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The rotation number for the Schrödinger operator with α-norm almost periodic measures

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Abstract

We introduce a new class of almost periodic measures, and consider one-dimensional almost periodic Schrödinger operators with measure-valued potentials. For operators of this kind we introduce a rotation number in the spirit of Johnson and Moser.

Contents

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1 Introduction

1.1 Background

The landmark paper [20] by Johnson and Moser introduced the concept of the *rotation number* for one-dimensional continuum Schrödinger operators with almost periodic potentials. The idea is to consider the solutions ψ of the associated differential equation and to study the average winding per unit of the associated two-vector given by the solution and its derivative around the origin in the (ψ', ψ)-plane. This number of course depends on the value of spectral parameter appearing in the differential equation, and hence the resulting function is an object that may be associated with the operator in question.

In fact, it is more natural to associate the rotation number with an operator family, namely the one that is obtained by letting the potential run through the hull (i.e., the closure of the set of translates in the uniform topology) of the given almost periodic function. Indeed, the unique ergodicity of the translation flow on the hull is the very reason for the existence of the limit defining the rotation number, and it will then work uniformly for all elements of the hull.

The rotation number plays a key role in the spectral analysis of this family of almost periodic Schrödinger operators, as shown by Johnson and Moser. The spectrum of the operator, which again is uniform across the hull, can be identified with the points of non-constancy of the rotation number. The gaps of the spectrum (i.e., the connected components of the complement of the spectrum in $\mathbb{R}$) are therefore such that the rotation number must be constant on any one of them. As the rotation number is monotone and locally increasing at each point of the spectrum, this constant value associated with a gap may be used to label it uniquely. Moreover, the possible values that can be taken in gaps belong to a countable set that can be determined by considering the frequency module of the initial almost periodic function. This is a special instance of the so-called *gap-labelling theorem*, which holds in a suitable formulation in a more general setting; see, for example, [3–5, 10, 19].

Our goal in this paper is to discuss the concept of the rotation number for almost periodic Schrödinger operators in a more general setting, namely in the case where the potential term is given by a measure. This is partly motivated by the significant recent interest in the study of almost periodic measures from the perspective of aperiodic order; see, for example, [2, 16, 22, 24–27, 36, 38, 39] and references therein. The theory of aperiodic order has grown out of the desire to study mathematical models of quasicrystals, which are structures that lack translational symmetry, but are sufficiently ordered so that their diffraction is pure point, and hence of the type exhibited by crystals. The concept of almost periodicity has been identified in the papers mentioned above as being intimately connected to pure point diffraction, and hence this naturally motivates the study of almost periodic measures.

1.2 The setting

Let us now describe the setting we consider in more detail. We are interested in Schrödinger operators with measure-valued potentials given by

$$H_\mu := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu. \tag{1.1}$$

Here, μ is a translation bounded Radon measure on $\mathbb{R}$ that will be defined precisely in Sect. 3. On the one hand, μ may be interpreted as a form of small perturbation of the classical *Dirichlet form* on $\mathbb{R}$; see the monograph by Ma and Röckner [32] where the higher-dimensional space/manifold is considered. On the other hand, for the one-dimensional space $\mathbb{R}$, H_μ can be understood along the lines of the classical Sturm–Liouville theory by the so-called *quasi-derivative* as

$$A_\mu\psi(x) := \psi^{(1)}(x) - \int_0^x \psi(s)\mathrm{d}\mu(s), \qquad \psi \in W^1_{1,\mathrm{loc}}(\mathbb{R}).$$

The operator H_μ is self-adjoint once it is defined on a suitable space. The spectral analysis of H_μ has been discussed in [6, 13, 23, 34, 35, 37, 41]. If the measure possesses some sort of recurrence, such as almost periodicity, then we may introduce the rotation number for (1.1) in the spirit of Johnson and Moser [20]. Let $E \in \mathbb{R}$. We define the solution of

$$H_\mu\psi = E\psi \tag{1.2}$$

in the weak sense. We understand that the solution $\psi(x)$ is continuous and $\psi'(x) = \psi'(x+)$ is right-continuous; see Appendix A. By the choice of a suitable homotopy class to deal with discontinuities of $\psi'(x)$, we have a well-defined argument as

$$\theta(x) = \theta_E(x,\mu) := \arg\left(\psi'(x) + \sqrt{-1}\psi(x)\right); \tag{1.3}$$

see Sect. 4. If the ergodic limit

$$\lim_{x\to+\infty} \frac{\theta(x)-\theta(0)}{x} \tag{1.4}$$

exists, then we call it the rotation number of (1.2), and denote it by ρ_E. The concept of the rotation number is due to Poincaré and is used to obtain a classification of orientation preserving self-homeomorphisms of the circle; see the monograph by Katok and Hasselblatt [30]. Extensions of this concept have been considered by many authors in the literature; see, for example, [1, 8, 17, 28, 31, 42] and references therein. The main result in this paper is the following, whose proof is given in Sect. 6.

Theorem 1.1 *Assume that μ is a α-norm almost periodic measure, and the pure point part μ_{pp} is uniformly away from the* 0*-measure. Then* (1.2) *admits a well-defined rotation number.*

Remark 1.2 (a) For the terminology used in this theorem, see Definition 3.5 and (3.12).

(b) The condition that μ_{pp} is uniformly away from the 0-measure is technical. A natural question is whether we can require a weaker condition such as supp(μ_{pp}) being weakly uniformly discrete instead of this condition, or whether we can even get rid of this condition entirely. We presently do not know the answer to this question. Compared with our recent work [11], these two papers are related but do not contain each other. For example, if the measure μ may be explicitly expressed as the sum of its absolutely continuous part and its pure point part, then the result [11, Theorem 5.3.] showed that the concept of the rotation number can be introduced without the condition on μ_{pp}. But the paper [11] did not discuss the case when the measure-valued potential includes a singular continuous part while Theorem 1.1 does.

(c) It is by now well established that almost periodicity plays a central role in the study of pure point diffraction. Lenz, Spindeler, and Strungaru have shown that a translation bounded measure has pure point diffraction if and only if it is mean almost periodic [25]. In fact they go on to also show in [25] that Besicovitch and Weyl almost periodicity are intimately related to other natural and fundamental questions in diffraction theory. See [2, 16, 24] for related work and [26] for a very accessible introduction to their work. The notion of almost periodicity that we assume, namely α-norm almost periodicity, is stronger than the almost periodicity notions mentioned above, and hence the Schrödinger operators we consider may be placed in the pure point diffractive realm.

(d) When the potential is a Bohr almost periodic function, Johnson and Moser showed that the spectrum of the Schrödinger operator may be identified with the set of points of non-constancy of the rotation number [20]. Moreover, there is a resulting gap labelling theory that puts the values taken in constancy intervals in correspondence with the frequency module. This has been generalized to general dynamically defined potentials by Johnson [19], with the Schwartzman group of the base dynamics providing the set of values of the rotation number in spectral gaps; see also the survey [10] and the monographs [9, 21]. When the potential is measure-valued, it can be reasonably expected that analogous results will hold. However, some of the tools that are used in establishing the characterization of the spectrum in terms of the rotation number and the resulting gap labelling theory do not yet exist in this generality. We plan to address these issues in future work.

(e) Furthermore, the Fourier–Bohr coefficients of α-norm almost periodic measures can be well defined; see [27]. It is interesting to study the connection between the coefficients and the operator H_μ. We also plan to discuss this issue in future work.

Notations. Throughout this paper, let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. Denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$, by $\mathcal{M}(\mathbb{R})$ the space of all signed Radon measures, by $C(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$, by $C_c(\mathbb{R})$ the space of all continuous functions has a compact support on $\mathbb{R}$, and by $L_{1,\text{loc}}(\mathbb{R})$ the space of all locally Lebesgue integrable functions on $\mathbb{R}$. $\lfloor x \rfloor$ denotes the maximal integer less than $x \in \mathbb{R}$, and δ_Γ denotes the Dirac measure supported on a point set $\Gamma \subset \mathbb{R}$. λ is the Lebesgue measure on $\mathbb{R}$, $\mathrm{i} := \sqrt{-1}$ is the imaginary unit and e is the Euler number.

2 Almost periodicity

In this section we recall some fundamental results on almost periodicity from [11, 12] and discuss the concept of a uniform almost periodic family in a formulation suitable to our setting.

2.1 Almost periodic point

Let (Y, dist) be a complete metric space. We consider a $\mathbb{Z}$ action on Y by shifts and denote for $y \in Y$ and $\tau \in \mathbb{Z}$ the corresponding shifted element in Y by $y \cdot \tau$. This shift action satisfies the following conditions:

- group structure:

$$y \cdot 0 = y, \text{ and } y \cdot (\tau_1 + \tau_2) = (y \cdot \tau_1) \cdot \tau_2, \qquad \forall\, y \in Y,\ \tau_1,\ \tau_2 \in \mathbb{Z}; \tag{2.1}$$

- isometry:

$$\mathrm{dist}(y_1 \cdot \tau, y_2 \cdot \tau) = \mathrm{dist}(y_1, y_2), \qquad \forall\, \tau \in \mathbb{Z},\ y_i \in Y,\ i = 1, 2. \tag{2.2}$$

For $y \in Y$, denote the *orbit* of y by

$$\mathrm{Orb}(y) := \{y \cdot \tau : \tau \in \mathbb{Z}\} \subset Y,$$

and the *hull* of y by

$$\mathrm{H}(y) := \overline{\mathrm{Orb}(y)}^{(Y,\mathrm{dist})}.$$

Denote by $\mathbb{K}$ either $\mathbb{Z}$ or $\mathbb{R}$. A set $A \subset \mathbb{K}$ is said to be *relatively dense* (with window size ℓ) if there exists $\ell \in \mathbb{K}^+$ such that

$$A \cap [a, a + \ell] \neq \emptyset, \qquad \forall\, a \in \mathbb{K}.$$

Definition 2.1 [12] We say that $y \in Y$ is almost periodic if one of the following equivalent conditions holds:

(i) for any $\varepsilon > 0$, $\mathrm{P}(y, \varepsilon) := \{\tau \in \mathbb{Z} : \mathrm{dist}(y \cdot \tau, y) < \varepsilon\}$ is relatively dense in $\mathbb{Z}$;
(ii) $\mathrm{H}(y)$ is compact;
(iii) for any sequence $\{\tilde{\tau}_k\}_{k\in\mathbb{N}} \subset \mathbb{Z}$, one can extract a subsequence $\{\tau_k\} \subset \{\tilde{\tau}_k\}$ such that $\{y \cdot \tau_k\}$ is convergent in (Y, dist), i.e., $\mathrm{Orb}(y)$ is relatively compact.

Denote the subset of Y consisting of all almost periodic points by Y_{ap}. We have

Lemma 2.2 [11] *Y_{ap} is closed in (Y, dist). Thus, $(Y_{\mathrm{ap}}, \mathrm{dist})$ is a complete metric space.*

We equip $\mathrm{H}(y)$ with a group operation as follows. For $i = 1,\ 2$ and $y \in Y_{\mathrm{ap}}$, let $\{\tau_k^i\}_{k\in\mathbb{N}} \subset \mathbb{Z}$ be a sequence, and

$$y_i := \lim_{k\to+\infty} y \cdot \tau_k^i \in \mathrm{H}(y). \tag{2.3}$$

Then we define the group operation by

$$y_1 \times y_2 := \lim_{k\to+\infty} y \cdot (\tau_k^1 + \tau_k^2), \tag{2.4}$$

and the inverse of y_1 is given by

$$(y_1)^{-1} := \lim_{k\to+\infty} y \cdot (-\tau_k^1). \tag{2.5}$$

Both $y_1 \times y_2$ and $(y_1)^{-1}$ are well defined and independent of the choice of the sequences $\{\tau_k^i\}_{k\in\mathbb{N}}$ in (2.3). Denote the one-time shift $\tilde{y} \cdot 1$ by $T(\tilde{y})$, where $\tilde{y} = \lim_{k\to+\infty} y \cdot \tilde{\tau}_k \in \mathrm{H}(y)$. Then we have the following results.

Lemma 2.3 [11] *For $y \in Y_{\mathrm{ap}}$, one has*

(i) $\mathrm{H}(\tilde{y}) = \mathrm{H}(y)$ *for each* $\tilde{y} \in \mathrm{H}(y)$*;*
(ii) $(\mathrm{H}(y), \times, ^{-1})$ *is a compact abelian topological group;*
(iii) $T : \mathrm{H}(y) \to \mathrm{H}(y)$ *is uniquely ergodic with the Haar measure, denoted by* ν_y*, being the only invariant measure.*

In particular, we consider the case $Y := \ell^\infty(\mathbb{Z})$. The norm $\|\cdot\|_\infty$ induces the metric $\ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z}) \to \mathbb{R}_0^+$ by

$$\operatorname{dist}(V_1, V_2) := \|V_1 - V_2\|_\infty = \sup_{i\in\mathbb{Z}} |v_i^1 - v_i^2|, \tag{2.6}$$

where $V_k := \{v_i^k\}_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}),\ k = 1, 2$. The following is well known.

Lemma 2.4 $(\ell^\infty(\mathbb{Z}), \operatorname{dist})$ *is a complete metric space.*

For $V = \{v_i\}_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and $\tau \in \mathbb{Z}$, the shift $V \cdot \tau$ of V is defined by

$$V \cdot \tau := \{v_{i+\tau}\}_{i\in\mathbb{Z}}. \tag{2.7}$$

Obviously for $V_k \in \ell^\infty(\mathbb{Z}),\ k = 1, 2$, we have

$$\operatorname{dist}(V_1 \cdot \tau, V_2 \cdot \tau) = \operatorname{dist}(V_1, V_2), \qquad \forall\, \tau \in \mathbb{Z}. \tag{2.8}$$

This means that $(\ell^\infty(\mathbb{Z}), \operatorname{dist})$ satisfies the isometry condition (2.2). Then Definition 2.1 defines *almost periodic bi-sequences*. We denote by $\ell_{\mathrm{ap}}(\mathbb{Z})$ the space of all almost periodic bi-sequences. By Lemma 2.2, we know that $(\ell_{\mathrm{ap}}(\mathbb{Z}), \operatorname{dist})$ is a complete space. Moreover we have

Lemma 2.5 [14] *Let* $V_k := \{v_i^k\}_{i\in\mathbb{Z}} \in \ell_{\mathrm{ap}}(\mathbb{Z})$ *be real-valued,* $k \in [1, k_0] \cap \mathbb{N}$. *Then*

$$\max_{1\le k\le k_0} \{V_k\} := \Big\{ \max_{1\le k\le k_0} \{v_i^k\}\Big\}_{i\in\mathbb{Z}} \in \ell_{\mathrm{ap}}(\mathbb{Z}).$$

2.2 Uniformly almost periodic family

The concept of a uniformly almost periodic family has been introduced for continuous functions defined on $\mathbb{R}$; see [14]. In this subsection we extend this concept to our case.

Definition 2.6 We say that a family $\mathfrak{Y} \subset Y$ of almost periodic points is uniformly almost periodic if for any $\varepsilon > 0$, $\mathrm{P}(\mathfrak{Y}, \varepsilon) := \bigcap_{y\in\mathfrak{Y}} \mathrm{P}(y, \varepsilon)$ is relatively dense in $\mathbb{Z}$.

Bochner's *translation functions* are well defined for almost periodic functions. In our setting, we introduce the so-called translation bi-sequence for almost periodic points, that is, a discrete version of Bochner's translation functions. For $y \in Y$, denote the *translation bi-sequence* of y by

$$V_y := \big\{v_y(\tau)\big\}_{\tau\in\mathbb{Z}} := \big\{ \operatorname{dist}(y \cdot \tau, y)\big\}_{\tau\in\mathbb{Z}}. \tag{2.9}$$

Obviously the translation bi-sequence is real-valued.

Lemma 2.7 *Assume that* $y \in Y$ *and* $V_y \in \ell^\infty(\mathbb{Z})$. *For* $\varepsilon > 0$ *and* $\tau_1,\ \tau_2 \in \mathbb{Z}$, *one has*

(i) $\mathrm{P}(y, \varepsilon) = \{\tau \in \mathbb{Z} : v_y(\tau) < \varepsilon\}$;
(ii) $v_y(\tau) \ge 0$, $v_y(-\tau) = v_y(\tau)$, $v_y(0) = 0$;
(iii) $v_y(\tau_1 + \tau_2) \le v_y(\tau_1) + v_y(\tau_2)$;
(iv) $v_y(\tau) = v_{V_y}(\tau)$;
(v) $y \in Y_{\mathrm{ap}}$ *if and only if* $V_y \in \ell_{\mathrm{ap}}(\mathbb{Z})$.

Proof $\boxed{\text{(i), (ii), and (iii)}}$ They are obvious from (2.2). $\boxed{\text{(iv)}}$ By (ii) and (iii) we have

$$\begin{aligned} v_y(\tau_2) &\geq \sup_{\tau_1 \in \mathbb{Z}} \left| v_y(\tau_1 + \tau_2) - v_y(\tau_1) \right| = \| V_y \cdot \tau_2 - V_y \|_\infty = v_{V_y}(\tau_2) \\ &\geq |v_y(\tau_2) - v_y(0)| = v_y(\tau_2). \end{aligned}$$

Then ((iv) follows immediately.
$\boxed{\text{v)}}$: Due to (i) and (iv), we have $\mathrm{P}(y, \varepsilon) = \mathrm{P}(V_y, \varepsilon)$. Thus (v) is deduced. □

Remark 2.8 From the proof of Lemma 2.7 (iv), we may conclude that if a bi-sequence $V \in \ell^\infty(\mathbb{Z})$ satisfies (ii) and (iii) in Lemma 2.7, then the translation bi-sequence of V is itself.

For a family $\mathfrak{Y} \subset Y$, denote

$$v_{\mathfrak{Y}}(\tau) := \sup_{y \in \mathfrak{Y}} \left\{ v_y(\tau) \right\}.$$

The following result gives a characterization of a uniformly almost periodic family.

Lemma 2.9 *Assume that* $V_{\mathfrak{Y}} := \left\{ v_{\mathfrak{Y}}(\tau) \right\}_{\tau \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. *Then we have*

(i) $v_{\mathfrak{Y}}(\tau) = v_{V_{\mathfrak{Y}}}(\tau)$;
(ii) $\mathfrak{Y}$ *is a uniformly almost periodic family if and only if* $V_{\mathfrak{Y}} \in \ell_{\mathrm{ap}}(\mathbb{Z})$.

Proof $\boxed{\text{(i)}}$ For any $y \in \mathfrak{Y}$, $v_y(\tau)$ satisfies (ii) and (iii) in Lemma 2.7. Taking the sup operation, we obtain that $v_{\mathfrak{Y}}(\tau)$ satisfies (ii) and (iii) in Lemma 2.7 as well. Then (i) is deduced by Remark 2.8. $\boxed{\text{(ii)}}$ We first show the implication $\Longrightarrow$. Let $\tau \in \mathrm{P}(\mathfrak{Y}, \varepsilon/2)$. By Definition 2.6 and Lemma 2.7 (i), we have $v_y(\tau) < \varepsilon/2$ for all $y \in \mathfrak{Y}$. It follows that $v_{\mathfrak{Y}}(\tau) \leq \varepsilon/2 < \varepsilon$. From (i) and Lemma 2.7 (i), one has

$$\mathrm{P}(\mathfrak{Y}, \varepsilon/2) \subset \mathrm{P}(V_{\mathfrak{Y}}, \varepsilon).$$

Since $\mathrm{P}(\mathfrak{Y}, \varepsilon/2)$ is relatively dense, then $V_{\mathfrak{Y}} \in \ell_{\mathrm{ap}}(\mathbb{Z})$.

Next we show the implication $\Longleftarrow$. Let $\tau \in \mathrm{P}(V_{\mathfrak{Y}}, \varepsilon)$. Again by (i) and Lemma 2.7 (i), we have $v_{\mathfrak{Y}}(\tau) < \varepsilon$. This implies that $v_y(\tau) < \varepsilon$ for all $y \in \mathfrak{Y}$. Due to Lemma 2.7 (i), one has that $\tau \in \mathrm{P}(y, \varepsilon)$, for all $y \in \mathfrak{Y}$. Thus we have

$$\mathrm{P}(V_{\mathfrak{Y}}, \varepsilon) \subset \bigcap_{y \in \mathfrak{Y}} \mathrm{P}(y, \varepsilon) = \mathrm{P}(\mathfrak{Y}, \varepsilon),$$

finishing the proof by the relative denseness of $\mathrm{P}(V_{\mathfrak{Y}}, \varepsilon)$. □

In particular, we may consider a family consisting of a finite number of elements, which will be useful in the following section.

Lemma 2.10 *Suppose that* $y_k \in Y_{\mathrm{ap}}$, *and* $V_{y_k} \in \ell^\infty(\mathbb{Z})$, $k \in [1, k_0] \cap \mathbb{N}$. *Then* $\mathfrak{Y} := \{y_k : 1 \leq k \leq k_0\} \subset Y$ *is a uniformly almost periodic family.*

Proof Since $V_{y_k} \in \ell^\infty(\mathbb{Z})$, then we have

$$V_{\mathfrak{Y}} = \max_{1 \leq k \leq k_0} \left\{ V_{y_k} \right\} \in \ell^\infty(\mathbb{Z}).$$

By Lemma 2.7 (v), we know that $V_{y_k} \in \ell_{\mathrm{ap}}(\mathbb{Z})$. It follows from Lemma 2.5 that $V_{\mathfrak{Y}} \in \ell_{\mathrm{ap}}(\mathbb{Z})$. Thus the desired result is deduced by Lemma 2.9 (ii). □

3 α-Norm almost periodic measure

There are different hierarchies of almost periodic measures under different topologies, such as weak almost periodicity, strong almost periodicity, and norm almost periodicity, where the topology is stronger in order; see [39, p. 273 and Proposition 5.3.3] and [15, 22, 27, 38]. However, when we consider the Schrödinger operator with measure-valued potentials, all topologies above are not strong enough to introduce the so-called rotation number of (1.1). In this section, to overcome this difficulty, we will use the argument in Sect. 2 to introduce a new class of almost periodic measures on $\mathbb{R}$ which is the so-called α-norm almost periodic measure; see Definition 3.5. This new class is included in that of norm almost periodic measures, but includes some interesting cases. For example, when the measure is absolutely continuous with respect to the Lebesgue measure, the α-norm almost periodic measure can be exactly regarded as the Stepanov almost periodic function; see Lemma 3.6 and Example 3.11.

3.1 Definition

We collect some facts about Radon measures here, which are standard and can be found in [18, 33, 36, 40]. For any $\mu \in \mathcal{M}(\mathbb{R})$, there is a unique closed subset $A \subset \mathbb{R}$ such that $\mu(A^c) = 0$ and $\mu(A \cap U) > 0$ if $U \subset \mathbb{R}$ is open and $A \cap U \neq \emptyset$. The set A is called the *support* of μ, denoted by $\mathrm{supp}(\mu)$. An element $x \in \mathbb{R}$ is said to be an *atom* of μ if $\mu(\{x\}) \neq 0$. The set of all atoms of μ is denoted by $\mathrm{atom}(\mu)$. With λ the Lebesgue measure on $\mathbb{R}$, write

$$\begin{aligned}
\mathcal{M}_{\mathrm{ac}}(\mathbb{R}) &:= \{\mu \in \mathcal{M}(\mathbb{R}) : \mu \ll \lambda\},\\
\mathcal{M}_{\mathrm{sc}}(\mathbb{R}) &:= \{\mu \in \mathcal{M}(\mathbb{R}) : \mu \perp \lambda \text{ and } \mu \text{ is continuous}\},\\
\mathcal{M}_{\mathrm{pp}}(\mathbb{R}) &:= \{\mu \in \mathcal{M}(\mathbb{R}) : \mu \text{ is pure point}\}.
\end{aligned}$$

It is well known that $\mu \in \mathcal{M}(\mathbb{R})$ has a unique Lebesgue decomposition

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}},$$

where $\mu_\bullet \in \mathcal{M}_\bullet(\mathbb{R})$, $\bullet \in \{\mathrm{ac}, \mathrm{sc}, \mathrm{pp}\}$; see [33, Theorem 6.14].

Lemma 3.1 *For any* $\mu \in \mathcal{M}(\mathbb{R})$, $\mathrm{atom}(\mu) \subset \mathbb{R}$ *is at most countable.*

Let $K \subset \mathbb{R}$ be any bounded subset with a non-empty interior. We define

$$\|\mu\|_K := \sup_{x\in\mathbb{R}} |\mu|(x + K), \tag{3.1}$$

where $|\mu| := \mu_+ + \mu_-$ is the *total variation* of $\mu \in \mathcal{M}(\mathbb{R})$. Due to [18, Theorem 14.11], we obtain

$$|\mu_1 + \mu_2| \le |\mu_1| + |\mu_2|, \qquad \mu_i \in \mathcal{M}(\mathbb{R}),\ i = 1, 2. \tag{3.2}$$

Denote $I := (-\frac{1}{2}, \frac{1}{2})$. Then there exist two constants $0 < c_1 \le c_2 < \infty$ such that

$$c_1\|\mu\|_K \le \|\mu\|_I \le c_2\|\mu\|_K.$$

This implies that $\|\mu\|_K$ and $\|\mu\|_I$ are equivalent [39, Remark 5.3.2]. The reason why we make use of the special interval I is given in [38, Remark 4.3].

Definition 3.2 [37, p. 6] [36, Definition 4.9.17.] [27, Definition 1.1.] A measure $\mu \in \mathcal{M}(\mathbb{R})$ is said to be translation bounded if $\|\mu\|_I < +\infty$.

Denote by $\mathcal{M}^\infty(\mathbb{R})$ the space of all translation bounded signed Radon measures on $\mathbb{R}$. For any $t \in \mathbb{R}$, we define the shift action T_t on $\mathbb{R}$ by $T_t x := x - t,\ x \in \mathbb{R}$. For any function f on $\mathbb{R}$, the shift $T_t f$ of f is defined by

$$(T_t f)(x) := f(T_{-t}x) = f(x+t).$$

For any $\mu \in \mathcal{M}^\infty(\mathbb{R})$, the shift $T_t\mu$ of μ is defined by

$$(T_t\mu)(f) := \mu(T_{-t}f) = \int_{\mathbb{R}} f(x-t)\,\mathrm{d}\mu(x), \qquad f \in C_c(\mathbb{R}). \tag{3.3}$$

Lemma 3.3 [38] *Let* $\mu \in \mathcal{M}^\infty(\mathbb{R})$*,* $t \in \mathbb{R}$ *and* $\bullet \in \{\mathrm{ac}, \mathrm{sc}, \mathrm{pp}\}$*.*

(i) $\|T_t\mu\|_I = \|\mu\|_I$.
(ii) $(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)$ *is a Banach space.*
(iii) $(T_t\mu)_\bullet = T_t(\mu_\bullet)$.
(iv) $T_t\delta_\Gamma = \delta_{\Gamma - t}$.
(v) $T_t\mu(S) = \mu(T_{-t}S) = \mu(S+t)$*, where* $S \in \mathcal{B}(\mathbb{R})$.
(vi) $\mathrm{supp}(T_t\mu) = T_t\,\mathrm{supp}(\mu)$.
(vii) $\|\mu_\bullet\|_I \le \|\mu\|_I \le \|\mu_{\mathrm{ac}}\|_I + \|\mu_{\mathrm{sc}}\|_I + \|\mu_{\mathrm{pp}}\|_I$.
(viii) $\mathcal{M}^\infty(\mathbb{R}) = \mathcal{M}^\infty_{\mathrm{ac}}(\mathbb{R}) \oplus \mathcal{M}^\infty_{\mathrm{sc}}(\mathbb{R}) \oplus \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R})$*, where* $\mathcal{M}^\infty_\bullet(\mathbb{R}) = \mathcal{M}_\bullet(\mathbb{R}) \cap \mathcal{M}^\infty(\mathbb{R})$.
(ix) $\mathcal{M}^\infty_\bullet(\mathbb{R}) \subset \mathcal{M}^\infty(\mathbb{R})$ *is closed, and hence* $(\mathcal{M}^\infty_\bullet(\mathbb{R}), \|\cdot\|_I)$ *is a Banach space.*

For $f \in C(\mathbb{R})$, we can use an $\mathbb{R}$ translation action or a $\mathbb{Z}$ translation action on $C(\mathbb{R})$ to define the so-called Bohr almost periodic function. Both approaches are equivalent because of uniform continuity of f. However this is different from the case $\mu \in \mathcal{M}^\infty(\mathbb{R})$. When considering an $\mathbb{R}$ translation action on $\mathcal{M}^\infty(\mathbb{R})$, we have the following known concept.

Definition 3.4 [38, Definition 2.4] A measure $\mu \in \mathcal{M}^\infty(\mathbb{R})$ is said to be norm almost periodic if, for any $\varepsilon > 0$,

$$\mathrm{P}_{\mathbb{R}}(\mu, \varepsilon) := \{t \in \mathbb{R} :\ \|T_t\mu - \mu\|_I < \varepsilon\} \tag{3.4}$$

is relatively dense in $\mathbb{R}$. The space of all norm almost periodic measures is denoted by $\mathcal{M}^\infty_{\mathrm{nap}}(\mathbb{R})$.

Let $\alpha > 0$ be a fixed step length. Now we consider a $\mathbb{Z}$ action on $\mathcal{M}^\infty(\mathbb{R})$ by shifts and denote for $\mu \in \mathcal{M}^\infty(\mathbb{R})$ and $\tau \in \mathbb{Z}$ the corresponding shifted element in $\mathcal{M}^\infty(\mathbb{R})$ by

$$\mu \cdot \tau := T_{\alpha\tau}\mu. \tag{3.5}$$

By Lemma 3.3 (i) and (ii), we know that the shift action (3.5) satisfies the isometry condition (2.2), and $(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)$ is a complete space. Then we are able to apply the argument on almost periodicity in Sect. 2 to introduce a new concept of almost periodic measures.

Definition 3.5 The elements of $\left(\mathcal{M}^\infty(\mathbb{R})\right)_{\mathrm{ap}}$ under the shift action (3.5) are called α-norm almost periodic measures.

We replace the notation $\left(\mathcal{M}^\infty(\mathbb{R})\right)_{\mathrm{ap}}$ by $\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$. The relation and difference between $\mathcal{M}^\infty_{\mathrm{nap}}(\mathbb{R})$ and $\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ are shown in the following lemmas.

Lemma 3.6 *For any* $\alpha > 0$*, one has* $\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R}) \subset \mathcal{M}^\infty_{\mathrm{nap}}(\mathbb{R})$.

Proof Assume that $\mu \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$. Due to Definition 2.1, we know that for any $\varepsilon > 0$, there exists $\ell := \ell_\varepsilon \in \mathbb{N}$ such that

$$\mathrm{P}(\mu, \varepsilon) \cap [n, n+\ell] \neq \emptyset, \qquad \forall\, n \in \mathbb{Z}.$$

By construction (3.5), we have

$$\mathrm{P}_{\mathbb{R}}(\mu, \varepsilon) \cap [\alpha n, \alpha n + \alpha\ell] \neq \emptyset, \qquad \forall\, n \in \mathbb{Z}. \tag{3.6}$$

Denote $\tilde{\ell} := \alpha(\ell + 1)$. We assert that

$$\mathrm{P}_{\mathbb{R}}(\mu, \varepsilon) \cap [a, a+\tilde{\ell}] \neq \emptyset, \qquad \forall\, a \in \mathbb{R}.$$

Indeed, we see that

$$\left[\alpha\left(\left\lfloor \frac{a}{\alpha} \right\rfloor + 1\right), \alpha\left(\left\lfloor \frac{a}{\alpha} \right\rfloor + \ell + 1\right)\right] \subset [a, a+\tilde{\ell}].$$

Then (3.6) yields the assertion. This implies that $\mu \in \mathcal{M}^\infty_{\text{nap}}(\mathbb{R})$. □

Lemma 3.7 $\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ *is closed under the operation of addition.*

Proof Assume that $\mu_i \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$, $i = 1, 2$. For any $\varepsilon > 0$, both $\mathrm{P}(\mu_1, \varepsilon/2)$ and $\mathrm{P}(\mu_2, \varepsilon/2)$ are relatively dense in $\mathbb{Z}$. Due to Lemma 2.10, we know that

$$\mathrm{P}(\mu_1, \varepsilon/2) \cap \mathrm{P}(\mu_2, \varepsilon/2) \text{ is relatively dense in } \mathbb{Z}. \tag{3.7}$$

For any $\tau \in \mathbb{Z}$, it follows from (3.2) that

$$\begin{aligned} \|(\mu_1+\mu_2)\cdot\tau - (\mu_1+\mu_2)\|_I &= \|(\mu_1\cdot\tau - \mu_1) + (\mu_2\cdot\tau - \mu_2)\|_I \\ &\leq \|\mu_1\cdot\tau - \mu_1\|_I + \|\mu_2\cdot\tau - \mu_2\|_I. \end{aligned}$$

Combining this with (3.7), we have that $\mathrm{P}(\mu_1+\mu_2, \varepsilon)$ is relatively dense in $\mathbb{Z}$. □

Remark 3.8 $\mathcal{M}^\infty_{\text{nap}}(\mathbb{R})$ is not closed under the operation of addition. For instance, $\delta_{\mathbb{Z}} \in \mathcal{M}^\infty_{\text{nap}}(\mathbb{R})$, and $\delta_{\sqrt{2}\mathbb{Z}} \in \mathcal{M}^\infty_{\text{nap}}(\mathbb{R})$, but $\delta_{\mathbb{Z}} + \delta_{\sqrt{2}\mathbb{Z}} \notin \mathcal{M}^\infty_{\text{nap}}(\mathbb{R})$. This is one of differences between $\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ and $\mathcal{M}^\infty_{\text{nap}}(\mathbb{R})$.

By Lemma 2.2 and Lemma 3.7, we have

Lemma 3.9 $\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R}) \subset \mathcal{M}^\infty(\mathbb{R})$ *is closed. Thus,* $(\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R}), \|\cdot\|_I)$ *is a Banach space.*

Another difference between $\mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ and $\mathcal{M}^\infty_{\text{nap}}(\mathbb{R})$ is stated as follows.

Lemma 3.10 $\mu \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ *if and only if* $\mu_\bullet \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$*, where* $\bullet \in \{\text{ac}, \text{sc}, \text{pp}\}$.

Proof The implication $\Longrightarrow$ is obvious by Lemma 3.3 (iii) and (vii). Now we prove the implication $\Longleftarrow$. For any $\varepsilon > 0$, due to Lemma 2.10, we know that

$$\mathrm{P}(\mu_{\text{ac}}, \varepsilon/3) \cap \mathrm{P}(\mu_{\text{sc}}, \varepsilon/3) \cap \mathrm{P}(\mu_{\text{pp}}, \varepsilon/3) \text{ is relatively dense in } \mathbb{Z}. \tag{3.8}$$

For any $\tau \in \mathbb{Z}$, by Lemma 3.3 (iii) and (vii) again, we have

$$\begin{aligned} &\|\mu\cdot\tau - \mu\|_I \\ &\leq \|(\mu\cdot\tau - \mu)_{\text{ac}}\|_I + \|(\mu\cdot\tau - \mu)_{\text{sc}}\|_I + \|(\mu\cdot\tau - \mu)_{\text{pp}}\|_I \\ &= \|\mu_{\text{ac}}\cdot\tau - \mu_{\text{ac}}\|_I + \|\mu_{\text{sc}}\cdot\tau - \mu_{\text{sc}}\|_I + \|\mu_{\text{pp}}\cdot\tau - \mu_{\text{pp}}\|_I. \end{aligned}$$

Combining this with (3.8), we conclude that $\mathrm{P}(\mu, \varepsilon)$ is relatively dense in $\mathbb{Z}$. □

Example 3.11 Let $q \in L_{1,\mathrm{loc}}(\mathbb{R})$ be a Stepanov almost periodic function, and $f_\alpha : \mathbb{R} \to \mathbb{R}$ is a non-decreasing and continuous function such that for any $n \in \mathbb{Z}$, $f_\alpha|_{[\alpha n, \alpha n+\alpha]} - f_\alpha(\alpha n)$ is the Devil's staircase. Then $\mu_{f_\alpha} \in \mathcal{M}^\infty_{\mathrm{sc}}(\mathbb{R})$, where f_α is the distribution function of μ_{f_α}. Due to Lemma 3.10, we have

$$\mu := q.\lambda + \mu_{f_\alpha} + \delta_{\alpha\mathbb{Z}} \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R}).$$

Indeed, it is obvious that $\mu_{f_\alpha} \in \mathcal{M}^\infty_{\mathrm{sc}}(\mathbb{R}) \cap \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ and $\delta_{\alpha\mathbb{Z}} \in \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R}) \cap \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$. By the definition of Stepanov almost periodic functions, for any $\varepsilon > 0$, $\mathrm{P}_\mathbb{R}(q, \varepsilon)$ is relatively dense in $\mathbb{R}$. Along with the idea of Lemma 2.10, $\{\alpha\tau : \tau \in \mathbb{Z}\} \cap \mathrm{P}_\mathbb{R}(q, \varepsilon)$ is relatively dense in $\mathbb{R}$. This implies that $q.\lambda \in \mathcal{M}^\infty_{\mathrm{ac}}(\mathbb{R}) \cap \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$. Note that the Stepanov norm $\|\cdot\|_{S^1}$ possesses uniform continuity, but $\|\cdot\|_I$ does not. For more details, see [7, 43].

3.2 The pure point case

For $\mu \in \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R})$, we assume by Lemma 3.1 that $\mathrm{atom}(\mu) := \{x_i\}_{i\in\mathbb{Z}}$. Then μ can be written as

$$\mu := \sum_{x\in \mathrm{atom}(\mu)} \mu(\{x\})\delta_x = \sum_{i\in\mathbb{Z}} \mu(\{x_i\})\delta_{x_i}. \tag{3.9}$$

We introduce a new norm on the space $\mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R})$ by

$$\|\mu\|_\infty := \sup_{x\in\mathbb{R}} \big\{|\mu(\{x\})|\big\}, \quad \text{for } \mu \in \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R}). \tag{3.10}$$

Obviously we have

$$\|\mu\|_\infty \le \|\mu\|_I, \quad \text{for } \mu \in \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R}); \tag{3.11}$$

see [39, Lemma 5.3.5].

A subset $\Lambda \subset \mathbb{R}$ is called *uniformly discrete* if there exists an open neighborhood U of $0 \in \mathbb{R}$ such that $(x+U) \cap (y+U) = \emptyset$ for all $x, y \in \Lambda$ with $x \ne y$. A subset $\Lambda \subset \mathbb{R}$ is called *weakly uniformly discrete* if for any compact subset $K \subset \mathbb{R}$ and any $x \in \mathbb{R}$, one has

$$\#(\Lambda \cap (x+K)) \le c_K,$$

where $\#(\cdot)$ is the function counting the number of elements in a set and c_K is a constant that depends on K. Obviously, $\Lambda = \mathbb{Z} \cup \{n + \frac{1}{n} : n \in \mathbb{Z}\}$ is weakly uniformly discrete but not uniformly discrete. We say that $\mu \in \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R})$ is *uniformly away from the* 0*-measure* if

$$\inf_{i\in\mathbb{Z}} \big\{|\mu(\{x_i\})|\big\} > 0. \tag{3.12}$$

The space of all above measures is denoted by $\mathcal{M}^\infty_{\mathrm{upp}}(\mathbb{R})$. Note that $\mathcal{M}^\infty_{\mathrm{upp}}(\mathbb{R})$ is not a closed subset in $(\mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R}), \|\cdot\|_I)$, because $\frac{1}{n}\delta_\mathbb{Z}$ converges to the 0-measure.

Lemma 3.12 *Let* $\mu \in \mathcal{M}^\infty_{\mathrm{upp}}(\mathbb{R}) \cap \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ *be fixed.*

(i) $\mathrm{supp}(\mu)$ *is weakly uniformly discrete, and thus totally ordered.*
(ii) *There exist a finite set* $F_\mu := \{x_i : 1 \le i \le k_\mu\}$ *and* $t_\mu \in \mathbb{N}$ *such that* $\mathrm{supp}(\mu) = \alpha t_\mu \mathbb{Z} + F_\mu$.
(iii) $\big\{\mu(\{x_i\})\big\}_{i\in\mathbb{Z}} \in \ell_{\mathrm{ap}}(\mathbb{Z})$.

Proof (i) By (3.2) and (3.12), we see that $\mathrm{atom}(\mu)$ is weakly uniformly discrete. It follows from the definition of the support of measures that $\mathrm{supp}(\mu) = \mathrm{atom}(\mu)$. (ii) Let $\mathrm{supp}(\mu) := \{x_i\}_{i\in\mathbb{Z}}$ and $\varepsilon_0 := \frac{1}{2}\inf_{i\in\mathbb{Z}}\left\{|\mu(\{x_i\})|\right\} > 0$. Since $\mu \in \mathcal{M}^{\infty}_{\alpha\text{-nap}}(\mathbb{R})$, $\mathrm{P}(\mu, \varepsilon_0)$ is relatively dense in $\mathbb{Z}$. By Lemma 3.3 (iv), (3.9), (3.10), and (3.11), for any $y \in \mathbb{R}$ and $\tau \in \mathrm{P}(\mu, \varepsilon_0)$, we have

$$\left|\sum_{i\in\mathbb{Z}} \mu(\{x_i\})\delta_{x_i-\alpha\tau}(\{y\}) - \sum_{i\in\mathbb{Z}} \mu(\{x_i\})\delta_{x_i}(\{y\})\right| \le \|T_{\alpha\tau}\mu - \mu\|_\infty < \varepsilon_0. \tag{3.13}$$

Let $y \in \mathrm{supp}(\mu)$. Then there exists $i_y \in \mathbb{Z}$ such that $x_{i_y} = y$. It follows from (3.13) that

$$\left|\sum_{i\in\mathbb{Z}} \mu(\{x_i\})\delta_{x_i-\alpha\tau}(\{y\}) - \mu(\{x_{i_y}\})\right| < \varepsilon_0.$$

This implies that $y \in \mathrm{supp}(\mu) - \alpha\tau$, and therefore $\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu) - \alpha\tau$. Conversely, using a similar argument, we have $\mathrm{supp}(\mu) - \alpha\tau \subset \mathrm{supp}(\mu)$. Denote $t_\mu := \min\limits_{\mathrm{P}(\mu,\varepsilon_0)\cap\mathbb{N}} \tau$. Hence

$$\mathrm{supp}(\mu) - \alpha t_\mu = \mathrm{supp}(\mu). \tag{3.14}$$

Due to (i), we may assume that $x_i < x_{i+1}$, for all $i \in \mathbb{Z}$. By (3.14), there exists $k_\mu \in \mathbb{N}$ such that

$$x_{i+k_\mu} - \alpha t_\mu = x_i. \tag{3.15}$$

Denote $F_\mu = \{x_1, x_2, \cdots, x_{k_\mu}\}$. The desired result is obtained.

(iii) For any $\varepsilon > 0$ and $t_\mu \in \mathbb{N}$ in (ii), due to Lemma 2.10, $\Lambda_\varepsilon := \mathrm{P}(\mu, \varepsilon) \cap t_\mu\mathbb{Z}$ is relatively dense in $\mathbb{Z}$. By (3.10) and (3.11), we have

$$\sup_{x\in\mathbb{R}}\left\{|(T_{\alpha t_\mu\tau}\mu - \mu)(\{x\})|\right\} < \varepsilon, \quad \text{for all } \tau \in t_\mu^{-1}\Lambda_\varepsilon \subset \mathbb{Z}. \tag{3.16}$$

Note that

$$\begin{aligned} T_{\alpha t_\mu\tau}\mu &= \sum_{i\in\mathbb{Z}} \mu(\{x_i\})T_{\alpha t_\mu\tau}\delta_{x_i} = \sum_{i\in\mathbb{Z}} \mu(\{x_i\})\delta_{x_i-\alpha t_\mu\tau} \\ &= \sum_{i\in\mathbb{Z}} \mu(\{x_i\})\delta_{x_{i-k_\mu\tau}} = \sum_{i\in\mathbb{Z}} \mu(\{x_{i+k_\mu\tau}\})\delta_{x_i}, \end{aligned}$$

where (3.9) and (3.15) are used. Then we have

$$\begin{aligned} &\sup_{x\in\mathbb{R}}\left\{|(T_{\alpha t_\mu\tau}\mu - \mu)(\{x\})|\right\} \\ &= \sup_{x\in\mathbb{R}}\left\{\left|\left(\sum_{i\in\mathbb{Z}}(\mu(\{x_{i+k_\mu\tau}\}) - \mu(\{x_i\}))\delta_{x_i}\right)(\{x\})\right|\right\} \\ &= \sup_{i\in\mathbb{Z}}\left\{|\mu(\{x_{i+k_\mu\tau}\}) - \mu(\{x_i\})|\right\}. \end{aligned}$$

Denote $V_\mu := \left\{\mu(\{x_i\})\right\}_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. It follows from (3.16) that

$$\|V_\mu \cdot k_\mu\tau - V_\mu\|_\infty = \sup_{i\in\mathbb{Z}}\left\{|\mu(\{x_{i+k_\mu\tau}\}) - \mu(\{x_i\})|\right\} < \varepsilon, \quad \text{for all } \tau \in t_\mu^{-1}\Lambda_\varepsilon.$$

This implies that $\frac{k_\mu}{t_\mu}\Lambda_\varepsilon \subset \mathrm{P}(V_\mu, \varepsilon)$. Since Λ_ε is relatively dense, we have the desired result. □

Remark 3.13 Conversely, let $\mu \in \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R})$ with the expression (3.9), and $\alpha > 0$. Suppose that there exist a finite set $F := \{x_1, x_2, \cdots, x_{k_0}\} \subset \mathbb{R}$ and $t_0 \in \mathbb{N}$ such that $\{x_i\}_{i\in\mathbb{Z}} = \alpha t_0\mathbb{Z} + F$, and $V_\mu = \left\{\mu(\{x_i\})\right\}_{i\in\mathbb{Z}} \in \ell_{\mathrm{ap}}(\mathbb{Z})$. We claim that $\mu \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$; see Appendix C. Note that in the proof of this implication, we do not require condition (3.12). A natural question is whether we can remove condition (3.12) in the proof of Lemma 3.12. We leave it to the reader.

3.3 The sub-hull

Applying the argument in Sect. 2 to $\mu \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ under the shift action (3.5), we see that

$$\mathrm{H}(\mu) = \overline{\{\mu \cdot \tau : \tau \in \mathbb{Z}\}}^{(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)}$$

is compact in $(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)$, where $\mu \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$. Then $\mathrm{H}(\mu)$ can be equipped with a group operation. The result in this subsection plays a fundamental role in the proof of our main theorem. We use the following notation,

$$\mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R}) := \{\mu \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R}) : \text{ the pure point part of } \mu \text{ belongs to } \mathcal{M}^\infty_{\mathrm{upp}}(\mathbb{R})\}.$$

Due to Lemma 3.10, we know that $\mu_{\mathrm{pp}} \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$ if $\mu \in \mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R})$. Thus the following notation is well defined.

Definition 3.14 For $\mu \in \mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R})$, the sub-hull of μ is defined by

$$\mathrm{H_s}(\mu) := \overline{\{\mu \cdot t_\mu\tau : \tau \in \mathbb{Z}\}}^{(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)} \subset \mathrm{H}(\mu),$$

where $t_\mu \in \mathbb{N}$ is introduced in Lemma 3.12 (ii).

Lemma 3.15 *Let $\mu \in \mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R})$ and $\tilde{\mu} \in \mathrm{H_s}(\mu)$. Then we have*

(i) $\mathrm{H_s}(\mu)$ *is closed, and hence compact in* $(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)$;
(ii) $\mathrm{H_s}(\tilde{\mu}) = \mathrm{H_s}(\mu)$;
(iii) $\|\tilde{\mu}\|_I = \|\mu\|_I$;
(iv) $\mathrm{supp}(\tilde{\mu}_{\mathrm{pp}}) = \mathrm{supp}(\mu_{\mathrm{pp}})$;
(v) $\tilde{\mu} \in \mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R})$.

Proof (i) This is obvious by Definition 2.1 (ii) and Definition 3.14. (ii) This is obvious by Lemma 2.3 (i). (iii) For $\tilde{\mu} \in \mathrm{H_s}(\mu)$, there exists a sequence $\{\tau_k\}_{k\in\mathbb{Z}} \subset \mathbb{Z}$ such that $\lim_{k\to+\infty} \|\mu \cdot t_\mu\tau_k - \tilde{\mu}\|_I = 0$. Due to (3.1) and (3.5), we have $\|\mu \cdot t_\mu\tau_k\|_I = \|\mu\|_I$ and $\left|\|\tilde{\mu}\|_I - \|\mu \cdot t_\mu\tau_k\|_I\right| \leq \|\tilde{\mu} - \mu \cdot t_\mu\tau_k\|_I$ for all $k \in \mathbb{Z}$. Then we have the desired result. (iv) Using the notation in (iii), we have

$$\lim_{k\to+\infty} \|T_{\alpha t_\mu\tau_k}\mu_{\mathrm{pp}} - \tilde{\mu}_{\mathrm{pp}}\|_I = 0,$$

where Lemma 3.3 (iii) and (vii) are used. Combing this with (3.11), we obtain

$$\lim_{k\to+\infty} \|T_{\alpha t_\mu\tau_k}\mu_{\mathrm{pp}} - \tilde{\mu}_{\mathrm{pp}}\|_\infty = 0. \tag{3.17}$$

Note that

$$\mathrm{supp}(T_{\alpha t_\mu \tau_k}\mu_{\mathrm{pp}}) = T_{\alpha t_\mu \tau_k}\,\mathrm{supp}(\mu_{\mathrm{pp}}) = \mathrm{supp}(\mu_{\mathrm{pp}}), \quad \text{for all } k \in \mathbb{Z}, \tag{3.18}$$

where Lemma 3.12 (ii) is used. By (3.18), we have

$$\begin{aligned} \inf_{x\in \mathrm{supp}(\mu_{\mathrm{pp}})} \left\{ |T_{\alpha t_\mu \tau_k}\mu_{\mathrm{pp}}(\{x\})| \right\} &= \inf_{x\in \mathrm{supp}(T_{\alpha t_\mu \tau_k}\mu_{\mathrm{pp}})} \left\{ |T_{\alpha t_\mu \tau_k}\mu_{\mathrm{pp}}(\{x\})| \right\} \\ &= \inf_{x\in \mathrm{supp}(\mu_{\mathrm{pp}})} \left\{ |\mu_{\mathrm{pp}}(\{x\})| \right\} > 0. \end{aligned}$$

It follows from (3.17) that $\mathrm{supp}(\tilde{\mu}_{\mathrm{pp}}) = \mathrm{supp}(\mu_{\mathrm{pp}})$. $\boxed{\text{(v)}}$ Due to Lemma 3.9, we obtain that $\tilde{\mu} \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$. Denote $\Gamma := \mathrm{supp}(\tilde{\mu}_{\mathrm{pp}})$. It follows from (3.17) that

$$\lim_{k\to+\infty} T_{\alpha t_\mu \tau_k}\mu_{\mathrm{pp}}(\{x\}) = \tilde{\mu}_{\mathrm{pp}}(\{x\}), \quad \text{uniformly for all } x \in \Gamma.$$

This implies that $\inf_{x\in\Gamma}\left\{|\tilde{\mu}_{\mathrm{pp}}(\{x\})|\right\} > \frac{1}{2}\inf_{x\in\Gamma}\left\{|\mu_{\mathrm{pp}}(\{x\})|\right\} > 0$, where (iii) and (3.12) are used. Thus $\tilde{\mu}_{\mathrm{pp}} \in \mathcal{M}^\infty_{\mathrm{upp}}(\mathbb{R})$. □

Similarly as for $\mathrm{H}(\mu)$, we equip $\mathrm{H_s}(\mu)$ with a group operation as follows. Let

$$\mu_i = \lim_{k\to+\infty} \mu \cdot t_\mu \tau_k^i \in \mathrm{H_s}(\mu), \qquad i = 1,\ 2. \tag{3.19}$$

We define the group operation by

$$\mu_1 \times \mu_2 := \lim_{k\to+\infty} \mu \cdot t_\mu(\tau_k^1 + \tau_k^2), \tag{3.20}$$

and the inverse of μ_1 is given by

$$(\mu_1)^{-1} := \lim_{k\to+\infty} \mu \cdot (-t_\mu \tau_k^1). \tag{3.21}$$

All the limits above are taken in the sense of $\|\cdot\|_I$. It is not difficult to check that both $\mu_1 \times \mu_2$ and $(\mu_1)^{-1}$ are well defined and independent of the choice of the sequences $\{\tau_k^i\}_{k\in\mathbb{N}}$. Then due to Lemma 2.3, we have

Lemma 3.16 *Let $\mu \in \mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R})$. Then $(\mathrm{H_s}(\mu), \times, ^{-1})$ is a compact abelian group with the Haar measure, denoted by $\nu = \nu_{\mathrm{H_s}(\mu)}$, being the unique invariant measure of $\mathrm{H_s}(\mu)$ under the shift $\mu \mapsto \mu \cdot t_\mu$.*

Remark 3.17 Recall the corresponding part in [20], Johnson and Moser used a continuous-time action $f(x) \to f(x+t)$ to equip the hull of Bohr almost periodic functions with the structure of a compact Abelian group. For our situation, we use a discrete-time action to do this. In fact, the existence of rotation numbers in [20] may be also established via the discrete-time approach, because the unique ergodicity of the Haar measure is preserved. For more details, see [43].

4 Argument

Denote by $\mathrm{M}(2,2)$ the space of all 2×2 real matrices. Let J_2 be the standard symplectic matrix

$$J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A matrix $D \in \mathrm{M}(2,2)$ is called to be *symplectic*, if $D^T J_2 D = J_2$, where D^T is the transpose matrix of D. It is easy to see that the collection of all 2×2 real symplectic matrices forms a group with respect to the matrix multiplication. Denote this group by $\mathrm{Sp}(2, \mathbb{R})$. We have

$$\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) = \{D \in \mathrm{M}(2,2) : \det(D) = 1\}.$$

From now on, let $\mu \in \mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R})$ be fixed. By Lemma 3.12 (ii), we denote $\mathrm{supp}(\mu_{\mathrm{pp}}) := \{x_i\}_{i\in\mathbb{Z}}$. Without loss of generality, we assume that 0 is one of the atoms of μ_{pp} and

$$x_0 := 0. \tag{4.1}$$

Recall the discussion in Appendix A. It follows from Lemma A.4 (iii) and Lemma A.3 (iv) that the planar system (A.4) is equivalent to

$$\begin{cases} \mathrm{d}\begin{pmatrix} \psi'(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}\mu(x) - E\mathrm{d}x \\ \mathrm{d}x & 0 \end{pmatrix} \begin{pmatrix} \psi'(x) \\ \psi(x) \end{pmatrix}, & x \in \mathbb{R} \setminus \{x_i\}_{i\in\mathbb{Z}}, \\ \begin{pmatrix} \psi'(x+) \\ \psi(x+) \end{pmatrix} = \begin{pmatrix} 1 & \mu(\{x\}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi'(x-) \\ \psi(x-) \end{pmatrix}, & x \in \{x_i\}_{i\in\mathbb{Z}}. \end{cases} \tag{4.2}$$

For definiteness, the solution of (4.2) is understood to be right-continuous with respect to $x \in \mathbb{R}$, that is, $(\psi'(x), \psi(x))^T \equiv (\psi'(x+), \psi(x+))^T$. Due to Lemma A.3 and Lemma A.4, $\psi'(x)$ and $\psi(x)$ are well defined on $\mathbb{R}$. By Lemma A.5, we have

Lemma 4.1 *For any $(a, b)^T \in \mathbb{R}^2$, there exists a unique global solution $(\psi'(x), \psi(x))^T$ of (4.2) with the initial value $(\psi'(0), \psi(0))^T = (a, b)^T$.*

Let $\Psi(x) := \Psi_E(x; \mu)$ be the fundamental matrix solution of (4.2) with the initial value $\Psi(0) = I_2$. Then we have the following result.

Lemma 4.2 *For any $x \in \mathbb{R}$, $\Psi(x) \in \mathrm{Sp}(2, \mathbb{R})$.*

Proof We only check the case $x \in (x_0, x_1]$. For the general case, we may obtain the result by induction. Consider the system (4.2) on (x_0, x_1). We have

$$\mathrm{d}\Psi(x) = \begin{pmatrix} 0 & \mathrm{d}\mu(x) - E\mathrm{d}x \\ \mathrm{d}x & 0 \end{pmatrix} \Psi(x), \qquad x \in (x_0, x_1).$$

It follows that

$$\begin{aligned} &\mathrm{d}\big(\Psi(x)^T J_2 \Psi(x)\big) \\ &\quad \big(\mathrm{d}\Psi(x)\big)^T J_2 \Psi(x) + \Psi(x)^T J_2 \mathrm{d}\Psi(x) \\ &\quad \Psi(x)^T \begin{pmatrix} 0 & \mathrm{d}x \\ \mathrm{d}\mu(x) - E\mathrm{d}x & 0 \end{pmatrix} J_2 \Psi(x) + \Psi(x)^T J_2 \begin{pmatrix} 0 & \mathrm{d}\mu(x) - E\mathrm{d}x \\ \mathrm{d}x & 0 \end{pmatrix} \Psi(x) \\ &\quad \equiv 0. \end{aligned}$$

Due to (4.1), we have $\Psi(x_0)^T J_2 \Psi(x_0) = J_2$. Since $\Psi(x)$ is right-continuous, we find that

$$\Psi(x)^T J_2 \Psi(x) \equiv J_2, \qquad x \in [x_0, x_1).$$

By the group property of $\mathrm{Sp}(2, \mathbb{R})$, we obtain that $\Psi(x_1) \in \mathrm{Sp}(2, \mathbb{R})$. □

If $(\psi'(x), \psi(x))^T$ has the initial value $(\psi'(0), \psi(0))^T = (a, b)^T$, we have $(\psi'(x), \psi(x))^T = \Psi(x)(a, b)^T$. Introduce the so-called Prüfer transformation as

$$\psi' + \mathrm{i}\,\psi = r\,\mathrm{e}^{\mathrm{i}\theta}. \tag{4.3}$$

Then the argument $\theta = \theta(x)$ may be denoted by

$$\theta(x) := \arg\big(\psi'(x) + \mathrm{i}\,\psi(x)\big),$$

where $(\psi'(x), \psi(x))^T$ is any non-trivial solution of (4.2). When the system (4.2) is restricted to $\mathbb{R} \setminus \{x_i\}_{i\in\mathbb{Z}}$, we understand $\arg(\cdot)$ as a continuous branch on (x_i, x_{i+1}), because $(\psi'(x), \psi(x))^T$ is continuous on this interval. It is easy to obtain that the equation for θ is

$$\mathrm{d}\theta(x) = \big(\cos^2\theta(x) + E\sin^2\theta(x)\big)\mathrm{d}x - \sin^2\theta(x)\mathrm{d}\mu(x), \qquad x \in \mathbb{R} \setminus \{x_i\}_{i\in\mathbb{Z}}.$$

As in [12], we use the homology class $[P_c]$ that is induced by (B.2) to deal with the jump of $\theta(x)$ at the atoms of μ_{pp}. Thus via the Prüfer transformation (4.3), the evolution of $\theta(x)$ is found to be

$$\begin{cases} \mathrm{d}\theta(x) = \big(\cos^2\theta(x) + E\sin^2\theta(x)\big)\mathrm{d}x - \sin^2\theta(x)\mathrm{d}\mu(x), & x \in \mathbb{R} \setminus \{x_i\}_{i\in\mathbb{Z}}, \\ \theta(x+) - \theta(x-) = J\big(\mu(\{x\}), \theta(x-)\big), & x \in \{x_i\}_{i\in\mathbb{Z}}, \end{cases} \tag{4.4}$$

where $J(\cdot,\cdot)$ is defined by (B.3). Similarly as for $(\psi'(x), \psi(x))^T$, we understand that $\theta(x)$ is right-continuous with respect to $x \in \mathbb{R}$ as well. It follows from Lemma 4.1 that

Lemma 4.3 *For any $\Theta \in \mathbb{R}$, there exists a unique solution of (4.4) defined for all $x \in \mathbb{R}$ passing through $(0, \Theta)$. Moreover, the solution is continuous at $x \in \mathbb{R}\setminus\{x_i\}_{i\in\mathbb{Z}}$. We denote it by $\theta(x) := \theta_E(x+; \mu, \Theta)$.*

Remark 4.4 The initial value problem of (4.4) is equivalent to the existence of the solution of an integral equation. Because μ is involved, (4.4) is not a classical ODE. Fortunately, the Lebesgue dominated convergence theorem is valid for any measure. We can prove Lemma 4.3 with the aid of the Lebesgue dominated convergence theorem and the fixed point theorem. When $\mu \in \mathcal{M}^\infty_{\mathrm{ac}}(\mathbb{R})$, the proof can be founded in [43, Lemma 3.1].

5 Reduction to skew-products

Let $\mu \in \mathcal{M}^*_{\alpha\text{-nap}}(\mathbb{R})$ and $\mathrm{supp}(\mu_{\mathrm{pp}}) = \alpha t_\mu\mathbb{Z} + F_\mu = \{x_i\}_{i\in\mathbb{Z}}$ where (4.1) is satisfied. We need to embed (4.4) in a family of systems as follows

$$\begin{cases} \mathrm{d}\theta(x) = \big(\cos^2\theta(x) + E\sin^2\theta(x)\big)\mathrm{d}x - \sin^2\theta(x)\mathrm{d}\tilde\mu(x), & x \in \mathbb{R} \setminus \alpha t_\mu\mathbb{Z} + F_\mu, \\ \theta(x+) - \theta(x-) = J\big(\tilde\mu(\{x\}), \theta(x-)\big), & x \in \alpha t_\mu\mathbb{Z} + F_\mu, \end{cases} \tag{5.1}$$

where $\tilde\mu \in \mathrm{H_s}(\mu)$ and Lemma 3.15 (iv) are used. In the following we state some results about $\theta_E(x; \tilde\mu, \Theta)$ that we need, which are not surprising. We nevertheless decide to sketch their proofs, because (5.1) is not a classical ODE. One should be careful of the difference caused by $\tilde\mu$.

Lemma 5.1 *Let $\tilde\mu \in \mathrm{H_s}(\mu)$ and $E \in \mathbb{R}$ be fixed.*

(i) *For any $x \in \mathbb{R}$, $\Theta \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have*

$$\theta_E(x; \tilde\mu, \Theta + 2k\pi) - (\Theta + 2k\pi) = \theta_E(x; \tilde\mu, \Theta) - \Theta.$$

(ii) *For $\Theta \in \mathbb{R}$ and $\tau_1, \tau_2 \in \mathbb{Z}$, we have*

$$\theta_E(\alpha t_\mu\tau_1 + \alpha t_\mu\tau_2; \tilde\mu, \Theta) = \theta_E(\alpha t_\mu\tau_1; \tilde\mu \cdot t_\mu\tau_2, \theta_E(\alpha t_\mu\tau_2; \tilde\mu, \Theta)),$$

where t_μ is introduced in Lemma 3.12 (ii), and $\tilde\mu \cdot t_\mu\tau_2$ is defined by (3.5).

(iii) *We have the following relation*

$$\lim_{x\to+\infty}\frac{\theta_E(x;\tilde{\mu},\Theta)-\Theta}{x}=\lim_{n\to+\infty}\frac{\theta_E(\alpha t_\mu n;\tilde{\mu},\Theta)-\Theta}{\alpha t_\mu n}, \tag{5.2}$$

that is, if one of limits exists, then the other one exists as well and they are equal.

Proof (i) By Lemma B.2, we know that the vector field of (5.1) is 2π-periodic with respect to θ. Then both $\check{\theta}_1(x) := \theta_E(x;\tilde{\mu},\Theta+2k\pi)$ and $\check{\theta}_2(x) := \theta_E(x;\tilde{\mu},\Theta)+2k\pi$ satisfy (5.1) with the same initial value $\check{\theta}_i(0)=\Theta+2k\pi,\ i=1,2$. By Lemma 4.3, we have

$$\theta_E(x;\tilde{\mu},\Theta+2k\pi)=\theta_E(x;\tilde{\mu},\Theta)+2k\pi,$$

finishing the proof of (i). (ii) Due to (3.5), Lemma 3.3 (iii), (v) and Lemma 3.12 (ii), we have

$$\mathrm{supp}((\tilde{\mu}\cdot t_\mu\tau_2)_{\mathrm{pp}})=\alpha t_\mu\mathbb{Z}+F_\mu.$$

Denote $\bar{\theta}_1(x) := \theta_E(x;\tilde{\mu}\cdot t_\mu\tau_2,\theta_E(\alpha t_\mu\tau_2;\tilde{\mu},\Theta))$. Then $\bar{\theta}_1(x)$ satisfies the following equation,

$$\begin{cases} \mathrm{d}\theta(x)=\big(\cos^2\theta(x)+E\sin^2\theta(x)\big)\mathrm{d}x-\sin^2\theta(x)\mathrm{d}\tilde{\mu}\cdot t_\mu\tau_2(x), & x\in\mathbb{R}\setminus\alpha t_\mu\mathbb{Z}+F_\mu,\\ \theta(x+)-\theta(x-)=J\big(\tilde{\mu}\cdot t_\mu\tau_2(\{x\}),\theta(x-)\big), & x\in\alpha t_\mu\mathbb{Z}+F_\mu, \end{cases} \tag{5.3}$$

with the initial value $\bar{\theta}_1(0)=\theta_E(\alpha t_\mu\tau_2;\tilde{\mu},\Theta)$. Denote $\bar{\theta}_2(x) := \theta_E(x+\alpha t_\mu\tau_2;\tilde{\mu},\Theta)$. Then $\bar{\theta}_2(x)$ satisfies the following equation,

$$\begin{cases} \mathrm{d}\theta(x)=\big(\cos^2\theta(x)+E\sin^2\theta(x)\big)\mathrm{d}x-\sin^2\theta(x)\mathrm{d}\tilde{\mu}(x+\alpha t_\mu\tau_2), & x\in\mathbb{R}\setminus\alpha t_\mu\mathbb{Z}+F_\mu,\\ \theta(x+)-\theta(x-)=J\big(\tilde{\mu}(\{x+\alpha t_\mu\tau_2\}),\theta(x-)\big), & x\in\alpha t_\mu\mathbb{Z}+F_\mu, \end{cases} \tag{5.4}$$

with the initial value $\bar{\theta}_2(0)=\theta_E(\alpha t_\mu\tau_2;\tilde{\mu},\Theta)$. It follows from (3.3), (3.5) and Lemma 3.3 (v) that

$$\mathrm{d}\tilde{\mu}\cdot t_\mu\tau_2(x)=\mathrm{d}\tilde{\mu}(x+\alpha t_\mu\tau_2),\qquad x\in\mathbb{R}\setminus\alpha t_\mu\mathbb{Z}+F_\mu,$$

and

$$\tilde{\mu}\cdot t_\mu\tau_2(\{x\})=\tilde{\mu}(\{x+\alpha t_\mu\tau_2\}),\qquad x\in\mathbb{R}.$$

Then (5.3) coincides with (5.4). Since $\bar{\theta}_1(0)=\bar{\theta}_2(0)$, we conclude from Lemma 4.3 that $\bar{\theta}_1(x)=\bar{\theta}_2(x)$ for all $x\in\mathbb{R}$. Taking $x=\alpha t_\mu\tau_1$, we have the desired result. (iii) Due to (4.1), we know that $x_{k_\mu\tau}=\alpha t_\mu\tau,\ \tau\in\mathbb{Z}$. For any $x\in\mathbb{R}$, there exists $\tau_x\in\mathbb{Z}$ such that $x\in(\alpha t_\mu\tau_x,\alpha t_\mu\tau_x+\alpha t_\mu]$. For any $0\le i\le k_\mu-1$, consider the following case,

$$x\in(x_{k_\mu\tau_x+i},x_{k_\mu\tau_x+i+1}]=(\alpha t_\mu\tau_x+x_i,\alpha t_\mu\tau_x+x_{i+1}].$$

By (5.1) and Lemma B.2, we have

$$\begin{aligned}
&|\theta_E(x;\tilde{\mu},\Theta)-\theta_E(\alpha t_\mu\tau_x+x_i;\tilde{\mu},\Theta)| \\
&\quad\le \int_{(\alpha t_\mu\tau_x+x_i,\alpha t_\mu\tau_x+x_{i+1})}\left|\cos^2\theta(x)+E\sin^2\theta(x)\right|\mathrm{d}x \\
&\qquad+\int_{(\alpha t_\mu\tau_x+x_i,\alpha t_\mu\tau_x+x_{i+1})}\sin^2\theta(x)\,\mathrm{d}|\tilde{\mu}|(x) \\
&\qquad+\left|J\left(\mu(\{x_{k_\mu\tau_x+i+1}\}),\theta(x_{k_\mu\tau_x+i+1}-)\right)\right| \\
&\quad\le (x_{i+1}-x_i)(1+|E|)+(x_{i+1}-x_i+1)\|\tilde{\mu}\|_I+\pi.
\end{aligned}$$

This implies that

$$\begin{aligned}
&|\theta_E(x;\tilde{\mu},\Theta)-\theta_E(\alpha t_\mu\tau_x;\tilde{\mu},\Theta)| \\
&\quad\le \sum_{i=0}^{k_\mu-1}\left((x_{i+1}-x_i)(1+|E|)+(x_{i+1}-x_i+1)\|\tilde{\mu}\|_I+\pi\right) \\
&\quad= \alpha t_\mu(1+|E|)+(k_\mu+\alpha t_\mu)\|\tilde{\mu}\|_I+k_\mu\pi<+\infty,\quad \text{for all } x\in(\alpha t_\mu\tau_x,\alpha t_\mu\tau_x+\alpha t_\mu].
\end{aligned}$$

Thus the relation (5.2) follows readily by the boundedness of $|\theta_E(x;\tilde{\mu},\Theta)-\theta_E(\alpha t_\mu\tau_x;\tilde{\mu},\Theta)|$.
□

Lemma 5.2 *Let $\tau\in\mathbb{Z}$ be fixed. Then, $\theta_E(\alpha t_\mu\tau;\tilde{\mu},\Theta):\mathrm{H_s}(\mu)\times\mathbb{R}\to\mathbb{R}$ is continuous.*

Proof It suffices to show that $\theta_E(x_1;\tilde{\mu},\Theta):\mathrm{H_s}(\mu)\times\mathbb{R}\to\mathbb{R}$ is continuous. Let $\theta_i(x):=\theta_E(x;\tilde{\mu}_i,\Theta_i)$, $i=1,\ 2$. Then for any $x\in(0,x_1)$, we have

$$\theta_i(x)=\Theta_i+\int_{(0,x)}\left(\cos^2\theta_i(s)+E\sin^2\theta_i(s)\right)\mathrm{d}s-\int_{(0,x)}\sin^2\theta_i(s)\,\mathrm{d}\tilde{\mu}_i(s),\qquad i=1,\ 2.$$

Denote $D(x):=\theta_2(x)-\theta_1(x)$, $x\in(0,x_1)$. It follows that

$$\begin{aligned}
D(x)=(\Theta_2-\Theta_1)&+\int_{(0,x)}\left(\cos^2\theta_2(s)+E\sin^2\theta_2(s)-\cos^2\theta_1(s)-E\sin^2\theta_1(s)\right)\mathrm{d}s \\
&+\int_{(0,x)}\left(\sin^2\theta_2(s)-\sin^2\theta_1(s)\right)\mathrm{d}\tilde{\mu}_2(s)+\int_{(0,x)}\sin^2\theta_1(s)\,\mathrm{d}(\tilde{\mu}_2-\tilde{\mu}_1)(s).
\end{aligned}$$

This implies that

$$\begin{aligned}
|D(x)|&\le|\Theta_2-\Theta_1|+\int_{(0,x)}(2+2|E|)|D(s)|\,\mathrm{d}s \\
&\quad+\int_{(0,x)}2|D(s)|\,\mathrm{d}|\tilde{\mu}_2|(s)+\int_{(0,x)}\mathrm{d}|\tilde{\mu}_2-\tilde{\mu}_1|(s) \\
&\le|\Theta_2-\Theta_1|+(\alpha t_\mu+1)\|\tilde{\mu}_2-\tilde{\mu}_1\|_I+\int_{(0,x)}|D(s)|\,\mathrm{d}\hat{\mu}(s),
\end{aligned}$$

where $\hat{\mu}:=(2+2|E|)\lambda+2|\tilde{\mu}_2|$. By Lemma A.6, we have

$$\begin{aligned}
|D(x)|&\le|\Theta_2-\Theta_1|+(\alpha t_\mu+1)\|\tilde{\mu}_2-\tilde{\mu}_1\|_I \\
&\quad+\int_{(0,x)}\left(|\Theta_2-\Theta_1|+(\alpha t_\mu+1)\|\tilde{\mu}_2-\tilde{\mu}_1\|_I\right)\mathrm{e}^{\hat{\mu}((s,x))}\,\mathrm{d}\hat{\mu}(s) \\
&\le(|\Theta_2-\Theta_1|+(\alpha t_\mu+1)\|\tilde{\mu}_2-\tilde{\mu}_1\|_I)(1+\mathrm{e}^C C),
\end{aligned}$$

where $C := (2 + 2|E|)\alpha t_\mu + 2(\alpha t_\mu + 1)\|\mu\|_I$ and Lemma 3.15 (iii) is used. It follows that $\theta(x_1-; \tilde{\mu}, \Theta) : \mathrm{H_s}(\mu) \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Furthermore, we have

$$\theta(x_1) = \theta(x_1-) + J\big(\tilde{\mu}(\{x_1\}), \theta(x_1-)\big).$$

Since $\big|\tilde{\mu}_2(\{x_1\}) - \tilde{\mu}_1(\{x_1\})\big| \le \|\tilde{\mu}_2 - \tilde{\mu}_1\|_I$, the desired result is deduced from Lemma B.2. □

Lemma 5.3 *Let $\tilde{\mu} \in \mathrm{H_s}(\mu)$ and $\tau \in \mathbb{Z}$ be fixed. Then, $\theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta) : \mathbb{R} \to \mathbb{R}$ is a strictly increasing self-homeomorphism.*

Proof Due to Lemma 4.3 and Lemma B.3, we know that $\theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta) : \mathbb{R} \to \mathbb{R}$ is strictly increasing. By Lemma 5.1 (ii), we have

$$\theta_E(\alpha t_\mu \tau; \tilde{\mu}, \theta_E(-\alpha t_\mu \tau; \tilde{\mu} \cdot t_\mu \tau, \Theta)) = \Theta = \theta_E(-\alpha t_\mu \tau; \tilde{\mu} \cdot t_\mu \tau, \theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta)).$$

This implies that the inverse of $\theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta)$ is given by $\theta_E(-\alpha t_\mu \tau; \tilde{\mu} \cdot t_\mu \tau, \Theta)$. By Lemma 5.2, we obtain the desired result. □

Let $\mathbb{S}_{2\pi} := \mathbb{R}/2\pi\mathbb{Z}$. We introduce the following product space

$$\mathrm{Z} := \mathrm{H_s}(\mu) \times \mathbb{S}_{2\pi},$$

which will play a central role in Sect. 6. The distance on Z is defined by

$$\mathrm{dist}\big((\tilde{\mu}_1, \vartheta_1), (\tilde{\mu}_1, \vartheta_1)\big) := \max\big\{\|\tilde{\mu}_1 - \tilde{\mu}_2\|_I, |\vartheta_1 - \vartheta_2|_{\mathbb{S}_{2\pi}}\big\} \tag{5.5}$$

where $(\tilde{\mu}_i, \vartheta_i) \in \mathrm{Z}$, $i = 1, 2$. We know that (Z, dist) is a compact metric space. For any $\tau \in \mathbb{Z}$, the skew-product transformation Φ^k on Z is defined by

$$\Phi_E^\tau(\tilde{\mu}, \vartheta) := (\tilde{\mu} \cdot t_\mu \tau, \theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta) \bmod 2\pi), \tag{5.6}$$

where $(\tilde{\mu}, \vartheta) \in \mathrm{Z}$ and $\Theta \in \mathbb{R}$ satisfies $\vartheta = \Theta \mod 2\pi$. By Lemma 5.1 (i), Φ_E^τ is well defined for any $\tau \in \mathbb{Z}$. Moreover we have

Lemma 5.4 $\{\Phi_E^\tau\}_{\tau \in \mathbb{Z}}$ *is a discrete-time and skew-product continuous dynamical system on the compact space Z.*

Proof Due to Lemma 3.3 (i) and Lemma 5.2, we know that Φ_E^τ is continuous on Z for any $\tau \in \mathbb{Z}$. Now we aim to prove that

$$\Phi_E^{\tau_1+\tau_2} = \Phi_E^{\tau_1} \circ \Phi_E^{\tau_2} \quad \text{for all } \tau_1, \ \tau_2 \in \mathbb{Z}.$$

In fact, assume that $(\tilde{\mu}, \vartheta) \in \mathrm{Z}$ and there exists $\Theta \in \mathbb{R}$ satisfying $\vartheta = \Theta \mod 2\pi$. By (5.6) and Lemma 5.1 (ii), we have

$$\begin{aligned}
&\Phi_E^{\tau_1} \circ \Phi_E^{\tau_2}(\tilde{\mu}, \vartheta) \\
&\quad = \Phi_E^{\tau_1}\big(\tilde{\mu} \cdot t_\mu \tau_2, \theta_E(\alpha t_\mu \tau_2; \tilde{\mu}, \Theta) \bmod 2\pi\big) \\
&\quad = \Big(\tilde{\mu} \cdot (t_\mu \tau_2 + t_\mu \tau_1), \theta_E\big(\alpha t_\mu \tau_1; \tilde{\mu} \cdot t_\mu \tau_2, \theta_E(\alpha t_\mu \tau_2; \tilde{\mu}, \Theta)\big) \bmod 2\pi\Big) \\
&\quad = \Phi_E^{\tau_1+\tau_2}(\tilde{\mu}, \vartheta).
\end{aligned}$$

The proof is completed. □

Introduce the observable $F_E : \mathrm{Z} \to \mathbb{R}$ by

$$F_E(\tilde{\mu}, \vartheta) := \theta_E(\alpha t_\mu; \tilde{\mu}, \Theta) - \Theta, \tag{5.7}$$

where $\Theta \in \mathbb{R}$ satisfies $\vartheta = \Theta \mod 2\pi$. By Lemma 5.1 (i), $F_E(\tilde{\mu}, \vartheta)$ is well defined on Z. Furthermore, by Lemma 5.2, we know that $F_E(\tilde{\mu}, \vartheta)$ is continuous on Z. By the construction above, we reduce the existence of (5.2) to that of the following ergodic limit with respect to the skew-product dynamical system $\{\Phi_E^\tau\}_{\tau\in\mathbb{Z}}$.

Lemma 5.5 *Assume that* $(\tilde{\mu}, \vartheta) \in \mathrm{Z}$ *and* $\Theta \in \mathbb{R}$ *satisfies* $\vartheta = \Theta \mod 2\pi$*. Then, the following relation holds,*

$$\lim_{x\to+\infty} \frac{\theta_E(x+; \tilde{\mu}, \Theta) - \Theta}{x} = \lim_{n\to+\infty} \frac{1}{\alpha t_\mu n} \sum_{\tau=0}^{n-1} F_E\big(\Phi_E^\tau(\tilde{\mu}, \vartheta)\big). \tag{5.8}$$

That is, if one of the limits exists, then the other one exists as well and they are equal.

Proof By Lemma 5.1 (ii) and (5.7), we have

$$\begin{aligned}
&\theta_E(\alpha t_\mu n; \tilde{\mu}, \Theta) - \Theta \\
&= \sum_{\tau=0}^{n-1} \big(\theta_E(\alpha t_\mu(\tau+1); \tilde{\mu}, \Theta) - \theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta)\big) \\
&= \sum_{\tau=0}^{n-1} \Big(\theta_E\big(\alpha t_\mu; \tilde{\mu}\cdot t_\mu \tau, \theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta)\big) - \theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta)\Big) \\
&= \sum_{\tau=0}^{n-1} F_E\big(\tilde{\mu}\cdot t_\mu \tau, \theta_E(\alpha t_\mu \tau; \tilde{\mu}, \Theta) \mod 2\pi\big) \\
&= \sum_{\tau=0}^{n-1} F_E\big(\Phi_E^\tau(\tilde{\mu}, \vartheta)\big).
\end{aligned}$$

Then the desired result is deduced via Lemma 5.1 (iii), provided one of the limits exists. □

6 Rotation number

To show the existence of rotation numbers, inspired by Lemma 5.5, we introduce the following notation

$$F_E^*(\tilde{\mu}, \vartheta) := \lim_{n\to+\infty} \frac{1}{n} \sum_{\tau=0}^{n-1} F_E\big(\Phi_E^\tau(\tilde{\mu}, \vartheta)\big), \qquad (\tilde{\mu}, \vartheta) \in \mathrm{Z},$$

provided the limit exists. For $\tilde{\mu} \in \mathrm{H_s}(\mu)$ and $\Theta \in \mathbb{R}$, denote

$$F_E^\diamond(\tilde{\mu}, \Theta) := \lim_{n\to+\infty} \frac{\theta_E(\alpha t_\mu n; \tilde{\mu}, \Theta) - \Theta}{\alpha t_\mu n}, \tag{6.1}$$

provided the limit exists.

Lemma 6.1 *If* $F_E^\diamond(\tilde{\mu}, \Theta_0)$ *exists for* $\Theta_0 \in \mathbb{R}$*, then* $F_E^\diamond(\tilde{\mu}, \Theta)$ *exists for all* $\Theta \in \mathbb{R}$ *and is independent of the choice of* $\Theta \in \mathbb{R}$.

Proof By Lemma 5.1 (i), we know that $F_E^{\diamond}(\tilde{\mu}, \Theta_0 + 2k\pi)$ exists for all $k \in \mathbb{Z}$. Then for any $\Theta \in \mathbb{R}$, there exists $k_\Theta \in \mathbb{Z}$ such that $\Theta \in [\Theta_0 + 2k_\Theta\pi, \Theta_0 + 2(k_\Theta + 1)\pi)$. By Lemma 5.3, for all $n \in \mathbb{N}$, we have

$$\theta_E(\alpha t_\mu n; \tilde{\mu}, \Theta_0 + 2k_\Theta\pi) \le \theta_E(\alpha t_\mu n; \tilde{\mu}, \Theta) < \theta_E(\alpha t_\mu n; \tilde{\mu}, \Theta_0 + 2k_\Theta\pi) + 2\pi.$$

This implies that $F_E^{\diamond}(\tilde{\mu}, \Theta) \equiv F_E^{\diamond}(\tilde{\mu}, \Theta_0)$, for all $\Theta \in \mathbb{R}$. □

The following uniform ergodic theorem due to Johnson and Moser plays a fundamental role in the proof of the main result.

Lemma 6.2 [20] *Let $\{\varphi^\tau\}_{\tau\in\mathbb{Z}}$ be a continuous discrete-time dynamical system on a compact metric space X. Then, for any continuous function f on X satisfying*

$$\int_X f \,\mathrm{d}\omega = 0$$

for all invariant Borel probability measures ω under $\{\varphi^\tau\}_{\tau\in\mathbb{Z}}$, one has

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{\tau=0}^{n-1} f\big(\varphi^k(x)\big) = 0$$

uniformly for all $x \in X$.

Proof of Theorem 1.1 By the Krylov–Bogoliubov theorem and Lemma 5.4, there exists an invariant Borel probability measure under $\big\{\Phi_E^\tau\big\}_{\tau\in\mathbb{Z}}$, denoted by ω. Then by the Birkhoff ergodic theorem, there exists a Borel set $\mathrm{Z}_\omega \subset \mathrm{Z}$, which depends on the measure ω, such that $\omega(\mathrm{Z}_\omega) = 1$ and $F_E^*(\tilde{\mu}, \vartheta)$ exists for all $(\tilde{\mu}, \vartheta) \in \mathrm{Z}_\omega$. Furthermore, F_E^* is integrable and satisfies

$$\int_{\mathrm{Z}} F_E^* \,\mathrm{d}\omega = \int_{\mathrm{Z}} F_E \,\mathrm{d}\omega =: \rho_{E,\omega}. \tag{6.2}$$

Due to Lemma 6.1, Z_ω can be written in the form $\mathrm{Z}_\omega = \mathrm{E}_\omega \times \mathbb{S}_{2\pi}$, where E_ω is a Borel set in $\mathrm{H_s}(\mu)$. Due to Lemma 3.16, let ν be the Haar measure on $\mathrm{H_s}(\mu)$. Then we have $\nu(\mathrm{E}_\omega) = 1$. By the unique ergodicity of the Haar measure, there exists a set $\hat{\mathrm{E}}_\omega \subset \mathrm{E}_\omega$ such that $\nu(\hat{\mathrm{E}}_\omega) = \omega(\hat{\mathrm{E}}_\omega \times \mathbb{S}_{2\pi}) = 1$ and $F_E^*(\tilde{\mu}, \vartheta)$ is a constant function on $\hat{\mathrm{E}}_\omega \times \mathbb{S}_{2\pi}$. It follows from (6.2) that the constant must be $\rho_{E,\omega}$.

By (6.1), we know that $\rho_{E,\omega}$ in (6.2) is independent of the choice of the measure ω. Set $\hat{F}_E := F_E - \rho_E$. By Lemma 5.2, $\hat{F}_E$ is continuous on Z. By (6.2), $\hat{F}_E$ satisfies the requirement of Lemma 6.2. Thus,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{\tau=0}^{n-1} \hat{F}_E\big(\Phi_E^\tau(\tilde{\mu}, \vartheta)\big) = \lim_{n\to+\infty} \frac{1}{n} \sum_{\tau=0}^{n-1} F_E\big(\Phi_E^\tau(\tilde{\mu}, \vartheta)\big) - \rho_E = 0, \tag{6.3}$$

uniformly for all $(\tilde{\mu}, \vartheta) \in \mathrm{Z}$.

At last, taking $\tilde{\mu} = \mu$ in (6.3), then by Lemma 5.2 and Lemma 5.5, we obtain the existence of the desired limit. □

Appendix A. Solutions

The way to understand the solutions of $H_\mu \psi = E\psi$ is twofold, where $\mu \in \mathcal{M}^\infty(\mathbb{R})$. One is to use the concept of weak derivatives from the point of view of PDEs, and the other is to extend the theory of ODEs to the so-called *measure differential equations* (MDEs). First, we recall from [6, 13, 37] some basic facts on solutions of (1.2) in the weak sense. For $k \in \mathbb{N}$, denote the Sobolev space $W^k_{1,\mathrm{loc}}(\mathbb{R})$ by

$$W^k_{1,\mathrm{loc}}(\mathbb{R}) := \{\psi \in L_{1,\mathrm{loc}}(\mathbb{R}) : \psi^{(i)} \in L_{1,\mathrm{loc}}(\mathbb{R}),\ \text{for } i = 1, 2, \ldots, k\},$$

where $\psi^{(i)}$ is the i^{th}-weak/distributional derivative of ψ. Due to the regularity, we know that every $\psi \in W^k_{1,\mathrm{loc}}(\mathbb{R})$ has a locally absolutely continuous representative.

Definition A.1 [37, p. 8] For $\psi \in W^1_{1,\mathrm{loc}}(\mathbb{R})$, the quasi-derivative $A_\mu \psi$ of ψ is defined by

$$A_\mu \psi(x) := \psi^{(1)}(x) - \int_0^x \psi(s)\, \mathrm{d}\mu(s) \in L_{1,\mathrm{loc}}(\mathbb{R}), \qquad \lambda\text{-a.e. } x \in \mathbb{R},$$

where $\int_0^x$ stands for

$$\int_0^x = \begin{cases} \int_{[0,x]} & x \geq 0, \\ -\int_{(x,0)} & x < 0, \end{cases}$$

and is understood as a Lebesgue-Stieltjes integral.

Definition A.2 A function $\psi \in L_{1,\mathrm{loc}}(\mathbb{R})$ is called a solution of (1.2) if $\psi \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ satisfies

$$-(A_\mu \psi)^{(1)} = E\psi \tag{A.1}$$

in the sense of distributions.

This implies that $A_\mu \psi \in W^2_{1,\mathrm{loc}}(\mathbb{R})$, and then $A_\mu \psi$ can be understood as a continuous representative. Since $\int_0^x$ is right-continuous, we may choose a right-continuous representative of $\psi^{(1)}$. Further properties of $\psi^{(1)}$ are listed in the following lemma.

Lemma A.3 *Let ψ be a solution of* (1.2). *Then with $x \in \mathbb{R}$ arbitrary, we have*

(i) $\psi \in C(\mathbb{R})$;
(ii) $\psi^{(1)}(x+)$ *and* $\psi^{(1)}(x-)$ *exist,* $\psi^{(1)}(x) = \psi^{(1)}(x+)$;
(iii) $\psi^{(1)}(x+)$ *is right-continuous, while* $\psi^{(1)}(x-)$ *is left-continuous;*
(iv) $\psi^{(1)}(x+) - \psi^{(1)}(x-) = \psi(x)\mu(\{x\})$.

Define the one-sided derivative of ψ by

$$D^\pm \psi(x) := \lim_{h \to 0\pm} \frac{\psi(x+h) - \psi(x)}{h}. \tag{A.2}$$

Note that $\psi^{(1)}$ is the derivative in the weak sense, whereas $D^\pm \psi$ are the derivatives in the classical sense. The relationship between $\psi^{(1)}$ and $D^\pm \psi$ is described in the following lemma.

Lemma A.4 *Let ψ be a solution of* (1.2). *Then we have:*

(i) *for all* $x \in \mathbb{R}$, $D^+\psi(x)$ *exists,* $D^+\psi(x) = \psi^{(1)}(x+)$, *and* $x \mapsto D^+\psi(x)$ *is right-continuous;*

(ii) *for all* $x \in \mathbb{R}$, $D^-\psi(x)$ *exists,* $D^-\psi(x) = \psi^{(1)}(x-)$, *and* $x \mapsto D^-\psi(x)$ *is left-continuous;*

(iii) *for all* $x \in \mathbb{R} \setminus \mathrm{atom}(\mu)$, $\psi'(x)$ *exists, and* $\psi'(x) = D^{\pm}\psi(x) = \psi^{(1)}(x)$.

Remark A.5 The property of $\psi^{(1)}(x+)$ and $D^+\psi(x)$ has been proved in [37]. By a similar way, we state the corresponding property of $\psi^{(1)}(x-)$ and $D^-\psi(x)$ in Lemma A.3 and Lemma A.4. A crucial point in the proof is that the Newton-Leibniz formula holds as well when ψ' is replaced by $D^{\pm}\psi$. At last, Lemma A.4 (iii) is deduced via Lemma A.3 (iv).

Due to $\mu \in \mathcal{M}^\infty(\mathbb{R})$ and the contraction mapping principle, we have

Lemma A.5 *Let* $a, b \in \mathbb{R}$ *be arbitrary. Then there exists a unique solution* $\psi_E(x; \mu)$ *of* (1.2) *defined for all* $x \in \mathbb{R}$ *such that* $\psi_E^{(1)}(0+; \mu) = a$ *and* $\psi_E(0; \mu) = b$.

The second way to understand the solutions of (1.2) is from the point of view of MDEs. Using the argument in [34, 41], we write (1.2) as the following second-order scalar linear MDE,

$$\mathrm{d}D^+\psi(x) - \psi(x)\,\mathrm{d}\mu(x) + E\psi(x)\,\mathrm{d}x = 0, \qquad x \in \mathbb{R}, \tag{A.3}$$

where $D^+\psi$ is the same as $\psi^\bullet$ in [34, 41]. The solution $(\psi(x), D^+\psi(x))$ of (A.3) possesses the same properties as those in Lemma A.3, Lemma A.4 and Lemma A.5. For our purpose, (A.3) can be written as the following planar system,

$$\mathrm{d}\begin{pmatrix} D^+\psi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}\mu(x) - E\,\mathrm{d}x \\ \mathrm{d}x & 0 \end{pmatrix} \begin{pmatrix} D^+\psi(x) \\ \psi(x) \end{pmatrix}, \qquad x \in \mathbb{R}. \tag{A.4}$$

A measure version of Gronwall's inequality is stated as follows.

Lemma A.6 [37, Lemma A.1] *Let* $x \geq 0$ *and let* $\mu \in \mathcal{M}(\mathbb{R})$ *be fixed. Assume that* $u : [0, x] \to \mathbb{R}$ *is continuous,* $a : [0, x] \to \mathbb{R}_0^+$ *is measurable with respect to* μ, *and*

$$u(x) \leq a(x) + \int_{(0,x)} u(s)\,\mathrm{d}\mu(s).$$

Then we have

$$u(x) \leq a(x) + \int_{(0,x)} a(s)\,\mathrm{e}^{\mu((s,x))}\,\mathrm{d}\mu(s).$$

Appendix B. Homotopy

The following result can be found in [29, Lemma 3, p. 5].

Lemma B.1 *For any* $D \in \mathrm{Sp}(2, \mathbb{R})$, *there exists a unique decomposition such that* $D = A\,U$, *where* $A \in \mathrm{Sp}(2, \mathbb{R})$ *is a symmetric and positive-definite matrix, and* $U \in \mathrm{Sp}(2, \mathbb{R})$ *is an orthogonal matrix. Explicitly, we have:*

$$D = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}, \tag{B.1}$$

where $(r, \vartheta, z) \in \mathbb{R}^+ \times \mathbb{R}/(2\pi\mathbb{Z} - \pi) \times \mathbb{R}$ *is uniquely determined by* D.

Due to the expression of (4.2), we consider the following sub-group

$$\mathrm{Trig}(2, \mathbb{R}) := \left\{ R_c := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \in \mathbb{R} \right\} \subset \mathrm{Sp}(2, \mathbb{R}).$$

The unique decomposition of R_c can be calculated as

$$R_c = \begin{pmatrix} \frac{c^2+2}{\sqrt{c^2+4}} & \frac{c}{\sqrt{c^2+4}} \\ \frac{c}{\sqrt{c^2+4}} & \frac{2}{\sqrt{c^2+4}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{c^2+4}} & \frac{c}{\sqrt{c^2+4}} \\ -\frac{c}{\sqrt{c^2+4}} & \frac{2}{\sqrt{c^2+4}} \end{pmatrix}.$$

Construct a continuous path $P_c(\cdot) : [0, 1] \to \mathrm{Sp}(2, \mathbb{R})$ as

$$P_c(\tau) = \begin{pmatrix} \frac{(\tau c)^2+2}{\sqrt{(\tau c)^2+4}} & \frac{\tau c}{\sqrt{(\tau c)^2+4}} \\ \frac{\tau c}{\sqrt{(\tau c)^2+4}} & \frac{2}{\sqrt{(\tau c)^2+4}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{(\tau c)^2+4}} & \frac{\tau c}{\sqrt{(\tau c)^2+4}} \\ -\frac{\tau c}{\sqrt{(\tau c)^2+4}} & \frac{2}{\sqrt{(\tau c)^2+4}} \end{pmatrix} = \begin{pmatrix} 1 & \tau c \\ 0 & 1 \end{pmatrix}. \tag{B.2}$$

$P_c(\cdot)$ connects I_2 and R_c. The path is shown in [12, Figure 1]. The homotopy class of $P_c(\cdot)$ is denoted by $[P_c]$. Let $\mathrm{V}(\mathbb{R}^2)$ be the set of all vectors starting from the origin in $\mathbb{R}^2$. The equivalence $\sim$ on $\mathrm{V}(\mathbb{R}^2)$ is defined by

$$\vec{v}_1 \sim \vec{v}_2 \iff \vec{v}_1 = k\vec{v}_2, \quad \text{for some } k \in \mathbb{R}^+.$$

Then $\mathrm{L}(\mathbb{R}) := \mathrm{V}(\mathbb{R}^2)/\sim$ is an orientable compact manifold of dimension one, and may be regarded as a two-covering of the real projective line $\mathbb{RP}^1$.

Let $\Theta \in \mathbb{R}$. By (B.2), we have

$$P_c(\tau)(\cos\Theta, \sin\Theta)^T = (\cos\Theta + \tau c \sin\Theta, \sin\Theta)^T.$$

Since the homotopy class $[P_c]$ is fixed and $\arg(\cdot)$ is understood as a continuous branch, the argument function

$$F(c, \tau, \Theta) = \arg(\cos\Theta + \tau c \sin\Theta + \mathrm{i} \sin\Theta)$$

is continuous with respect to $(c, \tau, \Theta) \in \mathbb{R} \times [0, 1] \times \mathbb{R}$. In particular, we may choose one continuous branch of $F(c, \tau, \Theta)$ such that when $\tau = 0$, we have

$$\arg(\cos\Theta + \mathrm{i}\sin\Theta) = \Theta.$$

Then $[P_c]$ yields the difference of arguments by

$$J(c, \Theta) := F(c, 1, \Theta) - F(c, 0, \Theta). \tag{B.3}$$

Lemma B.2 [12] *$J : \mathbb{R}^2 \to (-\pi, \pi)$ is continuous with respect to $(c, \Theta) \in \mathbb{R}^2$. Moreover, one has*

$$J(c, \Theta + 2\pi) = J(c, \Theta).$$

Lemma B.3 *Let $c \in \mathbb{R}$ be fixed. Then, $J(c, \Theta) + \Theta$ is strictly increasing with respect to $\Theta \in \mathbb{R}$.*

Appendix C. Proof of Remark 3.13

For any $\varepsilon > 0$, due to Lemma 2.10, $\Lambda_\varepsilon = \mathrm{P}(V_\mu, \varepsilon) \cap k_0\mathbb{Z}$ is relatively dense in $\mathbb{Z}$. This implies that

$$\|V_\mu \cdot k_0\tau - V_\mu\|_\infty = \|T_{\alpha t_0 \tau}\mu - \mu\|_\infty = \|\mu \cdot t_0\tau - \mu\|_\infty < \varepsilon, \quad \text{for } \tau \in k_0^{-1}\Lambda_\varepsilon.$$

Since $\mathrm{supp}(\mu)$ is uniformly discrete, there exists $c_0 > 0$ such that

$$\|T_t\mu - \mu\|_I \leq c_0\|T_t\mu - \mu\|_\infty, \quad \text{for all } t \in \mathbb{R}.$$

By the equivalence of the properties listed in Definition 2.1, we know that $\overline{\{\mu \cdot t_0\tau : \tau \in \mathbb{Z}\}}^{(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)}$ is compact. This implies that

$$\overline{\{\mu \cdot (t_0\tau + i) : \tau \in \mathbb{Z}\}}^{(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)} \text{ is compact}, \quad \text{for all } i \in [0, t_0 - 1] \cap \mathbb{N}.$$

Note that

$$\mathrm{H}(\mu) = \overline{\{\mu \cdot \tau : \tau \in \mathbb{Z}\}}^{(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)} = \bigcup_{0 \leq i \leq t_0 - 1} \overline{\{\mu \cdot (t_0\tau + i) : \tau \in \mathbb{Z}\}}^{(\mathcal{M}^\infty(\mathbb{R}), \|\cdot\|_I)}$$

It follows that $\mathrm{H}(\mu)$ is compact. By the equivalence of the properties listed in Definition 2.1 again, we obtain that $\mu \in \mathcal{M}^\infty_{\alpha\text{-nap}}(\mathbb{R})$. □

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References

1. Aliste-Prieto, J., Rand, B., Sadun, L.: Rotation numbers and rotation classes on one-dimensional tiling spaces. Ann. Henri Poincaré **22**, 2161–2193 (2021)
2. Baake, M., Moody, R.V.: Weighted Dirac combs with pure point diffraction. J. Reine Angew. Math. **573**, 61–94 (2004)
3. Bellissard, J.: K-theory of C^*-algebras in solid state physics, Statistical mechanics and field theory: mathematical aspects (Groningen, 1985), Lecture Notes in Physics, vol. 257. Springer, Berlin, pp. 99–156 (1986)
4. Bellissard, J.: Gap labelling theorems for Schrödinger operators, From number theory to physics (Les Houches, 1989). Springer, Berlin, pp. 538–630 (1992)
5. Bellissard, J.: The noncommutative geometry of aperiodic solids, Geometric and topological methods for quantum field theory (Villa de Leyva, 2001). World Sci. Publ., River Edge, pp. 86–156 (2003)
6. Ben Amor, A., Remling, C.: Direct and inverse spectral theory of one-dimensional Schrödinger operators with measures. Integral Equ. Oper. Theory **52**, 395–417 (2005)
7. Besicovitch, A.S.: Almost Periodic Functions. Dover Publications, New York (1954)
8. Bjerklöv, K., Jäger, T.: Rotation numbers for quasiperiodically forced circle maps-mode-locking vs. strict monotonicity. J. Am. Math. Soc. **22**, 353–362 (2009)
9. Damanik, D., Fillman, J.: One-Dimensional Ergodic Schrödinger Operators, I. General Theory, Graduate Studies in Mathematics, vol. 221. American Mathematical Society, Providence (2022)
10. Damanik, D., Fillman, J.: Gap labelling for discrete one-dimensional ergodic Schrödinger operators. From Complex Analysis to Operator Theory: A Panorama, pp. 341–404, Oper. Theory Adv. Appl., vol. 291. Birkhäuser/Springer, Cham (2023)
11. Damanik, D., Zhang, M., Zhou, Z.: The rotation number for almost periodic potentials with jump discontinuities and δ-interactions. Ann. Henri Poincaré **25**, 1359–1397 (2024)
12. Damanik, D., Zhou, Z.: The rotation number for the almost periodic Schrödinger operator with δ-potentials. J. Dyn. Differ. Equ. **34**, 155–177 (2022)
13. Eckhardt, J., Teschl, G.: Sturm–Liouville operators with measure-valued coefficients. J. Anal. Math. **120**, 151–224 (2013)
14. Fink, A.: Almost Periodic Differential Equations. Springer, New York (1974)

15. Gil de Lamadri, J., Argabright, L.N.: Almost periodic measures. Mem. Am. Math. Soc. **85** (428), vi+219 pp (1990)
16. Gouéré, J.-B.: Quasicrystals and almost periodicity. Commun. Math. Phys. **255**, 655–681 (2005)
17. Herman, M.R.: Une méthode pour minorer les exposants de Lyapunov et quelques examples montrant le caractère local d'un théorème d'Arnold er de Moser sur le tore de dimension 2 Comment. Math. Helv. **58**, 453–502 (1983)
18. Hewitt, E., Ross, K.A.: Abstract Harmonic Analysis, vol. I. Springer, Berlin (1963)
19. Johnson, R.: Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients. J. Differ. Equ. **61**, 54–78 (1986)
20. Johnson, R., Moser, J.: The rotation number for almost periodic potentials. Commun. Math. Phys. **84**, 403–438 (1982)
21. Johnson, R., Obaya, R., Novo, S., Núñez, C., Fabbri, R.: Nonautonomous Linear Hamiltonian Systems: Oscillation, Spectral Theory and Control, Developments in Mathematics 36. Springer, Cham (2016)
22. Lenz, D., Richard, C.: Pure point diffraction and cut and project schemes for measures: the smooth case. Math. Z. **256**, 347–378 (2007)
23. Lenz, D., Seifert, C., Stollmann, P.: Zero measure Cantor spectra for continuum one-dimensional quasicrystals. J. Differ. Equ. **256**, 1905–1926 (2014)
24. Lenz, D., Spindeler, T., Strungaru, N.: Pure point spectrum for dynamical systems and mean, Besicovitch and Weyl almost periodicity. Ergod. Theory Dyn. Syst. **44**, 524–568 (2024)
25. Lenz, D., Spindeler, T., Strungaru, N.: Pure point diffraction and mean, Besicovitch and Weyl almost periodicity, preprint. arXiv:2006.10821
26. Lenz, D., Spindeler, T., Strungaru, N.: Pure point diffraction and almost periodicity, preprint. arXiv:2312.12825
27. Lenz, D., Strungaru, N.: On weakly almost periodic measures. Trans. Am. Math. Soc. **371**, 6843–6881 (2019)
28. Li, W., Lu, K.: Rotation numbers for random dynamical systems on the cicle. Trans. Am. Math. Soc. **360**, 5509–5528 (2008)
29. Long, Y.: Index Theory for Symplectic Paths with Applications. Birkhäuser, Basel (2002)
30. Katok, A., Hasselblatt, B.: Introduction to the Modern Theory of Dynamical Systems. Cambridge Univ. Press, Cambridge (1995)
31. Kwapisz, J.: Poincaré rotation number for maps of the real line with almost periodic displacement. Nonlinearity **13**, 841–854 (2000)
32. Ma, Z., Röckner, M.: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Springer, Berlin (1992)
33. McDonald, J., Weiss, N.: A Course in Real Analysis. Academic Press (Beijing Branch), Beijing (2005)
34. Meng, G., Zhang, M.: Dependence of solutions and eigenvalues of measure differential equations on measures. J. Differ. Equ. **254**, 2196–2232 (2013)
35. Minami, N.: An extension of Kotani's theorem to random generalized Sturm–Liouville operators. Commun. Math. Phys. **103**, 387–402 (1986)
36. Moody, R.V., Strungaru, N.: Almost periodic measures and their Fourier transforms, Aperiodic order. vol. 2, pp. 173–270, Encyclopedia Math. Appl., vol. 166. Cambridge Univ. Press, Cambridge (2017)
37. Seifert, C.: Measure-perturbed one-dimensional Schrödinger operators—a continuum model for quasicrystals, Doctoral Dissertation thesis, Chemnitz University of Technology (2012)
38. Spindeler, T., Strungaru, N.: On norm almost periodic measures. Math. Z. **299**, 233–255 (2021)
39. Strungaru, N.: Almost periodic pure point measures, Aperiodic order, vol. 2, pp. 271–342. Encyclopedia Math. Appl., vol. 166. Cambridge Univ. Press, Cambridge (2017)
40. Yan, J.: Lectures on Measure Theory. Science Press, Beijing (2004) **(in Chinese)**
41. Zhang, M.: Extremal eigenvalues of measure differential equations with fixed variation. Sci. China Math. **53**, 2573–2588 (2010)
42. Zhang, M., Zhou, Z.: Uniform ergodic theorems for discontinuous skew-product flows and applications to Schrödinger equations. Nonlinearity **24**, 1539–1564 (2011)
43. Zhou, Z.: The rotation number of the linear Schrödinger equation with discontinuous almost periodic potentials. J. Differ. Equ. **259**, 4202–4228 (2015)

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