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n-Bridge braids and the braid index

Dane Gollero[*,‖], Siddhi Krishna[†,**], Marissa Loving[‡,††],
Viridiana Neri[†,‡‡], Izah Tahir[§,§§] and Len White[¶,¶¶]

[*]*Department of Mathematics,*
University of Utah, USA

[†]*Department of Mathematics,*
Columbia University, USA

[‡]*Department of Mathematics,*
University of Wisconsin - Madison, USA

[§]*School of Mathematics,*
Georgia Institute of Technology, USA

[¶]*Department of Mathematics,*
Portland State University, USA

[‖]*3rundane@gmail.com*
[**]*siddhi.krishna@bc.edu*
[††]*mloving2@wisc.edu*
[‡‡]*vjn2108@columbia.edu*
[§§]*itahir3@gatech.edu*
[¶¶]*len_white@live.com*

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ABSTRACT

In this work, we find a closed form formula for the braid index of an n-bridge braid, a class of positive braid knots which simultaneously generalizes torus knots, 1-bridge braids, and twisted torus knots. Our proof is elementary, effective, and self-contained, and partially recovers work of Birman–Kofman. Along the way, we show that the disparate definitions of twisted torus knots in the literature agree.

Keywords: Knots; links; braids; braid index; positive braids.

Mathematics Subject Classification 2020: 57K10

1. Introduction

1.1. *Motivation and summary*

Knots and links play an important role in low-dimensional topology. One simple way to measure the complexity of a link L in S^3 is the *braid index*, $i(L)$, which is the minimum number of strands required to represent L as the closure of a braid

[††]Corresponding author.

on as many strands. As every link is realized as the closure of some braid [1], the braid index is a well-defined link invariant. Even for knots, the braid index is often quite difficult to compute. The simplest infinite family for which the braid index is computed are the $T(p, q)$ torus knots for which $i(T(p, q)) = \min\{p, q\}$. Analogous formulas in the literature are rare.

It is natural to hope that generalizations of torus knots lend themselves to closed braid index formulas. One axis along which we can generalize comes from the Dehn surgery perspective. Dehn surgery is a powerful operation within 3-manifold topology: every 3-manifold is obtained by Dehn surgery along some link in S^3 [21, 29]. Despite the ubiquity of this technique, some of the most basic questions about Dehn surgery remain open. For example, the infamous *Berge Conjecture* predicts exactly which knots in S^3 admit a Dehn surgery to *lens spaces*, the rational homology 3-spheres admitting genus-1 Heegaard splittings [3]. Moser [25] showed that torus knots always admit Dehn surgeries to lens spaces. Lens spaces are examples of *L-spaces*: the closed, connected, oriented 3-manifolds with "small" Heegaard Floer homology [27]. It immediately follows that torus knots are examples of knots admitting a Dehn surgery to L-spaces. Thus, one way to generalize torus knots would be to identify other knots which *also* admit Dehn surgeries to L-spaces.

Perhaps surprisingly, there are infinitely many hyperbolic knots which admits surgeries to lens spaces: the first examples were identified by Fintushel and Stern a decade after Moser's work [12]. A decade later still, work of Berge and Gabai showed that an infinite sub-family of 1-*bridge braids* admit a Dehn surgery to a lens space [2, 14, 15] (a precise definition of these knots appears later in this paper). In fact, all 1-bridge braids admit a Dehn surgery to L-spaces [18]. Therefore, we see that 1-bridge braids are a generalization of torus knots from the Dehn surgery perspective — moreover, they are a natural extension from a braid-theoretic point of view as well (see Sec. 2 for more details). Besides 1-bridge braids, there are other braid theoretic ways to generalize torus knots, including n*-bridge braids* [15], *twisted torus knots* [4, 20, 28], and *T-links* [4]. Section 2 contains the definitions of these various families, and the relationships between them.

Braid theoretic definitions are valuable, in part, because they are explicit and concrete — however, it can be remarkably difficult to determine whether different braid theoretic definitions coincide. For example, twisted torus knots have received a lot of attention over the past few years [4, 8, 9, 19, 20, 28], yet there are multiple different braid theoretic definitions of twisted torus knots scattered throughout the literature. In this paper, on route to proving our main result, we prove that these various definitions of twisted torus knots coincide; see Sec. 3.

As mentioned above, n-bridge braids (which we define in Sec. 2) are one natural generalization of twisted torus knots from a braid theoretic standpoint. Torus knots are defined by using two parameters; in contrast, n-bridge braids are defined using four parameters. In this work, we compute the braid index of any n-bridge braid.

Theorem 1.1. *The braid index of an n-bridge braid*, $\mathcal{K}(w, b, t, n)$, *is determined by the defining parameters*; *namely*,

$$i(\mathcal{K}(w,b,t,n)) = \begin{cases} w, & t \geq w, \quad n \geq 1, \\ t, & w > t > b, \quad n \geq 1, \\ t+1, & w > b \geq t, \quad n = 1, \\ b+1, & w > b \geq t, \quad n+t \geq b+1, \quad n > 1, \\ n+t, & w > b \geq t, \quad n+t < b+1, \quad n > 1. \end{cases}$$

As an immediate consequence, we determine the braid index of a 1-bridge braid.

Corollary 1.2. *The braid index of a* 1*-bridge braid* $\mathcal{K}(w, b, t)$ *is*

$$i(\mathcal{K}(w,b,t)) = \begin{cases} w, & t \geq w, \\ t, & w > t > b, \\ t+1, & b \geq t. \end{cases}$$

The main proof strategy for Theorem 1.1, and Corollary 1.2 is elementary: we use the well-known *Markov moves* to manipulate the presentation of the braid, and then apply a result of Morton and Franks–Williams [13, 23, 24]. Their theorem says that if a positive braid β on k strands contains a *positive full twist*, then in fact, $i(\beta) = k$. Our proof is completely effective: we concretely apply Markov moves to produce an explicit positive braid which contains a full twist; we then apply the Morton–Franks–Williams result to this braid to know the braid index.

Theorem 1.1 partially recovers — using very different techniques — a result of Birman–Kofman [4]. In [4], the authors define *T-links* (these links are the closures of particular positive braids), and prove that the set of T-links coincides with the well studied *Lorenz links*, i.e. the set of links which can be embedded onto the "Lorenz template", which is seen in Fig. 1. Lorenz links are interesting in their own right as they exhibit rich dynamical and geometric properties [4, 5, 7, 8, 10, 11]. Notably, Birman–Kofman show that over half of the "simplest" hyperbolic knots are Lorenz knots [4].

We coarsely summarize the Birman–Kofman strategy for computing the braid index for T-links and then contrast it with the methods used in this paper. Birman–Williams [7] proved that Lorenz knots can always be realized as the closures of positive braids which contain a positive full twist — therefore, one can apply the Morton–Franks–Williams theorem to determine the braid index. So, Birman–Kofman first prove that T-links coincide with Lorenz knots, and then adapt the T-link presentation to a Lorenz presentation; applying Birman–Williams yields the final result. In contrast to their combinatorial and dynamical proof, our proof is self-contained, elementary, and explicit, as we bypass the Lorenz template and only utilize Markov moves. Moreover, unlike Birman–Kofman, our proof *produces an explicit braid* which is Markov equivalent to an n-bridge braid. This itself has

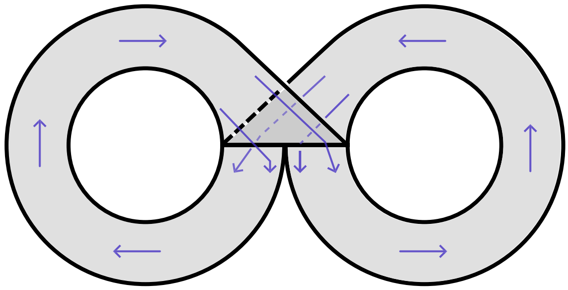

Fig. 1. In this figure, we see the *Lorenz template*, which is a 2-complex with some extra dynamical information. The arrows on the template dictate how a simple curve (or collection thereof) should flow around the surface. For example, a curve can flow from the left to the right by passing in the "front" of the branch locus, and it can flow from the right to the left by passing "behind" the branch locus. Simple closed curves that can be embedded on the Lorenz template are called *Lorenz links*.

value, and was utilized by Krishna–Morton to study 4-dimensional properties of Lorenz knots [19]; next, we briefly describe some of their works, and the ties to this paper.

Recently, Krishna–Morton showed that if a knot K can be realized as the closure of a positive braid with a full twist, then the braid index of K appears as the third exponent in the Alexander polynomial for K [19, Theorem 1.2]. This already yields applications for 1-bridge braids: in the proof of Corollary 1.2, we show that 1-bridge braids can be realized as the closure of a positive braid with a full twist and thus, by [19, Theorem 1.2], the third exponent of the Alexander polynomial for a 1-bridge braid can be determined directly from the braid index formula in Corollary 1.2. (We note that, in general, it is very hard to determine non-trivial terms in the Alexander polynomial of a positive braid knot.) Prior to our work, if one wanted to compute the braid index of a 1-bridge braid, one would have to do the following: (1) Show that a 1-bridge braid is a T-link (from Gabai's definition of 1-bridge braids, and Birman–Kofman's definition of T-links, this is not clear), and then (2) apply Birman–Kofman (and Birman–Williams) to determine the braid index.

Therefore, in addition to identifying a closed formula for the braid index, our paper accomplishes a few important goals: it unifies multiple viewpoints and definitions in the literature, and it is elementary and effective (and could be implemented by a computer for more complicated links). Perhaps most importantly, it produces an *explicit* positive braid word to which the Morton–Franks–Williams theorem applies.

1.2. *Outline of the paper*

In Sec. 2, we outline the definitions and foundational results that we will use throughout the paper and set some notational conventions for the remainder of the paper. In Sec. 3, we prove that the different definitions of twisted torus knots in the literature agree, and also show that n-bridge braids (as we defined them) are Lorenz knots. In Sec. 4, we establish a series of lemmas and propositions to be used in the proof of Theorem 1.1. The proof of Theorem 1.1 is contained in Sec. 5.

2. Background

We begin with some preliminaries.

Definition 2.1. The **braid group on** n **strands**, denoted B_n, is the group with the following presentation:

$$B_n := \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \,|\, \mathcal{R} \rangle,$$

where $\mathcal{R}$ denotes the following set of braid relations:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$,
(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, where $1 \leq i \leq n-2$.

This is also known as *Artin's presentation for the braid group*, and the generating set $\sigma_1, \ldots, \sigma_{n-1}$ are typically referred to as the *Artin generators* for the braid group. There are other group presentations for the braid group. The interested reader can consult [6] for a survey, and to discover some of the many connections between the braid group and topology, geometry, algebra, and dynamics.

Remark 2.2. In [16], Garside proves that the center of B_n is generated by the full twist; that is, the element $(\sigma_1\sigma_2 \ldots \sigma_{n-1})^n = (\sigma_{n-1} \ldots \sigma_2\sigma_1)^n$ commutes with every other element in B_n. In the same work, Garside defines the **Garside element**: for the braid group B_n, the Garside element Δ_n is defined as follows: $\Delta_n = (\sigma_1\sigma_2 \cdots \sigma_{n-1})(\sigma_1 \ldots \sigma_{n-2}) \ldots (\sigma_1\sigma_2)(\sigma_1)$. He notes that $(\Delta_n)^2$ is the full twist, and that $\sigma_i\Delta_n = \Delta_n\sigma_{n-i}$. These facts about the braid group will be useful in our proofs. For more about the Garside element, we recommend [17] as a reference.

Definition 2.3. A braid $\beta \in B_n$ is a **positive braid**, or **braid positive**, if it contains only positive Artin generators. A knot or link is **braid positive** if it can be realized as the closure of a positive braid.

Definition 2.4. A **T-link** is a link which is realized as the closure of a positive braid τ, where

$$\tau = (\sigma_1\sigma_2 \ldots \sigma_{p_1-1})^{q_1}(\sigma_1\sigma_2 \ldots \sigma_{p_2-1})^{q_2} \ldots (\sigma_1\sigma_2 \ldots \sigma_{p_s-1})^{q_s}. \tag{1}$$

Here, $2 \leq p_1 \leq p_2 \leq \cdots \leq p_s$, $0 < q_i$ for all i, and τ is a braid in B_{p_s}.

Definition 2.5 (à la Vafaee [28]). A **twisted torus knot** is realized as the closure of a positive braid ω on n strands, where

$$\omega = (\sigma_{n-1}\sigma_{n-2} \ldots \sigma_2\sigma_1)^p(\sigma_{n-1}\sigma_{n-2} \ldots \sigma_{n-k+1})^{qk}. \tag{2}$$

Here, $3 \leq n$, $2 \leq p$, $2 \leq k \leq n-1$, and $q \geq 1$. That is, adding q many positive full twists into k adjacent strands of a positive torus knot yields a twisted torus knot.

We note that we do not want to consider the case where $k = n$: if $k = n$, then the definition of ω in Definition 2.5 simplifies to the standard braid word for the torus link $T(n, p + qk) = T(n, p + qn)$.

Definition 2.6. An n**-bridge braid**, denoted $\mathcal{K}(w, b, t, n)$, is the link realized as the closure of the positive braid

$$(\sigma_b \sigma_{b-1} \dots \sigma_1)^n (\sigma_{w-1} \dots \sigma_2 \sigma_1)^t.$$

Here, $3 \leq w,\ 1 \leq b \leq w-2,\ t \leq 2$, and $1 \leq n$. Qualitatively, w is the number of strands on which the braid is presented, b is the bridge length, t is the number of twists, and n is the number of bridges.

Note: We do not want $b = w - 1$: if this were permitted, then the braid word in Definition 2.6 would simplify to the torus knot $T(w, n + t)$.

The family of 1-bridge braids (e.g. where $n = 1$ in Definition 2.6) are especially well studied: as we noted in Sec. 1, 1-bridge braids have been studied by Berge, Gabai, and Greene-Lewallen-Vafaee [2, 14, 15, 18], amongst others. Figure 2 organizes how 1-bridge braids, twisted torus knots, n-bridge braids, and T-links are related.

Note that we will use $\mathcal{K}(w, b, t, n)$ to denote both the link and the associated braid word

$$(\sigma_b \sigma_{b-1} \dots \sigma_1)^n (\sigma_{w-1} \dots \sigma_2 \sigma_1)^t.$$

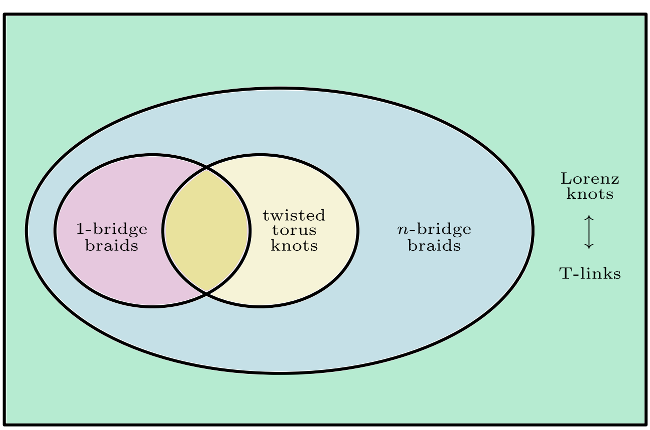

Fig. 2. A schematic explaining how the relevant families of knots are related. We emphasize that n-bridge braids can be viewed as a generalization of twisted torus knots: the hypothesis that there are q full twists on k adjacent strands is weakened to partial twists.

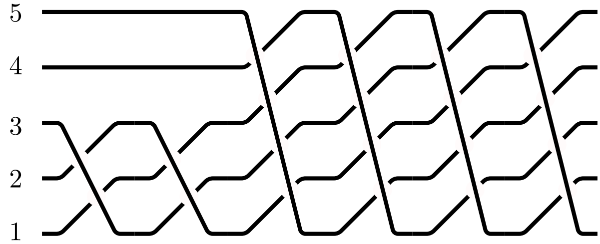

Fig. 3. The n-bridge braid $K(5,4,2,2)$ is realized as the closure of this braid.

To compute the braid index of a link L, we need a method for decreasing the number of strands in the braided presentation of L. This method is called *destabilization.*

Definition 2.7. Let ω be a braid word on n strands. A **stabilization** replaces ω with $\omega\sigma_n$ or $\omega\sigma_n^{-1}$, a braid word on $n+1$ strands. The reverse operation (of replacing $\omega\sigma_n$ or $\omega\sigma_n^{-1}$, where ω has no $\sigma_n^{\pm 1}$ letters) is called **destabilization**.

If two braids have the same closure, then the braids must be related in a particular way.

Theorem 2.8 (Markov [22]). *Let β_1 and β_2 be two braid words. Then, their braid closures are isotopic if and only if β_1 and β_2 are related by any combination of*: (1) *braid relations*, (2) *conjugations*, *and* (3) *(de)stabilizations.*

In particular, Markov's theorem tells us the following: if α and β are braids in B_n, then the braids $\alpha\sigma_n\beta$ and $\alpha\beta$ (which are braids in B_{n+1}) have isotopic closures as links in S^3. We will use this observation at various points throughout the proof of our main theorem.

Finally, to determine the braid index, we will use a result independently obtained by Morton and Franks–Williams.

Theorem 2.9 (Morton [23, 24]; Franks–Williams [13]). *Suppose $\beta \in B_n$ is a positive braid, and $\beta = \omega(\sigma_{n-1}\dots\sigma_1)^n$, where ω is a positive braid word. Then the braid index of β is n, i.e. $i(\beta) = n$.*

As noted in Remark 2.2, Garside proved that the positive full twist commutes with every other element in the braid group. In particular, combining with the Morton–Franks–Williams result, we see the following: if $\alpha, \beta \in B_n$, and α and β are both positive braid words, then $\alpha(\sigma_{n-1}\dots\sigma_1)^n\beta$ has braid index n.

2.1. *Conventions*

Throughout the paper, we will indicate how the braid word changes by underlining the letters of the braid word as they are changed by braid relations, conjugations, or de-stabilizations. When we draw our braids vertically, we read them top-to-bottom. When we draw our braids horizontally, we read them from left-to-right. For us, σ_i corresponds to strand $(i+1)$ crossing over strand (i). Given a braid β, the notation $\hat{\beta}$ will denote its closure. Finally, we will use "=" to denote that two sides of an equation are isotopic as braid closures and so are equal up to braid relations and Markov moves.

3. n-Bridge Braids are Lorenz Knots

Birman–Kofman [4] showed that the class of Lorenz links coincides with that of T-links. By the Birman–Kofman conventions [4, Eq. 1], a twisted torus knot on ℓ strands is realized as the closure of the following braid:

$$\beta_{BK} = (\sigma_1\sigma_2\ldots\sigma_r)^{kr}(\sigma_1\sigma_2\ldots\sigma_{\ell-1})^s.$$

However, our definition of twisted torus knots (in Definition 2.5) follows Vafaee's conventions [28]; he defines a twisted torus knot on n strands to be obtained by taking the braid closure of

$$\beta_V = (\sigma_{n-1}\sigma_{n-2}\ldots\sigma_2\sigma_1)^p(\sigma_{n-1}\sigma_{n-2}\ldots\sigma_{n-k})^{qk}.$$

It is not immediate that these braid words are Markov equivalent (and hence that their closures are isotopic knots in S^3). Given this discrepancy in the literature, we explicitly show that the Vafaee and Birman–Kofman twisted torus knots are Markov equivalent. The remainder of this section is devoted to this proof: we explicitly use Markov moves to put twisted torus knots and n-bridge braids into T-link form.

Lemma 3.1. *Fix some $w \geq 3$. Let $t \geq 2, a \geq 2$, and $c \geq 1$. Let $\alpha_1 = (\sigma_a\sigma_{a+1}\ldots\sigma_{a+c})(\sigma_1\sigma_2\ldots\sigma_{w-1})^t$ and let $\alpha_2 = (\sigma_{a-1}\sigma_a\ldots\sigma_{a+c-1})(\sigma_1\sigma_2\ldots\sigma_{w-1})^t$, where α_1 and α_2 are both elements of the braid group B_w. Then α_1 and α_2 are conjugate braids. In particular, $\widehat{\alpha_1}$ and $\widehat{\alpha_2}$ are isotopic links in S^3.*

Proof. We do some explicit braid moves to verify the claim. For clarity, we underline the portions of the braid that are being transformed from one line to the next. We set $\gamma_w := (\sigma_1\sigma_2\ldots\sigma_{w-1})$, a braid word in B_w. We begin by pushing some terms to the right:

$$\begin{aligned}\alpha_1 &= (\sigma_a\sigma_{a+1}\ldots\sigma_{a+c-1}\sigma_{a+c})(\sigma_1\sigma_2\ldots\sigma_{w-1})^t \\ &= (\sigma_a\sigma_{a+1}\ldots\sigma_{a+c-1}\sigma_{a+c})\ \gamma_w^t \\ &= (\sigma_a\sigma_{a+1}\ldots\sigma_{a+c-1}\ \underline{\sigma_{a+c}})(\sigma_1\sigma_2\ldots\sigma_{a+c-2}\sigma_{a+c-1}\sigma_{a+c}) \\ &\quad \times (\sigma_{a+c+1}\ldots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\end{aligned}$$

$$
\begin{aligned}
&= (\sigma_a\sigma_{a+1}\dots\sigma_{a+c-1})(\sigma_1\sigma_2\dots\sigma_{a+c-2}\ \underline{\sigma_{a+c}\sigma_{a+c-1}\sigma_{a+c}})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_a\sigma_{a+1}\dots\underline{\sigma_{a+c-1}})(\sigma_1\sigma_2\dots\sigma_{a+c-2}\ \sigma_{a+c-1}\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_a\sigma_{a+1}\dots\sigma_{a+c-2})(\sigma_1\sigma_2\dots\underline{\sigma_{a+c-1}\sigma_{a+c-2}\sigma_{a+c-1}})\\
&\quad\times(\sigma_{a+c}\sigma_{a+c-1})(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_a\sigma_{a+1}\dots\sigma_{a+c-2})(\sigma_1\sigma_2\dots\sigma_{a+c-2}\sigma_{a+c-1}\sigma_{a+c-2})(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_a\sigma_{a+1}\dots\underline{\sigma_{a+c-2}})(\sigma_1\sigma_2\dots\sigma_{a+c-2})(\sigma_{a+c-1}\sigma_{a+c-2})\\
&\quad\times(\sigma_{a+c}\sigma_{a+c-1})(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_a\sigma_{a+1}\dots\sigma_{a+c-3})(\sigma_1\dots\underline{\sigma_{a+c-2}\sigma_{a+c-3}\sigma_{a+c-2}})(\sigma_{a+c-1}\sigma_{a+c-2})\\
&\quad\times(\sigma_{a+c}\sigma_{a+c-1})(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_a\sigma_{a+1}\dots\sigma_{a+c-3})(\sigma_1\dots\sigma_{a+c-3})(\sigma_{a+c-2}\sigma_{a+c-3})(\sigma_{a+c-1}\sigma_{a+c-2})\\
&\quad\times(\sigma_{a+c}\sigma_{a+c-1})(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}.
\end{aligned}
$$

We repeat this process — of moving the last term of the left-most parenthetical as far into the braid as possible using commutation, and then applying the other braid relation — until we reach σ_a. Note that at the end of each iteration of this process, we produce a pair of adjacent terms of the form $(\sigma_{a+c-k}\sigma_{a+c-(k+1)})$. At the penultimate stage, we have

$$
\begin{aligned}
&= (\underline{\sigma_a})(\sigma_1\sigma_2\dots\sigma_{a-2}\sigma_{a-1})(\sigma_a\sigma_{a+1}\sigma_a)(\sigma_{a+2}\sigma_{a+1})\dots(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_1\sigma_2\dots\sigma_{a-2})(\underline{\sigma_a\sigma_{a-1}\sigma_a})(\sigma_{a+1}\sigma_a)(\sigma_{a+2}\sigma_{a+1})\dots(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_1\sigma_2\dots\sigma_{a-2})(\sigma_{a-1}\sigma_a\sigma_{a-1})(\sigma_{a+1}\sigma_a)(\sigma_{a+2}\sigma_{a+1})\dots(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}.
\end{aligned}
$$

Reassigning some parenthesis, we obtain

$$
\begin{aligned}
&= (\sigma_1\sigma_2\dots\sigma_{a-2}\sigma_{a-1}\sigma_a)(\sigma_{a-1})(\sigma_{a+1}\sigma_a)(\sigma_{a+2}\sigma_{a+1})\dots(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}.
\end{aligned}
$$

We observe that in each parenthetical of the form $(\sigma_{a+c-k}\sigma_{a+c-(k+1)})$, the left term has a larger index than the right term. Moreover, as we read the parentheticals from left to right, the index of the first term uniformly increases until we hit $(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^t$. Therefore, we can rewrite our braid by collecting terms towards the front of the braid. In the following set of moves, we push the underlined terms to the left:

$$\begin{aligned}
&= (\sigma_1\sigma_2\dots\sigma_{a-2}\sigma_{a-1}\sigma_a)(\sigma_{a-1})(\underline{\sigma_{a+1}}\sigma_a)(\sigma_{a+2}\sigma_{a+1})\dots(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_1\sigma_2\dots\sigma_{a-2}\sigma_{a-1}\sigma_a\sigma_{a+1})(\sigma_{a-1}\sigma_a)(\underline{\sigma_{a+2}}\sigma_{a+1})\dots(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}\\
&= (\sigma_1\sigma_2\dots\sigma_{a-2}\sigma_{a-1}\sigma_a\sigma_{a+1}\sigma_{a+2})(\sigma_{a-1}\sigma_a\sigma_{a+1})\dots(\sigma_{a+c}\sigma_{a+c-1})\\
&\quad\times(\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1})\ \gamma_w^{t-1}.
\end{aligned}$$

Repeating this leftwards operation eventually yields:

$$\begin{aligned}
&= (\sigma_1\sigma_2\dots\sigma_{a+1}\sigma_{a+2}\dots\sigma_{a+c})(\sigma_{a-1}\sigma_a\sigma_{a+1}\dots\sigma_{a+c-2}\sigma_{a+c-1})\\
&\quad\times(\underline{\sigma_{a+c+1}\dots\sigma_{w-2}\sigma_{w-1}})\ \gamma_w^{t-1}\\
&= (\sigma_1\dots\sigma_{w-1})(\sigma_{a-1}\sigma_a\sigma_{a+1}\dots\sigma_{a+c-2}\sigma_{a+c-1})\ \gamma_w^{t-1}\\
&= (\sigma_{a-1}\sigma_a\sigma_{a+1}\dots\sigma_{a+c-2}\sigma_{a+c-1})\ \gamma_w^{t}\\
&= \alpha_2.
\end{aligned}$$

In the last step, we conjugated by $(\sigma_1\dots\sigma_{w-1})$. We conclude that α_1 and α_2 are conjugate. □

Proposition 3.2. 1-*bridge braids are Lorenz knots.*

Proof. To prove that 1-bridge braids are Lorenz knots, it suffices to show that some sequence of Markov moves transforms β to a braid τ, as in Eq. (1).

Let $\beta = (\sigma_b\sigma_{b-1}\dots\sigma_2\sigma_1)(\sigma_{w-1}\sigma_{w-2}\dots\sigma_2\sigma_1)^t$ denote the standard braid presentation of a 1-bridge braid. Let $\beta' = (\sigma_{w-b}\sigma_{w-b+1}\dots\sigma_{w-1})(\sigma_1\sigma_2\dots\sigma_{w-1})^t$. We claim that $\widehat{\beta}$ and $\widehat{\beta'}$ are isotopic knots in S^3: view S^3 as $\mathbb{R}^3\cup\{\infty\}$, and fix the circle $C = z$-axis $\cup\{\infty\}$; we represent the z-axis by the purple dotted line in Fig. 4. We draw the braid β on the "left" side of C, and then rotate β about the purple line; this produces β', which is seen on the "right" side of C. In particular, if we take $\widehat{\beta}$ and follow it through the rotation isotopy, we will get $\widehat{\beta'}$. Therefore, $\widehat{\beta} = \widehat{\beta'}$ as knots in S^3. Alternatively, one can use some standard results in braid theory: if we conjugate β by the Garside element $\Delta \in B_w$, we produce β' (see [16, 17] for more details); since conjugation preserves the link type of the closure, $\widehat{\beta}$ and $\widehat{\beta'}$ present the same knot.

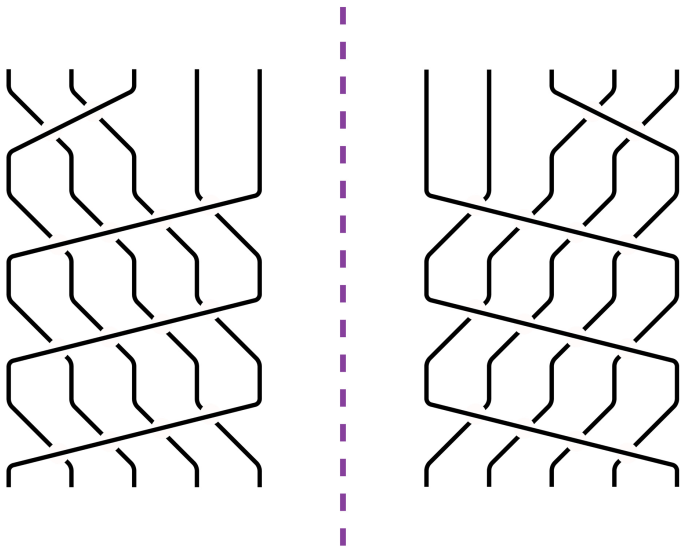

Fig. 4. Rotating the left braid about the purple line produces the right braid.

Next, we perform $(w-b-1)$ many applications of Lemma 3.1:

$$\begin{aligned}\beta &= (\sigma_b\sigma_{b-1}\dots\sigma_2\sigma_1)(\sigma_{w-1}\sigma_{w-2}\dots\sigma_2\sigma_1)^t\\ &= (\sigma_{w-b}\sigma_{w-b+1}\dots\sigma_{w-1})(\sigma_1\sigma_2\dots\sigma_{w-1})^t\\ &= (\sigma_{w-b-1}\sigma_{w-b}\dots\sigma_{w-2})(\sigma_1\sigma_2\dots\sigma_{w-1})^t\\ &= (\sigma_1\sigma_2\dots\sigma_b)(\sigma_1\sigma_2\dots\sigma_{w-1})^t.\end{aligned}$$

Thus, the 1-bridge braid β admits a T-link presentation. □

Lemma 3.3. *Twisted torus knots and n-bridge braids are Lorenz knots.*

Proof. Twisted torus knots are the closures of positive braids on w strands with the following form:

$$\rho = (\sigma_{w-1}\sigma_{w-2}\dots\sigma_1)^t(\sigma_{w-1}\sigma_{w-2}\dots\sigma_{w-k})^{sk}.$$

Rotating ρ as in Fig. 4 yields $\rho' = (\sigma_1\sigma_2\dots\sigma_{w-1})^t(\sigma_1\sigma_2\dots\sigma_k)^{sk}$. We know that $\widehat{\rho}$ and $\widehat{\rho'}$ are isotopic knots; since ρ' is presented as a T-link braid, we deduce that twisted torus knots are T-links.

Indeed, the braided presentation for n-bridge braids appears very similar to those of twisted torus knots (however, there is not required that b divides n). We quickly show that these, too, are T-links:

$$\begin{aligned}\eta &= (\sigma_b\sigma_{b-1}\dots\sigma_1)^n(\sigma_{w-1}\sigma_{w-2}\dots\sigma_1)^t\\ &= (\sigma_{w-b}\sigma_{w-b+1}\dots\sigma_{w-1})^n(\sigma_1\sigma_2\dots\sigma_{w-1})^t\\ &= (\sigma_{w-b}\sigma_{w-b+1}\dots\sigma_{w-1})^{n-1}(\sigma_{w-b}\sigma_{w-b+1}\dots\sigma_{w-1})\\ &\quad\times(\underline{\sigma_1\sigma_2\dots\sigma_{w-1}})(\sigma_1\sigma_2\dots\sigma_{w-1})^{t-1}.\end{aligned}$$

In the proof of Lemma 3.1, we only performed braid relationships — the only place we conjugated our braid is in the last step. Thus, applying the proof of Lemma 3.1, we see

$$\begin{aligned}&= (\sigma_{w-b}\sigma_{w-b+1}\dots\sigma_{w-1})^{n-1}(\underline{\sigma_1\sigma_2\dots\sigma_{w-1}})\\ &\quad\times(\sigma_{w-b-1}\sigma_{w-b}\dots\sigma_{w-2})(\sigma_1\sigma_2\dots\sigma_{w-1})^{t-1}\\ &= (\sigma_1\sigma_2\dots\sigma_{w-1})(\sigma_{w-b-1}\sigma_{w-b}\dots\sigma_{w-2})^n(\sigma_1\sigma_2\dots\sigma_{w-1})^{t-1}\\ &= (\sigma_{w-b-1}\sigma_{w-b}\dots\sigma_{w-2})^n(\sigma_1\sigma_2\dots\sigma_{w-1})^t.\end{aligned}$$

We repeat this process an additional $w-b-2$ times, yielding:

$$= (\sigma_1\sigma_2\dots\sigma_b)^n(\sigma_1\sigma_2\dots\sigma_{w-1})^t.$$

Thus, n-bridge braids are more general than twisted torus knots, and they are T-links. □

4. Preliminaries for the Proof of the Main Theorem

Remark. After this paper was posted to the arXiv, the author of [26] informed us that the results in this section are sketched in the body of the proof of Theorem 1.1 in [26]. This work contains the full proofs.

Definition 4.1. We define δ_n and γ_n to be the positive braid words $\delta_n := (\sigma_n\dots\sigma_2\sigma_1)$ and $\gamma_n := (\sigma_1\sigma_2\dots\sigma_n)$ in B_r, the braid group on r strands, where $r \geq n+1$.

Remark 4.2. Note that $\mathcal{K}(w,b,t) = \widehat{\delta_b\delta_{w-1}^t}$.

Lemma 4.3. *Let $\delta_k \in B_{k+1}$. Then $\delta_k\ \delta_k = \delta_{k-1}\ \delta_k\ \sigma_1$ as braid words in B_{k+1}.*

Proof. We begin by expanding the left-hand side.

$$\delta_k\ \delta_k = (\sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \dots\ \sigma_3\ \sigma_2\ \sigma_1)(\underline{\sigma_k}\dots\sigma_1) \tag{3}$$

$$= (\underline{\sigma_k\ \sigma_{k-1}\ \sigma_k}\ \sigma_{k-2}\ \dots\ \sigma_3\ \sigma_2\ \sigma_1)(\sigma_{k-1}\dots\sigma_1) \tag{4}$$

$$= (\sigma_{k-1}\ \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ \ldots\ \sigma_3\ \sigma_2\ \sigma_1)(\underline{\sigma_{k-1}}\ldots\sigma_1) \tag{5}$$

$$= (\sigma_{k-1}\ \sigma_k\ \underline{\sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-1}}\ \sigma_{k-3}\ \ldots\ \sigma_3\ \sigma_2\ \sigma_1)(\sigma_{k-2}\ldots\sigma_1) \tag{6}$$

$$= (\sigma_{k-1}\ \sigma_k\ \underline{\sigma_{k-2}}\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ \ldots\ \sigma_3\ \sigma_2\ \sigma_1)(\sigma_{k-2}\ldots\sigma_1) \tag{7}$$

$$= (\sigma_{k-1}\ \sigma_{k-2}\ \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ \ldots\ \sigma_3\ \sigma_2\ \sigma_1)(\underline{\sigma_{k-2}}\ldots\sigma_1) \tag{8}$$

$$= (\sigma_{k-1}\ \sigma_{k-2}\ \sigma_k\ \sigma_{k-1}\ \underline{\sigma_{k-2}\ \sigma_{k-3}\ \sigma_{k-2}}\ldots\ \sigma_3\ \sigma_2\ \sigma_1)(\sigma_{k-3}\ldots\sigma_1) \tag{9}$$

$$= (\sigma_{k-1}\ \sigma_{k-2}\ \sigma_k\ \sigma_{k-1}\ \underline{\sigma_{k-3}}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\ \sigma_3\ \sigma_2\ \sigma_1)(\sigma_{k-3}\ldots\sigma_1) \tag{10}$$

$$= (\sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\ \sigma_3\ \sigma_2\ \sigma_1)(\sigma_{k-3}\ldots\sigma_1). \tag{11}$$

We describe the operations at play: in line (3), we identify the σ_k letter that is furthest to the right, and apply $k-2$ commuting relations to push it as much to the left as possible. This creates the underlined subword $\sigma_k\sigma_{k-1}\sigma_k$ in line (4); applying the braid relation yields line (5). We repeat this procedure (of finding the largest letter in the right parenthetical subword, applying commuting relations to push it as far to the left as possible, and then applying a braid relation in lines (5)–(7). From lines (7) to (8), we identify and execute another commuting relation. We call this 4-step procedure a *left push*, and say that we perform a *left push on* σ_t when σ_t is the largest letter in the second parenthetical braid word. Note that after executing the left push operation on σ_{r+1}, the braid word decomposes into the three subwords $((\sigma_{k-1}\ldots\sigma_r)(\sigma_k\ldots\sigma_1))(\sigma_r\ldots\sigma_1)$; this is seen explicitly in lines (8) and (11). After repeating the *left push* operation another $k-5$ times from line (11) onwards, we get

$$= ((\sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\sigma_3\ \sigma_2)\ (\sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\ \sigma_3\ \sigma_2\ \sigma_1))(\sigma_2\ \sigma_1) \tag{12}$$

$$= (\sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\sigma_3\ \sigma_2\ \ \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\ \sigma_3\ \underline{\sigma_2\ \sigma_1\ \sigma_2}\ \sigma_1) \tag{13}$$

$$= (\sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\sigma_3\ \sigma_2\ \ \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\ \sigma_3\ \underline{\sigma_1}\ \sigma_2\ \sigma_1\ \sigma_1) \tag{14}$$

$$= ((\sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\sigma_3\ \sigma_2\ \sigma_1)\ (\sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \sigma_{k-3}\ldots\ \sigma_3\ \sigma_2\ \sigma_1))\ \sigma_1 \tag{15}$$

$$= \delta_{k-1}\ \delta_k\ \sigma_1. \tag{16}$$

This is exactly what we wanted to show. ☐

Lemma 4.4. *Let* $\delta_k \in B_{k+1}$. *Then* $\sigma_1\ \delta_k = \delta_k\ \sigma_2$.

Proof. We begin by expanding the left-hand side.

$$\begin{aligned}\sigma_1\ \delta_k &= \sigma_1\ \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ldots\sigma_3\ \sigma_2\ \sigma_1\\ &= \underline{\sigma_1}\ \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ \ldots\sigma_3\ \sigma_2\ \sigma_1\\ &= \sigma_k\ \sigma_{k-1}\ \sigma_{k-2}\ldots\sigma_3\ \underline{\sigma_1\ \sigma_2\ \sigma_1}\end{aligned}$$

$$= (\sigma_k\ \sigma_{k-1}\ \sigma_{k-2} \ldots \sigma_3\ \sigma_2\ \sigma_1)\ \sigma_2$$
$$= \delta_k \sigma_2.$$

This is what we wanted to prove. □

Lemma 4.5. *Let $\delta_k \in B_{k+1}$. Then $\sigma_j\ \delta_k = \delta_k\ \sigma_{j+1}$ when $1 < j < k$.*

Proof. Suppose $1 < j < k$. We begin by expanding $\sigma_j \delta_k$:

$$\begin{aligned}\sigma_j \delta_k &= \underline{\sigma_j}(\sigma_k \sigma_{k-1} \ldots \sigma_{j+2}\ \sigma_{j+1}\sigma_j\sigma_{j-1}\ \ldots \sigma_1)\\ &= \sigma_k\sigma_{k-1} \ldots \sigma_{j+2}\ \underline{\sigma_j\sigma_{j+1}\sigma_j}\sigma_{j-1}\ \ldots \sigma_1\\ &= \sigma_k\sigma_{k-1} \ldots \sigma_{j+2}\ \sigma_{j+1}\sigma_j\underline{\sigma_{j+1}}\sigma_{j-1}\ \ldots \sigma_1\\ &= (\sigma_k\sigma_{k-1} \ldots \sigma_{j+2}\ \sigma_{j+1}\sigma_j\sigma_{j-1}\ \ldots \sigma_1)\sigma_{j+1}\end{aligned}$$

which is $\delta_k \sigma_{j+1}$, as desired. □

Lemma 4.6. *Let $\delta_k \in B_{k+1}$. When $s < k$, we have $\sigma_1\ \delta_k^s = \delta_k^s\ \sigma_{s+1}$.*

Proof. We see that

$$\begin{aligned}\sigma_1 {\delta_k}^s &= \sigma_1 \delta_k {\delta_k}^{s-1}\\ &= \delta_k \sigma_2 \delta_k {\delta_k}^{s-2} \quad \text{by Lemma 4.4}\\ &= {\delta_k}^2 \sigma_3 \delta_k {\delta_k}^{s-3} \quad \text{by Lemma 4.5.}\end{aligned}$$

Applying Lemma 4.5 a total of $s-1$ times, we get

$$\delta_k^s\ \sigma_{s+1}.$$

Thus, $\sigma_1\ \delta_k^s = \delta_k^s\ \sigma_{s+1}$. □

Note that Lemma 4.5 generalizes Lemmas 4.4 and 4.6 combines Lemmas 4.4 and 4.5.

Proposition 4.7. *Let $\delta_j \in B_{j+1}$. Then $\delta_j^t = \delta_{j-1}\ \delta_j^{t-1}\ \sigma_{t-1}$, where $t < j$.*

Proof. We see that

$$\begin{aligned}\delta_j^t &= (\delta_j \delta_j)\delta_j^{t-2}\\ &= (\delta_{j-1}\delta_j\sigma_1)\delta_j^{t-2} \quad \text{by Lemma 4.3}\\ &= \delta_{j-1}\delta_j(\sigma_1\delta_j^{t-2})\\ &= \delta_{j-1}\delta_j(\delta_j^{t-2}\sigma_{(t-2)+1}) \quad \text{by Lemma 4.6}\\ &= \delta_{j-1}\delta_j\delta_j^{t-2}\sigma_{t-1}\\ &= \delta_{j-1}\delta_j^{t-1}\sigma_{t-1}.\end{aligned}$$

Thus, $\delta_j^t = \delta_{j-1}\ \delta_j^{t-1}\ \sigma_{t-1}$. □

Proposition 4.8. *Let $\delta_k, \gamma_{t-1} \in B_{k+1}$. Then, $\delta_k^t = \delta_{k-1}^{t-1}\ \delta_k\ \gamma_{t-1}$, where $t < k$.*

Proof. Note that

$$\begin{aligned}\delta_k^t &= \delta_{k-1}({\delta_k}^{t-1})\sigma_{t-1} \quad \text{by Proposition 4.7}\\ &= \delta_{k-1}(\delta_{k-1}{\delta_k}^{t-2}\sigma_{t-2})\sigma_{t-1} \quad \text{by Proposition 4.7}\\ &= \delta_{k-1}^2\ ({\delta_k}^{t-2})\ \sigma_{t-2}\sigma_{t-1}.\end{aligned}$$

Iteratively apply Proposition 4.7 an additional $t-3$ times to the rightmost $\delta_k^{t-\star}$ to obtain

$$\begin{aligned}\delta_k^t &= \delta_{k-1}^{t-1}\ \delta_k^1\ (\sigma_1\sigma_2\ldots\sigma_{t-1})\\ &= \delta_{k-1}^{t-1}\ \delta_k\ \gamma_{t-1}.\end{aligned}$$

This yields the desired conclusion. □

Proposition 4.9. *Let $\delta_k \in B_{k+1}$. Then, $\delta_k^k = \delta_{k-1}(\delta_k^{k-1})\sigma_{k-1}$.*

Proof.

$$\begin{aligned}\delta_k^k &= (\delta_k^{k-1})\delta_k\\ &= (\delta_{k-1}\delta_k^{k-2}\sigma_{k-2})\delta_k \quad \text{by Proposition 4.7}\\ &= \delta_{k-1}\delta_k^{k-2}(\sigma_{k-2}\delta_k)\\ &= \delta_{k-1}\delta_k^{k-2}(\delta_k\sigma_{k-1}) \quad \text{by Lemma 4.5}\\ &= \delta_{k-1}(\delta_k^{k-1})\sigma_{k-1}.\end{aligned}$$

This is what we wanted to show. □

Proposition 4.10. *Let $\delta_k \in B_{k+1}$, and suppose $\alpha \in B_k$ (so, in particular, α is a braid word on strictly fewer strands than δ_k). Then $\alpha\ \delta_k^k = \alpha\ \delta_{k-1}^k\gamma_{k-1}$.*

Proof. The proof requires straightforward applications of Propositions 4.8 and 4.9.

$$\begin{aligned}\alpha\ \delta_k^k &= \alpha\ (\delta_{k-1}\ \delta_k^{k-1}\ \sigma_{k-1}) \quad \text{by Propositions 4.9}\\ &= \alpha\ \delta_{k-1}\ (\delta_{k-1}^{k-2}\ \delta_k\ \gamma_{k-2})\ \sigma_{k-1} \quad \text{by Proposition 4.8}\\ &= \alpha\ \delta_{k-1}^{k-1}\ \delta_k\ \gamma_{k-1}\\ &= \alpha\ \delta_{k-1}^{k-1}\ \sigma_k\ \delta_{k-1}\ \gamma_{k-1}.\end{aligned}$$

Since α is a braid in B_k, there is a unique σ_k letter in this braid. So, we can destabilize to get

$$= \alpha\ \delta_{k-1}^k\ \gamma_{k-1}.$$

This is what we wanted to show. □

5. Proof of the Main Theorem

Theorem 1.1. *The braid index of an n-bridge braid*

$$\mathcal{K}(w, b, t, n)$$

is

$$i(\mathcal{K}(w,b,t,n)) = \begin{cases} w, & t \geq w, \;\; n \geq 1, \\ t, & w > t > b, \;\; n \geq 1, \\ t+1, & w > b \geq t, \;\; n = 1, \\ b+1, & w > b \geq t, \quad n+t \geq b+1, \;\; n > 1, \\ n+t, & w > b \geq t, \quad n+t < b+1, \;\; n > 1. \end{cases}$$

Proof. We use Propositions 4.7 and 4.8 and destabilizations to find a presentation of the knot which allows us to apply Theorem 2.9.

Case 1: $t \geq w, n \geq 1$**.**

Let $t \geq w$ and $n \geq 1$. Then, we know

$$\begin{aligned} \mathcal{K}(w,b,t,n) &= \delta_b^n \delta_{w-1}^t \\ &= \delta_b^n \delta_{w-1}^w \delta_{w-1}^{t-w}. \end{aligned}$$

Since $(\sigma_{w-1}\sigma_{w-2}\ldots\sigma_1)^w$ is a full twist on w strands, by applying Theorem 2.9, $i(\mathcal{K}(w,b,t,n)) = w$.

Case 2: $w > t > b, n \geq 1$**.**

Suppose $w > t > b$ and $n \geq 1$. If $w - 1 = t > b$, then we have

$$\begin{aligned} \mathcal{K}(w,b,t,n) &= \delta_b^n \delta_{w-1}^t \\ &= \delta_b^n \delta_{w-1}^{w-1} \\ &= \delta_b^n \;\delta_{w-2}^{w-1}\; \gamma_{w-2} \quad \text{by Proposition 4.10} \\ &= \delta_b^n \;\delta_{t-1}^t\; \gamma_{t-1} \quad \text{since } t = w-1. \end{aligned}$$

Therefore, the braid can be written to contain δ_{t-1}^t, which is a full twist on t strands. By Theorem 2.9, $i(\mathcal{K}(w,b,t,n)) = t$ if $w - 1 = t > b, n \geq 1$.

We now study what happens if $w - 1 > t > b$. We know

$$\begin{aligned} \mathcal{K}(w,b,t,n) &= \delta_b^n \underline{\delta_{w-1}^t} \\ &= \delta_b^n \underline{\delta_{w-2}^{t-1}\delta_{w-1}\gamma_{t-1}} \quad \text{by Proposition 4.8} \\ &= \delta_b^n \delta_{w-2}^{t-1}\underline{\sigma_{w-1}\delta_{w-2}}\gamma_{t-1}. \end{aligned}$$

Since $w > t > b$, then $w-1 \geq t > b$, hence there is a single σ_{w-1} in the braid word, which is currently in B_{w-1}. Thus, we can destabilize the braid to produce a new braid in B_{w-2}:

$$\begin{aligned}\mathcal{K}(w,b,t,n) &= \delta_b^n \delta_{w-2}^{t-1} \underline{\sigma_{w-1}\delta_{w-2}}\gamma_{t-1} \\ &= \delta_b^n \delta_{w-2}^{t-1} \underline{\delta_{w-2}}\gamma_{t-1} \\ &= \delta_b^n \delta_{w-2}^{t}\gamma_{t-1}.\end{aligned}$$

We iteratively: (1) apply Proposition 4.8 to the rightmost $\delta_{w-\star}^t$ term, and (2) destabilize the largest remaining Artin generator. Since $w > t > b$ we can repeat the above process a total of $w-t$ times, after which we have

$$\mathcal{K}(w,b,t) = \delta_b^n \delta_{t-1}^{t}\gamma_{t-1}^{w-t}.$$

As $t > b$, we know that $t-1 \geq b$, hence δ_b^n contains no σ_s letters, where $s \geq t-1$. Moreover, this is a braid word in B_t, and it contains a full twist on t strands. By Theorem 2.9, $i(\mathcal{K}(w,b,t,n)) = t$.

Case 3: $w > b \geq t, n = 1$**.**

Our definition of a 1-bridge braid requires that $w-2 \geq b$, so we may revise our assumptions to be $w-2 \geq b \geq t$. In particular, note that $t < w-1$. We have

$$\begin{aligned}\mathcal{K}(w,b,t) &= \delta_b \underline{\delta_{w-1}^t} \\ &= \delta_b \underline{\delta_{w-2}^{t-1}\delta_{w-1}\gamma_{t-1}} \quad \text{by Proposition 4.8} \\ &= \delta_b \delta_{w-2}^{t-1}\underline{\sigma_{w-1}\delta_{w-2}}\gamma_{t-1}.\end{aligned}$$

Since $w-2 \geq b$, there is a single σ_{w-1}, and we can destabilize the braid:

$$\begin{aligned}\mathcal{K}(w,b,t) &= \delta_b \delta_{w-2}^{t-1}\underline{\sigma_{w-1}}\delta_{w-2}\gamma_{t-1} \\ &= \delta_b \delta_{w-2}^{t-1}\delta_{w-2}\gamma_{t-1} \\ &= \delta_b \delta_{w-2}^{t}\gamma_{t-1}.\end{aligned}$$

We iteratively: (1) apply Proposition 4.8 to the rightmost $\delta_{w-\star}^t$ term, and (2) destabilize the largest remaining Artin generator. Since $w > b \geq t$, we can repeat the above process a total of $w-b-1$ times, after which we have

$$\begin{aligned}\mathcal{K}(w,b,t) &= \delta_b \underline{\delta_{w-2}^{t}\gamma_{t-1}} \\ &= \delta_b \underline{\delta_b^t \gamma_{t-1}^{w-b-1}}.\end{aligned}$$

This braid word is in B_{b+1}, the braid group on $b+1$ strands. If $b = t$, then

$$\mathcal{K}(w,b,t) = \delta_b^{t+1}\gamma_{t-1}^{w-b-1} = \delta_t^{t+1}\gamma_{t-1}^{w-b-1}.$$

Applying Theorem 2.9 allows us to conclude that $i(\mathcal{K}(w,b,t)) = t+1$.

Otherwise, $b > t$ and we iteratively: (1) apply Proposition 4.8 to the $\delta_{b-\star}^{t+1}$ term (note that $\star = 0$ to start), and (2) destabilize the largest remaining Artin generator. Since $b > t$, we can repeat this process a total of $b - t$ times to obtain

$$\begin{aligned}\mathcal{K}(w,b,t) &= \delta_b^{t+1}\gamma_{t-1}^{w-b-1}\\ &= \delta_t^{t+1}\gamma_t^{b-t}\gamma_{t-1}^{w-b-1}.\end{aligned}$$

Once again, by Theorem 2.9, we deduce $i(\mathcal{K}(w,b,t)) = t+1$.

Case 4: $w > b \geq t, n+t \geq b+1, n > 1$**.**

Suppose $w > b \geq t, n+t \geq b+1$, and $n > 1$. Our definition of a 1-bridge braid requires that $w - 2 \geq b$, so we may revise our assumptions to be $w - 2 \geq b \geq t$. In particular, note that $t < w - 1$. We have

$$\begin{aligned}\mathcal{K}(w,b,t,n) &= \delta_b^n(\delta_{w-1}^t)\\ &= \delta_b^n\delta_{w-2}^{t-1}(\delta_{w-1})\gamma_{t-1} \quad \text{by Proposition 4.8}\\ &= \delta_b^n\delta_{w-2}^{t-1}(\sigma_{w-1}\delta_{w-2})\gamma_{t-1}.\end{aligned}$$

Since $w - 2 \geq b$, there is a single σ_{w-1}, and we can destabilize the braid:

$$\begin{aligned}\mathcal{K}(w,b,t) &= \delta_b^n\delta_{w-2}^{t-1}(\sigma_{w-1})\delta_{w-2}\gamma_{t-1}\\ &= \delta_b^n\delta_{w-2}^{t-1}\delta_{w-2}\gamma_{t-1}\\ &= \delta_b^n\delta_{w-2}^t\gamma_{t-1}.\end{aligned}$$

We iteratively: (1) apply Proposition 4.8 to the rightmost $\delta_{w-\star}^t$ term, and (2) destabilize the largest remaining Artin generator. Since $w > b \geq t$, we can repeat the above process a total of $w - b - 1$ times, after which we have

$$\begin{aligned}\mathcal{K}(w,b,t) &= \delta_b^n\delta_{w-2}^t\gamma_{t-1}\\ &= \delta_b^n\delta_b^t\gamma_{t-1}^{w-b-1}.\end{aligned}$$

This is a braid word on $b+1$ strands. As $n + t \geq b + 1$, we get

$$\begin{aligned}\mathcal{K}(w,b,t) &= \delta_b^n\delta_b^t\gamma_{t-1}^{w-b-1}\\ &= \delta_b^{n+t}\gamma_{t-1}^{w-b-1}.\end{aligned}$$

Applying Theorem 2.9, we deduce $i(\mathcal{K}(w,b,t,n)) = b+1$.

Case 5: $w > b \geq t, n+t < b+1$**.**

Our definition of a 1-bridge braid requires that $w - 2 \geq b$, so we may revise our assumptions to be $w - 2 \geq b \geq t$. In particular, $t < w - 1$. We begin by applying Proposition 4.8 to the standard braided presentation of $\mathcal{K}(w,b,t,n)$:

$$\begin{aligned}\mathcal{K}(w,b,t,n) &= \delta_b^n\underline{\delta_{w-1}^t}\\ &= \delta_b^n\underline{\delta_{w-2}^{t-1}(\delta_{w-1})\gamma_{t-1}} \quad \text{by Proposition 4.8}\\ &= \delta_b^n\delta_{w-2}^{t-1}\underline{\sigma_{w-1}\delta_{w-2}}\gamma_{t-1}.\end{aligned}$$

Our definition of n-bridge braid required that $w-2 \geq b$. Therefore, there is a single σ_{w-1}, and we can destabilize the braid:

$$\begin{aligned}\mathcal{K}(w,b,t,n) &= \delta_b^n \delta_{w-2}^{t-1} \underline{\sigma_{w-1}} \delta_{w-2} \gamma_{t-1} \\ &= \delta_b^n \delta_{w-2}^{t-1} \delta_{w-2} \gamma_{t-1} \\ &= \delta_b^n \delta_{w-2}^{t} \gamma_{t-1}.\end{aligned}$$

We iteratively: (1) apply Proposition 4.8 to the rightmost $\delta_{w-\star}^t$ term, and (2) destabilize the largest remaining Artin generator. Since $w > b \geq t$, we can repeat the above process a total of $w-b-1$ times, after which we have

$$\begin{aligned}\mathcal{K}(w,b,t,n) &= \delta_b^n \delta_{w-2}^{t} \gamma_{t-1} \\ &= \delta_b^n \delta_b^t \gamma_{t-1}^{w-b-1}. \qquad (17)\end{aligned}$$

This braid word is on $b+1$ strands. We assumed that $n+t < b+1$, so namely, $n+t \leq b$. Suppose $n+t = b$. In this case,

$$\begin{aligned}\mathcal{K}(w,b,t,n) &= \delta_b^n \delta_b^t \gamma_{t-1}^{w-b-1} \\ &= \delta_b^b \gamma_{t-1}^{w-b-1} \\ &= \delta_{b-1} \underline{\delta_b^{b-1}} \sigma_{b-1} \gamma_{t-1}^{w-b+1} \quad \text{by Proposition 4.9} \\ &= \delta_{b-1} \underline{\delta_{b-1}^{b-2} \delta_b \gamma_{b-2}} \sigma_{b-1} \gamma_{t-1}^{w-b+1} \quad \text{by Proposition 4.8} \\ &= \delta_{b-1} \delta_{b-1}^{b-2} \underline{\delta_b} \gamma_{b-2} \sigma_{b-1} \gamma_{t-1}^{w-b+1} \\ &= \delta_{b-1} \delta_{b-1}^{b-2} \underline{\sigma_b \delta_{b-1}} \gamma_{b-2} \sigma_{b-1} \gamma_{t-1}^{w-b+1} \\ &= \delta_{b-1} \delta_{b-1}^{b-1} \gamma_{b-2} \sigma_{b-1} \gamma_{t-1}^{w-b+1} \quad \text{by destabilizing the unique } \sigma_b \text{ letter} \\ &= \delta_{b-1}^{b} \gamma_{b-1} \gamma_{t-1}^{w-b+1}.\end{aligned}$$

This braid word on b strands contains a full twist; thus, by Theorem 2.9, $i(\mathcal{K}(w,b,t,n)) = b = n+t$. Now suppose $n+t < b$. In particular, $n+t \leq b-1$. In this case, as in Eq. (17),

$$\begin{aligned}K(w,b,t,n) &= \delta_b^n \delta_b^t \gamma_{t-1}^{w-b-1} \\ &= \underline{\delta_b^{n+t}} \gamma_{t-1}^{w-b-1} \\ &= \underline{\delta_{b-1}^{n+t-1} \delta_b \gamma_{n+t-1}} \gamma_{t-1}^{w-b-1} \quad \text{by Proposition 4.8} \\ &= \delta_{b-1}^{n+t-1} \underline{\delta_b} \gamma_{n+t-1} \gamma_{t-1}^{w-b-1} \\ &= \delta_{b-1}^{n+t-1} \underline{\sigma_b \delta_{b-1}} \gamma_{n+t-1} \gamma_{t-1}^{w-b-1} \\ &= \delta_{b-1}^{n+t-1} \delta_{b-1} \gamma_{n+t-1} \gamma_{t-1}^{w-b-1} \quad \text{by destabilizing the unique } \sigma_b \text{ letter} \\ &= \delta_{b-1}^{n+t} \gamma_{n+t-1} \gamma_{t-1}^{w-b-1}.\end{aligned}$$

To simplify the right-hand side, we will need to (1) apply Proposition 4.8 to the $\delta_{b-\star}^{n+t}$ term, and then (2) destabilize the largest remaining Artin generator. We will need to repeat this process $(b-1)-(n+t)$ many times. Below, we write out explicitly what happens after applying steps (1) and (2) once, and then suppress the word for the remaining $(b-1)-(n+t)-1$ applications Proposition 4.8 and destabilization. We note: implicitly, we really are using that $n+t \leq b-1$.

$$\begin{aligned}
K(w,b,t,n) &= \delta_{b-1}^{n+t}\gamma_{n+t-1}\gamma_{t-1}^{w-b-1}\\
&= (\delta_{b-2}^{n+t-1}\delta_{b-1}\gamma_{n+t-1})\gamma_{n+t-1}\gamma_{t-1}^{w-b-1} \quad \text{by Proposition 4.8}\\
&= \delta_{b-2}^{n+t-1}\sigma_{b-1}\delta_{b-2}\gamma_{n+t-1}^{2}\gamma_{t-1}^{w-b-1} \quad \text{by the definition of } \delta_{b-1}\\
&= \delta_{b-2}^{n+t-1}\delta_{b-2}\gamma_{n+t-1}^{2}\gamma_{t-1}^{w-b-1} \quad \text{by destabilizing the } \sigma_{b-1} \text{ term}\\
&= \delta_{b-2}^{n+t}\ \gamma_{n+t-1}^{2}\ \gamma_{t-1}^{w-b-1}\\
&= \underline{\delta_{n+t}^{n+t}}\ \gamma_{n+t-1}^{b-1-(n+t)+1}\ \ \gamma_{t-1}^{w-b+1} \quad \text{after repeating this process another}\\
&\quad \times (b-1)-(n+t)-1 \text{ times}\\
&= \underline{\delta_{n+t-1}\delta_{n+t}^{n+t-1}\sigma_{n+t-1}}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1} \quad \text{by Proposition 4.9}\\
&= \delta_{n+t-1}\underline{\delta_{n+t}^{n+t-1}}\sigma_{n+t-1}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1}\\
&= \delta_{n+t-1}\underline{\delta_{n+t-1}^{n+t-2}\delta_{n+t}\gamma_{n+t-2}}\sigma_{n+t-1}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1} \quad \text{by Proposition 4.8}\\
&= \delta_{n+t-1}\delta_{n+t-1}^{n+t-2}\underline{\delta_{n+t}}\gamma_{n+t-2}\sigma_{n+t-1}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1}\\
&= \delta_{n+t-1}\delta_{n+t-1}^{n+t-2}\underline{\sigma_{n+t}\delta_{n+t-1}}\gamma_{n+t-2}\sigma_{n+t-1}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1}\\
&= \delta_{n+t-1}\delta_{n+t-1}^{n+t-2}\underline{\delta_{n+t-1}}\gamma_{n+t-2}\sigma_{n+t-1}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1}\\
&\quad \times \text{by destablizing the } \sigma_{n+t} \text{ term}\\
&= \delta_{n+t-1}^{n+t}\underline{\gamma_{n+t-2}\sigma_{n+t-1}}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1}\\
&= \delta_{n+t-1}^{n+t}\gamma_{n+t-1}\ \gamma_{n+t-1}^{b-n-t}\ \gamma_{t-1}^{w-b+1}\\
&= \delta_{n+t-1}^{n+t}\gamma_{n+t-1}^{b-n-t+1}\gamma_{t-1}^{w-b+1}.
\end{aligned}$$

This braid word in B_{n+t} contains a full twist; applying Theorem 2.9, we deduce the braid index is $n+t$. □

6. Future Directions

Our proof of Theorem 1.1 is self-contained and effective: we started with the definition of an n-bridge braid, and we produced a Markov equivalent positive braid containing a full twist. The algorithm we produce could be extended to all T-links. However, the computations are significantly more tedious, so we do not include them here. An interesting future direction would be to write a computer implementation of our algorithm for all T-links.

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