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Relations amongst twists along Montesinos twins in the 4-sphere

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Isotopy classes of diffeomorphisms of the 4-sphere can be described either from a Cerf-theoretic perspective in terms of loops of 5-dimensional handle attaching data, starting and ending with handles in canceling position, or via certain twists along submanifolds analogous to Dehn twists in dimension 2. The subgroup of the smooth mapping class group of the 4-sphere coming from loops of 5-dimensional handles of index 1 and 2 coincides with the subgroup generated by twists along Montesinos twins (pairs of 2-spheres intersecting transversely twice) in which one of the two 2-spheres in the twin is unknotted. We show that this subgroup is in fact trivial or cyclic of order 2.

57K40; 57K45

1 Introduction

By the smooth mapping class group of a smooth manifold X we mean $\pi_0(\mathrm{Diff}^+(X))$, where $\mathrm{Diff}^+(X)$ is the space of orientation-preserving self-diffeomorphisms of X. One way to describe a smooth mapping class is analogous to Dehn twists on surfaces: describe an explicit self-diffeomorphism of some standard neighborhood of some standard object, which is the identity on the boundary of that neighborhood, then implant this diffeomorphism via an embedding of this standard object into the ambient manifold and extend by the identity outside the neighborhood. Smooth mapping classes which are pseudoisotopic to the identity can also be described in a very different way via families of handlebodies, following Cerf [2]: If $\phi\colon X \to X$ is pseudoisotopic to the identity via a pseudoisotopy $\Phi\colon [0,1]\times X \to [0,1]\times X$, let f_t be a 1-parameter family of (generalized) Morse functions interpolating from $f_0 = \pi_{[0,1]}$, projection onto $[0,1]$, to $f_1 = f_0 \circ \Phi$; choosing an associated family of gradient-like vector fields then gives a family of handlebody structures on $X \times [0,1]$. The 1-parameter family of handle-attaching data in X then determines ϕ up to isotopy. Since f_0 and f_1 are Morse functions without critical points, this family starts and ends with canceling handle pairs.

Using the fact that every orientation-preserving diffeomorphism of S^4 is pseudoisotopic to the identity, the first author showed in [3] that every element of $\pi_0(\mathrm{Diff}^+(S^4))$ can be given in this Cerf-theoretic way by a 1-parameter family involving only 2-3-handle pairs, and that under favorable conditions (it is unclear whether these conditions might perhaps always be satisfied) such a family can be traded for a family involving a single 1-2-handle pair. Here we study the subgroup of $\pi_0(\mathrm{Diff}^+(S^4))$ coming from families of 1-2-handle pairs, the "1-2-subgroup". In [3] the first author gave a countable list of generators for the 1-2-subgroup, explicitly described as Dehn twist-like diffeomorphisms where the embedded object is a Montesinos twin, a pair of spheres intersecting transversely at two points. Here we go back and forth between the family of handlebodies perspective and the Montesinos twists perspective to show that this 1-2-subgroup is actually generated by a single element, and that the square of this element is trivial.

We now develop these two perspectives more carefully:

Definition 1 Given an embedding $f\colon S^1\times\Sigma\hookrightarrow X$, for some closed oriented surface Σ and some smooth oriented 4-manifold X, the *twist along* f is the isotopy class of diffeomorphisms τ_f obtained by choosing an orientation-preserving embedding $[-1,1]\times S^1\times\Sigma$ extending f, and performing a right-handed Dehn twist along $[-1,1]\times S^1$ and the identity along Σ. As an element of $\pi_0(\mathrm{Diff}^+(X))$, τ_f is uniquely determined by the isotopy class of the embedding f.

Note that in this definition the twist τ_f is sensitive to the separate orientations of the two factors S^1 and Σ in the embedded 3-dimensional submanifold $f(S^1\times\Sigma)\subset X$. In particular, if we reverse the orientation of $f(S^1\times\Sigma)$ by precomposing f with an orientation-reversing diffeomorphism of S^1 then the extension to an embedding of $[-1,1]\times S^1\times\Sigma$ needs to reverse the $[-1,1]$ direction, but in the end this reverses both factors in the annulus $[-1,1]\times S^1$, so we do not change the meaning of "right-handed Dehn twist". On the other hand, if we reverse the orientation of $f(S^1\times\Sigma)$ by precomposing f with an orientation-reversing diffeomorphism of Σ then we again need to reverse the $[-1,1]$ direction in the extension to $[-1,1]\times S^1\times\Sigma$, but since we did not reverse the orientation of S^1 we did in the end reverse the orientation of the annulus factor $[-1,1]\times S^1$ and right-handed Dehn twists become left-handed Dehn twists. In short, reversing the orientation of the S^1 factor does not change τ_f while reversing the orientation of the Σ factor turns τ_f into τ_f^{-1}.

Definition 2 A *Montesinos twin* in a 4-manifold X is a pair $W=(R,S)$ of embeddings $R,S\colon S^2\hookrightarrow X$, each with trivial normal bundle, which intersect transversely at two points. A *half-unknotted* Montesinos twin is one in which one of the two 2-spheres is unknotted.

As shown by Montesinos [5; 6] and discussed in [3], the boundary $\partial\nu(W)$ of a neighborhood of a Montesinos twin W in an oriented 4-manifold X is diffeomorphic to T^3, and if $X=S^4$ and we label the factors of T^3 as $S^1_l\times S^1_R\times S^1_S$, this parametrization of $\partial\nu(W)$ is canonically determined by the oriented isotopy class of W up to independent orientation-preserving reparametrizations of S^1_l, S^1_R and S^1_S and ambient isotopy in X. The S^1_l factor is homologically trivial in S^4, while S^1_R is a positive meridian for R and S^1_S is a positive meridian for S.

Definition 3 Given a Montesinos twin W in S^4, the *twist along* W, denoted by τ_W, is the twist along the embedding $S^1_l\times(S^1_R\times S^1_S)\hookrightarrow S^4$, as given in Definition 1. Let $\mathcal{M}$ be the subgroup of $\pi_0(\mathrm{Diff}^+(S^4))$ generated by twists along Montesinos twins. Let $\mathcal{M}_0\subset\mathcal{M}$ be the subgroup generated by twists along half-unknotted Montesinos twins $W=(R,S)$.

As we will discuss below, $\mathcal{M}_0$ is precisely the 1-2-subgroup of $\pi_0(\mathrm{Diff}^+(S^4))$.

The following is our main result:

Theorem 4 *The group $\mathcal{M}_0$ generated by twists along half-unknotted Montesinos twins is either trivial or cyclic of order* 2.

Besides the Cerf-theoretic perspective identifying $\mathcal{M}_0$ as the 1-2-subgroup, one can also think of $\mathcal{M}_0$ as the simplest class of "twist subgroups" of $\pi_0(\mathrm{Diff}^+(S^4))$, with $\mathcal{M}$, the subgroup generated by twists along arbitrary Montesinos twins, being the next interesting case. Continuing from there, one can consider subgroups generated by twists along general embeddings of $S^1 \times \Sigma_g$ for surfaces of various genus g. It would be interesting to know if Theorem 4 can be generalized to say something about these potentially more complicated subgroups.

Acknowledgments Both authors were supported in their work on this project by National Science Foundation grant DMS-2005554 *Smooth* 4*-manifolds*: 2-, 3-, 5- *and* 6*-dimensional perspectives*.

2 The proof modulo one main calculation

There are two main ingredients in the proof of Theorem 4. The first is part of [3, Lemma 3]:

Lemma 5 $$\tau_{(S,R)} = \tau^{-1}_{(R,S)}.$$

Proof Switching R and S changes the parametrization of the boundary of a tubular neighborhood of $R \cup S$ from $S^1_l \times S^1_R \times S^1_S$ to $S^1_l \times S^1_S \times S^1_R$. However, we are not changing the orientations of R or S, and thus the orientations of the meridians S^1_R and S^1_S do not change. Therefore to keep our parametrization of the boundary of $R \cup S$ correctly oriented, we need to switch the orientation of the longitudinal S^1_l factor. Then, when we turn this into a parametrization of a neighborhood of this 3-torus as $[-1,1] \times S^1_l \times T^2$, the $[-1,1]$ direction does not change orientation (being oriented by the outward normal convention), and thus the annulus $[-1,1] \times S^1_l$ in fact *does* change orientation. Therefore the Dehn twist along this annulus switches from a positive Dehn twist to a negative Dehn twist, and so $\tau_{(S,R)} = \tau^{-1}_{(R,S)}$. □

The second ingredient expands on the connection between twists along Montesinos twins and Cerf-theoretically described diffeomorphisms of S^4 developed in [3], so as to get a full set of generators and some relations for $\mathcal{M}_0$. We now briefly set up the Cerf-theoretic picture in a little more detail.

Let $\mathrm{Emb}(S^1, S^1 \times S^3)$ be the space of embeddings of S^1 into $S^1 \times S^3$, with basepoint taken to be the embedding $S^1 \times \{p\}$ for some $p \in S^3$. (In other words, without further comment, we will only be working in the component of $\mathrm{Emb}(S^1, S^1 \times S^3)$ containing $S^1 \times \{p\}$ and we will take that to be the basepoint for $\pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$.) There is a homomorphism

$$\mathcal{H}\colon \pi_1(\mathrm{Emb}(S^1, S^1 \times S^3)) \to \pi_0(\mathrm{Diff}^+(S^4)$$

discussed in [3] (where it is called $\mathcal{FH}_1$), which we describe briefly: Given $a \in \pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$, let $\alpha_t\colon S^1 \hookrightarrow S^1 \times S^3$, with $t \in [0,1]$, be a loop of embeddings representing a, with $\alpha_0 = \alpha_1 = S^1 \times \{p\}$. Extend this to a loop of *framed* embeddings (the fact that we can do this is also explained in [3]). For each t, let Z_t be the 5-dimensional cobordism built by starting with $[0,1] \times S^4$, attaching a 5-dimensional 1-handle along a fixed standard attaching map into $\{1\} \times S^4$, to give an upper boundary canonically identified with $S^1 \times S^3$, and then attaching a 2-handle along the framed circle α_t. Note that Z_0 and Z_1 are the same 5-manifold (ie built exactly the same way), so we can put these cobordisms together to build a 6-manifold W fibering over $S^1 = [0,1]/1 \sim 0$, with fiber over $t \in S^1$ equal to Z_t, so that W itself is a cobordism from $S^1 \times S^4$ to some 4-manifold bundle over S^1. In other words, each Z_t is a cobordism from S^4 to some 4-manifold X_t, and the top boundary of W is a 5-manifold fibering over S^1 with fiber over $t \in S^1$ equal to X_t. Furthermore, since our basepoint is $S^1 \times \{p\}$, the 2-handle at $t = 0 = 1$ cancels the 1-handle, and thus X_0 can be canonically identified with S^4. Hence the top of the cobordism W is in fact an S^4-bundle over S^1 with some potentially interesting monodromy which is determined by the (homotopy class of the) loop of attaching maps α_t. This monodromy, as an element of $\pi_0(\mathrm{Diff}^+(S^4))$, is by definition $\mathcal{H}(a)$.

There is an obvious subgroup of $\pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$ in the kernel of $\mathcal{H}$, namely the subgroup of homotopy classes represented by embeddings with image equal to $S^1 \times \{p\}$, ie the subgroup corresponding to reparametrizations of the domain S^1 (or "spinning the circle in place"). By multiplying by elements of this subgroup, we can thus always assume that our loops of embeddings $\alpha_t\colon S^1 \hookrightarrow S^1 \times S^3$ have the property that the circle $\{\alpha_t(z) \mid t \in [0,1]\}$, for a fixed $z \in S^1$, is homotopically trivial in $\pi_1(S^1 \times S^3) = \mathbb{Z}$.

The connection between Montesinos twins and loops of circles in $S^1 \times S^3$ is seen as follows: Given a loop $\alpha_t\colon S^1 \hookrightarrow S^1 \times S^3$ as in the preceding paragraph, suppose that the mapped in torus $T\colon S^1 \times S^1 \to S^1 \times S^3$ defined by $T(t,z) = \alpha_t(z)$ is actually an embedding. (Budney and Gabai [1] in fact show that every element of $\pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$ can be represented by such a loop α_t.) Let C be the basepoint circle $S^1 \times \{p\}$. Note that C lies on T. Surgery along C turns the triple $(S^1 \times S^3, T, C)$ into a triple (S^4, R, S) where R is an embedded S^2 in S^4 obtained by surgering the torus T along the essential simple closed curve C and S is the embedded S^2 which is the cocore of the surgery, or belt sphere of the associated 5-dimensional 2-handle (ie the surgery replaces $S^1 \times B^3$ with $B^2 \times S^2$, and S is $\{0\} \times S^2 \subset B^2 \times S^2$). Furthermore, R and S intersect transversely at two points, namely the two "scars" on R resulting from surgering the torus down to a sphere along C. In other words, $W = (R, S)$ is a Montesinos twin in S^4, which we will call the twin associated to the loop α_t. Note that S is unknotted, since it results from surgery along $S^1 \times \{p\} \subset S^1 \times S^3$, so that W is a half-unknotted Montesinos twin. Conversely, given a Montesinos twin $W = (R, S)$ in S^4, *if we assume that S is unknotted*, then surgering (S^4, R, S) along S yields $(S^1 \times S^3, T, C)$, where $C = S^1 \times \{p\}$ and T is an embedding of $S^1 \times S^1$ which is the trace of a loop of embeddings $\alpha_t\colon S^1 \hookrightarrow S^1 \times S^3$; we will call this the loop of circles associated to the twin W.

The following lemma is implicit in the proof of [3, Theorem 4]:

Lemma 6 *Let $W = (R, S)$ be a half-unknotted Montesinos twin in S^4; for the moment assume that S is unknotted. Let α_t be the loop of circles in $S^1 \times S^3$ associated to W. Then $\tau_W = \mathcal{H}([\alpha_t])$. If, on the other hand, R is unknotted, then let $\overline{\alpha}_t$ be the loop of circles associated to $\overline{W} = (S, R)$. In this case, $\tau_W = \tau_{\overline{W}}^{-1} = \mathcal{H}([\overline{\alpha}_t])^{-1} = \mathcal{H}([\overline{\alpha}_{1-t}])$.*

Note that there are orientation conventions hidden in the above statement. In particular, one needs to understand how the orientations of R and S determine the orientations both of each circle α_t, for each t, and of the t direction in the loop of circles; equivalently, one needs to understand how the orientations of R and S correspond to the orientations of meridian and longitude for the torus $T\colon S^1 \times S^1 \hookrightarrow S^1 \times S^3$. The truth is that it suffices to know that there exists some orientation convention that makes it correct, but in the end we will not need to nail down that convention to get our proofs correct, because we show that the whole group involved is trivial or cyclic of order 2.

Proof We have to show that, with appropriate orientation conventions, $\tau_W = \mathcal{H}([\alpha_t])$ when $W = (R, S)$ and S is unknotted. The second half of the statement of Lemma 6, for $\overline{W}$, follows directly from Lemma 5.

Given a loop of embeddings $\alpha_t\colon S^1 \hookrightarrow S^1 \times S^3$, a diffeomorphism representing $\mathcal{H}([\alpha_t])$ can be defined as follows. (Note that this idea goes back to Wall's proof [7] realizing automorphisms of intersection forms of 4-manifolds by diffeomorphisms.) Using the isotopy extension theorem, let $\psi_t\colon S^1 \times S^3 \to S^1 \times S^3$ be an ambient isotopy starting from $\psi_0 = \mathrm{id}$, for $t \in [0, 1]$, such that $\alpha_t = \psi_t \circ \alpha_0$. Since $\alpha_1 = \alpha_0 = S^1 \times \{p\} \subset S^1 \times S^3$ we can assume that ψ_1 is the identity on $S^1 \times U$ for U a 3-ball neighborhood of p. Then surgery along $\alpha_0 = S^1 \times \{p\}$ turns $S^1 \times S^3$ into S^4 in such a way that ψ_1 extends as the identity across the surgered region, and thus ψ_1 can be seen as a self-diffeomorphism of S^4, and the isotopy class of ψ_1 on S^4 is $\mathcal{H}([\alpha_t])$.

When the associated torus $T\colon S^1 \times S^1 \to S^1 \times S^3$ given by $T(t, b) = \alpha_t(b)$ is an embedding, then there is a standard construction of an explicit ambient isotopy ψ_t supported in a neighborhood $D^2 \times S^1 \times S^1$ of T as follows: Let $(r, \theta) \in [0, 1] \times [0, 2\pi]$ be polar coordinates on D^2, with coordinates (a, b) on $S^1 \times S^1$. (We have replaced the original t variable by a because now it represents a spatial coordinate, and t will be used for the time parameter in the isotopy.) Let $f\colon [0, 1] \to [0, 1]$ be a smooth nonincreasing function which is 1 on $[0, \epsilon]$ and 0 on $[1 - \epsilon, 1]$ for some suitably small positive ϵ. Then

$$\psi_t(r, \theta, a, b) = (r, \theta, a + tf(r), b)$$

is the desired isotopy. The complement of $\{r < \frac{1}{2}\epsilon\}$ in our neighborhood $D^2 \times S^1 \times S^1$ can now be parametrized (and oriented) as $\left[\frac{1}{2}\epsilon, 1\right] \times S^1_\theta \times S^1_a \times S^1_b$. This orientation agrees with the orientation as $\left[\frac{1}{2}\epsilon, 1\right] \times S^1_a \times S^1_b \times S^1_\theta$ and, with respect to this orientation, we see that ψ_1 is a *positive* Dehn twist on $\left[\frac{1}{2}\epsilon, 1\right] \times S^1_a$ crossed with the identity on $S^1_b \times S^1_\theta$.

We now need to see that the three circle parameters S^1_a, S^1_b and S^1_θ translate into S^1_l, S^1_R and S^1_S when we surger along C to turn T into a Montesinos twin (R, S). In particular we need to make sure that S^1_a becomes S^1_l so that the Dehn twist on $\left[\frac{1}{2}\epsilon, 1\right] \times S^1_a$ becomes the Dehn twist on $[-1, 1] \times S^1_l$

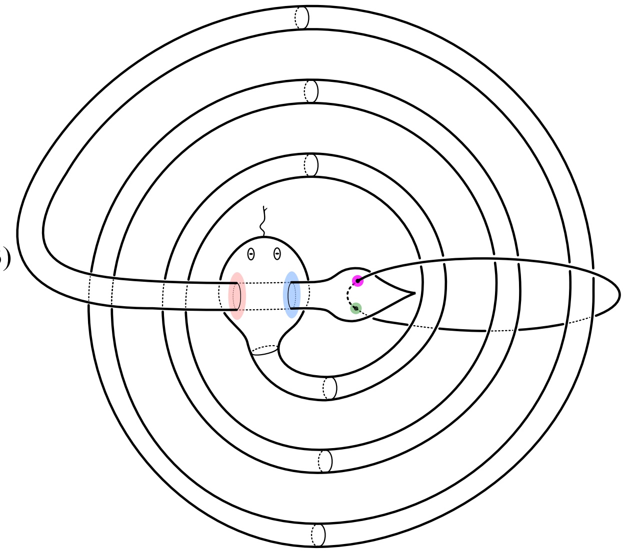

Figure 1: An illustration of $W(3)=(R(3),S)$, with the generalization to $W(i)$ being to wrap i times around instead of three times around. The red and blue disks in $R(3)$ ("ear holes" of the snake) are pushed forward and backwards in time to avoid self-intersection. We only show the equator of S, with the hemispheres lying in the past and future. The two intersection points are colored pink and green.

in our original parametrization of the neighborhood of a Montesinos twin. First of all, C is the circle $\{r=0,\, a=0,\, b\in S^1_b\}$ in $T=\{r=0,\, a\in S^1_a,\, b\in S^1_b\}$. The circle S^1_θ links T and thus, when we surger T to become the 2-sphere R, S^1_θ becomes the meridian S^1_R. The circle S^1_b essentially *is* the circle C, and thus after surgery becomes the meridian S^1_S to the new 2-sphere S. Finally, in order to see that S^1_a becomes the longitudinal circle S^1_l, we just need to see that S^1_a is homologically trivial in the complement of T. This follows from the fact that S^1_a is homotopically trivial in $S^1\times S^3$, which we arranged earlier by multiplying by an appropriate element of the domain reparametrization subgroup of $\pi_1(\mathrm{Emb}(S^1,S^1\times S^3))$. □

These facts, together with the fact that $\pi_1(\mathrm{Emb}(S^1,S^1\times S^3))$ is generated by loops which come from embedded tori, immediately give us the fact that the group of isotopy classes of diffeomorphisms of S^4 coming from loops of circles in $S^1\times S^3$ agrees with the group generated by twists along half-unknotted Montesinos twins:

Corollary 7 $$\mathcal{H}(\pi_1(\mathrm{Emb}(S^1,S^1\times S^3)))=\mathcal{M}_0.$$

One of the main results of [3] can be restated (combining Corollary 14 and Theorem 4 of [3] with Corollary 7) as:

Theorem 8 *$\mathcal{M}_0$ is generated by twists $\tau_{W(i)}$ for $i\in\mathbb{N}$, for the Montesinos twins $W(i)=(R(i),S)$ illustrated in* [3, Figures 1, 2 and 3]. *The loops of circles $\alpha(i)_t\colon S^1\hookrightarrow S^1\times S^3$ associated to these twins are described by the embedded tori $T(i)$ illustrated in* [3, Figure 8].

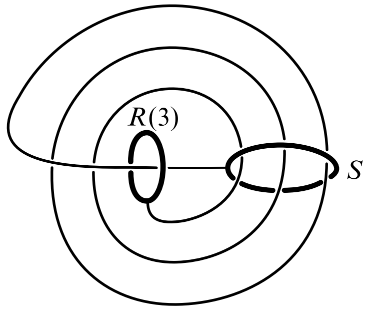

Figure 2: An alternative illustration of $W(3)$, involving two disjoint embedded 2-spheres in S^4 (the two thick circles capped off with hemispheres in past and future) and an arc connecting them. Pushing a finger from one of the spheres out along this arc and then doing a finger move when one encounters the other sphere, creating a pair of transverse intersections, gives $W(3)$. To recover Figure 1, push the finger from $R(3)$ until it meets S. However, this description is more "balanced" between $R(3)$, allowing the user to decide which sphere they prefer to draw as the complicated one.

Figures 1 and 2 reproduce two illustrations of $W(3)$ from [3]; the generalization to $W(i)$ is clear.

An important feature of the twins $W(i) = (R(i), S)$ is that *both* $R(i)$ and S are unknotted. Thus, in addition to the loops of circles $\alpha(i)_t$ associated to $W(i)$, we have loops of circles $\overline{\alpha(i)}_t$ associated to $\overline{W(i)} = (S, R(i))$. Then by Lemma 5, we know that

$$\tau^{-1}_{\overline{W(i)}} = \mathcal{H}([\overline{\alpha(i)}_t]).$$

Our main calculation in this paper is:

Proposition 9 *In* $\pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$ *we have* $[\overline{\alpha(i)}_t] = n_i[\alpha(1)_t]$ *for some integer* n_i*.*

(We use additive notation for $\pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$ because Budney and Gabai show that this group is abelian [1].) In fact one can show that $n_i = \pm i$, but we do not need such a precise result, and the result as stated is quicker and easier to prove. We combine the above result with the following observation:

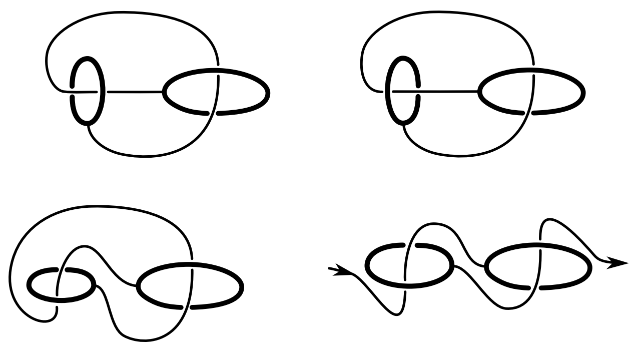

Figure 3: Isotoping $W(1)$ into a symmetric position so as to see that $W(1) = (R(1), S)$ is isotopic to $\overline{W(1)} = (S, R(1))$.

Lemma 10 *The twin $W(1) = (R(1), S)$ is isotopic (taking orientations into account) to $\overline{W(1)} = (S, R(1))$ and thus, by Lemma 5, $\tau_{W(1)} = \tau^{-1}_{\overline{W(1)}}$.*

Proof Figure 3 illustrates $W(1)$ using the "finger move" schematic of Figure 2, and then shows an isotopy to a diagram which is obviously symmetric between the two 2-spheres. □

Note that in [1] the authors discuss "barbell diffeomorphisms", clearly related to Montesinos twists, and give conditions under which barbell diffeomorphisms have order 2. Most likely Lemma 10 is a consequence of [1, Proposition 5.17].

From these three results we get:

Proof of Theorem 4 An immediate corollary of Proposition 9 is that, switching to additive notation for $\pi_0(\mathrm{Diff}^+(S^4))$,

$$-\tau_{W(i)} = \pm n_i \tau_{W(1)}.$$

From Lemma 10, we know that $\tau_{W(1)}$ has order 2 or is trivial. Since $\mathcal{M}_0$ is generated by $\{\tau_{W(i)}, i \in \mathbb{N}\}$, we conclude that $\mathcal{M}_0$ is generated by $\tau_{W(1)}$ and thus is either the trivial group or the cyclic group of order 2. □

The rest of this paper is devoted to proving Proposition 9.

3 Calculating the Budney–Gabai invariants

To prove Proposition 9, we need a picture of the loop of circles $\overline{\alpha(i)}_t$ in $S^1 \times S^3$ associated to the Montesinos twin $\overline{W(i)} = (S, R(i))$ in S^4. To get this, we need to first draw a picture of $\overline{W(i)} = (S, R(i))$ in which $R(i)$ appears as a standard unknotted S^2 and S appears as the interesting half of the twin. Then we can surger along $R(i)$ so as to draw a picture of the resulting embedded torus $\overline{T(i)}$ in $S^1 \times S^3$, from which we can understand the loop of embeddings $\overline{\alpha(i)}_t$. This will then be used to compute the W_2 invariant of $[\overline{\alpha(i)}_t] \in \pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$ defined in [1]. As the calculation will be sufficient to prove the proposition, we give a summary of Budney and Gabai's W_2 invariant below.

Figure 2 is most useful for performing the isotopy that standardizes $R(i)$ and leaves S looking complicated. To recover Figure 1 from Figure 2, one pushes the finger from the sphere labeled $R(3)$ along the arc until meeting S, and then performing a finger move there. However, one obtains an isotopic Montesinos twin by pushing the figure out along the arc starting from S until it meets $R(3)$ and then performing the finger move; this leaves $R(3)$ still "looking" like an unknot. In fact, we can first perform an isotopy to the diagram in Figure 2 to put $R(3)$ into exactly the position where S was, as in Figure 4. The final step in Figure 4 represents the result of surgering along the (now standardized) $W(3)$ to the torus $\overline{T(3)}$ in $S^1 \times S^3$. Figure 5 illustrates this final torus $\overline{T(3)}$ more explicitly in $S^1 \times S^3$, and this should be compared to Figure 6 (Figure 8 in [3]), which illustrates the original $T(3)$.

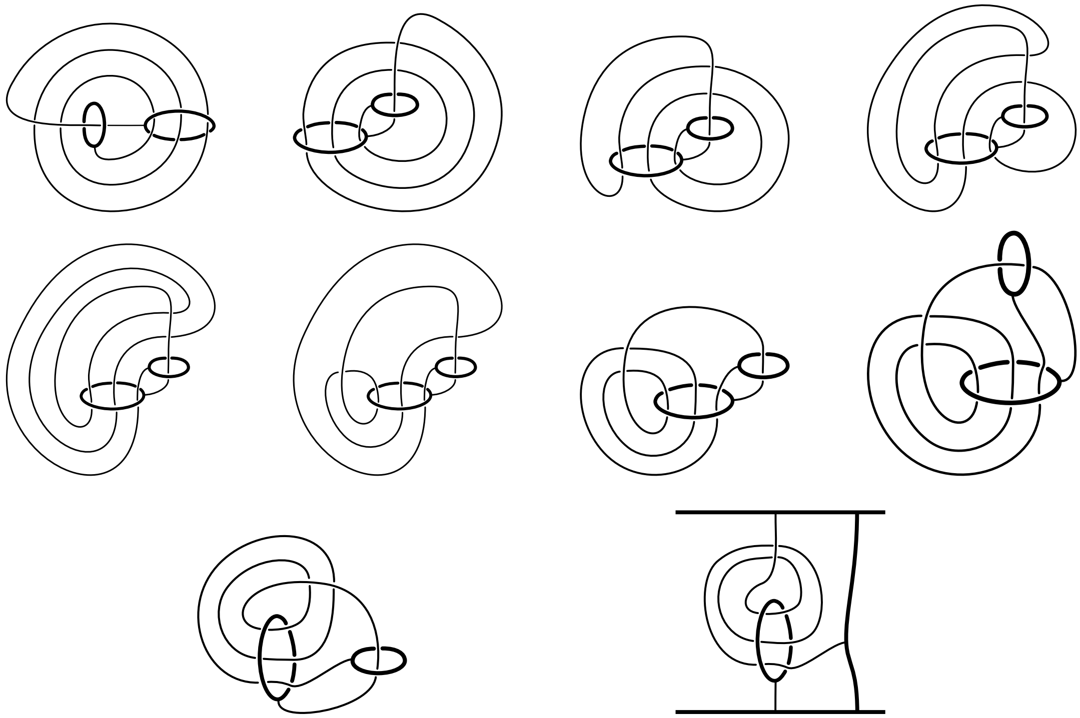

Figure 4: An isotopy of $W(3)$. The final frame should be interpreted as a diagram of an embedded torus in $S^1 \times S^3$, the result of surgering along $R(3)$. The interpretation of this diagram is made clearer in Figure 5.

Now we discuss Budney and Gabai's analysis of $\pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$ and the aforementioned W_2 invariant.

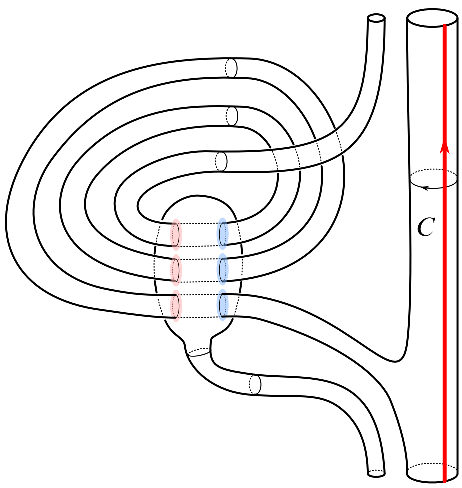

Figure 5: The embedded torus $\overline{T(3)}$ in $S^1 \times S^3$. The top is glued to the bottom, and horizontal slices are S^3's, with the "time" coordinate indicated in red/blue shading, as in Figure 1.

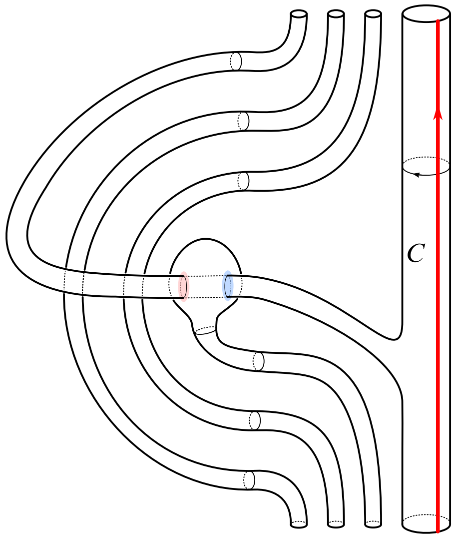

Figure 6: The embedded torus $T(3)$ in $S^1 \times S^3$, the obvious next member of the family of tori described in [1, Figure 4].

In [1, Theorem 2.9] the authors compute π_1 of all path components of $\pi_1(\mathrm{Emb}(S^1, S^1 \times S^3))$. Here we summarize their main result only for the component we care about, the component containing our chosen basepoint $S^1 \times \{p\}$: There is an isomorphism

$$W_1 \times W_2 \colon \pi_1(\mathrm{Emb}(S^1, S^1 \times S^3)) \to \mathbb{Z} \times \Lambda^0, \tag{1}$$

where

$$\Lambda^0 = \mathbb{Z}[x, x^{-1}]/\langle x^n - x^{-n}\ \forall n \in \mathbb{Z},\, x^0,\, x^{-1}\rangle.$$

The isomorphism $W_1 \times W_2$ is the product of two homomorphisms

$$W_1 \colon \pi_1(\mathrm{Emb}(S^1, S^1 \times S^3) \to \mathbb{Z} \quad \text{and} \quad W_2 \colon \pi_1(\mathrm{Emb}(S^1, S^1 \times S^3) \to \Lambda^0.$$

The invariant W_1 detects "spinning the circle in place" as discussed in the proof of Lemma 6. We have assumed already that our loops of embeddings $\alpha_t \colon S^1 \hookrightarrow S^1 \times S^3$ have the property that for a fixed $z \in S^1$, the loop $\{\alpha_t(z), t \in [0,1]\}$ is homotopically trivial. This directly translates to saying that we can always assume that $W_1([\alpha_t]) = 0$ for any of the loops of circles we will be considering.

Remark 11 As discussed in Lemma 6, we are free to reparametrize the domain of our loops, without affecting $\mathcal{H}$. Considering the isomorphism (1), we get that $\ker(W_2) \subset \ker(\mathcal{H})$.

The important invariant to discuss is thus W_2. We begin by reviewing the definition of W_2 found in [1]. Following the authors' notation, we will denote the two-point configuration space of a manifold M by $C_2(M)$. Let $\mathcal{CC} \subset C_2(S^1 \times S^3)$ be the submanifold of points of the form $((z_1, p), (z_2, p))$, diffeomorphic to $C_2(S^1) \times S^3$, with an orientation coming from the diffeomorphism

$$((z_1, z_2), p) \to ((z_1, p), (z_2, p)).$$

Note that $C_2(S^1)$ may be familiar to many low-dimensional topologists as the pillowcase S^2.

Given any loop of embeddings $\alpha_t: S^1 \hookrightarrow S^1 \times S^3$, the authors define a map $A: S^1 \times C_2(S^1) \to C_2(S^1 \times S^3)$ given by the formula

$$A(t,(z_1,z_2)) = (\alpha_t(z_1), \alpha_t(z_2)).$$

After a slight perturbation if necessary, we may assume A to be transverse to $\mathcal{CC}$, as the set of transverse maps is dense (but not open) [4]. However, to prove homotopy invariance, Budney and Gabai work instead with the Fulton–MacPherson compactification of the configuration spaces. Here transversality is both open and dense, and ultimately is where two of the relations defining Λ originate from. With A transverse to $\mathcal{CC}$, one sees that $A^{-1}(\mathcal{CC})$ is just a finite collection of points. On the set $A^{-1}(\mathcal{CC})$, there is a natural Σ_2 action given by permuting the $C_2(S^1)$ coordinates. On the quotient $A^{-1}(\mathcal{CC})/\Sigma_2$, assign to $[p] \in A^{-1}(\mathcal{CC})/\Sigma_2$ the monomial $\pm x^{k_p}$. Here the sign of the monomial is given by the signed intersection number of $A(p)$ and $\mathcal{CC}$, for some representative p of $[p]$, and the degree k_p of the monomial is given by the following procedure. First we consider the coordinates of the point p: $(t,(z_1,z_2))$. Next, take the path $[z_1,z_2]$ going from z_1 to z_2 in S^1 in the positively oriented direction. As $p \in \mathcal{CC}$, the S^3 coordinate of $\alpha_t(z_1)$ equals the S^3 coordinate of $\alpha_t(z_2)$. Let B_p denote the arc in $S^1 \times S^3$ which connects $\alpha_t(z_2)$ to $\alpha_t(z_1)$ by moving along the S^1 factor in the direction opposite to the orientation, while keeping the S^3 coordinate fixed. With this, we construct a map $K_p: S^1 \to S^1 \times S^3$ by concatenating the arc $\alpha_t([z_1,z_2])$ with the arc B_p. The degree k_p is then given by

$$k_p = \deg(\pi_{S^1} \circ K_p).$$

Alternatively we can calculate k_p first by counting the signed intersection of K_p with $\{z'\} \times S^3$, for a generic choice of $z' \in S^1$. If we choose our S^3 slice away from both $\alpha_t(z_1)$ and $\alpha_t(z_2)$, then k_p can be calculated by counting the signed intersection of the arcs $\alpha_t[z_1,z_2]$ and B_p with the S^3 slice, and adding the result. So

$$k_p = \alpha_t([z_1,z_2]) \cdot (\{z'\} \times S^3) + B_p \cdot (\{z'\} \times S^3),$$

where $\cdot$ is the signed count of transverse intersection points.

Adding the monomials up over all $[p] \in A^{-1}(\mathcal{CC})/\Sigma_2$ gives the formula for W_2:

$$W_2([\alpha_t]) = \sum_{[p] \in A^{-1}(\mathcal{CC})/\Sigma_2} \pm x^{k_p}.$$

It is shown in [1] that W_2 is well defined on homotopy classes of loops when considered as a map to Λ^0. Moreover, none of the choices made affect the result as long as they are made consistently.

Proof of Proposition 9 We begin by recalling some notation. First, $T(i)$ and $\overline{T(i)}$ are all embedded tori in $S^1 \times S^3$ (see Figures 5 and 6). Each embedded torus then corresponds to a loop of embeddings of circles $\alpha(i)_t: S^1 \hookrightarrow S^1 \times S^3$ for $t \in [0,1]$ and $\overline{\alpha(i)}_t: S^1 \hookrightarrow S^1 \times S^3$ for $t \in [0,1]$, respectively. Each loop starts at the embedding $t \mapsto (t,p)$, depicted by the red curve C in both Figures 5 and 6. Now in [1], the authors show that $W_2([\alpha(1)_t]) = \pm x^2$. As W_2 is a homomorphism, the proposition will follow once we show that $W_2([\overline{\alpha(i)}_t]) = nx^2$, for some $n \in \mathbb{Z}$.

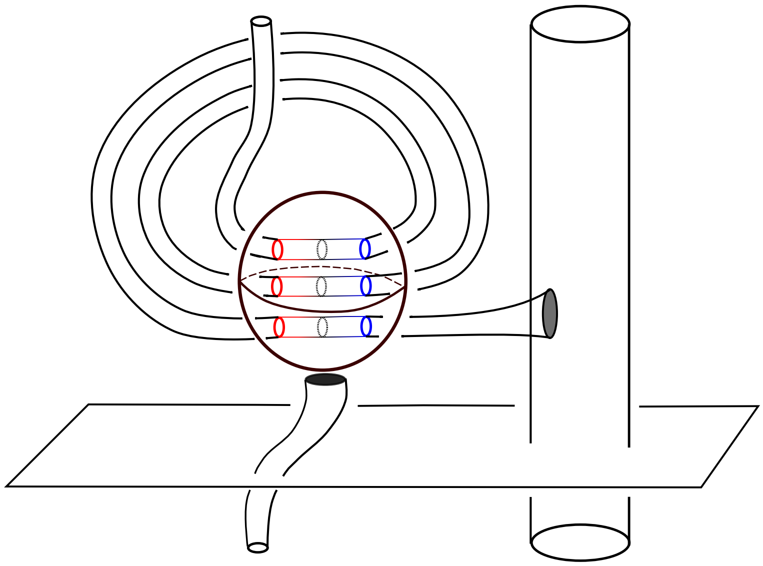

Figure 7: The tubing construction for the torus $\overline{T(3)}$. The general torus $\overline{T(i)}$ is given by using a tube which links the embedded sphere i times before it is attached to the sphere.

To begin our calculation, we note that the torus $\overline{T(i)}$ can be constructed by tubing together the "standard" torus with a 2-sphere as illustrated in Figure 7, with i equal to the number of times the tube spirals around and through the 2-sphere. The ambient space in the figure should be viewed as $[0,1]\times S^3$, with $\{0\}\times S^3$ being identified with $\{1\}\times S^3$ by the identity. Note that all the "spiraling" the tube does happens between $\{0\}\times S^3$ and $\{1\}\times S^3$. So, to calculate W_2, we will compute intersections with the sphere $\{0\}\times S^3$. Note that there are two values of t when $\alpha(i)_t$ intersects $\{0\}\times S^3$ nontransversely, but transversality of A with $\mathcal{CC}$ means that these values of t will not occur in the calculation of W_2.

Supposing (t,z_1,z_2) is a representative for $[p]\in A^{-1}(\mathcal{CC})/\Sigma_2$,

$$k_p=\overline{\alpha(i)}_t([z_1,z_2])\cdot(\{0\}\times S^3)+B_p\cdot(\{0\}\times S^3).$$

As $\overline{\alpha(i)}_t([z_1,z_2])$ is a subarc of the embedded circle $\overline{\alpha(i)}_t(S^1)$, and $\overline{\alpha(i)}_t(S^1)$ has either exactly one positive intersection or exactly two positive intersections and one negative intersection with $\{0\}\times S^3$, the signed intersection count of $\overline{\alpha(i)}_t([z_1,z_2])$ with $\{0\}\times S^3$ lies in the set $\{0,1,-1,2\}$. For the arc B_p, this is a subarc of $S^1\times\{v_p\}$ for some point $v_p\in S^3$, oriented opposite to the given orientation of S^1 in $S^1\times S^3$. As such, the intersection number of B_p with $\{0\}\times S^3$ is either 0 or -1. Putting these together, we get that

$$W_2([\overline{\alpha(i)}_t])=m_{-2}x^{-2}+m_{-1}x^{-1}+m_0x^0+m_1x^1+m_2x^2,$$

for some set of integers m_k for $k=-2,-1,0,1,2$. Finally, by considering the relations for Λ^0, we get

$$W_2([\overline{\alpha(i)}_t])=nx^2,$$

where $n=m_{-2}+m_2$. □

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