so-called "quantization condition". In the case of even r, the parameter $s \equiv s_{\frac{r}{2}} = \bar{s}_{\frac{r}{2}}$ may be any real number, but once it is fixed, the remaining s_μ, $\bar{s}_\mu$ with $\mu < \frac{r}{2}$ take a discrete set of admissible values, which is also defined by the quantization condition. In contrast, the pair of ODEs (7.1) and (7.15) correspond to the ground state. The coefficients c_μ with $\mu \geq \frac{r}{2}$ are related to the values of the RG invariants through eqs. (7.3)-(7.6). We expect that the values of the remaining c_μ with $\mu < \frac{r}{2}$ as well as $\bar{c}_\mu$ in (7.15) are a particular solution of a certain quantization condition. Determining the latter would be crucial for describing the scaling limit of the low energy spectrum for the spin chain with softly broken $\mathcal{Z}_r$ symmetry in the regime $\gamma \in (0, \frac{\pi}{r})$. However, this problem lies outside the scope of the paper.

8. Concluding remarks

In this work, we described within the ODE/IQFT approach the scaling behaviour of the Bethe roots for just the ground state of the spin - $\frac{1}{2}$ chain associated with the inhomogeneous six-vertex model. Even so, the results indicate the presence of a remarkable variety of multiparametric integrable structures in the underlying field theories. Further development of the ODE/IQFT correspondence would require obtaining the class of differential equations that describe the scaling limit of all the low energy Bethe states. In the $\mathcal{Z}_r$ invariant case with anisotropy parameter $\gamma \in (0, \frac{\pi}{r})$ this was done in ref. [31]. For the more general setup discussed in the present work, a preliminary analysis shows that a creative application of the original ideas from ref. [10] is required. An important physical question is uncovering the field theories governing the critical behaviours for $0 < \gamma < \pi$. It is relatively well understood now in the case $r = 2$ and for the domain $\pi(1 - \frac{1}{r}) < \gamma < \pi$ with arbitrary r due to the works [23,24,35–39] and [21], respectively.

Among the most interesting directions of research in the area of the ODE/IQFT correspondence is its extension to the spin chain associated with the higher spin - j generalization of the inhomogeneous six-vertex model and especially the $j \to \infty$ limit for the isotropic (XXX) spin chain, i.e., at the boundary of the domain of criticality, $\gamma = 0$ and $\gamma = \pi$. This may be useful for exploring the isotropic spin chain built from (infinite dimensional) unitary representations of $\mathfrak{sl}(2,\mathbb{C})$. The latter is relevant to high energy physics since, as was discovered in the pioneering paper of Lipatov [40], this spin chain (in a certain setup) describes the wave functions of compound states of reggeized gluons in multicolour QCD ($N_c \to \infty$) in the generalized leading logarithmic approximation. The idea was further developed by Faddeev and Korchemsky in [41]. In the works [42,43] the spectrum of the spin chain was studied for small lattice sizes N, which is interpreted as the number of gluons. The analysis becomes complicated for increasing N. This is where, one may hope, that the ODE/IQFT approach would be useful.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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Appendix A. Auxiliary functions appearing in the sum rules

Here we present the explicit formulae for f_1, f_2 and f_3 entering into eq. (2.41). We use the same notation as in ref. [39], where these functions have previously appeared.

The function f_1 is defined as

$$f_1(h,g) = \frac{\pi\Gamma(1-2g)}{\sin(\pi g)}\,\frac{\Gamma(g+2h)}{\Gamma(1-g+2h)}\,. \tag{A.1}$$

As for f_2 and f_3, the expression for these is more complicated. In particular, f_2 is given by the integral

$$f_2(h,g) = 2^{1-4g}\,\frac{\Gamma^2(1-g)}{\Gamma^2(\frac{1}{2}+g)}\,\frac{\Gamma(2g+2h)}{\Gamma(1-2g+2h)}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_1(x)}{x+\mathrm{i}h} \qquad \big(0<g<\tfrac{1}{2},\ \Re e(h)>0\big) \tag{A.2}$$

with

$$S_1(x) = \sinh(2\pi x)\,\Gamma(1-2g+2\mathrm{i}x)\,\Gamma(1-2g-2\mathrm{i}x)\,\big(\Gamma(g+2\mathrm{i}x)\Gamma(g-2\mathrm{i}x)\big)^2\,. \tag{A.3}$$

The above is applicable for $0<g<\frac{1}{2}$. Its analytic continuation to the interval $\frac{1}{2}<g<1$, yields

$$\begin{aligned} f_2(h,g) = 2^{1-4g}\,&\frac{\Gamma^2(1-g)}{\Gamma^2(\frac{1}{2}+g)}\,\frac{\Gamma(2g+2h)}{\Gamma(1-2g+2h)}\Bigg(\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_1(x)}{x+\mathrm{i}h} \\ &-\frac{\sin(2\pi g)\Gamma(3-4g)\Gamma^2(1-g)\Gamma^2(3g-1)}{(2h+1-2g)(2h-1+2g)}\Bigg) \qquad \big(\tfrac{1}{2}<g<1,\ \Re e(h)>0\big)\,. \end{aligned} \tag{A.4}$$

In the case of f_3, the following expression is valid for $0<g<\frac{1}{2}$ and $\Re e(h)>0$:

$$\begin{aligned} f_3(h,g) = 2^{2-6g}\,&\sqrt{\pi}\,\frac{\Gamma^3(1-g)}{\Gamma^3(\frac{1}{2}+g)}\,\frac{\Gamma(3g-1+2h)}{\Gamma(2-3g+2h)} \\ &\times\Bigg(-\frac{\sin(4\pi g)}{\pi^2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}y}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_2(x,y)}{(y+\mathrm{i}h)(x-y-\mathrm{i}0)}+\frac{1}{3}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_3(x)}{x+\mathrm{i}h}\Bigg), \end{aligned} \tag{A.5}$$

where the functions S_2 and S_3 read as

$$\begin{aligned} S_2(x,y) &= \sinh(2\pi y)\,\sinh(2\pi x)\,\Gamma(g+2\mathrm{i}y)\,\Gamma(g-2\mathrm{i}y)\,\Gamma(2g+2\mathrm{i}y)\,\Gamma(2g-2\mathrm{i}y) \\ &\quad\times\Gamma(2-3g+2\mathrm{i}y)\,\Gamma(2-3g-2\mathrm{i}y)\,\Gamma(1-2g+2\mathrm{i}x)\,\Gamma(1-2g-2\mathrm{i}x) \\ &\quad\times\big(\Gamma(g+2\mathrm{i}x)\,\Gamma(g-2\mathrm{i}x)\big)^2 \\ S_3(x) &= \sinh(2\pi x)\,\Gamma(2-3g+2\mathrm{i}x)\,\Gamma(2-3g-2\mathrm{i}x)\,\big(\Gamma(g+2\mathrm{i}x)\,\Gamma(g-2\mathrm{i}x)\big)^3 \\ &\quad\times\frac{\sin(4\mathrm{i}\pi x+2\pi g)-2\sin(2\pi g)}{\sin(2\mathrm{i}\pi x+2\pi g)}\,. \end{aligned} \tag{A.6}$$

For $\frac{1}{2}<g<\frac{2}{3}$ and $\Re e(h)>0$ one has

$$f_3(h,g) = 2^{2-6g}\,\sqrt{\pi}\,\frac{\Gamma^3(1-g)}{\Gamma^3(\frac{1}{2}+g)}\,\frac{\Gamma(3g-1+2h)}{\Gamma(2-3g+2h)} \tag{A.7}$$

$$\times\left[-\frac{\sin(4\pi g)}{\pi^2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}y}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_2(x,y)}{(y+\mathrm{i}h)(x-y-\mathrm{i}0)}+\frac{1}{3}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{\tilde{S}_3(x)}{x+\mathrm{i}h}\right],$$

where

$$\begin{aligned}\tilde{S}_3(x)=S_3(x)-3\pi^{-2}\sin(4\pi g)\sin(2\pi g)\sinh(2\pi x)\Gamma(3-4g)\Gamma^2(1-g)\Gamma^2(3g-1)\\ \times\Gamma(2-3g-2\mathrm{i}x)\Gamma(2-3g+2\mathrm{i}x)\\ \times\Gamma(g+2\mathrm{i}x)\Gamma(g-2\mathrm{i}x)\Gamma(2g+2\mathrm{i}x-1)\Gamma(2g-2\mathrm{i}x-1)\,.\end{aligned}\tag{A.8}$$

Finally if $\frac{2}{3}<g<1$ and $\Re e(h)>0$, the analytic continuation of (A.5) gives

$$\begin{aligned}f_3(h,g)=2^{2-6g}\sqrt{\pi}\,\frac{\Gamma^3(1-g)}{\Gamma^3(\frac{1}{2}+g)}\,\frac{\Gamma(3g-1+2h)}{\Gamma(2-3g+2h)}\\ \times\left[-\frac{\sin(4\pi g)}{\pi^2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}y}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_2(x,y)}{(y+\mathrm{i}h)(x-y-\mathrm{i}0)}+\frac{1}{3}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{\tilde{S}_3(x)}{x+\mathrm{i}h}+S_4\right],\end{aligned}\tag{A.9}$$

where S_4 is defined as

$$\begin{aligned}S_4=\frac{\sin(2\pi g)\sin(3\pi g)\Gamma(5-6g)\Gamma(2-2g)\Gamma^3(1-g)\Gamma^2(3g-1)\Gamma(5g-3)}{\pi\big((2-3g)^2-4h^2\big)}\\ +\frac{4\sin(3\pi g)\cos(\pi g)\Gamma(4-6g)\Gamma^3(2-2g)\Gamma^3(4g-2)}{3\big((2-3g)^2-4h^2\big)\big(2\cos(2\pi g)+2\cos(4\pi g)+1\big)}\\ \times\big(3g-6h-2+(15g-6h-10)\cos(2\pi g)+(9g-6h-6)\cos(4\pi g)+2(3g-2)\cos(6\pi g)\big)\,.\end{aligned}\tag{A.10}$$

Appendix B. Lagrange formula

Consider the algebraic equation

$$X^r+\sum_{m=1}^{r-1}G_mX^{r-m}=Y^r\,.\tag{B.1}$$

Let $X(Y)$ be the solution such that

$$X(Y)\to Y\qquad\text{as}\qquad Y\to\infty\,.\tag{B.2}$$

In fact, $X(Y)$ is a multivalued function of Y and this condition specifies its branch for sufficiently large Y. It was found by Lagrange [32], see also [33], that $X(Y)$ admits a convergent power series expansion of the form

$$X(Y)=Y+\sum_{\substack{k=1\\ k\neq 0\,\mathrm{mod}\,r}}^{\infty}R_k\,Y^{-k}\,,\tag{B.3}$$

where

$$R_k=\frac{1}{r}\sum_{\substack{\alpha_1,\ldots\alpha_{n-1}\geq 0\\ \alpha_1+2\alpha_2+\ldots+(r-1)\alpha_{r-1}=k+1}}\frac{(-1)^{\alpha_1+\ldots+\alpha_{r-1}}}{\alpha_1!\alpha_2!\ldots\alpha_{r-1}!}\,\frac{\Gamma(\alpha_1+\ldots+\alpha_{r-1}-\frac{k}{r})}{\Gamma(1-\frac{k}{r})}\,G_1^{\alpha_1}G_2^{\alpha_2}\ldots G_{r-1}^{\alpha_{r-1}}\,.\tag{B.4}$$

Appendix C. Relations between $\mathfrak{a}_{2j+1}$ and c_{2j+1} for $r=3,5,7$ and $A=\frac{r-1}{2}$

For r odd and $A=\frac{r-1}{2}$ the formula (2.83) was obtained in the main body of the text that expresses the RG invariants $\mathfrak{a}_{2j+1}$ (2.33) with $j=1,2,\ldots,\frac{r-1}{2}$ in terms of the coefficients c_{2j+1} of the ODE (2.18). The inversion of this relation leads to eq. (2.44). Here we write it out explicitly for $r=3,5,7$.

In the case $r=3$ there is only a single non-trivial RG invariant $\mathfrak{a}_1$. The coefficient c_1 entering into the differential equation is simply proportional to it:

$$c_1=C_0^{(0)}\,\mathfrak{a}_1\,,\tag{C.1}$$

where we use the notation $C_j^{(j)}$ from eq. (2.84), namely,

$$C_j^{(j)} = (-1)^j \, \frac{r\Gamma\big(\frac{r}{2n}\big)\Gamma\big(1-\frac{(2j+1)(n-r)}{2rn}\big)}{\Gamma\big(\frac{1}{2}+\frac{r}{2n}\big)\Gamma\big(\frac{1}{2}-\frac{(2j+1)(n-r)}{2rn}\big)} \, . \tag{C.2}$$

For $r=5$ the two coefficients c_1 and c_3 are expressed in terms of the two RG invariants $\mathfrak{a}_1$ and $\mathfrak{a}_3$ as:

$$\begin{aligned} c_1 &= C_0^{(0)}\,\mathfrak{a}_1 + \frac{n-5}{20n}\,\Big(\big(C_1^{(1)}\big)^2 - 5C_0^{(0)}\Big)\,\mathfrak{a}_3^2 \\ c_3 &= C_1^{(1)}\,\mathfrak{a}_3 \, . \end{aligned} \tag{C.3}$$

For $r=7$, one has

$$\begin{aligned} c_1 &= C_0^{(0)}\,\mathfrak{a}_1 - \frac{n-7}{14n}\,\Big(7C_0^{(0)} + C_1^{(1)}C_2^{(2)}\Big)\,\mathfrak{a}_3\mathfrak{a}_5 \\ &\quad + \frac{n-7}{n^2}\,\bigg(\frac{15n-7}{24}\,C_0^{(0)} + \frac{3(n-7)}{56}\,C_1^{(1)}\,C_2^{(2)} - \frac{3n+28}{588}\,\big(C_2^{(2)}\big)^3\bigg)\,\mathfrak{a}_5^3 \\ c_3 &= C_1^{(1)}\,\mathfrak{a}_3 - \frac{3(n-7)}{28n}\,\Big(7C_1^{(1)} + \big(C_2^{(2)}\big)^2\Big)\,\mathfrak{a}_5^2 \\ c_5 &= C_2^{(2)}\,\mathfrak{a}_5 \, . \end{aligned} \tag{C.4}$$

Appendix D. Scaling limit at the free fermion point with $\eta_\ell = (-1)^r\, e^{\frac{i\pi}{r}(2\ell-1)}$

Here we focus on the case with r odd and $q=\mathrm{i}$. In addition, the inhomogeneities are set to be $\eta_\ell = (-1)^r\, e^{\frac{i\pi}{r}(2\ell-1)}$ for which the spin chain possesses $\mathcal{Z}_r$ invariance. In this case the Bethe Ansatz equations take the especially simple form

$$\left(\frac{1+(\mathrm{i}\zeta_j)^r}{1-(\mathrm{i}\zeta_j)^r}\right)^{\frac{N}{r}} = -e^{\mathrm{i}\pi(2\mathtt{k}+S^z)} \, . \tag{D.1}$$

In sec. 2.1 we described the scaling limit of the eigenvalue of the Q - operator for the ground state of the spin chain. Formula (2.32) remains valid in the case of any low energy Bethe state provided $n>r$. However, at $n=r$ it should be modified as

$$\operatorname*{slim}_{N\to\infty} \left(\frac{N}{2r}\right)^{\frac{1}{2r}(-1)^A E^r} A_+\Big(\big(2N/\pi\big)^{-\frac{1}{r}}\,E\Big) = D_+(E) \qquad\qquad (n=r)\,, \tag{D.2}$$

where recall that the symbol “slim” is there as a reminder that the formula holds true only for the low energy states. The necessity of the N - dependent factor can be illustrated on the ground state eigenvalue. The scaling limit of the Bethe roots for the ground state, where $S^z=0$, yields

$$E_m^{(a)} = e^{\frac{i\pi}{r}(2a-\frac{r-1}{2})}\,\big((2m_a-1+2\mathtt{k})\,r\big)^{\frac{1}{r}} \qquad\qquad (a=1,\ldots,r) \tag{D.3}$$

with

$$m_a = m = 1, 2, \ldots \, . \tag{D.4}$$

Then, it is easy to see that the limit in the l.h.s. of (D.2) exists and leads to

$$D_+^{(\mathrm{vac})} = \frac{\Gamma\big(\frac{1}{2}+\mathtt{k}\big)}{\Gamma\big(\frac{1}{2}+\mathtt{k}+\epsilon^r\big)}\,, \tag{D.5}$$

where we use the short-cut notation

$$\epsilon = e^{-\frac{i\pi}{2r}(r+1)}\,(2r)^{-\frac{1}{r}}\, E \, . \tag{D.6}$$

Staying in the sector $S^z=0$, we now consider the scaling limit of $A_+(\zeta)$ for a low energy excited Bethe state. The chiral state $|\alpha\rangle$ appearing in eq. (2.108) can be characterized by the location of a finite number of “holes” at $n_{j,a}^-$ in the vacuum distribution of the integers m_a given by (D.4) as well as “particles” $n_{j,a}^+$. The positive integers $n_{j,a}^\pm$ are ordered as

$$1 \le n_{1,a}^\pm < n_{2,a}^\pm < \ldots < n_{M_a^\pm,a}^\pm \, . \tag{D.7}$$

Note that for the particles, $\mathtt{k}-n_{j,a}^+ + \frac{1}{2}$ is a negative number and we take its r-th root to be $e^{\frac{i\pi}{r}}\,(n_{j,a}^+ - \mathtt{k} - \frac{1}{2})^{\frac{1}{r}}$ so that the corresponding $E_m^{(a)}$ lies on the ray $\arg(E_m^{(a)}) = \frac{\pi}{2r}\,(4a-r-1)\ (\mathrm{mod}\ 2\pi)$. Then the scaling limit of $A_+(\zeta)$ yields

$$D_+ = \frac{\Gamma(\frac{1}{2}+\mathtt{k})}{\Gamma\big(\frac{1}{2}+\mathtt{k}+\epsilon^r\big)}\ \frac{\prod_{a=1}^r\prod_{j=1}^{M_a^-}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{r}a}\big(n_{j,a}^- -\frac{1}{2}+\mathtt{k}\big)^{\frac{1}{r}}}{\prod_{a=1}^r\prod_{j=1}^{M_a^+}\mathrm{e}^{\frac{\pi\mathrm{i}}{r}(2a+1)}\big(n_{j,a}^+ -\frac{1}{2}-\mathtt{k}\big)^{\frac{1}{r}}} \times\frac{\prod_{a=1}^r\prod_{j=1}^{M_a^+}\ \Big(\mathrm{e}^{\frac{\pi\mathrm{i}}{r}(2a+1)}\big(n_{j,a}^+ -\frac{1}{2}-\mathtt{k}\big)^{\frac{1}{r}}+\epsilon\Big)}{\prod_{a=1}^r\prod_{j=1}^{M_a^-}\ \Big(\mathrm{e}^{\frac{2\pi\mathrm{i}}{r}a}\big(n_{j,a}^- -\frac{1}{2}+\mathtt{k}\big)^{\frac{1}{r}}+\epsilon\Big)}\,. \tag{D.8}$$

As $\epsilon\to+\infty$, D_+ develops the asymptotic behaviour

$$\log D_+ \asymp \log(\mathfrak{C}_+) - \epsilon^r\ \big(r\log(\epsilon)-1\big) - \big(\mathtt{M}+r\mathtt{k}\big)\ \log(\epsilon) + \sum_{s=1}^{\infty}\frac{(-1)^s}{s}\ D_{+,s}\ \epsilon^{-s}\,. \tag{D.9}$$

Here $\mathtt{M}$ stands for the difference between the total number of holes and particles,

$$\mathtt{M}=\sum_{a=1}^r\big(M_a^- - M_a^+\big)\,, \tag{D.10}$$

while $\mathfrak{C}_+$ is given by

$$\mathfrak{C}_+ = \frac{\Gamma(\frac{1}{2}+\mathtt{k})}{\sqrt{2\pi}}\ \frac{\prod_{a=1}^r\prod_{j=1}^{M_a^-}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{r}a}\big(n_{j,a}^- -\frac{1}{2}+\mathtt{k}\big)^{\frac{1}{r}}}{\prod_{a=1}^r\prod_{j=1}^{M_a^+}\mathrm{e}^{\frac{\pi\mathrm{i}}{r}(2a+1)}\big(n_{j,a}^+ -\frac{1}{2}-\mathtt{k}\big)^{\frac{1}{r}}}\,. \tag{D.11}$$

The coefficients $D_{+,s}$ entering into the sum read as

$$D_{+,s} = r\delta_{s,0\,(\mathrm{mod}\,r)}\ \frac{B_{\frac{s}{r}+1}(\frac{1}{2}+\mathtt{k})}{\frac{s}{r}+1} + \sum_{a=1}^r\Bigg(\ \sum_{j=1}^{M_a^-}\mathrm{e}^{\frac{\mathrm{i}\pi}{r}2as}\Big(n_{j,a}^- -\tfrac{1}{2}+\mathtt{k}\Big)^{\frac{s}{r}} - \sum_{j=1}^{M_a^+}\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a+1)s}\Big(n_{j,a}^+ -\tfrac{1}{2}-\mathtt{k}\Big)^{\frac{s}{r}}\Bigg), \tag{D.12}$$

where $B_{m+1}(z)$ are the Bernoulli polynomials.

Within the usual ODE/IQFT interpretation [5], the coefficients $D_{+,s}$ in the expansion (D.9) would be the eigenvalues of certain integrals of motion. In particular, $D_{+,r}=I_1$ — the eigenvalue of the first local integral of motion,

$$\hat{I}_1=\int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\,T\,, \tag{D.13}$$

which is the chiral component of the CFT Hamiltonian (2.107). Formula (D.12), specialized to $s=r$ yields

$$I_1=\frac{r}{2}\left(\mathtt{k}+\frac{\mathtt{M}}{r}\right)^2-\frac{\mathtt{M}^2}{2r}-\frac{r}{24}+\sum_{a=1}^r\left(\sum_{j=1}^{M_a^-}\big(n_{j,a}^- -\tfrac{1}{2}\big)+\sum_{j=1}^{M_a^+}\big(n_{j,a}^+ -\tfrac{1}{2}\big)\right). \tag{D.14}$$

Let's write the integer $\mathtt{M}$ as

$$\mathtt{M}=\mathtt{m}+r\mathtt{w} \tag{D.15}$$

with

$$\mathtt{w}=0,\pm1,\pm2,\ldots\,,\qquad\qquad \mathtt{m}=0,1,2,\ldots,r-1\,. \tag{D.16}$$

Then the state with the lowest value of I_1 with given $\mathtt{M}\geq 0$ would have all $M_a^+=0$, while $\mathtt{m}$ members of the set $\{M_a^-\}_{a=1}^r$ would be equal to $\mathtt{w}+1$ and the rest would coincide with $\mathtt{w}$. The location of the holes is described by $n_{j,a}^-=j$ for $j\leq M_a^-$. For $\mathtt{M}<0$ all $M_a^-=0$, while the set $\{M_a^+\}_{a=1}^r$ is comprised of $r-\mathtt{m}$ integers $|\mathtt{w}|$ and $\mathtt{m}$ integers $|\mathtt{w}+1|$. Then one can re-write I_1 in the form which is applicable for any $\mathtt{M}\in\mathbb{Z}$,

$$I_1=\frac{r}{2}\left(\mathtt{k}+\mathtt{w}+\frac{\mathtt{m}}{r}\right)^2+\frac{\mathtt{m}\,(r-\mathtt{m})}{2r}-\frac{r}{24}+\mathtt{L}\,, \tag{D.17}$$

where $\mathtt{L}$ is a non-negative integer given by

$$\mathtt{L}=\sum_{a=1}^r\left(\sum_{j=1}^{M_a^-}\big(n_{j,a}^- -\tfrac{1}{2}\big)+\sum_{j=1}^{M_a^+}\big(n_{j,a}^+ -\tfrac{1}{2}\big)\right)-\mathtt{m}\sum_{j=1}^{|\mathtt{w}+1|}\big(j-\tfrac{1}{2}\big)-(r-\mathtt{m})\sum_{j=1}^{|\mathtt{w}|}\big(j-\tfrac{1}{2}\big)\,. \tag{D.18}$$

Note that the states with $\mathtt{L}=0$, having the lowest value of I_1, are $\frac{r!}{\mathtt{m}!(r-\mathtt{m})!}$ times degenerate. Analogous computations can be repeated for the chiral state $|\bar{\alpha}\rangle$. One should keep in mind that in the sector with fixed S^z, the total number of (left and right) holes should coincide with the total number of particles, i.e.,

$$\sum_{a=1}^{r}\left(M_a^-+\bar{M}_a^-\right)=\sum_{a=1}^{r}\left(M_a^++\bar{M}_a^+\right). \tag{D.19}$$

In the above considerations, we focused on the sector $S^z=0$ to make the discussion as simple as possible. This can be extended to the case of S^z being any (half-)integer. The final expressions for I_1 and $\bar{I}_1$ are given by formulae (2.125)-(2.127) where one should set $n=r$ therein. Then, the eigenvalues of the Hamiltonian $\hat{H}_{\rm CFT}$ (2.106) coincide with $I_1+\bar{I}_1$, while for the r - site lattice translation operator they are described by (2.128). This result motivated the conjecture contained in sec. 2.5.1. The latter was verified numerically on some specific examples away from the free fermion point.

Appendix E. Scaling limit for r, A even and $(\mu,j)=(\frac{r}{2},\frac{A}{2})$

Here we present the numerical scheme mentioned in sec. 5.1 which allows one to confirm that the scaling limit of the Bethe roots with the single non-trivial RG invariant taken as in (5.4) is described by the differential equation (5.1) with $\mu=\frac{r}{2}$.

As was explained in sec. 5.2, in the case $\mu=\frac{r}{2}$ the ground state Bethe roots, where the inhomogeneities obey the r - site periodicity condition $\eta_{J+r}=\eta_J$, are simply related to those for the spin chain with $\tilde{\eta}_{J+2}=\tilde{\eta}_J$. The parameters of the reduced system $(\tilde{N},\tilde{q},\tilde{\eta}_\ell)$ are expressed in terms of the original one (N,q,η_ℓ) as in eqs. (5.23) and (5.26) with $\sigma=\frac{r}{2}$. In particular,

$$\tilde{N}=2N/r\,,\qquad \tilde{q}=\exp\left(\frac{\mathrm{i}\pi}{\tilde{n}+2}\right),\qquad \text{where}\qquad \tilde{n}=2n/r\,. \tag{E.1}$$

From the definition of the RG invariant (5.4) and the normalization condition (2.34) one has

$$\begin{aligned}&\frac{1}{\tilde{\eta}_1}+\frac{1}{\tilde{\eta}_2}=(-1)^{\frac{r}{2}-1}\,\mathfrak{a}_{\frac{r}{2}}\left(\frac{4\tilde{N}_0}{\tilde{N}}\right)\,\log\left(\frac{r\tilde{N}}{4\tilde{N}_0}\right)\\ &\frac{1}{\tilde{\eta}_1^2}+\frac{1}{\tilde{\eta}_2^2}=-2\,,\end{aligned} \tag{E.2}$$

where $\tilde{N}_0$ is given by formula (1.17) with r and n replaced by 2 and $\tilde{n}$, respectively. In what follows it would be convenient to use the parameter α, defined as

$$\alpha=\arg(\tilde{\eta}_1)=-\arg(\tilde{\eta}_2)\,,\qquad \alpha\in(0,\pi)\,. \tag{E.3}$$

Then the system of equations (E.2) determining the inhomogeneities implies

$$\alpha=\frac{\pi}{2}+(-1)^{\frac{r}{2}}\,\mathfrak{a}_{\frac{r}{2}}\left(\frac{2\tilde{N}_0}{\tilde{N}}\right)\,\log\left(\frac{r\tilde{N}}{4\tilde{N}_0}\right)\,+O\big((\log\tilde{N})^3/\tilde{N}^3\big)\,. \tag{E.4}$$

The scaling limit of the 2 - site periodic spin chain in the regime with $\arg(\tilde{q})\in(0,\frac{\pi}{2})$ has attracted a lot of attention in the literature, see [23,24,35,37,38] and the more recent papers [20,39]. The interest originated following the work of Jacobsen and Saleur [35], which was devoted to the case when the parameter α in (E.3) is equal to $\frac{\pi}{2}$. In the subsequent papers [23] and [37], the energy spectrum for — in the terminology used in sec. 7 — primary Bethe states was studied. These were originally defined in [23] using the logarithmic form of the Bethe Ansatz equations. The corresponding Bethe roots turn out to be all real and the states can be characterized by the integer $\mathtt{m}$, which coincides with the difference between the number of positive ($\tilde{\zeta}_m>0$) and negative ($\tilde{\zeta}_m<0$) roots. In ref. [37] an important relation was found between $\mathtt{m}$ and the quantity

$$s=\frac{\tilde{n}}{2\pi}\log\left(\mathcal{K}^{(2)}\,/\,\mathcal{K}^{(1)}\right), \tag{E.5}$$

which is expressed in terms of the eigenvalues of the quasi-shift operators

$$\mathcal{K}^{(\ell)}=\mathrm{e}^{\mathrm{i}\pi\mathtt{k}}\,\tilde{q}^{-\frac{\tilde{N}}{2}}\prod_{m=1}^{\frac{\tilde{N}}{2}-\tilde{S}^z}\frac{\tilde{\zeta}_m+\tilde{\eta}_\ell\,\tilde{q}^{+1}}{\tilde{\zeta}_m+\tilde{\eta}_\ell\,\tilde{q}^{-1}}\qquad(\ell=1,2)\,. \tag{E.6}$$

In the case of the primary Bethe states there is a natural way of assigning them a dependence on the lattice size so that s becomes a function of $\tilde{N}$. Then it was observed from the numerical data that, with a properly chosen branch of the logarithm in the definition (E.5), the leading large - $\tilde{N}$ behaviour of s for fixed $\mathtt{m}$ is given by

$$s\sim\frac{\pi\mathtt{m}}{2\log(\tilde{N})}\qquad(\tilde{N}\to\infty)\,. \tag{E.7}$$

Moreover, the so-called "quantization condition" was proposed that provides a refinement to this leading asymptotic. The more precise version of the relation, which is also applicable to the case of quasi-periodic boundary conditions, appeared in ref. [20] and reads explicitly as

$$4s\log\left(\frac{\tilde{N}}{2\tilde{N}_0}\right)-\delta(s)-2\pi\,\mathtt{m}=O\big((\log\tilde{N})^{-\infty}\big)\ , \tag{E.8}$$

where

$$\delta(s)=-2\mathrm{i}\,\log\left(2^{-\frac{2\mathrm{i}s}{\tilde{n}}(\tilde{n}+2)}\,\frac{\Gamma(\frac{1}{2}+\tilde{p}+\frac{\mathrm{i}s}{2})\,\Gamma(\frac{1}{2}+\bar{\tilde{p}}+\frac{\mathrm{i}s}{2})}{\Gamma(\frac{1}{2}+\tilde{p}-\frac{\mathrm{i}s}{2})\,\Gamma(\frac{1}{2}+\bar{\tilde{p}}-\frac{\mathrm{i}s}{2})}\right) \tag{E.9}$$

and

$$\tilde{p}=\frac{\tilde{S}^z}{2}+\frac{\tilde{n}+2}{2}\,\big(\mathtt{k}+\mathtt{w}\big)\,,\qquad\qquad \bar{\tilde{p}}=\frac{\tilde{S}^z}{2}-\frac{\tilde{n}+2}{2}\,\big(\mathtt{k}+\mathtt{w}\big)\ . \tag{E.10}$$

The quantization condition holds true up to power law corrections in $\tilde{N}$, as indicated by the r.h.s. of (E.8). For given $\mathtt{m}$, it gives a remarkably accurate description for the value of s at large $\tilde{N}$ [44]. It follows that $s(\tilde{N})$ tends to zero as $\tilde{N}\to\infty$ if the integer $\mathtt{m}$ is left unchanged. A non-trivial scaling limit can be achieved by increasing $|\mathtt{m}|$ simultaneously with $\tilde{N}$ in such a way that s is kept fixed, i.e., treated as the RG invariant. Then (E.8) is interpreted as an equation specifying the $\tilde{N}$ - dependence of $\mathtt{m}$. This way, the states would be characterized by s, which may take any real value. The corresponding eigenvalue of the CFT Hamiltonian (2.106) is given by (7.20) with $r=2$. In the work [20] it was proposed that the differential equation

$$\Big[-\partial_{\tilde{y}}^2+\tilde{p}^2+\mathrm{e}^{(\tilde{n}+2)\tilde{y}}-\tilde{E}^2\,\mathrm{e}^{2\tilde{y}}-s\,\tilde{E}\,\mathrm{e}^{\tilde{y}}\Big]\psi=0 \tag{E.11}$$

describes the scaled Bethe roots for the primary Bethe states when the scaling limit is taken with s being treated as an RG invariant.

The quantization condition was originally found for the case when α from (E.3) is equal to $\frac{\pi}{2}$. It was generalized by Frahm and Seel in ref. [24] to $\alpha\in(\gamma,\pi-\gamma)$ being a fixed number independent of the lattice size. For our purposes, it is sufficient to use the results of that paper specialized to the ground state, i.e., the state of the spin chain Hamiltonian with the lowest possible energy. The important difference between $\alpha=\frac{\pi}{2}$ and $\alpha\neq\frac{\pi}{2}$ is that the value of s defined as in (E.5) for the ground state in the former case vanishes, while for the latter it is no longer zero. Moreover, as $\tilde{N}\to\infty$,

$$s_{\rm GS}(\tilde{N})=\tilde{N}\,b_\infty+O(1)\ , \tag{E.12}$$

where $s_{\rm GS}$ denotes the value of s computed for the ground state and

$$b_\infty=-\frac{n}{\pi}\int_0^\infty\frac{\mathrm{d}t}{t}\ \frac{\sinh(t)\,\sinh\big((\tilde{n}+1)t\big)}{\sinh(\tilde{n}t)\,\sinh\big((\tilde{n}+2)t\big)}\ \sinh\big((1-\tfrac{2\alpha}{\pi})(\tilde{n}+2)t\big)\ . \tag{E.13}$$

The remainder term in (E.12) admits a more accurate description. Namely, denoting it as

$$s_\alpha\equiv s_{\rm GS}(\tilde{N})-\tilde{N}\,b_\infty\ , \tag{E.14}$$

the following quantization condition holds true

$$4s_\alpha\,\log\left(\frac{\tilde{N}}{2\tilde{N}_0}\right)-\delta_{\mathtt{k}}(s_\alpha)=2\pi\mathtt{m}_\alpha+\frac{\tilde{n}+2}{\tilde{n}}\,(\pi-2\alpha)\,\tilde{N}+O\big((\log\tilde{N})^{-\infty}\big)\,. \tag{E.15}$$

Here $\delta_{\mathtt{k}}(s)$ is given by (E.9) with $\tilde{p}=-\bar{\tilde{p}}=\frac{1}{2}(\tilde{n}+2)\,\mathtt{k}$ and $\mathtt{m}_\alpha$ is the nearest integer to $-\frac{\tilde{n}+2}{2\pi\tilde{n}}\,(\pi-2\alpha)\,\tilde{N}$. Despite the similarity of the above with (E.8) one should keep in mind that the latter relation gives the $\tilde{N}$ dependence of s (E.5) for the primary Bethe states, while (E.15) describes how s_α, defined in (E.14) depends on $\tilde{N}$ for the ground state of the spin chain only.

In the relation (E.15) the parameter α is assumed to be fixed. Nevertheless, it inspired us to introduce the formal variable s as the solution of the equation

$$4s\,\log\left(\frac{\tilde{N}}{2\tilde{N}_0}\right)-\delta_{\mathtt{k}}(s)=(-1)^{\frac{r}{2}-1}\,\mathfrak{a}_{\frac{r}{2}}\,\frac{4\,(\tilde{n}+2)}{\tilde{n}}\,\tilde{N}_0\,\log\left(\frac{r\tilde{N}}{4\tilde{N}_0}\right)\,, \tag{E.16}$$

which is obtained from (E.15) by setting $\mathtt{m}_\alpha=0$; substituting α for the $\tilde{N}$ dependent expression in the r.h.s. of (E.4); and ignoring all correction terms. In the original scheme for performing the scaling limit $\mathfrak{a}_{\frac{r}{2}}$ was treated as the RG invariant. We now switch the schemes and suppose that s does not depend on the number of lattice sites, whereas the $\tilde{N}$ dependence of $\mathfrak{a}_{\frac{r}{2}}$ is dictated by (E.16). It turns out that in the new scheme the sums over the Bethe roots

$$\left(\frac{2\tilde{N}_0}{\tilde{N}}\right)^{\frac{\tilde{n}j}{\tilde{n}+2}}\ \frac{1}{j}\ \sum_{m=1}^{\tilde{N}/2}(\tilde{\zeta}_m)^{-j}$$

tend to their limiting values with corrections that decay as a power law in $\tilde{N}$. In contrast, if $\mathfrak{a}_{\frac{r}{2}}$ were to be kept fixed, the corrections would decay considerably more slowly — as the inverse power of $\log(\tilde{N})$. Thus, the new scheme allows one to observe the scaling behaviour for $\tilde{N}=\frac{2N}{r}\sim 1000$. On the other hand the definition (E.16) implies

$$s+(-1)^{\frac{r}{2}}\,\mathfrak{a}_{\frac{r}{2}}\,\frac{\tilde{n}+2}{\tilde{n}}\,\tilde{N}_0=O\big(1/\log(\tilde{N})\big)\,. \tag{E.17}$$

Hence, the change of scheme will not affect the scaling limit, but rather, the rate at which it is achieved as $\tilde{N}\to\infty$. In this regard, one may note that the choice of the constant rN_0 in the logarithm in (5.4) for the RG invariant $\mathfrak{a}_{\frac{r}{2}}$ is not essential: the constant can be replaced by any positive number.

This way, we were able to establish that the scaling limit of the Bethe roots is described by the ODE (E.11). Taking into account the relations (E.1) between the parameters of the original and reduced systems and performing the change of variables

$$\tilde{y}=\frac{r}{2}\,y-\frac{r}{n+r}\,\log\Big(\frac{r}{2}\Big) \tag{E.18}$$

and

$$\tilde{p}=\frac{2}{r}\,p\,,\qquad\qquad \tilde{E}=(-1)^{\frac{A}{2}}\,\Big(\frac{2}{r}\Big)^{\frac{n}{n+r}}\,E^{\frac{r}{2}} \tag{E.19a}$$

$$s=(-1)^{\frac{A}{2}}\,\frac{2c}{r} \tag{E.19b}$$

one arrives at the differential equation (5.1) with $\mu=\frac{r}{2}$. The $\mathfrak{a}_{\frac{r}{2}}$ — c relation (5.5) follows by combining (E.19b) with (E.17).

Appendix F. Supplementary tables

Here we give the specification of the RG invariants $\mathfrak{a}_s$ (1.22), such that the scaled Bethe roots (1.16) are described by the ODE

$$\Big[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-1)^A\,E^r\,\mathrm{e}^{ry}-c\,E^{\mu}\,\mathrm{e}^{\left((A\mu-rj)\,\frac{n+r}{r}+\mu\right)y}\Big]\,\psi=0 \tag{F.1}$$

with (μ,j) taken from the set (1.24).

For $r=3,4,\ldots,10$, $A=1,2,\ldots,r-2$ and all possible pairs (μ,j), presented in the tables are the values of the integer s corresponding to the minimal non-trivial set of RG invariants along with their exponents d_s. All other sums $\sum_{\ell=1}^{r}(\eta_\ell)^{-s}$ can be taken to be zero. Also, the normalization condition

$$\frac{1}{r}\sum_{\ell=1}^{r}(\eta_\ell)^{-r}=(-1)^{r-1} \tag{F.2}$$

is being assumed. The values of the non-vanishing RG invariants are expressed in terms of the coefficient c of the ODE as in formulae (4.22) - (4.26). Then, the scaling limit of the ground state Bethe roots is described by the differential equation for any $n>n_{\min}$, where $n_{\min}$ is listed in the last column of the tables.

	A	(μ,j)	s	d_s	$n_{\min}$
$r=3$	1	$(1,0)$	$\{1\}$	$d_1=\frac{2}{3}$	0
$r=4$	1	$(1,0)$	$\{1\}$	$d_1=1$	0
	2	$(1,0)$	$\{1,2\}$	$d_1=\frac{1}{2},\ d_2=1$	0
$r=5$	1	$(1,0)$	$\{1\}$	$d_1=\frac{6}{5}$	$\frac{5}{2}$
		$(2,0)$	$\{2,4,1\}$	$d_2=\frac{2}{5},\ d_4=\frac{4}{5},\ d_1=\frac{6}{5}$	$\frac{5}{2}$
	2	$(1,0)$	$\{1\}$	$d_1=\frac{4}{5}$	0
		$(3,1)$	$\{3,1\}$	$d_3=\frac{2}{5},\ d_1=\frac{4}{5}$	0
	3	$(1,0)$	$\{1,2\}$	$d_1=\frac{2}{5},\ d_2=\frac{4}{5}$	0
		$(2,1)$	$\{2\}$	$d_2=\frac{4}{5}$	0
$r=6$	1	$(1,0)$	$\{1\}$	$d_1=\frac{4}{3}$	6
		$(2,0)$	$\{2\}$	$d_2=\frac{2}{3}$	0
	2	$(1,0)$	$\{1\}$	$d_1=1$	0
	3	$(1,0)$	$\{1\}$	$d_1=\frac{2}{3}$	0
	4	$(1,0)$	$\{1,2,3\}$	$d_1=\frac{1}{3},\ d_2=\frac{2}{3},\ d_3=1$	0
		$(2,1)$	$\{2\}$	$d_2=\frac{2}{3}$	0
$r=7$	1	$(1,0)$	$\{1\}$	$d_1=\frac{10}{7}$	$\frac{21}{2}$
		$(2,0)$	$\{2\}$	$d_2=\frac{6}{7}$	0
		$(3,0)$	$\{3,6,2,5,1\}$	$d_3=\frac{2}{7},\ d_6=\frac{4}{7},\ d_2=\frac{6}{7},\ d_5=\frac{8}{7},\ d_1=\frac{10}{7}$	$\frac{21}{2}$
	2	$(1,0)$	$\{1\}$	$d_1=\frac{8}{7}$	$\frac{7}{4}$
		$(2,0)$	$\{2,4,6,1\}$	$d_2=\frac{2}{7},\ d_4=\frac{4}{7},\ d_6=\frac{6}{7},\ d_1=\frac{8}{7}$	$\frac{7}{4}$
		$(4,1)$	$\{4,1\}$	$d_4=\frac{4}{7},\ d_1=\frac{8}{7}$	$\frac{7}{4}$
	3	$(1,0)$	$\{1\}$	$d_1=\frac{6}{7}$	0

A	(μ,j)	s	d_s	$n_{\min}$
	$(3,1)$	$\{3\}$	$d_3=\frac{4}{7}$	0
	$(5,2)$	$\{5,3,1'\}$	$d_5=\frac{2}{7},\ d_3=\frac{4}{7},\ d_1=\frac{6}{7}$	0
4	$(1,0)$	$\{1,2\}$	$d_1=\frac{4}{7},\ d_2=\frac{8}{7}$	$\frac{7}{2}$
	$(2,1)$	$\{2\}$	$d_2=\frac{8}{7}$	$\frac{7}{2}$
	$(4,2)$	$\{4,1,5,2\}$	$d_4=\frac{2}{7},\ d_1=\frac{4}{7},\ d_5=\frac{6}{7},\ d_2=\frac{8}{7}$	$\frac{7}{2}$
5	$(1,0)$	$\{1,2,3\}$	$d_1=\frac{2}{7},\ d_2=\frac{4}{7},\ d_3=\frac{6}{7}$	0
	$(2,1)$	$\{2\}$	$d_2=\frac{4}{7}$	0
	$(3,2)$	$\{3\}$	$d_3=\frac{6}{7}$	0

	A	(μ,j)	s	d_s	$n_{\min}$
$r=8$	1	$(1,0)$	$\{1\}$	$d_1=\frac{3}{2}$	16
		$(2,0)$	$\{2\}$	$d_2=1$	0
		$(3,0)$	$\{3,6,1\}$	$d_3=\frac{1}{2},\ d_6=1,\ d_1=\frac{3}{2}$	16
	2	$(1,0)$	$\{1\}$	$d_1=\frac{5}{4}$	4
		$(2,0)$	$\{2,4\}$	$d_2=\frac{1}{2},\ d_4=1$	0
		$(5,1)$	$\{5,2,7,4\}$	$d_5=\frac{1}{4},\ d_2=\frac{1}{2},\ d_7=\frac{3}{4},\ d_4=1$	0
	3	$(1,0)$	$\{1\}$	$d_1=1$	0
		$(3,1)$	$\{3\}$	$d_3=1$	0
	4	$(1,0)$	$\{1\}$	$d_1=\frac{3}{4}$	0
		$(3,1)$	$\{3,6,1,4\}$	$d_3=\frac{1}{4},\ d_6=\frac{1}{2},\ d_1=\frac{3}{4},\ d_4=1$	0
	5	$(1,0)$	$\{1,2\}$	$d_1=\frac{1}{2},\ d_2=1$	0
		$(2,1)$	$\{2\}$	$d_2=1$	0
		$(5,3)$	$\{5,2\}$	$d_5=\frac{1}{2},\ d_2=1$	0
	6	$(1,0)$	$\{1,2,3,4\}$	$d_1=\frac{1}{4},\ d_2=\frac{1}{2},\ d_3=\frac{3}{4},\ d_4=1$	0
		$(2,1)$	$\{2,4\}$	$d_2=\frac{1}{2},\ d_4=1$	0
		$(3,2)$	$\{3\}$	$d_3=\frac{3}{4}$	0

	A	(μ,j)	s	d_s	$n_{\min}$
$r=9$	1	$(1,0)$	$\{1\}$	$d_1=\frac{14}{9}$	$\frac{45}{2}$
		$(2,0)$	$\{2\}$	$d_2=\frac{10}{9}$	$\frac{9}{4}$
		$(3,0)$	$\{3\}$	$d_3=\frac{2}{3}$	0
		$(4,0)$	$\{4,8,3,7,2,6,1\}$	$d_4=\frac{2}{9},\ d_8=\frac{4}{9},\ d_3=\frac{2}{3},\ d_7=\frac{8}{9},$ $d_2=\frac{10}{9},\ d_6=\frac{12}{9},\ d_1=\frac{14}{9}$	$\frac{45}{2}$
	2	$(1,0)$	$\{1\}$	$d_1=\frac{4}{3}$	$\frac{27}{4}$
		$(2,0)$	$\{2\}$	$d_2=\frac{2}{3}$	0
		$(5,1)$	$\{5,1\}$	$d_5=\frac{2}{3},\ d_1=\frac{4}{3}$	$\frac{27}{4}$
	3	$(1,0)$	$\{1\}$	$d_1=\frac{10}{9}$	$\frac{3}{2}$
		$(2,0)$	$\{2,4,6,8,1\}$	$d_2=\frac{2}{9},\ d_4=\frac{4}{9},\ d_6=\frac{2}{3},\ d_8=\frac{8}{9},\ d_1=\frac{10}{9}$	$\frac{3}{2}$
		$(4,1)$	$\{4,8,3\}$	$d_4=\frac{4}{9},\ d_8=\frac{8}{9}$	0
	4	$(1,0)$	$\{1\}$	$d_1=\frac{8}{9}$	0
		$(3,1)$	$\{3\}$	$d_3=\frac{2}{3}$	0
		$(5,2)$	$\{5,1\}$	$d_5=\frac{4}{9},\ d_1=\frac{8}{9}$	0
		$(7,3)$	$\{7,5,3,1\}$	$d_7=\frac{2}{9},\ d_5=\frac{4}{9},\ d_3=\frac{2}{3},\ d_1=\frac{8}{9}$	0
	5	$(1,0)$	$\{1\}$	$d_1=\frac{2}{3}$	0
		$(2,1)$	$\{2\}$	$d_2=\frac{4}{3}$	$\frac{27}{2}$
		$(4,2)$	$\{4\}$	$d_4=\frac{2}{3}$	0
	6	$(1,0)$	$\{1,2\}$	$d_1=\frac{4}{9},\ d_2=\frac{8}{9}$	0
		$(2,1)$	$\{2\}$	$d_2=\frac{8}{9}$	0
		$(5,3)$	$\{5,1,6,2\}$	$d_5=\frac{2}{9},\ d_1=\frac{4}{9},\ d_6=\frac{2}{3},\ d_2=\frac{8}{9}$	0
	7	$(1,0)$	$\{1,2,3,4\}$	$d_1=\frac{2}{9},\ d_2=\frac{4}{9},\ d_3=\frac{2}{3},\ d_4=\frac{8}{9}$	0
		$(2,1)$	$\{2,4\}$	$d_2=\frac{4}{9},\ d_4=\frac{8}{9}$	0
		$(3,2)$	$\{3\}$	$d_3=\frac{2}{3}$	0
		$(4,3)$	$\{4\}$	$d_4=\frac{8}{9}$	0

	A	(μ, j)	s	d_s	$n_{\min}$
$r = 10$	1	$(1,0)$	$\{1\}$	$d_1 = \frac{8}{5}$	30
		$(2,0)$	$\{2\}$	$d_2 = \frac{6}{5}$	5
		$(3,0)$	$\{3\}$	$d_3 = \frac{4}{5}$	0
		$(4,0)$	$\{4,8,2\}$	$d_4 = \frac{2}{5}$, $d_8 = \frac{4}{5}$, $d_2 = \frac{6}{5}$	5
	2	$(1,0)$	$\{1\}$	$d_1 = \frac{7}{5}$	10
		$(2,0)$	$\{2\}$	$d_2 = \frac{4}{5}$	0
		$(3,0)$	$\{3,6,9,2,5,1\}$	$d_3 = \frac{1}{5}$, $d_6 = \frac{2}{5}$, $d_9 = \frac{3}{5}$, $d_2 = \frac{4}{5}$, $d_5 = 1$, $d_1 = \frac{7}{5}$	10
		$(6,1)$	$\{6,2\}$	$d_6 = \frac{2}{5}$, $d_2 = \frac{4}{5}$	0
	3	$(1,0)$	$\{1\}$	$d_1 = \frac{6}{5}$	$\frac{10}{3}$
		$(2,0)$	$\{2,4\}$	$d_2 = \frac{2}{5}$, $d_4 = \frac{4}{5}$	0
		$(4,1)$	$\{4\}$	$d_4 = \frac{4}{5}$	0
		$(7,2)$	$\{7,4,1\}$	$d_7 = \frac{2}{5}$, $d_4 = \frac{4}{5}$, $d_1 = \frac{6}{5}$	$\frac{10}{3}$
	4	$(1,0)$	$\{1\}$	$d_1 = 1$	0
		$(3,1)$	$\{3\}$	$d_3 = 1$	0
	5	$(1,0)$	$\{1\}$	$d_1 = \frac{4}{5}$	0
		$(3,1)$	$\{3,6\}$	$d_3 = \frac{2}{5}$, $d_6 = \frac{4}{5}$	0
	6	$(1,0)$	$\{1,2\}$	$d_1 = \frac{3}{5}$, $d_2 = \frac{6}{5}$	5
		$(2,1)$	$\{2\}$	$d_2 = \frac{6}{5}$	5
		$(4,2)$	$\{4,8,2\}$	$d_4 = \frac{2}{5}$, $d_8 = \frac{4}{5}$, $d_2 = \frac{6}{5}$	5
		$(7,4)$	$\{7,4,1,8,5\}$	$d_7 = \frac{1}{5}$, $d_4 = \frac{2}{5}$, $d_1 = \frac{3}{5}$, $d_8 = \frac{4}{5}$, $d_5 = 1$	0
	7	$(1,0)$	$\{1,2\}$	$d_1 = \frac{2}{5}$, $d_2 = \frac{4}{5}$	0
		$(2,1)$	$\{2\}$	$d_2 = \frac{4}{5}$	0
		$(3,2)$	$\{3\}$	$d_3 = \frac{6}{5}$	10
		$(6,4)$	$\{6,2\}$	$d_6 = \frac{2}{5}$, $d_2 = \frac{4}{5}$	0
	8	$(1,0)$	$\{1,2,3,4,5\}$	$d_1 = \frac{1}{5}$, $d_2 = \frac{2}{5}$, $d_3 = \frac{3}{5}$, $d_4 = \frac{4}{5}$, $d_5 = 1$	0
		$(2,1)$	$\{2,4\}$	$d_2 = \frac{2}{5}$, $d_4 = \frac{4}{5}$	0
		$(3,2)$	$\{3\}$	$d_3 = \frac{3}{5}$	0
		$(4,3)$	$\{4\}$	$d_4 = \frac{4}{5}$	0

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