DISSIPATION ENHANCEMENT BY SHEAR FLOWS FOR GENERALIZED DIFFUSION OPERATORS IN HIGH DIMENSION *

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Abstract. We establish conditions for shear flows on the d-dimensional torus $\mathbb{T}^d$, $d\geq 2$, that give enhanced dissipation for the associated linear advection-diffusion equation for well-prepared data. The diffusion operator can be of fractional or high order and does not need to have constant coefficients. We then construct flows that satisfy these assumptions and obtain a quantitative estimate on the dissipation enhancement. Our examples generalize known examples in two space dimensions to the high-dimensional setting, which is relevant in applications to sampling a distribution and in optimization.

Keywords. generalized diffusion; advection; dissipation enhancement; shear flow; high-dimensional flows

AMS subject classifications. 35K25; 35K58; 76E06; 76F25

1. Introduction

This work concerns enhanced dissipation in arbitrarily high dimension for the (scalar) linear advection-diffusion equation, where the diffusion operator is a generalization of the standard Laplace operator to allow for fractional and higher-order operators and variable coefficients. The flow will be assumed incompressible, that is, the advecting velocity $\boldsymbol{v}$ is divergence free. For the case of the Laplace operator, the advection-diffusion equation can be written as a Fokker-Planck equation for a drift-diffusion process. The Fokker-Planck equation models the evolution of the probability density function associated to the process. High-dimensional drift-diffusion processes have been used in many applications, such as in diffusion approximation of Stochastic Gradient Descent methods for non-convex optimization [27]. As customary in fluid mechanics, we call the velocity $\boldsymbol{v}$ also the flow.

By *enhanced dissipation*, we mean in this context that the linear advection-diffusion operator acts on characteristic timescales that are faster than those of the diffusion alone. The characteristic timescale is defined as the smallest time it takes for the solution operator to reduce the size of the solution by a fixed fraction. We will measure the size of the solution in terms of its L^2 norm, which is well adapted to the use of Fourier analytic techniques. Because we work with linear parabolic equations, the characteristic timescale is determined also by the rate of decay in time of the solution operator as a function of the diffusion coefficient or *diffusivity* $\nu>0$. Equivalently, one can fix $\nu=1$ and obtain the decay in terms of the flow amplitude $A>0$.

We let $\mathbb{T}^d=[0,2\pi]^d$, $d\geq 2$, denote the standard d-dimensional torus. In the case of other periods L, there will be a dependence of constants and rates on L. We work with *shear flows*, which without loss of generality can be defined as follows. We denote a point $\boldsymbol{x}$ in $\mathbb{T}^d$ as $\boldsymbol{x}=(x_1,x_2,\ldots,x_{d-1},y)=:(\boldsymbol{x}',y)$, then a shear flow is an incompressible flow with velocity given by $v(\boldsymbol{x})=(\boldsymbol{u}(y),0)$. We restrict to considering smooth flows. A special case is given by a unidirectional shear, where $\boldsymbol{u}(y)=(u(y),0,\ldots,0)$. The scalar function u is called the *shear profile*, e.g., $u(y)=\sin(y)$, which gives the *Kolmogorov*

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flow in two dimensions. We will give specific examples later in the paper. It is an important question whether more general flows can be considered. Indeed, enhanced dissipation holds for certain steady flows in any dimension [9], for certain circularly symmetric flows in two space dimensions and pipe flows in three dimensions [10,12,19], and certain cellular flows in two dimensions [6,28] (see also [20]). Hence, a natural generalization of our results is to consider a flow of the form $\boldsymbol{v}(\boldsymbol{x})=(\boldsymbol{u}(\boldsymbol{y}),0)$, where now $\boldsymbol{x}=(x_1,x_2,\dots,x_{d-2},y_1,y_2)=:(\boldsymbol{x}',\boldsymbol{y})$. However, it is not clear that the methods used in this work extend to this case, in particular it is not clear what the analog of Assumption 1.1 is in this context (cf. [14]). Our focus here is on generalizing the diffusion operator rather than the convective term.

We then consider the following advection-diffusion equation:

$$\partial_t\theta(\boldsymbol{x}',y)=-\sum_{\ell=1}^{d-1}u_\ell(y)\partial_{x_\ell}\theta(\boldsymbol{x}',y)-\nu\sum_{\ell=1}^{d-1}\sigma_\ell D^{2\gamma}_{x_\ell}\theta(\boldsymbol{x}',y)-\nu D^\gamma_y(a(y)D^\gamma_y)\theta(\boldsymbol{x}',y) \tag{1.1}$$

$$=:-\tilde{\mathcal{L}}_\nu\theta(\boldsymbol{x}',y), \tag{1.2}$$

where $\sigma_\ell\in\{0,1\}$ for $\ell\in\{1,\cdots,d-1\}$, $\gamma\in(\frac{1}{2},+\infty)$, the function a is smooth, positive and bounded away from zero. In particular, there exist positive constants $c_i>0$, $i=1,2$, such that

$$a(y)\geq c_1>0,\quad \text{a.e. } y\in\mathbb{T},\qquad \|a\|_{L^\infty}<c_2. \tag{1.3}$$

We define the fractional derivative D^γ_z as a Riesz-type derivative of order $\gamma\in\mathbb{R}^+$, i.e., as a Fourier multiplier. Let $f\in L^2(\mathbb{T})$, then

$$D^\gamma_z f=\sum_{k\in\mathbb{Z}}|k|^\gamma\hat{f}(k)e^{ikz}, \tag{1.4}$$

where $\hat{f}$ is the Fourier Transform of f and k is the wavenumber. We remark that D^γ_z is a positive operator and $D^2_z=-\partial^2_z$. By Plancherel's Theorem, it holds that

$$\|D^\gamma_z f\|^2_{L^2}=\sum_{k\in\mathbb{Z}}|k|^{2\gamma}|\hat{f}(k)|^2=\|f\|^2_{\dot{H}^\gamma(\mathbb{T})},$$

where $\dot{H}^\gamma$ is a homogeneous Sobolev space. In particular, D^γ_z is a self-adjoint, unbounded operator on $L^2(\mathbb{T})$ with (maximal) domain $\mathcal{D}(D^\gamma)=H^\gamma(\mathbb{T})$. The diffusion operator $\nu\big(\sum_{\ell=1}^{d-1}\sigma_\ell D^{2\gamma}_{x_\ell}+D^\gamma_y(a(y)D^\gamma_y)\big)$ in (1.1) is then strongly elliptic, self-adjoint and positive with (maximal) domain $H^{2\gamma}(\mathbb{T}^d)$. The global well-posedness of (1.1) can be established, for instance, by semigroup methods (see e.g. [35]) for any initial data $\theta_0\in L^2(\mathbb{T}^d)$, yielding a unique solution $\theta\in C([0,\infty);L^2(\mathbb{T}^d))\cap C((0,\infty);H^{2\gamma}(\mathbb{T}^d))\cap C^1(0,+\infty);L^2(\mathbb{T}^d))$. In fact, well-posedness can be proved under much weaker assumptions on a and $\boldsymbol{u}$, but we do not seek the optimal regularity in this work.

Informally, enhanced dissipation is the result of the transfer of energy by the advection operator to small scales, where it is damped more efficiently by diffusion, as it can be seen by taking the Fourier Transform. This phenomenon has been well studied in the mathematics literature, especially for reaction-diffusion equations and quenching in combustion (see e.g. [16,31] and references therein). The flow of $\boldsymbol{v}$ will be called *dissipation enhancing* (*relaxation enhancing* for the steady case) if the associated advection-diffusion operator exhibits dissipation enhancement. For steady Lipschitz-continuous flows and Laplace's operator, Constantin *et Al.* [9] give a spectral characterization of relaxation-enhancing flows, namely, the advection operator cannot have any eigenfunctions in $H^1(M)$, where M is a smooth compact Riemannian manifold. For unsteady flows, it is known that sufficiently regular mixing flows are dissipation enhancing (we refer the reader to [17] and references therein for the definition of mixing flows and a more in-depth discussion). In the case of shear flows, the advection operator $\boldsymbol{v}\cdot\nabla$ has a large kernel, so enhanced dissipation occurs only on the L^2-orthogonal complement to this kernel and it depends on the shear profile, in particular on its critical points and their order (cf. [21,28] for the case of cellular flows). Then, dissipation enhancement has been established by using different tools, from hypocoercivity estimates (we mention in particular [1,2,14]) to probabilistic methods [13] to resolvent estimates [11,23,37] (see also [19] for shear flows in a circular geometry). The study of enhanced dissipation is more challenging in the case of unbounded domains or domains with boundaries. In both cases, the spectral properties of the advection operator are quite different from the periodic case. Therefore, we work only in the torus $\mathbb{T}^d$.

There are related phenomena to dissipation enhancement in fluid flows. We mention *Taylor dispersion*, which is an enhancement of the rate of spreading of a species in the direction of the shear flow (see again [14] and references therein), and the stability of linearized viscous and inviscid flows around shear flows with the associated mechanism of *inviscid damping* (among the several recent results, see [5,7,22,29,33]). In non-linear systems, enhanced dissipation can have important effects, such as delaying or preventing blow-up, as in aggregation models, for instance the Platak-Keller-Segel model for which addition of advection by a sufficiently strong flow prevents blow-up of solutions irrespective of the total mass [3,20,24–26,36]. Similarly, addition of a strong enough background flow leads to global existence for the Kuramoto-Sivashinsky equation, a model of front propagation in combustion [11,18,20]. There is also a close connection between enhanced dissipation and accelerated sampling in Langevin dynamics. In [8], the first author of this paper and her collaborators constructed an exponentially mixing drift to the overdamped Langevin equation on the d-dimensional torus and obtained a signicantly smaller mixing time for the modified Langevin dynamics.

While some of the above works have considered fractional dissipation and the higher-dimensional setting (e.g. [26], where mixing flows are used), to our knowledge, *our paper is the first to address both variable-coefficient fractional operators and steady shear flows with a rather general shear profile in arbitrarily high dimensions.* Proving enhanced dissipation for shear flow is significantly more difficult than for mixing flows, because the shear only acts in certain directions. This is why we cannot treat general variable coefficients operators.

In view of the geometry of the shear flow $\boldsymbol{v}$, we consider first a reduced form of the operator in (1.1), where the diffusion is only in the direction orthogonal to $\boldsymbol{v}$:

$$\partial_t\theta=-\sum_{\ell=1}^{d-1}u_\ell(y)\partial_{x_\ell}\theta-\nu D_y^\gamma(a(y)D_y^\gamma)\theta=:-\mathcal{L}_\nu\theta(\boldsymbol{x}',y). \tag{1.5}$$

We study both (1.5) and (1.1) by applying the Fourier Transform in $\boldsymbol{x}'=(x_1,\dots,x_{d-1})$. Let $\boldsymbol{k}=(k_1,k_2,\cdots,k_{d-1})\in\mathbb{Z}^{d-1}$ be the associated wavenumber, and let

$$\tilde{\mathcal{L}}_{\nu,\boldsymbol{k}}:=\nu\sum_{\ell=1}^{d-1}\sigma_\ell|k_\ell|^{2\gamma}+\nu D_y^\gamma(a(y)D_y^\gamma)+i\sum_{\ell=1}^{d-1}k_\ell u_\ell(y). \tag{1.6}$$

We similarly define

$$\mathcal{L}_{\nu,\boldsymbol{k}} := \nu D_y^\gamma(a(y)D_y^\gamma) + i\sum_{\ell=1}^{d-1} k_\ell u_\ell(y). \tag{1.7}$$

We view both $\mathcal{L}_{\nu,\boldsymbol{k}}$ and $\tilde{\mathcal{L}}_{\nu,\boldsymbol{k}}$ as unbounded operators on $L^2(\mathbb{T})$. Given an (unbounded) operator $\mathcal{L}$ on L^2, we denote the semigroup generated by $\mathcal{L}$ with $e^{t\mathcal{L}}$ and the operator norm with $\|\,\|_{\mathrm{op}}$. Since $\|e^{-t\tilde{\mathcal{L}}_{\nu,\boldsymbol{k}}}\|_{\mathrm{op}} \le \|e^{-t\mathcal{L}_{\nu,\boldsymbol{k}}}\|_{\mathrm{op}}$, it is enough to study the latter operator.

Generalizing the results in [11], inspired by [14], we impose the following condition on the shear $\boldsymbol{u}$. It can be seen as a condition on the level sets of the components u_ℓ of $\boldsymbol{u}$, in particular a condition of the degeneracy of the critical points.

Assumption 1.1. *There exist $m,N\in\mathbb{N}$, $c_0>0$ and $\delta_0\in(0,1)$, with the property that for any $\lambda\in\mathbb{R}$, $\delta\in(0,\delta_0)$ and $\boldsymbol{k}=(k_1,k_2,\cdots,k_{d-1})\in\mathbb{Z}^{d-1}$ with $|\boldsymbol{k}|\neq 0$, there exist $n\le N$ and points $\{y_1,\cdots,y_n\}\in\mathbb{T}$ such that*

$$\left|\sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda\right| > c_0\delta^m, \tag{1.8}$$

for any $|y-y_j|\ge\delta$, $j\in\{1,\cdots,n\}$.

The Kolmogorov flow satisfies the above assumption with $m=2$ in two space dimensions (see as in [11]). More generally, a unidirectional shear flow with profile $u(y)=\sin^m y$ satisfies Assumption 1.1, also in two space dimensions. We remark that, if Assumption 1.1 holds for some $\delta_0<1$, it also holds for any smaller δ_0. Therefore, in what follows we will be able to choose δ_0 as small as needed without loss of generality.

Under Assumption 1.1 on the flow, we are able to prove enhanced dissipation holds for (1.5) on the complement of the kernel of the advection operator $\boldsymbol{v}\cdot\nabla$. This kernel consists of functions independent of $\boldsymbol{x}'$. Therefore, given any function $g\in L^2(\mathbb{T}^d)$, we denote

$$\langle g\rangle(y) = \frac{1}{2\pi}\int_0^{2\pi} g(x_1,\cdots,x_{d-1},y)\,dx_1\cdots dx_{d-1}, \tag{1.9}$$

and let

$$g_{\neq} = g - \langle g\rangle. \tag{1.10}$$

$\langle g\rangle$ denotes the projection onto the kernel of $\boldsymbol{v}\cdot\nabla$ and $g_{\neq}$ the component of g in the orthogonal complement. We then state our main theorem as follows.

Theorem 1.1. *Let $\boldsymbol{u}$ satisfy Assumption 1.1. Let $\theta\in C([0,\infty);L^2(\mathbb{T}^d))\cap C((0,\infty);H^2(\mathbb{T}^d))\cap C^1(0,+\infty);L^2(\mathbb{T}^d))$ be the unique solution of (1.5) with initial data $\theta_0\in L^2(\mathbb{T}^d)$. There exist a universal constant $C>0$, independent of θ_0, and a number $\epsilon_0>0$ such that*

$$\|\theta_{\neq}(t)\|_{L^2} \le Ce^{-\epsilon_0\nu^{\frac{m}{m+2\gamma}}|\ln\nu|^{-2\alpha(\gamma)}t}\|(\theta_0)_{\neq}\|_{L^2}, \tag{1.11}$$

where $\alpha(\gamma)$ is given by

$$\alpha(\gamma) = \begin{cases} (1-\gamma), & \frac{1}{2}<\gamma<1,\\ 0, & \gamma\ge 1.\end{cases} \tag{1.12}$$

By interpolation and the Sobolev embedding, one can derive rates of decay in L^p for $1 \le p < dp/(d-\gamma p)$. The bound in (1.11) is a quantitative estimate of the dissipation enhancement, since diffusion without advection would have a characteristic time $\tau = O(\nu)$, while in our case $\tau = O(\nu^{\frac{m}{m+2\gamma}})$, which is much larger for ν small, as $m/(m+2\gamma) < 1$, neglecting the logarithmic correction.

We close the Introduction with some notation and a plan of the paper. In Section 2 we prove Theorem 1.1, using certain resolvent estimates for $\mathcal{L}_{\nu,\boldsymbol{k}}$, while Section 3 is devoted to examples of flows satisfying Assumptions 1.1.

Throughput, C or c will denote a generic constant that may change line to line. We use standard notation to denote function spaces, e.g. $H^s(\mathbb{T}^d)$ denote the L^2-based Sobolev space of order $s \in \mathbb{R}$. We also denote the image of $H^s(\mathbb{T}^d)$ under the Fourier Transform with $h^s(\mathbb{Z}^d)$; that is,

$$h^s(\mathbb{Z}^d) := \{(\hat{f}_{\boldsymbol{k}}), \boldsymbol{k} \in \mathbb{Z}^d; \|\hat{f}\|_{h^s} := \left(\sum_{\boldsymbol{k}} |\boldsymbol{k}|^{2s} |\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2} < \infty\}, \quad s \in \mathbb{R}. \tag{1.13}$$

The homogeneous Sobolev spaces are denoted, as usual, by $\dot{H}^s(\mathbb{T}^d)$, and correspondingly, their image under the Fourier Transform by $\dot{h}^s(\mathbb{Z}^d)$. When no confusion can arise, we will omit the domain and write H^s for $H^s(\mathbb{T}^d)$, for example. Finally, we let $\langle , \rangle$ be the L^2-inner product.

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2. Enhanced dissipation for shear flows in $\mathbb{T}^d$, $d \ge 2$

This section contains the proof of Theorem 1.1. As discussed in the Introduction, it is enough to estimate the operator norm of $e^{-t\mathcal{L}_{\nu,\boldsymbol{k}}}$, which in turn follows from a spectral estimate, employing a Gearhart-Prüss-Greiner type theorem (see e.g. [15] for more details), Theorem 1.3 in [37]. This result specializes the Gearhart-Prüss-Greiner theorem to m-accretive operators. An unbounded operator $\mathcal{L}$ on a Banach space X with dense domain $\mathcal{D}(\mathcal{L})$ is called *accretive* if $-\mathcal{L}$ is dissipative, that is, the positive half line is in the resolvent set of $-\mathcal{L}$ and we have the resolvent estimate:

$$\|(\lambda I + \mathcal{L})f\| \ge \lambda \|f\|, \qquad \lambda > 0, f \in \mathcal{D}(\mathcal{L}),$$

where I is the identity operator on X. When $X = H$ is a Hilbert space with inner product $(,)$, then accretivity is equivalent to coercivity of the induced bilinear form (see again [15]), i.e.,

$$\mathrm{Re}\langle \mathcal{L}f, f\rangle \ge 0, \quad \forall f \in \mathcal{D}(\mathcal{L}). \tag{2.1}$$

The operator $\mathcal{L}$ is called m or *maximally*-accretive if it is accretive and $(\lambda I + \mathcal{L})$ is surjective for any $\lambda > 0$. Again, for operators on Hilbert spaces, there is a useful characterization of m-accretive operator that applies in this context. Namely, an accretive operator $\mathcal{L}$ on a Hilbert space H has a closure that is m-accretive if and only if the adjoint $\mathcal{L}^*$ is accretive (see [32, Theorem I-4.4] and also [34, Lemma 1.3]). Theorem

1.3 in [37] gives an exponential bound on the semigroup generated by an m-accretive operator provided its spectral function Ψ discussed below satisfies a certain lower bound.

Our task in what follows is, therefore, to show that $\mathcal{L}_{\nu,\boldsymbol{k}}$ is m-accretive for fixed $\boldsymbol{k}\in\mathbb{Z}_+$ as an unbounded operator on $L^2(\mathbb{T})$ with maximal domain $H^{2\gamma}(\mathbb{T})$. In particular, $\mathcal{L}_{\nu,\boldsymbol{k}}$ is a closed operator. Moreover, since there is no enhanced dissipation in the kernel of the advection operator, we can assume that $\boldsymbol{k}\neq 0$ and work with functions that have average zero in $\boldsymbol{x}$. A straightforward calculation shows that (2.1) is satisfied, so $\mathcal{L}_{\nu,\boldsymbol{k}}$ is accretive. But $\mathcal{L}^*_{\nu,\boldsymbol{k}}$ is an operator of the same form and, hence, also accretive, so that we can conclude that $\mathcal{L}_{\nu,\boldsymbol{k}}$ is m-accretive. Next, we estimate the spectral function of this operator in Proposition 2.1, which implies the decay estimate for the semigroup.

We first state some auxiliary lemmata. The first lemma can be viewed as a generalized Leibnitz rule for the fractional derivative. This is a well-known result, at least in the case of the whole space $\mathbb{R}^d$ (see [30] and references therein). We omit the proof for brevity.

Lemma 2.1. *For any $f,g\in H^\gamma(\mathbb{T})\cap L^\infty(\mathbb{T})$, $\gamma>0$, the following holds:*

$$\|D_z^\gamma(fg)\|_{L^2}\le C\left(\|f\|_{H^\gamma}\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{H^\gamma}\right). \tag{2.2}$$

We will also need an interpolation inequality for fractional Sobolev spaces, which follows from the Sobolev embedding (see e.g. [4]). We include a short proof for completeness.

Lemma 2.2. *Let $f\in H^\gamma(\mathbb{T})$, $\gamma>\frac{1}{2}$, then*

$$\|f\|_{L^\infty}\le \frac{1}{\sqrt{2\pi}}\|f\|_{L^2}+C\|f\|_{\dot{H}^\gamma}^{\frac{1}{2\gamma}}\|f\|_{L^2}^{1-\frac{1}{2\gamma}}. \tag{2.3}$$

Proof. Let $\bar f=\frac{1}{2\pi}\int_0^{2\pi}f(y)\,dy$ denote the mean of f and observe that

$$\left\|f-\bar f\right\|_{H^\gamma}\approx\|f\|_{\dot{H}^\gamma}.$$

Next, fix $\Lambda\in\mathbb{Z}_+$ to be chosen later. Then

$$\begin{aligned}
&\|f\|_{L^\infty}\le|\bar f|+\left\|f-\bar f\right\|_{L^\infty}\le\frac{1}{\sqrt{2\pi}}\|f\|_{L^2}+\sum_{|k|\neq 0}|\hat f(k)|\\
&\le\frac{1}{\sqrt{2\pi}}\|f\|_{L^2}+\left(\sum_{0<|k|<\Lambda}+\sum_{|k|\ge\Lambda}\right)|\hat f(k)|\\
&\le\frac{1}{\sqrt{2\pi}}\|f\|_{L^2}+\Big(\sum_{0<|k|<\Lambda}1\Big)^{\frac{1}{2}}\Big(\sum_{0<|k|<\Lambda}|\hat f(k)|^2\Big)^{\frac{1}{2}}+\Big(\sum_{|k|\ge\Lambda}|\hat f(k)|^2|k|^{2\gamma}\Big)^{\frac{1}{2}}\Big(\sum_{|k|\ge\Lambda}|k|^{-2\gamma}\Big)^{\frac{1}{2}}\\
&\le\frac{1}{\sqrt{2\pi}}\|f\|_{L^2}+\Lambda^{\frac{1}{2}}\left\|f-\bar f\right\|_{L^2}+C\|f\|_{\dot{H}^\gamma}\Lambda^{\frac{1-2\gamma}{2}},
\end{aligned}$$

where in the last inequality we used that $\gamma>\frac{1}{2}$. Finally, inequality (2.3) follows by picking

$$\Lambda=\frac{\|f\|_{\dot{H}^\gamma}^{\frac{1}{\gamma}}}{\left\|f-\bar f\right\|_{L^2}^{\frac{1}{\gamma}}}.$$

□

For technical reasons that will be clear later in performing energy estimates, we will need to have a suitable smooth approximation for the signum of $\left(\sum_{\ell=1}^{d-1} k_\ell u_\ell(y)-\lambda\right)$.

LEMMA 2.3. *Let the vector field* $\boldsymbol{u}$ *satisfy Assumption 1.1. For any* $\lambda\in\mathbb{R}$ *and* $\boldsymbol{k}=(k_1,\cdots,k_{d-1})\in\mathbb{Z}^{d-1}$ *with* $|\boldsymbol{k}|\neq 0$, *let* $\{y_j, j=1,\cdots,n\}$ *denote the distinguished points in Assumption 1.1 and let* δ_0 *be as in Assumption 1.1. For any* $\delta\in(0,\delta_0)$, *define the set*

$$E:=\{y\in\mathbb{T}\,|\,|y-y_j|\geq\delta,\forall j=1,\cdots,n\}.$$

Then there exists a smooth function $\chi:\mathbb{T}\to[-1,1]$ *with the property that*

$$\chi(y)\left(\sum_{\ell=1}^{d-1}k_\ell u_\ell(y)-\lambda\right)\geq 0,\quad \forall y\in\mathbb{T}, \tag{2.4}$$

$$\chi(y)\left(\sum_{\ell=1}^{d-1}k_\ell u_\ell(y)-\lambda\right)=\left|\sum_{\ell=1}^{d-1}k_\ell u_\ell(y)-\lambda\right|,\quad \forall y\in E. \tag{2.5}$$

Moreover, for any fixed $\gamma>\frac{1}{2}$ *there exists a constant* C, *independent of* δ, *such that*

$$\|\chi\|_{H^\gamma}\leq C\delta^{\frac{1}{2}-\gamma}|\ln\delta|^{\alpha(\gamma)}, \tag{2.6}$$

where the function α *is defined as in* (1.12).

Proof. We begin by observing that, if $\boldsymbol{u}$ satisfies Assumption 1.1, then u_ℓ, $\ell=1,\ldots,d-1$ can change sign only on intervals of size δ. Therefore, by standard mollification argument we can construct a smooth function $\chi:\mathbb{T}\to[-1,1]$ that approximate $\operatorname{sign}\left(\sum_{\ell=1}^{d-1}k_\ell u_\ell(y)-\lambda\right)$ in the following sense:

$$\left\|\partial_y^s\chi\right\|_{L^2}\leq C\delta^{\frac{1}{2}-s},\quad \left\|\partial_y^s\chi\right\|_{L^\infty}\leq C\delta^{-s},\quad \forall s\in\mathbb{Z}_+, \tag{2.7}$$

$$\chi(y)\left(\sum_{\ell=1}^{d-1}k_\ell u_\ell(y)-\lambda\right)\geq 0,\quad \forall y\in\mathbb{T}, \tag{2.8}$$

$$\chi(y)\left(\sum_{\ell=1}^{d-1}k_\ell u_\ell(y)-\lambda\right)=\left|\sum_{\ell=1}^{d-1}k_\ell u_\ell(y)-\lambda\right|,\quad \forall y\in E. \tag{2.9}$$

The rest of the proof is devoted to estimating $\|\chi\|_{\dot H^\gamma}(\mathbb{T})$ for fractional exponent $\gamma>1/2$, using the Gagliardo-Nirenberg interpolation inequality. To obtain the best exponent, we will interpolate with $\|\chi\|_{\dot H^{1/2}}(\mathbb{T})$ as an endpoint. We claim that

$$\|\chi\|_{\dot H^{1/2}}\leq C|\ln\delta|. \tag{2.10}$$

To prove the claim, we recall the Fourier representation of the $\dot H^{1/2}$ norm

$$\|\chi-\bar\chi\|_{\dot H^{1/2}}^2=C\sum_{|\ell|\neq 0}|\hat\chi(\ell)|^2|\ell|, \tag{2.11}$$

where $\bar\chi=\frac{1}{2\pi}\int_0^{2\pi}\chi(y)\,dy$ and $\hat\chi(\ell)$ is the ℓ_{th} Fourier coefficient. Then integrating by parts and using the identity $e^{-i\ell y}=-\frac{1}{i\ell}\frac{d}{dy}e^{-i\ell y}$, we have that

$$|\hat\chi(\ell)|\leq\frac{1}{|\ell|}\int_{\mathbb{T}}|\partial_y\chi|\,dy\leq\frac{C}{|\ell|},\quad \ell\neq 0; \tag{2.12}$$

$$|\hat{\chi}(\ell)| \le \frac{1}{|\ell|^2}\int_{\mathbb{T}} \left|\partial_y^2 \chi\right| dy \le \frac{C}{\delta|\ell|^2}, \quad \ell \neq 0. \tag{2.13}$$

Thus the $\dot{H}^{1/2}$-seminorm can be bounded by

$$\|\chi\|_{\dot{H}^{1/2}}^2 = C\left(\sum_{0<|\ell|<\delta^{-1}} + \sum_{|\ell|\ge\delta^{-1}}\right)|\hat{\chi}(\ell)|^2|\ell| \tag{2.14}$$

$$\le C\left(\sum_{0<|\ell|<\delta^{-1}} \frac{1}{|\ell|} + \sum_{|\ell|\ge\delta^{-1}} \frac{1}{\delta^2|\ell|^3}\right) \tag{2.15}$$

$$\le C|\ln\delta| + C, \tag{2.16}$$

and estimate (2.10) follows by choosing $\delta_0 \ll 1$ in Assumption 1.1, which can be done whitout loss of generality. Applying the Gagliardo-Nirenberg interpolation inequality then gives:

$$\|\chi\|_{\dot{H}^\gamma} \le C\|\chi\|_{\dot{H}^{1/2}}^{2-2\gamma}\|\chi\|_{\dot{H}^1}^{2\gamma-1} \le C|\ln\delta|^{1-\gamma}\delta^{\frac{1}{2}-\gamma}, \quad \gamma \in (1/2,1); \tag{2.17}$$

$$\|\chi\|_{\dot{H}^\gamma} \le C\|\chi\|_{\dot{H}^{\lfloor\gamma\rfloor}}^{1-\gamma+\lfloor\gamma\rfloor}\|\chi\|_{\dot{H}^{\lfloor\gamma\rfloor+1}}^{\gamma-\lfloor\gamma\rfloor} \le C\delta^{\frac{1}{2}-\gamma}, \quad \gamma \ge 1, \tag{2.18}$$

where $\lfloor\gamma\rfloor$ denotes the greatest integer less than or equal to γ. Finally, since $\|\chi\|_{H^\gamma} \le \|\chi\|_{\dot{H}^\gamma} + \|\chi\|_{L^2}$, $\|\chi\|_{L^2} \le 2\pi$ and $\delta_0 \ll 1$, the desired result in (2.6) follows. □

We now turn to the key result, Proposition 2.1 below, in order to prove Theorem 1.1. We define the spectral function Ψ associated to the operator $\mathcal{L}_{\nu,\boldsymbol{k}}$, introduced in (1.7), as:

$$\Psi(\mathcal{L}_{\nu,\boldsymbol{k}}) = \inf\left\{\|(\mathcal{L}_{\nu,\boldsymbol{k}} - i\lambda)g\|_{L^2(\mathbb{T})} : g \in \mathcal{D}(\mathcal{L}_{\nu,\boldsymbol{k}}), \lambda \in \mathbb{R}, \|g\|_{L^2(\mathbb{T})} = 1\right\}. \tag{2.19}$$

By Theorem 1.3 in [37], since $\mathcal{L}_{\nu,\boldsymbol{k}}$ is m-accretive on $L^2(\mathbb{T})$,

$$\left\|e^{-t\mathcal{L}_{\nu,\boldsymbol{k}}}\right\|_{\text{op}} \le e^{-t\Psi(\mathcal{L}_{\nu,\boldsymbol{k}}) + \frac{\pi}{2}}, \quad \forall t \ge 0,$$

Therefore, it is enough to bound the spectral function $\Psi(\mathcal{L}_{\nu,\boldsymbol{k}})$.

Proposition 2.1. *Let $\boldsymbol{u}$ satisfy Assumption 1.1. Let $\nu < 1$ and $\boldsymbol{k} \neq 0$. There exists a positive constant $\epsilon_0 = \epsilon_0(N,m,\gamma)$ independent of ν and $\boldsymbol{k}$ such that*

$$\Psi(\mathcal{L}_{\nu,\boldsymbol{k}}) \ge \epsilon_0 \nu^{\frac{m}{m+2\gamma}}|\ln\nu|^{-2\alpha(\gamma)}, \tag{2.20}$$

where $\alpha(\gamma)$ is given in (1.12).

Remark 2.1. *As pointed in [23], the logarithmic correction appears only when $\gamma \in (1/2,1)$. When $\gamma \ge 1$, our result is consistent with the previous estimates for integer or fractional γ values in [11,23].*

Proof. For any given $\lambda \in \mathbb{R}$ and $g \in \mathcal{D}(\mathcal{L}_{\nu,\boldsymbol{k}}) = H^{2\gamma}(\mathbb{T})$ with $\|g\|_{L^2} = 1$, we denote

$$\mathcal{L}_\lambda := \nu D_y^\gamma(a(y)D_y^\gamma) + i\left(\sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda\right),$$

where for clarity we have suppressed the dependence on ν and $\boldsymbol{k}$ as they are fixed throughout the proof. We observe that

$$\text{Re}\langle\mathcal{L}_\lambda g, g\rangle = \nu\langle a(y)D_y^\gamma g, D_y^\gamma g\rangle$$

$$\geq c_1 \nu \left\| D_y^\gamma g \right\|_{L^2}^2,$$

which in turn gives

$$\left\| D_y^\gamma g \right\|_{L^2}^2 \leq \frac{1}{c_1 \nu} \left\| \mathcal{L}_\lambda g \right\|_{L^2} \left\| g \right\|_{L^2}. \tag{2.21}$$

Let the set E and the function χ be as in Lemma 2.3. We integrate by parts in the inner product $\langle \mathcal{L}_\lambda g, \chi g \rangle$:

$$\langle \mathcal{L}_\lambda g, \chi g \rangle = \nu \langle a(y) D_y^\gamma g, D_y^\gamma (\chi g) \rangle + i \Big\langle \left(\sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda \right) g, \chi g \Big\rangle.$$

Applying Holder's inequality, Lemma 2.1, Lemma 2.3 and the properties of χ in Lemma 2.3, we then have that

$$\begin{aligned}
|\langle (\sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda) g, \chi g \rangle| &= \left| \mathrm{Im} \langle \mathcal{L}_\lambda g, \chi g \rangle - \nu \mathrm{Im} \langle a(y) D_y^\gamma g, D_y^\gamma (\chi g) \rangle \right| \\
&\leq \left\| \mathcal{L}_\lambda g \right\|_{L^2} \left\| g \right\|_{L^2} + \nu \left\| a \right\|_{L^\infty} \left\| D_y^\gamma g \right\|_{L^2} \left\| D_y^\gamma (\chi g) \right\|_{L^2} \\
&\leq \left\| \mathcal{L}_\lambda g \right\|_{L^2} \left\| g \right\|_{L^2} + C c_2 \nu (\left\| D_y^\gamma g \right\|_{L^2} \left\| \chi \right\|_{H^\gamma} \left\| g \right\|_{L^\infty} + \left\| D_y^\gamma g \right\|_{L^2} \left\| g \right\|_{H^\gamma} \left\| \chi \right\|_{L^\infty}) \\
&\leq \left\| \mathcal{L}_\lambda g \right\|_{L^2} \left\| g \right\|_{L^2} + C c_2 \nu \delta^{\frac{1}{2} - \gamma} |\ln \delta|^{\alpha(\gamma)} \left\| g \right\|_{\dot{H}^\gamma} \left\| g \right\|_{L^2} \\
&\quad + C c_2 \nu \delta^{\frac{1}{2} - \gamma} |\ln \delta|^{\alpha(\gamma)} \left\| g \right\|_{\dot{H}^\gamma}^{1 + \frac{1}{2\gamma}} \left\| g \right\|_{L^2}^{1 - \frac{1}{2\gamma}} + C c_2 \nu \left\| g \right\|_{\dot{H}^\gamma}^2.
\end{aligned}$$

We further use the estimate in (2.21) and obtain

$$\begin{aligned}
\left| \Big\langle \left(\sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda \right) g, \chi g \Big\rangle \right| &\leq C \left(1 + \frac{c_2}{c_1} \right) \left\| \mathcal{L}_\lambda g \right\|_{L^2} \left\| g \right\|_{L^2} \\
&\quad + C c_1^{-1/2} c_2 \nu^{\frac{1}{2}} \delta^{\frac{1}{2} - \gamma} |\ln \delta|^{\alpha(\gamma)} \left\| \mathcal{L}_\lambda g \right\|_{L^2}^{\frac{1}{2}} \left\| g \right\|_{L^2}^{\frac{3}{2}} \\
&\quad + C c_1^{-\frac{4\gamma+1}{4\gamma}} c_2 \nu^{\frac{2\gamma-1}{4\gamma}} \delta^{\frac{1}{2} - \gamma} |\ln \delta|^{\alpha(\gamma)} \left\| \mathcal{L}_\lambda g \right\|_{L^2}^{\frac{2\gamma+1}{4\gamma}} \left\| g \right\|_{L^2}^{\frac{6\gamma-1}{4\gamma}}.
\end{aligned} \tag{2.22}$$

By utilizing again the properties of χ in Lemma 2.3, it follows that

$$\begin{aligned}
\left| \Big\langle \left(\sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda \right) g, \chi g \Big\rangle \right| &\geq \int_E \chi \left(\sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda \right) g^2 \, dy \\
&= \int_E \left| \sum_{\ell=1}^{d-1} k_\ell u_\ell(y) - \lambda \right| g^2 \, dy \\
&\geq c_0 \delta^m \int_E g^2 \, dy,
\end{aligned} \tag{2.23}$$

where Assumption 1.1 was used in the last inequality.

Combining estimate (2.22) and (2.23), we get

$$\begin{aligned}
\int_E g^2 \, dy &\leq \frac{C(c_1 + c_2)}{c_0 c_1 \delta^m} \left\| \mathcal{L}_\lambda g \right\|_{L^2} \left\| g \right\|_{L^2} + \frac{C c_2 \nu^{1/2} |\ln \delta|^{\alpha(\gamma)}}{c_0 c_1^{1/2} \delta^{m + \gamma - \frac{1}{2}}} \left\| \mathcal{L}_\lambda g \right\|_{L^2}^{1/2} \left\| g \right\|_{L^2}^{3/2} \\
&\quad + \frac{C c_2 \nu^{\frac{2\gamma-1}{4\gamma}} |\ln \delta|^{\alpha(\gamma)}}{c_0 c_1^{\frac{4\gamma+1}{4\gamma}} \delta^{m + \gamma - \frac{1}{2}}} \left\| \mathcal{L}_\lambda g \right\|_{L^2}^{\frac{2\gamma+1}{4\gamma}} \left\| g \right\|_{L^2}^{\frac{6\gamma-1}{4\gamma}}.
\end{aligned} \tag{2.24}$$

On the other hand, since the complement E^c of the set E has measure $|E^c|\le N\delta$, using Lemma 2.3, the integral of g^2 on E^c can be bounded by

$$\begin{aligned}\int_{E^c} g^2\,dy &\le N\delta\|g\|_{L^\infty}^2 \le CN\delta(\|g\|_{L^2}^2+\|g\|_{\dot H^\gamma}^{\frac{1}{\gamma}}\|g\|_{L^2}^{\frac{2\gamma-1}{\gamma}})\\ &\le CN\delta(\|g\|_{L^2}^2+c_1{}^{-\frac{1}{2\gamma}}\nu^{-\frac{1}{2\gamma}}\|\mathcal{L}_\lambda g\|_{L^2}^{\frac{1}{2\gamma}}\|g\|_{L^2}^{\frac{4\gamma-1}{2\gamma}}).\end{aligned} \tag{2.25}$$

Adding (2.24) and (2.25) yields

$$\begin{aligned}\|g\|_{L^2}^2 \le{}& \frac{C(c_1+c_2)}{c_0c_1\delta^m}\|\mathcal{L}_\lambda g\|_{L^2}\|g\|_{L^2}+\frac{Cc_2\nu^{1/2}|\ln\delta|^{\alpha(\gamma)}}{c_0c_1^{1/2}\delta^{m+\gamma-\frac{1}{2}}}\|\mathcal{L}_\lambda g\|_{L^2}^{1/2}\|g\|_{L^2}^{3/2}\\ &+\frac{Cc_2\nu^{\frac{2\gamma-1}{4\gamma}}|\ln\delta|^{\alpha(\gamma)}}{c_0c_1^{\frac{4\gamma+1}{4\gamma}}\delta^{m+\gamma-\frac{1}{2}}}\|\mathcal{L}_\lambda g\|_{L^2}^{\frac{2\gamma+1}{4\gamma}}\|g\|_{L^2}^{\frac{6\gamma-1}{4\gamma}}+CN\delta(\|g\|_{L^2}^2+c_1{}^{-\frac{1}{2\gamma}}\nu^{-\frac{1}{2\gamma}}\|\mathcal{L}_\lambda g\|_{L^2}^{\frac{1}{2\gamma}}\|g\|_{L^2}^{\frac{4\gamma-1}{2\gamma}}).\end{aligned}$$

By taking $\delta_0<\frac{1}{4CN}$ and applying Young's inequality, we have that

$$\begin{aligned}\frac{1}{4}\|g\|_{L^2}^2 \le C\left(\frac{c_1+c_2}{c_0c_1\delta^m}+\frac{c_2^2\nu|\ln\delta|^{2\alpha(\gamma)}}{c_0^2c_1\delta^{2m+2\gamma-1}}+\frac{c_2^{\frac{4\gamma}{2\gamma+1}}\nu^{\frac{2\gamma-1}{2\gamma+1}}|\ln\delta|^{\frac{4\gamma\alpha(\gamma)}{2\gamma+1}}}{c_0^{\frac{4\gamma}{2\gamma+1}}c_1^{\frac{4\gamma}{2\gamma+1}}\delta^{\frac{4\gamma^2-2\gamma+4m\gamma}{2\gamma+1}}}+\frac{N^{2\gamma}\delta^{2\gamma}}{c_1\nu}\right)&.\\ \cdot\|\mathcal{L}_\lambda g\|_{L^2}\|g\|_{L^2}&.\end{aligned}$$

Moreover, choosing

$$\delta=\delta_0\nu^{\frac{1}{m+2\gamma}},$$

and using the fact $\nu<1$, we have also

$$1=\|g\|_{L^2}\le C(N,m,\gamma)\nu^{-\frac{m}{m+2\gamma}}|\ln\delta|^{2\alpha(\gamma)}\|\mathcal{L}_\lambda g\|_{L^2}.$$

This last inequality finally implies that there exists a positive constant $\epsilon_0=\epsilon(N,m,\gamma)$, such that

$$\Psi(\mathcal{L}_{\nu,\boldsymbol{k}})\ge\epsilon_0\nu^{\frac{m}{m+2\gamma}}|\ln\delta|^{-2\alpha(\gamma)},$$

which concludes the proof. □

3. Examples of dissipation-enhancing high-dimensional shear flows

In this section, we give an example to illustrate that the set of flows in $\mathbb{T}^d$ satisfying Assumption 1.1 is not empty. Our example can be viewed as a multi-directional, higher-dimensional analog of the Kolmogorov flow.

PROPOSITION 3.1. *Let the flow velocity $\boldsymbol{v}=(\boldsymbol{u},0)$, where the components of the shear $\boldsymbol{u}$ are given by $u_\ell(y)=\sin^\ell(y)$ for $\ell=1,\cdots,d-1$, then Assumption 1.1 holds with $m=2(d-1)$.*

Proof. We begin with a claim that clarifies the meaning of Assumption 1.1 further.

Claim: Given $h\in\mathbb{N}$, for any $\delta>0$, $\alpha_\ell\in\mathbb{R}$, $\ell=1,\cdots,h-1$, and $\lambda\in\mathbb{R}$, we can find a set of at most h points $z_{j,h}\in\mathbb{R}$, $j=1,\cdots,h$, which depend on α_ℓ, $\ell=1,\cdots,h-1$, and λ, satisfying

$$\left|z^h+\alpha_{h-1}z^{h-1}+\cdots+\alpha_1 z-\lambda\right|>\delta^h, \tag{3.1}$$

for all $|z-z_{j,h}|>\delta$, $j=1,\cdots,h$.

Proof of the Claim: We note that there exist points $z_j\in\mathbb{C}$, $j=1,\cdots,h$, satisfying

$$z^h+\alpha_{h-1}z^{h-1}+\cdots+\alpha_1 z-\lambda=\Pi_{j=1}^h(z-z_j),$$

since the left-hand side is a polynomial of order h. We let $z_{j,h}=\operatorname{Re}z_j$ for $j=1,\cdots,h$. Then $z_{j,h}\in\mathbb{R}$ and we have the estimate

$$\left|z^h+\alpha_{h-1}z^{h-1}+\cdots+\alpha_1 z-\lambda\right|\geq\Pi_{j=1}^h|z-z_{j,h}| \tag{3.2}$$

Hence (3.1) follows as long as $|z-z_{j,h}|>\delta$ for all $j=1,\cdots,h$. The claim is proved.

We return to the proof of the Proposition. By hypothesis, $u_\ell(y)=\sin^\ell(y)$. We distinguish two cases. If $k_{d-1}\neq 0$, for any $\lambda\in\mathbb{R}$ and $\boldsymbol{k}=(k_1,k_2\cdots,k_{d-1})\in\mathbb{Z}^{d-1}$ with $|\boldsymbol{k}|\neq 0$ the left hand side of (1.8) can be estimated as

$$\begin{aligned}\left|\sum_{\ell=1}^{d-1}k_\ell\sin^\ell(y)-\lambda\right|&=|k_{d-1}|\left|\sin^{d-1}(y)+\frac{k_{d-2}}{k_{d-1}}\sin^{d-2}(y)+\cdots+\frac{k_1}{k_{d-1}}\sin(y)-\frac{\lambda}{k_{d-1}}\right| \\ &\geq\left|\sin^{d-1}(y)+\frac{k_{d-2}}{k_{d-1}}\sin^{d-2}(y)+\cdots+\frac{k_1}{k_{d-1}}\sin(y)-\frac{\lambda}{k_{d-1}}\right| \\ &\geq\Pi_{j=1}^{d-1}|\sin(y)-z_{j,d-1}|\end{aligned} \tag{3.3}$$

where $z_{j,d-1}$ is as in the proof of the Claim above with $h=d-1$ and we used (3.2).

Then following the argument in [11, Example 3.1], we observe that we can estimate $|\sin(y)-z_{j,d-1}|$ as follows. For each $j=1,\ldots,d-1$, there exists a positive integer M_j and points $y_{j,n}$, $n\leq M_j$, with the following property: there is a positive constant $\bar{c}_j$ such that $|\sin(y)-z_{j,d-1}|\geq\bar{c}_j\delta^2$ for any y satisfying $|y-y_{j,n}|\geq\delta$ for all $y_{j,n}$, $n\leq M_j$, and $\delta\in(0,\delta_0)$. Hence, (3.3) becomes

$$\left|\sum_{\ell=1}^{d-1}k_\ell\sin^\ell(y)-\lambda\right|\geq\delta^{2(d-1)}\Pi_{j=1}^{d-1}\bar{c}_j\,.$$

If $k_{d-1}=0$ instead, we let $\bar{\ell}=\arg\max_{1\leq\ell<d-1}k_\ell\neq 0$. Consequently,

$$\begin{aligned}\left|\sum_{\ell=1}^{d-1}k_\ell\sin^\ell(y)-\lambda\right|&=|k_{\bar{\ell}}|\left|\sin^{\bar{\ell}}(y)+\frac{k_{\bar{\ell}-1}}{k_{\bar{\ell}}}\sin^{\bar{\ell}-1}(y)+\cdots+\frac{k_1}{k_{\bar{\ell}}}\sin(y)-\frac{\lambda}{k_{\bar{\ell}}}\right| \\ &\geq\left|\sin^{\bar{\ell}}(y)+\frac{k_{\bar{\ell}-1}}{k_{\bar{\ell}}}\sin^{\bar{\ell}-1}(y)+\cdots+\frac{k_1}{k_{\bar{\ell}}}\sin(y)-\frac{\lambda}{k_{\bar{\ell}}}\right| \\ &\geq\Pi_{j=1}^{\bar{\ell}}\left|\sin(y)-z_{j,\bar{\ell}}\right|.\end{aligned} \tag{3.4}$$

Again, using the same argument as in [11, Example 3.1], we have that there exist $\bar{c}>0$, $\delta_0>0$, and finitely many points y_n, $n\leq M$, with M a positive integer, such that for any y satisfying $|y-y_n|\geq\delta$ for all y_n, $n\leq M$, and $\delta\in(0,\delta_0)$, the following holds

$$\left|\sum_{\ell=1}^{d-1}k_\ell\sin^\ell(y)-\lambda\right|\geq\bar{c}\delta^{2\bar{\ell}}.$$

We conclude that Assumption 1.1 is verified with $m=2(d-1)$ in this case. □

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