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ABSTRACT

Using the Zhu algebra for a certain category of $\mathbb{C}$-graded vertex algebras V, we prove that if V is finitely Ω-generated and satisfies suitable grading conditions, then V is rational, i.e., it has semi-simple representation theory, with a one-dimensional level zero Zhu algebra. Here, Ω denotes the vectors in V that are annihilated by lowering the real part of the grading. We apply our result to the family of rank one Weyl vertex algebras with conformal element ω_μ parameterized by $\mu \in \mathbb{C}$ and prove that for certain non-integer values of μ, these vertex algebras, which are non-integer graded, are rational, with a one-dimensional level zero Zhu algebra. In addition, we generalize this result to appropriate $\mathbb{C}$-graded Weyl vertex algebras of arbitrary ranks.

I. INTRODUCTION

In this paper, we study various subcategories of the category of $\mathbb{C}$-graded vertex algebras, including those with a conformal element imposing various grading structures. We illustrate the nature of these subcategories via the conformal flow for the family of $\mathbb{C}$-graded Weyl vertex algebras with conformal elements ω_μ parameterized by $\mu \in \mathbb{C}$. We prove two rationality results for certain $\mathbb{C}$-graded vertex algebras that admit a conformal structure with a "nice" grading property. We, then, apply these results to show that for $\mu \in \mathbb{C}$ in a certain simply closed region of the complex plane, the corresponding Weyl vertex algebras with conformal element ω_μ are rational (in the sense that the representation theory is semisimple) and, in fact, admit only one simple "admissible" module, where "admissible" here means having a grading compatible with that of the vertex algebra. These admissible modules are also the modules that are induced from the level zero Zhu algebra.

A large portion of the literature on vertex algebras and their representations from both a mathematical and physical standpoint has been devoted to the study of rational conformal vertex algebras that are non-negative integer graded (see, for instance, Sec. 1.1 of Ref. 1 for a list of these types of vertex algebras and references therein). It is a natural question to ask whether there are other significant classes of conformal vertex algebras that are well-behaved from the representation-theoretic point of view, for instance, either rational (have semi-simple representation theory) for some category of modules or irrational (have indecomposable modules that are not simple) for some category of modules, but the category has other nice properties. This is one of the motivations behind the concept of $\mathbb{C}$-graded vertex algebras.

Conformal flow consists of the deformation of the conformal vector ω associated with a vertex operator algebra V to obtain a new conformal structure ω_μ on V for $\mu \in \mathbb{C}$, a continuous parameter. All possible conformal structures associated with the Heisenberg vertex algebra (also known as the free bosonic vertex algebra) were formally classified in Ref. 2. One of these "shifted" conformal structures for the Heisenberg vertex algebra is used in the study of the triplet algebras,[3] an important example of C_2-cofinite but irrational vertex algebras. When deforming the conformal vector, the grading restrictions associated with the $L(0)$-operator are often lost. Namely, the new conformal vector ω_μ does not necessarily satisfy that its zero mode $L^\mu(0)$ acts semisimply on V or that each graded component of V must be finite dimensional. The appropriate framework to study the new conformal vertex algebra (V, ω_μ) is the theory of $\mathbb{C}$-graded vertex algebras developed in Refs. 4 and 5 as a continuation of the development of the theory of $\mathbb{Q}$-graded vertex algebras started in Ref. 6. Motivated, in part, by this work on $\mathbb{C}$-graded vertex algebras and conformal flow, where in Ref. 5 the notion of "$\mathbb{C}$-graded vertex algebra" is more specifically called "Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra" in our work, we establish a refinement of the various concepts of $\mathbb{C}$-grading for a vertex algebra.

The Weyl vertex algebra, to which we apply our results, admits a conformal flow. The Weyl vertex algebra has its origins in physics as fields of Faddeev–Popov ghosts in the early formulations of conformal field theory where it is also known in the physics literature (sometimes with some specific fixed central charge and, thus, conformal element) as the bosonic ghost system or the $\beta\gamma$-system (cf. Refs. 7–9 and references therein). The terminology bosonic ghost system for the Weyl vertex algebra refers to the fact that this vertex algebra comprises one of the four fundamental free field algebras, those being free bosons, free fermions, bosonic ghosts, and fermionic ghosts. Consequently, the Weyl vertex algebra has played a crucial role in many aspects of conformal field theory and the study of the various mathematical structures that conformal field theory involves. Conformal flow and the relationship between conformal flow for bosonic ghosts (i.e., the Weyl vertex algebra) and conformal flow of the free boson vertex subalgebra of bosonic ghosts was studied by Feigin and Frenkel in Ref. 10, and the BRST (Becchi, Rouet, Stora, and Tyutin) cohomology was calculated for certain Fock space representations of bosonic ghosts with $\mu = 2$ and central charge $c = 26$ associated with a 26-dimensional Minkowski space. The Weyl vertex algebra was used in the study of free field realizations of affine Lie algebras and the chiral de Rham complex (cf. Refs. 11–16), and more recently, Weyl vertex algebras have been used to describe relations between conformal field theory, topological invariants, and number theory through the study of the (twined) $K3$ elliptic genus and its connections to umbral and Conway moonshine.[17]

As discussed above, both free bosons and bosonic ghosts admit multiple conformal structures. In this paper, we give a detailed analysis of the nature of the conformal structures of the Weyl vertex algebra under conformal flow, classify the "admissible" modules for the Weyl vertex algebra for certain infinite families of conformal elements, and prove that the category of such admissible modules is semisimple for these conformal structures. We denote the Weyl vertex algebra by M and the Weyl vertex algebra with conformal element ω_μ, for the complex parameter $\mu \in \mathbb{C}$, by $({}_\mu M, \omega_\mu)$ or just ${}_\mu M$.

The Weyl vertex algebra with conformal element $\omega_\mu = \omega_0$, denoted by ${}_\mu M = {}_0 M$, gives a conformal vertex algebra with central charge $c = 2$ and has been studied intensively. For this conformal structure, the Weyl vertex algebra gives a distinguished example of an irrational $\mathbb{Z}$-graded conformal vertex algebra, of current interest in the setting of logarithmic conformal field theory. The term "irrational" refers to the fact that the conformal vertex algebra does not have semisimple representation theory, and logarithmic conformal field theory involves the study of such vertex algebras and the category structure of various types of modules for these algebras (cf. Refs. 18–20). In particular, categories of modules for which the zero mode of the conformal element $L(0)$ does not act semisimply even though the modules have certain nice $L(0)$-grading properties, often referred to as "admissible," are the categories of interest and specifically those closed under the tensor product and with graded characters that have modular invariance properties.

It was shown by Ridout and Wood in Ref. 21 that ${}_0 M$ is not C_2-cofinite and admits reducible yet indecomposable modules on which the Virasoro operator $L(0)$ acts non-semisimply. Moreover, in Ref. 21, the authors identified a module category $\mathcal{F}$ that satisfies three necessary conditions arising from logarithmic conformal field theory for the category to have a nice tensor structure. They also determined the modular properties of characters in that category and computed the Verlinde formulas. Then, in Ref. 22, Adamović and Pedić computed the dimension of the spaces of intertwining operators among simple modules in category $\mathcal{F}$ and gave a vertex-algebraic proof of the Verlinde type conjectures in Ref. 21. Recently, in Ref. 23, Allen and Wood classified all indecomposable modules in $\mathcal{F}$, showed that it is rigid, and determined the direct sum decompositions for all fusion products of its modules.

In Ref. 24, certain (nonadmissible) weak modules for the Weyl algebra with conformal element $\omega_{\frac{1}{2}}$ and central charge $c = -1$ were studied in the context of Whittaker modules and modules for the fixed point subalgebra of ${}_{\frac{1}{2}}M$ under a certain automorphism. Here, it was shown that the family of Whittaker modules described in Ref. 24 is irreducible for these orbifold (fixed point) subalgebras of the Weyl algebra at $\mu = 1/2$, while in a recent paper,[25] the opposite was proved for other orbifold subalgebras where these Whittaker modules were shown to be reducible.

A natural question to ask, then, is what is the nature of the category of admissible modules for the Weyl vertex algebra with a conformal element other than ω_0 under conformal flow and, more generally, what broader results concerning the modules for non-integer graded conformal vertex algebras hold? In particular, is the category of admissible modules semisimple or not?

In this paper, we answer these questions. In particular, we study the influence of the central charge, or equivalently the choice of conformal element, on the representation theory of Weyl vertex algebras of arbitrary rank in the case when the vertex algebra is not integer graded. More generally, we study non-integer graded conformal vertex algebras. We begin our investigation by studying the (level zero) Zhu algebra

$A(V)$ of a finitely Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra V, where Ω denotes the vectors in V that are annihilated by lowering the real part of the grading. In fact, we show that if V is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra that is finitely generated (in the usual sense) such that the generators do not have integer weights and V contains an $\mathbb{N}$-graded vertex subalgebra, then V is rational in the sense that the representation theory for admissible modules is semisimple. As an application, we prove that, in particular, a rank one Weyl vertex algebra M_c with $c \in \mathbb{R}$ and $-1 < c < 2$ is rational. Consequently, we prove that a rank n Weyl vertex algebra, which is a tensor product of n rank one Weyl vertex algebras, each with $c \in \mathbb{R}$ and $-1 < c < 2$, is rational. More generally, we show that, in fact, for certain complex values of the central charge under conformal flow, these rationality result holds as well.

This phenomenon of the change in the nature of the representation theory of the conformal Weyl vertex algebra for admissible modules (i.e., modules compatible with the grading arising from the conformal structure) under conformal flow is surprising in contrast to the lack of the change of the representation theory under conformal flow for the free boson vertex operator algebra. See Remark 37.

This paper is organized as follows: In Sec. II, we define various notions involving vertex algebras with gradings and/or with conformal vectors and their modules. In Sec. III, we study the rank one Weyl vertex algebra and the various graded structures imposed by the family of conformal vectors ω_μ, for $\mu \in \mathbb{C}$, under conformal flow with respect to μ. This family of conformal vertex algebras provides good examples and motivations for the various notions of vertex algebra defined in Sec. II.

In Sec. IV, we recall the notion of the Zhu algebra of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra as introduced in Ref. 5, where such vertex algebras were called $\mathbb{C}$-graded vertex algebras. We also present several results on the correspondence between modules for the Zhu algebra V and a certain class of V-modules, i.e., $\mathbb{C}_{Re>0}$-graded modules.

In Sec. V, we present our main results and applications to the Weyl vertex algebras. First, we prove a theorem on the rationality of Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebras satisfying certain conditions; see Theorem 46.

Then, in Subsection V A, motivated by the work of Zhu[26] and of Li,[27] we define a filtration on the Zhu algebra of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra and prove that under this filtration, we obtain a graded commutative associative algebra $grA(V)$. We show that there is an epimorphism from our Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra to this graded commutative associative algebra with the kernel of the epimorphism containing a set $C(V)$, which, in this setting, is an analog of the set $C_2(V)$ defining the C_2-cofinite condition for a vertex operator algebra. In Subsection V B, we give our main results on the rationality of certain $\mathbb{C}_{Re>0}$-graded vertex operator algebras with generators having non-integer weights by using the epimorphism from $V/C(V)$ to $grA(V)$; see Lemma 53 and Theorem 54.

In Subsection V C, we apply Theorems 35, 46, and 53 to the Weyl vertex algebras with conformal vectors ω_μ for μ in a certain region determined in Sec. IV that give these vertex algebras the structure of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra and prove that these are rational with only one $\mathbb{C}_{Re>0}$-graded module. We, then, apply this result to the rank n Weyl vertex algebras with a suitable conformal element. We also prove that, more generally, for $\mu \in \mathbb{C}\backslash\{0,1\}$ with $0 \le Re(\mu) \le 1$, then the Weyl vertex algebra ${}_\mu M$ admits a unique, up to isomorphism, irreducible $\mathbb{C}_{Re>0}$-graded module, namely, ${}_\mu M$ itself.

In Sec. VI, we summarize the results of this paper and also present a result giving the level one Zhu algebra for ${}_0M$, i.e., the Weyl vertex algebra with central charge $c = 2$.

II. $\mathbb{C}$-GRADED VERTEX ALGEBRAS AND THEIR MODULES

A. Vertex algebras and Ω-generated $\mathbb{C}$-graded vertex algebras

We recall the definitions of various types of vertex algebras, following, for instance, Refs. 1 and 28 for basic notions, but then also motivated by the work of Laber and Mason in Ref. 5 in the setting of $\mathbb{C}$-graded vertex algebras and related notions. However, it should be noted that we use different terminologies for some of the structures in Ref. 5; cf. Remarks 4 and 17.

Definition 1 (Ref. 28). A *vertex algebra* $(V, Y, \mathbf{1})$ consists of a vector space V together with a linear map,

$$Y : V \to (\operatorname{End} V)[[x, x^{-1}]],$$
$$v \mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1},$$

and a distinguished vector, $\mathbf{1} \in V$ (the vacuum vector), satisfying the following axioms:

(i) The lower truncation condition: for $v_1, v_2 \in V$, $Y(v_1, x)v_2$ has only finitely many terms with negative powers in x.
(ii) The vacuum property: $Y(\mathbf{1}, x)$ is the identity endomorphism 1_V of V.
(iii) The creation property: for $v \in V$, $Y(v, x)\mathbf{1} \in V[[x]]$ and $\lim_{x\to 0} Y(v, x)\mathbf{1} = v$.
(iv) The Jacobi identity: for $w, v \in V$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(v_1, x_1)Y(v_2, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v_2, x_2)Y(v_1, x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(v_2, x_0)v_1, x_2).$$

Definition 2. A vertex algebra equipped with a $\mathbb{C}$-grading $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$ is called a $\mathbb{C}$-*graded vertex algebra* if $\mathbf{1} \in V_0$ and if for $v \in V_\gamma$ with $\gamma \in \mathbb{C}$ and for $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$,

$$v_n V_\lambda \subset V_{\lambda+\gamma-n-1}. \tag{1}$$

Moreover, a homogeneous element in a $\mathbb{C}$-graded vertex algebra V is said to have *weight* λ if $v \in V_\lambda$. We denote this by $|v| = \lambda$, and we define the operator $L \in \mathrm{End}(V)$ as the linear extension of the map

$$\begin{aligned} V_\lambda &\to V_\lambda, \\ v &\mapsto \lambda v = |v| v. \end{aligned} \tag{2}$$

Remark 3.

1. Since we do not require the existence of a conformal element in a $\mathbb{C}$-graded vertex algebra, the map defined above is a natural tool to describe the weight of a homogeneous element.
2. In a $\mathbb{C}$-graded vertex algebra because of Definition 1, we have that for $v^1, v^2 \in V$,

$$|v^1_n v^2| = |v^1| + |v^2| - n - 1.$$

More generally, for $v, v^1, \dots, v^k \in V$,

$$|v^k_{n_k} \cdots v^1_{n_1} v| = \left(\sum_{j=1}^{k} |v^j| - n_j - 1\right) + |v|.$$

Remark 4. In Ref. 5, the notion of $\mathbb{C}$-graded vertex algebras has more conditions than what we require above in Definition 2. In our terminology, the Laber–Mason notion of a $\mathbb{C}$-graded vertex algebra is an Ω-generated $\mathbb{C}$-graded vertex algebra, as defined in Definition 8. Many of our results, in fact, make fine distinctions between these two notions.

Remark 5. Recall from Ref. 28 that for V, a vertex algebra, the endomorphism $D : V \to V$ defined as the linear map determined by $D(v) = v_{-2}\mathbf{1}$ satisfies the D-derivative property: $Y(Dv, x) = \frac{d}{dx} Y(v, x)$. Furthermore, $D(\mathbf{1}) = 0$ and $v = v_{-1}\mathbf{1}$. It then follows that for a $\mathbb{C}$-graded vertex algebra, by Eq. (1) and the D-derivative property, we have that if $v \in V_\lambda$, then $Dv = v_{-2}\mathbf{1} \in V_{\lambda+0-(-2)-1} = V_{\lambda+1}$.

Definition 6. Let $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$ be a $\mathbb{C}$-graded vertex algebra. We define

$$\Omega(V) = \{v \in V \mid \text{for any } u \in V_\gamma, n \in \mathbb{Z}, \text{ if } u_n v \neq 0 \text{ then either } n = \gamma - 1 \text{ or } n < Re(\gamma) - 1\},$$

where $Re(\gamma)$ denotes the real part of γ.

Remark 7.

1. The space $\Omega(V)$ consists of the vectors in V that are zero if they are acted on by any mode of V that lowers the real part of the weight. This space is often called the "vacuum space" or the "space of lowest weight vectors." However, the vacuum vector $\mathbf{1}$ is not necessarily in $\Omega(V)$. For instance, assume that $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$ such that $V_{-10} \neq 0$. Let $a \in V_{-10}$. Note that $a_{-1}\mathbf{1} = a \neq 0$. In addition, $-1 \neq -10 - 1$ and $-1 > \mathrm{Re}(-10) - 1$. Hence, in this case, $\mathbf{1} \notin \Omega(V)$. We give an example of such a vertex algebra in Sec. III, namely, the Weyl vertex algebra ${}_\mu M$ with $\mu \in \mathbb{R}$ and $\mu < 0$, for example, $\mu = -1/2$ and, thus, $c = 11$.
2. In addition, the term "lowest weight space" is misleading since there can be vectors in $\Omega(V)$ that are not of lowest weight in the sense of having any kind of minimality property with respect to their $\mathbb{C}$-grading in V; instead, these are the vectors that cannot be further lowered. An example of such a $\mathbb{C}$-graded vertex algebra is, for instance, the universal Virasoro vertex operator algebra of central charge $c = \frac{1}{2}$, denoted as $V_{Vir}(\frac{1}{2}, 0)$ (in the notation of Ref. 28). This $\mathbb{Z}$-graded vertex algebra is indecomposable but not irreducible, and it has a singular vector $v_{3,2}$ of weight 6 that satisfies $v_{3,2} \in \Omega(V_{Vir}(\frac{1}{2}, 0))$.

Next, we introduce the notion of an Ω-generated $\mathbb{C}$-graded vertex algebra motivated by Laber and Mason,[5] where this notion is called a $\mathbb{C}$-graded vertex algebra.

Definition 8. An Ω-*generated* $\mathbb{C}$-*graded vertex algebra* (or *a* $\mathbb{C}$-*graded vertex algebra generated by* Ω) is a $\mathbb{C}$-graded vertex algebra $(V, Y, \mathbf{1})$ such that every element $v \in V$ is a finite sum of elements of the form

$$v^k_{n_k} v^{k-1}_{n_{k-1}} \cdots v^1_{n_1} u^0$$

for $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{Z}$, $v^1, \ldots, v^k \in V$, and $u^0 \in \Omega(V)$.

The notions of an Ω-generated $\mathbb{R}$-graded, Ω-generated $\mathbb{Q}$-graded, Ω-generated $\mathbb{Z}$-graded, and Ω-generated $\mathbb{N}$-graded vertex algebra are defined in the obvious way.

Remark 9. We show in Sec. III that the collection of Ω-generated $\mathbb{C}$-graded vertex algebras forms a proper subset of the set of $\mathbb{C}$-graded vertex algebras. Namely, in Sec. III, we present a family of $\mathbb{C}$-graded Weyl vertex algebras, which are not Ω-*generated* $\mathbb{C}$-graded vertex algebras (see Theorem 35 III).

We will also need the notions of a *strongly generated* and *finitely strongly generated* vertex algebra given as follows:

Definition 10. A *strongly generated vertex algebra* is a vertex algebra $(V, Y, \mathbf{1})$ together with a subset $S \subset V$ such that every element $v \in V$ is a finite sum of elements of the form

$$v^k_{-n_k} v^{k-1}_{-n_{k-1}} \cdots v^1_{-n_1} \mathbf{1}$$

for $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{Z}_+$, and $v^1, \ldots, v^k \in S$. If V is strongly generated by a finite set S, then we say that V is *strongly finitely generated*.

Remark 11. Any Ω-generated $\mathbb{C}$-graded vertex algebra V is trivially a strongly generated vertex algebra with $S = V$. If V is also strongly finitely generated by a finite set of generators S acting on Ω and Ω is also finite, then we call V a *finitely* Ω-generated $\mathbb{C}$-graded vertex algebra. All Ω-generated $\mathbb{C}$-graded vertex algebras are strongly generated, but the converse is not true, even if we have finitely many strong generators. In Theorem 35 (III), we give examples of finitely strongly generated $\mathbb{C}$-graded Weyl vertex algebras, which are not Ω-generated.

For certain Ω-generated $\mathbb{C}$-graded vertex algebras, one can define a degree grading as follows in Definition 12, and we call such Ω-generated $\mathbb{C}$-graded vertex algebras Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebras. In Sec. III, we give examples of Ω-generated $\mathbb{C}$-graded Weyl vertex algebras that admit a grading as defined below.

Definition 12. An *Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra* is an Ω-generated $\mathbb{C}$-graded vertex algebra such that the following notion of degree is well defined: For V, an Ω-generated $\mathbb{C}$-graded vertex algebra, we define the *degree* of an element of V by setting the degree of elements in $\Omega(V)$ to be 0 and extending by linearity the following formula:

$$deg(v^k_{n_k} \cdots v^1_{n_1} u^0) = \sum_{j=1}^{k} (|v^j| - n_j - 1),$$

where $v^1, \ldots, v^k \in V$ for $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{Z}$, and $u^0 \in \Omega(V)$.

Remark 13. Note that this notion of degree is not necessarily well defined for every Ω-generated $\mathbb{C}$-graded vertex algebra. If $v_n u^0 \in \Omega(V)$, then by definition of $\Omega(V)$, if $v_n u^0 \neq 0$, then $deg(v_n u^0) = |v| - n - 1 = 0$ or $Re(deg(v_n u^0)) = Re(|v| - n - 1) = Re(|v|) - n - 1 > 0$. Therefore, by definition, this notion of degree, by setting all elements in $\Omega(V)$ to have degree zero, is precluding the possibility of elements in $\Omega(V)$ of the form $v_n u^0$ such that $u^0 \in \Omega(V)$ and $v_n u^0 \neq 0$ for some n satisfying $Re(|v|) - n - 1 > 0$. Thus, it is the requirement of well-definedness of this definition that is imposing the degree grading given below.

One can show (cf. Ref. 5) that it follows from Definitions 8 and 12.

Lemma 14. Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra. For $k \geq 1$, let $v^1, \ldots, v^k \in V$ be homogeneous, $n_1, \ldots, n_k \in \mathbb{Z}$, and $u^0 \in \Omega(V)$ such that

$$v^k_{n_k} v^{k-1}_{n_{k-1}} \cdots v^1_{n_1} u^0 \neq 0.$$

Then, for any given $v^j \in V$ and $n_j \in \mathbb{Z}$, either

$$\mathrm{Re}\left(\sum_{j=1}^{k} (|v^j| - n_j - 1)\right) > 0 \qquad \text{or} \qquad \sum_{j=1}^{k} (|v^j| - n_j - 1) = 0.$$

Proof. See the Appendix for a detailed proof of this fact. □

Remark 15. Note that if V is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra V, and we define $V(\lambda)$ to be the space of all $v \in V$ with $deg(v) = \lambda$, then, we have the following decomposition:

$$V = V(0) \bigoplus_{\substack{\lambda\in\mathbb{C} \\ \mathrm{Re}(\lambda)>0}} V(\lambda). \tag{3}$$

This motivates our use of the term $\mathbb{C}_{Re>0}$-graded vertex algebras to denote this particular family of $\mathbb{C}$-graded vertex algebras.

Proposition 16. Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra, and let deg *be as in Definition 12. Then, the homogeneous component $V(0)$ in (3) coincides with $\Omega(V)$.*

Proof. Note that $\Omega(V) \subset V(0)$ follows from Definition 12. Therefore, we need to show next that $V(0) \subset \Omega(V)$. Namely, we need to prove that if $v \in V$ satisfies $\deg(v) = 0$, then v must be a vector in $\Omega(V)$. We first prove this fact for vectors of the form

$$v^k_{n_k} \cdots v^1_{n_1} u^0,$$

where $v^1, \ldots, v^k \in V$ for $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{Z}$, and $u^0 \in \Omega(V)$:

Assume that $v^k_{n_k} \cdots v^1_{n_1} u^0 \neq 0$ and that $\deg(v^k_{n_k} \cdots v^1_{n_1} u^0) = \sum_{j=1}^k \big(|v^j| - n_j - 1\big) = 0$. We want to show that $v^k_{n_k} \cdots v^1_{n_1} u^0 \in \Omega(V)$. Let $u \in V$ and $n \in \mathbb{Z}$ be such that $u_n v^k_{n_k} \cdots v^1_{n_1} u^0 \neq 0$. Then, using Lemma 14 for $u_n v^k_{n_k} \cdots v^1_{n_1} u^0$, we have that either $|u| - n - 1 + \sum_{j=1}^k \big(|v^j| - n_j - 1\big) = 0$ or $\mathrm{Re}\big(|u| - n - 1 + \sum_{j=1}^k \big(|v^j| - n_j - 1\big)\big) > 0$. Since by assumption $\sum_{j=1}^k \big(|v^j| - n_j - 1\big) = 0$, it follows that either $n = |u| - 1$ or $n < \mathrm{Re}(|u| - 1)$. Therefore, $v^k_{n_k} \cdots v^1_{n_1} u^0 \in \Omega(V)$ if $\deg(v^k_{n_k} \cdots v^1_{n_1} u^0) = 0$.

Now, let v be any vector in V. Since V is an Ω-generated $\mathbb{C}$-graded vertex algebra, we know that v is a linear combination $v = \sum_{j=1}^m c_j \tilde{v}^j$, where $c_j \in \mathbb{C}$, and each $\tilde{v}^j$ is an element of the form $v^k_{n_k} \cdots v^1_{n_1} u^0$ with $n_1, \ldots, n_r \in \mathbb{Z}$ and $u^0 \in \Omega(V)$. If $\deg(v) = 0$, then we have that $\sum_{j=1}^m \deg(\tilde{v}^j) = 0$, where each $\deg(\tilde{v}^j)$ satisfies either

$$\deg(\tilde{v}^j) = 0 \ \text{ or } \ \mathrm{Re}(\deg(\tilde{v}^j)) > 0$$

by Lemma 14. Therefore, we obtain that $\deg(\tilde{v}^j) = 0$ for each $1 \le j \le m$. By the argument above, we have that each $\tilde{v}^j$ is an element in $\Omega(V)$, which implies that $v \in \Omega(V)$. □

Remark 17. In Ref. 5, all Ω-generated $\mathbb{C}$-graded vertex algebras are assumed to be $\mathbb{C}_{Re>0}$-graded and referred to as $\mathbb{C}$-graded vertex algebras instead.

An Ω-generated $\mathbb{C}$-graded vertex algebra resembles a vertex operator algebra (with a possibly weaker non-integer grading) in that it has a weight operator L defined as in Eq. (2), which generalizes the zero Virasoro mode $L(0)$. Since we need to work in the $\mathbb{C}$-graded vertex algebra setting, we introduce the definition of a $\mathbb{C}$-graded conformal vertex algebra and show how it generalizes the concept of a conformal vertex algebra.

Definition 18 (Ref. 1). A *$\mathbb{C}$-graded conformal vertex algebra* $(V, Y, \mathbf{1}, \omega)$ consists of a $\mathbb{C}$-graded vertex algebra,

$$V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda,$$

together with a distinguished vector $\omega \in V_2$ that satisfies the Virasoro relations:

(i) $[L(n), L(m)] = (n - m)L(m + n) + \frac{1}{12}(n^3 - n)\delta_{n,-m}c$ for $n, m \in \mathbb{Z}$, where $L(n) =: \omega_{n+1}$ for $n \in \mathbb{Z}$ and $c \in \mathbb{C}$, called the central charge of V.
(ii) The $L(-1)$-derivative property: for any $v \in V$, $Y(L(-1)v, x) = \frac{d}{dx} Y(v, x)$.
(iii) The $L(0)$-grading property: for $\mu \in \mathbb{C}$ and $v \in V_\mu$, $L(0)v = \mu v = (wt\ v)v$.

A $\mathbb{Z}$-graded conformal vertex algebra is defined in the obvious way.

Definition 19. A *vertex operator algebra* $(V, Y, \mathbf{1}, \omega)$ is a $\mathbb{Z}$-graded conformal vertex algebra,

$$V = \bigoplus_{n\in\mathbb{Z}} V_n,$$

such that

(i) $V_n = 0$ for n sufficiently negative and
(ii) $\dim V_n < \infty$ for $n \in \mathbb{Z}$.

Since the $\mathbb{Z}$-grading condition for a vertex operator algebra is too restrictive to work with the Weyl vertex algebras of all central charges, we will need the following modified concept of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

Definition 20. A *Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra* is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$ that is also a $\mathbb{C}$-graded conformal vertex algebra with the following additional properties:

(i) For $\lambda \in \mathbb{C}$, $V_\lambda = \{v \in V | L(0)v = \lambda v\}$ and $\dim V_\lambda < \infty$.
(ii) $\mathrm{Re}(\lambda) \geq |Im(\lambda)|$ for all but finitely many $\lambda \in Spec_V\, L(0)$.

Remark 21. Condition (ii) above, which may appear unnatural, guarantees that there are only finitely many eigenvalues λ of $L(0)$ such that $\mathrm{Re}(\lambda) < 0$ and $V_\lambda \neq 0$. As explained in Ref. 4, if an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra is $\mathbb{R}$-graded (namely, if $V_\lambda \neq 0$, then $\lambda \in \mathbb{R}$), condition (ii) guarantees the usual lower boundedness condition that $V_r = 0$ for all r sufficiently negative.

Remark 22. Since $\omega \in V_2$, we can conclude that any Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra contains the vertex operator algebra generated by ω.

The following are the relationships between the various types of vertex algebras introduced in this section:

$$\Omega VOA(\mathbb{C}_{Re>0}(\mathcal{V})) \subset (Conf(\mathbb{C}(\mathcal{V})) \cap \Omega(\mathbb{C}_{Re>0}(\mathcal{V}))) \subset \Omega(\mathbb{C}(\mathcal{V})) \subset \mathbb{C}(\mathcal{V}) \subset \mathcal{V},$$
$$\mathcal{VOA} \subset Conf(\mathbb{Z}(\mathcal{V})) \subset \mathbb{Z}(\mathcal{V}) \subset \mathcal{V}.$$

Here,

$\mathcal{V}$ = set of vertex algebras,
$\mathbb{C}(\mathcal{V})$ = set of $\mathbb{C}$-graded vertex algebras,
$\mathbb{Z}(\mathcal{V})$ = set of $\mathbb{Z}$-graded vertex algebras,
$\Omega(\mathbb{C}(\mathcal{V}))$ = set of Ω-generated $\mathbb{C}$-graded vertex algebras,
$\Omega(\mathbb{C}_{Re>0}(\mathcal{V}))$ = set of Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebras,
$\Omega(\mathbb{Z}(\mathcal{V}))$ = set of Ω-generated $\mathbb{Z}$-graded vertex algebras,
$Conf(\mathbb{C}(\mathcal{V}))$ = set of $\mathbb{C}$-graded conformal vertex algebras,
$Conf(\mathbb{Z}(\mathcal{V}))$ = set of $\mathbb{Z}$-graded conformal vertex algebras,
$\mathcal{VOA}$ = set of vertex operator algebras,
$\Omega VOA(\mathbb{C}_{Re>0}(\mathcal{V}))$ = set of Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebras.

B. Modules for $\mathbb{C}$-graded vertex algebras

Next, we introduce various types of representations of $\mathbb{C}$-graded vertex algebras, again following or motivated by, for instance, Refs. 1, 5, and 28. We begin by recalling the definition of a weak V-module for a fixed vertex algebra $(V, Y, \mathbf{1})$, as presented in Ref. 28.

Definition 23. Let V be a vertex algebra. A *weak V-module* is a vector space W equipped with a vertex operator map

$$Y_W : V \to (\mathrm{End}\, W)[[x, x^{-1}]],$$
$$v \mapsto Y_W(v, x) = \sum_{n\in\mathbb{Z}} v_n^W x^{-n-1},$$

satisfying the following axioms:

(i) The lower truncation condition: for $v \in V$ and $w \in W$, $Y_W(v, x)w$ has only finitely many terms with negative powers in x.
(ii) The vacuum property: $Y_W(\mathbf{1}, x)$ is the identity endomorphism 1_W of W.
(iii) The Jacobi identity: for $v_1, v_2 \in V$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(v_1, x_1) Y_W(v_2, x_2) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W(v_2, x_2) Y_W(v_1, x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(v_2, x_0)v_1, x_2).$$

Remark 24. In Ref. 28, the notion of a weak V-module given above is called a V-module for V, a vertex algebra, but if V has, for instance, the structure of a vertex operator algebra, then the structure V-module defined above is called in Ref. 28 a weak module for the vertex operator algebra structure of V. Since we will mainly be concerned with extra "vertex operator algebra"-type structures on V, to emphasize the differences between the weaker notions of a module for a vertex algebra versus a module for a vertex operator algebra, we have chosen to call these modules "weak" throughout.

Proposition 25 (Ref. 28). Let V be a vertex algebra, and let D be the linear map on V given by $Dv = v_{-2}\mathbf{1}$ as in Remark 5. Let W be a weak V-module.

1. *Then,*
$$Y_W(Dv, x) = \frac{d}{dx} Y_W(v, x).$$
2. *Let T be a subset of W, and let $\langle T\rangle$ denote the submodule generated by T. Then,*
$$\langle T\rangle = Span\{v_n t \mid v \in V,\ n \in \mathbb{Z}, t \in T\}.$$

Next, we recall the notion of a module over an Ω-generated $\mathbb{C}$-graded vertex algebra V, as introduced in Ref. 5.

Definition 26 (Ref. 5). Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra. A $\mathbb{C}_{Re>0}$*-graded V-module* W is a weak V-module with a grading of the form
$$W = W(0) \bigoplus_{\substack{\tau\in\mathbb{C},\\ \mathrm{Re}(\tau)>0}} W(\tau)$$
such that $W(0) \neq 0$, and for any homogeneous $v \in V_\lambda$, one has
$$v_n^W W(\tau) \subseteq W(\tau + \lambda - n - 1).$$
We say that a homogeneous element $w \in W(\tau)$ has *degree* τ.

Remark 27. In Ref. 5, $\mathbb{C}_{Re>0}$-graded modules are referred to as admissible modules.

Definition 28. Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra, and let
$$W = W(0) \bigoplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W(\tau)$$
be a $\mathbb{C}_{Re>0}$-graded V-module. We define
$$\Omega(W) = \{w \in W \mid \text{for any } v \in V \text{ if } v_n^W w \neq 0 \text{ then either } |w| = |v_n^W w| \text{ or } \mathrm{Re}(|w|) < \mathrm{Re}(|v_n^W w|)\}.$$

Note, in particular, that $W(0) \subset \Omega(W)$. Moreover, $\Omega(W)$ consists of the vectors in W that are annihilated by the action of any mode of V that lowers the real part of its weight, similarly to $\Omega(V)$ in Definition 6.

The following result was stated in Ref. 5 for Ω-generated $\mathbb{C}$-graded vertex algebras, where it was assumed that the degree grading is well defined for these types of vertex algebras. Here, we give the proof for the case in which V is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra.

Proposition 29 (cf. Ref. 5).

1. *Any Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra V is a $\mathbb{C}_{Re>0}$-graded V-module.*
2. *If W is a simple $\mathbb{C}_{Re>0}$-graded V-module, then $\Omega(W) = W(0)$.*

Proof. The first statement follows directly from the degree grading in Definition 12 on V together with Remark 15 and the definition of a $\mathbb{C}_{Re>0}$-graded V-module.

To prove the second statement, we first show that if $W = W(0) \oplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W(\tau)$ is a simple $\mathbb{C}_{Re>0}$-graded V-module, then $\Omega(W) \cap \left(\oplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W(\tau)\right) = 0$. To see this, let $w \in \Omega(W) \cap (\oplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W(\tau))$. Then, $\langle w\rangle = Span\{v_n^W w \mid v \in V, n \in \mathbb{Z}\} \subset \oplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W_\tau$ because $w \in \Omega(W)$, and so, in particular, $\mathrm{Re}(|v_n^W w|) \geq \mathrm{Re}(|w|) > 0$ for every $v \in V, n \in \mathbb{Z}$ such that $v_n^W w \neq 0$. Since $\langle w\rangle$ is a proper V-submodule $\langle w\rangle \subsetneq W$, we can conclude that $\langle w\rangle = \{0\}$. In particular, $w = 0$, and we have shown that $\Omega(W) \cap \left(\oplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W(\tau)\right) = 0$.

Finally, we show that $\Omega(W) = W(0)$. Let $u \in \Omega(W)$. Since $u \in W$, we can write $u = w' + w''$ for $w' \in W(0)$ and $w'' \in \oplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W(\tau)$. Since $w'' = u - w'$ and $W(0) \subseteq \Omega(W)$, we can conclude that $w'' \in \Omega(W)$. Moreover, $w'' \in \Omega(W) \cap \oplus_{\tau\in\mathbb{C},\mathrm{Re}(\tau)>0} W(\tau)$, which by our previous argument is 0. This implies that $w'' = 0$ and $u = w' \in W(0)$. Hence, $\Omega(W) = W(0)$. □

Definition 30. Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. An *ordinary V-module W* is a weak V-module that admits a decomposition into generalized eigenspaces via the spectrum of $L_W(0)$ as follows:

(i) $W = \bigoplus_{\lambda\in\mathbb{C}} W(\lambda)$ where $W(\lambda) = \{w \in W | L_W(0)w = \lambda w\}$.
(ii) $\dim W(\lambda) < \infty$ for all $\lambda \in \mathbb{C}$.
(iii) $\mathrm{Re}(\lambda) > 0$ for all but finitely many $\lambda \in SpecL_W(0)$.

Finally, we introduce the notion of rationality for the representations of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

Definition 31. Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. V is called *rational* if the category of $\mathbb{C}_{Re>0}$-graded V-modules is semisimple, i.e., every $\mathbb{C}_{Re>0}$-graded V-module is completely reducible, i.e., the sum of simple $\mathbb{C}_{Re>0}$ modules.

III. THE WEYL VERTEX ALGEBRA: CLASSIFICATION OF ITS $\mathbb{C}$-GRADED STRUCTURES

In this section, we introduce the rank one Weyl vertex algebra, denoted as M, with a family of conformal elements ω_μ parameterized by $\mu \in \mathbb{C}$, following, for instance, Ref. 22 (see also Ref. 29). We denote M with the conformal structure by $({}_\mu M, \omega_\mu)$ or just ${}_\mu M$. We discuss the various gradings and associated refined vertex algebra structures imposed on the rank one Weyl vertex algebra M by the choice of μ. The rank n Weyl vertex algebra, for $n \in \mathbb{Z}_+$, is then the n-fold tensor product of M.

Definition 32. Let $\mathcal{L}$ be the infinite-dimensional Lie algebra with generators $K, a(m)$ and $a^*(n)$ with $m, n \in \mathbb{Z}$ such that K is in the center and the bracket is given by

$$[a(m), a^*(n)] = \delta_{m+n,0}K.$$

We define the *rank one Weyl algebra* $\mathcal{A}_1$ to be the quotient,

$$\mathcal{A}_1 = \frac{\mathcal{U}(\mathcal{L})}{\langle K-1\rangle},$$

where $\mathcal{U}(\mathcal{L})$ denotes the universal enveloping algebra of $\mathcal{L}$ and $\langle K-1\rangle$ is the two sided ideal generated by $K-1$.

We have that $\mathcal{A}_1$ is an associative algebra with generators $a(m)$, $a^*(n)$, for $m, n \in \mathbb{Z}$, and relations

$$[a(m), a^*(n)] = \delta_{m+n,0}, \tag{4}$$
$$[a(m), a(n)] = [a^*(m), a^*(n)] = 0 \tag{5}$$

for all $m, n \in \mathbb{Z}$.

The Weyl algebra $\mathcal{A}_1$ has a countably infinite family of automorphisms, called *spectral flow* automorphisms given by

$$\rho_s : \mathcal{A}_1 \longrightarrow \mathcal{A}_1, \quad a(n) \mapsto a(n+s), \quad a^*(n) \mapsto a^*(n-s), \tag{6}$$

for $s \in \mathbb{Z}$, as well as the automorphism

$$\varphi_t : \mathcal{A}_1 \longrightarrow \mathcal{A}_1, \quad a(n) \mapsto t a^*(n), \quad a^*(n) \mapsto -t^{-1}a(n), \tag{7}$$

for $t \in \mathbb{C}^\times$.

The (rank one) Weyl vertex algebra M can be realized as an induced module for the Lie algebra $\mathcal{L}$ as follows. We first fix a triangular decomposition of $\mathcal{L} = \mathcal{L}^- \oplus \mathcal{L}^0 \oplus \mathcal{L}^+$ where

$$\begin{aligned}\mathcal{L}^- &= \mathrm{span}_\mathbb{C}\{a(-n), a^*(-m) \,|\, n \ge 1, m \ge 0\},\\ \mathcal{L}^0 &= \mathrm{span}_\mathbb{C}\{K\},\\ \mathcal{L}^+ &= \mathrm{span}_\mathbb{C}\{a(n), a^*(m+1) \in \mathcal{L} \,|\, n \ge 0, m \ge 0\}\end{aligned}$$

(see, for instance, Ref. 21 where this is called the normal triangular decomposition). Next, we give the one-dimensional vector space $\mathbb{C}\mathbf{1}$ the $\mathcal{L}^0 \oplus \mathcal{L}^+$-module structure, given by

$$\begin{aligned}a(0)\mathbf{1} &= 0,\\ K\mathbf{1} &= \mathbf{1},\\ a(n)\mathbf{1} &= 0 \quad \text{for } n > 0,\\ a^*(m+1)\mathbf{1} &= 0 \quad \text{for } m \ge 0,\end{aligned}$$

and we define M to be the induced module,

$$M = \mathcal{U}(\mathcal{L})\otimes_{\mathcal{U}(\mathcal{L}^0\oplus\mathcal{L}^+)}\mathbb{C}\mathbf{1}.$$

Then, M is a simple Weyl module and, as a vector space, $M \cong \mathbb{C}[a(-n), a^*(-m) \mid n > 0,\ m \geq 0]$. There is a unique vertex algebra structure on M (see, for instance, Theorem 5.7.1 in Ref. 28 or Lemma 11.3.8 in Ref. 14) given by $(M, Y, \mathbf{1})$ with vertex operator map $Y : M \to \mathrm{End}(M)[[z, z^{-1}]]$ such that

$$Y(a(-1)\mathbf{1}, z) = a(z),\ Y(a^*(0)\mathbf{1}, z) = a^*(z),$$
$$a(z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1},\ a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n)z^{-n}. \tag{8}$$

In particular,

$$Y(a(-1)a^*(0)\mathbf{1}, z) =: a(z)a^*(z):,$$

where: $a(z)a^*(z)$: denotes the ordered product of the fields $a(z)$ and $a^*(z)$ given by

$$: a(z)a^*(z) := a(z)^+a^*(z) + a^*(z)a(z)^-,$$

with $a(z)^+ = \sum_{n\leq -1} a(n)z^{-n-1}$, $a(z)^- = \sum_{n\geq 0} a(n)z^{-n-1}$.

In terms of the operator product expansion of the vertex operators, i.e., the corresponding fields, we have

$$a(z)a^*(w) = \frac{1}{z-w} + : a(z)a^*(w) : .$$

Moreover, the map $Y : M \to \mathrm{End}(M)[[z, z^{-1}]]$ is given by

$$Y(a(-m_1-1)a(-m_2-1)\ldots a(-m_k-1)a^*(-n_1)\ldots a^*(-n_l)\mathbf{1}, z)$$
$$= \prod_{i=1}^{k}\frac{1}{m_i!}\prod_{j=1}^{l}\frac{1}{n_j!} : \partial^{m_1}a(z)\cdots\partial^{m_k}a(z)\partial^{n_1}a^*(z)\cdots\partial^{n_l}a^*(z) :$$

for $m_1, \ldots m_k, n_1, \ldots n_l \in \mathbb{Z}_{\geq 0}$.

Remark 33.

1. The fields $a(z)$ and $a^*(z)$ defined in (8) are usually denoted by $\beta(z)$ and $\gamma(z)$ in the physics literature (up to a choice of sign) where the vertex algebra M is referred to as the $\beta\gamma$ vertex algebra or $\beta\gamma$-system.
2. Since for all $n \in \mathbb{Z}$, the n modes of the fields $Y(a(-1)\mathbf{1}, z) = a(z)$, $Y(a^*(0)\mathbf{1}) = a^*(z)$ satisfy

$$(a(-1)\mathbf{1})_n = a(n),$$
$$(a^*(0)\mathbf{1})_n = a^*(n+1),$$

we have that the set $T = \{a(-1)\mathbf{1}, a^*(0)\mathbf{1}\}$ is a set of strong generators for the vertex algebra M in the sense of Definition 10. Namely, M is spanned by the set of normally ordered monomials,

$$\{: \partial^{k_1}\alpha^{i_1}\ldots\partial^{k_l}\alpha^{i_k} : \mid k_1, \ldots, k_l \geq 0,\ \alpha^{i_j} \in T\}.$$

Therefore, M is strongly finitely generated as a vertex algebra in the sense of Definition 10.

From the simple relations between the modes of the strong generators $a(-1)\mathbf{1}$ and $a(0)\mathbf{1}$ given by (4) together with Remark 33, it is easy to see that M is a simple vertex algebra.

Let $\beta := a(-1)a^*(0)\mathbf{1}$. We set $\beta(z) = Y(\beta, z) = \sum_{n\in\mathbb{Z}} \beta(n)z^{-n-1}$. [We note that in Ref. 22, there was a typo in the exponent of z in the expansion of $\beta(z)$.] We note, in particular, that in this notation,

$$\beta(-2)\mathbf{1} = a(-2)a^*(0)\mathbf{1} + a(-1)a^*(-1)\mathbf{1}.$$

Then, β is a Heisenberg vector in M of level -1. Namely, for $n, m \in \mathbb{Z}$, we have

$$[\beta(m), \beta(n)] = -m\delta_{m+n,0}$$

as operators on M, and therefore,

$$\beta(z)\beta(w) = -\frac{1}{(z-w)^2} + : \beta(z)\beta(w) : .$$

In addition, we have

$$[\beta(m), a(n)] = -a(m+n) \quad \text{and} \quad [\beta(m), a^*(n)] = a^*(m+n).$$

We are interested in the possible $\mathbb{C}$-graded conformal vertex algebra structures on the vertex algebra M. The vertex algebra M admits a family of Virasoro vectors,

$$\begin{aligned}\omega_\mu &= (1-\mu)a(-1)a^*(-1)\mathbf{1} - \mu a(-2)a^*(0)\mathbf{1} \qquad \text{for } \mu \in \mathbb{C}\\ &= a(-1)a^*(-1)\mathbf{1} - \mu\big(a(-1)a^*(-1)\mathbf{1} + a(-2)a^*(0)\mathbf{1}\big)\\ &= a(-1)a^*(-1)\mathbf{1} - \mu\beta(-2)\mathbf{1},\end{aligned} \tag{9}$$

of central charge

$$c_\mu = 2(6\mu(\mu-1)+1). \tag{10}$$

The corresponding Virasoro field is

$$L^\mu(z) = (1-\mu) : a(z)\partial a^*(z) : -\mu : \partial a(z) a^*(z) :, \tag{11}$$

and it satisfies

$$L^\mu(z)L^\mu(w) = \frac{1-6\mu+6\mu^2}{(z-w)^4} + \frac{2L^\mu(w)}{(z-w)^2} + \frac{\partial_w L^\mu(w)}{z-w} + : L^\mu(z)L^\mu(w) : .$$

This gives a $\mathbb{C}$ grading on M as we give explicitly below, and we denote the particular $\mathbb{C}$-graded conformal vertex algebra structure on M by

$$({}_\mu M, Y, \mathbf{1}, \omega_\mu)$$

or just ${}_\mu M$.

Lemma 34. The composition of the spectral flow ρ_1 and φ_1 automorphisms of the Weyl algebra lifts to give the following isomorphisms of $\mathbb{C}$-graded conformal vertex algebras:

$$\varphi_1 \circ \rho_1 : ({}_\mu M, \omega_\mu) \xrightarrow{\cong} ({}_{1-\mu}M, \omega_{1-\mu}), \tag{12}$$

given explicitly on M by

$$\begin{aligned}&a(-m_1-1)\cdots a(-m_k-1)a^*(-n_1)\cdots a^*(-n_l)\mathbf{1} \mapsto\\ &(-1)^l a(-n_1-1)\cdots a(-n_l-1)a^*(-m_1)\cdots a^*(-m_k)\mathbf{1}\end{aligned}$$

for $k, l \in \mathbb{N}$ and $m_i, n_j \in \mathbb{N}$. Or, more generally, for the vertex algebra structure, letting $F = \varphi_1 \circ \rho_1$, we define

$$F(u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}) = [F(u^1)]_{n_1}\cdots[F(u^k)]_{n_k}\mathbf{1} \tag{13}$$

for $u^j = a(-1)\mathbf{1}$ or $a^*(0)\mathbf{1}$ for $j = 1, \ldots, k$ and $n_1, \ldots, n_k \in \mathbb{Z}$.

Moreover, this is the only $\mathbb{C}$-graded conformal vertex algebra isomorphism between $({}_\mu M, \omega_\mu)$ for distinct $\mu \in \mathbb{C}$. In particular, the central charge $c_\mu = c_{1-\mu}$ completely determines $({}_\mu M, \omega_\mu)$ up to isomorphism.

Proof. By definition, $F = \varphi_1 \circ \rho_1$ is a vector space isomorphism. Equation (13) implies that F is a vertex algebra homomorphism, as follows: By the definition of F, we have $F(u_n v) = F(u_n v_{-1}\mathbf{1}) = F(u)_n F(v)_{-1}\mathbf{1} = F(u)_n F(v)$ for $u, v \in \{a(-1)\mathbf{1}, a^*(0)\mathbf{1}\}$. By induction on k, we have that $F(u_n v) = F(u)_n F(v)$ for $v = u^1_{n_1}\cdots u^k_{n_k}\mathbf{1}$ for $u^1, \ldots, u^k \in \{a(-1)\mathbf{1}, a^*(0)\mathbf{1}\}$ and $n_1, \ldots, n_k \in \mathbb{Z}$.

Then, note that

$$\begin{aligned}F(Y(a(-1)\mathbf{1}, z)v) &= F\Bigg(\sum_{n\in\mathbb{Z}} a(n)vz^{-n-1}\Bigg) = \sum_{n\in\mathbb{Z}} F(a(n)v)z^{-n-1}\\ &= \sum_{n\in\mathbb{Z}} F((a(-1)\mathbf{1})_n v)z^{-n-1} = \sum_{n\in\mathbb{Z}} [F(a(-1)\mathbf{1})]_n F(v)z^{-n-1}\\ &= \sum_{n\in\mathbb{Z}} (a^*(0)\mathbf{1})_n F(v)z^{-n-1} = \sum_{j\in\mathbb{Z}} a^*(j)z^{-j}F(v)\\ &= Y(a^*(0)\mathbf{1}, z)F(v) = Y(F(a(-1)\mathbf{1}), z)F(v)\end{aligned}$$

and

$$
\begin{aligned}
F(Y(a^*(0)\mathbf{1},z)v) &= F\left(\sum_{n\in\mathbb{Z}} a^*(n)vz^{-n}\right) = \sum_{n\in\mathbb{Z}} F(a^*(n)v)z^{-n} \\
&= \sum_{n\in\mathbb{Z}} F((a^*(0)\mathbf{1})_{n-1}v)z^{-n} = \sum_{n\in\mathbb{Z}} [F(a^*(0)\mathbf{1})]_{n-1}F(v)z^{-n} \\
&= -\sum_{n\in\mathbb{Z}} (a(-1)\mathbf{1})_{n-1}F(v)z^{-n} = -\sum_{j\in\mathbb{Z}} (a(-1)\mathbf{1})_j F(v)z^{-j-1} \\
&= -Y(a(-1)\mathbf{1},z)F(v) = Y(F(a^*(0)\mathbf{1}),z)F(v).
\end{aligned}
$$

Therefore, by Proposition 5.7.9 in Ref. 28, we have $F(Y(u,z)v) = Y(F(u),z)F(v)$ for all $u, v \in M$ and F is a homomorphism of vertex algebras. Since it is a bijection, it is an isomorphism of vertex algebras.

Finally, for $\mu \in \mathbb{C}$,

$$
\begin{aligned}
\varphi_1 \circ \rho_1(\omega_\mu) &= \varphi_1 \circ \rho_1((1-\mu)a(-1)a^*(-1)\mathbf{1} - \mu a(-2)a^*(0)\mathbf{1}) \\
&= \varphi_1((1-\mu)a(0)a^*(-2)\mathbf{1} - \mu a(-1)a^*(-1)\mathbf{1}) \\
&= -(1-\mu)a^*(0)a(-2)\mathbf{1} + \mu a^*(-1)a(-1)\mathbf{1} \\
&= \mu a(-1)a^*(-1)\mathbf{1} - (1-\mu)a^*(0)a(-2)\mathbf{1} \\
&= (1-(1-\mu))a(-1)a^*(-1)\mathbf{1} - (1-\mu)a(-2)a^*(0)\mathbf{1} \\
&= \omega_{1-\mu},
\end{aligned}
$$

proving that this is an isomorphism of $\mathbb{C}$-graded conformal vertex algebras. Then, since $c_\mu = 2(6\mu(\mu-1)+1) = 2(6\nu(\nu-1)+1) = c_\nu$ for $\mu \neq \nu$ implies $\nu = 1-\mu$, this shows these are the only isomorphisms between the conformal structures on M. □

Let

$$L^\mu(z) = Y(\omega_\mu, z) = \sum_{n\in\mathbb{Z}} L^\mu(n)z^{-n-2}.$$

For $\mu = 0$, we set $\omega := \omega_0$, $L(n) := L^0(n)$, and, then, $c_0 = 2$. More generally, we have that for $\mu \in \mathbb{C}$,

$$\omega_\mu = \omega - \mu\beta(-2)\mathbf{1}. \tag{14}$$

Furthermore, since $(\beta(-2)\mathbf{1})_0 = (D\beta)_0 = 0$ and $(\beta(-2)\mathbf{1})_1 = (D\beta)_1 = -\beta(0)$, where D is the endomorphism described in Remark 5, we, thus, have that

$$
\begin{aligned}
L^\mu(-1) &= L(-1) \qquad \text{for all } \mu \in \mathbb{C}, \\
L^\mu(0) &= L(0) + \mu\beta(0).
\end{aligned}
$$

In addition, for all $m, n \in \mathbb{Z}$,

$$
\begin{aligned}
[L(m), a(n)] &= -na(m+n), \\
\left[L(m), a^*(n)\right] &= -(m+n)a^*(m+n).
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
[L(0), a(n)] &= -na(n), \\
\left[L(0), a^*(n)\right] &= -na^*(n), \\
\left[L^\mu(0), a(n)\right] &= [L(0) + \mu\beta(0), a(n)] = -na(n) - \mu a(n) = (-n-\mu)a(n), \\
\left[L^\mu(0), a^*(n)\right] &= [L(0) + \mu\beta(0), a^*(n)] = -na^*(n) + \mu a^*(n) = (-n+\mu)a^*(n).
\end{aligned}
$$

Note that for integers $m_1 \geq \cdots \geq m_k \geq 0$, $n_1 \geq \cdots \geq n_t \geq 0$, and $k, t \in \mathbb{Z}_+$, we have

$$
\begin{aligned}
L^\mu(0)a(-m_1-1)\cdots a(-m_k-1)\mathbf{1} &= ((m_1+\cdots+m_k+k) - k\mu)a(-m_1-1)\cdots a(-m_k-1)\mathbf{1} \\
&= ((m_1+\cdots+m_k) + k(1-\mu))a(-m_1-1)\cdots a(-m_k-1)\mathbf{1}, \\
L^\mu(0)a^*(-n_1)\cdots a^*(-n_t)\mathbf{1} &= ((n_1+\cdots+n_t) + t\mu)a^*(-n_1)\cdots a^*(-n_t)\mathbf{1},
\end{aligned}
$$

and

$$L^{\mu}(0)a(-m_1-1)\cdots a(-m_k-1)a^*(-n_1)\cdots a^*(-n_t)\mathbf{1}$$
$$=\left(\sum_{i=1}^{k} m_i+\sum_{j=1}^{t} n_j+k(1-\mu)+t\mu\right)a(-m_1-1)\cdots a(-m_k-1)a^*(-n_1)\cdots a^*(-n_t)\mathbf{1}. \tag{15}$$

Thus, from Eq. (15), an element $v=a(-m_1-1)\cdots a(-m_k-1)a^*(-n_1)\cdots a^*(-n_t)\mathbf{1}\in {}_{\mu}M$ has an $L^{\mu}(0)$-grading of the form

$$\mathrm{wt}\, v=r+s+k(1-\mu)+t\mu \qquad \text{for } r,s,k,t\in\mathbb{N},\ \text{with } k,t\in\mathbb{Z}_+ \text{ if } r,s\in\mathbb{Z}_+,\ \text{respectively.} \tag{16}$$

That is, the action of $L^{\mu}(0)$ on ${}_{\mu}M$ defines a $\mathbb{C}$-grading on ${}_{\mu}M$, which gives ${}_{\mu}M$ the structure of a $\mathbb{C}$-graded vertex algebra. It is also a $\mathbb{C}$-graded conformal vertex algebra with strong generators $a(-1)\mathbf{1}$, $a^*(0)\mathbf{1}$, which satisfy

$$|a(-1)\mathbf{1}|=1-\mu, \tag{17}$$
$$|a^*(0)\mathbf{1}|=\mu. \tag{18}$$

However, for only certain values of μ is ${}_{\mu}M$ an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra in the sense of Definition 20.

We are interested in these values of μ, which give ${}_{\mu}M$ an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra structure and the nature of the representations of these vertex algebras ${}_{\mu}M$.

To that end, we note that Eq. (16) implies that

$$\mathrm{wt}\, v=r+s+k+\mathrm{Re}(\mu)(t-k)+i\mathrm{Im}(\mu)(t-k) \tag{19}$$

for $r,s,k,t\in\mathbb{N}$ with $k,t\in\mathbb{Z}_+$ if $r,s\in\mathbb{Z}_+$, respectively.

The analysis of the structure of ${}_{\mu}M$ naturally falls into the following five cases:

Case 1: $\mu=0,1$, i.e., $c=2$. Then, with respect to $L^{\mu}(0)$ weight grading, we have that $({}_0M=\bigoplus_{n=0}^{\infty}{}_0M_n,\omega_0)=({}_0M,\omega)$ is an $\mathbb{N}$-graded conformal vertex algebra with central charge 2. Moreover, the space of vectors of $L^{\mu}(0)=L(0)$-weight zero is equal to $\Omega({}_0M)$ and is given by

$${}_0M_0=\mathrm{Span}\{a^*(0)\cdots a^*(0)\mathbf{1}=a^*(0)^t\mathbf{1}\mid t\in\mathbb{N}\},$$

which is an infinite-dimensional subspace of ${}_0M$. Analogously,

$$\Omega({}_1M)={}_1M_0=\mathrm{Span}\{a(-1)\cdots a(-1)\mathbf{1}=a(-1)^k\mathbf{1}\mid k\in\mathbb{N}\}.$$

${}_0M$ is not a vertex operator algebra (or for that matter, an Ω-generated $\mathbb{C}$-graded vertex operator algebra) since it has infinite-dimensional weight spaces.

${}_0M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra, and the $L(0)$-weight spaces of ${}_0M$ are also the degree spaces of ${}_0M$ viewed either as an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra or as a $\mathbb{C}_{Re>0}$-graded module over itself.

${}_0M$ is referred to as the Weyl vertex algebra with central charge 2 and is the unique rank 1 Weyl conformal vertex algebra with central charge 2 up to isomorphism by Lemma 34.

Case 2: $\mu\in\mathbb{R}$ and $0<\mu<1$, i.e., $c\in\mathbb{R}$, and $-1<c<2$. In this case, we have $0<\mu=\mathrm{Re}(\mu)<1$ and $0<\mathrm{Re}(1-\mu)<1$, and so from Eq. (16), we have that ${}_{\mu}M$ (or equivalently ${}_{1-\mu}M$) is an $\mathbb{R}$-graded conformal vertex algebra and is Ω-generated with $\Omega({}_{\mu}M)={}_{\mu}M_0=\mathbb{C}\mathbf{1}$ and

$${}_{\mu}M=\bigoplus_{\substack{\lambda\in\mathbb{R}\\ \lambda\geq 0}}{}_{\mu}M_{\lambda}={}_{\mu}M(0)\bigoplus_{\substack{\lambda\in\mathbb{R}\\ \lambda>0}}{}_{\mu}M(\lambda),$$

where the degree spaces and weight spaces coincide. Furthermore, ${}_{\mu}M_{\lambda}={}_{\mu}M(\lambda)=0$ unless $\lambda\in(\mathbb{N}+\mu\mathbb{Z})\cap\mathbb{R}_+$, and in fact,

$$Spec_{{}_{\mu}M}L^{\mu}(0)=\{r+s+k+\mu(t-k)\mid r,s,k,t\in\mathbb{N},\text{and } k,t\in\mathbb{Z}_+ \text{ if } r,s\in\mathbb{Z}_+,\ \text{resp.}\}$$
$$=\mu\mathbb{N}+(1-\mu)\mathbb{N}.$$

In this case, we have $\dim {}_{\mu}M_{\lambda}<\infty$ since ${}_{\mu}M$ is Ω-generated by the finite-dimensional set $\Omega({}_{\mu}M)={}_{\mu}M_0=\mathbb{C}\mathbf{1}$ and the generating set $S=\{a(-1)\mathbf{1},a^*(0)\mathbf{1}\}$ of positive non-integral weights μ and $1-\mu$, respectively, between 0 and 1.

Finally, noting that $\mathrm{Re}(\lambda)\geq 0=|\mathrm{Im}(\lambda)|$ for all $\lambda\in Spec_{\mu M}L^{\mu}(0)$, we conclude that ${}_{\mu}M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

Case 3: $\mu\in\mathbb{C}$, $\mathrm{Im}(\mu)\neq 0$, and $\mathrm{Re}(\mu)=0$ or 1. Since ${}_{\mu}M\cong{}_{1-\mu}M$, without loss of generality, we may assume $\mathrm{Re}(\mu)=0$. Setting $\mu=iq$, then $1-\mu=1-iq$; from Eq. (16), we have that ${}_{\mu}M_0=\mathbb{C}\mathbf{1}$, and with respect to the weight grading given by $L^{\mu}(0)$ and denoted as $|u|$, we have that $\mathrm{Re}(|u|)>0$ unless $\mathrm{Re}(|u|)=0$, in which case $u\in\mathrm{Span}\{a^*(0)\cdots a^*(0)\mathbf{1}=a^*(0)^t\mathbf{1}\mid t\in\mathbb{N}\}$. However, in this case, $|a^*(0)^t\mathbf{1}|=tiq$. Thus, ${}_{\mu}M_0=\mathbb{C}\mathbf{1}=\Omega({}_{\mu}M)$ and the degree grading and $L^{\mu}(0)$-weight grading coincide. Furthermore, $\dim({}_{\mu}M_{\lambda})<\infty$ for each $\lambda\in\mathbb{C}$.

Thus, in this case, ${}_{\mu}M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra and, therefore, a $\mathbb{C}_{Re>0}$-graded module over itself. However, as a module over itself, ${}_{\mu}M$ has an infinite number of $\lambda \in Spec_{{}_{\mu}M} L^{\mu}(0)$ with $\mathrm{Re}(\lambda) = 0$ as we now show below.

Since the weight spaces of ${}_{\mu}M$ in this case are finite dimensional, one might think it is a candidate for an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. Here, the question is does it satisfy $\mathrm{Re}(\lambda) \geq |\mathrm{Im}(\lambda)|$ for all but finitely many $\lambda \in Spec_{{}_{\mu}M} L^{\mu}(0)$. Here, the answer is no since $|a^*(0)\mathbf{1}| = tiq = \lambda$ implies $\mathrm{Re}(\lambda) = 0 < t|q| = |\mathrm{Im}(\lambda)|$ for $t \neq 0$. Thus, ${}_{\mu}M$ for $\mu \in i\mathbb{R}$ is an example of an Ω-generated $\mathbb{C}$-graded conformal vertex algebra that is not an Ω-generated $\mathbb{C}$-graded vertex operator algebra even though $\dim({}_{\mu}M_{\lambda}) < \infty$.

Analogous results hold for $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) = 1$ by Lemma 34.

Case 4: $\mu \in \mathbb{C}$, $\mathrm{Im}(\mu) \neq 0$, and $0 < \mathrm{Re}(\mu) < 1$. Setting $\mu = p + iq$, then $1 - \mu = (1-p) + i(-q)$ and $\mathrm{Re}(1-\mu) = 1 - p$ satisfies $0 < \mathrm{Re}(1-\mu) < 1$. Thus, from Eq. (16), we have that ${}_{\mu}M_0 = \mathbb{C}\mathbf{1}$, and with respect to the weight grading given by $L^{\mu}(0)$ and denoted as $|u|$, we have that $\mathrm{Re}(|u|) > 0$ unless $\mathrm{Re}(|u|) = 0$, in which case $u \in \mathbb{C}\mathbf{1}$. Thus, ${}_{\mu}M$ is $\mathbb{C}_{Re>0}$-graded with $\Omega({}_{\mu}M) = \mathbb{C}\mathbf{1}$. Therefore, ${}_{\mu}M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra and the $L^{\mu}(0)$-weight spaces correspond to the degree spaces. More precisely,

$$\begin{aligned} Spec_{{}_{\mu}M} L^{\mu}(0) &= \{r + s + k + p(t-k) + iq(t-k) \mid r, s, k, t \in \mathbb{N}, \text{and } k, t \in \mathbb{Z}_+ \text{ if } r, s \in \mathbb{Z}_+, \text{ resp.}\} \\ &\subset p\mathbb{N} + (1-p)\mathbb{N} + iq\mathbb{Z}. \end{aligned}$$

In this case, we also have that $\dim V_{\lambda} < \infty$ for all $\lambda \in Spec_{{}_{\mu}M} L^{\mu}(0)$. To see this, we observe that if we consider only the real part of the weight grading for ${}_{\mu}M$, then the grading is the same as that for case 2. That is, for $v \in M$ with $L^{\mu}(0)v = \lambda v$, $L^{Re(\mu)}(0)v = \mathrm{Re}(\lambda)v$. Thus, $\dim {}_{\mu}M_{\lambda} \leq \dim {}_{Re(\mu)}M_{Re(\lambda)} < \infty$.

To analyze when ${}_{\mu}M$ is also an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra, we need to determine if $\mathrm{Re}(\lambda) \geq |\mathrm{Im}(\lambda)|$ for all but finitely many weights $\lambda \in Spec_{{}_{\mu}M} L^{\mu}(0)$.

Here, we split into two subcases:

Case 4(a): If either $0 < \mathrm{Re}(\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(\mu)$ or $0 < \mathrm{Re}(1-\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(1-\mu)$ hold, then we claim that ${}_{\mu}M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. We first prove this for $0 < \mathrm{Re}(\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(\mu)$ and then note that since ${}_{\mu}M \cong {}_{1-\mu}M$, the result will hold for $0 < \mathrm{Re}(1-\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(1-\mu)$.

Hence, assume that $0 < \mathrm{Re}(\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(\mu)$, i.e., writing $\mu = p + iq$, we have $0 < p \leq 1/2$ and $|q| < p$. Then, for any $\lambda = r + s + k + p(t-k) + iq(t-k) \in Spec_{{}_{\mu}M} L^{\mu}(0)$, we have that $|\mathrm{Im}(\lambda)| = |q(t-k)| \leq |p(t-k)| \leq r + s + k + |p(t-k)|$, whereas $|\mathrm{Re}(\lambda)| = |r + s + k + p(t-k)|$. Thus, if $t - k \geq 0$, we have $|\mathrm{Im}(\lambda)| \leq |\mathrm{Re}(\lambda)|$. If $t - k < 0$, then $k \neq 0$ and $\mathrm{Re}(\lambda) \geq k + p(t-k) = k(1-p) + pt$. However, since $(1-p) \geq p$, we have $\mathrm{Re}(\lambda) \geq kp + pt = p(k+t)$ and thus $|\mathrm{Im}(\mu)| = |q(t-k)| \leq |p(t-k)| \leq |p(t+k)| \leq |\mathrm{Re}(\lambda)|$. Therefore, $\mathrm{Re}(\lambda) \geq |\mathrm{Im}(\lambda)|$ for all weights $\lambda \in Spec_{{}_{\mu}M} L^{\mu}(0)$.

Therefore, we have that in this case, ${}_{\mu}M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

Case 4(b): If $0 < \mathrm{Re}(\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| > \mathrm{Re}(\mu)$ or if $0 < \mathrm{Re}(1-\mu) < 1/2$ and $|\mathrm{Im}(\mu)| > \mathrm{Re}(1-\mu)$, then we claim that ${}_{\mu}M$ is not an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. To see this, we first prove the result for $0 < \mathrm{Re}(\mu) \leq 1/2$, but $|\mathrm{Im}(\mu)| > \mathrm{Re}(\mu)$, and we then note that since ${}_{\mu}M \cong {}_{1-\mu}M$, the result will hold for $0 < \mathrm{Re}(1-\mu) < 1/2$ and $|\mathrm{Im}(\mu)| > \mathrm{Re}(1-\mu)$.

Hence, assume that $0 < \mathrm{Re}(\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| > \mathrm{Re}(\mu)$. Then, $|a^*(0)^t\mathbf{1}| = t\mu = \lambda$ for $t \in \mathbb{N}$. $|\mathrm{Im}(\lambda)| = |tIm(\mu)| > tRe(\mu) = \mathrm{Re}(\lambda)$. Thus, for an infinite number of $\lambda \in Spec_{{}_{\mu}M} L^{\mu}(0)$, $\mathrm{Re}(\lambda) \geq |\mathrm{Im}(\mu)|$ is not satisfied, and thus, ${}_{\mu}M$ is not an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

Case 5: If $\mathrm{Re}(\mu) > 1$ or $\mathrm{Re}(\mu) < 0$, then ${}_{\mu}M$ has nonzero weight spaces ${}_{\mu}M_{\lambda}$ with both $\mathrm{Re}(\lambda)$ arbitrarily large negative and arbitrarily large positive. This can be seen by considering that $|a(-1)^k\mathbf{1}| = k(1-\mu)$ and $|a^*(0)^t\mathbf{1}| = t\mu$ for $k, t \in \mathbb{N}$. This then implies that $\mathbf{1} \notin \Omega({}_{\mu}M)$ even though $\mathbf{1} \in {}_{\mu}M_0$ since if $\mathrm{Re}(\mu) > 1$, then $a(-1)\mathbf{1} = (a(-1))_{-1}\mathbf{1} \neq 0$ with $|a(-1)\mathbf{1}| = 1 - \mu$, but $\mathrm{Re}(1-\mu) - (-1) - 1 = \mathrm{Re}(1-\mu) < 0$. Or analogously, if $\mathrm{Re}(\mu) < 0$, then $a^*(0)\mathbf{1} = (a^*(0))_{-1}\mathbf{1} \neq 0$ with $|a^*(0)\mathbf{1}| = \mu$, but $\mathrm{Re}(\mu) - (-1) - 1 = \mathrm{Re}(\mu) < 0$. Thus, in this case, ${}_{\mu}M_0 \notin \Omega({}_{\mu}M)$.

In fact, in this case, $\Omega({}_{\mu}M) = 0$. To see this, without loss of generality, assume that $\mathrm{Re}(\mu) > 1$. Then, consider $v = a(-m_1 - 1)\cdots a(-m_k - 1)a^*(-n_1)\cdots a^*(-n_t)\mathbf{1} \in {}_{\mu}M$. We have that $v \notin \Omega({}_{\mu}M)$ since for $u = a(-1)\mathbf{1} \in {}_{\mu}M_{1-\mu}$, we have $(a(-1)\mathbf{1})_{-1}v = a(-1)v \neq 0$ even though $-1 \neq 1 - \mu - 1 = -\mu$ and $-1 > -\mathrm{Re}(\mu) = \mathrm{Re}(1-\mu) - 1$. Extending by linearity, it follows that $\Omega({}_{\mu}M) = 0$.

Therefore, in this case, ${}_{\mu}M$ is not an Ω-generated $\mathbb{C}$-graded vertex algebra. Thus, in particular, it is also not an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

In summary, we have that, independently of its conformal structure, the Weyl vertex algebra is always strongly finitely generated [see Remark 33 (2)]. However, as shown above, this vertex algebra is not necessarily Ω-generated (as this does depend on the conformal structure). Thus, we have the following theorem:

Theorem 35. *The $\mathbb{C}$-graded Weyl vertex algebra with grading given by the conformal element ω_{μ} for $\mu \in \mathbb{C}$, denoted as ${}_{\mu}M = (M, \omega_{\mu})$, is a finitely strongly generated $\mathbb{C}$-graded vertex algebra that is also a conformal $\mathbb{C}$-graded vertex algebra. Furthermore, we have the following:*

I. *${}_{\mu}M$ is a finitely Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra,* i.e., *it is in $\Omega VOA(\mathbb{C}_{Re>0}(\mathcal{V}))$, if and only if one of the following holds:*
   (i) $0 < \mathrm{Re}(\mu) \leq 1/2$ *and* $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(\mu)$
   or
   (ii) $0 < \mathrm{Re}(1-\mu) < 1/2$ *and* $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(1-\mu)$.

In addition, in this case I, we have that $\Omega({}_\mu M) = \mathbb{C}\mathbf{1}$.

II. ${}_\mu M$ *is both a finitely* Ω*-generated* $\mathbb{C}_{Re>0}$*-graded vertex algebra and a conformal* $\mathbb{C}$*-graded vertex algebra,* i.e., *it is in* $Conf(\mathbb{C}(\mathcal{V})) \cap \Omega(\mathbb{C}_{Re>0}(\mathcal{V}))$, *if and only if* $0 \leq \mathrm{Re}(\mu) \leq 1$.

III. *If* $\mathrm{Re}(\mu) > 1$ *or* $\mathrm{Re}(\mu) < 0$, *then* ${}_\mu M$ *is a finitely strongly generated* $\mathbb{C}$*-graded conformal vertex algebra, but it is not an* Ω*-generated* $\mathbb{C}$*-graded vertex algebra.*

Finally, outside of the strip given by $0 \leq \mathrm{Re}(\mu) \leq 1$, *we have* $\Omega({}_\mu M) = 0$, *and inside of the strip,* $\Omega({}_0 M)$ *and* $\Omega({}_1 M)$ *are infinite, whereas elsewhere inside this strip,* $\Omega({}_\mu M) = \mathbb{C}\mathbf{1}$. *See Fig. 1 for a visual representation of the regions where these results apply.*

Proof. Cases 1–5 above exhaust the possibilities for $\mu \in \mathbb{C}$, and in each case, it is shown that ${}_\mu M$ is both a conformal $\mathbb{C}$-graded vertex algebra and an Ω-generated $\mathbb{C}$-graded vertex algebra with generating set $\{a(-1)\mathbf{1}, a^*(0)\mathbf{1}\}$.

Cases 2 and 4(a) show that when $0 < \mathrm{Re}(\mu) \leq 1/2$ and $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(\mu)$ or when $0 < \mathrm{Re}(1-\mu) < 1/2$ and $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(1-\mu)$, ${}_\mu M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

Cases 1, 3, 4(b), and 5 show that in the remaining cases, ${}_\mu M$ is not an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra but is still both a conformal $\mathbb{C}$-graded vertex algebra and an Ω-generated $\mathbb{C}$-graded vertex algebra.

Finally, we note that by cases 1–5, we have that if $0 \leq \mathrm{Re}(\mu) \leq 1$ (cases 1–4), then we have that ${}_\mu M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra, but outside of this region (case 5), it is not.

$\Omega({}_\mu M)$ are as given in cases 1–5. □

Remark 36. We note that the Weyl vertex algebras in case 3, ${}_\mu M$, with μ being a purely imaginary nonzero number, provide a family of examples of conformal $\mathbb{C}_{Re>0}$-graded vertex algebras, which are not $\mathbb{C}_{Re>0}$-graded vertex *operator* algebras.

Remark 37. Theorem 35 shows a stark contrast between the Weyl vertex algebras (i.e., free bosonic ghosts) under conformal flow in comparison to free bosons under conformal flow. Recall (cf. Ref. 2) that free bosons $V_{bos} = \mathbb{C}[\alpha(-n) \mid n \in \mathbb{Z}_+]$ admit a family of conformal vectors $\omega_\nu = \frac{1}{2}\alpha(-1)^2\mathbf{1} + \nu\alpha(-2)\mathbf{1}$ for $\nu \in \mathbb{C}$, which endow V_{bos} with a vertex operator algebra structure of central charge $1 - 12\nu^2$. However, under this conformal flow, we have that $L^\nu(0)$, and thus, the $\mathbb{Z}$-grading and vertex operator algebra structure of V_{bos} do not change. In fact, the vertex operator algebra (V_{bos}, ω_ν) differs from $(V_{bos}, \omega_{\nu'})$ only in its representation theory in that some indecomposable nonirreducible modules can have a $L^\nu(0)$-action that is semisimple on a module with a 2×2 Jordan block in the vacuum space $\Omega(W)$ for some values of ν, whereas the action of $L^{\nu'}(0)$ is not semi-simple for some other $\nu' \neq \nu$. However, the non-semi-simplicity of V_{bos}-modules under conformal flow does not change, i.e., (V_{bos}, ω_ν) is always irrational, whereas by comparison, we will show in Sec. V C that $({}_\mu M, \omega_\mu)$ can be rational or irrational depending on μ.

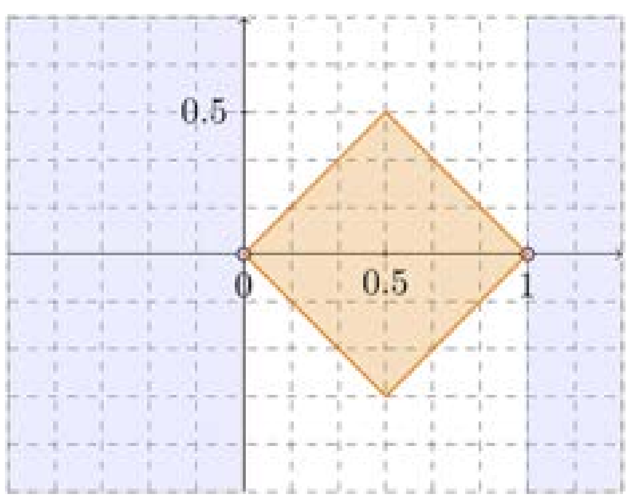

FIG. 1. Different $\mathbb{C}$-graded vertex algebra structures for ${}_\mu M$ under conformal flow. For $\mu \in \mathbb{C}$ satisfying Theorem 35, part I, i.e., for μ inside the diamond shaped region, ${}_\mu M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. In the regions where $\mathrm{Re}(\mu) > 1$ or $0 < \mathrm{Re}(\mu)$, ${}_\mu M$ is not an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra and, thus, is also not an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. In the remaining regions, ${}_\mu M$ has the structure of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra and a conformal vertex algebra *but not* of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

Remark 38. We note that the use of the term “rank” in the definition of the Weyl vertex algebra in Sec. III alludes to the number of pairs of fields of the form $a(z), a^*(z)$ [or pairs of $\beta(z), \gamma(z)$ fields in the physics literature] and not the rank of the vertex algebra in the sense of Ref. 30, which involves the action of the Virasoro algebra and is, instead, referred to as the central charge. Therefore, the rank n Weyl vertex algebra consists of the tensor product of n copies of the rank 1 Weyl vertex algebra. Its conformal structure is determined by the choice of the conformal structures associated with each of the tensor factors. Namely, the rank n Weyl vertex algebra ${}_{\mu_1}M \otimes \cdots \otimes {}_{\mu_n}M$ has conformal vector $\omega = \sum_{i=1}^n \omega_{\mu_i}$ [as in Eq. (9)] and central charge $c = \sum_i^n c_{\mu_i}$ [as in Eq. (10)].

IV. ZHU ALGEBRAS OF Ω-GENERATED $\mathbb{C}_{Re>0}$-GRADED VERTEX ALGEBRAS

In this section, we present some results from Ref. 5 on the (level zero) Zhu algebras of Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebras and the correspondence between simple modules for the Zhu algebra and simple $\mathbb{C}_{Re>0}$-graded modules for the vertex algebra. We remind the reader that in Ref. 5, our notion of Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra was called a $\mathbb{C}$-graded vertex algebra.

Let $V = (V, Y, \mathbf{1})$ be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra with grading $V = \oplus_{\lambda\in\mathbb{C}} V_\lambda$. For $u \in V_\lambda$, let $\overline{|u|}$ denote the ceiling of the real part of λ, i.e.,

$$\overline{|u|} := \min\{n \in \mathbb{Z} \mid n \geq \mathrm{Re}(\lambda) = \mathrm{Re}(|u|)\}.$$

Let

$$V^r := \mathrm{span}_{\mathbb{C}}\{v \in V \mid v \text{ is of homogeneous weight and } r = |v| - \overline{|v|}\}. \tag{20}$$

Then, we have that

$$V = \bigoplus_{r\in\mathbb{C}} V^r.$$

Remark 39. We observe that $u \in V^0$ if and only if $|u| = \overline{|u|}$ and, equivalently, if and only if $Lu = \overline{|u|}u$, where L is the operator defined in Eq. (2). We can then characterize V^0 as the vertex subalgebra of V consisting of all vectors of integer weight.

In Ref. 26, Zhu introduced an associative algebra, $A(V)$, associated with any vertex operator algebra V, which can be used to classify its irreducible representations. Laber and Mason in Ref. 5 studied the Zhu algebra associated with certain $\mathbb{C}$-graded vertex algebras by making the necessary modifications to the formulas introduced by Zhu. We will use the $\mathbb{C}$-graded Zhu algebra machinery to show that a particular family of Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebras are rational. We recall first the appropriate definition of the Zhu algebra in the $\mathbb{C}$-grading setting following Ref. 5.

Definition 40 (Ref. 5). Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra. Let $u \in V^r$ and $v \in V$. Define the products $\circ$ and $*$ on V as the linear extensions of the following equations:

$$u \circ v := \mathrm{Res}_z \frac{(1+z)^{\overline{|u|}+\delta_{r,0}-1}}{z^{1+\delta_{r,0}}} Y(u,z)v$$

and

$$u * v = \delta_{r,0}\,\mathrm{Res}_z \frac{(1+z)^{|u|}}{z} Y(u,z)v. \tag{21}$$

Define $O(V)$ to be the linear span of all elements of the form $u \circ v$ for $u, v \in V$.

Proposition 41 (Ref. 5). Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra, and define V^r as in Eq. (20). Then, we have the following:

(i) *If $r \neq 0$, then $V^r \subseteq O(V)$.*
(ii) *For $u \in V$ homogeneous, $(D + L)u \equiv 0 \bmod O(V)$, where L is the operator defined in Eq. (2).*
(iii) *For $u \in V^r$ homogeneous, $v \in V$, and any $m \geq n \geq 0$, we have*

$$\mathrm{Res}_z \frac{(1+z)^{\overline{|u|}+\delta_{r,0}-1+n}}{z^{1+\delta_{r,0}+m}} Y(u,z)v \in O(V).$$

(iv) *For $u, v \in V$ homogeneous, we have*

$$Y(u,z)v \equiv (1+z)^{-|u|-|v|} Y\left(v, -\frac{z}{1+z}\right)u \bmod O(V).$$

(v) *For $u, v \in V^0$ homogeneous, we have the identities*

$$u * v \equiv \mathrm{Res}_z \frac{(1+z)^{|v|-1}}{z} Y(v,z)u \bmod O(V)$$

and

$$u * v - v * u \equiv \mathrm{Res}_z (1+z)^{|u|-1} Y(u,z)v \bmod O(V).$$

(vi) *$O(V)$ is a two-sided ideal of V with respect to the $*$ product.*
(vii) *Define $A(V) := V/O(V)$. Then, $A(V)$ is an associative algebra with respect to the $*$ product.*

Remark 42. It follows from Proposition 41 (i) and (vii) that $A(V) = V^0/(O(V) \cap V^0)$.

Definition 43. For $v \in V$ being homogeneous, define the *zero mode $o(v)$ of v* as $o(v) = v_{|\overline{v}|-1}$. We extend this definition to all of V by linearity.

Remark 44. If $v \in V^r$ for some nonzero r, then $\overline{|v|} - 1 > Re(|v|) - 1$, and $o(v) = v_{\overline{|v|}-1}$ annihilates any element of $\Omega(V)$. In addition, if $v \in V^0$, then $|\overline{v}| = |v|$, and this definition of $o(v)$ reduces to the original zero mode definition given by Zhu in Ref. 26.

We conclude this section stating the expected correspondence between simple $A(V)$ modules and simple "admissible" V-modules in the $\mathbb{C}$-graded setting. We note that $\mathbb{C}_{Re>0}$-graded modules are the appropriate "admissible" representations in this context.

Proposition 45 (Ref. 5). Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra.

(i) *Let $W = W(0) \oplus_{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda)>0} W(\lambda)$ be a simple $\mathbb{C}_{Re>0}$-graded* V*-module with $\Omega(W) = W(0)$, as shown in Proposition 29. Then, $\Omega(W)$ is a simple $A(V)$-module.*
(ii) *There is a one-to-one correspondence between the categories of simple $A(V)$-modules and simple $\mathbb{C}_{Re>0}$-graded* V*-modules.*

V. RATIONALITY FOR CERTAIN $\mathbb{C}_{Re>0}$-GRADED VERTEX OPERATOR ALGEBRAS AND APPLICATIONS

In this section, we prove our main result on the rationality of finitely Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebras that are not $\mathbb{Z}$-graded and whose simple $\mathbb{C}_{Re>0}$-graded modules are all ordinary. We then apply this result to the Weyl vertex algebras with the central charges c_μ (or equivalently the conformal element ω_μ) that give $({}_\mu M, \omega_\mu)$ the structure of a $\mathbb{C}_{Re>0}$-graded vertex operator algebra.

The following theorem is analogous to Theorem 3.3 in Ref. 31 where a g-rationality for g-twisted modules of a vertex operator algebra V and for an automorphism g was studied. Here, we use the idea of their proof applied to the setting of Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebras.

Theorem 46. *Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra satisfying the following conditions:*

1. *Every simple $\mathbb{C}_{Re>0}$-graded V-module is an ordinary module.*
2. *$A(V)$ is a finite-dimensional semisimple associative algebra.*
3. *$\omega + O(V)$ acts via its zero mode $L(0)$ on all irreducible $A(V)$-modules as the same constant eigenvalue λ—that is, there is a fixed $\lambda \in \mathbb{C}$ such that for any $A(V)$-module U, U consists of generalized eigenvectors for $o(\omega) = L(0)$ with eigenvalue λ.*

Then, V is rational, i.e., every $\mathbb{C}_{Re>0}$-graded V*-module is completely reducible.*

Proof. Let W be a $\mathbb{C}_{Re>0}$-graded V-module. We will show that W is a completely reducible $\mathbb{C}_{Re>0}$-module by considering the following cases:

Case 1: $\Omega(W)$ is a simple $A(V)$-module, and W is generated by $\Omega(W)$. If $\tilde{W}$ is a V-submodule of W, then $\Omega(\tilde{W})$ is an $A(V)$-submodule of $\Omega(W)$, and therefore, by the assumption that $\Omega(W)$ is simple, $\Omega(\tilde{W})$ must be the trivial $A(V)$-module or $\Omega(W)$. Since W (and thus $\tilde{W}$) is generated by $\Omega(W)$, this implies that $\tilde{W} = 0$ or $\tilde{W}$ is generated by $\Omega(W)$ and is, thus, W, assuming $W \neq 0$. Therefore, W is a simple $\mathbb{C}_{Re>0}$-graded V-module.

Case 2: $\Omega(W)$ is not a simple $A(V)$-module, and W is generated by $\Omega(W)$. Since $\Omega(W)$ is not simple and $A(V)$ is finite-dimensional semisimple, $\Omega(W)$ is the direct sum of simple $A(V)$-modules, say, $\Omega(W) = \oplus_{i\in I} \Omega(W)^i$, where $\Omega(W)^i$ is simple for $i \in I$, for I some indexing set. Thus, if W is generated by $\Omega(W)$, we set $W^i := \langle \Omega(W)^i \rangle$ and we obtain $W = \oplus_{i\in I} W^i$, where each W^i is generated by the simple $A(V)$-module $\Omega(W)^i$ and is, thus, simple by the case 1 argument. Therefore, when W is generated by its lowest weight vectors, in this case, $\Omega(W)$, we have that W is a completely reducible $\mathbb{C}_{Re>0}$-graded V-module.

Case 3: W is not generated by $\Omega(W)$. We will show that W is completely reducible by further analyzing two subcases. First, we let $\tilde{W}$ be the submodule of W generated by $\Omega(W)$. Then, $\tilde{W}$ is completely reducible by case 2, i.e., $\tilde{W} = \oplus_{i\in I} \tilde{W}^i$ for $\tilde{W}^i$ irreducible. Thus,

$\tilde{W}(0) = \bigoplus_{i\in I}\tilde{W}^i(0) = \bigoplus_{i\in I}\Omega(\tilde{W}^i) = \Omega(W)$ by Proposition 29. Then, we have $\tilde{W} = \tilde{W}(0)\bigoplus_{\mu\in\mathbb{C},\mathrm{Re}(\mu)>0}\tilde{W}(\mu)$, with $\tilde{W}(0)$ being a generalized eigenspace for $L(0) = \omega_1^{\tilde{W}}$ with eigenvalue λ, so that $\mathrm{Spec}_{\tilde{W}(0)}L(0) = \lambda$. Moreover, we have that $\tilde{W}(0) = \Omega(W)$ generates $\tilde{W}$, and for $m \in \mathbb{Z}$ and $v \in V$, we have that $L(0)$ acts on $v_m\tilde{W}(0)$ via the $L(0)$-eigenvalue wt $v - m - 1 +$ wt w for $w \in \tilde{W}(0)$. Thus, $\tilde{W}$ is graded by $L(0)$-generalized eigenspaces with eigenvalues of the form $(\mathrm{Spec}_V L(0) + \lambda + \mathbb{N}) \cap \{\mu + \lambda \in \mathbb{C} \mid \mu \in \mathrm{Spec}_V L(0)\} = \mathrm{Spec}_V L(0) + \lambda$, i.e.,

$$\tilde{W} = \tilde{W}_\lambda \bigoplus_{\mu\in\mathbb{C},\ \mathrm{Re}(\mu)>0} \tilde{W}_{\lambda+\mu} = \tilde{W}_\lambda \bigoplus_{\substack{\mu\in\mathrm{Spec}_V L(0)\\ \mu\neq 0}} \tilde{W}_{\lambda+\mu},$$

with $\tilde{W}_\lambda = \tilde{W}(0)$ [where we have used the fact that $v_m\tilde{W}(0) = 0$ for $m > 0$ since $\tilde{W}(0) = \Omega(W)$]. Next, we consider the module $W/\tilde{W}$. Since we are assuming that W is not generated by $\Omega(W)$, we have that $W/\tilde{W} \neq 0$. This implies $\Omega(W/\tilde{W}) \neq 0$. We analyze the following two subcases to show that under the assumptions of case 3, W must be completely reducible:

Case 3I: Suppose that $W/\tilde{W}$ is completely reducible. Then, as above, $(W/\tilde{W})(0) = \Omega(W/\tilde{W})$ and every $w + \tilde{W} \in W/\tilde{W}$ is contained in some $L(0)$-generalized eigenspace, i.e., $w + \tilde{W} \in (W/\tilde{W})_{\lambda+\mu}$ for some $\mu \in \mathrm{Spec}_V L(0)$. However, then W itself is $[\mathrm{Spec}_V L(0) + \lambda]$-graded by $L(0)$-generalized eigenspaces. In addition, W_λ is an $A(V)$-module and, thus, completely reducible. Thus, the submodule of W generated by W_λ is completely reducible. Denote this by W'. However, then, if $W \neq W'$, there exist elements in $\Omega(W/W')$ that are not in W_λ and, thus, are in a generalized eigenspace for $L(0)$ of the form $\lambda + \mu$ for $\mu \neq 0$, which contradicts the fact that $\Omega(W/W')$ is an $A(V)$-module and, thus, in the λ generalized eigenspace for $L(0)$. Therefore, $W = W'$ and W is completely reducible.

Case 3II: Suppose that $W/\tilde{W}$ is not completely reducible. Then, replace $W/\tilde{W}$ with the submodule of $W/\tilde{W}$ generated by $\Omega(W/\tilde{W})$, which is $U/\tilde{W}$ for some submodule U of W. Then, by the argument above, since $U/\tilde{W}$ is completely reducible, every $u + \tilde{W} \in U/\tilde{W}$ is contained in some $L(0)$-generalized eigenspace, i.e., $u + \tilde{W} \in (U/\tilde{W})_{\lambda+\mu}$ for some $\mu \in \mathrm{Spec}_V L(0) \setminus 0$. However, then, U itself is $[\mathrm{Spec}_V L(0) + \lambda]$-graded by $L(0)$-generalized eigenspaces. In addition, U_λ is an $A(V)$-module and, thus, completely reducible. Thus, the submodule of W generated by U_λ is completely reducible. Denote this by U'. However, then, if $U \neq U'$, there exist elements in $\Omega(U/U')$ that are not in U_λ and, thus, are in a generalized eigenspace for $L(0)$ of the form $\lambda + \mu$ for $\mu \neq 0$, which contradicts the fact that $\Omega(U/U')$ is an $A(V)$-module and, thus, in the λ generalized eigenspace for $L(0)$. Therefore, $U = U'$ and is completely reducible.

Finally, we will show below that $\tilde{W}$ the submodule of W generated by $\Omega(W)$ is a maximal completely reducible submodule of W. This would then imply that $\tilde{W} \subset U \subset \tilde{W}$, implying that $\Omega(W/\tilde{W})$ generates the trivial module $\tilde{W}/\tilde{W}$, and thus, it must be the trivial $A(V)$-module, which implies $W = \tilde{W}$, and so W is completely reducible.

Therefore, we only have left to show that $\tilde{W}$ is a maximal completely reducible submodule of W. Indeed, if U is also a maximal completely reducible submodule, then both $\tilde{W}$ and U are generated by $\Omega(W)$ and, thus, equal.

This completes the proof. □

Remark 47. Note that in the proof above, one of the key facts used repeatedly is that for the class of $\mathbb{C}$-graded vertex algebras that we are working with, namely, Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebras, we have that $\Omega(W) = W(0)$ for all simple $\mathbb{C}_{Re>0}$-graded V-modules W, i.e., Proposition 29 (ii) holds.

A. Filtration of Zhu algebras

In Ref. 26, Zhu introduced two associative algebras related to a vertex operator algebra V, the Zhu algebra $A(V)$ and the C_2 algebra $V/C_2(V)$. Moreover, to prove that $V/C_2(V)$ is a Poisson algebra, Zhu built a filtration further studied and generalized by Li in Ref. 27. In this section, we use analogous constructions to describe $A(V)$ in the $\mathbb{C}_{Re>0}$-grading setting.

Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra with grading $V = \bigoplus_{\mu\in\mathbb{C}} V_\mu$. We continue with the notation from Sec. IV and, in particular, of $\overline{|u|}$ for the ceiling of the real part of μ if $u \in V_\mu$ and V^r for the set of all elements $u \in V$ with $r = |u| - \overline{|u|}$. In addition, recall that $V = \bigoplus_{r\in\mathbb{C}} V^r$, and for $r \neq 0$, we have $V^r \subseteq O(V)$. Observe that

$$A(V) = V/O(V) = V^0 + O(V)/O(V) = \bigoplus_{n\in\mathbb{Z}} V_n + O(V)/O(V) \tag{22}$$

by Remarks 39 and 42.

Now, we further assume that the integer grading for V^0 is bounded below. For the purposes of the exposition, we write $V^0 = \bigoplus_{n\in\mathbb{N}} V_n$ although the results will follow with little modification if $V^0 = \bigoplus_{n=N}^\infty V_n$ for some $N \in \mathbb{Z}$. Consider the filtration $\{F_t A(V)\}_{t\in\mathbb{N}}$ where

$$F_t A(V) = \bigoplus_{j=0}^t V_j + O(V)/O(V) \subset A(V).$$

From the definition of $*$, i.e., Eq. (21), and by taking the residue, we have that for $u \in V^r, v \in V$,

$$u * v = \delta_{r,0} \operatorname{Res}_z \frac{(1+z)^{|u|}}{z} Y(u,z)v = \delta_{r,0} \sum_{i \geq 0} \binom{|u|}{i} u_{i-1} v. \tag{23}$$

In addition, note that for homogeneous elements $u, v \in V$, we have $|u_{i-1}v| = |u| + |v| - i$. Hence, we can conclude that

$$F_s A(V) * F_t A(V) \subseteq F_{s+t} A(V) \qquad \text{for } s,t \in \mathbb{N}. \tag{24}$$

Letting $F_{-1}A(V) = 0$, we define an $\mathbb{N}$-grading on $A(V)$ by

$$gr_t A(V) = F_t A(V)/F_{t-1}A(V) \quad \text{for } t \in \mathbb{N}$$

so that

$$grA(V) = \bigoplus_{t=0}^{\infty} gr_t A(V).$$

Observe that by Eq. (24), the multiplication of $A(V)$ induces an associative multiplication on the graded vector space $grA(V)$. In addition, we have the following lemma.

Lemma 48. Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra such that $V^0 = \oplus_{n=0}^{\infty} V_n$. Then, $grA(V)$ is a commutative associative algebra.

Proof. Let $u \in V^r$, $v \in V^s$. By the fact that $V^r \subset O(V)$ if $r \neq 0$, i.e., Proposition 41(i) if either $r \neq 0$ or $s \neq 0$, then $u * v = 0 = v * u$ since either u or v is in $O(V)$. Therefore, assume that $u, v \in V^0$ are homogeneous elements, and $u \in F_t A(V)$, $v \in F_s A(V)$. Since by Proposition 41(v),

$$\begin{aligned} u * v - v * u &= Res_z (1+z)^{|u|-1} Y(u,z)v \bmod O(V) \\ &= Res_z \sum_{i \geq 0} \binom{|u|-1}{i} z^i Y(u,z)v \bmod O(V) \\ &= Res_z \sum_{i \geq 0} \binom{|u|-1}{i} u_i v \bmod O(V) \end{aligned}$$

and $|u_i v| = |u| + |v| - i - 1$, we can conclude that modulo $O(V)$, $u * v - v * u \in F_{s+t-1}A(V)$, i.e., is zero in $gr_{t+s}A(V)$. Hence, $gr(V)$ is commutative. □

Remark 49. $grA(V)$ is isomorphic to $A(V)$ as a vector space.

Next, we study an upper bound for $\dim A(V)$ when V is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra. For simplicity, we write $[u]$ for $u + O(V)$. Consider the linear epimorphism

$$\begin{aligned} f : V &\longrightarrow grA(V) \\ u &\mapsto [u] + F_{k-1}A(V) \qquad \text{for } u \in V_k. \end{aligned} \tag{25}$$

Note that if $u \in V^r$ and $r > 0$, then $f(u) = [0] + F_{k-1}A(V)$. Consequently, $u \in Ker(f)$.

Now, let u, v be homogeneous elements in V.

Case 1: $u \in V^0$.

Since $u \circ v = \operatorname{Res}_z \frac{(1+z)^{\overline{|u|}}}{z^2} Y(u,z)v = \sum_{i \geq 0} \binom{\overline{|u|}}{i} u_{i-2} v \in O(V)$ and

$$|u_{j-2}v| = |u| + |v| - j + 1 \leq |u| + |v| \text{ when } j \geq 1,$$

we have

$$f(u_{-2}v) = [u_{-2}v] + F_{|u|+|v|}A(V) = [0] + F_{|u|+|v|}A(V).$$

Moreover, $u_{-2}v \in Ker(f)$.

Case 2: $u \in V^r$ such that $r > 0$.

Since $u \circ v = \operatorname{Res}_z \frac{(1+z)^{\overline{|u|}-1}}{z} Y(u,z)v = \sum_{j \geq 0} \binom{\overline{|u|}-1}{j} u_{j-1} v \in O(V)$ and

$$|u_{j-1}v| = |u| + |v| - j < |u| + |v| \quad \text{for all } j \geq 1,$$

these imply that

$$f(u_{-1}v) = [u_{-1}v] + F_{|u|+|v|-1}A(V) = [0] + F_{|u|+|v|-1}A(V).$$

In addition, $u_{-1}v \in Ker(f)$.

In conclusion, we have the following theorem.

Theorem 50. *Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra with integer graded part V^0 as in Eq. (20) satisfying $V^0 = \bigoplus_{n=0}^{\infty} V_n$. Let $f : V \to grA(V)$ be defined by (25). Set*

$$C(V) = Span_{\mathbb{C}}\{a, u_{-2}v, b_{-1}w \mid a, b \in V^r \, with \, r \neq 0, \; u \in V^0, \; and \; v, w \in V\}.$$

Then, $C(V) \subseteq Ker(f)$. In addition, f induces a linear epimorphism $\bar{f}$ from $V/C(V)$ to $grA(V)$, and therefore, $\dim A(V) \leq \dim V/C(V)$.

Recall that an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra V is endowed with the endomorphism D as defined in Remark 5. Using the fact that $(Du)_n = -nu_{n-1}$ for $u \in V$, $n \in \mathbb{Z}$, we have the following corollary:

Corollary 51. Let V be a $\mathbb{C}_{Re>0}$-graded vertex algebra such that $V^0 = \bigoplus_{n=0}^{\infty} V_n$. Then, we have the following:

1. *For $u \in V^0$, $v \in V$, we have $u_{-n}v \in C(V)$ for all $n \geq 2$.*
2. *For $b \in V^r$ with $r \neq 0$ and $w \in V$, then $b_{-m}w \in C(V)$ for all $m \geq 1$.*

Remark 52. It is necessary to assume that $V^0 = \bigoplus_{n\in\mathbb{Z}} V_n$ is bounded below. Otherwise, the theorem above is false. For instance, when L is a nondegenerate non-positive definite even lattice of an arbitrary rank, it was shown in Ref. 32 that $A(V_L^+) \neq 0$ and in Ref. 33 that $\dim(V_L^+/C(V_L^+)) = 0$. In that context, $C(V_L^+) = C_2(V_L^+)$, the C_2 space originally defined by Zhu[26] for $\mathbb{Z}$-graded vertex algebras. However, as noted earlier, it is enough to just assume that the grading for V^0 is bounded from below, not necessarily by zero.

B. Main results on rationality for certain $\mathbb{C}_{Re>0}$-graded vertex operator algebras

In this section, we use the construction of the Zhu algebra $A(V)$ presented above to prove that under some mild conditions, an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra admits only one $\mathbb{C}_{Re>0}$-graded simple module.

Theorem 53. *Let V be a finitely Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra generated as in Remark 11 by $u^1, \ldots, u^k$. Let V^0 be the integer graded part of V as in Eq. (20). Assume that we have the following:*

1. *For $j \in \{1, \ldots, k\}$, $|u^j|$ is not an integer. Namely, the strong generators satisfy $u^j \in V \setminus V^0$ for $1 \leq j \leq k$.*
2. *$V^0 = \bigoplus_{n=0}^{\infty} V_n$.*

Then, $\dim A(V) = 1$ *and $A(V) \cong \mathbb{C}$.*

Proof. By Corollary 51, using that $u^j \in V^r$ for $r \neq 0$, we can conclude that $u^{j_1}_{-n_1} \cdots u^{j_t}_{-n_t}\mathbf{1} \in C(V)$ for all $u^{j_i} \in \{u^1, \ldots, u^k\}$, $n_i \geq 0$. Hence, $V/C(V) = \mathbb{C}\mathbf{1} + C(V)$. Moreover, in light of Theorem 50, this implies that $\dim A(V) = 1$ and $A(V) \cong \mathbb{C}$ as desired. □

Theorem 54. *Let V be a finitely Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra that is finitely generated by $u^1, \ldots, u^k$ as in Remark 11 and, in addition, satisfies the following:*

1. *For each $j \in \{1, \ldots, k\}$, $|u^j|$ is not an integer.*
2. *$V^0 = \bigoplus_{n=0}^{\infty} V_n$.*
3. *Every simple $\mathbb{C}_{Re>0}$-graded V-module is ordinary.*

Then, V is rational and has only one simple $\mathbb{C}_{Re>0}$-graded V-module.

Proof. Since V is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra, using Theorem 53, we have that $\dim A(V) = 1$. By Proposition 45 (ii), we can conclude that V has only one simple $\mathbb{C}_{Re>0}$-graded V-module. Finally, by Theorem 46 and condition (3), we can conclude that V is rational. □

C. On rationality of $\mathbb{C}_{Re>0}$-graded Weyl vertex operator algebras

In this section, we apply Theorems 46 and 53 to Weyl vertex algebras ${}_{\mu}M$ with certain conformal structures as classified in Theorem 35 to prove the rationality of ${}_{\mu}M$ for those values of μ that give ${}_{\mu}M$ the structure of an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra, including, for instance, when $0 < \mathrm{Re}(\mu) < 1$ and $Im(\mu) = 0$, which corresponds to the case of the central charge c real and in the range $-1 < c < 2$.

Theorem 55. *Let $\mu \in \mathbb{C}$ such that one of the following holds:*

(i) $0 < Re(\mu) \le 1/2$ *and* $|Im(\mu)| \le Re(\mu)$
or
(ii) $0 < Re(1-\mu) < 1/2$ *and* $|Im(\mu)| \le Re(1-\mu)$.

Then, $({}_{\mu}M, \omega_{\mu})$ is a rational Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra and has only one simple $\mathbb{C}_{Re>0}$-graded module, which is, in fact, a simple ordinary ${}_{\mu}M$-module, namely, ${}_{\mu}M$ itself.

Proof. Theorem 35 implies that ${}_{\mu}M$ is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex operator algebra. Theorem 53 and the fact that $|a(-1)\mathbf{1}| = 1-\mu$ and $|a^*(0)\mathbf{1}| = \mu$ imply that $A({}_{\mu}M) \cong \mathbb{C}$, and thus, ${}_{\mu}M$ has only one irreducible $\mathbb{C}_{Re>0}$-graded module. Since ${}_{\mu}M$ is a $\mathbb{C}_{Re>0}$-graded irreducible module over itself, we conclude that the only simple $\mathbb{C}_{Re>0}$-module is ${}_{\mu}M$. We observe that, in fact, ${}_{\mu}M$ is an ordinary ${}_{\mu}M$-module as well. Thus, by Theorem 46, we have that ${}_{\mu}M$ is rational. □

Corollary 56. For $i \in \{1, \ldots, n\}$, if $\mu_i \in \mathbb{C}$ and one of the following holds for each μ_i,

(i) $0 < Re(\mu_i) \le 1/2$ *and* $|Im(\mu_i)| \le Re(\mu_i)$
or
(ii) $0 < Re(1-\mu_i) < 1/2$ *and* $|Im(\mu_i)| \le Re(1-\mu_i)$.

Then, $({}_{\mu_1}M, \omega_{\mu_1}) \otimes \cdots\cdots \otimes ({}_{\mu_n}M, \omega_{\mu_n})$ is rational.

Proof. This follows immediately from Theorem 55. □

For more general values of μ, namely, in the range $0 \le \mathrm{Re}(\mu) \le 1$, but not necessarily in the subregion defined by (i) and (ii) in Theorem 55 and Corollary 56, we do not necessarily obtain a $\mathbb{C}_{Re>0}$ graded vertex *operator* algebra structure on ${}_{\mu}M$, but we still have that ${}_{\mu}M$ is a finitely Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra (see Theorem 35 Case II). We conclude this section by showing that these families of Weyl vertex algebras admit only one irreducible $\mathbb{C}_{Re>0}$-graded simple module.

Theorem 57. *Let $\mu \in \mathbb{C}\backslash\{0,1\}$ be such that $0 \le \mathrm{Re}(\mu) \le 1$. Then, the Weyl vertex algebra ${}_{\mu}M$ admits a unique, up to isomorphism, irreducible $\mathbb{C}_{Re>0}$-graded module, which is ${}_{\mu}M$ itself.*

Proof. By Theorem 35 (II), for $\mu \in \mathbb{C}$ such that $0 \le \mathrm{Re}(\mu) \le 1$, the Weyl vertex algebra ${}_{\mu}M$ is a finitely Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra. Moreover, because $\mu \ne 0, 1$, we have from Eqs. (17) and (18) that the strong generators of ${}_{\mu}M$ have non-integer degree so that condition (1) of Theorem 53 is satisfied. In addition, since $\Omega({}_{\mu}M) = \mathbb{C}\mathbf{1}$, it is clear that condition (2) of the Theorem also holds. Therefore, we obtain that $A({}_{\mu}M) \cong \mathbb{C}$. Finally, Proposition 45 (2) implies that ${}_{\mu}M$ admits only one irreducible $\mathbb{C}_{Re>0}$-graded module, which must be ${}_{\mu}M$ itself. □

Remark 58.

1. We note that in light of Theorem 57, we obtain a family of conformal vertex algebras in which the Zhu algebra is one dimensional. In particular, the class of the conformal vector $[\omega] \in A(V)$ must be a multiple of the class of the vacuum vector $[\mathbf{1}]$ as in the classical setting of vertex operator algebras constructed from self-dual lattices.[34]
2. The Weyl vertex algebras admit many non-isomorphic irreducible *weak* modules such as the relaxed highest weight modules studied in Ref. 21. We note, however, that those modules are independent of the conformal structure on the Weyl vertex algebra and are not $\mathbb{C}_{Re>0}$-modules because they have infinite-dimensional graded components. In particular, they are not "admissible" modules, namely, modules induced from the level zero Zhu algebra and, thus, possessing a $\mathbb{C}_{Re>0}$-grading. Other such examples of (non-admissible) non-isomorphic weak modules are the (generalized) Whittaker modules, for which the reducibility was studied in Refs. 24 and 25.

VI. SUMMARY OF APPLICATIONS AND FUTURE WORK

In this work, we classified the $\mathbb{C}$-graded conformal structures associated with the Weyl vertex algebra. Moreover, we showed that a large family of these vertex algebras admits a unique irreducible "admissible" module in the appropriate sense. We also described in detail which families of Weyl vertex algebras admit the $\mathbb{C}$-graded notion of a vertex operator algebra and proved that non-integer $\mathbb{C}$-graded Weyl vertex operator algebras are rational. In the literature, the Weyl vertex algebra at central charge 2 has been studied in detail (see, for instance, Refs. 21–23 and 29). This vertex algebra, ${}_0M$ in our notation, is not a vertex operator algebra because its graded components fail to be finite dimensional. Linshaw showed in Ref. 29 that the (level zero) Zhu algebra $A({}_0M)$ is isomorphic to the rank one Weyl algebra $\mathcal{A}_1$. Higher level Zhu algebras introduced by Dong, Li, and Mason in Ref. 35, can be used to study indecomposable nonirreducible modules. Using the theory and methods developed by Barron, along with Vander Werf and Yang in Refs. 36 and 37, and by Addabbo and Barron in Refs. 38 and 39, preliminary calculations by Addabbo together with the authors of the current paper indicate that the level one Zhu algebra for this Weyl vertex algebra satisfies

$$A_1({}_0M) \cong \mathcal{A}_1 \oplus (\mathcal{A}_1 \otimes Mat_2(\mathbb{C})),$$

with $\mathcal{A}_1$ being the rank one Weyl algebra. In particular, the injective image of the level zero Zhu algebra $\mathcal{A}_1$ inside the level one Zhu algebra has a direct sum complement, namely, $\mathcal{A}_1 \otimes Mat_2(\mathbb{C})$, and this complement is Morita equivalent to the level zero Zhu algebra $\mathcal{A}_1$. Therefore, there are no new $\mathbb{N}$-gradable ${}_0M$-modules detected by the level one Zhu algebra for ${}_0M$ that were not already detected by the level zero Zhu algebra. Thus, this agrees with the work of Ref. 23 on category $\mathcal{F}$ as discussed in the Introduction. Although this shows that the structure of the level one Zhu algebra gives no new information for the admissible ${}_0M$-modules, we expect that the study of higher level Zhu algebras for ${}_0M$ and in the more general $\mathbb{C}$-graded setting will shed light on the difficult open problem of describing the Zhu algebra for an orbifold vertex algebra in which twisted modules are expected to be detected.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Katrina Barron: Conceptualization (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Karina Batistelli**: Conceptualization (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Florencia Orosz Hunziker**: Conceptualization (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Veronika Pedić Tomić**: Conceptualization (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Gaywalee Yamskulna**: Conceptualization (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: PROOF OF LEMMA 14

We will prove that if V is an Ω-generated $\mathbb{C}_{Re>0}$-graded vertex algebra, then it satisfies the following:

For $r \geq 1$, $v^1, \ldots, v^r$ homogeneous elements in V, $n_1, \ldots, n_r$ integers, and u^0 being a vector in $\Omega(V)$ such that

$$v^r_{n_r} v^{r-1}_{n_{r-1}} \cdots v^1_{n_1} u^0 \neq 0,$$

either

$$\sum_{j=1}^{r} (|v^j| - n_j - 1) = 0 \qquad \text{or} \qquad \mathrm{Re}\left(\sum_{j=1}^{r} (|v^j| - n_j - 1)\right) > 0.$$

Proof. We prove the proposition by induction on r. If $r = 1$ and $v^1_{n_1} u^0 \neq 0$, then because $u^0 \in \Omega(V)$, we have that either $n_1 = |v^1| - 1$ or $n < \mathrm{Re}(|v^1| - 1)$. Equivalently, either $|v^1| - n_1 - 1 = 0$ or $\mathrm{Re}(|v^1| - 1 - n) > 0$, so the proposition holds for $r = 1$.

Next, assume that $r \geq 2$ and that

$$v^r_{n_r} v^{r-1}_{n_{r-1}} \cdots v^1_{n_1} u^0 \neq 0.$$

Using the inductive hypothesis on

$$v^{r-1}_{n_{r-1}} \cdots v^1_{n_1} u^0,$$

we know that either

$$\sum_{j=1}^{r-1} (|v^j| - n_j - 1) = 0 \qquad \text{or} \qquad \mathrm{Re}\left(\sum_{j=1}^{r-1} (|v^j| - n_j - 1)\right) > 0.$$

We consider the following two cases:

Case 1: If either $|v^r| - n_r - 1 = 0$ or $\mathrm{Re}(|v^r| - n_r - 1) > 0$, we can conclude immediately that either $\sum_{j=1}^{r} (|v^j| - n_j - 1) = 0$ or $\mathrm{Re}\left(\sum_{j=1}^{r} (|v^j| - n_j - 1)\right) > 0$ and we are done with this case.

Case 2: If $|v^r| - n_r - 1 \neq 0$ and $\mathrm{Re}(|v^r| - n_r - 1) \leq 0$, before presenting the proof of the lemma in this case, we recall the commutator formula [cf. Eq. (3.1.9) in Ref. 28], which holds for $n, m \in \mathbb{Z}$ and any two elements v, v' in a vertex algebra,

$$[v_n, v'_m] = \sum_{i \geq 0} \binom{n}{i} (v_i v')_{m+n-i}. \tag{A1}$$

Using (A1), we can rewrite

$$\begin{aligned} &v^r_{n_r} v^{r-1}_{n_{r-1}} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0 \\ &\quad = v^{r-1}_{n_{r-1}} v^r_{n_r} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0 + \sum_{i \geq 0} \binom{n_r}{i} (v^r_i v^{r-1})_{n_r + n_{r-1} - i} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0. \end{aligned} \tag{A2}$$

We further analyze the following two subcases:

Case 2.I: There exists $i \geq 0$ such that $(v^r_i v^{r-1})_{n_r + n_{r-1} - i} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0 \neq 0$. By the inductive hypothesis, we have that

$$|v^r_i v^{r-1}| - n_r - n_{r-1} + i - 1 + \sum_{j=1}^{r-2} (|v^j| - n_j - 1) = 0$$

or

$$\mathrm{Re}\left(|v^r_i v^{r-1}| - n_r - n_{r-1} + i - 1 + \sum_{j=1}^{r-2} (|v^j| - n_j - 1)\right) > 0.$$

Using Remark 3 (2), we have that $|v^r_i v^{r-1}| = |v^r| + |v^{r-1}| - i - 1$, so we can conclude that either $\sum_{j=1}^{r} (|v^j| - n_j - 1) = 0$ or $\mathrm{Re}\left(\sum_{j=1}^{r} (|v^j| - n_j - 1)\right) > 0$, and the lemma holds in case 2.I.

Case 2.II: For all $i \geq 0$, $(v^r_i v^{r-1})_{n_r + n_{r-1} - i} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0 = 0$. Then, by (A2), we have that

$$v^r_{n_r} v^{r-1}_{n_{r-1}} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0 = v^{r-1}_{n_{r-1}} v^r_{n_r} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0.$$

Using the commutator formula (A1) again on the right-hand side of the equation above, we get

$$v^{r-1}_{n_{r-1}} \quad v^r_{n_r} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0 = v^{r-1}_{n_{r-1}} \left(v^{r-2}_{n_{r-2}} v^r_{n_r} v^{r-3}_{n_{r-2}} \cdots v^1_{n_1} u^0 + \sum_{i\geq 0} \binom{n_r}{i} (v^r_i v^{r-2})_{n_r+n_{r-2}-i} v^{r-3}_{n_{r-3}} \cdots v^1_{n_1} \right) u^0.$$

If there exists $i \geq 0$ such that $v^{r-1}_{n_{r-1}} (v^r_i v^{r-2})_{n_r+n_{r-2}-i} v^{r-3}_{n_{r-3}} \cdots v^1_{n_1} u^0 \neq 0$ using the inductive hypothesis, we have that either

$$\sum_{j=1}^{r} (|v^j| - n_j - 1) = 0 \qquad \text{or} \qquad \mathrm{Re}\left(\sum_{j=1}^{r} (|v^j| - n_j - 1) \right) > 0.$$

Moreover, this reasoning applies as long as there exists $1 < j < r$ and $i \geq 0$ such that

$$v^{r-1}_{n_{r-1}} v^{r-2}_{n_{r-2}} \cdots (v^r_i v^{r-j})_{n_r+n_{r-j}-i} \cdots v^1_{n_1} u^0 \neq 0.$$

To finish the proof, we show that there must exist such j and i: Otherwise, the commutator formula applied r times implies that

$$v^r_{n_r} v^{r-1}_{n_{r-1}} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} u^0 = v^{r-1}_{n_{r-1}} v^{r-2}_{n_{r-2}} \cdots v^1_{n_1} v^r_{n_r} u^0 \neq 0.$$

In particular, $v^r_{n_r} u^0 \neq 0$, which contradicts the fact that $u^0 \in \Omega(V)$ since by assumption $|v^r| - n_r - 1 \neq 0$ and $\mathrm{Re}(|v^r| - n_r - 1) \leq 0$ in case 2. Therefore, the lemma holds in case 2.II. □

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