Product Quantization for Limited Feedback MISO and RIS-Aided Wireless Communication Systems

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***Abstract*—Recently, scalar quantization (SQ) and vector quantization (VQ) have been widely adopted in wireless communication systems. VQ outperforms SQ, making it an attractive choice for low-dimensional channel vectors. Despite the dominance, VQ demands a large storage capacity for codevectors and a fast-searching algorithm. Thus, VQ may not be the most desirable solution in many applications. To address these concerns, we introduce a novel product quantization (PQ) in this letter. Our PQ has applications in reconfigurable intelligent surface (RIS)-aided systems as well as multiple-input single-output antenna systems. It differs from the original PQ, designed for source coding, in three key aspects: (i) the encoded vector is complex-valued and comprises a single feature, (ii) a shared codebook is used for quantization, and (iii) the quantization of one sub-vector is influenced by other previously quantized sub-vectors. By properly adjusting the quantization parameters, the proposed PQ needs significantly less hardware and computational resources compared with VQ. Further, it outperforms SQ with a comparable hardware and computational complexity. The numerical assessments reveal the superiority of the proposed PQ.**

***Index Terms*—limited feedback, beamforming, reconfigurable intelligent surface, quantized CSI, product quantization, MISO.**

I. Introduction

In wireless communications, downlink (DL) and uplink (UL) signals are transmitted in time division duplex (TDD) and frequency division duplex (FDD) modes. In TDD transmission, if the channel reciprocity is ensured, the transmitter obtains channel information by sending a pilot signal in the reverse direction. However, obtaining channel information in FDD mode is more complex due to the absence of channel reciprocity. The transmission in FDD mode causes signals in DL and UL directions to experience different channels. In particular, the DL channel is first estimated at the receiver, quantized, and fed back to the transmitter.

Quantization approaches in limited feedback wireless systems have been discussed in [1], [2]. In narrowband multiple-input single-output (MISO) antenna systems, the optimal beamforming scheme, i.e., maximum ratio transmission (MRT), necessitates quantizing both the magnitude and phase of the channel vector. A common approach involves designing a codebook containing codevectors known to both the transmitter and receiver. The channel vector is then quantized to the closest codevector and the index of the codevector is fed back using B bits [1], [3]. This approach, utilizing predefined codevectors for quantization, is named vector quantization (VQ). Various methods exist for designing such codebooks, including random VQ (RVQ), Grassmannian line packing, successive beamforming, and Lloyd-type VQ designs [4]–[8]. While the number of feedback bits per channel vector in these beamforming approaches remains fixed, designing variable-rate feedback beamforming is advantageous [9]. The codebook for variable-rate VQ is designed based on a Lagrangian formulation to satisfy an entropy constraint while minimizing the distortion [10]. However, there are two significant drawbacks to the VQ approach. First, a considerable amount of memory is needed to store the codebook at the transmitter and receiver. Second, in a fixed-rate VQ, finding the closest codevector to the channel vector requires an exhaustive search. Notably, with an increase in the number of antenna elements, both the codebook storage size and the computational complexity grow exponentially.

Another beamforming scheme is equal gain transmission (EGT) where all elements have the same magnitude [1]. Equal gain beamforming slightly compromises the performance in terms of capacity and signal-to-noise ratio (SNR) compared with the optimal MRT [11]–[13]. Both scalar quantization (SQ) and VQ can be used for equal gain beamforming. In SQ, the phase of each element in the channel vector is quantized using a uniform quantizer. SQ holds an advantage over VQ as it does not require codebook storage or exhaustive search. However, SQ has two main drawbacks compared with VQ. First, as the number of transmit antennas increases, SQ's performance relative to VQ further degrades [12]. Second, the number of feedback bits is constrained to be an integer multiple of the number of transmit antennas. Therefore, unlike VQ, it is not possible to allocate an arbitrary number of feedback bits with SQ. However, similar to MRT, applying VQ to EGT introduces memory and computational challenges. To enjoy the benefits of VQ while mitigating its drawbacks, a lattice quantizer has been introduced in [14]. This approach generates codevectors using a structured algorithm which eliminates the need for codebook storage. Additionally, a fast algorithm is employed to expedite the nearest codevector search process [15]. Despite great implementation advantages, similar to SQ, the feedback bit rate is limited to an integer multiple of the number of transmit antennas.

SQ and VQ have been predominantly designed and evaluated for conventional MISO systems, typically with up to 16 transmit antennas. In the next generation of cellular communications, named 6G, reconfigurable intelligent surface (RIS) is envisioned as a promising technology to improve various aspects of wireless communications such as SNR and beamforming [16]. To this end, a RIS is equipped with tens or even hundreds of antenna elements and phase shifters enabling precise control over the propagation environment by adjusting the direction of incident signals [17], [18]. Another system capable of modifying the channel conditions is a reconfigurable multiple-input multiple-output (MIMO) system, widely studied for both sub-6 GHz and millimeter-wave frequen-

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cies [19]–[21]. A RIS can be located between the transmitter and receiver, but it cannot obtain the channel state information (CSI). Instead, CSI is calculated at the receiver and then sent back to the transmitter, resulting in limited CSI at the transmitter. In recent years, numerous studies have addressed the limited feedback problem in RIS-aided systems [22]–[28]. For instance, a cascaded codebook using a bit partitioning approach has been employed in [22]. A learning method for codebook design is used in [23]. Refs. [24], [26] leverage the sparsity and low-rank properties of higher frequency bands to facilitate angle information feedback. Further, [27] reduces the overhead by selecting several dominant paths from all available paths. In general, [22]–[27] concentrate on spatially correlated channels, where the dimension of the quantized vectors is very small and equal to the number of independent paths. These methods cannot be adopted for RIS phase vector quantization because of the large vector dimensions. Moreover, RVQ is used in a small-size RIS for precise phase information feedback [28]. However, employing traditional or learning-based codebooks for VQ in RIS-aided systems with numerous antenna elements is impractical due to memory and computational complexity constraints. Other methods struggle to provide precise and accurate RIS phase information.

This letter addresses the memory and search challenges associated with VQ by introducing a novel product quantization (PQ) approach. PQ was initially introduced as a shape-gain vector quantizer [29] and has since found widespread use in speech and image coding for dealing with large-scale real-valued vectors with many features [30], [31]. Despite its success in real-valued data, PQ is overlooked in the context of complex-valued data in wireless communications. To address this gap, we propose a new PQ encoder. Our approach involves partitioning a large complex-valued CSI vector into multiple sub-vectors. We then use an arbitrary small-size codebook to quantize the sub-vectors. This enables the proposed PQ to effectively quantize high-dimension information vectors, impossible by conventional VQs, while maintaining the low storage capacity and computational complexity at the receiver. Simulation results for RIS-aided and MISO systems under different fading conditions are provided. The results verify the superiority of the proposed PQ in terms of the achievable rate.

The letter is organized as follows: Section II describes the system model. Section III represents the proposed encoder and decoder of PQ. The proposed PQ is further discussed in Section IV. Numerical results are presented in Section V. Finally, the letter is concluded in Section VI.

Notations: In this paper, $j = \sqrt{-1}$. Regular letters, bold letters, and bold capital letters represent scalars, vectors, and matrices, respectively. Superscripts $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and the transpose-conjugate operations, respectively. Further, $|x|$ denotes the absolute value of x and the operation $\angle \mathbf{x}$ calculates the element-wise angle of the vector $\mathbf{x}$.

II. System Model

The system model comprises a single-antenna transmitter/receiver and a RIS consisting of K antenna elements and K phase shifters between them. The setup is known as RIS-aided single-input single-output (SISO). The communication is operated in FDD mode. The vectors $\mathbf{f} \in \mathbb{C}^{K\times 1}$ and $\mathbf{g} \in \mathbb{C}^{K\times 1}$ represent the transmitter-RIS and RIS-receiver fading channels, respectively. Accordingly, the effective overall channel is $h = \mathbf{g}^T \boldsymbol{\Theta} \mathbf{f}$ where $\boldsymbol{\Theta} = \mathrm{diag}(\boldsymbol{\theta})$, in which $\boldsymbol{\theta} = [e^{j\theta_1}, \cdots, e^{j\theta_K}]^T$, with $\theta_k \in [-\pi, \pi)$, reflects the impact of the RIS. The channels are perfectly available at the receiver. RIS with a two-dimensional rectangular shape is subject to spatially correlated fading, largely depending on the inter-element spacing, with larger spacing resulting in less correlated channels. We do not make any assumption on the shape of the RIS and the inter-element spacing. The RIS could be a one-dimensional ULA or two-dimensional UPA with a sufficiently large inter-element spacing modeled by an independent and identically distributed (i.i.d.) Rayleigh fading channel. Before initiating data transmission in the DL direction, the phase vector is quantized and fed back to the transmitter using B bits. The feedback channel is assumed to be error-free and with no delay. The quantized phase information is sent to the RIS via a wired channel [28]. We define the cascade channel vector as

$$\mathbf{h} = \mathbf{G}\mathbf{f}, \tag{1}$$

where $\mathbf{G} = \mathrm{diag}(\mathbf{g})$. To maximize the beamforming gain, the optimal value of θ_k is $\theta_k = \angle h_k$ for $k = 1, \cdots, K$ which results in a maximum overall channel gain of $H = |\boldsymbol{\theta}^H \mathbf{h}|^2 = \left(\sum_{k=1}^{K} |h_k|\right)^2$. Due to the limited feedback channel, only the quantized θ_k is possible, resulting in $H_Q = |\boldsymbol{\theta}_Q^H \mathbf{h}|^2 \leq H$.

III. Proposed Product Quantization

A. Product Quantization: Overview

PQ is a powerful approach for quantizing a real-valued high-dimensional vector with many features. It is composed of two components, encoder and decoder. At the encoder, the data is partitioned into multiple sub-vectors. The encoder quantizes sub-vectors independently by using a specific sub-codebook for each sub-vector. The original vector can be approximately reconstructed from the quantized sub-vectors at the decoder [32].

Our proposed PQ has three distinct characteristics that set it apart from the original PQ. First, the encoded vector is complex-valued and comprises only one feature. Second, a shared codebook is used for quantizing the sub-vectors. Third, the quantization of the current sub-vector is influenced by the value of the previously quantized sub-vectors. More detail on the structure of the proposed PQ is provided below.

B. Encoder

The overall channel vector $\mathbf{h}$ in (1) is divided into N sub-vectors, expressed as

$$\mathbf{h} = [\mathbf{h}_1^T, \mathbf{h}_2^T, \cdots, \mathbf{h}_N^T]^T, \tag{2}$$

where $\mathbf{h}_n \in \mathbb{C}^{\frac{K}{N}\times 1}$ denotes the overall channel vector corresponding to the sub-vector n. The choice of the number of elements per sub-vector, i.e., $\frac{K}{N}$, is flexible and can be determined based on the receiver's computational capability and memory consideration. Then, each sub-vector is quantized using a common codebook defined as

$$\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_{2^{B'}}\}, \tag{3}$$

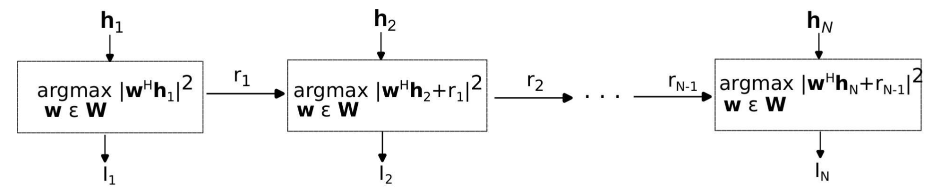

Fig. 1: The proposed encoder. The overall channel vector is partitioned into N sub-vectors $\mathbf{h}_n$. Only one codebook, i.e., $\mathbf{W}$, is used for the quantization. The encoding of each sub-vector depends on the codebook as well as the complex-valued r_{n-1} defined in (5).

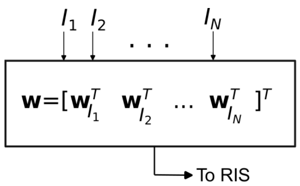

Fig. 2: Schematic of the decoder. The overall quantization vector is reconstructed by concatenating the codevectors.

where

$$\mathbf{w} = [e^{j\phi_1}, e^{j\phi_2}, \cdots, e^{j\phi_{\frac{K}{N}}}]^T \in \mathcal{W}, \tag{4}$$

and $B^{'} = \frac{qK}{N}$ denotes the number of feedback bits required for reporting the index of the chosen codevector for each sub-vector. Further, $q \in \mathbb{R}^+$ denotes the number of quantization bits per antenna element. In total, $B = NB^{'}$ bits are needed to feed back all the selected codevectors. Determination of $\mathcal{W}$ is directly related to the codebook design problem. As mentioned previously, there are several codebook design methods [4], [6]–[8]. In this paper, we adopt the Lloyd-type codebook design.

Figure 1 illustrates the encoder for the proposed product quantizer deployed at the receiver. The encoding procedure is explained as follows. First, the inner product of the prior quantized sub-vectors and their selected codevectors are calculated. Let r_{n-1} denote the inner product value if $\mathbf{h}_n$ is the current sub-vector to be quantized. The term r_{n-1} is calculated as

$$r_{n-1} = \mathbf{w}_{I_1}^H \mathbf{h}_1 + \cdots + \mathbf{w}_{I_{n-1}}^H \mathbf{h}_{n-1}, \quad n = 2, \cdots, N, \tag{5}$$

where $\mathbf{w}_{I_{n-1}}$ denotes the selected codevector by the quantizer of the sub-vector $n-1$. Also, I_{n-1} is the index of the corresponding selected codevector. Next, r_{n-1} is added to the objective function of the quantization of the current sub-vector, and the codevector is determined as

$$\mathbf{w}_{I_n} = \underset{\mathbf{w} \in \mathcal{W}}{\text{argmax}} |\mathbf{w}^H \mathbf{h}_n + r_{n-1}|^2, \tag{6}$$

as done in [12] for $r_{n-1} = 0$. The procedure continues until all sub-vectors are quantized. The index vector of the selected codevectors $[I_1, I_2, \cdots, I_N]$ is fed back to the transmitter.

C. Decoder

Each index represents a codevector in the codebook. The codevectors are concatenated to build the quantized phase vector. In particular, the transmitter uses the indices to reconstruct the overall quantized phase vector as

$$\mathbf{w} = [\mathbf{w}_{I_1}^T, \mathbf{w}_{I_2}^T, \cdots, \mathbf{w}_{I_N}^T]^T, \tag{7}$$

where $\mathbf{w}_{I_n} = \mathcal{W}[I_n]$. The transmitter then sends the quantized phase information to RIS using a wired feedback channel. The decoder is shown in Fig. 2.

D. Role of r_{n-1}

Introducing r_{n-1} is one of the main contributions of this letter. The motivation behind it is to capture the impact of the error caused by the quantization of the previous sub-vectors in the quantization of the current sub-vector. Without r_{n-1}, i.e., setting $r_{n-1} = 0$, the encoder quantizes each sub-vector $\mathbf{h}_n$ using VQ and the codebook $\mathcal{W}$ as $\mathbf{w}_{I_n} = \underset{\mathbf{w}_0 \in \mathcal{W}}{\text{argmax}} |{\mathbf{w}_0}^H \mathbf{h}_n|^2$ for $n = 1, \cdots, N$. The decoder reconstructs the quantized vector as $\mathbf{w} = [\mathbf{w}_{I_1}^T, \mathbf{w}_{I_2}^T, \cdots, \mathbf{w}_{I_N}^T]^T$. Hence, the quantized overall channel gain is given as $H_{Q,\text{w}} = |\mathbf{w}^H \mathbf{h}|^2 \equiv |\mathbf{w}_{I_1}^H \mathbf{h}_1 + \cdots + \mathbf{w}_{I_N}^H \mathbf{h}_N|^2$. Since each sub-vector is quantized independently, it is possible that the terms $\mathbf{w}_{I_1}^H \mathbf{h}_1$, $\mathbf{w}_{I_2}^H \mathbf{h}_2, \cdots$ are added destructively, resulting in the unfavorable outcome $H_{Q,\text{w}} \ll H_{Q,\text{v}}$. To prevent the destructive effect, we have introduced the parameter r_{n-1} such that the phase information of the previously quantized sub-vectors is considered in the current sub-vector quantization. Note that although r_{n-1} is a constant, since the two components of (6) are complex, it will affect the outcome.

IV. Discussion

This section first compares the proposed PQ with SQ and VQ in terms of the number of feedback bits. Then, it discusses the application of the proposed PQ for beamforming in MISO systems. While we apply the proposed PQ to EGT and RIS-aided systems with constant-amplitude RIS elements, we should note that a similar structure can be applied to MRT, which is left as future work. Finally, we discuss the RIS-aided systems with multiple antennas.

A. Comparison with SQ and VQ

Using SQ, each phase is quantized separately, requiring an integer number of feedback bits per RIS element. Consequently, for an integer m bits per element, mK feedback bits are needed in the underlying system model. For instance, if $K = 100$ and $m = 2$, the total number of feedback bits is $B = 200$ bits. This will result in a significant overhead in practical RIS-aided systems. VQ demonstrates better performance compared to SQ when considering the same number of feedback bits. Nevertheless, as mentioned earlier, VQ's main drawback is its restriction on storage and searching, resulting in an exponential increase in memory requirements and computational complexity. For instance, for $K = 100$ and one feedback bit per antenna element, the total number of feedback bits for SQ is $B = 100$. As a result, the receiver needs to store and search through 2^{100} codevectors which may not be feasible in reality.

Our proposed product quantizer requires to store $2^{B^{'}}$ codevectors, and to perform $N2^{B^{'}}$ searches for all N sub-vectors.

For instance, for $K = 100$, $q = 1$, and $N = 10$, the proposed PQ needs to store 2^{10} codevectors, significantly far less than that of VQ, i.e., $2^{10} \ll 2^{100}$. It is worth mentioning that, in this example, both quantizers need $B = 100$ feedback bits. Clearly, q does not have to be an integer for PQ. For example, $q = 0.7$ results in $B = 70$ and 2^7 codevectors in the codebook.

B. Beamforming in MISO

In the conventional MISO systems with K transmit antennas, the vector $\mathbf{h} \in \mathbb{C}^{K\times 1}$ represents the wireless channel between the transmitter and receiver. To perform EGT, the phase information must be fed back to the transmitter. Our proposed PQ can be utilized for the quantization in MISO systems such that, at the receiver, the vector $\mathbf{h}$ is divided into N sub-vectors similar to (2). Then, the encoder shown in Fig. 1 is used for the quantization. At the transmitter, the decoder in Fig. 2 is used to reconstruct the quantized phase vector. Particularly, the choice of $N = K$ results in an element-wise quantization, similar to scalar quantization, that like the SQ method in [12] does not require any memory storage. Nevertheless, the proposed PQ remarkably outperforms SQ when the same number of feedback bits is used. This will be demonstrated in the simulation results.

C. Extension to Multiple Transmit Antennas in RIS-aided Systems

For more than one transmit antenna, two well-known transmission techniques are antenna selection and beamforming. In the case of antenna selection, the receiver first determines the transmit antenna resulting in the highest overall channel gain. Then, the receiver applies the proposed PQ to quantize the RIS phase vector of the selected cascade channel. The antenna index and the phase vector information are fed back to the transmitter. For beamforming, the system includes active beamforming and passive beamforming that refer to multiple transmit antenna beamforming and the RIS-aided beamforming, respectively. Let N_t denote the number of transmit antennas and the cascade channel vectors $\mathbf{h}_{n_t}$ for $n_t = 1, \cdots, N_t$ be available at the receiver side. Note that the RIS phase vector is common for all cascade channels. To determine the quantized RIS phase vector $\mathbf{w}$, the proposed PQ is applied as follows. Each cascade channel vector is partitioned into multiple sub-vectors. The number of sub-vectors is the same for all cascade channel vectors. To obtain $\mathbf{w}_{I_n}$, the objective function in (6) is the summation of the terms $|\mathbf{w}^H\mathbf{h}_{n_t,n} + r_{n_t,n-1}|^2\ \forall n_t$. Concatenating the vectors $\mathbf{w}_{I_n}$ yields the overall quantized phase vector $\mathbf{w}$. Next, using $\mathbf{w}$, the overall effective quantized channel is obtained from its elements, i.e., $h_Q = \mathbf{g}^T\mathbf{W}\mathbf{f}$ where $\mathbf{W} = \text{diag}(\mathbf{w})$. Now, using a proper VQ, the active beamforming is designed.

V. Simulation Results

The numerical simulations are conducted for two types of systems: RIS-aided and conventional MISO. The main differences between these two systems lie in the channel fading characteristics and the number of antenna elements employed for beamforming. In the RIS-aided system, the channels experience i.i.d. Rician fading [23], [28] with the Rician factor $\nu = -5$ dB. The distance between the transmitter and the RIS, and between the RIS and the receiver, are set to 10 m and 40 m, respectively. Additionally, the path loss exponent between the transmitter and the RIS is 3.5, while between the RIS and the receiver is 2.5. On the other hand, in the MISO system, the channel fading is assumed to be i.i.d. Rayleigh fading [2], [6], [9]. The number of antenna elements in the RIS-aided system is $K = 100$ while in the MISO system, it is set to $K = 2, 4, 8, 16$.

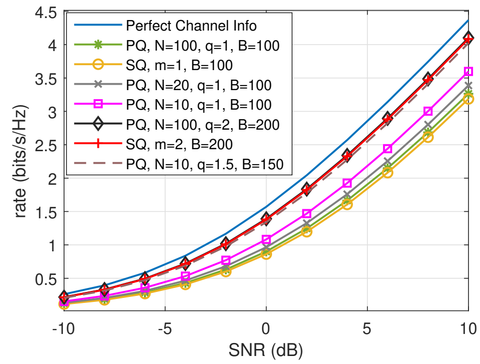

Fig. 3: The RIS-aided rate performance vs SNR for $K = 100$.

The achievable rate versus SNR is used as the evaluation criterion. In a system where perfect CSI is available at the transmitter, the achievable rate is expressed as $R = \log_2(1 + \text{SNR} \times H)$ bits/s/Hz where $H = \left(\sum_{k=1}^{K} |h_k|\right)^2$. With the quantized CSI, the rate becomes $R_Q = \log_2(1 + \text{SNR} \times H_Q)$ bits/s/Hz where $H_Q = |\mathbf{w}^H\mathbf{h}|^2$ as explained in Section II.

The codebooks for the proposed PQ are generated using the Lloyd algorithm. Furthermore, in RIS-aided systems, our comparison focuses solely on SQ due to the impracticality of generating and storing a suitable codebook for VQ, particularly for large K. However, in MISO systems, we compare the achievable rate of the proposed PQ with that of VQ and SQ for a small K.

A. RIS-Aided Systems

In Fig. 3, for $K = 100$ RIS elements and for $N = 100$, $N = 20$ and $N = 10$ sub-vectors, the codebook sizes are $2^1 = 2$, $2^5 = 32$, and $2^{10} = 1024$, respectively. This indicates that the system with $N = 10$ requires more memory capacity and computational capability. Furthermore, we observe that, for a fixed number of feedback bits $B = 100$, as N decreases, the performance improves due to a larger codebook size. For instance, the proposed PQ outperforms SQ by 1.4 dB for $N = 10$. For $B = 200$, the performance is similar for $B = 100$. Additionally, we simulate the proposed PQ for $B = 150$ with $q = 1.5$ and $N = 10$ which is not feasible with SQ. This highlights the advantages of the proposed PQ over SQ in RIS-aided systems. Instead of element-wise quantization, the proposed PQ performs sub-vector-wise quantization while the required storage and computational complexity are kept low.

B. MISO Systems

The rate comparison between the proposed PQ, VQ, and SQ in a MISO system employing EGT with $K = 2, 4, 8, 16$

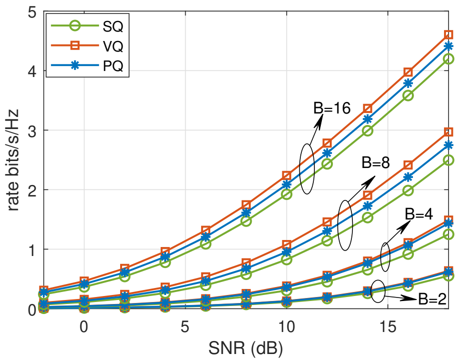

Fig. 4: The MISO rate performance vs SNR for $K = 2, 4, 8, 16$.

transmit antennas and $q = m = 1$ is shown in Fig. 4. Further, we set $N = K$ in the proposed PQ, meaning element-wise quantization. Therefore, similar to SQ, the proposed PQ does not need memory storage. The objective is to demonstrate that for a small number of transmit antennas, by using the encoder depicted in Fig. 1, instead of independently quantizing each element in SQ, the performance improves. The main difference is that the proposed PQ needs to calculate (5) after the quantization of each element. It is observed that for $K = 2$ and 4, the performance of the proposed PQ is similar to that of VQ and better than SQ. By increasing K to 8 and 16, the proposed PQ shows about 1 dB superiority over SQ. Compared to VQ, the PQ performance is worse due to element-wise quantization in PQ and vector-wise quantization in VQ. Note that the codebook size in VQ is 2^8 and 2^{16} for $K = 8$ and $K = 16$, respectively.

VI. Conclusion

This letter introduces a novel encoder for PQ, facilitating its application in wireless systems with limited feedback, i.e., quantized beamforming. Notably, the wireless channel vectors are complex-valued and directly applying the existing PQ approaches, developed for source coding, is ineffective for phase quantization. Additionally, with the emergence of new technologies such as RIS, the conventional VQ and SQ may not be practical or efficient due to the high dimension of RIS-aided overall channel vectors and hardware and computational restrictions. The numerical results confirm the findings. Furthermore, the proposed PQ in MISO systems outperforms SQ with the same number of feedback bits and memory requirements. By changing the length of sub-vectors, PQ provides a tradeoff between performance and complexity.

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