FUSION RULES AND RIGIDITY FOR WEIGHT MODULES OVER THE SIMPLE ADMISSIBLE AFFINE $\mathfrak{sl}(2)$ AND $\mathcal{N}=2$ SUPERCONFORMAL VERTEX OPERATOR SUPERALGEBRAS

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Abstract. We prove that the categories of weight modules over the simple $\mathfrak{sl}(2)$ and $\mathcal{N}=2$ superconformal vertex operator superalgebras at fractional admissible levels and central charges are rigid (and hence the categories of weight modules are braided ribbon categories) and that the decomposition formulae of fusion products of simple projective modules conjectured by Thomas Creutzig, David Ridout and collaborators hold (including when the decomposition involves summands that are indecomposable yet not simple). In addition to solving this old open problem, we develop new techniques for the construction of intertwining operators by means of integrating screening currents over certain cycles, which are expected to be of independent interest, due to their applicability to many other algebras. In the example of $\mathfrak{sl}(2)$ these new techniques allow us to give explicit formulae for a logarithmic intertwining operator from a pair of simple projective modules to the projective cover of the tensor unit, namely, the vertex operator algebra as a module over itself.

1. Introduction

Within the body of literature on conformal field theory and vertex operator algebras the class of rational theories (satisfying a number of technical niceness conditions including C_2-cofiniteness and the category of admissible modules being semisimple) stand out as being exceptionally intensively studied and well understood. One particularly compelling aspect of these rational theories is the abundance of rich mathematical structures they exhibit. For example, the fact that their categories of modules are modular tensor categories and that the categorical action of the modular group via Hopf links and twists matches the modular transformations of characters [43]. Apart from their intrinsic mathematical beauty these modularity results also have practical implications: they allow the efficient computation of tensor products (also called fusion) of vertex operator algebra modules in terms of the modular transformation formulae of characters. In practice, this provides an enormous reduction in the computational effort required to understand fusion products.

There is ample evidence to suggest that the modularity properties enjoyed by rational theories generalise to large classes of non-rational ones. The purpose of this paper is to prove that these conjectured modularity properties do indeed hold for so called admissible $\mathfrak{sl}(2)$ and $\mathcal{N}=2$ superconformal theories. In order to better understand these conjectures, we begin with a historical overview. This paper makes use of a coset realisation of the $\mathcal{N}=2$ superconformal algebra in terms of $\mathfrak{sl}(2)$ and a fermionic ghost (or bc) system [57, 56, 22, 35]. Because of this the representation theories of $\mathcal{N}=2$ and $\mathfrak{sl}(2)$ are closely intertwined and any statement about the representation theory for one algebra has a corresponding version for the other.

A first hint of modularity beyond rationality was discovered by Kac and Wakimoto [52, 51, 53] when they computed modular transformation formulae for characters of simple highest weight modules over affine Lie algebras at admissible levels and weights. Since the non-negative integral levels (which are the only ones giving rise to rational theories) are a proper subset of all admissible levels, it is somewhat surprising for such formulae to exist at all. Shortly afterwards Koh and Sorba

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[59] plugged these modular transformation properties for $\mathfrak{sl}(2)$ into the Verlinde formula, which, rather startlingly, predicted negative multiplicities for some summands appearing in certain fusion products. This in turn led to some concern within the academic community that these non-integral admissible (also called fractional admissible) theories may suffer from some intrinsic "sickness" [33].

New light was later shed on this riddle by Ridout after a careful analysis of $\mathfrak{sl}(2)$ at level $k = -\frac{1}{2}$ [65] led him to note that the characters of highest weight modules at this level were only convergent in certain domains and that the modular transformation formulae of Kac and Wakimoto hold only for the analytic continuations of characters rather then their series expansions. This is because these modular transformations do not preserve domains of convergence. He also noted that in the category of weight modules the analytic continuation of characters of certain highest weight modules was the negative of the analytic continuation of characters of certain other weight modules. This gave a first hint as to why signs were appearing in the Verlinde formula of Koh and Sorba.

Creutzig and Ridout then studied the category of weight modules over affine $\mathfrak{gl}(1|1)$ and the modular properties of characters [32]. While category $\mathcal{O}$ is semisimple and finite, the category of weight modules is neither (this is also true for affine $\mathfrak{sl}(2)$ at non-integral admissible levels and weights). However, the span of characters (taken as specific series expansions rather than their analytic continuations) of a distinguished class of weight modules (called *standard modules*) carries an action of the modular group. Further, evaluating the Verlinde formula (in a generalised version conjectured to hold for infinite categories of modules) using this action predicts non-negative fusion multiplicities. This new generalised Verlinde formula sheds light on the riddle of negative fusion multiplicities of Koh and Sorba: there are linear relations between the analytic continuations of characters of simple weight modules. So a negative multiplicity in the Verlinde formula for category $\mathcal{O}$ can be interpreted as a positive multiplicity of a different weight module outside of category $\mathcal{O}$. This work was then generalised to $\mathfrak{sl}(2)$ at all admissible levels [30, 31] and together with other authors to many other families of algebras [28, 67, 68, 12, 13, 29, 10, 66, 55, 38, 37].

Two key conjectures or hopes that crystallised from Creutzig and Ridout's work in [32, 30, 31] were that the category of weight modules is rigid (this implies that the fusion product is exact, a necessary condition for anything akin to a Verlinde formula to be well defined) and that the fusion product decomposition formulae predicted by the generalised Verlinde formulae of Creutzig and Ridout are true (at levels $k = -\frac{1}{2}, -\frac{4}{3}$ the fusion product formulae were explicitly checked using the Nahm-Gaberdiel-Kausch algorithm). Two of the main results of this paper are that both of these conjectures hold in the sense that all weight modules are rigid and that that predicted decomposition formulae for fusion products of projective modules are true. Another important result of our work is that the analogous conjectures for the category of weight modules for the $\mathcal{N} = 2$ fractional minimal models also hold. That is, all weight modules are rigid and the conjectured fusion product decompositions of [23] hold.

The description of the logarithmic tensor structures for categories of modules over Virasoro algebras and its $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super extensions is an important problem that has been intensively studied in the literature in the past years [18, 64, 63, 24, 62, 23]. Since the monoidal structure on categories of vertex operator algebra modules automatically come with a braiding and a twist that is balanced with respect to this braiding, the rigidity of these categories implies that they are braided ribbon categories. The fusion rules for the strongly rational theories arising from unitary $\mathcal{N} = 2$ minimal models were studied by Adamović in [2] while the tensor category structure for their module categories was established by Huang and Milas in [47] using the theory of Lepowsky and Huang developed in [44]. In the Virasoro and $\mathcal{N} = 1$ cases, the central charges in which the universal vertex superalgebras are non-simple are exactly the central charges in which the simple quotients are strongly rational vertex operator superalgebras. Interestingly, the $\mathcal{N} = 2$ vertex superalgebras have richer behavior since the central charges in which the universal $\mathcal{N} = 2$

vertex superalgebra admits a strongly rational simple quotient vertex superalgebra form a proper subset of the central charges in which the universal vertex superalgebra is non-simple. While the logarithimic tensor product theory developed by Huang, Lepowsky and Zhang in [46] has recently been used to establish the tensor category structure on natural module categories for the universal Virasoro [18] and $\mathcal{N} = 1$ vertex superalgebras [24], in the case of the $\mathcal{N} = 2$ vertex superalgebras the logarithmic tensor structure has not been applied to the universal algebra, but only to the simple fractional minimal model central charges, namely, to the intermediate family of central charges in which the universal $\mathcal{N} = 2$ vertex superalgebra admits a non-trivial ideal such that its simple quotient yields an irrational representation theory [14]. In our work, we make use of the logarithmic tensor category structure established by Creutzig in [14] for the category of weight modules for the fractional $\mathcal{N} = 2$ minimal models and for the fractional admissible $\mathfrak{sl}(2)$ weight modules to prove the rigidity of both categories.

This paper is organised as follows. In Section 2 we review all the vertex operator superalgebras that will be needed as well as their corresponding categories of weight modules. In particular, we recall some $\mathfrak{sl}(2)$ fusion product decompositions, which will be needed later, that have already been proved and state the conjectured ones (Conjecture 2.9) which we shall prove in later sections.

In Section 3 we review the coset construction of the $\mathcal{N} = 2$ superconformal algebra in terms of affine $\mathfrak{sl}(2)$ and how the fusion products of these two algebras are interrelated. We prove the first result, Lemma 3.4, of the paper: a sufficient condition for transporting rigidity results from the $\mathfrak{sl}(2)$ side to the $\mathcal{N} = 2$ side. In Lemma 3.6, we give a sufficient condition for the semisimple fusion products of Conjecture 2.9 to hold in terms of the dimensions of certain spaces of intertwining operators.

Sections 4 and 5 are then dedicated to proving Lemma 3.6 by proving upper and lower bounds, and observing that these bounds are equal. The upper bounds are computed in Section 4 by use of the Zhu algebra formalism on the $\mathcal{N} = 2$ side. The lower bounds are computed in Section 5 on the $\mathfrak{sl}(2)$ side by means of a free field realisation and screening operators. The appearance of screening operators and their associated integrals over intricate cycles necessitates the use of $P(w)$ rather than logarithmic intertwining operators, that is, intertwining operators where the variable is considered as a complex number rather than a formal variable.

The non-semisimple fusion products of Conjecture 2.9 are tackled in Sections 6 and 7. Specifically, the projective cover of the tensor unit for $\mathfrak{sl}(2)$ is constructed in Section 6.1 using the free field realisation that was also used in Section 5. In Section 6.2 this projective module is used to analyse one of the fusion products appearing in Conjecture 2.9 and to construct the evaluation morphisms for a certain family of simple projective modules, $E_{\mu;1,1}$, that will be needed to prove their rigidity. In Section 6.2 we prove that the $E_{\mu;1,1}$ modules are rigid. Finally, in Section 7 the results of Section 6 are combined to prove that the categories of weight modules over simple admissible affine $\mathfrak{sl}(2)$ and $\mathcal{N} = 2$ are rigid, as well as that the non-semisimple fusion product decomposition of Conjecture 2.9 hold.

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As this manuscript was nearing completion, the authors became aware of other work [27] also proving the rigidity of simple admissible affine $\mathfrak{sl}(2)$ and [15] proving the conjectured Verlinde formula for $\mathfrak{sl}(2)$ at admissible levels. Since the methods in these other works are completely different from those used here, it was agreed to complete all manuscripts independently and submit them to arXiv simultaneously.

2. Algebras and categories of modules

We review and fix notation for the four families of vertex algebras that will be considered repeatedly below.

2.1. Affine vertex algebras.

Let $\mathfrak{g}$ be a simple (possibly abelian) finite dimensional complex Lie algebra with invariant symmetric non-degenerate bilinear form κ and choice of Cartan subalgebra $\mathfrak{h}$. Consider the affinisation of $\mathfrak{g}$,

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K, \tag{2.1}$$

where K is central and $x \otimes f(t), y \otimes g(t) \in \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ satisfy the commutation relation

$$[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t)g(t) + \kappa\big(x, y\big) \operatorname{Res}_t f(t)dg(t)K. \tag{2.2}$$

We do not include the degree operator in the construction of $\hat{\mathfrak{g}}$ because we will always identify it with the negative for the Virasoro L_0 operator obtained from the Sugawara construction. Denote $x_n = x \otimes t^n$ for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$ and define

$$\hat{\mathfrak{g}}_\pm = \operatorname{span}_\mathbb{C}\{x_{\pm n} \mid x \in \mathfrak{g},\ n \geq 1\}, \quad \hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}K, \quad \hat{\mathfrak{g}}_{\geq 0} = \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+, \tag{2.3}$$

to obtain the triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$.

Definition 2.1. Let M be a $\hat{\mathfrak{g}}$-module on which the central element K acts as $k \cdot \mathrm{id}$ for some $k \in \mathbb{C}$.

(1) The module M is called *smooth*, if for every $m \in M$

$$x_n m = 0 \quad \forall x \in \mathfrak{g},\ n \gg 0. \tag{2.4}$$

Denote the full subcategory of the category of all $\hat{\mathfrak{g}}$-modules whose objects are all smooth modules by $\hat{\mathfrak{g}}_k\text{-}\mathsf{mod}^{\mathrm{smth}}$.

(2) The module M is called *weight*, if it is smooth, finitely generated, simultaneously graded by generalized L_0 eigenvalues and $\mathfrak{h}$ eigenvalues, and all homogeneous spaces are finite dimensional. That is, M decomposes into a direct sum

$$M = \bigoplus_{\substack{\lambda \in \mathfrak{h}^* \\ h \in \mathbb{C}}} M_{\lambda,h}, \qquad M_{\lambda,h} = \{m \in M \mid (x_0 - \lambda(x))m = 0 = (L_0 - h)^N m,\ N \gg 0,\ x \in \mathfrak{h}\}, \tag{2.5}$$

where $\dim M_{\lambda,h} < \infty$ for all $\lambda \in \mathfrak{h}^*$, $h \in \mathbb{C}$ and for fixed $\lambda \in \mathfrak{h}^*$, $M_{\lambda,h} = 0$ for $\operatorname{Re}(h) \ll 0$.

Denote the full subcategory of $\hat{\mathfrak{g}}_k\text{-}\mathsf{mod}^{\mathrm{smth}}$ whose objects are all weight modules by $\hat{\mathfrak{g}}_k\text{-}\mathsf{mod}^{\mathrm{wt}}$.

(3) A weight module M is called *positive energy*, if there exists a real number $h_{\min}$ such that for all $\lambda \in \mathfrak{h}^*$ and all $h \in \mathbb{C}$ satisfying $\operatorname{Re}(h) < h_{\min}$ we have $M_{\lambda,h} = 0$. Denote the full subcategory of $\hat{\mathfrak{g}}_k\text{-}\mathsf{mod}^{\mathrm{wt}}$ whose objects are all positive energy modules by $\hat{\mathfrak{g}}_k\text{-}\mathsf{mod}^{\mathrm{wt}}_{\geq 0}$.

(4) Let M be weight. A non-zero homogeneous vector $v \in M$ is called *relaxed highest weight*, if it is homogeneous and $\hat{\mathfrak{g}}_+ v = 0$. If $v \in M$ is relaxed highest weight and generates M, then M is called a *relaxed highest weight module*.

We can now easily construct examples of smooth $\hat{\mathfrak{g}}$-modules. For example, generalised Verma modules and their simple quotients are smooth. These are constructed as follows. Let $\overline{M}$ be a $\mathfrak{g}$-module, which becomes a $\hat{\mathfrak{g}}_{\geq 0}$-module by defining K to act as $k \cdot \mathrm{id}$, $k \in \mathbb{C}$ and $\hat{\mathfrak{g}}_+$ to act trivially. The generalised Verma module $\mathfrak{V}(k, \overline{M})$, is the induced module

$$\mathfrak{V}(k, \overline{M}) = \operatorname{Ind}_{\hat{\mathfrak{g}}_{\geq 0}}^{\hat{\mathfrak{g}}} \overline{M}. \tag{2.6}$$

If $\overline{M}$ is simple, then $\mathfrak{V}(k, \overline{M})$ has a unique maximal ideal hence a unique simple quotient

$$\mathfrak{L}(k, \overline{M}) = \frac{\mathfrak{V}(k, \overline{M})}{\langle \text{maximal submodule} \rangle}. \tag{2.7}$$

Proposition 2.2 (Frenkel, Zhu [39]). *Let $\mathfrak{g}$ be simple (possibly abelian), $k \in \mathbb{C}$ and let $\mathbb{C}$ be the trivial $\mathfrak{g}$-module. Then the parabolic Verma module $\mathfrak{V}(k,\mathbb{C})$ admits the structure of a vertex operator algebra by defining the field map on $x_{-1}\mathbf{1}$, $x \in \mathfrak{g}$ to be*

$$Y(x_{-1}\mathbf{1}, z) = x(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n-1} \tag{2.8}$$

and continuing by derivatives and normal ordering. This vertex operator algebra is called the universal affine $\mathfrak{g}$ vertex operator algebra at level k. *Any choice of basis of $\mathfrak{g}$ corresponds to a set of strong generators and any two such generators $x, y \in \mathfrak{g}$ satisfy the operator product expansion relations*

$$x(z)y(w) \sim \frac{\kappa\big(x,y\big)k}{(z-w)^2} + \frac{[x,y](w)}{z-w}. \tag{2.9}$$

There is a distinguished choice of conformal vector, called the Sugawara vector

$$\omega_k = \frac{1}{2(k+h^\vee)} \sum_i (x^i)_{-1}(y^i)_{-1}\mathbf{1}, \tag{2.10}$$

where $h^\vee$ is the dual Coxeter number (defined to be 0, if $\mathfrak{g}$ is abelian) $\{x^i\}$ is any choice of basis of $\mathfrak{g}$ and $\{y^i\}$ is its dual with respect to κ, of central charge $c = \frac{k \dim \mathfrak{g}}{k+h^\vee}$. Finally, $\hat{\mathfrak{g}}_k\text{-}\mathsf{mod}^{smth}$ is the category of all $\mathfrak{V}(k,\mathbb{C})$-modules.

2.1.1. *The Heisenberg vertex algebra.* Let $\mathbb{C}$ be the trivial Lie algebra with non-degenerate invariant bilinear form characterised by $\kappa\big(1,1\big) = 2$. The rank one Heisenberg vertex algebra $\mathsf{H}(t)$ is the affine vertex operator algebra constructed by applying Proposition 2.2 at level $t \in \mathbb{C}^\times$ to the trivial Lie algebra $\mathbb{C}$. Higher rank Heisenberg vertex algebras are constructed by taking tensor products of the rank one Heisenberg vertex algebras. Note also that any two Heisenberg vertex algebras of equal rank are isomorphic regardless of the choice of level.

The (parabolic) Verma modules for the Heisenberg algebra are just Fock spaces. In particular, we denote by $\mathcal{F}_p$ the Fock space of highest weight $p \in \mathbb{C}$ (as a module of the Heisenberg Lie algebra $\mathsf{H}(t) = \mathcal{F}_0$).

Proposition 2.3. *Let $t \in \mathbb{R}^\times$. As a linear abelian category $\hat{\mathbb{C}}_t\text{-}\mathsf{mod}^{smth}$ is semisimple with Fock spaces forming a complete set of representatives for all simple isomorphism classes. Let $\hat{\mathbb{C}}_t\text{-}\mathsf{mod}^{smth}$ be the full subcategory generated by Fock spaces with real weights. Then $\mathsf{H}(t)$ furnishes $\hat{\mathbb{C}}_t\text{-}\mathsf{mod}^{smth}$ with the structure of a rigid braided monoidal category, which is braided equivalent to the category of finite dimensional $\mathbb{R}$-graded vector spaces.*

The linear category structure of $\hat{\mathbb{C}}_t\text{-}\mathsf{mod}^{\mathrm{smth}}$ (in particular the semisimplicity and classification of simples) is due to [60, Prop 3.6] and the monoidal structure is due to [34]. See [6, Prop 3.11, Thm 3.12] for a discussion of both.

2.1.2. *The $\mathfrak{sl}(2)$ vertex operator algebras.* Consider the smallest non-abelian simple complex Lie algebra $\mathfrak{sl}(2) = \mathrm{span}_{\mathbb{C}}\{e,h,f\}$ with choice of Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$. We spell out the well known relations and normalisations of $\mathfrak{sl}(2)$ to fix notation. The non-vanishing commutation relations are

$$[e,f] = h, \qquad [h,e] = 2e, \qquad [h,f] = -2f. \tag{2.11}$$

We normalise the Killing form such that its non-vanishing parings are

$$\kappa\big(h,h\big) = 2, \qquad \kappa\big(e,f\big) = \kappa\big(f,e\big) = 1 \tag{2.12}$$

Recall that the dual Coxeter number of $\mathfrak{sl}(2)$ is $h^\vee = 2$. The vertex operator algebra $\mathsf{A}_1(k)$ constructed from $\mathfrak{sl}(2)$ by applying Proposition 2.2 at level $k \in \mathbb{C} \setminus \{-2\}$ is called the *universal* $\mathfrak{sl}(2)$ *vertex operator algebra at level* k.

Recall that $\widehat{\mathfrak{sl}}(2)$ admits a family of automorphisms called *spectral flow*.

Definition 2.4.

(1) The *spectral flow* automorphisms $\sigma^\ell, \ell \in \mathbb{Z}$, of $\widehat{\mathfrak{sl}}(2)$ are determined on basis vectors and the Virasoro operators constructed by the Sugawara conformal vector by

$$\sigma^\ell(e_n) = e_{n-\ell}, \qquad \sigma^\ell(h_n) = h_n - \ell\delta_{n,0}K, \qquad \sigma^\ell(f_n) = f_{n+\ell},$$
$$\sigma^\ell(L_n) = L_n - \frac{1}{2}\ell h_n + \frac{1}{4}\ell^2 K, \qquad \sigma^\ell(K) = K. \tag{2.13}$$

(2) Let $g \in \mathrm{Aut}\Big(\widehat{\mathfrak{sl}}(2)\Big)$ and M an $\widehat{\mathfrak{sl}}(2)$-module, then the g-twist of M is an $\widehat{\mathfrak{sl}}(2)$-module whose underlying vectors space is that of M be with the action

$$x \cdot_g m = g^{-1}(x) \cdot m, \qquad \forall x \in \widehat{\mathfrak{sl}}(2),\ \forall m \in M, \tag{2.14}$$

where $\cdot$ is the original action of $\widehat{\mathfrak{sl}}(2)$ on M.

Note that the categories $\widehat{\mathfrak{sl}}(2)_k\text{-}\mathsf{mod}^{\mathrm{smth}}$ and $\widehat{\mathfrak{sl}}(2)_k\text{-}\mathsf{mod}^{\mathrm{wt}}$ are both closed under spectral flow automorphisms.

Proposition 2.5. *The vertex operator algebra* $\mathsf{A}_1(k)$ *admits a non-trivial proper ideal if and only if there exist positive integers* $u, v \in \mathbb{Z}_{>0}$, $\gcd(u,v) = 1$, $u \geq 2$ *such that* $k + 2 = \frac{u}{v}$. *The ideals at these levels are unique and hence maximal. The simple quotient of* $\mathsf{A}_1\big(\frac{u}{v} - 2\big)$ *will be denoted* $\mathsf{A}_1(u,v)$ *and is called the* $\mathfrak{sl}(2)$ (u,v)*-minimal model.*

For $u \geq 2$, $\mathsf{A}_1(u,1)$ is the much studied $\mathfrak{su}_2$ Wess-Zumino-Witten model at non-negative integral level. Here we shall primarily be interested in $\mathsf{A}_1(u,v)$ with $v \geq 2$. Modules over $\mathsf{A}_1(u,v)$ are of course naturally identified with modules over $\mathsf{A}_1\big(\frac{u}{v} - 2\big)$ on which the maximal ideal acts trivially, which leads to the following definition.

Definition 2.6. Let $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{smth}}$ be the full subcategory of $\mathsf{A}_1\big(\frac{u}{v} - 2\big)\text{-}\mathsf{mod}^{\mathrm{smth}}$ whose objects are all smooth modules on which the maximal ideal of $\mathsf{A}_1\big(\frac{u}{v} - 2\big)$ acts trivially.

The category of weight modules $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ is defined to be the full subcategory of all weight modules in $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{smth}}$ for which the eigenvalues of the Cartan generator h_0 are real. The category of positive energy modules $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}_{\geq 0}$ is the full subcategory of all positive energy modules in $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$.

Proposition 2.7. *For* $r, s \in \mathbb{Z}$ *and* $t \in \mathbb{C}^*$, *let*

$$\lambda_{r,s,t} = r - 1 - ts, \qquad \Delta^{\mathrm{aff}}_{r,s,t} = \frac{(r - st)^2 - 1}{4t}. \tag{2.15}$$

(1) (Gabriel [41]). *Every simple weight module over* $\mathfrak{sl}(2)$ *(modules on which* h *acts semisimply and each weight space is finite dimensional) is isomorphic to one of the following mutually inequivalent modules.*
   (i) $\overline{L(\mu)}$, $\mu \in \mathbb{Z}_{\geq 0}$, *the simple finite dimensional module of highest weight* μ *and dimension* $\mu + 1$.
   (ii) $\overline{D^+(\mu)}$, $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, *the Verma module of highest weight* μ.
   (iii) $\overline{D^-(\mu)} \cong \overline{D^+(-\mu)}^*$, $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, *the lowest weight modules which are the duals of the Verma modules above (they are also Verma modules for the choice of Borel* $\mathrm{span}_{\mathbb{C}}\{h, f\}$*).*

(iv) $\overline{E(\mu,q)}$, $\mu\in\mathbb{C}/2\mathbb{Z}$, $q\in\mathbb{C}\setminus\{\frac{\nu}{2}(\nu+2)\mid\nu\in\mu\}$, *the dense module with weight support* μ *on which the quadratic Casimir* $\frac{1}{2}h^2+ef+fe$ *acts as* $q\cdot\mathrm{id}$.

(2) (Adamović, Milas [5]). *For coprime* $u,v\in\mathbb{Z}_{>0}$, $u\geq 2$, *every simple relaxed highest weight module over* $\mathsf{A}_1(u,v)$ *(equivalently every simple object in* $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{wt}_{\geq 0}$*) is isomorphic to one of the following mutually inequivalent simple quotients of an induction of a simple* $\mathfrak{sl}(2)$ *module.*
  (i) $L_r=\mathfrak{L}\Big(\frac{u}{v}-2,\overline{L(r-1)}\Big)$, $r\in\{1,\dots,u-1\}$.
  (ii) $D^+_{r,s}=\mathfrak{L}\Big(\frac{u}{v}-2,\overline{D^+(\lambda_{r,s,\frac{u}{v}})}\Big)$, $1\leq r\leq u-1$, $1\leq s\leq v-1$.
  (iii) $D^-_{r,s}=D^{+\,\prime}_{r,s}$, $1\leq r\leq u-1$, $1\leq s\leq v-1$.
  (iv) $E_{\mu;r,s}=\mathfrak{L}\Big(\frac{u}{v}-2,\overline{E(\mu,2\frac{u}{v}\Delta^{\mathrm{aff}}_{r,s,\frac{u}{v}})}\Big)$, $1\leq r\leq u-1$, $1\leq s\leq v-1$, $vr+us<uv$, $\mu\in\mathbb{R}/2\mathbb{Z}$, $\lambda_{r,s},\lambda_{u-r,v-s}\notin\mu$. *The mixed inequality involving both* r *and* s *is required due to the isomorphism* $E_{\mu;r,s}\cong E_{\mu;u-r,v-s}$.

Further, there are the spectral flow relations $\sigma^{\pm1}L_r\cong D^{\pm}_{u-r,v-1}$ *and* $\sigma D^-_{r,s}\cong D^+_{u-r,v-s-1}$, *where* $1\leq r\leq u-1$ *and* $1\leq s\leq v-2$.

(3) (Futorny [40]; Adamović, Kawasetsu, Ridout [4]). *Every simple module in* $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{wt}$ *is isomorphic to one of the following mutually inequivalent spectral flow twists of simple modules in* $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{wt}_{\geq 0}$.
  (i) $\sigma^\ell D^+_{r,s}$, $1\leq r\leq u-1$, $1\leq s\leq v-1$, $\ell\in\mathbb{Z}$.
  (ii) $\sigma^\ell E_{\mu;r,s}$, $1\leq r\leq u-1$, $1\leq s\leq v-1$, $vr+us<uv$, $\mu\in\mathbb{R}/2\mathbb{Z}$, $\lambda_{r,s},\lambda_{u-r,v-s}\notin\mu$.

(4) (Adamović, Kawasetsu, Ridout [4]). *At each of the two disallowed weights* μ *above there are two mutually inequivalent reducible yet indecomposable relaxed highest weight modules uniquely characterised by the non-split short exact sequences*

$$\begin{aligned}0\to D^{\pm}_{r,s}\to &E^{\pm}_{r,s}\to D^{\mp}_{u-r,v-s}\to 0,\\ 0\to D^{\pm}_{u-r,v-s}\to &E^{\pm}_{u-r,v-s}\to D^{\mp}_{r,s}\to 0,\end{aligned}\tag{2.16}$$

for $1\leq r\leq u-1$, $1\leq s\leq v-1$, $vr+us<uv$, *where* $E^+_{r,s}$ *and* $E^-_{u-r,v-s}$ *correspond to* $\mu=[\lambda_{r,s}]$, *and* $E^-_{r,s}$ *and* $E^+_{u-r,v-s}$ *correspond to* $\mu=[\lambda_{u-r,v-s}]$.

(5) (Arakawa, Creutzig, Kawasetsu [9]). *The simple modules* $\sigma^\ell E_{\mu;r,s}$, $1\leq r\leq u-1$, $1\leq s\leq v-1$, $vr+us<uv$, $\ell\in\mathbb{Z}$, $\mu\in\mathbb{R}/2\mathbb{Z}$, $\lambda_{r,s},\lambda_{u-r,v-s}\notin\mu$, *are projective and injective in* $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{wt}$.

(6) (Adamović [3]; Arakawa, Creutzig, Kawasetsu [9]). *For* $1\leq r\leq u-1$, $1\leq s\leq v-1$, $\ell\in\mathbb{Z}$ *the projective cover and injective hull of* $D^+_{r,s}$ *are isomorphic and denoted* $P_{r,s}$. *These modules are uniquely characterised by the non-split exact sequences*

$$\begin{aligned}0\to\sigma^\ell E^+_{r,s}\to&\sigma^\ell P_{r,s}\to\sigma^{\ell+1}E^+_{r,s+1}\to 0,\qquad 1\leq s\leq v-2\\ 0\to\sigma^\ell E^+_{r,v-1}\to&\sigma^\ell P_{r,v-1}\to\sigma^{\ell+2}E^+_{u-r,1}\to 0,\end{aligned}\tag{2.17}$$

and they are also uniquely characterised by the non-split exact sequences

$$\begin{aligned}0\to\sigma^\ell E^-_{u-r,v-s-1}\to&\sigma^{\ell-1}P_{r,s}\to\sigma^{\ell-1}E^-_{u-r,v-s}\to 0,\qquad 1\leq s\leq v-2\\ 0\to\sigma^\ell E^-_{r,v-1}\to&\sigma^{\ell-2}P_{r,v-1}\to\sigma^{\ell-2}E^-_{u-r,1}\to 0.\end{aligned}\tag{2.18}$$

Since the L_r, $D^-_{r,s}$ *modules are related to the* $D^+_{r,s}$ *by spectral flow their projective covers and injective hulls satisfy the same spectral flow identifications. In particular the projective cover of* $L_1\cong\sigma^{-1}(D^+_{u-1,v-1})$, *(namely,* $\mathsf{A}_1(u,v)$ *as a module over itself and hence the tensor unit) is* $\sigma^{-1}P_{u-1,v-1}$.

We recall the following result of Creutzig that shows that the category of weight modules for $\mathsf{A}_1(u,v)$ admits a braided tensor category structure, with the framework of Lepowsky, Huang and Zhang [46].

Theorem 2.8 (Creutzig [14]; Creutzig, Huang, Yang [17]). *For coprime $u \geq 2$, $v \geq 1$ the category $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{wt}$ admits a tensor product induced from intertwining operators and thereby is a braided monoidal category. Further, for $a,b,c,d \in \mathbb{N}_0$, $b,c,d \leq a-1$, define the integers*

$$N_{c,d}^{(a)\,b} = \begin{cases} 1 & \text{if } |c-d|+1 \leq b \leq \min\{c+d-1, 2a-c-d-1\},\ b+c+d \equiv 1 \pmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

The simple $\mathsf{A}_1(u,v)$ weight modules L_r satisfy the fusion product decompositions

$$\begin{aligned} L_r \boxtimes L_{r'} &\cong \bigoplus_{r''=1}^{u-1} N_{r,r'}^{(u)\,r''} L_{r''}, \quad 1 \leq r, r' \leq u-1, \\ L_r \boxtimes E_{\mu';r',s'} &\cong \bigoplus_{r''=1}^{u-1} N_{r,r'}^{(u)\,r''} E_{\mu'+r-1;r'',s'}, \quad \mu' \in \mathbb{R}/2\mathbb{Z},\ \lambda_{r,s}, \lambda_{u-r,v-s} \notin \mu', \\ &\qquad 1 \leq r, r' \leq u-1,\ 1 \leq s' \leq v-1,\ vr+us < uv. \end{aligned} \tag{2.20}$$

Further, the simple modules L_r, $1 \leq r \leq u-1$ are rigid.

Conjecture 2.9 (Creutzig, Ridout [31]; Creutzig, Kanade, Liu, Ridout [20]). *For $1 \leq r, r' \leq u-1$, $1 \leq s, s' \leq v-1$, $vr+us < uv$, $\mu, \mu' \in \mathbb{R}/2\mathbb{Z}$, $\lambda_{r,s}, \lambda_{u-r,v-s} \notin \mu$, $\lambda_{r',s'}, \lambda_{u-r',v-s'} \notin \mu'$,*

$$\begin{aligned} E_{\mu;r,s} \boxtimes E_{\mu';r',s'} \cong &\bigoplus_{r'',s''} N_{r,r'}^{(u)\,r''} \Big(N_{s,s'-1}^{(v)\,s''} + N_{s,s'+1}^{(v)\,s''} \Big) E_{\mu+\mu';r'',s''} \\ &\oplus \bigoplus_{r'',s''} N_{r,r'}^{(u)\,r''} N_{s,s'}^{(v)\,s''} \big(\sigma^{-1} E_{\mu+\mu'+\frac{u}{v};r'',s''} \oplus \sigma E_{\mu+\mu'-\frac{u}{v};r'',s''} \big), \end{aligned} \tag{2.21}$$

where additionally $\mu+\mu'$ must be such that all of the relaxed highest weight modules appearing in the direct sum decomposition on the right-hand side are simple (this only excludes a finite number of values for the sum $\mu+\mu'$). For $1 \leq r \leq u-1$, $\mu, \mu' \in \mathbb{R}/2\mathbb{Z}$, $\pm\lambda_{1,1} \notin \mu$, $\pm\lambda_{r,1} \notin \mu'$ and $\lambda_{r,0} = r-1 \in \mu+\mu'$

$$E_{\mu;1,1} \boxtimes E_{\mu';r,1} \cong \sigma^{-1} P_{u-r,v-1} \oplus (1-\delta_{v,2}) E_{\mu+\mu';r,2}, \tag{2.22}$$

while for $1 \leq r \leq u-1$, $2 \leq s \leq v-2$, $\mu, \mu' \in \mathbb{R}/2\mathbb{Z}$, $\pm\lambda_{1,1} \notin \mu$, $\pm\lambda_{r,s} \notin \mu'$,

$$E_{\mu;1,1} \boxtimes E_{\mu';r,s} \cong \begin{cases} P_{r,s-1} \oplus \sigma^{-1} E_{\mu+\mu'+\frac{u}{v};r,s} \oplus E_{\mu+\mu';r,s+1}, & \text{if } \lambda_{r,s-1} \in \mu+\mu', \\ P_{u-r,v-s-1} \oplus \sigma^{-1} E_{\mu+\mu'+\frac{u}{v};r,s} \oplus E_{\mu+\mu';r,s-1} & \text{if } \lambda_{u-r,v-s-1} \in \mu+\mu', \\ \sigma^{-1} P_{r,s} \oplus \sigma E_{\mu+\mu'-\frac{u}{v};r,s} \oplus E_{\mu+\mu';r,s-1}, & \text{if } \lambda_{r,s+1} \in \mu+\mu', \\ \sigma^{-1} P_{u-r,v-s} \oplus \sigma E_{\mu+\mu'-\frac{u}{v};r,s} \oplus E_{\mu+\mu';r,s+1} & \text{if } \lambda_{u-r,v-s+1} \in \mu+\mu'. \end{cases} \tag{2.23}$$

Note that from (2.21) we can see that all simple relaxed highest weight modules of the form $E_{\mu;r,s}$ are conjecturally generated as direct summands by repeated application of $E_{\mu';1,1}$ and $E_{\mu';2,1}$ to $E_{\mu'';1,1}$, and further that all other simple modules appear as subquotients of these repeated tensor products, due to the appearance of the indecomposable reducible projective modules $P_{r,s}$. Since the $E_{\mu;r,s}$ are projective, if they are simple, a sufficient condition for their rigidity is that $E_{\mu';1,1}$ and $E_{\mu';2,1}$ are rigid. Further, from (2.20), we see that $E_{\mu';2,1} \cong L_2 \boxtimes E_{\mu'+1;1,1}$ and hence by the rigidity of L_2, the rigidity of $E_{\mu';1,1}$ implies the rigidity of $E_{\mu';2,1}$. Thus sufficient conditions for the rigidity of $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ are given by Conjecture 2.9 being true and $E_{\mu';1,1}$ being rigid.

2.2. **The $\mathcal{N}=2$ superconformal Lie algebra.** In order to define the $\mathcal{N}=2$ superconformal vertex algebra we first introduce the superconformal Lie algebra. It is the infinite dimensional Lie superalgebra with even and odd components of the basis respectively given by $\{L_n, J_n, C \mid n \in \mathbb{Z}\}$ and $\{G^\pm_r \mid r \in \frac{1}{2}+\mathbb{Z}\}$ subject to the relations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}C, & [L_n, G^\pm_r] &= \left(\frac{1}{2}n - r\right)G^\pm_{n+r}, \\ \{G^+_r, G^-_s\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{4r^2-1}{12}\delta_{r,-s}C, & [L_n, J_m] &= -mJ_{n+m}, \\ [J_n, J_m] &= \frac{n}{3}\delta_{n,-m}C, \qquad [J_n, G^\pm_r] = \pm G^\pm_{n+r}, & \{G^\pm_r, G^\pm_s\} &= 0, \end{aligned} \tag{2.24}$$

for all $n, m \in \mathbb{Z}$, $r, s \in \frac{1}{2}+\mathbb{Z}$ and where C is central. We denote

$$\begin{aligned} \mathfrak{N}_\pm &= \mathrm{span}_\mathbb{C}\left\{L_{\pm(n+1)}, J_{\pm(n+1)}, G^+_{\pm(n+\frac{1}{2})}, G^-_{\pm(n+\frac{1}{2})} \mid n \in \mathbb{N}_0\right\} \\ \mathfrak{N}_0 &= \mathrm{span}_\mathbb{C}\{L_0, J_0, C\} \\ \mathfrak{N}_{\geq 0} &= \mathfrak{N}_+ \oplus \mathfrak{N}_0. \end{aligned} \tag{2.25}$$

so that we have a triangular decomposition $\mathfrak{N} = \mathfrak{N}_- \oplus \mathfrak{N}_0 \oplus \mathfrak{N}_+$.

Definition 2.10. Let M be a $\mathbb{Z}_2$-graded $\mathfrak{N}$-module on which C acts as $c \cdot \mathrm{id}$ for some $c \in \mathbb{C}$.

(1) M is called *smooth*, if for every $m \in M$

$$E_n m = 0 = O_{n+\frac{1}{2}} m, \qquad n \gg 0 \tag{2.26}$$

where $E \in \{L, J\}$, $O \in \{G^+, G^-\}$. Denote the full subcategory of all $\mathfrak{N}$-modules whose objects are all smooth modules by $\mathfrak{N}_c\text{-}\mathsf{mod}^{\mathrm{smth}}$.

(2) M is called *weight*, if it is smooth, finitely generated, $\mathfrak{N}_{>0}$ acts locally nilpotently (for every $m \in M$, $\dim(\mathcal{U}(\mathfrak{N}_{>0})m) < \infty$) and J_0 acts semisimply. This implies that M is graded by J_0-eigenvalues and generalised L_0-eigenvalues, that is, it decomposes into a direct sum of homogeneous spaces

$$M = \bigoplus_{q,h\in\mathbb{C}} M_{q,h}, \qquad M_{q,h} = \{m \in M \mid (J_0 - q)m = 0 = (L_0 - h)^N m,\ N \gg 0\}, \tag{2.27}$$

where $\dim M_{q,h} < \infty$, the weight support $\mathrm{supp}(M) = \{(q,h), \in \mathbb{C}^2 \mid M_{q,h} \neq 0\} \subset \mathbb{C}^2$ is contained within a finite number of $\mathbb{Z}^2$-cosets of $\mathbb{C}^2$ and $M_{q,h} = 0$ for $\mathrm{Re}(h) \ll 0$. Denote the full subcategory of $\mathfrak{N}_c\text{-}\mathsf{mod}^{\mathrm{smth}}$ whose objects are all weight modules by $\mathfrak{N}_c\text{-}\mathsf{mod}^{\mathrm{wt}}$.

(3) Let M be weight. A non-zero homogeneous vector $v \in M$ is called *singular* if $\mathfrak{N}_+ v = 0$. Note that $G^\pm_{\frac{1}{2}} v = 0 = J_1 v$ is a necessary and sufficient condition for singularity. If v is singular and generates M, then v is called *highest weight* and M is called a *highest weight module*.

Recall that $\mathfrak{N}$ admits a family of automorphisms called *spectral flow*.

Definition 2.11.

(1) The *spectral flow* automorphisms $\sigma^\ell, \ell \in \mathbb{Z}$ of $\mathfrak{N}$, are determined on basis vectors by

$$\sigma^\ell(L_n) = L_n - \ell J_n + \frac{1}{6}\ell^2\delta_{n,0}C, \quad \sigma^\ell(J_n) = J_n - \frac{1}{3}\ell\delta_{n,0}C, \quad \sigma^\ell(G^\pm_s) = G^\pm_{s+\ell}, \quad \sigma^\ell(C) = C. \tag{2.28}$$

(2) Let $g \in \mathrm{Aut}(\mathfrak{N})$ and M a $\mathfrak{N}$-module, then the g-twist of M is a $\mathfrak{N}$-module whose underlying vectors space is that of M but with the action

$$x \cdot_g m = g^{-1}(x) \cdot m, \qquad \forall x \in \mathfrak{N},\ \forall m \in M, \tag{2.29}$$

where $\cdot$ is the original action of $\mathfrak{N}$ on M.

Note that the categories $\mathfrak{N}_c\text{-}\mathsf{mod}^{\mathrm{smth}}$ and $\mathfrak{N}_c\text{-}\mathsf{mod}^{\mathrm{wt}}$ are both closed under spectral flow automorphisms. Note that in principle the spectral flow parameter could be a half odd integer, however, the resulting twisted module would then no longer satisfy the locality axiom. That is, the series expansion of fields can contain half odd integers. Such twisted modules are called *Ramond* (as opposed to *Neveu-Schwarz*) modules. Half odd integral spectral flow interchanges the Neveu-Schwarz and Ramond sectors.

We can now easily construct examples of smooth $\mathfrak{N}$-modules. For example, the Verma modules and their simple quotients are smooth. These are constructed as follows. For $q, h, c \in \mathbb{C}$, let $\mathbb{C}\mathbf{1}_{q,h,c}$ denote the even 1-dimensional $\mathfrak{N}_{\geq 0}$-module characterised by

$$J_0\mathbf{1}_{q,h,c} = q\mathbf{1}_{q,h,c}, \qquad C\mathbf{1}_{q,h,c} = c\mathbf{1}_{q,h,c}$$
$$L_0\mathbf{1}_{q,h,c} = h\mathbf{1}_{q,h,c},, \qquad \mathfrak{N}_+\mathbf{1}_{q,h,c} = 0. \tag{2.30}$$

The Verma module $\mathcal{M}_{q,h,c}$, is the induced module

$$\mathcal{M}_{q,h,c} = \mathcal{U}(\mathfrak{N}) \otimes_{\mathcal{U}(\mathfrak{N}_{\geq 0})} \mathbb{C}\mathbf{1}_{q,h,c}, \tag{2.31}$$

while the simple quotient of $\mathcal{M}_{q,h,c}$ is

$$\mathcal{L}_{q,h,c} = \frac{\mathcal{M}_{q,h,c}}{\langle \text{maximal submodule} \rangle}. \tag{2.32}$$

Proposition 2.12. *Let $c \in \mathbb{C}$. Then the $\mathfrak{N}$-module $\mathsf{N}(c) = \mathcal{M}_{0,0,c}/\langle G^{\pm}_{-\frac{1}{2}}\mathbf{1}_{0,0,c}\rangle$ admits the structure of a vertex operator superalgebra by defining the field map on the strong generators*

$$Y(L_{-2}\mathbf{1}, z) = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \qquad Y(J_{-1}\mathbf{1}, z) = J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$$
$$Y(G^{\pm}_{-\frac{3}{2}}\mathbf{1}, z) = G^{\pm}(z) = \sum_{n\in\mathbb{Z}} G^{\pm}_{n+\frac{1}{2}} z^{-n-2}, \tag{2.33}$$

and continuing by derivatives and normal ordering. This vertex operator superalgebra is called the universal $\mathcal{N} = 2$ superconformal vertex algebra. *The strong generators above satisfy the operator product expansion relations*

$$L(z)L(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} \qquad L(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w}$$
$$L(z)G^{\pm}(w) \sim \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial_w G^{\pm}(w)}{z-w} \qquad J(z)J(w) \sim \frac{\frac{c}{3}}{(z-w)^2}$$
$$J(z)G^{\pm}(w) \sim \frac{\pm G^{\pm}(w)}{(z-w)} \qquad G^{\pm}(z)G^{\pm}(w) \sim 0$$
$$G^{\pm}(z)G^{\mp}(w) \sim \frac{\frac{2c}{3}}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2L(w) \pm \partial J(w)}{z-w}. \tag{2.34}$$

Finally, $\mathfrak{N}_c\text{-}\mathsf{mod}^{smth}$ is the category of all $\mathsf{N}(c)$-modules.

Proposition 2.13 (Gorelik, Kac [42])**.** *The universal $\mathcal{N} = 2$ vertex algebra $\mathsf{N}(c)$ is not simple if and only if there exist $u, v \in \mathbb{Z}$ with $u \geq 2$, $v \geq 1$ and $gcd(u, v) = 1$ such that the central charge satisfies*

$$c = 3 - \frac{6v}{u}. \tag{2.35}$$

For these central charges $\mathsf{N}(c)$ admits a non-trivial maximal ideal generated by a singular vector of conformal weight $(u-1)v$ and J-weight 0.

When $\mathsf{N}(c)$ is not simple, so that $c = 3 - \frac{6v}{u}$ for some u, v as in Proposition 2.13, we denote its simple quotient by $\mathsf{N}(u,v)$ and refer to it as the $\mathcal{N}=2$ minimal model at central charge $3-\frac{6v}{u}$.

Definition 2.14. For coprime $u, v \in \mathbb{Z}_{>0}$, $u \geq 2$, let $\mathsf{N}(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ be the full subcategory of $\mathsf{N}\big(3-\frac{6v}{u}\big)\text{-}\mathsf{mod}^{\mathrm{wt}}$ of modules annihilated by the maximal ideal of $\mathsf{N}\big(3-\frac{6v}{u}\big)$ and for which the eigenvalues of J_0 are real.

Proposition 2.15 (Adamović [1])**.** *For coprime $u, v \in \mathbb{Z}_{>0}$, $u \geq 2$, denote*

$$h_{r,s;q} = \frac{\big(r - \frac{u}{v}s\big)^2 - 1}{4\frac{u}{v}} - \frac{u}{4v}q^2, \qquad r, s \in \mathbb{Z},\ q \in \mathbb{R}. \tag{2.36}$$

Then the simple highest weight modules $\mathcal{L}_{q,h,3-\frac{6v}{u}}$ and their parity reversals $\Pi\mathcal{L}_{q,h,3-\frac{6v}{u}}$ whose conformal and J-weight (q,h) lie in one of the following sets

(1) $\Big\{\Big(\frac{pv}{u}, h_{r,0;\frac{pv}{u}}\Big) \mid 1 \leq r \leq u-1,\ 1-r \leq p \leq r-1, p+r \equiv 1 \pmod 2\Big\}$,

(2) $\{(q, h_{r,s;q}) \mid 1 \leq r \leq u-1,\ 1 \leq s \leq v-1,\ vr + us < uv, q \in \mathbb{R}\}$,

form a complete set of representatives of simple isomorphism classes in $\mathsf{N}(u,v)\text{-}\mathsf{mod}^{wt}$. We denote $\mathcal{L}_{q,h_{r,s;q},3-\frac{6v}{u}} = \mathcal{L}_{q,r,s}$.

Proposition 2.16 (Creutzig, [14, Thm 8.2])**.** *The category $\mathsf{N}(u,v)\text{-}\mathsf{mod}^{wt}$ admits a tensor product induced from intertwining operators and thereby is a braided monoidal category.*

2.3. **The fermionic ghost vertex algebra.** Ghost systems date back to the early days in string theory in the physics literature. For a formal mathematical description of them, we refer to Kac's book [50]. To define the fermionic ghost vertex algebra BC we first introduce its underlying Lie super algebra $\mathfrak{bc}$ with even and odd components of the basis respectively given by $\{\mathbf{1}\}$ and $\{b_r, c_r \mid r \in \frac{1}{2}+\mathbb{Z}\}$. These basis vectors satisfy the commutation relations

$$\{b_r, c_s\} = \delta_{r+s,0}\mathbf{1}, \qquad r, s \in \frac{1}{2} + \mathbb{Z}, \tag{2.37}$$

where $\mathbf{1}$ is central (and will always be taken to act as the identity on modules). We then have the triangular decomposition

$$\mathfrak{bc}_{\pm} = \operatorname{span}_{\mathbb{C}}\Big\{b_{\pm r}, c_{\pm r} \mid r \in \frac{1}{2} + \mathbb{N}_0\Big\}, \qquad \mathfrak{bc}_0 = \mathbb{C}\mathbf{1}, \qquad \mathfrak{bc}_{\geq} = \mathfrak{bc}_+ \oplus \mathfrak{bc}_0. \tag{2.38}$$

Since we require $\mathbf{1}$ to act as the identity there is only one Verma module (up to parity reversal) for this decomposition.

$$\mathsf{BC} = \operatorname{Ind}_{\mathfrak{bc}_{\geq}}^{\mathfrak{bc}} \mathbb{C}\Omega, \qquad b_r\Omega = c_r\Omega = 0, \quad \forall r \in \frac{1}{2} + \mathbb{N}_0. \tag{2.39}$$

Proposition 2.17. *The $\mathfrak{bc}$-module BC admits the structure of a vertex operator super algebra by defining the field map on the odd strong generators*

$$Y(b_{-\frac{1}{2}}\Omega, z) = b(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} b_r z^{-r-\frac{1}{2}}, \qquad Y(c_{-\frac{1}{2}}\Omega, z) = c(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} c_r z^{-r-\frac{1}{2}}, \tag{2.40}$$

and continuing by derivatives and normal ordering. This vertex algebra has a number of names in the literature including BC vertex algebra, fermionic ghost system *and* charged free fermions*. The strong generators above satisfy the operator product expansion relations*

$$b(z)c(w) \sim \frac{1}{z-w}, \qquad b(z)b(w) \sim c(z)c(w) \sim 0. \tag{2.41}$$

The BC vertex algebra *admits a conformal vector*

$$\omega_{\mathsf{BC}} = \frac{1}{2}\Big(b_{-\frac{3}{2}}c_{-\frac{1}{2}} + c_{-\frac{3}{2}}b_{-\frac{1}{2}}\Big)\Omega, \quad Y(\omega_{\mathsf{BC}}, z) = T^{\mathsf{BC}}(z) = \frac{1}{2}(:(\partial b(z))c(z): + :(\partial c(z))b(z):). \quad (2.42)$$

of central charge 1 *and an additional distinguished vector*

$$Q = b_{-\frac{1}{2}}c_{-\frac{1}{2}}\Omega, \qquad Y(Q,z) = Q(z) = :b(z)c(z):, \qquad (2.43)$$

such that $b(z)$, $c(z)$ *are conformal weight* $\frac{1}{2}$ *primaries,* $Q(z)$ *is a conformal weight* 1 *primary and* $Q(z)$ *satisfies the operator product expansion relations*

$$Q(z)b(w) \sim \frac{b(w)}{z-w}, \qquad Q(z)c(w) \sim \frac{-c(w)}{z-w}, \quad Q(z)Q(w) \sim \frac{1}{(z-w)^2}, \qquad (2.44)$$

that is $Q(z)$ *generates a subalgebra isomorphic to the Heisenberg vertex algebra* $\mathsf{H}\big(\frac{1}{2}\big)$ *and its zero mode* Q_0 *(the coefficient of* z^{-1}*) gives a* $\mathbb{Z}$*-grading to* BC*,* called ghost weight*, which assigns weight* 0 *to* Ω*, weight* 1 *to* $b(z)$ *modes and weight* -1 *to* $c(z)$ *modes.*

Definition 2.18. Let M be a $\mathbb{Z}_2$-graded $\mathfrak{bc}$-module on which $\mathbf{1}$ acts as the identity.

(1) M is called *smooth*, if for every $m \in M$

$$b_r m = c_r m = 0, \qquad r \gg 0. \qquad (2.45)$$

Denote the full subcategory of all $\mathfrak{bc}$-modules whose objects are all smooth modules by $\mathfrak{bc}\text{-}\mathsf{mod}^{\mathrm{smth}}$. This category is also the category of all $\mathbb{Z}_2$-graded BC-modules.

(2) M is called *weight*, if it is smooth, finitely generated, $\mathfrak{bc}_{\geq 0}$ acts locally nilpotently (for every $m \in M$, $\dim(\mathcal{U}(\mathfrak{bc}_{\geq 0})m) < \infty$) and Q_0 acts semisimply. This implies that M is graded by Q_0-eigenvalues and generalised L_0-eigenvalues, that is, it decomposes into a direct sum of homogeneous spaces

$$M = \bigoplus_{q,h\in\mathbb{C}} M_{q,h}, \qquad M_{q,h} = \{m \in M \mid (Q_0 - q)m = 0 = (L_0 - h)^N m,\ N \gg 0\}, \qquad (2.46)$$

where $\dim M_{q,h} < \infty$, the weight support $\mathrm{supp}(M) = \{(q,h), \in \mathbb{C}^2 \mid M_{q,h} \neq 0\} \subset \mathbb{C}^2$ is contained within a finite number of $\mathbb{Z}^2$-cosets of $\mathbb{C}^2$ and $M_{q,h} = 0$ for $\mathrm{Re}(h) \ll 0$. Denote the full subcategory of $\mathfrak{bc}\text{-}\mathsf{mod}^{\mathrm{smth}}$ whose objects are all weight modules by $\mathfrak{bc}\text{-}\mathsf{mod}^{\mathrm{wt}}$.

We conclude this section by recalling the known equivalence between the category of $\mathfrak{bc}$ weight modules and super vector spaces.

Proposition 2.19. *The category* $\mathfrak{bc}\text{-}\mathsf{mod}^{wt}$ *is semisimple with two simple isomorphism classes* BC *and its parity reversal* $\Pi\mathsf{BC}$*. As a braided monoidal category (with the tensor product constructed from intertwining operators)* $\mathfrak{bc}\text{-}\mathsf{mod}^{wt}$ *is equivalent to* sVec *the symmetric monoidal category of finite dimensional super vector spaces. In particular* BC *corresponds to* $\mathbb{C}^{1|0}$ *the even one dimensional super vector space and* $\Pi\mathsf{BC}$ *corresponds to* $\mathbb{C}^{0|1}$ *the odd one dimensional super vector space.*

3. The coset realisation of the $\mathcal{N} = 2$ superconformal algebra

In this section we recall a well known coset construction of the universal $\mathcal{N} = 2$ superconformal vertex operator superalgebras $\mathsf{N}(c)$ and their simple quotients $\mathsf{N}(u,v)$ which appeared in the physics literature in [57, 56, 35] and was more recently studied in detail in [22].

Proposition 3.1. *Consider the vertex operator superalgebra* $\mathsf{A}_1(t-2) \otimes \mathsf{BC}$*,* $t \in \mathbb{C}^\times$*, then there exists an embedding* $\phi_h : \mathsf{H}(2t) \hookrightarrow \mathsf{A}_1(t-2) \otimes \mathsf{BC}$ *characterised on the strong generator* $a_{-1}|0\rangle$ *and continued to the conformal vector* $\frac{1}{4t}a_{-1}^2|0\rangle$ *by*

$$a = a_{-1}|0\rangle \xmapsto{\phi_h} h \otimes \Omega + 2\mathbf{1} \otimes Q = \Big(h_{-1} + 2b_{-\frac{1}{2}}c_{-\frac{1}{2}}\Big)\mathbf{1} \otimes \Omega$$

$$\frac{1}{4t}a_{-1}^2|0\rangle \xmapsto{\phi_h} \frac{1}{4t}\Big(h_{-1}^2 + 4\Big(b_{-\frac{3}{2}}c_{-\frac{1}{2}} + c_{-\frac{3}{2}}b_{-\frac{1}{2}}\Big) + 4h_{-1}b_{-\frac{1}{2}}c_{-\frac{1}{2}}\Big)\mathbf{1}\otimes\Omega. \tag{3.1}$$

There exists an additional embedding $\phi_n : \mathsf{N}\big(3-\frac{6}{t}\big) \hookrightarrow \mathsf{A}_1(t-2)\otimes\mathsf{BC}$

$$\begin{aligned}
J_{-1}\mathbf{1} = J &\xmapsto{\phi_n} \frac{h\otimes\Omega - (t-2)\mathbf{1}\otimes Q}{t} = \frac{h_{-1} - (t-2)b_{-\frac{1}{2}}c_{-\frac{1}{2}}}{t}\mathbf{1}\otimes\Omega \\
G^+ = G^+_{-\frac{3}{2}}\mathbf{1} &\xmapsto{\phi_n} \sqrt{\frac{2}{t}}e\otimes c = \sqrt{\frac{2}{t}}e_{-1}c_{-\frac{1}{2}}\mathbf{1}\otimes\Omega \\
G^- = G^-_{-\frac{3}{2}}\mathbf{1} &\xmapsto{\phi_n} \sqrt{\frac{2}{t}}f\otimes b = \sqrt{\frac{2}{t}}f_{-1}b_{-\frac{1}{2}}\mathbf{1}\otimes\Omega \\
\omega = L_{-2}\mathbf{1} &\xmapsto{\phi_n} \frac{1}{2t}\Big(e_{-1}f_{-1} + f_{-1}e_{-1} + (t-2)\Big(b_{-\frac{3}{2}}c_{-\frac{1}{2}} + c_{-\frac{3}{2}}b_{-\frac{1}{2}}\Big) - 2h_{-1}b_{-\frac{1}{2}}c_{-\frac{1}{2}}\Big)\mathbf{1}\otimes\Omega. \qquad (3.2)
\end{aligned}$$

Further, the image of ϕ_n *is the commutant (also known as a coset in the physics literature) of the image of* ϕ_h*, that is*

$$\phi_n\left(\mathsf{N}\left(3-\frac{6}{t}\right)\right) = \mathrm{Com}(\phi_h(\mathsf{H}(2t)), \mathsf{A}_1(t-2)\otimes\mathsf{BC}). \tag{3.3}$$

Further, if $t=\frac{u}{v}$*, for coprime* $u\geq 2$*,* $v\geq 1$*, then* $\mathsf{N}\big(3-\frac{6}{t}\big)$ *and* $\mathsf{A}_1(t-2)$ *admit non-trivial ideals, and the embedding* ϕ_n *and commutant factor through the respective simple quotients, that is*

$$\phi_n(\mathsf{N}(u,v)) = \mathrm{Com}(\phi_h(\mathsf{H}(2t)), \mathsf{A}_1(u,v)\otimes\mathsf{BC}). \tag{3.4}$$

Proposition 3.2. *For* $t\in\mathbb{C}^\times$ *consider the embedding* $\phi=\phi_h\otimes\phi_n : \mathsf{H}(2t)\otimes\mathsf{N}\big(3-\frac{6}{t}\big)\hookrightarrow \mathsf{A}_1(t-2)\otimes\mathsf{BC}$*, where* ϕ_h,ϕ_n *were characterised in Proposition 3.1. Let* M *be an indecomposable* $\mathsf{A}_1(t-2)$ *weight module with* $\mathfrak{sl}(2)$*-weight support* $\mathrm{supp}(M)=\lambda+2\mathbb{Z}, \lambda\in\mathbb{C}$*. Then the following hold.*

(1) $M\otimes\Pi^i\mathsf{BC}$*,* $i=0,1$ *is an* $\mathsf{H}(2t)\otimes\mathsf{N}\big(3-\frac{6}{t}\big)$*-module by restriction (that is, pulling back along* ϕ*) and decomposes as*

$$\mathrm{Res}\big(M\otimes\Pi^i\mathsf{BC}\big) \cong \bigoplus_{p\in\mathrm{supp}(M)} \mathcal{F}_p\otimes C_p^{[i]}(M), \tag{3.5}$$

where the $C_p^{[i]}(M)$ *are indecomposable* $\mathsf{N}\big(3-\frac{6}{t}\big)$*-modules.*

(2) *A vector* $m\in M\otimes\Pi^i\mathsf{BC}$ *is homogeneous for* $\mathsf{A}_1(t-2)\otimes\mathsf{BC}$ *if and only if it is homogeneous for* $\mathsf{H}(2t)\otimes\mathsf{N}\big(3-\frac{6}{t}\big)$*. The grading operators (and hence weights) for the four algebras are interrelated by*

$$L_0^{\mathfrak{sl}(2)} + L_0^{\mathsf{BC}} = L_0^{\mathsf{H}} + L_0^{\mathcal{N}=2}, \qquad a_0 = h_0 + 2Q_0, \qquad J_0 = \frac{h_0-(t-2)Q_0}{t}. \tag{3.6}$$

(3) *For every* $\widehat{\mathfrak{sl}}(2)$ *relaxed highest weight vector* $v\in M$ *of* $\mathfrak{sl}(2)$ *weight* $\mu\in\mathrm{supp}(M)$ *and conformal weight* h *there exists a* $\mathfrak{N}$ *highest weight vector* $\chi\in C_\mu^{[i]}(M)$ *whose conformal weight is* $h-\frac{\mu^2}{4t}$ *and whose* J*-weight is* $\frac{\mu}{t}$*. In particular, in the decomposition (3.5) above* $v\otimes\Omega = |\mu\rangle\otimes\chi$*.*

(4) *If* $t=\frac{u}{v}$ *for coprime* $u\geq 2$*,* $v\geq 1$ *and* M *is one of the simple relaxed highest weight modules* $E_{\mu;r,s}$*,* $\mu\in\mathbb{C}/2\mathbb{Z}$*,* $\pm\lambda_{r,s}\notin\mu$ *of Proposition 2.7, then*

$$C_p^{[i]}(\sigma^\ell E_{\mu;r,s}) = \Pi^{i+\ell}\mathcal{L}_{\frac{pv}{u}-\ell;r,s}, \qquad p\in\mu+\ell\tfrac{u}{v}, \tag{3.7}$$

with $\mathcal{L}_{q,r,s}$ *as in Proposition 2.15.*

Proof. With the exception of Part 3 above Proposition 3.2 is well known. Most of the proof will therefore focus on Part 3.

The form of the restriction (3.5) and the range of the Fock space weights p follows by applying [19, Thm 3.8] to the embedding $\phi = \phi_h \otimes \phi_n$. The indecomposability of $C_p(M)$ then follows from the indecomposability of M also by using [19, Thm 3.8].

The relations between grading operators follow directly from the embedding formulae (3.1) and (3.2).

Let $v \in M$ be a relaxed highest weight vector of $\mathfrak{sl}(2)$ weight μ and conformal weight h. The formula for the embedding (3.1) implies that

$$a_n v \otimes \Omega = (h_n + 2Q_n) v \otimes \Omega = \mu \delta_{n,0} v \otimes \Omega, \qquad n \geq 0. \tag{3.8}$$

Note that we have suppressed the embedding ϕ. Thus $v \otimes \Omega$ is a Heisenberg highest weight vector of Heisenberg weight μ, that is, $v \otimes \Omega \in |\mu\rangle \otimes \chi$, $\chi \in C_\mu(M)$. Further, the formulae for the embedding (3.2) imply that

$$\begin{aligned} G_r^+ v \otimes \Omega &= \sqrt{\frac{2}{t}} \sum_{n \in \mathbb{N}_0} e_{-n} c_{r+n} v \otimes \Omega = 0, \quad r \geq \frac{1}{2} \\ G_r^- v \otimes \Omega &= \sqrt{\frac{2}{t}} \sum_{n \in \mathbb{N}_0} f_{-n} b_{r+n} v \otimes \Omega = 0, \quad r \geq \frac{1}{2} \\ J_n v \otimes \Omega &= \frac{h_n - kQ_n}{t} v \otimes \Omega = \delta_{n,0} \frac{\mu}{t} v \otimes \Omega, \quad n \geq 0. \end{aligned} \tag{3.9}$$

Thus χ is an $\mathfrak{N}$ singular vector with J-weight $\frac{\mu}{t}$. Finally, since the image of the sum of the conformal vectors of $\mathsf{H}(2t)$ and $\mathsf{N}\big(3 - \frac{6}{t}\big)$ is the conformal vector of $\mathsf{A}_1(t-2) \otimes \mathsf{BC}$ the conformal weight h of v must be the sum of the conformal weights of $|\mu\rangle$ and χ. Hence the conformal weight of χ is $h - \frac{\mu^2}{4t}$.

The relevant formulae for $C_p^{[i]}(M)$ are given in [23, Eq 4.16] (note that i here corresponds to $2i$ in [23]).

□

For coprime $u \geq 2$, $v \geq 1$, the categories $\mathsf{H}\big(2\frac{u}{v}\big)\text{-}\mathsf{mod}^{\mathrm{wt}}$, $\mathsf{N}(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$, $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ and $\mathsf{BC}\text{-}\mathsf{mod}^{\mathrm{wt}}$ are all locally finite abelian categories whose objects all have finite Jordan-Hölder length (see [9, Thm 1.2] and [14, Cor 5.1]), thus by [61, Thm 4.11] we have the following equivalences of braided monoidal categories.

$$\begin{aligned} \Big(\mathsf{H}\Big(2\frac{u}{v}\Big) \otimes \mathsf{N}(u,v)\Big)\text{-}\mathsf{mod}^{\mathrm{wt}} &\cong \mathsf{H}\Big(2\frac{u}{v}\Big)\text{-}\mathsf{mod}^{\mathrm{wt}} \widehat{\boxtimes} \mathsf{N}(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}, \\ (\mathsf{A}_1(u,v) \otimes \mathsf{BC})\text{-}\mathsf{mod}^{\mathrm{wt}} &\cong \mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}} \widehat{\boxtimes} \mathsf{BC}\text{-}\mathsf{mod}^{\mathrm{wt}}, \end{aligned} \tag{3.10}$$

where $\widehat{\boxtimes}$ denotes the Deligne tensor product of abelian categories. Further, $\mathsf{A}_1(u,v) \otimes \mathsf{BC}$ is a commutative algebra object in (a direct limit completion of) $\mathsf{H}\big(2\frac{u}{v}\big) \otimes \mathsf{N}(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ (see [25] for details on direct limit completions in the context of vertex operator superalgebras). We can therefore consider the category $\operatorname{Rep} \mathsf{A}_1(u,v) \otimes \mathsf{BC}$, the category of $\mathsf{A}_1(u,v) \otimes \mathsf{BC}$-modules in $\big(\mathsf{H}\big(2\frac{u}{v}\big) \otimes \mathsf{N}(u,v)\big)\text{-}\mathsf{mod}^{\mathrm{wt}}$. These will generally be twisted modules, the full subcategory of non-twisted (or local) modules is denoted $\operatorname{Rep}^0 \mathsf{A}_1(u,v) \otimes \mathsf{BC} \cong (\mathsf{A}_1(u,v) \otimes \mathsf{BC})\text{-}\mathsf{mod}^{\mathrm{wt}}$. Further, the restriction functor $\operatorname{Res} : \operatorname{Rep} \mathsf{A}_1(u,v) \otimes \mathsf{BC} \to \big(\mathsf{H}\big(2\frac{u}{v}\big) \otimes \mathsf{N}(u,v)\big)\text{-}\mathsf{mod}^{\mathrm{wt}}$ has a left adjoint, the induction functor $\operatorname{Ind} : \big(\mathsf{H}\big(2\frac{u}{v}\big) \otimes \mathsf{N}(u,v)\big)\text{-}\mathsf{mod}^{\mathrm{wt}} \to \operatorname{Rep} \mathsf{A}_1(u,v) \otimes \mathsf{BC}$. This functor admits a monoidal structure (see [21, Sec 2&3] for details) which makes it braided monoidal.

Since the categories $\mathsf{H}\big(2\frac{u}{v}\big)\text{-}\mathsf{mod}^{\mathrm{wt}}$ and $\mathsf{BC}\text{-}\mathsf{mod}^{\mathrm{wt}}$ are semisimple and pointed (that is, tensoring by any simple object is invertible) many hom spaces are 1-dimensional which leads to convenient factorisations in the Deligne tensor product categories. For example,

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{N}(u,v)\otimes\mathsf{H}(2\frac{u}{v})}((M\otimes\mathcal{F}_p)\boxtimes(N\otimes\mathcal{F}_q),(P\otimes\mathcal{F}_s))\\
&\quad\cong \mathrm{Hom}_{\mathsf{N}(u,v)}(M\boxtimes N,P)\otimes\mathrm{Hom}_{\mathsf{H}(2\frac{u}{v})}(F_p\boxtimes F_q,F_s)\cong\delta_{p+q,s}\,\mathrm{Hom}_{\mathsf{N}(u,v)}(M\boxtimes N,P)\\
&\mathrm{Hom}_{\mathsf{A}_1(u,v)\otimes\mathsf{BC}}\big((R\otimes\Pi^i\mathsf{BC})\boxtimes(S\otimes\pi^j\mathsf{BC}),(T\otimes\Pi^k\mathsf{BC})\big)\\
&\quad\cong \mathrm{Hom}_{\mathsf{A}_1(u,v)}(R\boxtimes S,T)\otimes\mathrm{Hom}_{\mathsf{BC}}\big(\Pi^i\mathsf{BC}\boxtimes\Pi^j\mathsf{BC},\Pi^k\mathsf{BC}\big)\cong\delta_{i+j,k}\,\mathrm{Hom}_{\mathsf{A}_1(u,v)}(R\boxtimes S,T),
\end{aligned}\tag{3.11}$$

where $M,N,P\in\mathsf{N}(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$, $p,q,s\in\mathbb{R}$, $R,S,T\in\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ and $i,j,k\in\{0,1\}$.

Theorem 3.3. *The right adjoint of the restriction functor $\mathsf{H}\big(2\frac{u}{v}\big)\otimes\mathsf{N}(u,v)\text{-}\mathsf{mod}^{wt}$ is the induction functor. Moreover, let M,N,O be weight $\mathsf{A}_1(u,v)$-modules, let $p\in\operatorname{supp}M$, $q\in\operatorname{supp}N$, and let $C_p(M)$, $C_q(N)$ be respective counterparts to $\mathcal{F}_p$ and $\mathcal{F}_q$ in the restriction (3.5), then there exists a functorial linear isomorphism*

$$\begin{pmatrix} & O & \\ M, & & N\end{pmatrix}_{\mathsf{A}_1(u,v)}\cong\begin{pmatrix} & C_{p+q}^{[i+j]}(O) & \\ C_p^{[i]}(M), & & C_q^{[j]}(N)\end{pmatrix}_{\mathsf{N}(u,v)},\tag{3.12}$$

and further for $s\in\operatorname{supp}O$

$$\begin{pmatrix} & C_s^{[n]}(O) & \\ C_p^{[i]}(M), & & C_q^{[j]}(N)\end{pmatrix}_{\mathsf{N}(u,v)}=0,\qquad \text{if } s\neq p+q \text{ or } n\not\equiv i+j\pmod 2.\tag{3.13}$$

The above theorem was implicit in the presentation of the conjectured fusion rules in [23, Sec 6] and has also been generalised for larger families of algebras in [16]. Nevertheless, we provide a specialised proof for clarity.

Proof. The theorem follows from induction and restriction being adjoint functors and induction being a monoidal functor [21, Thm 2.59]). More specifically,

$$\begin{aligned}
\begin{pmatrix} & O & \\ M, & & N\end{pmatrix}_{\mathsf{A}_1(u,v)}&\cong\mathrm{Hom}_{\mathsf{A}_1(u,v)}(M\boxtimes N,O)\\
&\cong\mathrm{Hom}_{\mathsf{A}_1(u,v)\otimes\mathsf{BC}}\big((M\otimes\Pi^i\mathsf{BC})\boxtimes(N\otimes\Pi^j\mathsf{BC}),O\otimes\Pi^{i+j}\mathsf{BC}\big)\\
&\cong\mathrm{Hom}_{\mathsf{A}_1(u,v)\otimes\mathsf{BC}}\Big(\mathrm{Ind}\big((\mathcal{F}_p\otimes C_p^{[i]}(M))\boxtimes(\mathcal{F}_q\otimes C_q^{[j]}(N))\big),O\otimes\Pi^{i+j}\mathsf{BC}\Big)\\
&\cong\mathrm{Hom}_{\mathsf{A}_1(u,v)\otimes\mathsf{BC}}\Big(\big(\mathcal{F}_p\otimes C_p^{[i]}(M)\big)\boxtimes\big(\mathcal{F}_q\otimes C_q^{[j]}(N)\big),\mathrm{Res}\big(O\otimes\Pi^{i+j}\mathsf{BC}\big)\Big)\\
&\cong\bigoplus_{s\in\operatorname{supp}O}\mathrm{Hom}_{\mathsf{A}_1(u,v)\otimes\mathsf{BC}}\Big(\mathcal{F}_{p+q}\otimes\big(C_p^{[i]}(M)\boxtimes C_q^{[j]}(N)\big),\mathcal{F}_s\otimes C_s^{[i+j]}(O)\Big)\\
&\cong\mathrm{Hom}_{\mathsf{A}_1(u,v)}\Big(C_p^{[i]}(M)\boxtimes C_q^{[j]}(N),C_{p+q}^{[i+j]}(O)\Big)\cong\begin{pmatrix} & C_{p+q}^{[i+j]}(O) & \\ C_p^{[i]}(M), & & C_q^{[j]}(N)\end{pmatrix}_{\mathsf{N}(u,v)},
\end{aligned}\tag{3.14}$$

where we have used that $\Pi^i\mathsf{BC}\boxtimes\Pi^j\mathsf{BC}\cong\Pi^{i+j}\mathsf{BC}$ and $\mathcal{F}_p\boxtimes\mathcal{F}_q\cong\mathcal{F}_{p+q}$. □

While it is known that monoidal functors such as induction preserve duals (see [36, Exe 2.10.6]), sufficient conditions for the converse to hold will also prove helpful below.

Lemma 3.4. *Let $M \in \mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{wt}$ be simple and let $C_p^{[i]}(M) \otimes \mathcal{F}_p$, $p \in \operatorname{supp}(M)$, $i \in \{0,1\}$ be a direct summand of the restriction* $\operatorname{Res} M \otimes \mathsf{BC}$. *If M is rigid with dual $M^\vee$, then $C_p^{[i]}(M)$ is too, with rigid dual $C_p^{[i]}(M)^\vee = C_{-p}^{[i]}(M^\vee)$.*

Proof. Since all tensor categories involved are braided, it is sufficient to only consider left duals. Further, since all the vertex operator superalgebras involved are isomorphic to their contragredients, it is sufficient to show that only one of the rigidity zig zag relations is non-zero by [26, Lem 4.2.1 and Cor 4.2.2]. The vertex operator algebra $\mathsf{A}_1(u,v)$ is self contragredient, thus, the rigid dual $M^\vee$ of M, which exists by assumption, must be isomorphic to the contragredient M'. Hence the weight support of $M^\vee$ must be the negative of the weight support of M, that is, $\operatorname{supp}(M^\vee) = -\operatorname{supp}(M)$. Therefore the restriction of $M^\vee \otimes \mathsf{BC}$ must admit a non-trivial direct summand of the form $C_{-p}^{[i]}(M^\vee) \otimes \mathcal{F}_{-p}$. Since BC is rigid and self dual $M \otimes \mathsf{BC}$ is also rigid with the first of its zig zag relations given by

$$\begin{aligned}\mathrm{id} =& M \otimes \mathsf{BC} \xrightarrow{l^{-1}} (\mathsf{A}_1(u,v) \otimes \mathsf{BC}) \boxtimes M \otimes \mathsf{BC} \xrightarrow{i_{M\otimes\mathsf{BC}}\boxtimes} ((M \otimes \mathsf{BC}) \boxtimes (M^\vee \otimes \mathsf{BC})) \boxtimes (M \otimes \mathsf{BC}) \\ &\xrightarrow{\mathcal{A}^{-1}} (M \otimes \mathsf{BC}) \boxtimes ((M^\vee \otimes \mathsf{BC}) \boxtimes (M \otimes \mathsf{BC})) \\ &\xrightarrow{1\boxtimes e_E} (M \otimes \mathsf{BC}) \boxtimes \mathsf{A}_1(u,v) \otimes \mathsf{BC} \xrightarrow{r} (M \otimes \mathsf{BC}) \end{aligned} \tag{3.15}$$

Applying the restriction functor to this composition of maps, using that induction is monoidal and discarding all but the $C_p^{[i]}(M)$ summand then yields the following composition.

$$\begin{aligned}\mathrm{id} =& C_p^{[i]}(M) \otimes \mathcal{F}_p \xrightarrow{l^{-1}} (\mathsf{N}(u,v) \otimes \mathcal{F}_0) \boxtimes C_p^{[i]}(M) \otimes \mathcal{F}_p \\ &\xrightarrow{?} ((C_p^{[i]}(M) \otimes \mathcal{F}_p) \boxtimes \Big(C_{-p}^{[i]}(M^\vee) \otimes \mathcal{F}_{-p}\Big)) \boxtimes \big(C_p^{[i]}(M) \otimes \mathcal{F}_p\big) \\ &\xrightarrow{\mathcal{A}^{-1}} \big(C_p^{[i]}(M) \otimes \mathcal{F}_p\big) \boxtimes \Big(\Big(C_{-p}^{[i]}(M^\vee) \otimes \mathcal{F}_{-p}\Big) \boxtimes \big(C_p^{[i]}(M) \otimes \mathcal{F}_p\big)\Big) \\ &\xrightarrow{?} \big(C_p^{[i]}(M) \otimes \mathcal{F}_p\big) \boxtimes \mathsf{N}(u,v) \otimes \mathcal{F}_0 \xrightarrow{r} \big(C_p^{[i]}(M) \otimes \mathcal{F}_p\big), \end{aligned} \tag{3.16}$$

where the arrows marked by a question mark are to be determined. Next we use that $\mathsf{N}(u,v) \otimes \mathsf{H}\big(2\frac{u}{v}\big)\text{-}\mathsf{mod}^{\mathrm{wt}}$ is braided equivalent to the Deligne tensor product $\mathsf{N}(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}} \widehat{\boxtimes} \mathsf{H}\big(2\frac{u}{v}\big)\text{-}\mathsf{mod}^{\mathrm{wt}}$ to factor out all the Fock spaces to obtain the composition

$$\begin{aligned}\mathrm{id} =& C_p^{[i]}(M) \widehat{\boxtimes} \mathcal{F}_p \xrightarrow{l^{-1}} \big(\mathsf{N}(u,v) \boxtimes C_p^{[i]}(M)\big) \widehat{\boxtimes} (\mathcal{F}_0 \boxtimes \mathcal{F}_p) \\ &\xrightarrow{?} \Big(\Big(C_p^{[i]}(M) \boxtimes C_{-p}^{[i]}(M^\vee)\Big) \boxtimes C_p^{[i]}(M)\Big) \widehat{\boxtimes} ((\mathcal{F}_p \boxtimes \mathcal{F}_{-p}) \boxtimes \mathcal{F}_p) \\ &\xrightarrow{\mathcal{A}^{-1}} \Big(C_p^{[i]}(M) \boxtimes \Big(C_{-p}^{[i]}(M^\vee) \boxtimes C_p^{[i]}(M)\Big)\Big) \widehat{\boxtimes} (\mathcal{F}_p (\boxtimes \mathcal{F}_{-p} \boxtimes \mathcal{F}_p)) \\ &\xrightarrow{?} \big(C_p^{[i]}(M) \boxtimes \mathsf{N}(u,v)\big) \widehat{\boxtimes} (\mathcal{F}_p \boxtimes \mathcal{F}_0) \xrightarrow{r} C_p^{[i]}(M) \widehat{\boxtimes} \mathcal{F}_p. \end{aligned} \tag{3.17}$$

Note that each of the products of Fock spaces above is simple (and isomorphic to $\mathcal{F}_p$) and so each of the arrows in the composition above lies in a Deligne tensor product hom space of the form $\operatorname{Hom}_{\mathsf{N}(u,v)}(A,B) \otimes \operatorname{Hom}_{\mathsf{H}(2\frac{u}{v})}(\mathcal{F}_p, \mathcal{F}_p)$. Since the Heisenberg hom space is one dimensional, every $f \in \operatorname{Hom}_{\mathsf{N}(u,v)}(A,B) \otimes \operatorname{Hom}_{\mathsf{H}(2\frac{u}{v})}(\mathcal{F}_p, \mathcal{F}_p)$ can be written as a tensor product $f_N \otimes f_H$, $f_N \in \operatorname{Hom}_{\mathsf{N}(u,v)}(A,B)$, $f_H \in \operatorname{Hom}_{\mathsf{H}(2\frac{u}{v})}(\mathcal{F}_p, \mathcal{F}_p)$ as opposed to some linear combination of such tensor products. We can therefore factor out all the Heisenberg morphisms to obtain the composition

$$\begin{aligned}\mathrm{id} =& C_p^{[i]}(M) \xrightarrow{l^{-1}} \mathsf{N}(u,v) \boxtimes C_p^{[i]}(M) \xrightarrow{?} ((C_p^{[i]}(M)) \boxtimes C_{-p}^{[i]}(M^\vee)) \boxtimes C_p^{[i]}(M) \\ &\xrightarrow{\mathcal{A}^{-1}} C_p^{[i]}(M) \boxtimes \Big(C_{-p}^{[i]}(M^\vee) \boxtimes C_p^{[i]}(M)\Big) \xrightarrow{?} C_p^{[i]}(M) \boxtimes \mathsf{N}(u,v) \xrightarrow{r} C_p^{[i]}(M). \end{aligned} \tag{3.18}$$

The two arrows marked by a question mark lie in the hom spaces $\mathrm{Hom}_{\mathsf{N}(u,v)}\Big(\mathsf{N}(u,v), C_p^{[i]}(M)\boxtimes C_{-p}^{[i]}(M^\vee)\Big)$ and $\mathrm{Hom}_{\mathsf{N}(u,v)}\Big(C_p^{[i]}(M^\vee)\boxtimes C_{-p}^{[i]}(M), \mathsf{N}(u,v)\Big)\otimes\mathrm{id}$. Thus there exists a pair of morphisms from these hom spaces and such that the composition yields a non-zero morphism $C_p^{[i]}(M)\to C_p^{[i]}(M)$.

□

Theorem 3.5. *For coprime $u\geq 2$, $v\geq 1$, $1\leq r\leq u-1$ and $1\leq s\leq v-1$ and $q\in\mathbb{R}$, let $\mathcal{L}_{q;r,s}$ be the simple highest weight $\mathsf{N}(u,v)$-module $\mathcal{L}_{q,h_{r,s;q},3-\frac{6v}{u}}$ as given in the second part of Proposition 2.15. Additionally consider the simple relaxed highest weight $\mathsf{A}_1(u,v)$-modules $E_{[q\frac{u}{v}];r,s}$, where $[q\frac{u}{v}]=q\frac{u}{v}+2\mathbb{Z}$. Then for all $1\leq r',r''\leq u-1$, $1\leq s',s''\leq v-1$, $q'\in\mathbb{R}$, $\frac{u}{v}q\notin\pm\lambda_{r,s}+2\mathbb{Z}$ the non-vanishing simple fusion rules between $\mathcal{L}_{q;r,s},\mathcal{L}_{q';r',s'}$ and between $E_{[q\frac{u}{v}];r,s},E_{[q'\frac{u}{v}];r',s'}$ are*

$$\dim\begin{pmatrix}\mathcal{L}_{q+q';r'',s''}\\ \mathcal{L}_{q;r,s},\ \mathcal{L}_{q';r',s'}\end{pmatrix}=\dim\begin{pmatrix}E_{[(q+q)'\frac{u}{v}];r'',s''}\\ E_{[q\frac{u}{v}];r,s},\ E_{[q'\frac{u}{v}];r',s'}\end{pmatrix}=N_{r,r'}^{(u)\,r''}\Big(N_{s,s'-1}^{(v)\,s''}+N_{s,s'+1}^{(v)\,s''}\Big),$$
$$\dim\begin{pmatrix}\Pi\mathcal{L}_{q+q'\pm1;r'',s''}\\ \mathcal{L}_{q;r,s},\ \mathcal{L}_{q';r',s'}\end{pmatrix}=\dim\begin{pmatrix}\sigma^{\mp1}E_{[(q+q'\pm1)\frac{u}{v}];r'',s''}\\ E_{[q\frac{u}{v}];r,s},\ E_{[q'\frac{u}{v}];r',s'}\end{pmatrix}=N_{r,r'}^{(u)\,r''}N_{s,s'}^{(v)\,s''},\tag{3.19}$$

where the N-coefficients are defined in (2.19).

The equality of the dimensions of $\mathfrak{sl}(2)$ and $\mathcal{N}=2$ intertwiner spaces (3.19) is just Theorem 3.3 applied to the relaxed highest weight modules $E_{[q\frac{u}{v}];r,s}$. The computation of these dimensions will form the focus of the next two sections. Note that when q,q' in (3.19) are such that $\frac{u}{v}(q+q'+\epsilon),\notin\pm\lambda_{r,s}+2\mathbb{Z}$, $\epsilon=-1,0,1$, then (3.19) together with the projectivity of the simple modules $\sigma^\ell E_{\mu;r,s}$ established in [9] implies that the tensor products of $E_{[q\frac{u}{v}];r,s},E_{[q'\frac{u}{v}];r',s'}$ and of $\mathcal{L}_{q;r,s},\mathcal{L}_{q';r',s'}$ are semisimple with direct sum decomposition given by

$$\begin{aligned}E_{[q\frac{u}{v}];r,s}\boxtimes E_{[q'\frac{u}{v}];r',s'}&=\bigoplus_{r'',s''}N_{r,r'}^{(u)\,r''}\Big(N_{s,s'-1}^{(v)\,s''}+N_{s,s'+1}^{(v)\,s''}\Big)E_{[(q+q')\frac{u}{v}];r'',s''}\\&\quad\oplus\bigoplus_{r'',s''}N_{r,r'}^{(u)\,r''}N_{s,s'}^{(v)\,s''}\Big(\sigma^{-1}E_{[(q+q'+1)\frac{u}{v}];r'',s''}\oplus\sigma E_{[(q+q'-1)\frac{u}{v}];r'',s''}\Big),\\ \mathcal{L}_{q;r,s}\boxtimes\mathcal{L}_{q';r',s'}&=\bigoplus_{r'',s''}N_{r,r'}^{(u)\,r''}\Big(N_{s,s'-1}^{(v)\,s''}+N_{s,s'+1}^{(v)\,s''}\Big)\mathcal{L}_{q+q';r',s'}\\&\quad\oplus\bigoplus_{r'',s''}N_{r,r'}^{(u)\,r''}N_{s,s'}^{(v)\,s''}(\Pi\mathcal{L}_{q+q'+1;r'',s''}\oplus\Pi\mathcal{L}_{q+q'-1;r'',s''})\end{aligned}\tag{3.20}$$

By associativity these tensor products are generated by $E_{[q\frac{u}{v}];1,1},E_{[q\frac{u}{v}];2,1}$ and by $\mathcal{L}_{q;1,1},\mathcal{L}_{q;2,1}$. Moreover, by specialising (2.20), we observe $E_{[q\frac{u}{v}];2,1}\cong L_2\boxtimes E_{[q\frac{u}{v}]-1;1,1}$. Hence all of the relaxed fusion rules can be obtained from the $E_{[q\frac{u}{v}];1,1}$ fusion rules and by tensoring those with L_2. That is, Theorem 3.5 holds if and only if the following lemma holds.

Lemma 3.6. *For coprime $u\geq 2$, $v\geq 1$, and $1\leq r,r'\leq u-1$, $1\leq s,s'\leq v-1$, $q,q'\in\mathbb{R}$, $\frac{u}{v}q'\notin\pm\lambda_{r,s}+2\mathbb{Z}$. If $q\notin\pm\lambda_{1,1}+2\mathbb{Z}$,*

$$\dim\begin{pmatrix}\mathcal{L}_{q+q';r',s'}\\ \mathcal{L}_{q;1,1},\ \mathcal{L}_{q';r,s}\end{pmatrix}=\dim\begin{pmatrix}E_{[(q+q')\frac{u}{v}];r',s'}\\ E_{[q\frac{u}{v}];1,1},\ E_{[q'\frac{u}{v}];r,s}\end{pmatrix}=\delta_{r,r'}(\delta_{s-1,s'}+\delta_{s+1,s'}),$$
$$\dim\begin{pmatrix}\Pi\mathcal{L}_{q+q'\pm1;r',s'}\\ \mathcal{L}_{q;1,1},\ \mathcal{L}_{q';r,s}\end{pmatrix}=\dim\begin{pmatrix}\sigma^{\mp1}E_{[(q+q'\pm1)\frac{u}{v}];r',s'}\\ E_{[q\frac{u}{v}];1,1},\ E_{[q'\frac{u}{v}];r,s}\end{pmatrix}=\delta_{r,r'}\delta_{s,s'},$$
$$\dim\begin{pmatrix}X\\ \mathcal{L}_{q;1,1},\ \mathcal{L}_{q';r,s}\end{pmatrix}=\dim\begin{pmatrix}Y\\ E_{[q\frac{u}{v}];1,1},\ E_{[q'\frac{u}{v}];r,s}\end{pmatrix}=0,\tag{3.21}$$

where X and Y are any simple modules other than those appearing in the first two lines of (3.21).

We will prove Lemma 3.6 by computing upper bounds for the dimensions in (3.21) on the $\mathcal{N}=2$ side using Zhu's algebra and its bimodules in Section 4 (see Proposition 4.15) and lower bounds on the $\mathfrak{sl}(2)$ side in Section 5 (see Propositions 5.4 and 5.5) by explicitly constructing certain intertwining operators.

4. Upper bounds for Lemma 3.6 via Zhu algebras

We recall some definitions and results from Zhu and Frenkel's work [39, 75] and the generalisation by Kac and Wang [54] to vertex superalgebras where all odd vectors with respect to the $\mathbb{Z}_2$ grading have half odd conformal weights. We then apply these results to $\mathsf{N}(u,v)$ to compute upper bounds for Lemma 3.6 in Proposition 4.15.

4.1. **Zhu algebras, Zhu modules and intertwining operators.** For any super vector space V, we denote the subspaces of even and odd vectors by $V^{\bar{0}}$ and $V^{\bar{1}}$ respectively. For a vertex operator superalgebra V in which the space of odd vectors coincides with the space of vectors of half odd integer conformal weight, the Zhu algebra is constructed from the two bilinear operations on V characterised for homogenous $a,b\in V$ by

$$\begin{aligned} a*b &= \begin{cases} \operatorname{Res}_z\Big(Y(a,z)\frac{(1+z)^{\mathrm{wt}a}}{z}b\Big), & a,b\in V^{\bar{0}},\\ 0, & a\in V^{\bar{1}} \text{ or } b\in V^{\bar{1}}, \end{cases}\\ a\circ b &= \begin{cases} \operatorname{Res}_z\Big(Y(a,z)\frac{(1+z)^{\mathrm{wt}a}}{z^2}b\Big), & a\in V^{\bar{0}},\\ \operatorname{Res}_z\Big(Y(a,z)\frac{(1+z)^{\mathrm{wt}a-\frac{1}{2}}}{z}b\Big), & a\in V^{\bar{1}}. \end{cases} \end{aligned} \tag{4.1}$$

Proposition 4.1 (Zhu [75]; Kac, Wang [54])**.** *Let V be a vertex operator superalgebra for which the space of odd vectors and the space of vectors of half odd conformal weight coincide and define $O(V)=\operatorname{span}_{\mathbb{C}}\{a\circ b\mid a,b\in V\}$. Then $A(V)=V/O(V)$ is an associative unital algebra with multiplication given by $*$. The identity element is $[\mathbf{1}]=\mathbf{1}+O(V)$, the equivalence class of the vacuum vector, and the class $[\omega]$ of the conformal vector is central.*

Note that for odd $a\in V^{\bar{1}}$, $a\circ\mathbf{1}=a$ and hence $V^{\bar{1}}\subset O(V)$. We next recall some standard Zhu algebra results.

Lemma 4.2 (Zhu [75]; Kac, Wang [54])**.** *Let V be a vertex operator superalgebra. For all homogeneous elements $a,b\in V$ and $m\geq n\geq 0$ we have that*

(1) $(L_{-1}+L_0)a\in O(V)$, $a\in V^{\bar{0}}$,
(2) $\operatorname{Res}_x\Big(Y(a,x)\frac{(1+x)^{\mathrm{wt}a+n}}{x^{2+m}}b\Big)\in O(V)$, $a\in V^{\bar{0}}$,
(3) $\operatorname{Res}_x\Big(Y(a,x)\frac{(1+x)^{\mathrm{wt}a+n-\frac{1}{2}}}{x^{1+m}}b\Big)\in O(V)$, $a\in V^{\bar{1}}$,
(4) $a*b=Res_x\Big(Y(b,x)\frac{(x+1)^{\mathrm{wt}b-1}}{x}a\Big)+O(V)$, $a,b\in V^{\bar{0}}$.

Theorem 4.3 (Zhu [75]; Kac, Wang [54])**.** *Let V be a vertex operator superalgebra and let M be a V-module. Define the* top space *of M to be $M^{top}=\{m\in M\mid v_nm=0\ \forall v\in V,\forall n>0\}$.*

(1) *Any element $[a]\in A(V)$ acts on M^{top} by the zero mode a_0 where $Y(a,z)=\sum_{n\in\mathbb{Z}}a_nz^{-wt(a)-n}$, hence giving M^{top} the structure of a left $A(V)$-module. Further, if M is simple as a module over V, then M^{top} is simple over $A(V)$.*

(2) *Any left $A(V)$-module $M^{\rm top}$ can be induced to a V module with conformal weights bounded below and with $M^{\rm top}$ contained in the top space. If $M^{\rm top}$ is simple over $A(V)$, then the induction admits a unique simple quotient.*
(3) *Simple $A(V)$ modules are in bijection with simple V-modules with conformal weights bounded below.*

Ideals in a vertex operator superalgebra and in its corresponding Zhu algebra are interrelated as follows.

Lemma 4.4 (Zhu [75]; Kac, Wang [54])**.** *Let V be a vertex operator superalgebra and I a $\mathbb{Z}_2$-graded ideal of V (with grading consistent with that of V). Assume that $\mathbf{1} \notin I, \omega \notin I$. Then the Zhu algebra of the quotient $A(V/I)$ is isomorphic to the quotient $A(V)/[I]$ where $[I]$ is the image of I in $A(V)$.*

If M is a V-module, we define an $A(V)$-bimodule $A(M)$ as follows. First, we set the left and right actions of V on M by

$$a * m = \begin{cases} \mathrm{Res}_z\Big(Y(a,z)\frac{(1+z)^{\mathrm{wt}a}}{z}m\Big), & \text{if } a \in V^{\bar{0}},\\ 0 & \text{if } a \in V^{\bar{1}};\end{cases}$$
$$m * a = \begin{cases} \mathrm{Res}_z\Big(Y(a,z)\frac{(1-z)^{\mathrm{wt}a-1}}{z}m\Big), & \text{if } a \in V^{\bar{0}},\\ 0 & \text{if } a \in V^{\bar{1}}.\end{cases} \tag{4.2}$$

where $m \in M$. Next, we let $O(M)$ be the subspace of M linearly spanned by elements of the form

$$\mathrm{Res}_z\left(Y(a,z)\frac{(1+z)^{\mathrm{wt}a}}{z^2}m\right), \qquad a \in V^{\bar{0}}$$
$$\mathrm{Res}_z\left(Y(a,z)\frac{(1+z)^{\mathrm{wt}a-\frac{1}{2}}}{z}m\right), \qquad a \in V^{\bar{1}}, \tag{4.3}$$

where $m \in M$.

Proposition 4.5 (Zhu [75]; Kac, Wang [54])**.** *Let V be a vertex operator superalgebra and M a V-module. Then $A(M) = M/O(M)$ is an $A(V)$-bimodule with left and right actions given by (4.2). For any homogeneous elements $a \in V$, $m \in M$ and $p \geq q \geq 0$ we have that:*

(1) $\mathrm{Res}_x\Big(Y(a,x)\frac{(1+x)^{\mathrm{wt}a+q}}{x^{2+p}}m\Big) \in O(M)$, *$a \in V^{\bar{0}}$, and*
(2) $\mathrm{Res}_x\Big(Y(a,x)\frac{(1+x)^{\mathrm{wt}a+q-\frac{1}{2}}}{x^{1+p}}m\Big) \in O(M)$, *$a \in V^{\bar{1}}$.*

As in Lemma 4.4 the Zhu bimodule structure is compatible with quotients by submodules as described in the following result.

Lemma 4.6 (Zhu [75]; Kac, Wang [54])**.** *Let V be a vertex operator superalgebra and let M be a V-module.*

(1) *If M' is a submodule of the V-module M, then $A(M/M') \cong A(M)/[M']$, where $[M']$ denotes the image of M' under the projection $M \twoheadrightarrow A(M)$.*
(2) *If I is an ideal of V, $\mathbf{1} \notin I, \omega \notin I$ and $I.M \subset M'$, then the $A(V/I)$-module $A(M)/A(M')$ is isomorphic to $A(M/M')$.*

We will denote the vector space of intertwining operators of type $\binom{W_3}{W_1\ W_2}$ by the same symbol $\binom{W_3}{W_1\ W_2}$ and for the moment postpone the definition of intertwining operators until they are explicitly used in Section 5. One of the main benefits of Zhu bimodules is that they provide a means for computing dimensions of spaces of intertwining operators.

Lemma 4.7 (Huang, Yang [48, Prop 5.8])**.** *Let V be a vertex operator superalgebra and let M_1, M_2 and M_3 be V-modules with M_2 and M_3' generated from their top spaces. Then*

$$\dim \begin{pmatrix} M_3 \\ M_1 \quad M_2 \end{pmatrix} \leq \dim \mathrm{Hom}_{A(V)}\big(A(M_1) \otimes_{A(V)} M_2^{\mathrm{top}}, M_3^{\mathrm{top}}\big). \tag{4.4}$$

The relationship between spaces of intertwining operators and hom spaces was, of course, also considered in Zhu [75]; Kac, Wang [54], but not with the assumptions that we require here.

4.2. **Determining the Zhu algebra for $\mathsf{N}(u,v)$.** Let c be any complex number and let $\mathsf{N}(c)$ denote the universal $\mathcal{N}=2$ vertex operator super algebra of central charge c as defined in Proposition 2.12.

Theorem 4.8 (Gaberdiel, Eholzer [35])**.**

(1) *For any $c \in \mathbb{C}$, the Zhu algebra $A(\mathsf{N}(c))$ is isomorphic to the polynomial algebra $\mathbb{C}[\Delta, \eta]$, where Δ is the image of the conformal vector $[\omega]$ and η is the image of $[J_{-1}\mathbf{1}]$.*

(2) *For u, v coprime $u \geq 2$, $v \geq 1$, and $c = 3 - \frac{6v}{u}$, the image in $A(\mathsf{N}(c))$ of the non-trivial maximal ideal of $\mathsf{N}(c)$ generated by a singular vector $\chi \in \mathsf{N}(c)$ of conformal weight $(u-1)v$ and J-weight 0 is $\langle p_1(\Delta,\eta), p_2(\Delta,\eta)\rangle$, where p_1 is the image of χ and p_2 is the image of $G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}}\chi$.*

(3) *The Zhu algebra of $\mathsf{N}(u,v)$ admits the presentation*

$$A(\mathsf{N}(u,v) \cong \frac{\mathbb{C}[\Delta.\eta]}{\langle p_1(\Delta,\eta), p_2(\Delta,\eta)\rangle} \tag{4.5}$$

The goal of this section is to compute the polynomials p_1, p_2.

Theorem 4.9. *Fix $c = 3 - \frac{6v}{u}$ for coprime $u \geq 2$, $v \geq 1$ and let $\mathcal{J}$ denote the maximal ideal $\mathcal{J} \subset \mathsf{N}(c)$. Then, as an ideal of $A_0(\mathsf{N}(c)) \cong \mathbb{C}[\Delta, \eta]$ under the isomorphism given in Theorem 4.8, $[\mathcal{J}]$ is generated by the following two polynomials*

$$p_1(\eta,\Delta) = f_u\Big(\eta, \Delta, \frac{u}{v}\Big) \prod_{(r,s)\in K(u,v)} (\Delta - h_{r,s;\eta}), \tag{4.6}$$

$$p_2(\eta,\Delta) = \left(f_u\Big(\eta + \frac{2v}{u}, \Delta - \eta - \frac{v}{u}, \frac{u}{v}\Big) - f_u\Big(\eta, \Delta, \frac{u}{v}\Big)\right)(2\Delta - \eta) \prod_{(r,s)\in K(u,v)} (\Delta - h_{r,s,\eta}) \tag{4.7}$$

where $h_{r,s;\eta}$ is the conformal weight defined in (2.36), $K(u,v)$ is the set of pairs $(r,s) \in \{1,\dots,u-1\} \times \{1,\dots v-1\}$ satisfying the additional constraint $vr+us < uv$, and the polynomials $f_n(x,y,z)$, $n \geq 2$ are defined recursively by

$$f_{n+2}(x,y,z) = \frac{(2n+1)xz}{(n+1)^2} f_{n+1}(x,y,z) - \frac{4yz + x^2z^2 - (n-1)(n+1)}{(n+1)^2} f_n(x,y,z), \tag{4.8}$$

with

$$f_2(x,y,z) = xz, \qquad f_3(x,y,z) = \frac{x^2z^2}{2} - yz. \tag{4.9}$$

In particular, the Zhu algebra for the minimal model superconformal algebra at central charge $c = 3 - \frac{6v}{u}$ admits the presentation

$$A(\mathsf{N}(u,v)) \cong \frac{\mathbb{C}[\Delta,\eta]}{\langle p_1(\Delta,\eta), p_2(\Delta,\eta)\rangle}. \tag{4.10}$$

Proof. Eholzer and Gaberdiel established in [35] that, under the isomorphism φ in Theorem 4.8, the ideal $[\mathcal{J}] \subset \mathbb{C}[\Delta,\eta]$ is generated by the images of the singular vector χ, which generates $\mathcal{J}$ and its descendant $G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\chi$. Thus all that is left to show is that these images are equal to the polynomials p_1, p_2. For this we use the coset realisation in Proposition 3.1 and the relations between singular vectors in Proposition 3.2(3) for the vacuum module $\mathsf{A}_1\big(\frac{u}{v}-2\big)$. Let $w_{u,v}$ denote the singular vector generating the maximal $\widehat{\mathfrak{sl}}(2)$ submodule in $\mathsf{A}_1\big(\frac{u}{v}-2\big)$. Then $f_0^{u-1}w_{u,v}$ is a relaxed singular vector in $\mathsf{A}_1\big(\frac{u}{v}-2\big)$ whose image in the Zhu algebra of $\mathsf{A}_1\big(\frac{u}{v}-2\big)$, up to a non-zero multiple, was shown in [69] to be given by

$$I\big([h],[T_{\mathfrak{sl}(2)}]\big) = \prod_{(r,s)\in K(u,v)} \big([T_{\mathfrak{sl}(2)}]-\Delta^{\mathrm{aff}}_{r,s,t}\big) g_{u,v}\big([h],[T_{\mathfrak{sl}(2)}]\big) \tag{4.11}$$

where $[h]$ and $[T_{\mathfrak{sl}(2)}]$ denote the images of $h_{-1}\mathbf{1}$ and and the conformal vector $T_{\mathfrak{sl}(2)} = \omega_{\frac{u}{v}-2} = \frac{1}{2\frac{u}{v}}\big(\frac{1}{2}h_{-1}^2 - e_{-1}f_{-1} - f_{-1}e_{-1}\big)\mathbf{1}$ in $A\big(\mathsf{A}_1\big(\frac{u}{v}-2\big)\big) \cong \mathcal{U}(\mathfrak{sl}(2))$ and $g_{u,v}(x,y)$ are polynomials recursively defined in [69] by

$$g_{2,v}(x,y) = x \tag{4.12}$$

$$g_{3,v}(x,y) = \frac{3}{4}x^2 - ty \tag{4.13}$$

and

$$g_{u+2,v}(x,y) = \frac{(2u+1)x}{(u+1)^2} g_{u+1,v}(x,y) - \frac{4ty-(u-1)(u+1)}{(u+1)^2} g_{u,v}(x,y), \tag{4.14}$$

where t is a parameter that will be later specialised to $t=\frac{u}{v}$. By Proposition 3.2.(3) we have that

$$\phi(|0\rangle\otimes\chi) = f_0^{u-1}w_{u,v}\otimes\Omega, \tag{4.15}$$

where ϕ is the coset embedding of Proposition 3.1, $|0\rangle$ is the vacuum vector in $\mathsf{H}\big(\frac{u}{v}\big)$ and Ω is the vacuum vector in BC.

In particular, the zero modes $(|0\rangle\otimes\chi)_0 = \mathrm{id}\otimes\chi_0$ and $(f^{u-1}w_{u,v}\otimes\Omega)_0 = (f^{u-1}w_{u,v})_0\otimes\mathrm{id}$ must act equally on any $\mathsf{A}_1\big(\frac{u}{v}-2\big)\otimes\mathsf{BC}$ module. Thus, for an $\mathsf{A}_1\big(\frac{u}{v}-2\big)$ relaxed highest weight vector $m(\lambda,\Delta^{\mathrm{aff}})$ of $\mathfrak{sl}(2)$ weight λ and conformal weight Δ^{aff} we have

$$\chi_0\big(m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega\big) = \big((f^{u-1}w_{u,v})_0 m(\lambda,\Delta^{\mathrm{aff}})\big)\otimes\Omega = I(\lambda,\Delta^{\mathrm{aff}})m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega, \tag{4.16}$$

where in the second step we have used that Zhu's algebra models the algebra of zero modes acting on relaxed highest weight vectors. Recall that the $\mathfrak{sl}(2)$ data $(\lambda,\Delta^{\mathrm{aff}})$ corresponding to the J_0 eigenvalue η and $\mathcal{N}=2$ conformal weight Δ are interrelated via $\lambda=t\eta$ and $\Delta^{\mathrm{aff}}=\Delta+\frac{t}{4}\eta^2$. Thus $p_1(\eta,\Delta) = I(t\eta,\Delta+\frac{t}{4}\eta^2)$, which matches the formula given in the Theorem after identifying $f_u(\eta,\Delta,t) = g_{u,v}(t\eta,\Delta+\frac{t}{4}\eta^2)$ and $t=\frac{u}{v}$.

Next we compute the image of $G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\chi$ in the Zhu algebra, again via the coset realisation. Following on from the identification (4.15) we have

$$\begin{aligned}\phi(|0\rangle\otimes G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\chi) &= \phi(G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}})f_0^{u-1}w_{u,v}\otimes\Omega \\ &= \frac{2}{t}\Big(e_0\otimes c_{-\frac{1}{2}} + e_{-1}\otimes c_{\frac{1}{2}}\Big) f_0 f_0^{u-1}w_{u,v}\otimes b_{-\frac{1}{2}}\Omega \\ &= -\frac{2}{t}e_0 f_0 f_0^{u-1}w_{u,v}\otimes Q + \frac{2}{t}e_{-1}f_0f_0^{u-1}w_{u,v}\otimes\Omega,\end{aligned} \tag{4.17}$$

where Q is the Heisenberg vector within BC. Applying both sides of the above identity to the same test vector $m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega$ as before yields

$$\Big(G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\chi\Big)_0 m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega$$
$$= -\frac{2}{t}\big(e_0f_0f_0^{u-1}w_{u,v}\big)_0\otimes Q_0\big(m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega\big)+\frac{2}{t}\big(e_{-1}f_0f_0^{u-1}w_{u,v}\big)_0\big(m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega\big). \quad (4.18)$$

Note that the first summand vanishes due to $Q_0\Omega=0$, while the second can be simplified using the Zhu relations (4.2) to give

$$\begin{aligned}\Big(G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\chi\Big)_0 m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega &= \frac{2}{t}\big(f*\big(f_0^{u-1}w_{u,v}\big)*e-\big(f_0^{u-1}w_{u,v}\big)*f*e\big)_0 m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega\\ &= \frac{2}{t}\big(f_0\big(f_0^{u-1}w_{u,v}\big)_0e_0-\big(f_0^{u-1}w_{u,v}\big)_0f_0e_0\big)m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega\\ &= \frac{2}{t}\big(I\big(\lambda+2,\Delta^{\mathrm{aff}}\big)-I\big(\lambda,\Delta^{\mathrm{aff}}\big)\big)\left(2t\Delta^{\mathrm{aff}}-\frac{\lambda^2}{2}-\lambda\right)m(\lambda,\Delta^{\mathrm{aff}})\otimes\Omega, \quad (4.19)\end{aligned}$$

where for the final equality we used that $m(\lambda,\Delta^{\mathrm{aff}})$ is a relaxed highest weight vector and hence $f_0e_0m(\lambda,\Delta^{\mathrm{aff}})=\big(2tL_0-\frac{1}{2}h_0^2-h_0\big)m(\lambda,\Delta^{\mathrm{aff}})$. The $\mathfrak{sl}(2)$ and $\mathcal{N}=2$ weights are again interrelated by $\lambda=t\eta$ and $\Delta^{\mathrm{aff}}=\Delta+\frac{t}{4}\eta^2$ and thus the eigenvalue of $\Big(G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\chi\Big)_0$ is the polynomial $p_2(\eta,\Delta)$. □

Using Theorem 4.9 we recover the classification of $\mathsf{N}(u,v)$-irreducible modules obtained by Adamović in [1].

Theorem 4.10. *A complete set of inequivalent simple modules over the Zhu algebra* (4.10) *is given by the 1-dimensional vector spaces on which* (η,Δ) *act as any of the pairs of scalarswithin the following sets.*

(1) $\Big\{\Big(\frac{pv}{u},h_{r,0;\frac{pv}{u}}\Big)\mid 1\le r\le u-1,\ 1-r\le p\le r-1, p+r\equiv 1 \pmod 2\Big\}$,

(2) $\{(q,h_{r,s;q})\mid 1\le r\le u-1,\ 1\le s\le v-1,\ vr+us<uv,\ q\in\mathbb{C}\}$.

where $h_{r,s;q}$ *is given in* (2.36).

Proof. The theorem follows by showing that the pairs of scalars above constitute all the simultaneous zeros of the polynomials $p_1(\eta,\Delta)$, $p_2(\eta,\Delta)$ of Theorem 4.9. Note first that both polynomials contain the divisors $(\Delta-h_{r,s;\eta})$. These correspond to the second set of pairs $(q,h_{r,s;q})$ above.

If a simultaneous zero of p_1, p_2 is not of the form $(q,h_{r,s;q})$, then it must necessarily be a simultaneous zero of $f_u\big(\eta,\Delta,\frac{u}{v}\big)$ and $f_u\big(\eta,\Delta,\frac{u}{v}\big)-f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)$, or of $f_u\big(\eta,\Delta,\frac{u}{v}\big)$ and $(2\Delta-\eta)$. Using Bézout's Theorem, which in its simplest form states that two bivariate polynomials can have at most as many simultaneous zeros as the product of their degrees we will show that there are at most $\frac{u(u-1)}{2}$ such simultaneous zeros. This is precisely the number of pairs of scalars in the first set above. After that has been established, all that will remain to be shown is that the above pairs are indeed simultaneous zeros, which we will show inductively.

We start by giving on upper bound on the simultaneous zeros of $f_u\big(\eta,\Delta,\frac{u}{v}\big)$ and $f_u\big(\eta,\Delta,\frac{u}{v}\big)-f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)$. From the given formulae for f_2 and f_3, and the recursion relation for f_u, we see that $f_u\big(\eta,\Delta,\frac{u}{v}\big)$ has degree $u-1$ in η and degree at most $\frac{u-1}{2}$ in Δ, in particular the monomial term with the largest power of η has no factor of Δ and the monomial term with the largest power of Δ has no factor of η. Further the shifted version $f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)$ preserves the power of η in its first argument and Δ in its second, so it too will have degree $u-1$ in η and degree at most $\frac{u-1}{2}$ in Δ, with the monomial term with the largest power of η having no factor of Δ and the monomial term with the largest power of Δ having no factor of η. Crucially, these respective greatest monomial terms of $f_u\big(\eta,\Delta,\frac{u}{v}\big)$ and $f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)$ have identical coefficients so the difference $\tilde{f}_u(\eta,\Delta)=f_u\big(\eta,\Delta,\frac{u}{v}\big)-f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)$ has degree at most

$u-2$ in η and at most $\frac{u-1}{2}-1$ in Δ. So if we introduce an additional variable γ, which is to take on the role of a square root of Δ, then $f_u\big(\eta,\gamma^2,\frac{u}{v}\big)$ is a polynomial in η,γ of degree $u-1$ and $\tilde{f}_u(\eta,\gamma^2)$ is a polynomial of degree $u-2$ in η,γ, that is, replacing all the Δ by squares does not increase the total degrees of the polynomials. Thus by Bézout's Theorem they have at most $(u-1)(u-2)$ simultaneous zeros. However, these zeros come in pairs, as if (η_0,γ_0) is a zero, then $(\eta_0,-\gamma_0)$ is too, yet both of these zeros correspond to the same zero in η,Δ given by (η_0,γ_0^2). So f_u and $\tilde{f}_u$ have at most $\frac{(u-1)(u-2)}{2}$ simultaneous zeros in η,Δ. Bézout's Theorem also immediately bounds the number of simultaneous zeros of f_u and $(2\Delta-\eta)$ from above by $u-1$. Thus there are at most $\frac{(u-1)(u-2)}{2}+u-1=\frac{u(u-1)}{2}$ simultaneous zeros of $f_u\big(\eta,\Delta,\frac{u}{v}\big)$ and $f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)(2\Delta-\eta)$.

Next we show by induction that all pairs in the first set listed in the theorem are indeed simultaneous zeros of $f_u\big(\eta,\Delta,\frac{u}{v}\big)$ and $f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)(2\Delta-\eta)$. For our base case we need to consider $u=2$ and $u=3$. First $u=2$ and v odd. Then

$$f_2\left(\eta,\Delta,\frac{2}{v}\right)=\eta\frac{2}{v}=0 \quad\text{and}\quad f_2\left(\eta+v,\Delta-\eta-\frac{v}{2},\frac{2}{v}\right)(2\Delta-\eta)=(\eta+v)\frac{2}{v}(2\Delta-\eta)=0. \tag{4.20}$$

The first equation requires $\eta=0$ and then the second can only hold if $\Delta=0$, which reproduces the first set of zeros for $u=2$. Similarly for $u=3$ and v not a multiple of 3, one can easily verify that the three candidate simultaneous zeros for $u=3$ solve the equations

$$\begin{aligned} f_3\left(\eta,\Delta,\frac{3}{v}\right)&=\eta^2\frac{9}{2v^2}-\frac{3}{v}\Delta=0,\\ f_3\left(\eta+\frac{2v}{3},\Delta-\eta-\frac{v}{3},\frac{3}{v}\right)(2\Delta-\eta)&=\left(\left(\eta+\frac{2v}{3}\right)^2\frac{9}{2v^2}-\frac{3}{v}\left(\Delta-\eta-\frac{v}{3}\right)\right)(2\Delta-\eta)=0. \end{aligned}\tag{4.21}$$

Next for $u\geq 3$ and v coprime to u assume that the pairs $(\frac{pv}{u},h_{r,0;\frac{pv}{u}})$, $1\leq r\leq k-1$, $1-r\leq p\leq r-1$, $p+r\equiv 1 \pmod 2$ are zeros of $f_k(\eta,\Delta,\frac{u}{v})$ for $2\leq k<u$. Then the recursion relation (4.8) implies that $(\frac{pv}{u},h_{r,0;\frac{pv}{u}})$ with $r\leq u-3$ is a zero, because it is a zero of f_{u-1} and f_{u-2}. For $r=u-2$, note that plugging $(\frac{pv}{u},h_{u-2,0;\frac{pv}{u}})$ into the recursion relation (4.8) leads to the coefficient in front of f_{u-2} vanishing and hence $(\frac{pv}{u},h_{u-2,0;\frac{pv}{u}})$ is a zero of f_u because it is a zero of f_{u-1}. Finally, $f_u\Big(\frac{pv}{u},h_{u-1,0;\frac{pv}{u}},\frac{u}{v}\Big)$ was computed in [69, Eq (4.30)] in terms of $\mathfrak{sl}(2)$ data. Expressed in terms of $\mathcal{N}=2$ data this becomes

$$f_u\left(\frac{pv}{u},h_{u-1,0;\frac{pv}{u}},\frac{u}{v}\right)=\binom{2(u-1)}{u-1}\frac{(p+u-2)(p+u-4)\cdots(p-u+2)}{2^{u-1}(u-1)!}, \tag{4.22}$$

which is zero for $2-u\leq p\leq u-2$, $p+u\equiv 0 \pmod 2$.

Finally we consider the zeros of $f_u\big(\eta+\frac{2v}{u},\Delta-\eta-\frac{v}{u},\frac{u}{v}\big)(2\Delta-\eta)$. The base case of $u=2$ and $u=3$ has already been established. For the induction assume again that the pairs $(\frac{pv}{u},h_{r,0;\frac{pv}{u}})$, $1\leq r\leq k-1$, $1-r\leq p\leq r-1$, $p+r\equiv 1 \pmod 2$ are zeros of $f_k(\eta+\frac{2v}{u},\Delta-\eta-\frac{u}{v},\frac{u}{v})(2\Delta-\eta)$ for $2\leq k<u$. Then the pairs $(\frac{pv}{u},h_{r,0;\frac{pv}{u}})$ are for zeros of $f_u(\eta+\frac{2v}{u},\Delta-\eta-\frac{u}{v},\frac{u}{v})(2\Delta-\eta)$ for $r\leq u-3$ by the recursion relation (4.8). For $r=u-2$, plugging $(\frac{pv}{u},h_{u-2,0;\frac{pv}{u}})$ into the recursion relation (4.8) for $f_u(\eta+\frac{2v}{u},\Delta-\eta-\frac{u}{v},\frac{u}{v})(2\Delta-\eta)$ leads to the coefficient in front of $f_{u-2}(\eta+\frac{2v}{u},\Delta-\eta-\frac{u}{v},\frac{u}{v})(2\Delta-\eta)$ vanishing and hence $(\frac{pv}{u},h_{u-2,0;\frac{pv}{u}})$ is a zero for $f_u(\eta+\frac{2v}{u},\Delta-\eta-\frac{u}{v},\frac{u}{v})(2\Delta-\eta)$ because it is a zero for $f_{u-1}(\eta+\frac{2v}{u},\Delta-\eta-\frac{u}{v},\frac{u}{v})(2\Delta-\eta)$. Finally we can again use [69, Eq (4.30)] to compute

$$f_u\left(\frac{pv}{u}+\frac{2v}{u},h_{u-1,0;\frac{pv}{u}}-\frac{pv}{u}-\frac{v}{u},\frac{u}{v}\right)=\binom{2(u-1)}{u-1}\frac{(p+u)(p+u-2)\cdots(p-u+4)}{2^{u-1}(u-1)!}, \tag{4.23}$$

and note that

$$2h_{u-1,0;\frac{vp}{u}} - \frac{vp}{u} = \frac{v}{2u}\big(u^2 - 2u - p^2 - 2p\big) = -\frac{vp}{2u}(p-u+2)(p+u). \tag{4.24}$$

Thus $f_u\Big(\frac{pv}{u} + \frac{2v}{u}, h_{u-1,0;\frac{pv}{u}} - \frac{pv}{u} - \frac{v}{u}, \frac{u}{v}\Big)\Big(2h_{u-1,0;\frac{vp}{u}} - \frac{vp}{u}\Big)$ vanishes for $2-u \leq p \leq u-2$, $u \equiv p \pmod 2$.

□

4.3. **Fusion rules from Zhu bimodules.** Using the theory on Zhu algebras outlined in Section 4.1 we are now in a position to describe the Zhu bimodules associated to $\mathsf{N}(c)$-Verma modules $\mathcal{M}_{q,h,c}$.

We first introduce the following auxiliary spaces:

$$\begin{aligned}\overline{W}_{q,h} &= \mathrm{span}_{\mathbb{C}}\{\mathbf{1}_{q,h}, G^+_{-\frac{1}{2}}\mathbf{1}_{q,h}, G^-_{-\frac{1}{2}}\mathbf{1}_{q,h}, G^-_{-\frac{1}{2}}G^+_{-\frac{1}{2}}\mathbf{1}_{q,h}\} \subset \mathcal{M}_{q,h,c}\\ W_{q,h} &= \mathbb{C}[x,y,z] \otimes \overline{W}_{q,h}.\end{aligned} \tag{4.25}$$

Next, we endow $W_{q,h}$ with a $\mathbb{C}[\Delta,\eta]$-bimodule structure as follows.

$$\begin{aligned}\Delta.(f(x,y,z)\otimes v) &= x\cdot f(x,y,z)\otimes v\\ (f(x,y,z)\otimes v).\Delta &= y\cdot f(x,y,z)\otimes v\\ \eta.(f(x,y,z)\otimes v) &= z\cdot f(x,y,z)\otimes v + f(x,y,z)\otimes J_0 v\\ (f(x,y,z)\otimes v).\eta &= z\cdot f(x,y,z)\otimes v\end{aligned} \tag{4.26}$$

Using standard arguments in Zhu theory (see [75, 39, 54]) we obtain the following description for the Zhu bimodules $A(\mathcal{M}_{q,h,c})$.

Proposition 4.11. *As an $A(\mathsf{N}(c)) \cong \mathbb{C}[\Delta,\eta]$-bimodule*

$$A(\mathcal{M}_{q,h,c}) \cong W_{q,h} \tag{4.27}$$

where the isomorphism is given by

$$\begin{aligned}\mathbb{C}[x,y,z]\otimes\overline{W}_{q,h} &\longrightarrow A(\mathcal{M}_{q,h,c})\\ f(x,y,z)\otimes w &\longmapsto [f(L_{-2}+2L_{-1}+L_0, L_{-2}+L_{-1}, J_{-1})w].\end{aligned} \tag{4.28}$$

Denote the left $A(\mathsf{N}(c))$-module formed by the space of least conformal weight of a Verma module $\mathcal{M}_{q,h,c}$ by $N_{q,h}$. That is $N_{q,h} = \mathbb{C}n_{q,h}$, with $\Delta.n_{q,h}{=}hn_{q,h}$ and $\eta.n_{q,h} = qn_{q,h}$. We describe $A(\mathcal{M}_{q,h,c}) \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2}$ as a left $A(\mathsf{N}(c))$-module.

Lemma 4.12. *Let $h_1,h_2,q_1,q_2 \in \mathbb{C}$.*

(1) *Then, the set*

$$\Big\{x^a \otimes {G^+_{-\frac{1}{2}}}^i {G^-_{-\frac{1}{2}}}^j \mathbf{1}_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} n_{q_2,h_2} \,\big|\, a\in\mathbb{Z}, i,j\in\{0,1\}\Big\} \tag{4.29}$$

is a $\mathbb{C}$-basis of $W_{q_1,h_1,c} \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2}$.

(2) *The action of Δ and η on the $\mathbb{C}$-basis above is*

$$\begin{aligned}\Delta.x^a\otimes w\otimes_{A(\mathsf{N}(c))} n_{q_2,h_2} &= x^{a+1}\otimes w\otimes_{A(\mathsf{N}(c))} n_{q_2,h_2}\\ \eta.x^a\otimes w\otimes_{A(\mathsf{N}(c))} n_{q_2,h_2} &= (q_2+\tilde{q})x^a\otimes w\otimes_{A(\mathsf{N}(c))} n_{q_2,h_2},\end{aligned} \tag{4.30}$$

where $w = {G^+_{-\frac{1}{2}}}^i {G^-_{-\frac{1}{2}}}^j \mathbf{1}_{q_1,h_1}$ and $\tilde{q} = q_1 + i - j$ is the J_0-weight of w.

Theorem 4.13. *Let $h_i, q_i \in \mathbb{C}$, $i = 1,2,3$, then*

$$\dim \mathrm{Hom}\big(W_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2},\ N_{q_3,h_3}\big) = \begin{cases} 1 & \text{if} \quad |q_1+q_2-q_3| = 1,\\ 2 & \text{if} \quad q_1+q_2 = q_3,\\ 0 & \text{else.}\end{cases} \tag{4.31}$$

Proof. Since the action of Δ and η on $W_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2}$ does not mix the basis vectors $G^{+}_{-\frac{1}{2}}{}^{i} G^{-j}_{\frac{1}{2}} \mathbf{1}_{q_1,h_1}$ of $\overline{W}_{q_1,h_1}$, we have the direct sum decomposition

$$W_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2} = \bigoplus_{i=0,j=0}^{2} \mathbb{C}[x] \otimes \mathbb{C} G^{+}_{-\frac{1}{2}}{}^{i} G^{-}_{-\frac{1}{2}}{}^{j} \mathbf{1}_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} \mathbb{C} n_{q_2,h_2}, \tag{4.32}$$

where each summand is the η-eigenspace of respective eigenvalue q_1+q_2+i-j. This immediately implies the "else" line of the theorem.

Next assume there exists a homomorphism $\rho : W_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2} \to\ N_{q_3,h_3}$, then

$$0 = (\Delta - h_3)\rho(-) = \rho((\Delta - h_3)-), \tag{4.33}$$

that is, the image of $\Delta - h_3$ in $W_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2}$ lies in the kernel of ρ. The image of $(\Delta - h_3)$ is just the submodule

$$S_{h_3} = \big\langle (x-h_3) \otimes w \otimes_{A(\mathsf{N}(c))} n_{q_2,h_2} | w \in \overline{W}_{q_1,h_1} \big\rangle. \tag{4.34}$$

Consider the quotient by S_{h_3} and we immediately obtain

$$\begin{aligned}\frac{W_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} N_{q_2,h_2}}{S_{h_3}} &\cong \bigoplus_{i=0,j=0}^{2} \frac{\mathbb{C}[x]}{\langle x-h_3\rangle} \otimes \mathbb{C} G^{+}_{-\frac{1}{2}}{}^{i} G^{-}_{-\frac{1}{2}}{}^{j} \mathbf{1}_{q_1,h_1} \otimes_{A(\mathsf{N}(c))} \mathbb{C} n_{q_2,h_2}\\ &\cong N_{h_3,q_1+q_2-1} \oplus 2N_{h_3,q_1+q_2} \oplus N_{h_3,q_1+q_2+1},\end{aligned} \tag{4.35}$$

which implies the theorem. ☐

Proposition 4.14. *Let $t, q \in \mathbb{C}$ and consider the Verma module $\mathcal{M}_{q,h_{1,1;q}(t),c(t)}$, where $h_{1,1;q}(t) = \frac{(1-t)^2-1-q^2}{4t}$. Then the vector*

$$w_{1,1}(t,q) = \left((q-1)L_{-1} + \frac{t}{2}(q^2-1)J_{-1} + G^{+}_{-\frac{1}{2}} G^{-}_{-\frac{1}{2}}\right)\mathbf{1}_{q,h_{1,1;q}(t),c(t)} \tag{4.36}$$

is singular in $\mathcal{M}_{q,h_{1,1;q}(t),c(t)}$. Furthermore, under the isomorphism in (4.28), the images of the singular vector $[w_{1,1}] \in A(\mathcal{M}_{q,h_{1,1;q}(t),c(t)})$ and its $G^{\pm}_{-\frac{1}{2}}$ descendants are

$$\begin{aligned}
[w_{1,1}] &\mapsto f^1(x,y,z,q) \otimes \big[\mathbf{1}_{q,h_{1,1;q}(t)}\big] + 1 \otimes \Big[G^{+}_{-\frac{1}{2}} G^{-}_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1;q}(t)}\Big]\\
\Big[G^{+}_{-\frac{1}{2}} w_{1,1}\Big] &\mapsto f^{+}(x,y,z,q) \otimes \Big[G^{+}_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1;q}(t)}\Big]\\
\Big[G^{-}_{-\frac{1}{2}} w_{1,1}\Big] &\mapsto f^{-}(x,y,z,q) \otimes \Big[G^{-}_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1;q}(t)}\Big],\\
\Big[G^{+}_{-\frac{1}{2}} G^{-}_{-\frac{1}{2}} w_{1,1}\Big] &\mapsto \frac{t}{2}(q^2-1)(2y-z) \otimes \big[\mathbf{1}_{q,h_{1,1;q}(t)}\big]\\
&\qquad + f^{G}(x,y,z,q) \otimes \Big[G^{+}_{-\frac{1}{2}} G^{-}_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1;q}(t)}\Big],
\end{aligned} \tag{4.37}$$

where

$$f^1(x,y,z,q) = (q-1)\left(x - y + \frac{1}{2} + \frac{t}{4}(q+1)(2z+q-1)\right),$$

$$f^+(x,y,z,q) = (q-1)\Big(x - y + \frac{t}{4}(q+1)(2z+q+1)\Big),$$
$$f^-(x,y,z,q) = (q+1)\Big(x - y + \frac{t}{4}(q-1)(2z+q-1)\Big),$$
$$f^G(x,y,z,q) = (q+1)\Big(x - y - \frac{1}{2} + \frac{t}{4}(q-1)(2z+q+1)\Big). \tag{4.38}$$

Proof. By direct computation we see that $G^{\pm}_{\frac{1}{2}} w_{1,1} = 0 = J_1 w_{1,1}$ and hence $w_{1,1}$ is singular. The simplifications within the Zhu bimodule follow from the relations in Proposition 4.11, which allows us to make the following replacements

$$L_{-2} \mapsto 2y - x + L_0, \quad L_{-1} \mapsto x - y - L_0, \quad J_{-1} \mapsto z,$$
$$[J_{-2}v] = -[J_{-1}v], \qquad \Big[G^{\pm}_{-\frac{3}{2}} v\Big] = -\Big[G^{\pm}_{-\frac{1}{2}} v\Big]. \tag{4.39}$$

□

Proposition 4.15. *For $q,t \in \mathbb{C}$, $q \neq \pm 1$, let $c(t) = 3 - \frac{6}{t}$. Let r,s be positive integers, $q_2 \in \mathbb{C}$ and $h_{r,s;q_2}(t) = \frac{(r-ts)^2 - 1 - (tq_2)^2}{4t}$. Finally, let $q_3, h_3 \in \mathbb{C}$. Then,*

$$\dim \operatorname{Hom}\bigg(A\bigg(\frac{\mathcal{M}_{q,h_{1,1;q}(t),c(t)}}{\langle w_{1,1}\rangle}\bigg) \otimes_A N_{q_2,h_{r,s}},\ N_{q_3,h_3}\bigg) \leq \begin{cases} 1, \text{ if } q_3 = q + q_2 + 1,\ h_3 = h_{r,s,q_3}, \\ 1, \text{ if } q_3 = q + q_2 - 1,\ h_3 = h_{r,s,q_3}, \\ 1, \text{ if } q_3 = q + q_2,\ h_3 = h_{r,s\pm 1,q_3}, \\ 0, \text{ otherwise}, \end{cases} \tag{4.40}$$

where $\otimes_A$ denotes the tensor product over $A(\mathsf{N}(c))$.

Proof. We use the canonical identification of vector spaces

$$\operatorname{Hom}\big(A(\mathcal{M}_{q,h_{1,1;q}(t),c(t)}) \otimes_A N_{q_2,h_{r,s;q_2}(t)},\ N_{q_3,h_3}\big) \cong N^*_{q_3,h_3} \otimes_A A(\mathcal{M}_{q,h_{1,1;q}(t),c(t)}) \otimes_A N_{q_2,h_{r,s;q}(t)} \tag{4.41}$$

to compute dimensions, where $N^*_{q_3,h_3}$ is the dual of N_{q_3,h_3}. By Lemma 4.6, the dimension of the hom space in (4.40) is therefore the codimension of $N^*_{q_3,h_3} \otimes_A A(\langle w_{1,1}\rangle) \otimes_A N_{q_2,h_{r,s;q}(t)}$ in $N^*_{q_3,h_3} \otimes_A A(\mathcal{M}_{,q,h_{1,1;q}(t),c(t)}) \otimes_A N_{q_2,h_{r,s;q}(t)}$. If $q_3 - q - q_2 \neq 0, \pm 1$, then the dimension of the hom space (4.40) must be zero by Theorem 4.13. So we first consider the case $q_3 = q + q_2 + 1$. After tensoring the vectors in (4.37) with the basis vector n_{q+q_2+1,h_3} of $N^*_{q_3,h_3}$ we see that only the second vector in (4.37) does not vanish and yields

$$n_{q+q_2+1,h_3} \otimes_A G^+_{-\frac{1}{2}} w_{1,1} \otimes_A n_{q_2,h_2} = f^+(h_3, h_{r,s;q_2}(t), q_2, q) n_{q+q_2+1,h_3} \otimes_A G^+_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1}} \otimes_A n_{q_2,h_2}, \tag{4.42}$$

where

$$\begin{aligned} f^+(h_3, h_{r,s;q_2}(t), q_2, q) &= (q-1)\Big(h_3 - h_{r,s;q_2}(t) - \frac{t}{4}(1-q^2) + \frac{t}{2}(q+1)(q_2+1)\Big) \\ &= (q-1)(h_3 - h_{r,s;q+q_2+1}(t)), \end{aligned} \tag{4.43}$$

that is, the variables x, y, z evaluate to $h_3, h_{r,s;q_2}(t), q_2$, respectively. Hence $N^*_{q_3,h_3} \otimes_A A(\langle w_{1,1}\rangle) \otimes_A N_{q_2,h_{r,s;q}(t)}$ has codimension 0 in $N^*_{q_3,h_3} \otimes_A A(\mathcal{M}_{q,h_{1,1;q}(t),c(t)}) \otimes_A N_{q_2,h_{r,s;q}(t)}$ unless $h_3 = h_{r,s;q+q_2+1}(t)$, which proves the first line of (4.40). Similarly, for $q_3 = q + q_2 - 1$ only the third vector in (4.37) does not vanish when tensoring with n_{q+q_2-1,h_3} and yields

$$n_{q+q_2-1,h_3} \otimes_A G^-_{-\frac{1}{2}} w_{1,1} \otimes_A n_{q_2,h_2} = f^-(h_3, h_{r,s;q_2}(t), q_2, q) n_{q+q_2-1,h_3} \otimes_A G^+_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1}} \otimes_A n_{q_2,h_2}, \tag{4.44}$$

where

$$f^-(h_3, h_{r,s;q_2}(t), q_2, q) = (q+1)\Big(h_3 - h_{r,s;q_2}(t) - \frac{t}{4}(1-q^2) + \frac{t}{2}(q-1)(q_2-1)\Big) \\ = (q+1)(h_3 - h_{r,s;q+q_2-1}(t)). \tag{4.45}$$

For $q_3 + q + q_2$ the second and third vectors in (4.37) vanish when tensoring with n_{q+q_2,h_3}, while the others yield

$$\begin{aligned} n_{q+q_2,h_3} \otimes_A w_{1,1} \otimes_A n_{q_2,h_2} =& f^1(h_3, h_{r,s;q_2}(t), q_2, q) n_{q+q_2,h_3} \otimes_A \mathbf{1}_{q,h_{1,1}} \otimes_A n_{q_2,h_2} \\ &+ n_{q+q_2,h_3} \otimes_A G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1}} \otimes_A n_{q_2,h_2}, \\ n_{q+q_2,h_3} \otimes_A G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} w_{1,1} \otimes_A n_{q_2,h_2} =& \frac{t}{2}(q^2-1)(2h_{r,s;q_2}(t) - q_2) \otimes_A \mathbf{1}_{q,h_{1,1}} \otimes_A n_{q_2,h_2} \\ &+ f^G(h_3, h_{r,s;q_2}(t), q_2, q) \otimes G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} \mathbf{1}_{q,h_{1,1}} \otimes_A n_{q_2,h_2}. \end{aligned} \tag{4.46}$$

The constant coefficient in front of one of the summands above means that $N^*_{q+q_2,h_3} \otimes_A A(\langle w_{1,1}\rangle) \otimes_A N_{q_2,h_{r,s;q}(t)}$ must always have codimension at most 1 in $N^*_{q+q_2,h_3} \otimes_A A(\mathcal{M}_{q,h_{1,1;q}(t),c(t)}) \otimes_A N_{q_2,h_{r,s;q}(t)}$. To determine when the codimension can be greater than 0 we compute the determinant of the coefficients (4.46) to obtain a polynomial in h_3. The roots of this polynomial are precisely where the codimension can be greater than 0.

$$\det\begin{pmatrix} f^1(h_3, q, h_{r,s;q_2}(t), q_2) & 1 \\ \frac{t}{2}(q^2-1)(2h_{r,s;q_2}(t) - q_2) & f^G(h_3, q, h_{r,s;q_2}(t), q_2) \end{pmatrix} \\ = \big(q^2-1\big)\big(h_3 - h_{r,s+1;q+q_2}(t)\big)\big(h_3 - h_{r,s-1;q+q_2}(t)\big). \tag{4.47}$$

Therefore $h_3 = h_{r,s\pm1;q+q_2}(t)$ are the only values for which the codimension need not be 0.

□

Note that these bounds on hom space dimensions in particular provide an upper bound for the fusion rules (3.21) in Lemma 3.6 after setting $t = \frac{u}{v}$.

5. Lower bounds for Lemma 3.6 via free field realisations

In this section we prove lower bounds for the dimensions appearing in Lemma 3.6 by explicitly constructing suitable intertwining operators in a free field realisation of $\mathsf{A}_1(u,v)$. To this end we recall some facts regarding $P(w)$-intertwining operators following [21, 45] before discussing the free field realisation of $\mathsf{A}_1(u,v)$ and its screening operators.

Given a generalised V-module $M = \bigoplus_{h\in\mathbb{C}} M_{[h]}$, its *algebraic completion* $\overline{M}$ is defined as the superspace $\overline{M} = \prod_{h\in\mathbb{C}} M_{[h]}$, where $\overline{M}^i = \overline{M^i}$ for $i = \bar{0}, \bar{1}$ while its *contragredient* or *graded dual* module M' is defined as the vector space

$$M' = \bigoplus_{h\in\mathbb{C}} M^*_{[h]}, \quad M^*_{[h]} = \mathrm{Hom}(M_{[h]}, \mathbb{C}) \tag{5.1}$$

together with the action $Y_{M'}$ characterised by

$$\langle Y_{M'}(v,z)m', m\rangle = \langle m', Y^{\mathrm{opp}}_M(v,z)m\rangle, \tag{5.2}$$

where $m' \in M', m \in M$, $v \in V$, and where

$$Y^{\mathrm{opp}}_M(v,z) = Y_M(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1}) \tag{5.3}$$

is the *opposed action* (note that the module M needs to satisfy some lower boundedness conditions, which will never be an issue here, in order for the action on M' to be well defined). Similarly, for any triple of modules M_1, M_2, M_3 and an intertwining operator $\mathcal{Y} \in \binom{M_3}{M_1,\ M_2}$ the opposed intertwining operator

$$\mathcal{Y}^{\mathrm{opp}}(m_1, z) = \mathcal{Y}(e^{zL_1}(-z^{-2})^{L_0} m_1, z^{-1}) \tag{5.4}$$

characterises an intertwining operator $\widetilde{\mathcal{Y}} \in \binom{M_2'}{M_1,\ M_3'}$ via

$$\left\langle m_2, \widetilde{\mathcal{Y}}(m_1, z) m_3' \right\rangle = \langle m_3', \mathcal{Y}^{\mathrm{opp}}(m_1, z) m_2 \rangle. \tag{5.5}$$

Definition 5.1. Let $w \in \mathbb{C}^\times$ and let M_1, M_2, M_3 be modules over a vertex operator superalgebra V. A *$P(w)$-intertwining map of type* $\binom{M_3}{M_1 \quad M_2}$ is a bilinear map $I\colon M_1 \otimes M_2 \to \overline{M_3}$, for $\overline{M_3}$ the algebraic completion of M_3, that satisfies the following properties:

(1) *Lower truncation*: For any $\psi_1 \in M_1$, $\psi_2 \in M_2$, $\pi_h(I(\psi_1 \otimes \psi_2)) = 0$ for all $\mathrm{Re}(h) \ll 0$, where π_h denotes the projection onto the generalised eigenspace $(M_3)_{[h]}$ of $L(0)$-eigenvalue h;

(2) *Convergence*: For any $\psi_1 \in M_1$, $\psi_2 \in M_2$, $\psi_3 \in M_3'$, the contragredient of M_3, the series defined by

$$\begin{aligned}&\langle \psi_3, Y_3(v, z) I(\psi_1, w) \psi_2 \rangle,\\ &\langle \psi_3, I(Y_1(v, z - w)\psi_1, w)\psi_2 \rangle, \quad \text{and}\\ &\langle \psi_3, I(\psi_1, w) Y_2(v, z)\psi_2 \rangle\end{aligned} \tag{5.6}$$

are absolutely convergent in the regions $|z| > |w| > 0$, $|w| > |z - w| > 0$ and $|w| > |z| > 0$, respectively, where the subscript under each Y indicates the module being acted on.

(3) *Cauchy-Jacobi identity*: Given any $f(t) \in R_{P(w)} = \mathbb{C}\big[t, t^{-1}, (t - w)^{-1}\big]$, the field of rational functions whose poles lie in some subset of $\{0, w, \infty\}$, we have the following identity.

$$\begin{aligned}\oint_{0,w} f(z) \langle \psi_3, Y_3(v, z) I(\psi_1, w)\psi_2 \rangle \mathrm{d}z &= \oint_{w} f(z) \langle \psi_3, I(Y_1(v, z - w)\psi_1, w)\psi_2 \rangle \mathrm{d}z\\ &\quad + \oint_{0} f(z) \langle \psi_3, I(\psi_1, w) Y_2(v, z)\psi_2 \rangle \mathrm{d}z\end{aligned} \tag{5.7}$$

where the subscript on each integral indicates the points in $\{0, w, \infty\}$ that must be enclosed by simple positively oriented contours.

Next we recall a free field construction of $\mathsf{A}_1(u, v)$ due to Semikhatov [70], and Adamović [3]. Consider the 2-dimensional trivial Lie algebra with basis denoted $\{a, b\}$ and with invariant non-degenerate symmetric form normalised such that

$$\langle a, a \rangle = -\langle b, b \rangle = \frac{t}{2} - 1, , \qquad \langle a, b \rangle = 0. \tag{5.8}$$

We denote the rank 2 Heisenberg vertex algebra constructed from the affinisation of this Lie algebra by $\mathsf{H}(2, t)$. This vertex algebra is isomorphic to the tensor product of two rank 1 Heisenberg algebras of respective levels $\frac{t-2}{4}$ and $\frac{2-t}{4}$, that is $\mathsf{H}(2, t) \cong \mathsf{H}\big(\frac{t-2}{4}\big) \otimes \mathsf{H}\big(\frac{2-t}{4}\big)$. Let $\{a(z), b(z)\}$ be the generating currents corresponding to the basis $\{a, b\}$ whose operator product expansions therefore satisfy

$$a(z)a(w) \sim -b(z)b(w) \sim \frac{\frac{t}{2} - 1}{(z - w)^2}, \qquad a(z)b(w) \sim 0. \tag{5.9}$$

Further, consider the rank 1 lattice $\mathbb{L} = \mathbb{Z}\frac{2}{t-2}(a - b)$ (note that the pairing (5.8) restricted to this lattice vanishes) and corresponding lattice vertex algebra

$$\mathsf{L}(t) = \bigoplus_{p \in \mathbb{L}} \mathcal{F}_p, \tag{5.10}$$

where $\mathcal{F}_p$ denotes the rank 2 Fock space over $\mathsf{H}(2,t)$ whose highest weight vector $|p\rangle$ satisfies

$$a_0|p\rangle = \langle a,p\rangle|p\rangle, \qquad b_0|p\rangle = \langle b,p\rangle|p\rangle. \tag{5.11}$$

Note that even though $\mathsf{H}(2,t)=\mathcal{F}_0$ has two independent generating currents (that is, it is rank 2) the lattice $\mathbb{L}$ is only rank 1. The hence dual of $\mathbb{L}$

$$\mathbb{L}^* = \left\{ v \in \mathbb{R}\otimes_{\mathbb{Z}}\mathbb{L} \mid \frac{2}{t-2}\langle v, a-b\rangle \in \mathbb{Z} \right\} = \mathbb{R}(a-b)\oplus\mathbb{Z}b \tag{5.12}$$

is not discrete. Finally, for any $\mu\in\mathbb{L}^*/\mathbb{L}$, the sum of Fock spaces

$$\mathbb{F}_\mu = \bigoplus_{p\in\mu}\mathcal{F}_p \tag{5.13}$$

is naturally an $\mathsf{L}(t)$ module.

Theorem 5.2 (Adamović [3, Thm 5.5, Prop 9.2])**.** *For coprime $u,v\geq 2$, let $\mathsf{M}(u,v)$ denote the Virasoro minimal model vertex operator algebra at central charge $c_{u,v}=1-6\frac{(u-v)^2}{uv}$. Denote the conformal vector of $\mathsf{M}(u,v)$ by $\omega_{u,v}=L_{-2}^{(u,v)}\mathbf{1}$.*

(1) *There exists an embedding of vertex operator algebras*

$$\psi : \mathsf{A}_1(u,v) \hookrightarrow \mathsf{M}(u,v)\otimes\mathsf{L}\Big(\frac{u}{v}\Big) \tag{5.14}$$

characterised on generators by

$$e_{-1}\mathbf{1} \mapsto \mathbf{1}\otimes\left|\frac{2v}{u-2v}(a-b)\right\rangle, \tag{5.15}$$

$$h_{-1}\mathbf{1} \mapsto 2a_{-1}\mathbf{1}\otimes|0\rangle, \tag{5.16}$$

$$f_{-1}\mathbf{1} \mapsto \Big(\frac{u}{v}L_{-2}^{(u,v)} - b_{-1}^2 - \Big(\frac{u}{v}-1\Big)b_{-2}\Big)\mathbf{1}\otimes\left|-\frac{2v}{u-2v}(a-b)\right\rangle. \tag{5.17}$$

The image of the conformal vector in $\mathsf{A}_1(u,v)$ under this embedding is

$$\frac{1}{2\frac{u}{v}}\Big(\frac{1}{2}h_{-1}^2 + e_{-1}f_{-1} + f_{-1}e_{-1}\Big)\mathbf{1} \mapsto \Big(L_{-2}^{(u,v)} + \frac{v}{u-2v}\big(a_{-1}^2 - b_{-1}^2\big) - b_{-2}\Big)\mathbf{1}\otimes|0\rangle. \tag{5.18}$$

(2) *Let $\mathsf{S}_{r,s}$, $1\leq r\leq u-1$, $1\leq s\leq v-1$ denote the simple $\mathsf{M}(u,v)$-module of highest conformal weight*

$$h_{r,s} = \frac{(us-vr)^2-(u-v)^2}{4uv}. \tag{5.19}$$

Recall that $\mathsf{A}_1(u,v)$ admits an automorphism γ called conjugation, *which is characterised by*

$$e_{-1}\mathbf{1}\mapsto f_{-1}\mathbf{1}, \qquad h_{-1}\mathbf{1}\mapsto -h_{-1}\mathbf{1}, \qquad f_{-1}\mathbf{1}\mapsto e_{-1}\mathbf{1}. \tag{5.20}$$

Thus $\psi\circ\gamma$ is also an embedding $\mathsf{A}_1(u,v)\hookrightarrow\mathsf{M}(u,v)\otimes\mathsf{L}\big(\frac{u}{v}\big)$. For any $\mathsf{M}(u,v)\otimes\mathsf{L}\big(\frac{u}{v}\big)$-module we can therefore pull back along ψ and $\psi\circ\gamma$ to obtain $\mathsf{A}_1(u,v)$ modules. Under these pullbacks, for $\ell\in\mathbb{Z}$, $1\leq r\leq u-1$, $1\leq s\leq v-1$ we have the identifications

$$\begin{aligned}
\psi^*\mathsf{S}_{r,s}\otimes\mathbb{F}_{\mu_{r,s;l}} &\cong \sigma^\ell E^-_{u-r,v-s}, & \mu_{r,s;l} &= \Big[\Big(\tfrac{v\lambda_{r,s}}{u-2v}+\ell\Big)(a-b)+(l-1)b\Big],\\
(\psi\circ\gamma)^*\mathsf{S}_{r,s}\otimes\mathbb{F}_{\tilde{\mu}_{r,s;l}} &\cong \sigma^\ell E^+_{r,s}, & \tilde{\mu}_{r,s;l} &= \Big[\Big(-\tfrac{v\lambda_{r,s}}{u-2v}-\ell\Big)(a-b)-(l+1)b\Big],\\
\psi^*\mathsf{S}_{r,s}\otimes\mathbb{F}_\mu &\cong \sigma^{\left\langle\frac{2v}{u-2v}(a-b),\mu+b\right\rangle}E_{\langle 2b,\mu+b\rangle;r,s}, & \mu&\in\mathbb{L}^*/\mathbb{L},\ \langle 2b,\mu+b\rangle\neq[\pm\lambda_{r,s}],\\
(\psi\circ\gamma)^*\mathsf{S}_{r,s}\otimes\mathbb{F}_\mu &\cong \sigma^{\left\langle-\frac{2v}{u-2v}(a-b),\mu+b\right\rangle}E_{-\langle 2b,\mu+b\rangle;r,s}.
\end{aligned} \tag{5.21}$$

(3) *Consider the simple* $\mathsf{M}(u,v)$ *module* $\mathsf{S}_{1,2}$ *of highest conformal weight* $h_{1,2} = \frac{3u}{4v} - \frac{1}{2}$ *generated by a highest weight vector* $v_{1,2}$*. Tensor products with* $\mathsf{S}_{1,2}$ *decompose, for* $1 \le r \le u-1$*, as*

$$\mathsf{S}_{1,2} \boxtimes \mathsf{S}_{r,s} \cong \begin{cases} \mathsf{S}_{r,2} & \text{if } s=1, \\ \mathsf{S}_{r,s-1} \oplus \mathsf{S}_{r,s+1} & \text{if } 2 \le s \le v-2 \\ \mathsf{S}_{r,v-2} & \text{if } s = v-1. \end{cases} \tag{5.22}$$

Further, let $I^+_{r,s} \in \binom{\mathsf{S}_{r,s+1}}{\mathsf{S}_{1,2},\ \mathsf{S}_{r,s}}$*,* $1 \le s \le v-2$ *and* $I^-_{r,s} \in \binom{\mathsf{S}_{r,s-1}}{\mathsf{S}_{1,2},\ \mathsf{S}_{r,s}}$*,* $2 \le s \le v-1$ *be surjective* $\mathsf{M}(u,v)$*-intertwining operators and let* I_μ *a surjective* $\mathsf{L}(\frac{u}{v})$*-intertwining operator of type* $\binom{\mathbb{F}_{\mu+b}}{\mathbb{F}_{b+\mathbb{L}},\ \mathbb{F}_\mu}$ *such that their tensor product* $\mathcal{Y}^\pm_{\mu;r,s} = I^\pm_{r,s} \otimes I_\mu$ *is an* $\mathsf{M}(u,v) \otimes \mathsf{L}(\frac{u}{v})$*-intertwining operator of type* $\binom{\mathsf{S}_{r,s\pm1} \otimes \mathbb{F}_{b+\mu}}{\mathsf{S}_{1,2} \otimes \mathbb{F}_{b+\mathbb{L}},\ \mathsf{S}_{r,s} \otimes \mathbb{F}_\mu}$*. Then* $v_{1,2} \otimes |b\rangle \in \mathsf{S}_{1,2} \otimes \mathbb{F}_{b+\mathbb{L}}$ *is a screening vector for the free field realisations* ψ *and* $\psi \circ \gamma$*, that is*

$$\operatorname{im} \psi = \operatorname{im} \psi \circ \gamma \subset \ker \operatorname{Res}_z \mathcal{Y}^+_{0;1,1}(v_{1,2} \otimes |b\rangle, z). \tag{5.23}$$

We denote the corresponding screening currents by

$$\mathcal{Q}^\pm(z) = \mathcal{Y}^\pm_{\mu;r,s}(v_{1,2} \otimes |b\rangle, z), \tag{5.24}$$

where the indices μ, r, s *will be determined by the module the screening current is applied to. In particular, the* $\mathsf{A}_1(u,v)$ *generators satisfy the following operator product expansions with the screening currents.*

$$Y(\psi(e), w)\mathcal{Q}^\pm(z) \sim 0, \qquad Y(\psi(h), w)\mathcal{Q}^\pm(z) \sim 0$$
$$Y(\psi(f), w)\mathcal{Q}^\pm(z) \sim \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{I_\mu(|\tau\rangle, z)\partial_z I^\pm_{r,s}(v_{1,2}, z) + I^\pm_{r,s}(v_{1,2}, z) I_\mu\big(\big(\frac{u}{u-2v}b_{-1} + 2\frac{v-u}{u-2v}a_{-1}\big)|\tau\rangle, z\big)}{z-w}\right.$$
$$\left. \frac{-\big(\frac{u}{v}-1\big) I^\pm_{r,s}(v_{1,2}, z) I_\mu(|\tau\rangle, z)}{(z-w)^2}\right], \tag{5.25}$$

where $|\tau\rangle = \big|\frac{u}{u-2v}b - \frac{2v}{u-2v}a\big\rangle$*.*

Remark 5.3. There is a minor typographical error in [3, Sec 9.1] where the module label $(1,2)$ is stated as $(2,1)$ for the highest weight vector used to construct the screening current.

Proposition 5.4. *For coprime* $u, v \ge 2$*, let* I *be a surjective* $\mathsf{L}(\frac{u}{v})$ *intertwining operator of type* $\binom{\mathbb{F}_{\frac{v}{u-2v}(p+p')(a-b)-2b+\mathbb{L}}}{\mathbb{F}_{\frac{v}{u-2v}p(a-b)-b+\mathbb{L}},\ \mathbb{F}_{\frac{v}{u-2v}p'(a-b)-b+\mathbb{L}}}$*. Recall that vertex operator algebra actions on modules are surjective intertwining operators and so we denote the* $\mathsf{M}(u,v)$*-action on* $\mathsf{S}_{r,s}$ *by* $Y_{r,s}$ *to obtain an intertwining operator of type* $\binom{\mathsf{S}_{r,s}}{\mathsf{S}_{1,1},\ \mathsf{S}_{r,s}}$ *and by tensoring an* $\mathsf{M}(u,v) \otimes \mathsf{L}(\frac{u}{v})$ *intertwining operator of type*

$$Y_{r,s} \otimes I \in \binom{\mathsf{S}_{r,s} \otimes \mathbb{F}_{\frac{v}{u-2v}(p+p')(a-b)-2b+\mathbb{L}}}{\mathsf{S}_{1,1} \otimes \mathbb{F}_{\frac{v}{u-2v}p(a-b)-b+\mathbb{L}},\ \mathsf{S}_{r,s} \otimes \mathbb{F}_{\frac{v}{u-2v}p'(a-b)-b+\mathbb{L}}}. \tag{5.26}$$

Then the pullback of $Y_{r,s} \otimes I$ *along* ψ *gives surjective* $\mathsf{A}_1(u,v)$*-intertwining operators of type*

$$\binom{\sigma^{-1}E^-_{u-r,v-s}}{E_{[p];1,1},\ E_{[p'];r,s}},\ p+p'+\tfrac{u}{v} \in \lambda_{r,s} + 2\mathbb{Z},\quad \binom{\sigma^{-1}E^-_{r,s}}{E_{[p];1,1},\ E_{[p'];r,s}},\ p+p'+\tfrac{u}{v} \in \lambda_{u-r,-vs} + 2\mathbb{Z},$$
$$\binom{\sigma^{-1}E_{[p+p'+\frac{u}{v}];r,s}}{E_{[p];1,1},\ E_{[p'];r,s}},\ p+p'+\tfrac{u}{v} \notin \pm\lambda_{r,s} + 2\mathbb{Z}, \tag{5.27}$$

while the pullback along $\psi\circ\gamma$ gives surjective $\mathsf{A}_1(u,v)$-intertwining operators of type

$$\begin{pmatrix} \sigma E^+_{r,s} \\ E_{[-p];1,1},\ E_{[-p'];r,s}\end{pmatrix},\ -\tfrac{u}{v}-p-p'\in\lambda_{r,s}+2\mathbb{Z},\ \begin{pmatrix} \sigma E^+_{u-r,v-s} \\ E_{[-p];1,1},\ E_{[-p'];r,s}\end{pmatrix},\ -\tfrac{u}{v}-p-p'\in\lambda_{u-r,v-s}+2\mathbb{Z},$$
$$\begin{pmatrix} \sigma E_{[-\frac{u}{v}-p-p'];r,s} \\ E_{[-p];1,1},\ E_{[-p'];r,s}\end{pmatrix},\ -\tfrac{u}{v}-p-p'\notin\pm\lambda_{r,s}+2\mathbb{Z}. \tag{5.28}$$

In particular, these spaces of intertwining operators are therefore at least 1-dimensional.

Proof. The free field realisations ψ, $\psi\circ\gamma$ in Theorem 5.2 construct $\mathsf{A}_1(u,v)$ as a subalgebra of $\mathsf{M}(u,v)\otimes\mathsf{L}(\frac{u}{v})$. Hence any $\mathsf{M}(u,v)\otimes\mathsf{L}(\frac{u}{v})$ intertwining operator is also a $\mathsf{A}_1(u,v)$ intertwining operator by restriction. The identification of modules given in Part 2 of Theorem 5.2 then immediately implies the identifications of intertwining operators in (5.27). □

Proposition 5.4 proves the lower bound on the dimensions of intertwining operator spaces in the second line of (3.21). We devote the rest of the section to establishing the remaining fusion rules in (3.21).

Proposition 5.5. *Let $w\in\mathbb{C}^\times$ and let $\Gamma_{0,w}$ be a Pochhammer contour (or its homotopy class) about 0 and w (see Figure 1 for a visualisation). Define linear maps*

$$\Phi^\tau_{r,s}(w)=\int_{\Gamma_{0,w}}\mathcal{Q}^\tau(x)Y_{r,s}(-,w)-\otimes I(-,w)-\,\mathrm{d}x, \tag{5.29}$$

where $\tau=\pm$, with respective domains and codomains

$$\Phi^\tau_{r,s}(w):\mathsf{S}_{1,1}\otimes\mathbb{F}_{p(a-b)-b+\mathbb{L}}\otimes\mathsf{S}_{r,s}\otimes\mathbb{F}_{p'(a-b)-b+\mathbb{L}}\to\overline{\mathsf{S}_{r,s+\tau1}\otimes\mathbb{F}_{(p+p')(a-b)-b+\mathbb{L}}}, \tag{5.30}$$

where $\pm\lambda_{1,1}\notin\left[(\frac{u}{v}-2)p\right]$, $\pm\lambda_{r,s}\notin\left[(\frac{u}{v}-2)p'\right]$ and $\pm\lambda_{r,s+\tau1}\notin\left[(\frac{u}{v}-2)(p+p')\right]$. Then $\Phi^\tau_{r,s}(w)$ is a surjective $\mathsf{A}_1(u,v)$ $P(w)$-intertwining operator of type $\begin{pmatrix} E_{[(\frac{u}{v}-2)(p+p')];r,s+\tau1} \\ E_{[(\frac{u}{v}-2)p];1,1},\ E_{[(\frac{u}{v}-2)p'];r,s}\end{pmatrix}$. In particular, these intertwining operator spaces are at least 1-dimensional.

Proof. Note that since $Y_{r,s}$ and I are intertwining operators the lower truncation and convergence properties follow immediately for $\Phi^\tau_{r,s}$. Thus in order to conclude that they are intertwining operators we only need to verify the Cauchy-Jacobi identity (5.7). Specifically, for any $o\in\mathsf{S}'_{r,s+\tau1}\otimes\mathbb{F}'_{(p+p')(a-b)-b+\mathbb{L}}$, $m\in\mathsf{S}_{1,1}\otimes\mathbb{F}_{(p+p')(a-b)-b+\mathbb{L}}$, $n\in\mathsf{S}_{r,s}\otimes\mathbb{F}_{(p+p')(a-b)-b+\mathbb{L}}$, $v\in\mathsf{A}_1(u,v)$ and $f(y)\in\mathbb{C}[y,y^{-1},(y-z)^{-1}]$ we need to show the identity

$$\begin{aligned}\oint_{0,w}f(y)\langle o,Y(v,y)\Phi^\tau_{r,s}(m,w)n\rangle\frac{\mathrm{d}y}{2\pi\mathsf{i}}&=\oint_{w}f(y)\langle o,\Phi^\tau_{r,s}(Y(v,y-w)m,z)n\rangle\frac{\mathrm{d}y}{2\pi\mathsf{i}}\\&\quad+\oint_{0}f(y)\langle o,\Phi^\tau_{r,s}(m,w)Y(v,y)n\rangle\frac{\mathrm{d}y}{2\pi\mathsf{i}},\end{aligned} \tag{5.31}$$

where the respective integration contours are counterclockwise circles that encircle 0 and w, w but not 0, and 0 but not w. We therefore consider the complex function

$$\begin{aligned}A(w)&=\oint_{0,w}f(y)\langle o,Y(v,y)\Phi^\tau_{r,s}(m,w)n\rangle\frac{\mathrm{d}y}{2\pi\mathsf{i}}-\oint_{w}f(y)\langle o,\Phi^\tau_{r,s}(Y(v,y-w)m,w)n\rangle\frac{\mathrm{d}y}{2\pi\mathsf{i}}\\&\quad-\oint_{0}f(y)\langle o,\Phi^\tau_{r,s}(m,w)Y(v,y)n\rangle\frac{\mathrm{d}y}{2\pi\mathsf{i}}\\&=\oint_{0,w,x}\int_{\Gamma_{0,w}}f(y)\langle o,Y(v,y)\mathcal{Q}^\tau(x)(Y_{r,s}\otimes I)(m,w)n\rangle\mathrm{d}x\frac{\mathrm{d}y}{2\pi\mathsf{i}}\end{aligned}$$

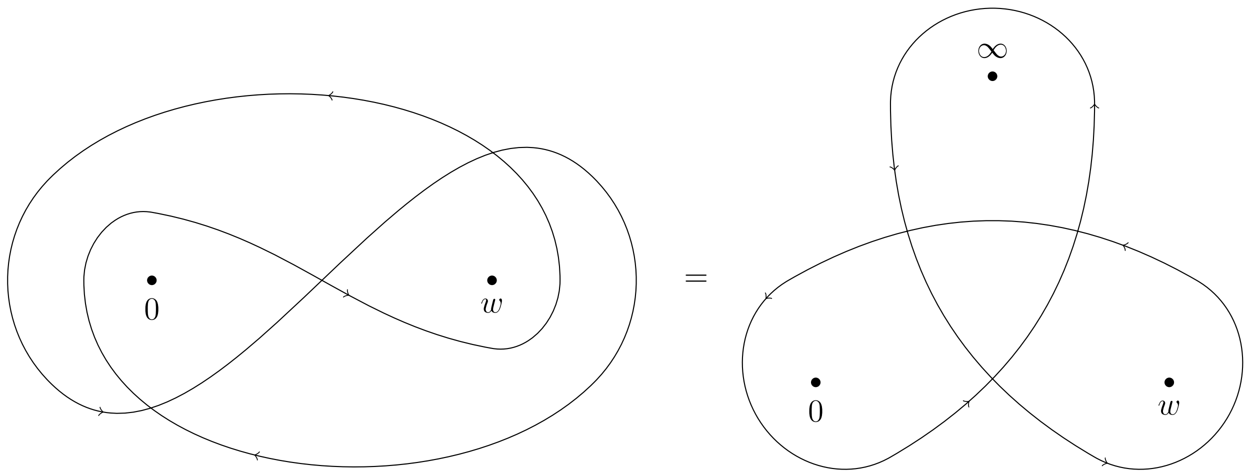

FIGURE 1. Left: A Pochhammer contour about the points 0 and w. The contour is non-contractible, runs counter-clockwise around both points once and then clockwise around both points so that the total winding number for each point is 0.
Right: The Pochammer contour can be moved around the back of the Riemann sphere to form a trefoil contour about 0, w and ∞. When considered on the Riemann sphere the clockwise winding about $0, w$ is equivalent to a counterclockwise winding about ∞.

$$
\begin{aligned}
&- \oint_{w} \int_{\Gamma_{0,w}} f(y)\langle o, \mathcal{Q}^{tau}(x)(Y_{r,s}\otimes I)(Y(v,y-w)m,w)n\rangle \mathrm{d}x\frac{\mathrm{d}y}{2\pi\mathsf{i}} \\
&- \oint_{0} \int_{\Gamma_{0,w}} f(y)\langle o, \mathcal{Q}^{\tau}(x)(Y_{r,s}\otimes I)(m,w)Y(v,y)n\rangle \mathrm{d}x\frac{\mathrm{d}y}{2\pi\mathsf{i}}. \qquad (5.32)
\end{aligned}
$$

Since $Y_{r,s}\otimes I$ is an intertwining operator, it satisfies (5.7) and hence

$$
\begin{aligned}
A(z) &= \oint_{0,w,x} \int_{\Gamma_{0,w}} f(y)\langle o, Y(v,y)\mathcal{Q}^{\tau}(x)(Y_{r,s}\otimes I)(m,w)n\rangle \mathrm{d}x\frac{\mathrm{d}y}{2\pi\mathsf{i}} \\
&\qquad - \int_{\Gamma_{0,w}} \oint_{0,w} f(y)\langle o, \mathcal{Q}^{\tau}(x)Y(v,y)(Y_{r,s}\otimes I)(m,w)n\rangle \mathrm{d}x\frac{\mathrm{d}y}{2\pi\mathsf{i}} \\
&= \oint_{0,w} \mathrm{Res}_{x=y}\, f(y)\langle o, \mathcal{Q}^{\tau}(x)Y(v,y)(Y_{r,s}\otimes I)(m,w)n\rangle \mathrm{d}y = 0, \qquad (5.33)
\end{aligned}
$$

where the residue vanishes (that is, the third equality holds) because $\mathcal{Q}^{\tau}(y)$ is a screening current, that is, because of (5.23). The second equality follows by carefully manipulating the contours and keeping track of winding numbers. These manipulations are illustrated in Figure 2. Thus $A(z) = 0$ and $\Phi^{\tau}_{r,s}$ satisfies (5.7). All that remains now is showing that $\Phi^{\tau}_{r,s}$ is non-zero by evaluating it on suitable vectors. Denote the highest weight vector of $\mathsf{S}_{r,s}$ by $v_{r,s}$ and its conformal weight by $h_{r,s}$. Further, consider the vectors

$$
\begin{aligned}
&v_{1,1}\otimes|p(a-b)-b\rangle \in \mathsf{S}_{1,1}\otimes\mathbb{F}_{p(a-b)-b+\mathbb{L}}, \quad v_{r,s}\otimes|p'(a-b)-b\rangle \in \mathsf{S}_{r,s}\otimes\mathbb{F}_{p(a-b)-b+\mathbb{L}}, \\
&v_{r,s+\tau 1}\otimes\langle p(a-b)-b| \in \mathsf{S}'_{r,s+\tau 1}\otimes\mathbb{F}'_{p(a-b)-b+\mathbb{L}}. \qquad (5.34)
\end{aligned}
$$

Then

$$
\big\langle v_{r,s+\tau 1}\otimes\langle p(a-b)-b|, \Phi^{\tau}_{r,s}(v_{1,1}\otimes|p(a-b)-b\rangle, w)v_{r,s}\otimes|p'(a-b)-b\rangle\big\rangle
$$

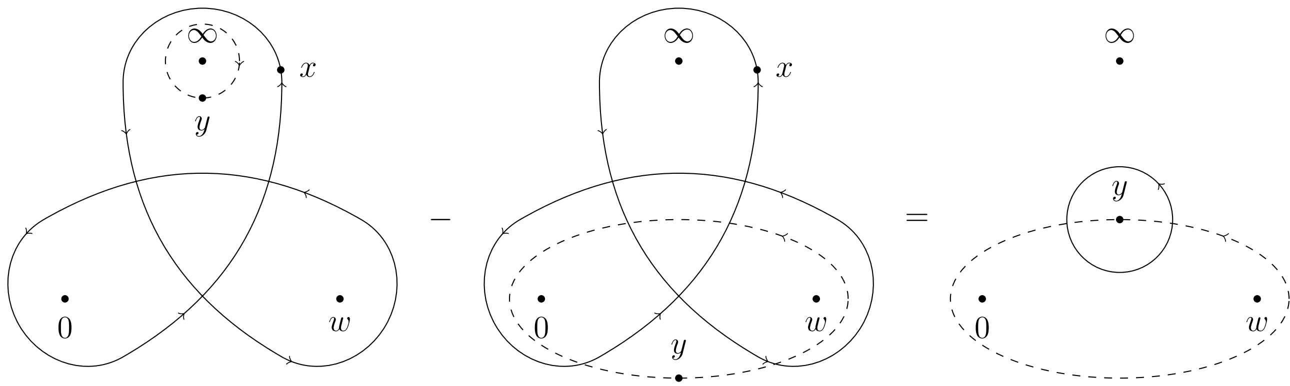

FIGURE 2. By taking the difference of the Pochhammer contours we see that the windings about 0, w and ∞ cancel out and only a circle about y is left.

$$\begin{aligned}
&= \int_{\Gamma_{0,w}} w^{(1+p+p')\left(1-\frac{u}{2v}\right)} x^{\left(\frac{u}{2v}-1\right)(p'+1)+h_{r,s+\tau 1}-h_{1,2}-h_{r,s}} (x-w)^{\left(\frac{u}{2v}-1\right)(p+1)} \mathrm{d}x \\
&= w^{\frac{-1-\tau r+\tau s\frac{u}{v}}{2}} e^{\pi\mathrm{i}\left(\frac{u}{2v}-1\right)(1+p)} \int_{\Gamma_{0,1}} y^{(1+p')\left(\frac{u}{2v}-1\right)+\frac{1-\tau r-\frac{u}{v}(1-\tau s)}{2}} (1-y)^{\left(\frac{u}{2v}-1\right)(1+p)} \mathrm{d}y \\
&= w^{\frac{-1-\tau r+\tau s\frac{u}{v}}{2}} e^{\pi\mathrm{i}\left(\frac{u}{2v}-1\right)(1+p)} \left(1-e^{2\pi\mathrm{i}\left((1+p')\left(\frac{u}{2v}-1\right)+\frac{1-\tau r-\frac{u}{v}(1-\tau s)}{2}\right)}\right) \\
&\quad \left(1-e^{2\pi\mathrm{i}\left(\left(\frac{u}{2v}-1\right)(1+p)\right)}\right) \frac{\Gamma\left(\frac{u}{2v}(1+p')-p'+\frac{1-\tau r-\frac{u}{v}(1-\tau s)}{2}\right)\Gamma\left(\frac{u}{2v}(1+p)-p\right)}{\Gamma\left(\frac{u}{2v}(1+p+p')-p-p'+\frac{1-\tau r-\frac{u}{v}(1-\tau s)}{2}\right)} \neq 0, \qquad (5.35)
\end{aligned}$$

where we used the following well known relation between Pochhammer contours and and the beta function $B(x,y)$.

$$\begin{aligned}
B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \Re(x), \Re(y) > 0, \\
&\int_{\Gamma_{0,1}} t^{x-1}(1-t)^{y-1}\mathrm{d}t = \left(1-e^{2\pi\mathrm{i}x}\right)\left(1-e^{2\pi\mathrm{i}y}\right)B(x,y). \qquad (5.36)
\end{aligned}$$

Hence $\Phi^{\tau}_{r,s}$ is non-zero. □

Note that Proposition 5.5 proves the lower bound on the dimensions of intertwining operator spaces in the first line of (3.21) of Lemma 3.6. Thus, combining Proposition 5.4, Proposition 5.5, Proposition 4.15 and Theorem 3.3 we obtain a proof for Lemma 3.6 and equivalently, a proof for Theorem 3.5. This settles the semisimple fusion rules, namely the fusion product decomposition formula (2.21) in Conjecture 2.9.

6. Intertwining operators and rigidity

The purpose of this section is to prove that the simple projective modules $E_{\mu;1,1}$ are rigid. To do this we will need to construct the projective cover of the tensor unit, a surjective intertwining operator taking values in this projective cover and a candidate for the evaluation map.

6.1. **Constructing the projective cover of the tensor unit.** In this subsection, we construct the projective cover $\sigma^{-1}P_{u-1,v-1}$ of the tensor unit, that is, $\mathsf{A}_1(u,v)$ as a module over itself, by constructing an indecomposable module which satisfies the second non-split exact sequence of (2.18) with $r=u-1$ and $\ell=1$. This projective module will be required to show that the simple members of the family of modules $E_{\mu;1,1}$ are rigid. This projective module will also turn out to be the first summand in the conjectured decomposition of the fusion product (2.22), when $r = 1$. The construction presented here is similar to that of [3, Prop 9.5] in that we consider a sum of $\mathsf{M}(u,v)\otimes \mathsf{L}\big(\frac{u}{v}\big)$ modules and then twist the action of $\mathsf{M}(u,v)\otimes\mathsf{L}\big(\frac{u}{v}\big)$ by screening currents. The details of how the twisting is done here will differ from [3], yet result in isomorphic indecomposable modules. The calculations in [3] involve taking residues of screening currents, while our considerations here require us to integrate over Pochhammer contours.

Recall the screening currents $\mathcal{Q}^{\pm}$ of Part 3 of Theorem 5.2 and set

$$\mathbb{H}^0 = \mathsf{S}_{1,1}\otimes \mathbb{F}_{[-2b]} \cong \sigma^{-1}E^{-}_{1,1},\qquad \mathbb{H}^1 = \mathsf{S}_{1,1}\otimes\mathbb{F}_0 \cong \sigma E^{-}_{u-1,v-1},\qquad \mathbb{H} = \mathbb{H}^0\oplus\mathbb{H}^1, \tag{6.1}$$

where we recall that that $\sigma^{-1}E^{-}_{1,1}$ and $\sigma E^{-}_{u-1,v-1}$ satisfy the non-split exact sequences

$$\begin{aligned}&0\to\sigma^{-1}(D^{-}_{1,1})\to\sigma^{-1}E^{-}_{1,1}\to\sigma^{-1}D^{+}_{u-1,v-1}\cong L_1\to 0,\\ &0\to\sigma D^{-}_{u-1,v-1}\cong L_1\to\sigma E^{-}_{u-1,v-1}\to\sigma(D^{+}_{1,1})\to 0.\end{aligned} \tag{6.2}$$

We denote

$$\big\|\theta^i\big\rangle = v_{1,1}\otimes|2(i-1)b\rangle\in\mathbb{H}^i,\quad i=0,1 \tag{6.3}$$

and the corresponding dual vector in the contragredient module $(\mathbb{H})'$ by $\langle\theta^i\|$. Consider the four point function

$$\big\langle\theta^1\big\|\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\big\|\theta^0\big\rangle = \big\langle v_{1,1}, I^{-}_{1,2}(v_{1,2},z_1)I^{+}_{1,1}(v_{1,2},z_2)v_{1,1}\big\rangle\langle 0|I_{-b}(|b\rangle,z_1)I_{-2b}(|b\rangle,z_2)|{-2b}\rangle, \tag{6.4}$$

where we have factorised the screening currents $\mathcal{Q}^-(z_1)$ and $\mathcal{Q}^+(z_2)$ into their Virasoro and Heisenberg parts. Note that the Virasoro intertwining operators can be normalised such that

$$\big\langle v_{1,1}, I^{-}_{1,2}(v_{1,2},z_1)I^{+}_{1,1}(v_{1,2},z_2)v_{1,1}\big\rangle = (z_1-z_2)^{-2h_{1,2}} = (z_1-z_2)^{1-\frac{3u}{2v}}, \tag{6.5}$$

and the Heisenberg intertwining operators such that

$$\langle 0|I_{-b}(|b\rangle,z_1)I_{-2b}(|b\rangle,z_2)|{-2b}\rangle = z_1^{\frac{u}{v}-2}z_2^{\frac{u}{v}-2}(z_1-z_2)^{1-\frac{u}{2v}}. \tag{6.6}$$

With these choices of normalisation, we therefore find

$$\big\langle\theta^1\big\|\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\big\|\theta^0\big\rangle = z_1^{-2}\left(\frac{z_2}{z_1}\right)^{\frac{u}{v}-2}\left(1-\frac{z_2}{z_1}\right)^{-2\frac{u}{v}+2}. \tag{6.7}$$

Following the arguments of [73, 74] on choosing appropriate contours for integrating screening currents, we introduce new variables $(z,y)\in\mathbb{C}^*\times(\mathbb{C}\setminus\{0,1\})$ and set $z_1 = z, z_2 = zy$. Then in these new variables, we have

$$\big\langle\theta^1\big\|\mathcal{Q}^-(z)\mathcal{Q}^+(zy)\big\|\theta^0\big\rangle = z^{-2}y^{\frac{u}{v}-2}(1-y)^{2-2\frac{u}{v}} = \frac{G(y)}{z^2y^2},\qquad G(y) = y^{\frac{u}{v}}(1-y)^{2-2\frac{u}{v}}. \tag{6.8}$$

Let $C_{z=0}$ be the homology class of a counter-clockwise circle about origin $z=0$ rescaled by $(2\pi\mathsf{i})^{-1}$ (so that $\int_{C_{z=0}}\frac{\mathrm{d}z}{z} = 1$) and let $\Gamma_{0,1}$ be the Pochhammer contour about $y=0$ and $y=1$. Then the contour

$$\Gamma = C_{z=0}\times\Gamma_{0,1} \tag{6.9}$$

is a twisted cycle with respect to the multivaluedness of $\langle\theta^1\|\mathcal{Q}^-(z)\mathcal{Q}^+(zy)\|\theta^0\rangle$ (or $G(y)$). That is, for any $E,F\in\mathbb{C}[y^{\pm1},(1-y)^{-1},z^{\pm1}]$, we have

$$0 = \int_\Gamma \mathrm{d}\big(\big\langle\theta^1\big\|\mathcal{Q}^-(z)\mathcal{Q}^+(zy)\big\|\theta^0\big\rangle(E\mathrm{d}z+F\mathrm{d}y)\big), \tag{6.10}$$

where d is the exterior derivative with respect to the variables (z,y). Note that

$$\big\langle v', \mathcal{Q}^+(z)\mathcal{Q}^-(zy)v\big\rangle\in G(y)\mathbb{C}\big[y^{\pm1},(1-y)^{-1},z^{\pm1}\big] \tag{6.11}$$

for any $v \in \mathbb{H}^0$, $v' \in (\mathbb{H}^1)'$. Then we can define the following operators

$$\mathcal{Q}^{[2]}(z) = \int_{\Gamma_{0,1}} \mathcal{Q}^-(z)\mathcal{Q}^+(zy)z\mathrm{d}y \ : \ \mathbb{H}^0 \to \mathbb{H}^1[[z,z^{-1}]],$$
$$\mathcal{Q}^{[2]} = \int_{\Gamma} \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathrm{d}z_1\mathrm{d}z_2 \ : \ \mathbb{H}^0 \to \mathbb{H}^1. \tag{6.12}$$

These operators are non trivial, since, for example, we have

$$\int_\Gamma \langle \theta^1 \| \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2) \| \theta^0 \rangle \mathrm{d}z_1\mathrm{d}z_2 = \int_\Gamma y^{\frac{u}{v}-2}(1-y)^{2-2\frac{u}{v}}\mathrm{d}y\frac{\mathrm{d}z}{z} = \int_{\Gamma_{0,1}} y^{\frac{u}{v}-2}(1-y)^{-2\frac{u}{v}+2}\mathrm{d}y$$
$$= \big(1-e^{2\pi\mathrm{i}\frac{u}{v}}\big)\big(1-e^{-2\pi\mathrm{i}2\frac{u}{v}}\big)\frac{\Gamma(\frac{u}{v}-1)\Gamma(3-2\frac{u}{v})}{\Gamma(1-\frac{u}{v})} \neq 0. \tag{6.13}$$

Since Γ is a twisted cycle and the $\mathcal{Q}^\pm(z)$ are screening currents, we obtain the following proposition.

Proposition 6.1. *The operator $\mathcal{Q}^{[2]} : \mathbb{H}^0 \to \mathbb{H}^1$ commutes with the $\mathsf{A}_1(u,v)$-action (that is, it is an $\mathsf{A}_1(u,v)$-module homomorphism). Thus $\mathcal{Q}^{[2]}$ defines a non trivial screening operator.*

Proof. We verify that $\mathcal{Q}^{[2]}$ commutes with the action of $\mathsf{A}_1(u,v)$ generators and consider the f-generator first. Let $R^\pm(z,w)$ denote the right-hand side of the operator product expansion relation (5.25) for the f-generator with the screening current without the total derivative, that is,

$$R^\pm(z,w) = \frac{I_\mu(|\tau\rangle,z)\partial_z I^\pm_{r,s}(v_{1,2},z) + I^\pm_{r,s}(v_{1,2},z)I_\mu\big(\big(\frac{u}{u-2v}b_{-1} + 2\frac{v-u}{u-2v}a_{-1}\big)|\tau\rangle,z\big)}{z-w}$$
$$\frac{-\big(\frac{u}{v}-1\big)I^\pm_{r,s}(v_{1,2},z)I_\mu(|\tau\rangle,z)}{(z-w)^2} \tag{6.14}$$

Then from the definition (6.12) of $\mathcal{Q}^{[2]}(z)$, we have

$$[Y(\psi(f),w),\mathcal{Q}^{[2]}] = Y(\psi(f),w)\mathcal{Q}^{[2]}(z) \sim \int_{\Gamma_{0,1}} (\partial_z R^-(z,w))\mathcal{Q}^+(zy)z\mathrm{d}y$$
$$+ \int_{\Gamma_{0,1}} \mathcal{Q}^-(z)(\partial_{\tilde{z}} R^+(\tilde{z},w))|_{\tilde{z}=zy}z\mathrm{d}y. \tag{6.15}$$

Note that $\mathrm{d}z \wedge (z\mathrm{d}y) = \mathrm{d}z \wedge \mathrm{d}(zy)$. Then, from (6.15), we have

$$[\mathcal{Q}^{[2]}, Y(\psi(f),w)] = \mathrm{Res}_{z=w}\, Y(\psi(f),w)\mathcal{Q}^{[2]}(z)$$
$$= \int_{C_{z=w}}\int_{\Gamma_{0,1}} \mathrm{d}\Big(R^-(z,w)\mathcal{Q}^+(\tilde{z})\mathrm{d}\tilde{z} + \mathcal{Q}^-(z)R^+(\tilde{z},w)\mathrm{d}z\Big), \tag{6.16}$$

where $C_{z=w}$ is a counter clock-wise circle (scaled by $(2\pi\mathrm{i})^{-1}$ about w, and d is the exterior derivative with respect to the variables $(z,\tilde{z})$. Since $C_{z=w} \times \Gamma_{0,1}$ is a twisted cycle and the integrand is exact (6.16) must vanish. Thus $\mathcal{Q}^{[2]}$ commutes with the f-generator. For the the e and h-generators, note that the operator product expansions (5.25) are regular and hence the integral calculations analogous to those above vanish trivially. Thus $\mathcal{Q}^{[2]}$ commutes with the action of $\mathsf{A}_1(u,v)$. □

Consider the operator

$$\Delta_\epsilon(-;z) : \mathsf{A}_1(u,v) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0,\mathbb{H}^1)[\![z^{\pm1},\epsilon^{\pm1}]\!]$$
$$g \mapsto \frac{1}{\epsilon}\oint_z \mathcal{Q}^{[2]}(w)\mathsf{e}^{-\epsilon}Y(g,z)\mathsf{e}^{\epsilon}\mathrm{d}w \tag{6.17}$$

where $\mathsf{e}^{\pm\epsilon}$ is the Heisenberg weight shifting operator characterised by

$$[a_0, \mathsf{e}^{\pm\epsilon}] = \pm\frac{\epsilon}{2}\mathsf{e}^{\pm\epsilon}, \qquad [b_0, \mathsf{e}^{\pm\epsilon}] = \mp\frac{\epsilon}{2}\mathsf{e}^{\pm\epsilon},$$
$$[a_n, \mathsf{e}^{\pm\epsilon}] = [b_n, \mathsf{e}^{\pm\epsilon}] = 0,\ n \neq 0, \qquad \mathsf{e}^{\pm\epsilon}|\mu\rangle = \left|\mu \pm \epsilon\frac{v}{u-2v}(a-b)\right\rangle. \tag{6.18}$$

Note that the integral in (6.17) can also be expressed as the commutator

$$\Delta_\epsilon(g;z) = \frac{1}{\epsilon}[\mathcal{Q}^{[2]}, \mathsf{e}^{-\epsilon}Y(g,z)\mathsf{e}^{\epsilon}]. \tag{6.19}$$

Proposition 6.2. *Consider the operator $\Delta_\epsilon(-;z)$ defined above.*

(1) *For all $g \in \mathsf{A}_1(u,v)$, $\Delta_\epsilon(g;z) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0, \mathbb{H}^1)[\![z^{\pm1}, \epsilon]\!]$, that is, no negative powers of ϵ appear.*

(2) *For all $g \in \mathsf{A}_1(u,v)$, set $\Delta(g;z) = \Delta_\epsilon(g;z)|_{\epsilon=0}$. Then on the generators of $\mathsf{A}_1(u,v)$ and the conformal vector we have*

$$\Delta(e;z) = 0, \qquad \Delta(h;z) = 0,$$
$$\Delta(f;z) = -\frac{1}{z}\left[\mathcal{Q}^{[2]}, Y\left(b_{-1}\left|\frac{-2v}{u-2v}\right\rangle, z\right)\right] + \frac{1}{z^2}\frac{u-v}{2v}\left[\mathcal{Q}^{[2]}, Y\left(\left|\frac{-2v}{u-2v}\right\rangle, z\right)\right],$$
$$\Delta(\omega_{\frac{u-2v}{v}};z) = \frac{1}{z}\frac{v}{u-2v}[\mathcal{Q}^{[2]}, Y((a_{-1}-b_{-1})|0\rangle, z)]. \tag{6.20}$$

Proof. For any $v \in \mathbb{H}^0$, $u \in (\mathbb{H}^1)'$ and $g \in \mathsf{A}_1(u,v)$, the matrix element

$$\mathrm{Res}_{z=1}\langle u, \mathcal{Q}^{[2]}(z)\mathbf{e}^{-\epsilon}Y(g,1)\mathbf{e}^{\epsilon}v\rangle \in \epsilon\mathbb{C}[\epsilon] \tag{6.21}$$

is a power series in non-negative powers of ϵ, because the Heisenberg weight shift operators will only introduce non-negative powers by the relations (6.17). Next, recall from the proof of Proposition 6.1 that $\mathrm{Res}_{z=w}\,\mathcal{Q}^{[2]}(z)Y(g,w) = 0$ ($g \in \mathsf{A}_1(u,v)$), hence the power series (6.21) has vanishing constant term, that is,

$$\mathrm{Res}_{z=1}\langle u, \mathcal{Q}^{[2]}(z)\mathbf{e}^{-\epsilon}Y(g,1)\mathbf{e}^{\epsilon}v\rangle \in \epsilon\mathbb{C}[\epsilon]. \tag{6.22}$$

Thus we have $\Delta_\epsilon(g;z) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0, \mathbb{H}^1)[\![z^{\pm1}, \epsilon]\!]$ and hence $\Delta(g;z)$ is well defined.

We evaluate $\Delta(-;z)$ on generators and the conformal vector by direct computation. First note that

$$\mathsf{e}^{-\epsilon}Y(a_{-1}|0\rangle, z)\mathsf{e}^{\epsilon} = Y(a_{-1}|0\rangle, z) + \frac{\epsilon/2}{z}, \qquad \mathsf{e}^{-\epsilon}Y(b_{-1}|0\rangle, z)\mathsf{e}^{\epsilon} = Y(b_{-1}|0\rangle, z) + \frac{\epsilon/2}{z},$$
$$\mathsf{e}^{-\epsilon}Y\left(\left|\frac{\pm2v}{u-2v}(a-b)\right\rangle, z\right)\mathsf{e}^{\epsilon} = Y\left(\left|\frac{\pm2v}{u-2v}(a-b)\right\rangle, z\right). \tag{6.23}$$

For images under the free field realisation ψ of Part 1 of Theorem 5.2 we therefore have

$$\mathsf{e}^{-\epsilon}Y(\psi(e_{-1}\mathbf{1}), z)\mathsf{e}^{\epsilon} = Y(\psi(e_{-1}\mathbf{1}), z), \qquad \mathsf{e}^{-\epsilon}Y(\psi(h_{-1}\mathbf{1}), z)\mathsf{e}^{\epsilon} = Y(\psi(h_{-1}\mathbf{1}), z) + \frac{\epsilon}{z},$$
$$\mathsf{e}^{-\epsilon}Y(\psi(f_{-1}\mathbf{1}), z)\mathsf{e}^{\epsilon} = Y(\psi(f_{-1}\mathbf{1}), z) - \frac{\epsilon}{z}Y\left(b_{-1}\left|\frac{-2v}{u-2v}\right\rangle, z\right) - \frac{\epsilon^2/4}{z^2}Y\left(\left|\frac{-2v}{u-2v}\right\rangle, z\right)$$
$$+ \frac{u-v}{2v}\frac{\epsilon}{z^2}Y\left(\left|\frac{-2v}{u-2v}\right\rangle, z\right),$$
$$\mathsf{e}^{-\epsilon}Y\left(\psi(\omega_{\frac{u-2v}{v}}), z\right)\mathsf{e}^{\epsilon} = Y\left(\psi(\omega_{\frac{u-2v}{v}}), z\right) + \frac{\epsilon}{z}\frac{v}{u-2v}Y((a_{-1}-b_{-1})|0\rangle, z) + \frac{\epsilon/2}{z^2}. \tag{6.24}$$

The evaluations of $\Delta(-,z)$ are therefore just the commutator of $\mathcal{Q}^{[2]}$ with the terms linear in ϵ above. □

Proposition 6.3. *Consider the $\mathsf{M}(u,v)\otimes\mathsf{L}\big(\frac{u}{v}\big)$ module $\mathbb{H}$ and denote the $\mathsf{M}(u,v)\otimes\mathsf{L}\big(\frac{u}{v}\big)$-action by $Y_{\mathbb{H}}$ (which is also an $\mathsf{A}_1(u,v)$ action by restriction). For $d\in\mathbb{C}^\times$ define the linear operator $\widetilde{Y}_d(-,z):\mathsf{A}_1(u,v)\to\mathrm{Hom}_{\mathbb{C}}(\mathbb{H},\mathbb{H})[\![z^{\pm1}]\!]$ by*

$$\widetilde{Y}_d(-,z)\Big|_{\mathbb{H}^0}=Y_{\mathbb{H}}(-,z)+d\cdot\Delta(-;z),\qquad \widetilde{Y}_d(-,z)\Big|_{\mathbb{H}^1}=Y_{\mathbb{H}}(-,z).\tag{6.25}$$

Then $\widetilde{Y}_d$ is an action of $\mathsf{A}_1(u,v)$ on $\mathbb{H}$. Further, $(\mathbb{H},\widetilde{Y}_d)$ is isomorphic to the projective cover $\sigma^{-1}P_{u-1,v-1}$ of $L_1\cong\mathsf{A}_1(u,v)$.

The construction above is similar to [3, Prop 9.5], but our choice of constructing will turn out to be more convenient for the intertwining operators to be considered below. Note that the extension group $\mathrm{Ext}^1\big(\sigma E^-_{u-1,v-1},\sigma^{-1}E^-_{1,1}\big)$ is one dimensional. The parameter d above hence reflects this one degree of freedom.

Proof. To show that $\widetilde{Y}_d$ defines an $\mathsf{A}_1(u,v)$-action, we need to show that $\widetilde{Y}_d(\mathbf{1},z)=\mathrm{id}_{\mathbb{H}}$, that $\widetilde{Y}_d(w,z)h$ has at most a finite order pole for any $w\in\mathsf{A}_1(u,v)$ and any $h\in\mathbb{H}$, and that $\widetilde{Y}_d$ satisfies the Jacobi identity. The first two conditions hold by construction and so we only need to consider the Jacobi identity. Note that the factor d can be absorbed by rescaling the entire space $\mathbb{H}^1$ by d, hence without loss of generality it is sufficient to verify the Jacobi identity for $d=1$. We thus set $\widetilde{Y}=\widetilde{Y}_1$. Let $f(z)\in\mathbb{C}[z^{\pm1},(z-w)^{-1}]$, $\phi\in\mathbb{H}$, $\psi\in\mathbb{H}'$ and $a,b\in\mathsf{A}_1(u,v)$, then we need to show that the following sum of integrals vanishes.

$$\oint_{0,w}f(z)\Big\langle\psi,\widetilde{Y}(a,z)\widetilde{Y}(b,w)\phi\Big\rangle\mathrm{d}z-\oint_{w}f(z)\Big\langle\psi,\widetilde{Y}(Y(a,z-w)b,w)\phi\Big\rangle\mathrm{d}z\\ -\oint_{0}f(z)\Big\langle\psi,\widetilde{Y}(b,w)\widetilde{Y}(a,z)\phi\Big\rangle\mathrm{d}z\tag{6.26}$$

Note that for $\phi\in\mathbb{H}^1,\psi\in\mathbb{H}^{1\prime}$ or $\phi\in\mathbb{H}^0,\psi\in\mathbb{H}^{0\prime}$ the above expression reduces to the Jacobi identity for the action $Y_{\mathbb{H}}$ and hence vanishes and that it is trivially zero for $\phi\in\mathbb{H}^1,\psi\in\mathbb{H}^{0\prime}$. So we only need to consider the case $\phi\in\mathbb{H}^0,\psi\in\mathbb{H}^{1\prime}$, where the above expression specialises to

$$\oint_{0,w}f(z)\langle\psi,(\Delta(a,z)Y_{\mathbb{H}}(b,w)+Y_{\mathbb{H}}(a,z)\Delta(b,w))\phi\rangle\mathrm{d}z\\ -\oint_{w}f(z)\langle\psi,\Delta(Y(a,z-w)b,w)\phi\rangle\mathrm{d}z\\ -\oint_{0}f(z)\langle\psi,(Y_{\mathbb{H}}(b,w)\Delta(a,z)+\Delta(b,w)Y_{\mathbb{H}}(a,z))\phi\rangle\mathrm{d}z.\tag{6.27}$$

The first and third integrands above can be simplified by noting

$$\Delta(a,z)Y_{\mathbb{H}}(b,w)+Y_{\mathbb{H}}(a,z)\Delta(b,w)=\frac{1}{\epsilon}\oint_{z,w}\int_{\Gamma_{0,1}}\mathcal{Q}^-(x)\mathcal{Q}^+(xy)\mathsf{e}^{-\epsilon}Y_{\mathbb{H}}(a,z)Y_{\mathbb{H}}(b,w)\mathsf{e}^{\epsilon}\mathrm{d}x\mathrm{d}y\Bigg|_{\epsilon=0}.\tag{6.28}$$

Let $\{\gamma_i\}$ be a basis of $\mathbb{H}^1$ with dual basis $\{\gamma^i\}$, then (6.27) is equal to

$$\oint_{z,w}\int_{\Gamma_{0,1}}\sum_i\big\langle\psi,\mathcal{Q}^-(x)\mathcal{Q}^+(xy)\gamma_i\big\rangle\Bigg(\oint_{0,w}f(z)\big\langle\gamma^i,Y_{\mathbb{H}}(a,z)Y_{\mathbb{H}}(b,w)\phi\big\rangle\mathrm{d}z\\ -\oint_{w}f(z)\big\langle\gamma^i,Y_{\mathbb{H}}(Y(a,z-w)b,w)\phi\big\rangle\mathrm{d}z$$

$$- \oint_0 f(z)\langle \gamma^i, Y_{\mathbb{H}}(b,w)Y_{\mathbb{H}}(a,z)\phi\rangle \mathrm{d}z\Big), \tag{6.29}$$

which vanishes, because $Y_{\mathbb{H}}$ satisfies the Jacobi identity. Further, from the formulae in Part 2 of Proposition 6.2 we see that the h_0-grading remains unchanged, while the L_0 operator acquires a nilpotent part but its generalised eigenvalues do not change, hence $(\mathbb{H}, \widetilde{Y})$ is a weight module. To conclude that $(\mathbb{H}, \widetilde{Y})$ is isomorphic to $\sigma^{-1}P_{u-1,v-1}$ note that $\mathbb{H}^1$ is a submodule isomorphic to $\sigma E^-_{u-1,v-1}$, that the quotient of $(\mathbb{H}, \widetilde{Y})$ by $\mathbb{H}^1$ is isomorphic to $\mathbb{H}^0$ and thus $\sigma^{-1}E^-_{1,1}$ and that $(\mathbb{H}, \widetilde{Y})$ is indecomposable due to Δ mapping between $\mathbb{H}^0$ and $\mathbb{H}^1$. Hence $(\mathbb{H}, \widetilde{Y})$ satisfies the characterising non-split exact sequence (2.18) of $\sigma^{-1}P_{u-1,v-1}$.

□

6.2. **Constructing logarithmic intertwining operators.** This section assumes some familiarity with twisted cycles and twisted De Rham cohomology, and we refer readers unfamiliar with these notions to Aomoto and Kita's book on hypergeometric functions [8] for an comprehensive account or to [74, Sec 3.2] for a short summary.

Let $w, u \in \mathbb{R}_{\geq 0}$, $w > u$ and consider the complex manifold $Z_{w,u} = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq z_2,\ z_1, z_2 \neq w,\ z_1, z_2 \neq u\}$. For $\alpha, \beta, \gamma \in \mathbb{C}$, let

$$\mathcal{U}_w(\alpha,\beta,\gamma,z) = \prod_{i=1}^{2} z_i^{\alpha} \prod_{j=1}^{2} (z_j - w)^{\beta}(z_1 - z_2)^{\gamma}(z_2 - z_1)^{\gamma}. \tag{6.30}$$

be a multivalued function on $Z_{w,0}$. The logarithmic derivative of $\mathcal{U}_w(\alpha,\beta,\gamma,z)$

$$\omega_w(\alpha,\beta,\gamma,z) = \mathrm{d}\ \log\mathcal{U}_w(\alpha,\beta,\gamma,z) = \sum_{i=1,2}\Big(\frac{\alpha}{z_i} + \frac{\beta}{z_i - w}\Big)\mathrm{d}z_i + 2\gamma\frac{\mathrm{d}z_1 - \mathrm{d}z_2}{z_1 - z_2} \tag{6.31}$$

defines the twisted differential $\nabla_{\omega_w} = \mathrm{d} + \omega_w(\alpha,\beta,\gamma,z)\wedge$. Let $\mathcal{L}_w(\alpha,\beta,\gamma)$ be the local system defined by the local solutions of $\nabla_{\omega_w} g(z_1,z_2) = 0$, and let $\mathcal{L}^\vee_w(\alpha,\beta,\gamma) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{L}_w(\alpha,\beta,\gamma), \mathbb{C})$ be the dual local system. The twisted homology groups with coefficients in $\mathcal{L}^\vee_w(\alpha,\beta,\gamma)$ are denoted by $H_p(Z_{w,0}, \mathcal{L}^\vee_w(\alpha,\beta,\gamma))$ and the twisted cohomology groups by

$$H^p(Z_{w,0}, \mathcal{L}_w(\alpha,\beta,\gamma)) = \mathrm{Hom}_{\mathbb{C}}(H_p(Z_{w,0}, \mathcal{L}^\vee_w(\alpha,\beta,\gamma)), \mathbb{C}). \tag{6.32}$$

It is known that the twisted cohomology groups are isomorphic to the twisted de Rham cohomology groups (see [8, Sec 2])

$$H^p(Z_{w,0}, \mathcal{L}_w(\alpha,\beta,\gamma)) \simeq H^p(Z_{w,0}, \nabla_{\omega_w}). \tag{6.33}$$

By permuting the variables z_1, z_2 there is a natural action of the symmetric group. We denote the subspace that is invariant this action by $H^2(Z_{w,0}, \nabla_{\omega_w})^{\mathfrak{S}}$ and the subspace on which the transposition $(1,2)$ acts by -1 by $H^2(Z_{w,0}, \nabla_{\omega_w})^{\mathfrak{A}}$.

Lemma 6.4. *Consider the hyperplane arrangement*

$$\begin{aligned}
\mathcal{H} &= \bigcup_{i=1}^{2}\{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid i(\alpha + (i-1)\gamma) \in \mathbb{Z}\} \\
&\quad \cup \bigcup_{j=1}^{2}\{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid j(\beta + (j-1)\gamma) \in \mathbb{Z}\} \cup \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 2\gamma \in \mathbb{Z}\}, \\
\mathcal{G} &= \mathcal{H} \cup \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \alpha + \beta + 2\gamma \notin \mathbb{Z}\},
\end{aligned} \tag{6.34}$$

and let $(\alpha,\beta,\gamma) \in \mathbb{C}^3 \backslash \mathcal{G}$. The skew symmetric 2-form $\mathcal{U}_w(\alpha,\beta,\gamma,z)$ represents a non-zero cohomology class within the cohomology group $H^2(Z_{w,0}, \nabla_{\omega_w})^{\mathfrak{A}}$ and $\dim H^2(Z_{w,0}, \nabla_{\omega_w})^{\mathfrak{A}} = 1$. Further, for *any symmetric Laurent polynomial in* $F(z_1,z_2)\in\mathbb{C}[z_1^{\pm1},z_2^{\pm1},(z_1-w)^{-1},(z_2-w)^{-1}]^{\mathfrak{S}}\left[(z_1-z_2)^{-2}\right]$ *there exist a rational function* $c(\alpha,\beta,\gamma)\in\mathbb{C}(\alpha,\beta,\gamma)$ *such that*

$$\mathcal{U}_w(\alpha,\beta,\gamma,z_1,z_2)F(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2=c(\alpha,\beta,\gamma)\mathcal{U}_w(\alpha,\beta,\gamma,z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2+\mathrm{d}(\cdots),\tag{6.35}$$

where $\mathrm{d}(\cdots)$ *denotes a total derivative, that is, the two form* $\mathcal{U}_w(\alpha,\beta,\gamma,z_1,z_2)F(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2$ *is cohomologous to a scalar multiple of* $\mathcal{U}_w(\alpha,\beta,\gamma,z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2$*. Finally, the rational function* $c(\alpha,\beta,\gamma)$ *is holomorphic on* $\mathbb{C}^3\setminus\mathcal{H}$*, that is, all of its singularities lie in* $\mathcal{H}$*.*

By [8, Sec 2] Lemma 6.4 is known for generic values of α,β,γ, however, as we shall see below, here we are primarily interested in the case $\alpha+\beta+\gamma=-1$, which is not generic.

Proof. By [8, Eq. (2.53) and Thm 2.5] local systems cohomology can be identified with twisted de Rham cohomology of rational forms. Hence we restrict our attention to rational forms. Fix $(\alpha,\beta,\gamma)\in\mathbb{C}^3\setminus\mathcal{G}$ and denote $\mathcal{U}_w=\mathcal{U}_w(\alpha,\beta,\gamma,z_1,z_2)$. Consider the 1-form $\mathcal{U}_w((z_1-w)(z_1^kz_2^l)\mathrm{d}z_2-(z_2-w)(z_2^kz_1^l)\mathrm{d}z_1)$, $k,l\in\mathbb{Z}$, $k\geq l$ and compute its exterior derivative.

$$\begin{aligned}
&\mathrm{d}\big(\mathcal{U}_w((z_1-w)(z_1^kz_2^l)\mathrm{d}z_2-(z_2-w)(z_2^kz_1^l)\mathrm{d}z_1)\big)\\
&\quad=(\alpha+\beta+2\gamma+k+1)(z_1^kz_2^l+z_2^kz_1^l)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2\\
&\qquad-w(\alpha+2\gamma+k)(z_1^{k-1}z_2^l+z_2^{k-1}z_1^l)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2\\
&+2\gamma\left(\sum_{i=1}^{k-l-1}z_1^{i+l}z_2^{k-i}-w\sum_{j=1}^{k-l-2}z_1^{j+l}z_2^{k-1-j}\right)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2, && k\geq l+2.\\
&\mathrm{d}\big(\mathcal{U}_w((z_1-w)(z_1^{l+1}z_2^l)\mathrm{d}z_2-(z_2-w)(z_2^{l+1}z_1^l)\mathrm{d}z_1)\big)\\
&\quad=(\alpha+\beta+2\gamma+l+2)(z_1^{l+1}z_2^l+z_2^{l+1}z_1^l)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2\\
&\qquad-w2(\alpha+\gamma+l+1)z_!^lz_2^l\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2, && k=l+1.\\
&\mathrm{d}\big(\mathcal{U}_w((z_1-w)(z_1^lz_2^l)\mathrm{d}z_2-(z_2-w)(z_2^lz_1^l)\mathrm{d}z_1)\big)\\
&\quad=2(\alpha+\beta+\gamma+l+1)z_1^lz_2^l\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2\\
&\qquad-w(\alpha+l)(z_1^{l-1}z_2^l+z_2^{l-1}z_1^l)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2, && k=1.
\end{aligned}\tag{6.36}$$

Note that with the exception of the coefficient on the second to last line, the coefficients in the above identities cannot vanish because of the assumptions on α,β,γ. The above exterior derivative implies that a skew symmetric 2-form of the form

$$(z_1^nz_2^m+z_2^nz_1^m)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2,\qquad m,n\in\mathbb{Z}\tag{6.37}$$

is cohomologous to linear combinations of such two forms, where the difference $|m-n|$ of the exponents m,n has been reduced by 1 or more. This procedure can be iterated until the exponents are equal. Once the exponents are equal we can apply the relations generated from (6.36) twice to shift both exponents by 1, that is,

$$\begin{aligned}
&(\alpha+\beta+\gamma+n+2)z_1^{n+1}z_2^{n+1}\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2\\
&\qquad\qquad=w^2\frac{(\alpha+\gamma+n+1)(\alpha+n+1)}{\alpha+\beta+2\gamma+n+2}\mathcal{U}_wz_1^nz_2^n\mathrm{d}z_1\wedge\mathrm{d}z_2+\mathrm{d}(\cdots).
\end{aligned}\tag{6.38}$$

For $\alpha+\beta+\gamma\notin\mathbb{Z}$ this allows one to conclude that any 2-form of the form (6.37) is cohomologous to a scalar multiple of $\mathcal{U}_w$. If $\alpha+\beta+\gamma=p\in\mathbb{Z}$, then $z_1^{n+1}z_2^{n+1}\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2$ is cohomologous to a scalar multiple of $\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2$ for $n\geq -p-1$ and to 0 for $n\leq -p-2$ and so the same conclusion holds. Further since $\mathcal{U}_w$ is symmetric under interchanging α,β and factors of z_i with z_i-w analogous arguments to those above imply that any 2-form of the form

$$((z_1-w)^n(z_2-w)^m+(z_2-w)^n(z_1-w)^m)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2,\qquad m,n\in\mathbb{Z} \tag{6.39}$$

is also cohomologous to a scalar multiple of $\mathcal{U}_w$. Next we consider more general skew symmetric 2-forms made up of linear combinations of 2-forms of the form

$$\left(\frac{z_1^k z_2^l}{(z_1-w)^n(z_2-w)^m}+\frac{z_2^k z_1^l}{(z_2-w)^n(z_1-w)^m}\right)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2,\qquad m,n\in\mathbb{Z}_{\geq 0},\quad k,l\in\mathbb{Z}, \tag{6.40}$$

however, using partial fractal decomposition these can be reduced linear combinations of (6.37) and of

$$\begin{aligned}&\left(\frac{z_1^k}{(z_2-w)^m}+\frac{z_2^k}{(z_1-w)^m}\right)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2,\\ &\left(\frac{1}{z_1^k(z_2-w)^m}+\frac{1}{z_2^k(z_1-w)^m}\right)\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2,\qquad k,m\in\mathbb{Z}_{\geq 0}.\end{aligned} \tag{6.41}$$

So we consider these two new cases. The first case of (6.41) reduced to the case (6.39) after noting

$$\frac{z_i^k}{(z_j-w)^m}=\frac{(z_i-w+w)^k}{(z_j-w)^m}=\frac{1}{(z_j-w)^m}\sum_{p=0}^{k}\binom{k}{p}(z_i-w)^p w^{k-p}. \tag{6.42}$$

The final case can be reduced to the case (6.37) after noting the relation

$$\begin{aligned}&\left(\frac{1}{z_1^k(z_2-w)^m}+\frac{1}{z_2^k(z_1-w)^m}\right)\mathcal{U}_w(\alpha,\beta,\gamma,z)\mathrm{d}z_1\wedge\mathrm{d}z_2\\ &=\left(\frac{(z_1-w)^m}{z_1^k}+\frac{(z_2-w)^m}{z_2^k}\right)\mathcal{U}_w(\alpha,\beta-m,\gamma,z)\mathrm{d}z_1\wedge\mathrm{d}z_2\\ &=\left(\sum_{j\geq 0}\binom{m}{j}z_1^{j-k}(-w)^{m-j}+\sum_{j\geq 0}\binom{m}{j}z_2^{j-k}(-w)^{m-j}\right)\mathcal{U}_w(\alpha,\beta-m,\gamma,z)\mathrm{d}z_1\wedge\mathrm{d}z_2\end{aligned} \tag{6.43}$$

and that (α,β,γ) satisfies the assumptions of the lemma if and only if $(\alpha,\beta+m,\gamma)$ does. Finally, the most general form that a skew symmetric 2-form can take is a linear combination of 2-forms of the form

$$\frac{F(z_1,z_2)}{(z_1-z_2)^{2n}}\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2,\quad n\in\mathbb{Z}_{\geq 0},\ F(z_1,z_2)\in\mathbb{C}\big[z_1^{\pm},z_2^{\pm},(z_1-w)^{-1},(z_2-w)^{-1}\big]. \tag{6.44}$$

The factor of $(z_1-z_2)^{2n}$ can be absorbed in to the multivalued function $\mathcal{U}_w$ so that, from the reasoning above, we can conclude the existence of a rational functions $c(\alpha,\beta,\gamma)$ satisfying

$$\begin{aligned}\frac{F(z_1,z_2)}{(z_1-z_2)^{2n}}\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2&=F(z_1,z_2)\mathcal{U}_w(\alpha,\beta,\gamma-2n,z)\mathrm{d}z_1\wedge\mathrm{d}z_2\\ &=c(\alpha,\beta,\gamma)\mathcal{U}_w(\alpha,\beta,\gamma-2n,z)\mathrm{d}z_1\wedge\mathrm{d}z_2+\mathrm{d}(\cdots).\end{aligned} \tag{6.45}$$

This shift in the γ-argument can be converted to a shift in the α and β arguments by repeatedly using the identity

$$\begin{aligned}&\mathrm{d}((z_1-z_2)\mathcal{U}_w(\alpha,\beta,\gamma-2,z_1,z_2)\mathrm{d}z_2+(z_1-z_2)\mathcal{U}_w(\alpha,\beta,\gamma-2,z_1,z_2)\mathrm{d}z_1)\\ &=2(\gamma-1)\mathcal{U}_w(\alpha,\beta,\gamma-2,z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2-\alpha z_1^{-1}z_2^{-1}\mathcal{U}_w(\alpha,\beta,\gamma,z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2\\ &\qquad-\beta(z_1-w)^{-1}(z_2-w)^{-1}\mathcal{U}_w(\alpha,\beta,\gamma,z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2.\end{aligned} \tag{6.46}$$

which can then be simplified using all the previously discussed identities. Hence any 2-form of the form (6.44) is cohomologous to a scalar multiple of $\mathcal{U}_w\mathrm{d}z_1\wedge\mathrm{d}z_2$. Thus $\dim H^2(Z_{w,0},\nabla_{\omega_w})^{\mathfrak{A}}=1$.

The rationality of the functions $c(\alpha,\beta,\gamma)$ is immediate from the identity (6.36), its corresponding version for the case (6.39) and (6.46), because the coefficients that appear in these identities have zeros only in $\mathcal{G}$. The rationality of $c(\alpha,\beta,\gamma)$ is also a special case of [71, Cor 1.1.1]. □

For real numbers $a>b$, let $\Delta_{a,b}=\{a>z_1>z_2>b\}$. Given $f(z_1,z_2)\in\mathbb{C}[z_1^{\pm1},z_2^{\pm1},(z_1-w)^{-1},(z_2-w)^{-1}]$, we see that for appropriate (α,β,γ), the integral

$$\mathcal{I}_w[f](\alpha,\beta,\gamma)=\int_{\Delta_{w,0}}\mathcal{U}_w(\alpha,\beta,\gamma,z)f(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \tag{6.47}$$

converges. In the case $f=1$, the explicit formula is given by

$$\mathcal{I}_w[1](\alpha,\beta,\gamma)=w^{2(\alpha+\beta+1)}\prod_{i=1,2}\frac{\Gamma(1+i\gamma)\Gamma(1+\alpha+(i-1)\gamma)\Gamma(1+\beta+(i-1)\gamma)}{\Gamma(1+\gamma)\Gamma(2+\alpha+\beta+i\gamma)}. \tag{6.48}$$

In [72, Cor 1.1.1], it was shown that $\mathcal{I}_w[f](\alpha,\beta,\gamma)$ admits an analytic continuation to $\mathbb{C}^3\setminus\mathcal{H}$, where $\mathcal{H}$ is the hyperplane arrangement (6.34) in Lemma 6.4. In [73, Def 3.6], a non-zero twisted cycle $[\Delta_{w,0}(\alpha,\beta,\gamma)]\in H_2(Z_{w,0},\mathcal{L}_w^\vee(\alpha,\beta,\gamma))$ was constructed from the simplex $\Delta_{w,0}$. This cycle $[\Delta_{w,0}(\alpha,\beta,\gamma)]$ satisfies

$$\int_{[\Delta_{w,0}(\alpha,\beta,\gamma)]}\mathcal{U}_w(\alpha,\beta,\gamma,z)f(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2=\mathcal{I}_w[f](\alpha,\beta,\gamma) \tag{6.49}$$

for $f(z_1,z_2)\in\mathbb{C}[z_1^{\pm1},z_2^{\pm1},(z_1-w)^{-1},(z_2-w)^{-1}]$. We shall abbreviate this cycle as $[\Delta_{w,0}]$ when the parameters α,β,γ are clear from context.

Let $\lambda,\epsilon\in\mathbb{R}$ such that $\lambda\neq\pm\lambda_{u-1,v-1} \pmod 2=\pm(\frac{u}{v}-2)\pmod 2$ and $\lambda+\epsilon\neq\pm\lambda_{u-1,v-1}\pmod 2$. We specialise the parameters of $\mathcal{U}_w(\alpha,\beta,\gamma,z)$ to $\alpha=\frac{u}{2v}-1-\frac{\lambda+\epsilon}{2}$, $\beta=\frac{u}{2v}-1+\frac{\lambda+\epsilon}{2}$, $\gamma=1-\frac{u}{v}$ and define the shorthand

$$\begin{aligned}\mathcal{U}_w^\epsilon(z)&=\mathcal{U}_w\Big(\tfrac{u}{2v}-1-\tfrac{\lambda+\epsilon}{2},\tfrac{u}{2v}-1+\tfrac{\lambda+\epsilon}{2},1-\frac{u}{v},z\Big),\\ \mathcal{L}_{w,\epsilon}&=\mathcal{L}_w\Big(\tfrac{u-2v-(\lambda+\epsilon)v}{2v},\tfrac{u-2v+(\lambda+\epsilon)v}{2v},-\tfrac{u}{v}+1\Big),\\ \mathcal{L}_{w,\epsilon}^\vee&=\mathcal{L}_w^\vee\Big(\tfrac{u-2v-(\lambda+\epsilon)v}{2v},\tfrac{u-2v+(\lambda+\epsilon)v}{2v},-\tfrac{u}{v}+1\Big).\end{aligned} \tag{6.50}$$

Now consider a non-zero $\mathsf{L}\big(\frac{u}{v}\big)$-intertwining operator

$$I_{\lambda+\epsilon,-\lambda}\in\begin{pmatrix}\mathbb{F}_{\epsilon\frac{v}{u-2v}(a-b)-2b}\\ \mathbb{F}_{(\lambda+\epsilon)\frac{v}{u-2v}(a-b)-b},\ \mathbb{F}_{-\lambda\frac{v}{u-2v}(a-b)-b}\end{pmatrix}, \tag{6.51}$$

and recall that $E_{[\lambda];1,1}\cong\mathsf{S}_{1,1}\otimes\mathbb{F}_{\left[\lambda\frac{v}{u-2v}(a-b)-b\right]}$ and $\sigma^{-1}E_{[\lambda_{u-1,v-1}+\epsilon];1,1}\cong\mathsf{S}_{1,1}\otimes\mathbb{F}_{\left[\epsilon\frac{v}{u-2v}(a-b)-2b\right]}=\mathsf{e}^\epsilon\mathbb{H}^0$. Tensoring $I_{\lambda+\epsilon,-\lambda}$ with the action of $\mathsf{M}(u,v)$ on $\mathsf{S}_{1,1}$ yields an $\mathsf{A}_1(u,v)$-intertwining operator

$$\mathcal{Y}_{\lambda+\epsilon,-\lambda}\in\begin{pmatrix}\sigma^{-1}E_{[\lambda_{u-1,v-1}+\epsilon];1,1}\\ E_{[\lambda+\epsilon];1,1},\ E_{[-\lambda];1,1}\end{pmatrix} \tag{6.52}$$

by restriction.

Corollary 6.5. *For any $o\in(\mathbb{H}^1)'$, $m\in\mathsf{S}_{1,1}\otimes\mathbb{F}_{[\lambda\frac{v}{u-2v}(a-b)-b]}\cong E_{[\lambda];1,1}$ and $n\in\mathsf{S}_{1,1}\otimes\mathbb{F}_{[-\lambda\frac{v}{u-2v}(a-b)-b]}\cong E_{[-\lambda];1,1}$*

$$\big\langle o,\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathbf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathbf{e}^\epsilon m,w)n\big\rangle\in\mathcal{U}_w^\epsilon(z)\mathbb{C}[z_1^{\pm1},z_2^{\pm1},(z_1-w)^{-1},(z_2-w)^{-1}]^{\mathfrak{S}}. \tag{6.53}$$

Specifically, $o_0=\tilde{v}_{1,1}\otimes\langle0|$, $m_0=v_{1,1}\otimes\big|\frac{2v\lambda}{u-2v}(a-b)-b\big\rangle$ and $n_0=v_{1,1}\otimes\big|\frac{-2v\lambda}{u-2v}(a-b)-b\big\rangle$ we have

$$\big\langle o_0,\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathbf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathbf{e}^\epsilon m_0,w)n_0\big\rangle=w^{-\frac{u-2+\epsilon v}{2v}}\mathcal{U}_w^\epsilon(z) \tag{6.54}$$

up to a phase.

Then we can define a bilinear operator $\mathcal{Q}^{[2]} *^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(-,w)-\colon E_{\lambda;1,1} \times E_{-\lambda;1,1} \to \overline{\mathbb{H}^1}$. as

$$\mathcal{Q}^{[2]} *^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(m,w)n = \int_{[\Delta_{w,0}]} \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathbf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathbf{e}^{\epsilon}m,w)n\mathrm{d}z_1\mathrm{d}z_2, \tag{6.55}$$

where $m \in E_{\lambda;1,1}$, $n \in E_{-\lambda;1,1}$.

Let $\iota\colon Z_{w,0} \to Z_{w^{-1},0}$ be a diffeomorphism defined by $\iota(z_1,z_2) = (z_1^{-1}, z_2^{-1})$. Let $u_1 = z_1^{-1}$, $u_2 = z_2^{-1}$. Then we have

$$\iota_*(\mathcal{U}_{w,\epsilon}(z_1,z_2)\mathrm{d}z_1 \wedge \mathrm{d}z_2) = (u_2-u_1)^{-2\frac{u}{v}+2} \prod_{i=1,2} u_i^{\frac{u}{v}-2}(1-u_iw)^{\frac{u-2v+(\lambda+\epsilon)v}{2v}}\mathrm{d}u_1 \wedge \mathrm{d}u_2. \tag{6.56}$$

Thus we can pair this differential form with the twisted cycle Γ defined by (6.9). By pulling back Γ to the original variables (z_1,z_2), we have a twisted cycle $[\iota^*(\Gamma)] \in H_2(Z_{w,0}, \mathcal{L}_w^\vee)$. The cycle $\Gamma_\infty = \iota^*(\Gamma)$ allows us to define a transpose of the operator $\mathcal{Q}^{[2]}$.

Lemma 6.6. *Let* ${}^t\mathcal{Q}^{[2]} : (\mathbb{H}^1)' \to (\mathbb{H}^0)'$ *be the transpose of* $\mathcal{Q}^{[2]}$*. That is, the a linear map uniquely characterised by the identity*

$$\langle {}^t\mathcal{Q}^{[2]}o, m\rangle = \langle o, \mathcal{Q}^{[2]}m\rangle. \tag{6.57}$$

Then an explicity integral formula for ${}^t\mathcal{Q}^{[2]}$ *is given by*

$$\int_{[\Gamma_\infty]} o(\mathcal{Q}^{-\ \mathrm{opp}}(z_1)\mathcal{Q}^{+\ \mathrm{opp}}(z_2)-)\mathrm{d}z_1\mathrm{d}z_2, \tag{6.58}$$

where $-$ *indicates the argument from* $\mathbb{H}^0$*, and where* $\mathcal{Q}^{\pm\ \mathrm{opp}}$ *denotes the application the opposition formula* (5.4)*, that is,* $\mathcal{Q}^{\pm\ \mathrm{opp}}(z) = -z^{-2}\mathcal{Q}^\pm(z^{-1})$*.*

Proof. The lemma follows immediate from the definition of $[\Gamma_\infty]$ as

$$\begin{aligned}\langle {}^t\mathcal{Q}^{[2]}o, m\rangle &= \int_{[\Gamma_\infty]} \langle o, \mathcal{Q}^-(z_1^{-1})\mathcal{Q}^+(z_2^{-1})m\rangle \frac{\mathrm{d}z_1\mathrm{d}z_2}{z_1z_2} = \int_{[\Gamma]} \langle o, \mathcal{Q}^-(u_1)\mathcal{Q}^+(u_2)m\rangle \mathrm{d}u_1\mathrm{d}u_2 \\ &= \langle o, \mathcal{Q}^{[2]}m\rangle\end{aligned} \tag{6.59}$$

□

Next we consider the operator analogous to $\mathcal{Q}^{[2]} *_\infty^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(-,w)-$, but integrated over $[\Gamma_\infty]$ instead of $[\Gamma]$.

Lemma 6.7. *Consider linear operator* $\mathcal{Q}^{[2]} *_\infty^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(-,w)-\colon E_{\lambda;1,1} \times E_{-\lambda;1,1} \to \overline{\mathbb{H}^1}$ *defined by*

$$\mathcal{Q}^{[2]} *_\infty^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(m,w)n = \int_{[\Gamma_\infty]} \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\mathrm{d}z_1\mathrm{d}z_2, \tag{6.60}$$

for $m \in E_{\lambda;1,1}$ *and* $n \in E_{-\lambda;1,1}$*. Then*

$$Q^{[2]} *^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda} = c_{\lambda,\epsilon} Q^{[2]} *_\infty^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}, \tag{6.61}$$

where

$$c_{\lambda,\epsilon} = \frac{e^{3\pi \mathrm{i}\frac{u}{v}}}{4\sin(\pi\frac{u}{v})\sin(2\pi\frac{u}{v})} \frac{\Gamma(1-\frac{u}{v})}{\Gamma(\frac{u}{v}-1)\Gamma(3-2\frac{u}{v})} I_1[1]\Big(\tfrac{u-(\lambda+\epsilon)v}{2v}-1, \tfrac{u+(\lambda+\epsilon)v}{2v}-1, 1-\tfrac{u}{v}\Big). \tag{6.62}$$

Further,

$$\langle o, \mathcal{Q}^{[2]} *^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(m,w)n\rangle = c_{\lambda,\epsilon}\langle \mathcal{Q}^{[2]}o, \mathbf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathbf{e}^{\epsilon}m,w)n\rangle. \tag{6.63}$$

Proof. The lemma follows by direct computation. By Lemma 6.4, we know that for any $o \in (\mathbb{H}^1)'$, $m \in E_{\lambda;1,1}$ and $n \in E_{-\lambda;1,1}$, there exists a constant $c \in \mathbb{C}$ such that

$$\begin{aligned} w^{\frac{u-2v+\epsilon v}{2v}} &\big\langle o, \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m, w)n\big\rangle \mathrm{d}z_1 \wedge \mathrm{d}z_2 \\ &= c\,\big\langle o_0, \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m_0, w)n_0\big\rangle \mathrm{d}z_1 \wedge \mathrm{d}z_2 + \mathrm{d}(\cdots) \\ &= c\,\mathcal{U}^{\epsilon}_w(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2 + \mathrm{d}(\cdots), \end{aligned} \tag{6.64}$$

where $\mathrm{d}(\cdots)$ is some total derivative, that is, an exact 2-form, and where $m_0 = v_{1,1}\otimes\big|\frac{2v\lambda}{u-2v}(a-b)-b\big\rangle$, $n_0 = v_{1,1}\otimes\big|\frac{-2v\lambda}{u-2v}(a-b)-b\big\rangle$ and $o_0 = \tilde{v}_{1,1}\otimes\langle 0|$. By integrating (6.64) over $[\Gamma]$ and $[\Gamma_\infty]$ we obtain the identities

$$\begin{aligned} \big\langle o, Q^{[2]} *^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle &= c w^{\frac{2v-u-\epsilon v}{2v}} \int_{[\Delta_{w,0}]} \mathcal{U}^{\epsilon}_w(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2, \\ \big\langle o, Q^{[2]} *^{(\epsilon)}_{\infty} \mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle &= c w^{\frac{2v-u-\epsilon v}{2v}} \int_{[\Gamma_\infty]} \mathcal{U}^{\epsilon}_w(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2. \end{aligned} \tag{6.65}$$

The integral over $[\Delta_{w,0}]$ is just (6.49). While for the $[\Gamma_\infty]$-integral we use (6.56)

$$\begin{aligned} \int_{[\Gamma_\infty]} \mathcal{U}^{\epsilon}_w(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2 &= \int_{[\Gamma]} (u_2-u_1)^{2-2\frac{u}{v}} \prod_{i=1}^{2} u_i^{\frac{u}{v}-2}(1-u_i w)^{\frac{u}{2v}-1+\frac{\lambda+\epsilon}{2}} \mathrm{d}u_1\wedge\mathrm{d}u_2 \\ &= \oint_0 \int_{\Gamma_{0,1}} (xy-x)^{2-2\frac{u}{v}} x^{\frac{u}{v}-2}(xy)^{\frac{u}{v}-2}(1-xw)^{\frac{u}{2v}-1+\frac{\lambda+\epsilon}{2}} x\mathrm{d}x\mathrm{d}y \\ &= (-1)^{-2\frac{u}{v}} \int_{\Gamma_{0,1}} y^{\frac{u}{v}-2}(1-y)^{2-2\frac{u}{v}}\mathrm{d}y \\ &= e^{-2\pi\mathrm{i}\frac{u}{v}}\big(1-e^{2\pi\mathrm{i}\frac{u}{v}}\big)\big(1-e^{-2\pi\mathrm{i}2\frac{u}{v}}\big)\frac{\Gamma(\frac{u}{v}-1)\Gamma(3-2\frac{u}{v})}{\Gamma(2-\frac{u}{v})}, \\ &= 4e^{-3\pi\mathrm{i}\frac{u}{v}}\sin(\pi\tfrac{u}{v})\sin(2\pi\tfrac{u}{v})\frac{\Gamma(\frac{u}{v}-1)\Gamma(3-2\frac{u}{v})}{\Gamma(2-\frac{u}{v})} \end{aligned} \tag{6.66}$$

Thus

$$c_{\lambda,\epsilon} = 4e^{-3\pi\mathrm{i}\frac{u}{v}}\sin(\pi\tfrac{u}{v})\sin(2\pi\tfrac{u}{v})\frac{\Gamma(\frac{u}{v}-1)\Gamma(3-2\frac{u}{v})}{\Gamma(2-\frac{u}{v})} I_1[1]\Big(\tfrac{u-(\lambda+\epsilon)v}{2v}-2, \tfrac{u+(\lambda+\epsilon)v}{2v}-2, 1-\frac{u}{v}\Big). \tag{6.67}$$

Hence

$$\begin{aligned} c_{\lambda,\epsilon} &= \frac{\big\langle o_0, \mathcal{Q}^{[2]} *^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(m_0,w)n_0\big\rangle}{\Big\langle o_0, \mathcal{Q}^{[2]} *^{(\epsilon)}_{\infty} \mathcal{Y}_{\lambda,-\lambda}(m_0,w)n_0\Big\rangle} \\ &= \frac{e^{3\pi\mathrm{i}\frac{u}{v}}}{4\sin(\pi\frac{u}{v})\sin(2\pi\frac{u}{v})}\frac{\Gamma(1-\frac{u}{v})}{\Gamma(\frac{u}{v}-1)\Gamma(3-2\frac{u}{v})} I_1[1]\Big(\tfrac{u-(\lambda+\epsilon)v}{2v}-1, \tfrac{u+(\lambda+\epsilon)v}{2v}-1, 1-\tfrac{u}{v}\Big). \end{aligned} \tag{6.68}$$

To obtain (6.63) note

$$\begin{aligned} \big\langle o, \mathcal{Q}^{[2]} *^{(\epsilon)} \mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle &= c_{\lambda,\epsilon}\big\langle o, \mathcal{Q}^{[2]} *^{(\epsilon)}_{\infty} \mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \\ &= c_{\lambda,\epsilon}\big\langle {}^{t}\mathcal{Q}^{[2]}o, \mathbf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathbf{e}^{\epsilon}m,w)n\big\rangle. \end{aligned} \tag{6.69}$$

□

We set $\mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda} = \mathcal{Q}^{[2]} *^{(0)} \mathcal{Y}_{\lambda,-\lambda}$.

Proposition 6.8. *The operator $Q^{[2]} * \mathcal{Y}_{\lambda,-\lambda}$, constructed above, is a non-zero intertwining operator of type*

$$\begin{pmatrix} & \mathbb{H}^1 & \\ E_{\lambda;1,1} & & E_{-\lambda;1,1} \end{pmatrix}. \tag{6.70}$$

Furthermore, the image of this intertwining operator is the submodule $L_1 \subset \mathbb{H}^1_1$ (recall that L_1 is the vertex operator algebra $\mathsf{A}_1(u,v)$ as a module over itself and hence the tensor unit). denote the intertwining operator $Q^{[2]} * \mathcal{Y}_{\lambda,-\lambda}$ with the codomain restricted to L_1 by e_E.

We will show later that e_E defines an evaluation morphism for $E_{\lambda;1,1}$.

Proof. Since $\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}$ clearly satisfies the truncation condition, we begin by showing that it also satisfies the convergence condition. Let $m \in E_{\lambda;1,1}$, $n \in E_{-\lambda;1,1}$, $o \in (\mathbb{H}^1)'$ and $g \in \mathsf{A}_1(u,v)$ and consider

$$\begin{aligned} \big\langle o, \mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle &= \sum_{k\in\mathbb{Z}} \big\langle o, \mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)g_k n\big\rangle z^{-k-1}, \\ \big\langle o, \mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\big\rangle &= \sum_{k\in\mathbb{Z}} \big\langle o, \mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(g_k m,w)n\big\rangle (z-w)^{-k-1}, \\ \big\langle o, Y(g,z)\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle &= \sum_{k\in\mathbb{Z}} \big\langle o, g_k\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle z^{-k-1}. \end{aligned} \tag{6.71}$$

The convergence of the first two series in their respective domains follows applying the identity (6.69) to transpose $\mathcal{Q}^{[2]}$ and move it to the left side of the pairing and noting that $\mathcal{Y}_{\lambda,-\lambda}$ is an intertwining operator and hence satisfies the convergence condition. The convergence of the third series follows from

$$\begin{aligned} \big\langle o, Y(g,z)\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle &= \big\langle Y(g,z)^{\mathrm{opp}}o, \mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \\ &= c_{\lambda,0}\big\langle {}^t\mathcal{Q}^{[2]}Y(g,z)^{\mathrm{opp}}o, \mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \\ &= c_{\lambda,0}\big\langle Y(g,z)^{\mathrm{opp}}\,{}^t\mathcal{Q}^{[2]}o, \mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \\ &= c_{\lambda,0}\big\langle {}^t\mathcal{Q}^{[2]}o, Y(g,z)\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle, \end{aligned} \tag{6.72}$$

where the third equality uses that $\mathcal{Q}^{[2]}$ and hence also its transpose commutes with the action of the vertex operator algebra, and convergence then follows from $\mathcal{Y}_{\lambda,-\lambda}$ satisfying the convergence condition.

The $P(w)$-compatibility condition for $\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}$ also follows by transposing $\mathcal{Q}^{[2]}$ moving it to the left side of the pairing. That is for $f(z) \in \mathbb{C}[z, z^{-1}, (z-w)^{-1}]$, consider

$$\begin{aligned} &\oint_{0,w} f(z)\big\langle o, Y(g,z)\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \mathrm{d}z - \oint_{w} f(z)\big\langle o, \mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\big\rangle \mathrm{d}z \\ &\qquad - \oint_{w} f(z)\big\langle o, \mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle \mathrm{d}z \\ &= \oint_{0,w} c_{\lambda,\epsilon} f(z)\big\langle {}^t\mathcal{Q}^{[2]}o, Y(g,z)\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \mathrm{d}z - \oint_{w} c_{\lambda,\epsilon} f(z)\big\langle {}^t\mathcal{Q}^{[2]}o, \mathcal{Y}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\big\rangle \mathrm{d}z \\ &\qquad - \oint_{w} c_{\lambda,\epsilon} f(z)\big\langle {}^t\mathcal{Q}^{[2]}o, \mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle \mathrm{d}z. \end{aligned} \tag{6.73}$$

The right-hand side of the above equality vanishes due to $\mathcal{Y}_{\lambda,-\lambda}$ satisfying the Jacobi identity and thus so does $\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}$. □

Lemma 6.9. *Consider the bilinear operator* $\mathrm{Ad}_{\mathsf{e}^\epsilon}\mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(-,w)-\colon E_{\lambda;1,1}\otimes E_{-\lambda;1,1}\to \overline{\mathbb{H}}^1$ *defined by*

$$\mathrm{Ad}_{\mathsf{e}^\epsilon}\mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m,w)n = w^\epsilon \int_{[\Delta_{w,0}\left(\frac{u-2v-(\lambda+\epsilon)v}{2v},\frac{u-2v+\lambda v}{2v},-\frac{u}{v}+1\right)]} \mathsf{e}^\epsilon\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda,-\lambda}(m,w)n, \tag{6.74}$$

where $m\in E_{\lambda;1,1}$, $n\in E_{-\lambda;1,1}$. *Then the operator*

$$\mathcal{Y}^{\Delta}_{\lambda,-\lambda} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathrm{Ad}_{\mathsf{e}^\epsilon}\mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}\bigg|_{\epsilon=0} : E_{\lambda;1,1}\otimes E_{-\lambda;1,1}\to\overline{\mathbb{H}}^1 \tag{6.75}$$

is well defined and non-trivial.

Proof. By [72, Cor 1.1.1], we see that

$$\left\langle o, \mathrm{Ad}_{\mathsf{e}^\epsilon}\mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m,w)n\right\rangle, \qquad m\in E_{\lambda;1,1},\ n\in E_{-\lambda;1,1},\ o\in(\mathbb{H}^1)' \tag{6.76}$$

is holomorphic in the variable ϵ in a neighbourhood of $\epsilon=0$. This can also be verified by expanding the image of $\mathrm{Ad}_{\mathsf{e}^\epsilon}\mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}$ in a monomial basis of $\mathbb{H}^1$ (a basis given by monomials of negative Virasoro and Heisenberg modes applied to $v_{1,1}\otimes\left|\frac{2nv}{u-2v}(a-b)\right\rangle$ with an appropriate choice of ordering), where we see that each coefficient of this expansion is holomorphic in a neighbourhood of $\epsilon=0$. Thus $\mathcal{Y}^{\Delta}_{\lambda,-\lambda}$ is well defined.

To conclude non-triviality consider the vectors $m_0 = v_{1,1}\otimes\left|\frac{2v\lambda}{u-2v}(a-b)-b\right\rangle$, $n_0 = v_{1,1}\otimes\left|\frac{-2v\lambda}{u-2v}(a-b)-b\right\rangle$ and $o_0=\tilde{v}_{1,1}\otimes\langle 0|$ and note

$$\begin{aligned} w^{\frac{u-2v}{2v}}\left\langle o_0,\mathcal{Y}^{\Delta}_{\lambda,-\lambda}(m_0,w)n_0\right\rangle &= w^{\frac{u-2v}{2v}}\partial_\epsilon\left\langle o_0,\mathrm{Ad}_{\mathsf{e}^\epsilon}\mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m_0,w)n_0\right\rangle|_{\epsilon=0} \\ &= \frac{\partial}{\partial\epsilon}\prod_{i=1,2}\frac{\Gamma(1+i(1-\frac{u}{v}))\Gamma(\frac{u-(\lambda+\epsilon)v}{2v}+(i-1)(1-\frac{u}{v}))\Gamma(\frac{u+\lambda v}{2v}+(i-1)(1-\frac{u}{v}))}{\Gamma(1+i(1-\frac{u}{v}))\Gamma(2+\frac{u-2v-\epsilon v}{v}+i(1-\frac{u}{v}))}\Bigg|_{\epsilon=0}. \end{aligned} \tag{6.77}$$

Then by using the asymptotic expansion of the digamma function

$$\frac{\Gamma'(z)}{\Gamma(z)}\sim\log(z)-\frac{1}{2z}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}} \qquad (|\mathrm{arg}z|<\pi,\ z\to\infty), \tag{6.78}$$

we have an asymptotic behaviour

$$\frac{w^{\frac{u-2v}{2v}}\left\langle o_0,\mathcal{Y}^{\Delta}_{\lambda,-\lambda}(m_0,w)n_0\right\rangle}{\Gamma(\frac{u}{2v}+\frac{\lambda}{2})\Gamma(1-\frac{u}{2v}+\frac{\lambda}{2})}\sim C\Gamma(-2^{-1}\lambda)^2\log(-\lambda) \qquad (\lambda\to-\infty) \tag{6.79}$$

where B_{2n} are the Bernoulli numbers and C is a some nonzero constant. Thus $\mathcal{Y}^{\Delta}_{\lambda,-\lambda}$ is non-trivial. ∎

With the definition of $\mathcal{Y}^{\Delta}_{\lambda,-\lambda}$ in place, we are now able to give an explicit construction of an intertwining operator with non-trivial logarithmic parts.

Theorem 6.10. *Recall the bilinear operators* $\mathcal{Y}_{\lambda,-\lambda} : E_{\lambda;1,1}\otimes E_{-\lambda;1,1}\to\overline{\mathbb{H}^0}$ *and* $\mathcal{Y}^{\Delta}_{\lambda,-\lambda} : E_{\lambda;1,1}\otimes E_{-\lambda;1,1}\to\overline{\mathbb{H}^1}$ *defined above and consider* $\mathcal{Y}^{\log}_{\lambda,-\lambda} = \mathcal{Y}_{\lambda,-\lambda}+\mathcal{Y}^{\Delta}_{\lambda,-\lambda} : E_{\lambda;1,1}\otimes E_{-\lambda;1,1}\to\overline{\mathbb{H}}$. *Then* $\mathcal{Y}^{\log}_{\lambda,-\lambda}$ *is a* $P(w)$*-intertwining operator of type*

$$\begin{pmatrix} (\mathbb{H},\widetilde{Y}_{c_{\lambda,0}}) \\ E_{\lambda;1,1},\ E_{-\lambda;1,1}\end{pmatrix}. \tag{6.80}$$

The above theorem implies that $\sigma^{-1}P_{u-1,v-1}$, the projective cover of the tensor unit, is a direct summand of the fusion product $E_{\lambda;1,1} \boxtimes E_{-\lambda;1,1}$. To prove this theorem we prepare the following lemmas.

Lemma 6.11. *For all $m \in E_{\lambda;1,1}$, $n \in E_{-\lambda;1,1}$, $o \in (\mathbb{H}^1)'$ and $g \in \mathsf{A}_1(u,v)$ the series*

$$\begin{aligned} \big\langle o, \mathrm{Ad}_{\mathsf{e}^\epsilon} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m,w) Y(g,z) n \big\rangle &= \sum_{k\in\mathbb{Z}} \big\langle o, \mathrm{Ad}_{\mathsf{e}^\epsilon} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m,w) g_k n \big\rangle z^{-k-1} \\ \big\langle o, \mathrm{Ad}_{\mathsf{e}^\epsilon} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(Y(g,z-w)m,w) n \big\rangle &= \sum_{k\in\mathbb{Z}} \big\langle o, \mathrm{Ad}_{\mathsf{e}^\epsilon} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(g_k m,w) n \big\rangle (z-w)^{-k-1} \\ \big\langle o, Y(g,z) \mathrm{Ad}_{\mathsf{e}^\epsilon} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m,w) n \big\rangle &= \sum_{k\in\mathbb{Z}} \big\langle o, g_k \mathrm{Ad}_{\mathsf{e}^\epsilon} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m,w) n \big\rangle z^{-k-1} \end{aligned} \tag{6.81}$$

are absolutely convergent on $w > |z| > 0$, $w > |z-w| > 0$ and $|z| > w$, respectively.

Proof. We only show the first case, as the other cases can be proved in the same way. Consider the correlation function

$$\big\langle o, \mathsf{e}^\epsilon \mathcal{Q}^-(z_1) \mathcal{Q}^+(z_1) \mathsf{e}^{-\epsilon} \mathcal{Y}_{\lambda,-\lambda}(m,w) Y(g,z) n \big\rangle = \\ z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \big\langle o, \mathcal{Q}^-(z_1) \mathcal{Q}^+(z_1) \mathcal{Y}_{\lambda,-\lambda}(m,w) Y(g,z) n \big\rangle, \tag{6.82}$$

which is absolutely convergent on $w > |z| > 0$ for each fixed (z_1, z_2). By expanding

$$z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \big\langle o, \mathcal{Q}^-(z_1) \mathcal{Q}^+(z_2) \mathcal{Y}_{\lambda,-\lambda}(m,w) Y(g,z) n \big\rangle = \\ \sum_{k\in\mathbb{Z}} z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \big\langle o, \mathcal{Q}^-(z_1) \mathcal{Q}^+(z_2) \mathcal{Y}_{\lambda,-\lambda}(m,w) g_k n \big\rangle z^{-k-1} \tag{6.83}$$

we see that

$$\begin{aligned} &w^{\frac{u-2v}{2v}} z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \big\langle o, \mathcal{Q}^-(z_1) \mathcal{Q}^+(z_2) \mathcal{Y}_{\lambda,-\lambda}(m,w) g_k n \big\rangle \\ &\in \mathcal{U}_w\Big(\tfrac{u-(\lambda+\epsilon)v}{2v} - 1, \tfrac{u+\lambda v}{2v} - 1, 1 - \tfrac{u}{v}, z_1, z_2\Big) \mathbb{C}[w^{\pm1}][z_1^{\pm1}, z_2^{\pm1}, (z_1-w)^{-1}, (z_2-w)^{-1}]^{\mathfrak{S}_2}. \end{aligned} \tag{6.84}$$

Then by Lemma 6.4, there exists a rational function $c_k(\lambda, w) \in \mathbb{C}(\lambda) \otimes \mathbb{C}[\epsilon, w^{\pm1}]$ such that

$$\begin{aligned} &w^{\frac{u-2v}{2v}} z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \big\langle o, \mathcal{Q}^-(z_1) \mathcal{Q}^+(z_2) \mathcal{Y}_{\lambda,-\lambda}(m,w) g_k n \big\rangle \mathrm{d}z_1 \wedge \mathrm{d}z_2 \\ &= c_k(\lambda, w) \mathcal{U}_w\Big(\tfrac{u-(\lambda+\epsilon)v}{2v} - 1, \tfrac{u+\lambda v}{2v} - 1, 1 - \tfrac{u}{v}, z_1, z_2\Big) \mathrm{d}z_1 \wedge \mathrm{d}z_2 + \mathrm{d}(\cdots). \end{aligned} \tag{6.85}$$

By taking sufficiently small absolute value of z, we see that z is contained in the tubular neighbourhood of the twisted cycle $[\Delta_{w,0}\big(\frac{u-2v-(\lambda+\epsilon)v}{2v}, \frac{u-2v+\lambda v}{2v}, -\frac{u}{v} + 1\big)]$ (see [8, Sec 3.2.4]). Thus, the first of (6.81) converges for small $|z| > 0$ and satisfies

$$w^{\frac{u-2v}{2v}} \big\langle o, \mathrm{Ad}_{\mathsf{e}^\epsilon} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(m,w) Y(g,z) n \big\rangle \\ = I_1[1]\Big(\tfrac{u-(\lambda+\epsilon)v}{2v} - 1, \tfrac{u+\lambda v}{2v} - 1, 1 - \frac{u}{v}\Big) \sum_{k\in\mathbb{Z}} c_k(\lambda, w) z^{-k-1}. \tag{6.86}$$

After having show convergence for small $|z| > 0$, we next need to extend to the domain $|w| > |z| > 0$. Let $\iota : Z_{w,0} \to Z_{w^{-1},0}$ be a diffeomorphism defined by $\iota(z_1, z_2) = (z_1^{-1}, z_2^{-1})$. Let $u_1 = z_1^{-1}$, $u_2 = z_2^{-1}$. Then we have

$$\iota_*\Big(\mathcal{U}_w\big(\tfrac{u-(\lambda+\epsilon)v}{2v} - 1, \tfrac{u+\lambda v}{2v} - 1, 1 - \frac{u}{v}, z_1, z_2\big) \mathrm{d}z_1 \wedge \mathrm{d}z_2\Big)$$

$$= (u_2-u_1)^{-2\frac{u}{v}+2}\prod_{i=1,2} u_i^{\frac{u}{v}-2+\frac{\epsilon}{2}}(1-u_iw)^{\frac{u-2v+\lambda v}{2v}}\mathrm{d}u_1\wedge\mathrm{d}u_2$$
$$= w^{\frac{u-2v+\lambda v}{2v}}\mathcal{U}_{w^{-1}}\big(\tfrac{u}{v}-2+\tfrac{\epsilon}{2},\tfrac{u-2v+\lambda v}{2v},1-\tfrac{u}{v},u_2,u_1\big)\mathrm{d}u_1\wedge\mathrm{d}u_2, \tag{6.87}$$

where the multivalued function in the pushforward ι_* is the multivalued function on the right-hand side of (6.84). Then by pulling back the twisted cycle

$$\big[\Delta_{w^{-1},0}\big(\tfrac{u}{v}-2+\tfrac{\epsilon}{2},\tfrac{u-2v+\lambda v}{2v},1-\tfrac{u}{v}\big)\big]\in H_2\big(Z_{w^{-1},0},\mathcal{L}^\vee_{w^{-1}}\big(\tfrac{u}{v}-2+\tfrac{\epsilon}{2},\tfrac{u+\lambda v}{2v}-1,1-\tfrac{u}{v}\big)\big) \tag{6.88}$$

by ι, we have a twisted cycle

$$\iota^*\big[\Delta_{w^{-1},0}\big(\tfrac{u}{v}-2+\tfrac{\epsilon}{2},\tfrac{u+\lambda v}{2v}-1,1-\tfrac{u}{v}\big)\big]\in H_2\Big(Z_{w,0},\mathcal{L}^\vee_{w}\Big(\tfrac{u-(\lambda+\epsilon)v}{2v}-1,\tfrac{u+\lambda v}{2v}-1,1-\tfrac{u}{v}\Big)\Big). \tag{6.89}$$

This twisted cycle is a regularisation of the contour $\{\infty>z_1>z_2>w\}$. Then by (6.83) and by (6.85), we have

$$w^{\frac{u-2v}{2v}}\int_{\iota^*\left[\Delta_{w^{-1},0}\left(\frac{u}{v}-2+\frac{\epsilon}{2},\frac{u+\lambda v}{2v}-1,1-\frac{u}{v}\right)\right]} z_1^{-\frac{\epsilon}{2}}z_2^{-\frac{\epsilon}{2}}\big\langle o,\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle\mathrm{d}z_1\wedge\mathrm{d}z_2$$
$$= I_1[1]\big(\tfrac{u}{v}-2+\tfrac{\epsilon}{2},\tfrac{u+\lambda v}{2v}-1,1-\tfrac{u}{v}\big)\sum_{k\in\mathbb{Z}}c_k(\lambda,w)z^{-k-1} \tag{6.90}$$

which is absolutely convergent on $w>|z|>0$. Thus by (6.86), we see that the first series of (6.81) is absolutely convergent on $w>|z|>0$. □

Lemma 6.12. *The series*

$$\big\langle o,\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle=\sum_{k\in\mathbb{Z}}\big\langle o,\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)g_kn\big\rangle z^{-k-1}$$
$$\big\langle o,\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(\mathsf{e}^{-\epsilon}Y(g,z-w)\mathsf{e}^{\epsilon}m,w)n\big\rangle=\sum_{k\in\mathbb{Z}}\big\langle o,\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(\mathsf{e}^{-\epsilon}g_k\mathsf{e}^{\epsilon}m,w)n\big\rangle(z-w)^{-k-1}$$
$$\big\langle o,Y(g,z)\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle=\sum_{k\in\mathbb{Z}}\big\langle o,g_k\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle z^{-k-1} \tag{6.91}$$

are absolutely convergent on the domains $w>|z|>0$, $w>|z-w|>0$ and $|z|>w$, respectively, for any $m\in E_{\lambda;1,1}$, $n\in E_{-\lambda;1,1}$, $o\in(\mathbb{H}^1)'$ and $g\in\mathsf{A}_1(u,v)$.

Proof. The proof follows by a very similar argument to the proof of Lemma 6.11. □

Lemma 6.13. *For $m\in E_{\lambda;1,1}$, $n\in E_{-\lambda;1,1}$, $o\in(\mathbb{H}^1)'$, $g\in\mathsf{A}_1(u,v)$ and $f(z)\in\mathbb{C}[z,z^{-1},(z-w)^{-1}]$, we have*

$$\oint_{w,0}f(z)\big\langle o,Y(g,z)\mathcal{Y}^{\Delta}_{\lambda,-\lambda}(m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}$$
$$=\oint_{w}f(z)\big\langle o,\mathcal{Y}^{\Delta}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}+\oint_{0}f(z)\big\langle o,\mathcal{Y}^{\Delta}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle\frac{\mathrm{d}z}{2\pi i}$$
$$+c_{\lambda,-\lambda}\oint_{w,0}f(z)\langle o,\Delta_1(g;z)\mathcal{Y}_{\lambda,-\lambda}(m,w)n\rangle\frac{\mathrm{d}z}{2\pi i}. \tag{6.92}$$

Proof. From the definition of $\mathcal{Y}^{\Delta}_{\lambda,-\lambda}$, it is enough to show the equality

$$\oint_{w}f(z)\big\langle o,\mathrm{Ad}_{\mathsf{e}^{\epsilon}}\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}+\oint_{0}f(z)\big\langle o,\mathrm{Ad}_{\mathsf{e}^{\epsilon}}\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle\frac{\mathrm{d}z}{2\pi i}$$
$$-\oint_{w,0}f(z)\big\langle o,Y(g,z)\mathrm{Ad}_{\mathsf{e}^{\epsilon}}\mathcal{Q}^{[2]}*\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}$$

$$= -\epsilon c_{\lambda+\epsilon,-\lambda} \oint_{w,0} f(z)\langle o, \Delta_1(g;z;\epsilon)\mathcal{Y}_{\lambda,-\lambda}(m,w)n\rangle \frac{\mathrm{d}z}{2\pi i} + (\dots) \tag{6.93}$$

where $(\dots)$ denotes functions which are homomorphic near $\epsilon = 0$ with vanishing derivatives at $\epsilon = 0$, since (6.92) is obtained by differentiating both sides of (6.93) by ϵ and setting $\epsilon = 0$. By the $P(w)$-compatibility condition of $\mathcal{Y}_{\lambda,-\lambda}$ and by the proof of Proposition 6.1, we see that there exist $F_1, F_2 \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, (z_1-w)^{-1}, (z_2-w)^{-1}, w, w^{-1}]$ satisfying

$$\begin{aligned}
&\oint_{w} f(z)\big\langle o, \mathsf{e}^{\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad + \oint_{0} f(z)\big\langle o, \mathsf{e}^{\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad - \oint_{w,0} f(z)\big\langle o, Y(g,z)\mathsf{e}^{\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&= \oint_{w,0} f(z)\big\langle o, \mathsf{e}^{\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}Y(g,z)\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad - \oint_{w,0} f(z)\big\langle o, Y(g,z)\mathsf{e}^{\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&= w^{-\frac{u-2v}{2v}} z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \mathrm{d}_{z_1,z_2}\big(\mathcal{U}_w\big(\tfrac{u-\lambda v}{2v}-1, \tfrac{u+\lambda v}{2v}-1, 1-\tfrac{u}{v}, z_1, z_2\big)\big(F_1\mathrm{d}z_2 + F_2\mathrm{d}z_1\big)\big).
\end{aligned} \tag{6.94}$$

Similarly, we see that there exist $F_1^{(\epsilon)}, F_2^{(\epsilon)} \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, (z_1-w)^{-1}, (z_2-w)^{-1}, w, w^{-1}, \epsilon]$ satisfying

$$\begin{aligned}
&\oint_{w} f(z)\big\langle o, \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(Y(g,z-w)\mathsf{e}^{\epsilon}m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad + \oint_{0} f(z)\big\langle o, \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)Y(g,z)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad - \oint_{w,0} f(z)\big\langle o, \mathsf{e}^{-\epsilon}Y(g,z)\mathsf{e}^{\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&= \oint_{w,0} f(z)\big\langle o, \mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}Y(g,z)\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad - \oint_{w,0} f(z)\big\langle o, \mathsf{e}^{-\epsilon}Y(g,z)\mathsf{e}^{\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&= z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \oint_{w,0} f(z)\big\langle o, \mathsf{e}^{-\epsilon}\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)Y(g,z)\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad - z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \oint_{w,0} f(z)\big\langle o, \mathsf{e}^{-\epsilon}Y(g,z)\mathcal{Q}^-(z_1)\mathcal{Q}^+(z_2)\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&= w^{-\frac{u-2v+\epsilon v}{2v}} z_1^{-\frac{\epsilon}{2}} z_2^{-\frac{\epsilon}{2}} \mathrm{d}_{z_1,z_2}\Big(\mathcal{U}_w\Big(\tfrac{u-\lambda v}{2v}-1, \tfrac{u+(\lambda+\epsilon)v}{2v}-1, 1-\tfrac{u}{v}, z_1, z_2\Big)\Big(F_1^{(\epsilon)}\mathrm{d}z_2 + F_2^{(\epsilon)}\mathrm{d}z_1\Big)\Big)
\end{aligned} \tag{6.95}$$

and $F_1^{(0)} = F_1$, $F_2^{(0)} = F_2$. Thus, noting (6.91), by (6.94) and by (6.95), we have

$$\begin{aligned}
&\oint_{w} f(z)\big\langle o, \mathrm{Ad}_{\mathsf{e}^{\epsilon}}\mathcal{Q}^{[2]}{*}\mathcal{Y}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i} + \oint_{0} f(z)\big\langle o, \mathrm{Ad}_{\mathsf{e}^{\epsilon}}\mathcal{Q}^{[2]}{*}\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle \frac{\mathrm{d}z}{2\pi i}\\
&\qquad - \oint_{w,0} f(z)\big\langle o, Y(g,z)\mathrm{Ad}_{\mathsf{e}^{\epsilon}}\mathcal{Q}^{[2]}{*}\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle \frac{\mathrm{d}z}{2\pi i}
\end{aligned}$$

$$
\begin{aligned}
&= w^{\frac{\epsilon}{2}}\Big(\oint_w f(z)\big\langle o, \mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(\mathsf{e}^{-\epsilon}Y(g,z-w)\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&\quad+\oint_0 f(z)\big\langle o, \mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&\quad-\oint_{w,0} f(z)\big\langle o, \mathsf{e}^{-\epsilon}Y(g,z)\mathsf{e}^{\epsilon}\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\Big)+(\dots)
\end{aligned}\tag{6.96}
$$

where $(\dots)$ denotes functions which are homomorphic near $\epsilon=0$ with vanishing derivatives at $\epsilon=0$. By (6.63), we have

$$
\begin{aligned}
&\oint_w f(z)\big\langle o, \mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(\mathsf{e}^{-\epsilon}Y(g,z-w)\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}+\oint_0 f(z)\big\langle o, \mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)Y(g,z)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&\quad-\oint_{w,0} f(z)\big\langle o, \mathsf{e}^{-\epsilon}Y(g,z)\mathsf{e}^{\epsilon}\mathcal{Q}^{[2]}*^{(\epsilon)}\mathcal{Y}_{\lambda,-\lambda}(m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&= c_{\lambda,\epsilon}\Big(\oint_w f(z)\big\langle \mathcal{Q}^{[2]}o, \mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(Y(g,z-w)\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&\quad+\oint_0 f(z)\big\langle \mathcal{Q}^{[2]}o, \mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)Y(g,z)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&\quad-\oint_{w,0} f(z)\big\langle \mathcal{Q}^{[2]}\mathsf{e}^{-\epsilon}Y(g,z)^{\mathrm{opp}}\mathsf{e}^{\epsilon}o, \mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\Big)\\
&= c_{\lambda,\epsilon}\Big(\oint_{w,0} f(z)\big\langle \mathcal{Q}^{[2]}o, \mathsf{e}^{-\epsilon}Y(g,z)\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&\quad-\oint_{w,0} f(z)\big\langle \mathcal{Q}^{[2]}\mathsf{e}^{-\epsilon}Y(g,z)^{\mathrm{opp}}\mathsf{e}^{\epsilon}o, \mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\Big)\\
&= c_{\lambda,\epsilon}\Big(\oint_{w,0} f(z)\big\langle \mathsf{e}^{-\epsilon}Y(g,z)^{\mathrm{opp}}\mathsf{e}^{\epsilon}\mathcal{Q}^{[2]}o, \mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&\quad-\oint_{w,0} f(z)\big\langle \mathcal{Q}^{[2]}\mathsf{e}^{-\epsilon}Y(g,z)^{\mathrm{opp}}\mathsf{e}^{\epsilon}o, \mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\Big)\\
&= -c_{\lambda,\epsilon}\oint_{w,0} f(z)\,\mathrm{Res}_{y=z}\big\langle \mathcal{Q}^{[2]}(y)\mathsf{e}^{-\epsilon}Y(g,z)^{\mathrm{opp}}\mathsf{e}^{\epsilon}o, \mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}.
\end{aligned}\tag{6.97}
$$

By (6.57), we see that the last term of the above identity becomes

$$
\begin{aligned}
&-c_{\lambda,\epsilon}\oint_{w,0} f(z)\,\mathrm{Res}_{y=z}\big\langle o, \mathcal{Q}^{[2]}(y)\mathsf{e}^{-\epsilon}Y(g,z)\mathsf{e}^{\epsilon}\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}\\
&= -\epsilon c_{\lambda,\epsilon}\oint_{w,0} f(z)\big\langle o, \Delta_1(g;z;\epsilon)\mathsf{e}^{-\epsilon}\mathcal{Y}_{\lambda+\epsilon,-\lambda}(\mathsf{e}^{\epsilon}m,w)n\big\rangle\frac{\mathrm{d}z}{2\pi i}.
\end{aligned}\tag{6.98}
$$

Thus by (6.96), we see that the identity (6.93) holds. □

Proof of Theorem 6.10. The operator $\mathcal{Y}^{log}_{\lambda,-\lambda}$ clearly satisfies the truncation condition and the convergence condition follows from Lemma 6.11. Therefore, all that remains to be shown is that $\mathcal{Y}^{log}_{\lambda,-\lambda}$ satisfies the Cauchy-Jacobi identity. By Lemma 6.13, for $m\in E_{\lambda;1,1}$, $n\in E_{-\lambda;1,1}$, $o\in(\mathbb{H}^1)'$, $g\in\mathsf{A}_1(u,v)$ and $f(z)\in\mathbb{C}[z,z^{-1},(z-w)^{-1}]$, $\mathcal{Y}^{log}_{\lambda,-\lambda}$ satisfies

$$
\oint_{w,0} f(z)\Big\langle o, \widetilde{Y}_{c_{\lambda,0}}(g,z)\mathcal{Y}^{log}_{\lambda,-\lambda}(m,w)n\Big\rangle\frac{\mathrm{d}z}{2\pi i}
$$

$$= \oint_w f(z)\Big\langle o, \mathcal{Y}^{log}_{\lambda,-\lambda}(Y(g,z-w)m,w)n\Big\rangle \frac{\mathrm{d}z}{2\pi i} + \oint_0 f(z)\Big\langle o, \mathcal{Y}^{log}_{\lambda,-\lambda}(m,w)Y(g,z)n\Big\rangle \frac{\mathrm{d}z}{2\pi i}, \tag{6.99}$$

where $\widetilde{Y}_{c_{\lambda,0}}(g,z)$ is the $\mathsf{A}_1(u,v)$-action on $\mathbb{H}$ defined by

$$\widetilde{Y}_{c_{\lambda,-\lambda}}(g,z) = \begin{cases} Y(g,z) - c_{\lambda,0}\Delta_1(g;z) & \text{on } \mathbb{H}^0 \\ Y(g,z) & \text{on } \mathbb{H}^1 \end{cases}. \tag{6.100}$$

Therefore, from (6.99), $\mathcal{Y}^{log}_{\lambda,-\lambda}$ satisfies the Cauchy-Jacobi identity. □

Remark 6.14. Because the action of L_0 on $\mathbb{H}$ has Jordan blocks of size 2, the $P(w)$ intertwining operator $\mathcal{Y}^{log}_{\lambda,-\lambda}$ has first order logarithmic terms (that is, $\log z$ appears but $(\log z)^n$, $n \geq 2$ does not) once the complex variable w is replaced by a formal variable z by using the identity [45, Eq. (4.17)], that is,

$$\mathcal{Y}^{\log}_{\lambda,-\lambda}(v_1,z)v_2 = y^{L(0)}z^{L(0)}\mathcal{Y}^{\log}_{\lambda,-\lambda}(y^{-L(0)}z^{-L(0)}v_1 \otimes y^{-L(0)}z^{-L(0)}v_2)\Big|_{y=e^{-\log w}}, \tag{6.101}$$

where z^{L_0} is defined to the formal series

$$z^{L(0)} = z^n \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{(L(0)-n)^i v}{i!}(\log z)^i. \tag{6.102}$$

6.3. **The rigidity of simple $E_{\mu;1,1}$ modules.** In this subsection, following the same principles as in [7, Prop 5.13], we will show that $E_{[\lambda];1,1}$, $\lambda \in \mathbb{R}\setminus\{\pm\lambda_{1,1}+\mathbb{Z}\}$, $[\lambda] = \lambda + 2\mathbb{Z}$ are rigid in $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$, with rigid duals $E_{[-\lambda];1,1}$ (we use explicit representatives for the coset $[\lambda]$, because it will be convenient in calculations below). Note that since the category of weight modules is braided, there is no need to distinguish left and right duals. Since the integration cycles required here are more complicated than in [7] the details of the arguments are considerably more subtle. In particular, the methods of [8, Sec 3.2] will be crucial.

Let $w_1, w_2 \in \mathbb{R}_{>0}$ be real numbers satisfying $w_1 > w_1 - w_2 > 0$ and let $E = E_{[\lambda];1,1}$, $E^\vee = E_{[-\lambda];1,1}$. By [26, Lem 4.2.1 and Cor 4.2.2] a sufficient condition for rigidity is the existence of two morphisms $i_{E_{\lambda;1,1}}\colon L_1 \to E_{\lambda;1,1}\boxtimes_w E_{-\lambda;1,1}$ and $e_{E_{\lambda;1,1}}\colon E_{-\lambda;1,1}\boxtimes_w E_{\lambda;1,1} \to L_1$, where L_1 is the unit of the category (the vertex operator algebra as a module over itself), such that the composition

$$R_E = E \xrightarrow{l^{-1}} L_1 \boxtimes E \xrightarrow{i_E\boxtimes 1} (E\boxtimes_{w_1} E')\boxtimes_{w_2} E \xrightarrow{\mathcal{A}^{-1}} E\boxtimes_{w_1}(E'\boxtimes_{w_2} E) \xrightarrow{1\boxtimes e_E} E\boxtimes L_1 \xrightarrow{r} E \tag{6.103}$$

is non-zero. Here, l and r are the left and right unitors, respectively, and $\mathcal{A}$ the associator of the category. Since $E_{[\lambda];1,1}$ is simple R_E will be a scalar multiple of the identity and hence the pair $(e_{E_{\lambda;1,1}}, i_{E_{\lambda;1,1}})$ define evaluation and coevaluation morphisms after dividing one of them by this scalar. We begin by constructing candidates for these evaluation and coevaluation maps. For e_E, we take,

$$e_E = \mathcal{Q}^{[2]} * \mathcal{Y}_{-\lambda,\lambda}, \tag{6.104}$$

as noted after Proposition 6.8. To characterise the coevaluation morphism we prepare the notation

$$\theta_\lambda = v_{1,1}\otimes\left|\frac{v\lambda}{u-2v}(a-b)-b\right\rangle \in E_{[\lambda];1,1}, \qquad \overline{\theta_\lambda} = \left|\frac{v\lambda}{u-2v}(a-b)-b\right\rangle. \tag{6.105}$$

Note that from Theorem 6.10 we know that $E\boxtimes E'$ has a direct summand isomorphic to $(\mathbb{H}, \widetilde{Y}_{c_{\lambda,0}})$, the projective cover of L_1. We then characterise the coevaluation map i_E by the image of the vacuum vector $\Omega \in L_1$.

$$i_E\colon \Omega \to v_{1,1}\otimes|0\rangle \xrightarrow{(\mathcal{Q}^{[2]})^{-1}} v_{1,1}\otimes|-2b\rangle \to \mathcal{Y}_{\lambda,-\lambda}(|\theta_\lambda\rangle, w)|\theta_{-\lambda}\rangle \xrightarrow{\mathcal{Q}^{[2]}*} \mathcal{Q}^{[2]} * \mathcal{Y}_{\lambda,-\lambda}(|\theta_\lambda\rangle, w)|\theta_{-\lambda}\rangle, \tag{6.106}$$

where the first arrow is the inclusion of L_1 into $\sigma E_{\lambda_{1,1};1,1} \subset (\mathbb{H}, \widetilde{Y})$, $(\mathcal{Q}^{[2]})^{-1}$ denotes picking preimages of $\mathcal{Q}^{[2]}$. Note that the ambiguity of picking preimages of $\mathcal{Q}^{[2]}$ is undone by applying $\mathcal{Q}^{[2]}*$ and hence the map is well-defined. The morphism R_E is thus non-zero (and hence E is rigid) if and only if $R_E(\theta_\lambda)$ is non-zero. This in turn is the case if and only if the matrix element $\langle \mu_\lambda, R_E(\theta_\lambda)\rangle$ is non-zero, where

$$\mu_\lambda = v_{1,1} \otimes \left\langle \frac{v\lambda}{u-2v}(a-b)+b\right|, \qquad \overline{\mu_\lambda} = \left\langle \frac{v\lambda}{u-2v}(a-b)+b\right| \tag{6.107}$$

and $v_{1,1}$ is the vector dual to the vacuum vector of the Virasoro minimal model. The remainder of this section will be dedicated to show that $R(\lambda, w_1, w_2)$ is non-zero.

From the characterisations (6.104) and (6.106) above of e_E and i_E, we see that

$$\begin{aligned}\langle \mu_\lambda, R_E(\theta_\lambda)\rangle &= \int\int \big\langle \mu_\lambda, Q^-(z_1)Q^+(z_2)Q^-(z_3)Q^+(z_4)\mathcal{Y}(\theta_\lambda; w_1)\mathcal{Y}(\theta_{-\lambda}; w_2)\theta_\lambda\big\rangle \mathrm{d}z_1\cdots\mathrm{d}z_4\\ &= \int\int \big\langle I^-_{1,2}(v_{1,2}; z_1) I^+_{1,2}(v_{1,2}; z_2) I^-_{1,2}(v_{1,2}; z_1) I^+_{1,2}(v_{1,2}; z_4)\big\rangle\\ &\qquad \cdot \big\langle \overline{\mu_\lambda}, \mathcal{Y}(b; z_1)\mathcal{Y}(b; z_2)\mathcal{Y}(b; z_3)\mathcal{Y}(b; z_4)\mathcal{Y}\big(\overline{\theta_\lambda}; w_1\big)\mathcal{Y}\big(\overline{\theta_{-\lambda}}; w_2\big)\overline{\theta_\lambda}\big\rangle \mathrm{d}z_1\cdots\mathrm{d}z_4,\end{aligned} \tag{6.108}$$

where, for the momement, we deliberately suppress the integration cycles. Naively one would want to integrate the z_3, z_4 variables over the regularised symplex

$$[\Delta_{w_2,0}] = \left[\Delta_{w_2,0}\left(\tfrac{u}{2v} - 1\tfrac{\lambda}{2}, \tfrac{u}{2v} - 1 + \tfrac{\lambda}{2}, 1 - \tfrac{u}{v}\right)\right] \tag{6.109}$$

considered in Section 6.2 and the z_1, z_2 variables over $[\Delta_{w_1,w_2}]$, that is, the translate of $[\Delta_{w_1-w_2,0}]$ by w_2. However, these two cycles intersect at $z_1 = z_2 = z_3 = z_4 = w_2$ leading to a number of subtle technical problems. We side step this issure entirely by using the freedom to shift the weight λ by arbitrary even integers to go the domain $\lambda \le -2 - 2\frac{u}{v}$. There we can construct cycles homologous to $[\Delta_{w_1,w_2}]$ and $[\Delta_{w_2,0}]$ which avoid these interesction subtilties and on which the integrand admits an expansion that can be evaluated term by term. Before doing this we need to understand the integrand in greater detail.

The free field part of the matrix element above is easily evaluated to

$$\begin{aligned}&\big\langle \overline{\mu_\lambda}, \mathcal{Y}(b; z_1)\mathcal{Y}(b; z_2)\mathcal{Y}(b; z_3)\mathcal{Y}(b; z_4)\mathcal{Y}\big(\overline{\theta_\lambda}; w_1\big)\mathcal{Y}\big(\overline{\theta_{-\lambda}}; w_2\big)\overline{\theta_\lambda}\big\rangle\\ &\quad= \prod_{1\le i<j\le 4}(z_i - z_j)^{1-\frac{u}{2v}} \prod_{i=1}^{4}(z_i - w_1)^{\frac{u+v\lambda}{2v}-1} \prod_{i=1}^{4}(z_i - w_2)^{\frac{u-v\lambda}{2v}-1} \prod_{i=1}^{4} z_i^{\frac{u+v\lambda}{2v}-1}\\ &\qquad \cdot (w_1 - w_2)^{1-\frac{u}{2v}} w_1^{\frac{u+v\lambda}{2v}-1} w_2^{1-\frac{u}{2v}}\end{aligned} \tag{6.110}$$

To compute the 4-point function involving the Virasoro intertwining operators, we need to solve the corresponding BPZ equation [11].

Lemma 6.15. *The intertwining operators $I^{\pm}_{1,2}$ can be normalised such that*

$$\begin{aligned}\Psi(z) &= \big\langle I^-_{1,2}(v_{1,2}; z_1) I^+_{1,2}(v_{1,2}; z_2) I^-_{1,2}(v_{1,2}; z_1) I^+_{1,2}(v_{1,2}; z_4)\big\rangle\\ &= \left(\frac{z_{1,2}z_{3,4}z_{1,4}z_{2,3}}{z_{1,3}z_{2,4}}\right)^{1-\frac{3u}{2v}} {}_2F_1\left(2 - \tfrac{3u}{v}, 1 - \tfrac{u}{v}, 2 - \tfrac{2u}{v}; \tfrac{z_{1,4}z_{2,3}}{z_{1,3}z_{2,4}}\right),\end{aligned} \tag{6.111}$$

for $|z_1| > |z_2| > |z_3| > |z_4|$, where $z_{ij} = z_i - z_j$.

Proof. By global conformal covariance any 4-point function of conformal highest weight (also known as primary) vectors can be factorised into a part depending on the differences of variables $z_i - z_j$

and a part depending only on the cross ratio

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \tag{6.112}$$

For this particular 4-point function this yields

$$\Psi(z) = z_{1,3}^{-2h_{1,2}} z_{2,4}^{-h_{1,2}} G(x), \tag{6.113}$$

where $h_{1,2} = \frac{3u}{4v} - \frac{1}{2}$ is the conformal weight of the highest weight vector $v_{1,2}$. Using the Ward identities to convert Virasoro generators into differential operators (see [33, Sec 8.3] for an overview), the singular vector relation $\big(L_{-1}^2 - \frac{u}{v} L_{-2}\big) v_{1,2} = 0$ then implies that $G(x)$ satisfies the differential equation

$$\left[\partial_x^2 + \frac{u}{v}\left(\frac{1}{x-1} + \frac{1}{x}\right)\partial_x - h_{1,2}\frac{u}{v}\left(\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) - 2h_{1,2}\frac{u}{v}\left(\frac{1}{x} - \frac{1}{x-1}\right)\right] G(x) = 0. \tag{6.114}$$

This is a Fuchsian differential equation (see, for example, [49, Sec 2.1] for an overview on solving such equations). The characteristic exponents at $x = 0$ and at $x = 1$ for this differential equation are the roots of the equation

$$\rho^2 + \left(\frac{u}{v} - 1\right)\rho - h_{1,2}\frac{u}{v} = 0, \tag{6.115}$$

that is $\rho \in \{1 - \frac{3u}{2v}, \frac{u}{2v}\}$. The characteristic exponents at $x = \infty$ are the roots of

$$\gamma^2 + (1 - \frac{2u}{v})\gamma = 0 \tag{6.116}$$

and hence $\gamma \in \{0, \frac{2u}{v} - 1\}$. For each choice of root ρ in (6.115) there is a solution to (6.114) given by a series expansion in $x^{\rho+n}$, $n \in \mathbb{Z}_{\geq 0}$. Further, the types of the intertwining operators $I_{1,2}^+$ and $I_{1,2}^-$ are $\binom{\mathsf{S}_{1,2}}{\mathsf{S}_{1,2},\ \mathsf{S}_{1,1}}$ and $\binom{\mathsf{S}_{1,1}}{\mathsf{S}_{1,2},\ \mathsf{S}_{1,2}}$ respectively, so $I_{1,2}^-(v_1, 2; z_3) I_{1,2}^+(v_1, 2; z_4)$ can be expanded as a series in $(z_3 - z_4)^{-2h_{1,2}+n}$, $n \in \mathbb{Z}_{\geq 0}$, which implies that we need to pick the root $1 - \frac{3u}{2v} = -2h_{1,2}$ as our solution to (6.115). Hence

$$G(x) = x^{1-\frac{3u}{2v}} (x-1)^{1-\frac{3u}{2v}} {}_2F_1\big(2 - 3\tfrac{u}{v}, 1 - \tfrac{u}{v}, 2 - 2\tfrac{u}{v}; x\big), \tag{6.117}$$

which proves (6.111). □

To more easily group the various factors appearing in integrand defining $\langle \mu_\lambda, R_E(\theta_\lambda)\rangle$, we introduce the notation

$$\begin{aligned}
H(z,w) &= w_1^{-2\lambda} z_1^{\frac{u+v\lambda}{2v}-1} z_2^{\frac{u+v\lambda}{2v}-1} (z_1 - z_3)^{\frac{u}{v}} (z_2 - z_4)^{\frac{u}{v}} (z_1 - z_4)^{2-2\frac{u}{v}} (z_2 - z_3)^{2-2\frac{u}{v}} \\
&\quad \cdot (w_1 - z_3)^{\frac{u+v\lambda}{2v}-1} (w_1 - z_4)^{\frac{u+v\lambda}{2v}-1} \\
&= w_1^{-2\lambda} z_1^{1+\frac{v\lambda-u}{2v}} z_2^{1+\frac{v\lambda-u}{2v}} \left(1 - \frac{z_3}{z_1}\right)^{\frac{u}{v}} \left(1 - \frac{z_4}{z_2}\right)^{\frac{u}{v}} \left(1 - \frac{z_4}{z_1}\right)^{2-2\frac{u}{v}} \left(1 - \frac{z_3}{z_2}\right)^{2-2\frac{u}{v}} \\
&\quad \cdot (w_1 - z_3)^{\frac{u+v\lambda}{2v}-1} (w_1 - z_4)^{\frac{u+v\lambda}{2v}-1} \\
\mathcal{G}_{u_1,u_2}^{\lambda}(y_1, y_2) &= (y_1 - y_2)^{2-2\frac{u}{v}} \prod_{i=1}^{2} (y_i - u_1)^{\frac{u+v\lambda}{2v}-1} \prod_{i=1}^{2} (y_i - u_2)^{\frac{u-v\lambda}{2v}-1} \\
f(w_1, w_2) &= w_1^{1-\lambda-\frac{u}{2v}} w_2^{1-\frac{u}{2v}} (w_1 - w_2)^{1-\frac{u}{2v}}.
\end{aligned} \tag{6.118}$$

Then we can write the integrand defining $\langle \mu_\lambda, R_E(\theta_\lambda)\rangle$ as

$$M(\lambda, w, z) = f(w_1, w_2) H(z, w)$$

$$ {}_2F_1\left(2-3\tfrac{u}{v},1-\tfrac{u}{v},2-2\tfrac{u}{v};\tfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\right)\mathcal{G}^{\lambda}_{w_1,w_2}(z_1,z_2)\mathcal{G}^{-\lambda}_{w_2,0}(z_3,z_4). \quad (6.119)$$

The factor of $w_1^{-2\lambda}$ in the definition of H is there to simplify the following expansion formula.

$$\begin{aligned} &H(z,w)\,{}_2F_1\left(2-\tfrac{3u}{v},1-\tfrac{u}{v},2-\tfrac{2u}{v};\tfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\right)\\ &\quad= w_2^{\frac{v\lambda-u}{v}+2}w_1^{\frac{u+v\lambda}{v}-2}\sum_i H^{(i)}_{1,2}(z_1,z_2,w_2)H^{(i)}_{3,4}(z_3,z_4,w_1),\\ &H^{(i)}_{1,2}\in\mathbb{C}\left[w_2^{\pm1}\right]\otimes\mathbb{C}[\![z_i-w_2\mid i=1,2]\!]^{\mathfrak{S}},\\ &H^{(i)}_{3,4}\in\mathbb{C}\left[w_1^{\pm1}\right]\otimes\mathbb{C}[z_3,z_4]^{\mathfrak{S}}, \end{aligned} \quad (6.120)$$

where the superscript $\mathfrak{S}$ indicates invariance under permuting z_1 with z_2 or z_3 with z_4, respectively, and where the factor $w_2^{\frac{v\lambda-u}{v}+2}w_1^{\frac{u+v\lambda}{v}-2}$ comes from

$$\begin{aligned} z_1^{1+\frac{v\lambda-u}{2v}}z_2^{1+\frac{v\lambda-u}{2v}}(w_1-z_3)^{\frac{u+v\lambda}{2v}-1}(w_1-z_4)^{\frac{u+v\lambda}{2v}-1} &= w_2^{\frac{v\lambda-u}{v}+2}w_1^{\frac{u+v\lambda}{v}-2}\left(1+\frac{z_1-w_2}{w_2}\right)^{1+\frac{v\lambda-u}{2v}}\\ &\cdot\left(1+\frac{z_2-w_2}{w_2}\right)^{1+\frac{v\lambda-u}{2v}}\left(1-\frac{z_3}{w_1}\right)^{\frac{u+v\lambda}{2v}-1}\left(1-\frac{z_4}{w_1}\right)^{\frac{u+v\lambda}{2v}-1}. \end{aligned} \quad (6.121)$$

The non-integral exponents of the factors mixing the z_1,z_2 variables with the z_3,z_4 make the integral (6.108) very difficult to evaluate head on. So instead we will construct regularised cycles which are homologous (in appropriate homology groups) to $[\Delta_{w_1,w_2}]$, $[\Delta_{w_2,0}]$. These will allow us to decompose the integral into more manageable parts. Specifically, they will allow us to pull the sum appearing in (6.120) out of the integral.

Definition 6.16. Let $\left[\Delta^0_{w_1,w_2}\right]$ and $\left[\Delta^0_{w_2,0}\right]$ be the surfaces constructed in Figure 3.

Lemma 6.17. *Consider the surfaces $\left[\Delta^0_{w_1,w_2}\right]$ and $\left[\Delta^0_{w_2,0}\right]$ constructed in Definition 6.16. Then $[\Delta^0_{w_1,w_2}]\in H_2^{\mathrm{lf}}(\mathcal{Z}_{w_1,w_2},\mathcal{L}^\vee(\mathcal{G}^\lambda_{w_1,w_2}))$ and $[\Delta^0_{w_2,0}]\in H_2^{\mathrm{lf}}(\mathcal{Z}_{w_2,0},\mathcal{L}^\vee(\mathcal{G}^{-\lambda}_{w_2,0}))$ and these cycles satisfy*

$$\partial[\Delta^0_{w_1,w_2}]=\{(z_1,z_2)=(w_2,w_2)\},\qquad \partial[\Delta^0_{w_2,0}]=\{(z_3,z_4)=(w_2,w_2)\}. \quad (6.122)$$

Further, $\left[\Delta^0_{w_1,w_2}\right]$ and $\left[\Delta^0_{w_2,0}\right]$ are, respectively, homologous to $[\Delta_{w_1,w_2}]$ and $[\Delta_{w_2,0}]$.

Proof. This lemma is a combination of the reasoning and constructions in Sections 3.2.4 and 3.2.5 of [8]: The fact that $\left[\Delta^0_{w_1,w_2}\right]$ and $\left[\Delta^0_{w_2,0}\right]$ lie in the stated homology groups is [8, Lem 3.3] and these newly constructed cycles being homologous to $[\Delta_{w_1,w_2}]$ and $[\Delta_{w_2,0}]$, respectively, is [8, Thm 3.1]. ∎

Proposition 6.18. *Let $\lambda\in\mathbb{C}$ with $\Re\lambda\le -2-2\frac{u}{v}$, then*

$$\begin{aligned} \langle\mu_\lambda,R_E(\theta_\lambda)\rangle &= \int_{[\Delta^0_{w_1,w_2}]}\left(\int_{[\Delta^0_{w_2,0}]}M(\lambda,w,z)\mathrm{d}z_3\wedge\mathrm{d}z_4\right)\mathrm{d}z_1\wedge\mathrm{d}z_2,\\ &= f(w_1,w_2)\sum_i\int_{\left[\Delta^0_{w_1,w_2}\right]}H^{(i)}_{1,2}(z_1,z_2,w_2)\mathcal{G}^\lambda(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2\\ &\qquad\int_{\left[\Delta^0_{w_2,0}\right]}H^{(i)}_{3,4}(z_3,z_4,w_1)\mathcal{G}^{-\lambda}(z_3,z_4)\mathrm{d}z_3\wedge\mathrm{d}z_4. \end{aligned} \quad (6.123)$$

In particular the right-hand side is well defined and the sum converges for $\Re(\lambda)\le -2-2\frac{u}{v}$.

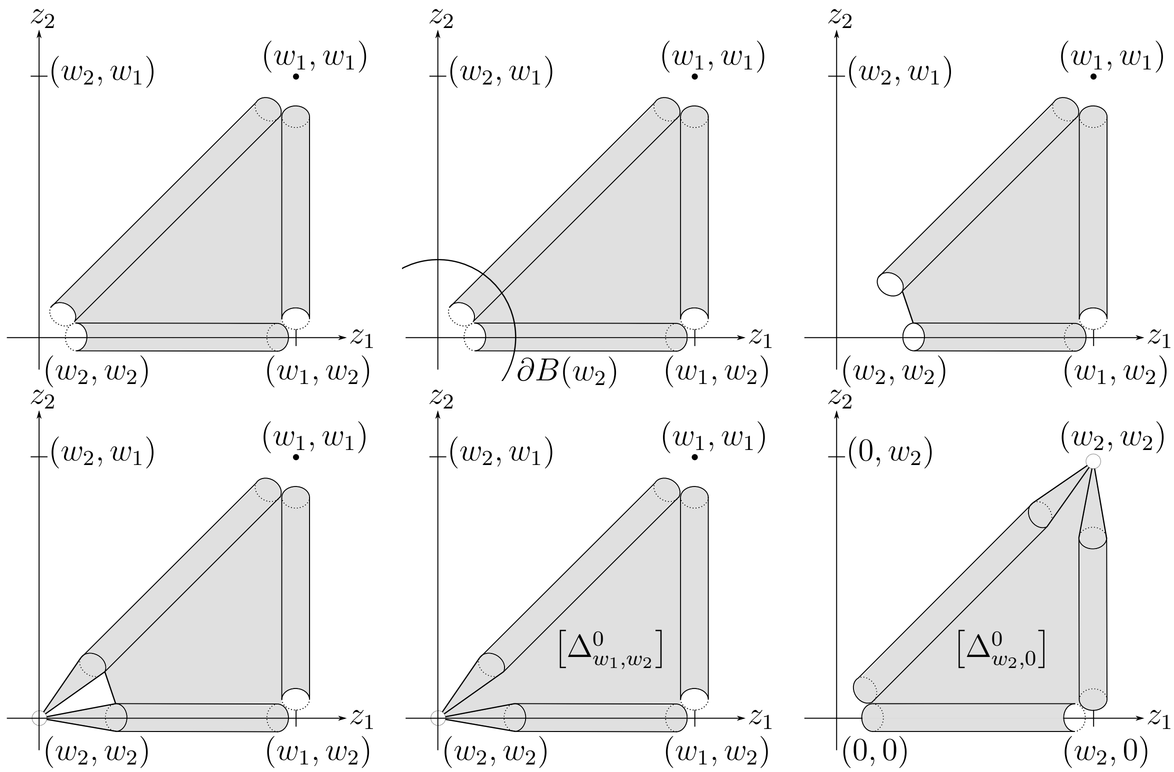

FIGURE 3. Constructing the regularised cycles $\left[\Delta^0_{w_1,w_2}\right]$ and $\left[\Delta^0_{w_2,0}\right]$: Starting from the top left with the standard twisted cycle built from the 2-simplex $\{w_1 > |z_1| > |z_2| > w_2\}$ by replacing the edges by tubular neighbourhoods. Note that the tubes do not intersect. This is just an artifact of presenting the image in two real (as opposed to complex) dimensions. In the second image $B(w_2) = \{4|z_1 - w_2|^2 + 4|z_2 - w_2|^2 \leq (w_1 + w_2)^2\}$ denotes a ball centred at (w_2, w_2) with radius $\frac{w_1^2+w_2^2}{4}$. We cut away all of the twisted cycle that is within the ball leaving two circles on the tubes connected by a line and obtain the third image (see [73, Sec 5.4] and [8, Sec 3.2.4]). The circles are then connected to (w_2, w_2) by a pair of cones, whose orientation matches that of the circle they are attached to. This leaves a wedge shaped hole and is depicted in the fourth image. The hole is filled in to give $\left[\Delta^0_{w_1,w_2}\right]$ in the fifth image. The same procedure but with (w_1, w_2) replaced by $(w_2, 0)$ yields $\left[\Delta^0_{w_2,0}\right]$.

Proof. Let $[\Delta^\eta_{x_i,x_{i+1}}] = [\Delta^0_{x_i,x_{i+1}}] \cap \{|z_i - w_2|^2 + |z_{i+1} - w_2|^2 \geq \eta^2\}$, where $\frac{w_1+w_2}{2} > \eta > 0$. Let us denote $\langle \mu_\lambda, R_E(\theta_\lambda)\rangle_\eta$ the integral (6.108) integrated over $[\Delta^\eta_{w_1,w_2}]$ and $[\Delta^\eta_{w_2,0}]$. From the definition of $[\Delta^\eta_{x_i,x_{i+1}}]$, for any points $(z_1, z_2) \in [\Delta^\eta_{x_1,x_2}]$, $(z_3, z_4) \in [\Delta^\eta_{x_3,x_4}]$, we see that $|z_k| > |z_l|$, $k = 1, 2,\ l = 3, 4$. That is the cycle $[\Delta^\eta_{w_1,w_2}] \times [\Delta^\eta_{w_2,0}]$ is contained in the domain

$$\{w_1 - w_2 > |z_1 - w_2| > |z_2 - w_2| > 0\} \cap \{|z_2| > |z_3| > |z_4| > 0\}. \tag{6.124}$$

Further, the right-hand side of the expansion (6.120) is absolutely convergent on the domain (6.124), hence we can pull the sum out of the integral. That is, we have

$$\begin{aligned}\langle \mu_\lambda, R_E(\theta_\lambda)\rangle_\eta &= \int_{[\Delta^\eta_{w_1,w_2}]\otimes[\Delta^\eta_{w_2,0}]} M(\lambda, w, z)\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4 \\ &= \int_{[\Delta^\eta_{w_1,w_2}]} \left(\int_{[\Delta^\eta_{w_2,0}]} M(\lambda, w, z)\mathrm{d}z_3 \wedge \mathrm{d}z_4 \right) \mathrm{d}z_1 \wedge \mathrm{d}z_2\end{aligned}$$

$$= f(w_1,w_2)\int_{[\Delta^\eta_{w_1,w_2}]}\int_{[\Delta^\eta_{w_2,0}]}\sum_i\Big(H^{(i)}_{1,2}(z_1,z_2,w_2)H^{(i)}_{3,4}(z_3,z_4,w_1)\mathcal{G}^\lambda(z_1,z_2)$$
$$\mathcal{G}^{-\lambda}(z_3,z_4)\mathrm{d}z_3\wedge\mathrm{d}z_4\Big)\mathrm{d}z_1\wedge\mathrm{d}z_2$$
$$= f(w_1,w_2)\sum_i\int_{[\Delta^\eta_{w_1,w_2}]}H^{(i)}_{1,2}(z_1,z_2,w_2)\mathcal{G}^\lambda(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2$$
$$\int_{[\Delta^\eta_{w_2,0}]}H^{(i)}_{3,4}(z_3,z_4,w_1)\mathcal{G}^{-\lambda}(z_3,z_4)\mathrm{d}z_3\wedge\mathrm{d}z_4. \tag{6.125}$$

Since $\Re\lambda\le -2-2\frac{u}{v}$, we see that $M(\lambda,w,z)$ is bounded on the closure of $U_{w_2}\cap([\Delta^0_{w_1,w_2}]\times[\Delta^0_{w_2,0}])$ where U_{w_2} is a small neighbourhood of w_2. Thus from (6.125), we have

$$\langle\mu_\lambda,R_E(\theta_\lambda)\rangle_0 = \int_{[\Delta^0_{w_1,w_2}]}\left(\int_{[\Delta^0_{w_2,0}]}M(\lambda,w,z)\mathrm{d}z_3\wedge\mathrm{d}z_4\right)\mathrm{d}z_1\wedge\mathrm{d}z_2$$
$$= f(w_1,w_2)\sum_i\int_{[\Delta^0_{w_1,w_2}]}H^{(i)}_{1,2}(z_1,z_2,w_2)\mathcal{G}^\lambda(z_1,z_2)\mathrm{d}z_1\wedge\mathrm{d}z_2$$
$$\int_{[\Delta^0_{w_2,0}]}H^{(i)}_{3,4}(z_3,z_4,w_1)\mathcal{G}^{-\lambda}(z_3,z_4)\mathrm{d}z_3\wedge\mathrm{d}z_4. \tag{6.126}$$

□

Next we show that $\langle\mu_\lambda,R_E(\theta_\lambda)\rangle$ is non-zero by showing that the integral (6.123) is non-zero.

Lemma 6.19. *Let $\lambda\in\mathbb{C}$, then for $\Re\lambda\le -2-2\frac{u}{v}$ and $\lambda-3\frac{u}{v}\notin 2\mathbb{Z}$ we have that $\langle\mu_\lambda,R_E(\theta_\lambda)\rangle\ne 0$.*

Proof. We set $R(\lambda,w_1,w_2)=(w_1^{\lambda+\frac{u}{v}-2}w_2^{\lambda+2-\frac{u}{v}}f(w_1,w_2))^{-1}\langle\mu_\lambda,R_E(\theta_\lambda)\rangle$. We will see shortly that $R(\lambda,w_1,w_2)$ admits an expansion in $\mathbb{C}[\![\frac{w_1-w_2}{w_2},\frac{w_2}{w_1}]\!]$ and we will show that this expansion is non-zero by showing that the constant term is non-zero. Computing this expansion is most easily done in a new set of variables $y_1,\dots,y_4$ related to the $z_1,\dots,z_4$ as follows.

$$z_1=(w_1-w_2)y_1+w_2,\ \ z_2=(w_1-w_2)y_2+w_2,\ \ z_3=w_2y_3,\ \ z_4=w_2y_4. \tag{6.127}$$

In these new variables the $\mathcal{G}$ functions are expressed as

$$\mathcal{G}^\lambda_{w_1.w_2}(z_1,z_2)=(w_1-w_2)^{-2}\mathcal{G}^\lambda_{1,0}(y_1,y_2),\qquad \mathcal{G}^{-\lambda}_{w_2,0}(z_3,z_4)=w_2^{-2}\mathcal{G}^{-\lambda}_{1,0}(y_3,y_4). \tag{6.128}$$

In particular, the right-hand side of (6.123) becomes

$$(w_1-w_2)^2w_2^2\int_{[\Delta^0_{1,0}]}\left(\int_{[\Delta^0_{1,0}]}\overline{M}(\lambda,w,y)\mathrm{d}y_3\wedge\mathrm{d}y_4\right)\mathrm{d}y_1\wedge\mathrm{d}y_2, \tag{6.129}$$

where $\overline{M}(\lambda,w,y)$ is $M(\lambda,w,z)$ after the change of variables (6.127). We can then integrate (6.129) term by term as a series in $\frac{w_1-w_2}{w_2},\frac{w_2}{w_1}$. Thus we consider how the factors H and ${}_2F_1$ of $M(\lambda,w,z)$ are expand after the change of variables (6.127). The function H can then be expressed as

$$(w_2^{\frac{v\lambda-u}{v}+2}w_1^{\frac{u+v\lambda}{v}-2})^{-1}H(z,w)$$
$$=\left(1+\frac{z_1-w_2}{w_2}\right)^{1+\frac{v\lambda-u}{2v}}\left(1+\frac{z_2-w_2}{w_2}\right)^{1+\frac{v\lambda-u}{2v}}\left(1-\frac{z_3}{w_1}\right)^{\frac{u+v\lambda}{2v}-1}\left(1-\frac{z_4}{w_1}\right)^{\frac{u+v\lambda}{2v}-1}$$
$$\cdot\left(1-\frac{z_3}{w_2\Big(1+\frac{z_1-w_2}{w_2}\Big)}\right)^{\frac{u}{v}}\left(1-\frac{z_4}{w_2\Big(1+\frac{z_2-w_2}{w_2}\Big)}\right)^{\frac{u}{v}}$$

$$
\begin{aligned}
&\cdot\left(1-\frac{z_4}{w_2\left(1+\frac{z_1-w_2}{w_2}\right)}\right)^{2-2\frac{u}{v}}\left(1-\frac{z_3}{w_2\left(1+\frac{z_2-w_2}{w_2}\right)}\right)^{2-2\frac{u}{v}}.\\
&=\left(1+\frac{w_1-w_2}{w_2}y_1\right)^{1+\frac{v\lambda-u}{2v}}\left(1+\frac{w_1-w_2}{w_2}y_2\right)^{1+\frac{v\lambda-u}{2v}}\left(1-\frac{w_2}{w_1}y_3\right)^{\frac{u+v\lambda}{2v}-1}\left(1-\frac{w_2}{w_1}y_4\right)^{\frac{u+v\lambda}{2v}-1}\\
&\cdot\left(1-\frac{y_3}{1+\frac{w_1-w_2}{w_2}y_1}\right)^{\frac{u}{v}}\left(1-\frac{y_4}{1+\frac{w_1-w_2}{w_2}y_2}\right)^{\frac{u}{v}}\\
&\cdot\left(1-\frac{y_4}{1+\frac{w_1-w_2}{w_2}y_1}\right)^{2-2\frac{u}{v}}\left(1-\frac{y_3}{1+\frac{w_1-w_2}{w_2}y_2}\right)^{2-2\frac{u}{v}}.\\
&=(1-y_3)^{2-\frac{u}{v}}(1-y_4)^{2-\frac{u}{v}}+o(\tfrac{w_1-w_2}{w_2},\tfrac{w_2}{w_1}) \qquad (6.130)
\end{aligned}
$$

Before we consider the hypergeometric function note that

$$
\begin{aligned}
1-\frac{z_1-z_2}{z_1-z_3}\frac{z_3-z_4}{z_2-z_4}=\frac{z_1-z_4}{z_1-z_3}\frac{z_2-z_3}{z_2-z_4}&=\frac{\left(\frac{w_1-w_2}{w_2}y_1+1-y_4\right)}{\left(\frac{w_1-w_2}{w_2}y_1+1-y_3\right)}\frac{\left(\frac{w_1-w_2}{w_2}y_2+1-y_3\right)}{\left(\frac{w_1-w_2}{w_2}y_2+1-y_4\right)}\\
&=1+o(\tfrac{w_1-w_2}{w_2},\tfrac{w_2}{w_1}) \qquad (6.131)
\end{aligned}
$$

and hence we need to expand around 1 rather than 0. This can be done using the connection formula

$$
\begin{aligned}
{}_2F_1\left(2-3\tfrac{u}{v},1-\tfrac{u}{v},2-2\tfrac{u}{v};x\right)&=\frac{\Gamma(2-2\frac{u}{v})\Gamma(2\frac{u}{v}-1)}{\Gamma(\frac{u}{v})\Gamma(1-\frac{u}{v})}{}_2F_1\left(2-3\tfrac{u}{v},1-\tfrac{u}{v},2-2\tfrac{u}{v};1-x\right)\\
&+(1-x)^{2\frac{u}{v}-1}\frac{\Gamma(2-2\frac{u}{v})\Gamma(1-2\frac{u}{v})}{\Gamma(2-3\frac{u}{v})\Gamma(1-\frac{u}{v})}{}_2F_1\left(\tfrac{u}{v},1-\tfrac{u}{v},2\tfrac{u}{v};1-x\right). \qquad (6.132)
\end{aligned}
$$

Recalling the well known formulae for hypergeometric functions evaluated at 1 we therefore obtain

$$
\begin{aligned}
&{}_2F_1\left(2-3\tfrac{u}{v},1-\tfrac{u}{v},2-2\tfrac{u}{v};\tfrac{z_1-z_2}{z_1-z_3}\tfrac{z_3-z_4}{z_2-z_4}\right)\\
&=\frac{\Gamma(2-2\frac{u}{v})\Gamma(2\frac{u}{v}-1)}{\Gamma(\frac{u}{v})\Gamma(1-\frac{u}{v})}\frac{\Gamma(2-2\frac{u}{v})\Gamma(2\frac{u}{v}-1)}{\Gamma(\frac{u}{v})\Gamma(1-\frac{u}{v})}\\
&\quad+\frac{\Gamma(2-2\frac{u}{v})\Gamma(1-2\frac{u}{v})}{\Gamma(2-3\frac{u}{v})\Gamma(1-\frac{u}{v})}\frac{\Gamma(2\frac{u}{v})\Gamma(2\frac{u}{v}-1)}{\Gamma(\frac{u}{v})\Gamma(3\frac{u}{v}-1)}+o(\tfrac{w_1-w_2}{w_2},\tfrac{w_2}{w_1})\\
&=\frac{\sin(\pi\frac{u}{v})^2+\sin(\pi3\frac{u}{v})\sin(\pi\frac{u}{v})}{\sin(\pi2\frac{u}{v})^2}+o(\tfrac{w_1-w_2}{w_2},\tfrac{w_2}{w_1})=1+o(\tfrac{w_1-w_2}{w_2},\tfrac{w_2}{w_1}), \qquad (6.133)
\end{aligned}
$$

where in the second equality we have used Euler's reflection formula $\Gamma(1-x)\Gamma(x)=\frac{\pi}{\sin(\pi x)}$. Thus

$$
\begin{aligned}
R(\lambda,w_1,w_2)&=\int_{[\Delta_{1,0}]}\mathcal{G}^{\lambda}_{1,0}(y_1,y_2)\mathrm{d}y_1\wedge\mathrm{d}y_2\\
&\quad\cdot\int_{[\Delta_{1,0}]}\mathcal{G}^{-\lambda}_{1,0}(y_3,y_4)(1-y_3)^{2-\frac{u}{v}}(1-y_4)^{2-\frac{u}{v}}\mathrm{d}y_3\wedge\mathrm{d}y_4+o(\tfrac{w_1-w_2}{w_2},\tfrac{w_2}{w_1})\\
&=\mathcal{I}_1[1]\left(\tfrac{u-v\lambda}{2v}-1,\tfrac{u+v\lambda}{2v}-1,1-\tfrac{u}{v}\right)\cdot\mathcal{I}_1[1]\left(\tfrac{u+v\lambda}{2v}-1,\tfrac{-u-v\lambda}{2v}+1,1-\tfrac{u}{v}\right)+o(\tfrac{w_1-w_2}{w_2},\tfrac{w_2}{w_1}), \qquad (6.134)
\end{aligned}
$$

where we have evaluated the integrals using (6.48). From the formula (6.48) for $\mathcal{I}$ in terms of Γ-functions, we see that the Γ functions in the numerator have poles when $\lambda-3\frac{u}{v}\in\mathbb{Z}$, but that $\langle\mu_\lambda,R_E(\theta_\lambda)\rangle\neq 0$ when $\lambda-3\frac{u}{v}\notin\mathbb{Z}$. Hence the lemma follows. □

Theorem 6.20. *The simple projective modules $\sigma^\ell E_{\mu;1,1}$, $\mu \in \mathbb{R}/2\mathbb{Z}$, $\mu \neq \pm\lambda_{1,1}+2\mathbb{Z}$, $\lambda \in \mathbb{Z}$ are rigid, with rigid dual $\sigma^{-\ell}E_{-\mu;1,1}$.*

Proof. Without loss of generality we can restrict ourselves to only a single value of the spectral flow parameter and hence we choose $\ell = 0$. As mentioned at the beginning of this section by [26, Lem 4.2.1 and Cor 4.2.2] a sufficient condition for $E_{-\mu;1,1}$ being the rigid dual of $E_{\mu;1,1}$ is the non-vanishing of the matrix element (6.108). So initially assume that $\mu \neq 3\frac{u}{v}+2\mathbb{Z}$ and pick a representative $\lambda \in \mu$ which is sufficiently negative for Lemma 6.19 to apply. Hence the matrix element $\langle \mu_\lambda, R_E(\theta_\lambda)\rangle$ is non-vanishing and $E_{\mu;1,1}$ is rigid. To extend this conclusion to the previously excluded case, let $\mu_1,\mu_2 \in \mathbb{R}/2\mathbb{Z}$, $\mu_1,\mu_2 \notin \{\pm\frac{u}{v}+2\mathbb{Z}, 3\frac{u}{v}+2\mathbb{Z}\}$ satisfying $\mu_1+\mu_2 = 3\frac{u}{v}+2\mathbb{Z}$. Then the $E_{\mu_i;1,1}$ are both rigid (by our reasoning so far) and projective. From the tensor product formula (2.21), which we have already proved, we see that the tensor product $E_{\mu_1;1,1}\boxtimes E_{\mu_2;1,1}$ contains $\sigma E_{3\frac{u}{v}+2\mathbb{Z};1,1}$ and $\sigma^{-1}E_{5\frac{u}{v}+2\mathbb{Z};1,1}$ as direct summands. Hence $E_{3\frac{u}{v}+2\mathbb{Z};1,1}$is also rigid and the theorem follows. ☐

7. Non-semisimple fusion decompositions

Theorem 7.1. *The non-semisimple fusion product decomposition of Conjecture 2.9 hold. That is, for $1\le r\le u-1$, $\mu,\mu' \in \mathbb{R}/2\mathbb{Z}$, $\pm\lambda_{1,1}\notin\mu$, $\pm\lambda_{r,1}\notin\mu'$ and $r-1\in\mu+\mu'$*

$$E_{\mu;1,1}\boxtimes E_{\mu';r,1} \cong \sigma^{-1}P_{u-r,v-1}\oplus(1-\delta_{v,2})E_{\mu+\mu';r,2}, \tag{7.1}$$

while for $1\le r\le u-1$, $2\le s\le v-2$, $\mu,\mu'\in\mathbb{R}/2\mathbb{Z}$, $\pm\lambda_{1,1}\notin\mu$, $\pm\lambda_{r,s}\notin\mu'$,

$$E_{\mu;1,1}\boxtimes E_{\mu';r,s}\cong\begin{cases} P_{r,s-1}\oplus\sigma^{-1}E_{\mu+\mu'+\frac{u}{v};r,s}\oplus E_{\mu+\mu';r,s+1}, & \text{if } \lambda_{r,s-1}\in\mu+\mu',\\ P_{u-r,v-s-1}\oplus\sigma^{-1}E_{\mu+\mu'+\frac{u}{v};r,s}\oplus E_{\mu+\mu';r,s-1} & \text{if } \lambda_{u-r,v-s-1}\in\mu+\mu',\\ \sigma^{-1}P_{r,s}\oplus\sigma E_{\mu+\mu'-\frac{u}{v};r,s}\oplus E_{\mu+\mu';r,s-1}, & \text{if } \lambda_{r,s+1}\in\mu+\mu',\\ \sigma^{-1}P_{u-r,v-s}\oplus\sigma E_{\mu+\mu'-\frac{u}{v};r,s}\oplus E_{\mu+\mu';r,s+1} & \text{if } \lambda_{u-r,v-s+1}\in\mu+\mu'.\end{cases} \tag{7.2}$$

Moreover, the category $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{wt}$ is rigid.

Proof. Note that a module is rigid if and only if any of its spectral flows are rigid. We begin by proving (7.1) for $r=1$. Note that from Propositions 4.15, 5.4 and 5.5 we have the equality $\dim\mathrm{Hom}_{\mathsf{A}_1(u,v)}(E_{\mu;1,1}\boxtimes E_{-\mu;1,1},E_{0;1,2}) = 1-\delta_{v,2}$, hence since $E_{0;1,2}$ is simple and projective, if it exists (that is, if $v>2$) it must be a direct summand. From Propositions 4.15, 5.4 and 5.5 we also see that there can be no simple projective summands other than $E_{0;1,2}$. Further, from Theorem 6.10 we also know that $\sigma^{-1}P_{u-1,v-1}$ is a direct summand of $E_{\mu;1,1}\boxtimes E_{-\mu;1,1}$. Thus

$$E_{\mu;1,1}\boxtimes E_{-\mu;1,1}\cong\sigma^{-1}P_{u-1,v-1}\oplus(1-\delta_{v,2})E_{0;1,2}\oplus M, \tag{7.3}$$

where M is a projective module whose composition factors must all be spectral flows of highest weight modules, that is of the form $\sigma^\ell D^+_{r,s}$. We then choose $\nu\in\mathbb{R}/2\mathbb{Z}$ such that $\nu\notin\{2\mathbb{Z},-\mu,[\pm 2t],[\pm t\pm'\lambda_{1,2}],[\pm\lambda_{1,3}]\}$, and by associativity we can evaluate $E_{\nu;1,1}\boxtimes E_{\mu;1,1}\boxtimes E_{-\mu;1,1}$ as

$$\begin{aligned}(E_{\nu;1,1}\boxtimes E_{\mu;1,1})\boxtimes E_{-\mu;1,1}\cong{}&\sigma^{-2}E_{\nu+2\frac{u}{v};1,1}\oplus\sigma^2E_{\nu-2\frac{u}{v};1,1}\oplus(3-\delta_{v,2})E_{\nu;1,1}\\ &\oplus(1-\delta_{v,2})\Big(\sigma^{-1}E_{\nu+\frac{u}{v};1,2}\oplus E_{\nu-\frac{u}{V};1,2}\oplus(1-\delta_{v,3})E_{\nu;1,3}\Big)\end{aligned} \tag{7.4}$$

and note that the right-hand side of (7.4) is semisimple with $5-\delta_{v,2}+(1-\delta_{v,2})(3-\delta_{v,3})$ simple projective summands. Evaluating the fusion product with the other bracketing yields

$$\begin{aligned}E_{\nu;1,1}\boxtimes(E_{\mu;1,1}\boxtimes E_{-\mu;1,1})&\cong E_{\nu;1,1}\boxtimes\big(\sigma^{-1}P_{u-1,v-1}\oplus(1-\delta_{v,2})E_{0;1,2}\oplus M\big)\\ &=E_{\nu;1,1}\boxtimes\big(\sigma^{-1}P_{u-1,v-1}\oplus M\big)\end{aligned}$$

$$\oplus (1-\delta_{v,2})\Big(\sigma^{-1}E_{\nu+\frac{u}{v};1,2} \oplus \sigma E_{\nu-\frac{u}{v};1,2} \oplus E_{\nu;1,1} \oplus (1-\delta_{v,3})E_{\nu;1,3}\Big). \quad (7.5)$$

The projective module $P_{u-1,v-1}$ has four composition factors and since the right-hand side of (7.4) is semisimple, $E_{\nu;1,1} \boxtimes P_{u-1,v-1}$ must contribute at least four simple summands in addition to the $(1-\delta_{v,2})(3-\delta_{v,3})$ contributed by $E_{\nu;1,1} \boxtimes E_{0;1,2}$, which brings the total up to $5-\delta_{v,2}+(1-\delta_{v,2})(3-\delta_{v,3})$. Hence $E_{\nu;1,1} \boxtimes M = 0$, that is $M = 0$.

The decomposition formula (7.1) for general r now follows from the formula for $r = 1$ by repeatedly applying $L_2\boxtimes$.

Next we show (7.2) for $r = 1$ (once this formula has been establish for $r = 1$, the formula for general r follows again by applying $L_2\boxtimes$). We consider first the case $\mu + \mu' = [\lambda_{r,s-1}]$, then Propositions 4.15, 5.4 and 5.5 imply that $E_{\mu;1,1} \boxtimes E_{\mu';1,s}$ has exactly two direct summands isomorphic to $\sigma^{-1}E_{[\lambda_{r,s-1}+\frac{u}{v}];1,s}$ and $E_{[\lambda_{r,s-1}+\frac{u}{v}];1,s+1}$. From Proposition 5.4 we also see that there exists a surjective intertwining operator of type $\binom{\sigma E^+_{1,s}}{E_{\mu;1,1},\,E_{\mu';1,s}}$. Hence $P_{1,s-1}$ (which by (2.18) is the projective cover of $\sigma^{-1}E^-_{1,s}$) must be a direct summand of $E_{\mu;1,1} \boxtimes E_{\mu';1,s}$. Thus

$$E_{\mu;1,1} \boxtimes E_{\mu';1,s} = \sigma^{-1}P_{1,s-1} \oplus \sigma^{-1}E_{[\lambda_{r,s-1}+\frac{u}{v}];1,s} \oplus E_{[\lambda_{r,s-1}+\frac{u}{v}];1,s+1} \oplus M, \quad (7.6)$$

where again M is a projective module whose composition factors must all be spectral flows of highest weight modules, that is of the form $\sigma^\ell D^+_{r,s}$. We can now use the same reasoning for concluding that $M = 0$ as we did for (7.3) by applying $E_{\nu;1,1}$ to (7.6) with $\nu \in \mathbb{R}/2\mathbb{Z}$ chosen such that the fusion product will be semisimple. Counting the number of simple summands contributed by each summand in (7.6) to $E_{\nu;1,1} \boxtimes E_{\mu;1,1} \boxtimes E_{\mu';1,s}$ will again imply that $M = 0$. Similarly for $\mu + \mu' = [\lambda_{u-1,v-s-1}]$ Proposition 5.4 implies the existence of a surjective intertwining operator of type $\binom{\sigma E^+_{u-1,v-s}}{E_{\mu;1,1},\,E_{\mu';1,s}}$ with $P_{u-1,v-s-1}$ as the corresponding projective summand. While for $\mu + \mu' = [\lambda_{1,s+1}]$ and $\mu + \mu' = [\lambda_{u-1,v-s+1}]$ the corresponding intertwining operator types and projective summands are $\binom{\sigma^{-1}E^-_{1,s+1}}{E_{\mu;1,1},\,E_{\mu';1,s}}$ and $\sigma^{-1}P_{1,s}$, and $\binom{\sigma^{-1}E^-_{u-1,v-s+1}}{E_{\mu;1,1},\,E_{\mu';1,s}}$ and $\sigma^{-1}P_{u-1,v-s}$, respectively.

With the fusion product decomposition formulae of Conjecture 2.9 now proved, we see that all projective modules appear as direct summands in repeated products of the simple projective modules $E_{1;1,\mu}$ and $E_{2;1,\mu}$. Hence all projective modules are rigid. To conclude that all the composition factors of the non-simple indecomposable projectives are rigid, we note that in the nomenclature of [58, App A] $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ is weakly rigid [58, Def A.4] and semirigid [58, Def A.6] because the $\mathsf{A}_1(u,v)$ is isomorphic to itself and is hence a dualising object in the sense of Grothendieck-Verdier categories (see [6, Thm 2.12]). Further, $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ is Frobenius [58, Def A.9] because there are sufficiently many injectives and projectives, all projectives are injective and vice versa, and all projectives are rigid. Thus we can use [58, Prop A.2], which states that if two terms in an exact sequence in a Frobenius category are rigid, then the third is as well. We already know that the simple modules $L_r \cong \sigma^{-1}D^+_{u-r,v-1}$, $1 \le r \le u-1$ are rigid. From Proposition 2.7 we know that $P_{r,v-1}$ admits a socle filtration $P_{r,v-1} = M_2 \supset M_1 \supset M_0$ satisfying

$$M_0 \cong D^+_{r,v-1} \cong M_2/M_1, \quad M_1/M_0 \cong \sigma^{-1}D^+_{u-r,v-2} \oplus \sigma^2 D^+_{u-r,1}. \quad (7.7)$$

Thus [58, Prop A.2] implies that M_2/M_0 is rigid because M_0 and $M_2 = P_{r,v-1}$ are rigid and in turn M_1/M_0 is rigid because $M_2/M_1 \cong (M_2/M_0)/(M_1/M_0)$ and M_2/M_0 are. Hence $D^+_{u-r,v-2}$ and $D^+_{u-r,1}$ are rigid because they are (spectral flows) of direct summands of rigid modules. By the same argument the rigidity of the remaining $D^+_{r,s}$ can be deduced inductively from the socle filtration of $P_{r,s}$, $1 \le s \le v-2$, which satisfies

$$M_0 \cong D^+_{r,s} \cong M_2/M_1, \quad M_1/M_0 \cong \sigma^{-1}D^+_{r,s-1} \oplus \sigma D^+_{r,s+1}. \quad (7.8)$$

and hence the rigidity of $D^+_{r,s}$ implies the rigidity of $D^+_{r,s+1}$. Thus $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$ is rigid. □

Theorem 7.2. *The category* $\mathsf{N}(u,v)\text{-}\mathsf{mod}^{wt}$ *is rigid.*

Proof. This follows immediately from Lemma 3.4 and the rigidity of $\mathsf{A}_1(u,v)\text{-}\mathsf{mod}^{\mathrm{wt}}$. □

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